A STUDY OF STABILITY AND REDUCED ORDER APPROXIMATIONS OF CONTINUOUS-TIME LINEAR SYSTEMS VIA ORTHOGONAL POLYNOMIAL MATRICES

A DOCTORAL DISSERTATION
PRESENTED TO OSAKA UNIVERSITY

BY
HIROSHI NAGAOKA

MARCH, 1987

CONTENTS

ACKNOWLEDGEMENTS

My first thanks go to Professor S. Arimoto, whose interest in the Levinson algorithm and related topics inspired me to investigate the continuous-time counterparts of them. He constantly encouraged me in my studies. I am also indebted to Professor Y. Monden for teaching me the importance of orthogonal polynomial matrices in the linear system theory and for many useful comments.

I would like to thank Professor N. Suda and Professor Y. Sakawa for critically reading the manuscript. I am quite grateful to Professor H. Kimura for his warm advice on my research. I also thank Dr. Y. Inouye for valuable discussions.

Finally, I wish to express my sincere gratitude to many members of the laboratory, where I spent four years, for sustaining me in the endeavour to write up the thesis. I would particularly like to thank Dr. S. Kawamura for his frienship, Mr. H. Suzuki and Mr. T. Noda for teaching me patiently how to use a word processor.

1. INTRODUCTION

A linear time-invariant system is represented by its transfer function denoted as H(s) in the continuous-time (abbreviated to "c.t.") case and as H(z) in the discrete-time (abbreviated to "d.t.") case. The transfer function has two different structures. One is concerned with the 'form' of H(s) and is called the "algebraic structure". State space representations, matrix fraction descriptions, the McMillan degree, the controllability and the observability, etc. are algebraic notions. The other is the structure of H(s) as a complex function, which characterizes the frequency response of the system, and is called the "analytic structure". It is the algebraic structure that can be accessed directly, and therefore we must investigate the mutual relationship between the two structures in order to access the analytic structure. The aim of the stability theory for linear systems is to study the notion of stability, which belongs to the analytic structure, in terms of algebraic notions. On the other hand, in a reduced order approximation problem (or model reduction problem) it is required to find a system $\hat{H}(s)$ 'near to' a given system H(s) in a criterion, which is usually defined in terms of analytic notions, under the algebraic restriction that $\hat{H}(s)$ should have a prescribed order. Thus we can see that the stability theory and a reduced order approximation problem treat similar situations.

In the d.t. theory, it is widely recognized that the notion of "orthogonal polynomial matrices (abbreviated to "orthogonal

PM's") on the unit circle" provides us with a theoretical framework unifying several different problems. The notion, which is equivalent to the AR (auto-regressive) fitting of a stationary stochastic process, has a long history in the linear filtering theory ([16]), not only because the AR fitting problem is practically important, but also because the notion is deeply concerned with the basis of the theory through the so-called innovation approach. Furthermore, the notion is closely related to the d.t. stability of PM's. Indeed, it is well known that the LWR (Levinson-Whittle-Wiggins-Robinson) algorithm, which is a matrix version of the Levinson-Durbin algorithm, for generating the orthogonal PM's yields a stability criterion for PM's (see, for instance, [5],[10] and [20]). In particular the Levinson-Durbin algorithm is equivalent to the Schur-Cohn stability test. The notion is also related to reduced order approximation problems because the AR fitting itself is an approximation. In addition, the work of Mullis and Roberts [21] on some approximation problems can be regarded as an extension of the AR fitting method to the ARMA case, and can be situated in the field of orthogonal PM's on the unit circle as a variation.

In the c.t. case, we can define the notion of "orthogonal PM's on the imaginary axis" correspondingly. However, owing to the fact that the roles occupied by the powers $\{z^k; k=0,1,...\}$ in the d.t. theory are shared between $\{s^k; k=0,1,...\}$ and $\{e^{ts}; t\geq 0\}$ in the c.t. theory, the notion has been given very little opportunity for playing an active role so far. Indeed, we might say that the notion is useless in the basic fields of the linear filtering theory from which the notion of orthogonal PM's on the unit circle was developed. But this does not deny the possibility that the notion is used effectively in some other fields. The aim of this thesis is to establish the usefulness of the notion in the c.t. theory of stability and reduced order approximations.

The thesis is organized in three major parts of some chapters each. In Part I (Chap.2-Chap.5), the stability of PM's is studied in the framework of orthogonal PM's chiefly for the c.t. case. Our guiding principle is to seek the c.t. counterparts of the d.t. notions such as the orthogonal PM's on the unit circle, the LWR algorithm, etc. However, the existing d.t. theory can be applied only to strictly regular PM's, i.e., PM's of the form $A(z) = \sum_{k=0}^{n} z^k A_k$ with A_n nonsingular. This limitation is a defect of the theory in view of generality, and hence we attempt to construct the stability theory applicable to every column reduced PM. (It should be noted that the column reducedness can be assumed without loss of generality (see [17]).) For this purpose it is necessary to develop a general theory for column reduced PM's at first, which will be pursued in Chap.2. In Sec.2.1, a new basis for representing column reduced PM's will be introduced. In Sec.2.2, we will show that a transformation group consisting of unimodular PM's acts on the totality of column reduced PM's having prescribed column degrees, and will investigate the structure of the action. These results will be used for studying controllability indices and the problem of canonical forms in Sec.2.3, where the so-called polynomial-echelon (or Popov) form ([26],see also [17]) will be derived in our framework. It might be said that the contents of Chap.2 are not essentially new, but the method developed there will prove to be quite useful in treating column reduced PM's. In Chap.3, associated with a given PM C(s), an inner product $\langle P(s),Q(s)\rangle_C$ for arbitrary PM's P(s) and Q(s) will be defined. The definition will be presented first for a c.t. stable C(s) by an integral on the imaginary axis (Sec.3.1), which is also represented in the time domain (Sec.3.2), and next for a general C(s) via a Lyapunov equation or via an equivalent PM equation (Sec.3.3). The inner product allows us to introduce the notion of orthogonal PM's associated with C(s). Succeeding the definition, a criterion for the c.t. stability of C(s) in terms of the orthogonal PM's will be presented in Sec.4.1. We will also derive a recursive algorithm producing the orthogonal PM's, which combines the stsbility criterion with the algebraic structure of C(s). In particular, the algorithm is shown to be equivalent to the Routh-Hurwitz stability test in the scalar case. These results will be applied to construction of Schwarz matrices and of the Routh approximation for multivariable systems in Sec.4.2. Sec.4.3 treats the duality of PM's, which was first introduced by Anderson and Bitmead [1] for relating the stability of PM's to some circuit theoretic notions. Our approach to the duality is as follows. First, the definition by [1] of the dual PM D(s) of a given PM C(s) is rewritten in terms of the inner product associated with C(s). Next, D(s) is constructed by the use of orthogonal PM's.

Finally, a circuit theoretic interpretation of the recursive algorithm in Sec.4.1 is presented via D(s). In Sec.4.4, we will investigate the behavior of the orthogonal PM's under the action of unimodular PM's introduced in Sec.2.2, where the meaning of the quantities appeared in the orthogonal PM's will be studied further from a geometric point of view. Chap.5 is mainly devoted to a survey of known results in the d.t. theory of orthogonal PM's. However, from a comparative viewpoint on the d.t. case and the c.t. case, some new results will be obtained there. The LWR algorithm will be extended to the "generalized LWR algorithm" which applies to every column reduced PM. We will also derive a version of the LWR algorithm called the "polar-type LWR algorithm". This algorithm achieves a simplification of the LWR algorithm at the sacrifice of the symmetry between the forward PM's and the backward PM's.

In Part II (Chap.6 and Chap.7) we will study the Mullis-Roberts type approximations, which consist of the modified least squares approximation (MLSA) and the interpolatory approximation (IA). The MLSA, which is also called the equation error method, has a long history itself (see, for instance, [19] and [27]). However, it is Mullis and Roberts [21] who first elucidated the properties of the MLSA theoretically in the d.t. scalar case. Furthermore, they formulated the d.t. IA problem, in which it is required to find reduced order systems preserving a part of the impulse response sequence and the autocovariance sequence of a given system, and noticed a similarity between the MLSA and the IA. Actually they solved the IA problem in the d.t. scalar case with the aid of the results on the MLSA, and showed that there exists precisely two solutions to the IA problem. These results were extended to the d.t. multivariable case by Inouye [12]. Chap.6 in this thesis is mostly devoted to a survey and a refinement of the results in [12] and [21]. On the other hand, almost all the results on the MLSA and the IA in the c.t. case are newly obtained in Chap.7. The c.t. MLSA can scarcely be found in the past literature, except that the first order case, in which approximants are restricted to the form $\hat{H}(s) = (sI-A)^{-1}B$, was treated by some authors ([7],[25]). Recently Anderson and Skelton [4] studied a problem similar to the IA from somewhat different viewpoint. Their result in the d.t. case is almost the same as that of Mullis and Roberts. In the c.t. case, their formulation of the problem differs from ours in that the condition imposed on a solution of their problem is weaker than the condition imposed on a solution of the IA problem. It will be shown in Sec.7.1 that the unique solution of the MLSA problem turns out to be a special solution of the IA problem. This is a remarkable difference from the d.t. case. In particular, the solutions of the IA problem for a scalar system are unique and coincide with the solution of the MLSA problem. In Sec.7.2, a recursive structure of the MLSA will be elucidated, which will lead to a block diagram representation of the approximation. Sec.7.3 treats the "weighted MLSA problem" with a "weighting PM" W(s). We will present there two apparently differnt methods to solve the problem, both of which reduce the problem to the (weightless) MLSA problem by the use of some factors of W(s).

In part III (Chap.8), we will consider the situation where a c.t. system H(s) is approximated by a d.t. system $G^{(t)}(z)$ with the unit time length (or the sampling period) t, and will investigate on the contents of the previous chapters how the d.t. results applied to $G^{(t)}(z)$ 'converge to' the c.t. results applied to H(s) as t tends to 0. In Sec.8.1, some preliminary considerations on the convergence $G^{(t)}(z) \longrightarrow H(s)$ $(t \downarrow 0)$, including the definition of the convergence, will be made. Sec.8.2 studies the 'convergence' of the d.t. results on orthogonal PM's in Chap.5 to the corresponding results in Chap.4. We will see there how the LWR algorithm and the recursive algorithm in Sec.4.1, which have apparently different structures, are linked to each other in a limiting process. In Sec.8.3, we will investigate the limiting behaviors of the d.t. MLSA and of the d.t. IA in connection with the c.t. MLSA and the c.t. IA. It will be shown that if $G^{(t)}(z) \longrightarrow H(s)$ $(t \downarrow 0)$ then the unique solution of the d.t. MLSA problem for $G^{(t)}(z)$ converges to the unique solution of the MLSA problem for H(s) as $t \downarrow 0$. We will also derive the condition for a solution of the d.t. IA problem to converge to a solution of the c.t. IA problem. In particular, it will be seen that in the scalar case one of the two solutions of the d.t. IA problem converges to the unique solution of the c.t. IA (= MLSA) problem while the other solution has no c.t. limit.

PART I

THE STABILITY OF POLYNOMIAL MATRICES

2. COLUMN REDUCED POLYNOMIAL MATRICES

2.1. A basis for column reduced polynomial matrices

Throughout this thesis, the term 'PM' is used as an abbreviation for 'polynomial matrix'.

Let $\tilde{\mathcal{U}}$ be the totality of $p \times p$ unimodular PM's ; i.e.,

$$\tilde{\mathcal{U}} \triangleq \{ U(s) : p \times p \text{ PM} \mid \det U(s) = \text{const.} \neq 0 \} \quad .$$

Since $\tilde{\mathcal{U}}$ is a group in the sense that

$$\text{if} \quad U_1(s),\ U_2(s) \in \tilde{\mathcal{U}} \quad \text{then} \quad U_1(s)U_2^{-1}(s) \in \tilde{\mathcal{U}} \quad ,$$

an equivalence relation between two $p \times p$ PM's $C(s)$ and $\bar{C}(s)$ can be defined by

$$C(s) \sim \bar{C}(s) \overset{\Delta}{\Longleftrightarrow} \exists U(s) \in \tilde{\mathcal{U}} \quad \text{s.t.} \quad \bar{C}(s) = C(s)U(s). \tag{1}$$

In the above, $\bar{C}(s)$ is said to be right similar to $C(s)$, and the mapping $C(s) \mapsto C(s)U(s)$ is called the right similarity transformation by $U(s)$. This equivalence relation corresponds to the arbitrariness of right irreducible matrix fraction descriptions (MFD's) of an arbitrary rational matrix $H(s) : q \times p$; i.e., if $C(s)$ is the denominator of a right irreducible MFD

$$H(s) = D(s)\ C^{-1}(s), \tag{2}$$

then the condition that $C(s) \sim \bar{C}(s)$ is necessary and sufficient for $\bar{C}(s)$ to be the denominator of another right irreducible MFD of $H(s)$.

Suppose that $C(s) : p \times p$ is a nonsingular (i.e. $\det C(s) \not\equiv 0$) PM, and let n_i ($1 \leqq i \leqq p$) be the i-th column degree (i.e. the highest degree of the polynomials in the i-th column) of $C(s)$. Then it holds in general that

$$\deg \det C(s) \leqq \sum_{i=1}^{p} n_i. \tag{3}$$

When the equality holds in the above, $C(s)$ is said to be <u>column reduced</u>. It is known that every nonsingular PM is right similar to a column reduced PM ([17]). The concepts of <u>left similarity</u>, <u>row reducedness</u>, etc., are also defined in a similar way.

In the sequel, we work with fixed column degrees $(n_1, \ldots, n_p)$ and denote by $\mathcal{C}$ the totality of $p \times p$ column reduced PM's having the prescribed column degrees (n_i). The aim of the present chapter is to investigate the structure of $\mathcal{C}$. Let

$$N \triangleq \sum_{i=1}^{p} n_i \tag{4}$$

$$T(s) \triangleq \mathrm{diag}\{s^{n_1}, \ldots, s^{n_p}\} : p \times p \tag{5}$$

$$L'(s) \triangleq \text{block diag}\{[s^{n_1-1}, \ldots, s, 1], \ldots, [s^{n_p-1}, \ldots, 1]\} : p \times N, \tag{6}$$

where diag $\{\ldots\}$ (block diag $\{\ldots\}$) denotes the (block) diagonal matrix with (block) diagonal elements $\{\ldots\}$, and the prime denotes the transpose of a matrix. Then an arbitrary $p \times p$ PM $C(s)$ having the prescribed column degrees is written as

$$C(s) = C_{high}T(s) + C_{low}L(s), \tag{7}$$

where C_{high} and C_{low} are constant matrices whose sizes are $p \times p$ and $p \times N$ respectively. Furthermore it can be shown ([17]) that $C(s)$ is column reduced if and only if

$$\det C_{high} \neq 0. \tag{8}$$

Thus we have

$$\mathfrak{C} = \{ C_{high}T(s)+C_{low}L(s) \mid C_{high} \in GL(p),\ C_{low} \in \mathbb{R}^{p\times N} \}, \tag{9}$$

where $GL(p)$ denotes the totality of $p \times p$ regular constant matrices. ('GL' stands for 'general linear group'.)

Although it might be a rule to represent a column reduced PM as (7), the character of this representation that a PM $C(s)$ is decomposed into only two terms —— the higher degree term $C_{high}T(s)$ and the lower degree term $C_{low}L(s)$ —— is not suitable for some applications. Indeed, when we treat a recursive algorithm for PM's for instance, we usually do not use (7) but represent a PM as

$$C(s) = s^n C_n + s^{n-1} C_{n-1} + \dots + C_0 . \tag{10}$$

However, the representation (10) is not fit to be used as a parametrization of $\mathfrak{C}$. In the sequel we will derive a new representation of PM's in $\mathfrak{C}$ which can be regarded as a modification of (10).

Since the columns of $C(s)$ can be arbitrarily rearranged by a right similarity transformation, we can assume without loss of generality that

$$(n \triangleq)\ n_1 \geqq n_2 \geqq \cdots \geqq n_p. \tag{11}$$

Now, consider the following matrix for j = 0,1,...,n:

$$T(s)/s^{n-j} = \text{diag}\ \{s^{j-n+n_i};\ i=1,2,\ldots,p\}\ .$$

This is not generally a PM. Indeed, defining the integer r(j) for each j=0,1,...,n as

$$r(j) \triangleq \max\{\, i \mid 1 \leqq i \leqq p \ \text{ and } \ j-n+n_i \geqq 0 \,\}, \tag{12}$$

we see that the first r(j) elements among the p diagonal elements of $T(s)/s^{n-j}$ are non-negative powers of s, while the other diagonal elements are negative powers when $r(j)<p$. Hence, picking up the first r(j) rows of $T(s)/s^{n-j}$, we can define a $r(j) \times p$ PM $T_j(s)$ as follows:

$$T_j(s) \triangleq \left[\begin{array}{cccc|c} s^j & & & & \\ & s^{j-n+n_2} & & & \\ & & \ddots & & O \\ & & & s^{j-n+n_{r(j)}} & \end{array}\right] \tag{13}$$

The following properties are immediate from the definition.

$$\begin{cases} 1 \leqq r(0) \leqq r(1) \leqq \cdots\cdot \leqq r(n) = p & (14) \\ \displaystyle\sum_{j=0}^{n-1} r(j) = \sum_{i=1}^{p} n_i = N & (15) \\ s\ T_j(s) = \Lambda_j\ T_{j+1}(s) \quad (0 \leqq j \leqq n-1) & (16) \end{cases}$$

where

$$\Lambda_j \triangleq [I_{r(j)} \mid 0] \quad : r(j) \times r(j+1) \tag{17}$$

Let us investigate the properties of $\{ T_0(s),\ldots,T_n(s) \}$ as a basis for representing PM's. Suppose that a $k \times p$ PM $P(s)$ is written as

$$P(s) = \sum_{j=0}^{d} P_j T_j(s), \qquad P_j \in \mathbb{R}^{k\times r(j)}. \tag{18}$$

Then the i-th column degree of $P(s)$ is not greater than that of $T_d(s)$, i.e. $d-n+n_i$. Here a negative column degree means that the corresponding column of $P(s)$ is a zero vector, and therefore the last $p-r(d)$ columns of $P(s)$ constitute a zero matrix when $r(d)<p$. Conversely, if the i-th column degree of an arbitrary $k \times p$ PM $P(s)$ is not greater than $d-n+n_i$ for $\forall i=1,2,\ldots,p$, then $P(s)$ is uniquely expressed as (18). When $P_d \neq 0$ in (18), d is called the <u>degree</u> of $P(s)$ and written as $d = \deg P(s)$. Especially when $P_d = I_{r(d)}$, $P(s)$ is said to be <u>monic</u>.

Let $C(s)$ be an arbitrary element of $\mathcal{C}$. Since the column degrees of $C(s)$ is $(n_1,\ldots,n_p)$, the degree of $C(s)$ in the above sense is n, and $C(s)$ is represented as

$$C(s) = \sum_{j=0}^{n} C_j T_j(s), \qquad C_j \in \mathbb{R}^{p\times r(j)}. \tag{19}$$

Evidently C_n in the above is equal to C_{high} in (7), and therefore we have

$$\mathcal{C} = \{ \sum_{j=0}^{n} C_j T_j(s) \mid C_n \in GL(p), \quad C_j \in \mathbb{R}^{p\times r(j)} \quad (0 \leqq j \leqq n-1) \} \tag{20}$$

Example 2.1.1 (Stictly regular case) In the case where $n_i = n$ for $\forall i$, it turns out that $r(j) = p$ and that $T_j(s) = s^j I_p$ for $\forall j$. In this case, every element $C(s)$ of $\mathcal{C}$ is written as (10) with C_n being regular. Such a PM is said to be strictly regular.

Example 2.1.2 Let $p = 3$ and let $(n_1, n_2, n_3) = (4,2,1)$. Then we have : $r(4) = r(3) = 3$, $r(2) = 2$, $r(1) = r(0) = 1$,

$$T_4(s) = \begin{bmatrix} s^4 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s \end{bmatrix}, \qquad T_3(s) = \begin{bmatrix} s^3 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_2(s) = \begin{bmatrix} s^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad T_1(s) = \begin{bmatrix} s & 0 & 0 \end{bmatrix},$$

$$T_0(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Every element $C(s)$ of $\mathcal{C}$ is writeen as

$$C(s) = \underset{C_4}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} T_4(s) + \underset{C_3}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} T_3(s)$$

$$+ \underset{C_2}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}} T_2(s) + \underset{C_1}{\begin{bmatrix} * \\ * \\ * \end{bmatrix}} T_1(s) + \underset{C_0}{\begin{bmatrix} * \\ * \\ * \end{bmatrix}} T_0(s) .$$

It is known that the representation (7) yields the controller form realization of the MFD (2). We will show in the following that the representation (19) yields a version of the realization. From (16) and (19), we have

$$s\, T_{n-1}(s) = \Lambda_{n-1}\, T_n(s)$$

$$= \Lambda_{n-1}\, C_n^{-1}\, C(s) - \sum_{j=0}^{n-1} \hat{C}_j\, T_j(s)\ , \tag{21}$$

where $\hat{C}_j \triangleq \Lambda_{n-1}\, C_n^{-1}\, C_j\ :\ r(n-1)\times r(j)$. Let

$$\Gamma \triangleq \begin{bmatrix} 0 & \Lambda_0 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \Lambda_{n-2} \\ -\hat{C}_0 & -\hat{C}_1 & \cdots & -\hat{C}_{n-1} \end{bmatrix} \quad : \quad N\times N \tag{22}$$

$$B \triangleq [\ O \mid C_n'^{-1}\ \Lambda_{n-1}'\ ]' \quad : \quad N\times p \tag{23}$$

$$T(s) \triangleq [T_0'(s), T_1'(s),\ \ldots\ , T_{n-1}'(s)]' \quad : \quad N\times p \tag{24}$$

Then it follows from (16) and (21) that

$$(sI - \Gamma)\, T(s) = B\, C(s)\ . \tag{25}$$

We call Γ the block-companion matrix defined from C(s) and denote it by

$$\Gamma = \mathrm{comp}\left\{ C(s) \right\} .$$

It is obvious that the pair (Γ, B) is controllable, and we call it the companion pair defined from C(s).

Example 2.1.3 (Continued from Ex.2.1.2.)

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Suppose that we are given a $q \times p$ rational matrix H(s) which is strictly proper ($\lim_{s \to \infty} H(s) = 0$) and is represented by a right irreducible MFD as (2) with a $C(s) \in \mathcal{C}$ and a $q \times p$ PM D(s). Since C(s) is column reduced, the strictly-properness of H(s) implies that the i-th column degree of D(s) is less than n_i for $\forall i = 1,2,\ldots,p$ ([17]). This means that $\deg D(s) \leqq n-1$, and therefore D(s) can be written as

$$\begin{aligned} D(s) &= \sum_{j=0}^{n-1} D_j T_j(s) , \qquad D_j : q \times r(j) \\ &= D\, T(s) , \end{aligned} \tag{26}$$

where $D \triangleq [D_0, \ldots, D_{n-1}] : q \times N$. It follows from (2), (25) and (26) that

$$H(s) = D(sI - \Gamma)^{-1} B , \tag{27}$$

i.e., (Γ, B, D) is a realization of H(s). We call it the block-companion type controller form realization of the MFD H(s) =

$D(s)C^{-1}(s)$. This realization and the usual controller form realization only differ in arrangement of rows and columns, and of course there is no essential difference between the two.

2.2. The action of unimodular polynomial matrices

For an arbitrary $C(s) \in \mathcal{C}$, define

$$\mathcal{U} \triangleq \{ U(s) \in \tilde{\mathcal{U}} \mid C(s)U(s) \in \mathcal{C} \}.$$

As we will show in the sequel, the definition of $\mathcal{U}$ is independent of a choice of $C(s) \in \mathcal{C}$ and therefore we have

$$\mathcal{U} = \{ U(s) \in \tilde{\mathcal{U}} \mid C(s)U(s) \in \mathcal{C} \quad \text{for } \forall C(s) \in \mathcal{C} \}. \tag{1}$$

It is obvious that $\mathcal{U}$ is a subgroup of $\tilde{\mathcal{U}}$. In this section we will investigate the structure of $\mathcal{U}$ and the action of $\mathcal{U}$ on $\mathcal{C}$.

First, let us derive a representation for an element of $\mathcal{U}$. Suppose that C(s)U(s) has column degrees $(\bar{n}_1, \ldots, \bar{n}_p)$, where C(s) and U(s) is an arbitrary $p \times p$ nonsingular PM. Then it can be shown that the condition

$$\bar{n}_i \leqq n_i \qquad \text{for } \forall i. \tag{2}$$

is equivalent to

$$\deg\, [U(s)]_{ij} \leqq n_j - n_i \quad \text{for } \forall i,\ \forall j, \tag{3}$$

where $[\ldots]_{ij}$ denotes the (i,j) element of a matrix ... . Obviously these are necessary conditions for C(s)U(s) to belong to $\mathcal{C}$. On the other hand, it generally holds that (see (2.1.3))

$$\sum_{i=1}^{p} \bar{n}_i \geqq \deg \det C(s)U(s)$$

$$= \sum_{i=1}^{p} n_i + \deg \det U(s).$$

From the above inequality, we can see that (2) implies the following equations:

$$\begin{cases} \bar{n}_i = n_i \quad \text{for } \forall i \\ \deg \det C(s)U(s) = \sum_{i=1}^{p} \bar{n}_i \\ \deg \det U(s) = 0. \end{cases}$$

These equations mean that $C(s)U(s) \in \mathcal{C}$ and $U(s) \in \mathcal{U}$. Hence we get

$$\mathcal{U} = \{U(s) : p \times p \text{ PM} \mid U(s) \text{ is nonsingular and satisfies (3)}\}. \tag{4}$$

From the above result, an arbitrary $U(s) \in \mathcal{U}$ is written as

$$[U(s)]_{ij} = \begin{cases} \sum_{k=0}^{n_j - n_i} u_{ij}^{(k)} s^k & \text{if } n_j \geq n_i \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

The quantities $\{u_{ij}^{(k)}\}$ constitute a coordinate system of $\mathcal{U}$ and we have

$$\dim \mathcal{U} = \sum_{(i,j) \text{ s.t. } n_j \geq n_i} (n_j - n_i + 1), \tag{6}$$

or equivalently

$$\dim \mathcal{U} = p^2 + N\,p - \sum_{j=0}^{n-1} r(j+1)r(j). \tag{7}$$

Remark 2.2.1 (See Ex.2.1.1.) In the strictly regular case where $n = n_i$ for $\forall i$, it turns out that $\mathcal{U} = GL(p)$ and the arguments in the present section yield only trivial results.

Example 2.2.2 (Continued from Ex.2.1.3.) Every element $U(s)$ of $\mathcal{U}$ is written as

$$U(s) = \begin{bmatrix} u_{11}(s) & 0 & 0 \\ u_{21}(s) & u_{22}(s) & 0 \\ u_{31}(s) & u_{32}(s) & u_{33}(s) \end{bmatrix}$$

$$\begin{cases} u_{11}(s) = u_{11}^{(0)}, \quad u_{22}(s) = u_{22}^{(0)}, \quad u_{33}(s) = u_{33}^{(0)}, \\ u_{21}(s) = u_{21}^{(2)}s^2 + u_{21}^{(1)}s + u_{21}^{(0)}, \\ u_{31}(s) = u_{31}^{(3)}s^3 + u_{31}^{(2)}s^2 + u_{31}^{(1)}s + u_{31}^{(0)}, \\ u_{32}(s) = u_{32}^{(1)}s + u_{32}^{(0)}, \end{cases}$$

and hence $\dim \mathcal{U} = 12$.

Now, let us investigate how the coefficient matrices of $C(s)$ are transformed to those of $C(s)U(s)$ by $U(s) \in \mathcal{U}$. First, we consider the case where $C(s) = T_n(s)$. Since $T_n(s)$ belongs to $\mathcal{C}$, $T_n(s)U(s)$ also belongs to $\mathcal{C}$ and is written as

$$T_n(s)U(s) = \sum_{j=0}^{n} F_j T_j(s), \tag{8}$$

where $F_n \in GL(p)$ and $F_j \in \mathbb{R}^{p\times r(j)}$ for $j < n$. The matrices $\{F_j\}$ are determined by $U(s)$ and we write $F_j = F_j(U(s))$. It is obvious that the mapping

$$U(s) \longmapsto (F_0(U(s)),\dots,F_n(U(s)))$$

is injective. Using the expression (5), we obtain

$$[F_k]_{ij} = \begin{cases} u_{ij}^{(n_j-n_i-n+k)} & \text{if } n_j - n_i \geqq n - k \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Example 2.2.3 (Continued from Ex.2.2.2.)

$$F_4 = \begin{bmatrix} u_{11}^{(0)} & 0 & 0 \\ u_{21}^{(2)} & u_{22}^{(0)} & 0 \\ u_{31}^{(3)} & u_{32}^{(1)} & u_{33}^{(0)} \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 0 & 0 & 0 \\ u_{21}^{(1)} & 0 & 0 \\ u_{31}^{(2)} & u_{32}^{(0)} & 0 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 0 & 0 \\ u_{21}^{(0)} & 0 \\ u_{31}^{(1)} & 0 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0 \\ 0 \\ u_{31}^{(0)} \end{bmatrix} \qquad F_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $0 \leqq \forall k \leqq n$ and for $n-k \leqq \forall j \leq n$, let $F_j^{(k)}$ be the $r(k) \times r(j-n+k)$ left-upper submatrix of F_j. Then it follows from (8) that

$$T_k(s)U(s) = \sum_{j=0}^{k} F_{n-k+j}^{(k)} T_j(s) \tag{10}$$

Hence, if two PM's $C(s)$ and $\bar{C}(s)$ in $\mathcal{C}$ are written as

$$\begin{cases} C(s) = \sum_{j=0}^{n} C_j T_j(s) \\ \bar{C}(s) = \sum_{j=0}^{n} \bar{C}_j T_j(s) \end{cases}$$

and if they satisfy

$$\bar{C}(s) = C(s)U(s), \tag{11}$$

then it holds that

$$\bar{C}_j = \sum_{k=j}^{n} C_k F_{n-k+j}^{(k)} \qquad (0 \leqq \forall j \leqq n). \tag{12}$$

This is the transformation rule for coefficient matrices of PM's in $\mathcal{C}$ under the right similarity transformation by $U(s) \in \mathcal{U}$.

We can see from (12) that, for each $k = 0,1,\ldots,n$, $\{\bar{C}_n,\ldots,\bar{C}_k\}$ are determined from $\{C_n,\ldots,C_k \,;\, F_n,\ldots,F_k\}$ and are independent of $\{C_{k-1},\ldots,C_0 \,;\, F_{k-1},\ldots,F_0\}$. Indeed, defining

$$G_k = G_k(U(s))$$

$$\triangleq \begin{bmatrix} F_n^{(n)} & F_{n-1}^{(n)} & \cdots\cdots & F_k^{(n)} \\ & F_n^{(n-1)} & \cdots\cdots & F_{k+1}^{(n-1)} \\ & & \ddots & \vdots \\ \bigcirc & & & F_n^{(k)} \end{bmatrix} : N_k \times N_k \tag{13}$$

$$N_k \triangleq \sum_{j=k}^{n} r(j), \tag{14}$$

we have

$$[\bar{C}_n, \ldots, \bar{C}_k] = [C_n, \ldots, C_k]\, G_k. \tag{15}$$

It is shown from (15) that the mapping $U(s) \longmapsto G_k(U(s))$ is a homomorphism from $\mathcal{U}$ into $GL(N_k)$; i.e., for $\forall U(s), \forall V(s) \in \mathcal{U}$,

$$G_k(U(s)V^{-1}(s)) = G_k(U(s))G_k(V(s))^{-1}.$$

Therefore, for each $k=0,1,\ldots,n$, we can define a subgroup of $\mathcal{U}$, say $\mathcal{U}_k$, as

$$\begin{aligned} \mathcal{U}_k &\triangleq \{ U(s) \in \mathcal{U} \mid G_k(U(s)) = I \} \\ &= \{ U(s) \in \mathcal{U} \mid F_n(U(s)) = I, \\ &\qquad F_j(U(s)) = 0 \quad \text{for } \forall j = k, \ldots, n-1 \}. \end{aligned} \tag{16}$$

Furthermore $\mathcal{U}_k$ is a subgroup of $\mathcal{U}_{k+1}$, because $G_k(U(s)) = I$ implies $G_{k+1}(U(s)) = I$. Thus we have obtained a sequence of subgroups:

$$\mathcal{U} \supset \mathcal{U}_n \supset \mathcal{U}_{n-1} \supset \cdots \supset \mathcal{U}_0 = \{I\}.$$

The group $\mathcal{U}_k$ represents the degree of freedom of $[\bar{C}_{k-1},\ldots,\bar{C}_0]$ in (11) under the constraint $[\bar{C}_n,\ldots,\bar{C}_k] = [C_n,\ldots,C_k]$. We can show that

$$\dim \mathcal{U}_k = \sum_{j=0}^{k-1} r(j)\,\{\,p - r(j+n-k)\,\}\,, \tag{17}$$

Remark 2.2.4 We can see from (10) that

$$\deg P(s)U(s) = \deg P(s)$$

for $\forall P(s)$ and $\forall U(s) \in \mathcal{U}$. Furthermore, if P(s) is monic and if $U(s) \in \mathcal{U}_n$, then P(s)U(s) is also monic.

Example 2.2.5 (Continued from Ex.2.2.3.)

$$\dim \mathcal{U}_4 = 6, \quad \dim \mathcal{U}_3 = 3, \quad \dim \mathcal{U}_2 = 1,$$

$$\dim \mathcal{U}_1 = \dim \mathcal{U}_0 = 0.$$

Example 2.2.6 (See Rem.2.2.1.) In the strictly regular case, $\mathcal{U}_j = \{I\}$ for $\forall j = 0,1,\ldots,n$.

2.3. On the controllability indices and canonical forms

Suppose that a matrix pair (X,Y) is controllable, where X : $N \times N$ and Y : $N \times p$. Then $W \triangleq [Y, XY, \ldots, X^{N-1}Y]$: $N \times Np$ has full row rank N. Hence, searching the columns of X from left to right, we can find a set of N linearly independent column vectors $\{X^j y_i$; $1 \leqq i \leqq p$ and $0 \leqq j \leqq \nu_i - 1\}$, where y_i is the i-th column of Y and $\{\nu_i\}$ are nonnegative integers such that $\sum_{i=1}^{p} \nu_i = N$. It can be shown ([17]) that the ordered set $(\nu_1, \ldots, \nu_p)$ is invariant under the similarity transformation $(X,Y) \longmapsto (TXT^{-1}, TY)$ for an arbitrary nonsingular matrix T, and ν_i is called the <u>i-th controllability index</u> of (X,Y).

For an arbitrary nonsingular square PM $C(s)$, there exist matrices X and Y and a PM $R(s)$, with (X,Y) controllable and with $(C(s), R(s))$ right coprime, such that

$$(sI - X)^{-1}Y = R(s)\, C^{-1}(s). \tag{1}$$

Let $\{\nu_i\}$ be the controllability indices of (X,Y). Then, owing to the invariance property of the indices, $\{\nu_i\}$ is independent of a choice of $(X,Y,R(s))$ and is uniquely determined from $C(s)$. So, we call $\{\nu_i\}$ the controllability indices of $C(s)$. Note that the indices are invariant under the right similarity transformation $C(s) \longmapsto C(s)U(s)$ for anarbitrary unimodular PM $U(s)$.

Let us consider the controllability indices $\{\nu_i\}$ of a PM $C(s)$ in $\mathfrak{C}$;i.e., $C(s)$ is column reduced with column degrees $(n_1, \ldots, n_p)$. In this case, we can adopt the companion pair (Γ, B) defined from $C(s)$ as a matrix pair (X,Y) in (1) (see (2.1.25)).

Owing to a special feature of the companion pair, $\{\nu_i\}$ are obtained in the following way. For each $j = 0,1,\ldots,n$, let

$$Q^{(j)} \triangleq [I_{r(j)} \mid 0]C_n^{-1} \ : \ r(j) \times p$$

$$= [q_1^{(j)},\ldots,q_p^{(j)}] \ ,$$

where C_n is the n-th degree coefficient matrix defined by (2.1.19). Serching the columns of $Q^{(j)}$ from left to right, we can find a set of $r(j)$ linearly independet column vectors $\{ q_i^{(j)} \ ; \ i \in \mathcal{L}^{(j)} \}$, where $\mathcal{L}^{(j)}$ is a subset of $\{1,2,\ldots,p\}$ and contains $r(j)$ elements. It is noted that $\mathcal{L}^{(j)} \subset \mathcal{L}^{(j+1)}$. In this situation, ν_i is given by

$$\nu_i = n - \min\{ j \mid i \in \mathcal{L}^{(j)} \} .$$

It is shown from the above equation that, as an unordered set, $\{\nu_i\}$ coincides with the set of column degrees $\{n_i\}$. On the other hand, we can see that the order of $\{\nu_i\}$ is determined by certain rank conditions on submatrices of C_n^{-1}. For instance, consider the following two extreme cases:

$$\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_p \tag{2}$$

and

$$\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_p \ . \tag{3}$$

A necessary and sufficient condition for (2) (which is equivalent to $(n_1,\ldots,n_p) = (\nu_1,\ldots,\nu_p)$) is that

$$\left\{ \begin{array}{l} \text{the } r(j)\times r(j) \text{ left-upper submatrix of} \\ C_n^{-1} \text{ is nonsingular for every } j = 0,1,\ldots,n, \end{array} \right. \tag{4}$$

while a necessary and sufficient condition for (3) is that

$$\begin{cases} \text{the } r(j)\times(p-r(j)) \text{ left-upper submatrix of} \\ C_n^{-1} \text{ is a zero matrix for every } j = 0,1,\ldots,n. \end{cases} \tag{5}$$

These conditions are also expressed in terms of C_n instead of C_n^{-1} as

$$\begin{cases} \text{the } r(j)\times r(j) \text{ right-lower submatrix of } C_n \\ \text{is nonsingular for every } j = 0,1,\ldots,n \end{cases} \tag{4$'$}$$

and

$$\begin{cases} \text{the } r(j)\times(p-r(j)) \text{ right-lower submatrix of } C_n \\ \text{is a zero matrix for every } j = 0,1,\ldots,n, \end{cases} \tag{5$'$}$$

respectively.

Next, we study the problem of canonical PM's in our framework. Let $\mathfrak{D}$ be a subset of $\mathfrak{C}$, and let $\mathcal{E}$ be a subset of $\mathfrak{D}$. Let us consider the following problem.

<u>Problem A</u> Given $C(s) \in \mathfrak{D}$, find $E(s) \in \mathcal{E}$ which is right similar to $C(s)$.

If the problem has the unique solution $E(s)$ for $^\forall C(s) \in \mathfrak{D}$, it is said that $\mathcal{E}$ is a <u>canonical form</u> on $\mathfrak{D}$. In the sequel we restrict our selves to the case where $\mathcal{E}$ is written as

$$\mathcal{E} = \{ \sum_{j=0}^{n} E_j T_j(s) \mid E_j \in \mathcal{E}_j \quad (0 \leqq {}^\forall j \leqq n) \},$$

where $\mathcal{E}_j$ is a subset of $\mathbb{R}^{p\times r(j)}$.

For each $k \in \{0,1,\ldots,n\}$, consider the following problem.

Problem B_k Given $C(s) = \sum_{j=0}^{n} C_j T_j(s) \in \mathfrak{D}$, find $\bar{C}(s) = \sum_{j=0}^{n} \bar{C}_j T_j(s)$ which is right similar to $C(s)$ and satisfies

$$\begin{cases} [\bar{C}_n,\ldots,\bar{C}_{k+1}] = [C_n,\ldots,C_{k+1}] & (6) \\ \bar{C}_k \in \mathcal{E}_k . & (7) \end{cases}$$

If $\mathcal{E}$ is a canonical form on $\mathfrak{D}$, and if $\mathfrak{D}$ is closed under the action of $\mathcal{U}$ in the sense that $C(s)U(s) \in \mathfrak{D}$ for $\forall C(s) \in \mathfrak{D}$ and for $\forall U(s) \in \mathcal{U}$, then the solution of Prob.A for given $C(s)$ is obtained by solving Prob.B_n, Prob.B_{n-1}, ... , Prob.B_0, successively.

When $\bar{C}(s)$ is right similar to $C(s)$ and written as $\bar{C}(s) = C(s)U(s)$ for $U(s) \in \mathcal{U}$, the condition (6) is equivalent to $U(s) \in \mathcal{U}_{k+1}$ (see (2.2.15,16)). In this case, it follows from (2.2.12) that

$$\bar{C}_k = \begin{cases} C_n\, F_n(U(s)) & \text{if } k=n \\ C_k + C_n\, F_k(U(s)) & \text{otherwise.} \end{cases}$$

Thus, defining for $k=0,1,\ldots,n$

$$\mathcal{F}_k \triangleq F_k(\mathcal{U}_{k+1}) = F_k(\mathcal{U}) \subset \mathbb{R}^{p \times r(k)},$$

we see that Prob.B_k is equivalent to Problem C_k in the following.

Problem C_n Given $C_n \in GL(p)$, find $F_n \in \mathcal{F}_n$ such that

$$C_n\, F_n \in \mathcal{E}_n .$$

Problem C_k $(0 \leqq k \leqq n-1)$ Given $C_n \in GL(p)$ and $C_k \in \mathbb{R}^{p \times r(k)}$,

find $F_k \in \mathcal{F}_k$ such that

$$C_k + C_n F_k \in \mathcal{E}_k .$$

Consequently, a necessary and sufficient condition for $\mathcal{E}$ to be a canonical form on $\mathcal{D}$ is that, for $\forall C(s) = \sum_{j=0}^{n} C_j T_j(s) \in \mathcal{D}$,

$$| C_n \cdot \mathcal{F}_n \cap \mathcal{E}_n | = 1 \tag{8}$$

$$| (C_k + C_n \cdot \mathcal{F}_k) \cap \mathcal{E}_k | = 1 \qquad (0 \leqq \forall k \leqq n-1), \tag{9}$$

where $| \ |$ denotes the number of elements of a set.

In the strictly regular case, $\mathcal{F}_n = GL(p)$ and $\mathcal{F}_k = \{0\}$ (see Rem.2.2.1 and Ex.2.2.6). Hence a canonical form on $\mathcal{C}$ is obtained by choosing

$$\mathcal{E}_n = \{I_p\}, \qquad \mathcal{E}_k = \mathbb{R}^{p \times p} \quad (0 \leqq k \leqq n-1).$$

In other cases, however, the subset $C_n \cdot \mathcal{F}_k \subset \mathbb{R}^{p \times r(k)}$ depends on C_n, and it is impossible to choose $\{\mathcal{E}_k\}$ so as to satisfy (8) and (9) for $\forall C_n \in GL(p)$ and $\forall C_k \in \mathbb{R}^{p \times r(k)}$. We can see that the domain $\mathcal{D}$ of a canonical form should be defined by a certain condition on C_n to restrict the range of $C_n \cdot \mathcal{F}_k$ adequetly. Actually, if we define $\mathcal{D}$ by specifying the order of the controllability indices $\{\nu_i\}$, a canonical form on $\mathcal{D}$, which is called the <u>polynomial echelon (or Popov) form</u> ([26],[17]), can be obtained. For instance, let

$$\mathcal{D} = \{ \sum_{j=0}^{n} C_j T_j(s) \in \mathcal{C} \mid C_n \text{ satisfies } (4)' \},$$

which corresponds to the condition $\nu_1 \geqq \cdots \geqq \nu_p$. Then a canonical form on $\mathcal{D}$ is obtained by choosing

$$\mathcal{E}_n = \{ E_n : p\times p \mid [E_n]_{ii} = 1 \text{ for } \forall i, \text{ and } [E_n]_{ij} = 0 \text{ for } \forall (i,j) \text{ s.t. } i \neq j \text{ and } n_j \geqq n_i \}$$

$$\mathcal{E}_k = \{ E_k : p\times r(k) \mid [E_k]_{ij} = 0 \quad (0 \leqq k \leqq n-1) \quad \text{for } \forall (i,j) \text{ s.t. } n_j - n_i \geqq n-k \}.$$

Note that the sparseness of $E_k \in \mathcal{E}_k$ is complementary to that of $F_k \in \mathcal{F}_k$ (see (2.2.9)). It is also noted that $E(s) \in \mathcal{E}$ is written as

$$E(s) = \sum_{k=0}^{n} E_k T_k(s) = T_n'(s) + \sum_{k=0}^{n-1} T_k'(s) \tilde{E}_k , \qquad (10)$$

where $\tilde{E}_k$ is a $r(k) \times p$ matrix defined by

$$[\tilde{E}_k]_{ij} = \begin{cases} [E_{k+n_i-n_j}]_{ij} & \text{if } k \leqq n+n_j-n_i \\ 0 & \text{otherwise.} \end{cases}$$

This means that $E(s) \in \mathcal{E}$ is at once column reduced and row reduced, which is a distinctive feature of the polynomial echelon form.

<u>Example 2.3.1</u> (Continued from Ex.2.2.5) When $(\nu_1, \nu_2, \nu_3) = (n_1, n_2, n_3) = (4,2,1)$, the polynomial echelon form $E(s) \in \mathcal{E}$ is written as

$$E(s) = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} T_4(s) + \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} T_3(s)$$

$$+ \begin{pmatrix} * & * \\ 0 & * \\ 0 & * \end{pmatrix} T_2(s) + \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} T_1(s) + \begin{pmatrix} * \\ * \\ * \end{pmatrix} T_0(s$$

$$= T'_4(s) + T'_3(s) \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} + T'_2(s) \begin{pmatrix} * & * & 0 \\ * & * & * \end{pmatrix}$$

$$+ T'_1(s) \begin{pmatrix} * & * & * \end{pmatrix} + T'_0(s) \begin{pmatrix} * & * & * \end{pmatrix} .$$

3. INNER PRODUCTS AND THE CONTINUOUS-TIME STABILITY OF POLYNOMIAL MATRICES

3.1. The inner product on the imaginary axis

In this section, we will work with a given $p \times p$ positive-definite matrix Π and a given $p\times p$ PM $C(s)$ which is (continuous-time) stable, i.e., all the zeros of det $C(s)$ have negative real parts. Suppose that $P(s) : k\times p$ and $Q(s) : \ell\times p$ are PM's such that $P(s)C^{-1}(s)$ and $Q(s)C^{-1}(s)$ are both strictly proper; for convenience we denote the condition by

$$P(s) \dashv C(s) \quad \text{and} \quad Q(s) \dashv C(s). \tag{1}$$

Then, $P(s)C^{-1}(s)$ and $Q(s)C^{-1}(s)$ are both square integrable on the imaginary axis, and hence we can define a matrix-valued inner product as follows:

$$\langle P(s),Q(s)\rangle \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} P(i\omega)C^{-1}(i\omega)\,\Pi\, C_*^{-1}(i\omega)Q_*(i\omega)d\omega \quad : k\times \ell, \tag{2}$$

where $C_*(s) \triangleq C'(-s)$ and $Q_*(s) \triangleq Q'(-s)$. It is evident from the definition that the inner product satisfies the following properties.

(i) $\langle P_1(s)+P_2(s),Q(s)\rangle = \langle P_1(s),Q(s)\rangle + \langle P_2(s),Q(s)\rangle$.

(ii) $\langle KP(s),Q(s)\rangle = K\langle P(s),Q(s)\rangle$,
where K is a constant matrix.

(iii) $\langle P(s),Q(s)\rangle = \langle Q(s),P(s)\rangle'$.

(iv) $\begin{cases} \langle P(s),P(s)\rangle \geqq 0. \\ \langle P(s),P(s)\rangle = 0 \quad \text{iff} \quad P(s)=0. \\ \langle P(s),P(s)\rangle > 0 \quad \text{iff} \quad P(s) \text{ has full row rank.} \end{cases}$

(v) $$\langle sP(s),Q(s)\rangle + \langle P(s),sQ(s)\rangle = 0 \tag{3}$$

if

$$sP(s) \prec C(s) \quad \text{and} \quad sQ(s) \prec C(s). \tag{4}$$

We note that the property (v) is a distinctive feature of inner products on the imaginary axis.

As the next step, we will extend the domain of the definition of the inner product $\langle P(s),Q(s)\rangle$ from (1) to

$$P(s) \sim C(s) \quad \text{and} \quad Q(s) \prec C(s), \tag{5}$$

where $P(s) \sim C(s)$ means that $P(s)C^{-1}(s)$ is proper (i.e., tends to a finite value as $s \to \infty$). Suppose that $P(s) \sim C(s)$ and that $P(s) \not\prec C(s)$. Then $P(s)C^{-1}(s)$ is bounded on the imaginary axis, although it is not square integrable. Hence, if $Q(s)C^{-1}(s)$ is absolutely integrable, the integrand $PC^{-1}\Pi C_*^{-1}Q_*$ in (2) is also absolutely integrable, and the inner product $\langle P(s),Q(s)\rangle$ can be defined. We note that a necessary and sufficient condition for $Q(s)C^{-1}(s)$ to be absolutely integrable is that $sQ(s) \prec C(s)$. On the other hand, the situation is more delicate if $Q(s) \prec C(s)$ and if $sQ(s) \not\prec C(s)$. In this case, $PC^{-1}\Pi C_*^{-1}Q_*$ is not absolutely integrable. Nevertheless, we can define $\langle P(s),Q(s)\rangle$ by taking the Cauchy's principal value of the integral (2), on the basis of the following lemma.

Lemma 3.1.1   If a rational function f(s) is analytic in Re s $\geqq$ 0 and is strictly proper, then

$$\lim_{r\to\infty}\int_{-r}^{r} f(i\omega)d\omega = \pi[f(s)]_{-1} \tag{6}$$

where

$$[f(s)]_{-1} \triangleq \lim_{s\to\infty} sf(s).$$

(Proof)   Integrating f(s) along the semicircular closed curve shown in Fig.1 and applying Cauchy's integral theorem, we have

$$\int_{-r}^{r} f(i\omega)d\omega = r\int_{-\pi/2}^{\pi/2} f(re^{i\omega})e^{i\omega}d\omega. \tag{7}$$

Since

$$f(s) = [f(s)]_{-1}s^{-1} + O(s^{-2}),$$

it follows that

$$\int_{-\pi/2}^{\pi/2} f(re^{i\omega})e^{i\omega}d\omega = \pi[f(s)]_{-1}r^{-1} + O(r^{-2}). \tag{8}$$

The desired result (6) is obtained from (7) and (8).   (QED)

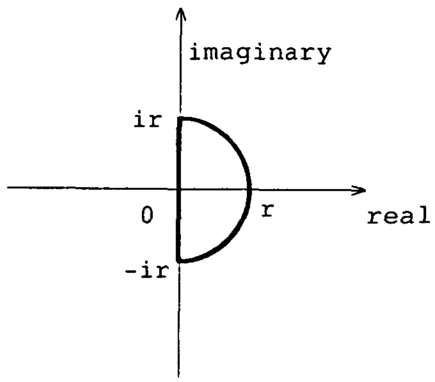

Fig.1 The path of integration

The following is immediate from the above lemma.

(vi) if $Q(s) \prec C(s)$, then

$$\langle C(s),Q(s)\rangle = \frac{1}{2}\Pi\,[Q(s)C^{-1}(s)]'_{-1}. \tag{9}$$

It is concluded from (vi) that C(s) is orthognal to every Q(s) such that $sQ(s) \prec C(s)$.

Generally, an arbitrary PM P(s) such that P(s) C(s) is written as

$$P(s) = K\,C(S) + R(s), \tag{10}$$

where K is a constant matrix and R(s) is a PM such that $R(s) \prec C(s)$. Hence we have

$$\langle P(s),Q(s)\rangle = \frac{1}{2}K\,\Pi\,[Q(s)C^{-1}(s)]'_{-1} + \langle R(s),Q(s)\rangle \tag{11}$$

Thus, $\langle P(s),Q(s)\rangle$ has been defined for arbitrary P(s) and Q(s) satifying (5). It is obvious that the fundamental properties (i)-(iv) are valid for the extended inner product also and that the assumption (4) in the property (v) can be replaced with the weakened assumption

$$P(s) \prec C(s) \quad \text{and} \quad Q(s) \prec C(s). \tag{12}$$

Now, let us consider how to calculate $\langle P(s),Q(s)\rangle$ for given P(s) and Q(s).

<u><Method I></u> We use the fact that there exist matrices F and G and a PM S(s) such that

$$\begin{cases} (F,G) \text{ is controllable,} & (13) \\ C(s) \text{ and } S(s) \text{ are right coprime,} & (14) \\ (sI-F)^{-1}G = S(s)C^{-1}(s). & (15) \end{cases}$$

Given such a triplet $(F,G,S(s))$, then

$$X \triangleq \langle S(s),S(s)\rangle$$

is obtained as the unique solution of the Lyapunov equation

$$FX + XF' + G\Pi G' = 0. \tag{16}$$

If $P(s)$ and $Q(s)$ satisfy (1), they are written as

$$P(s) = P\,S(s) \quad \text{and} \quad Q(s) = Q\,S(s), \tag{17}$$

where P and Q are constant matrices, and we have

$$\langle P(s),Q(s)\rangle = PXQ'. \tag{18}$$

The case of (5) is reduced to the case of (1) by the use of (10) and (11).

<Method II> It is known ([1]) that under the condition that $C(s)$ is stable there exists a PM $W(s)$ satisfying

$$C(s)W_*(s) + W(s)C_*(s) = \Pi. \tag{19}$$

This equation is also written as

$$C^{-1}(s)\,\Pi\,C_*^{-1}(s) = C^{-1}(s)W(s) + W_*(s)C_*^{-1}(s). \tag{20}$$

Since the integrand $PC^{-1}\,\Pi\,C_*^{-1}Q_*$ of the integral (2) is strictly proper under the condition (5), it follows from (20) that

$$P(s)C^{-1}(s)\,\Pi\, C_*^{-1}(s)Q_*(s)$$

$$= [P(s)C^{-1}(s)W(s)Q_*(s)]_{sp} + [P(s)W_*(s)C_*^{-1}(s)Q_*(s)]_{sp}, \tag{21}$$

where $[\ ]_{sp}$ denotes the strictly proper part of a rational function defined as follows.

<u>Definition 3.1.2</u> An arbitrary rational matrix F(s) is uniquely represented as

$$F(s) = P(s) + G(s),$$

where P(s) is a PM and G(s) is strictly proper. We write

$$\begin{cases} P(s) = [F(s)]_{pol} \\ G(s) = [F(s)]_{sp}\ . \end{cases}$$

Noting that the first term in the right-hand side of (21) is analytic in Re s $\geqq 0$ while the second term is analytic in Re s $\leqq 0$, we obtain from Lemma 3.1.1

$$\langle P(s),Q(s)\rangle = \frac{1}{2}[P(s)\{C^{-1}(s)W(s)-W_*(s)C_*^{-1}(s)\}Q_*(s)]_{-1}, \tag{22}$$

where we define

$$[F(s)]_{-1} \triangleq \lim_{s\to\infty} s[F(s)]_{sp} \tag{23}$$

for a rational matrix F(s).

3.2. Time-domain representations of the inner product

In this section, the meaning of the inner product defined in the frequency domain in the previous section will be investigated in the time domain, both in a deterministic framework and in a stochastic framework.

<u><The deterministic representation></u>

Let $\{h(t);\ t \in \mathbb{R}\}$ be the impulse response matrix of $C^{-1}(s)$; i.e.,

$$\begin{cases} C(d/dt)h(t) = \delta(t)I & (1) \\ h(t) = 0 \quad \text{if} \quad t<0, & (2) \end{cases}$$

where d/dt denotes the differentiation with respect to t and $\delta(t)$ is Dirac's delta function. It is noted that $C^{-1}(s)$ is the Laplace transform of $h(t)$; i.e.,

$$C^{-1}(s) = \int_0^\infty h(t)e^{-st}dt. \tag{3}$$

Since $P(s)C^{-1}(s)$ and $Q(s)C^{-1}(s)$ are the Laplace transforms of $P(d/dt)h(t)$ and $Q(d/dt)h(t)$ respectively, application of Parseval's theorem to eq.(3.1.2) yields

$$\langle P(s),Q(s)\rangle = \int_{0-}^\infty \{P(d/dt)h(t)\}\, \Pi\, \{Q(d/dt)h(t)\}'\, dt. \tag{4}$$

This is the deterministic representation of the inner product in the time domain. In the case where $P(s) \dashv C(s)$ and $Q(s) \dashv C(s)$, we can use $\int_0^\infty$ instead of $\int_{0-}^\infty$ in (4). On the other hand, in the case where $P(s) \not\dashv C(s)$, $\int_{0-}^\infty$ cannot be replaced with $\int_0^\infty$, because $P(d/dt)h(t)$ has a δ-function like singularity at $t=0$. It is noted that the property (v) in the previous section (with the weakened assumption (3.1.12)) is obtained from (4) by the use of integration by parts.

Let us see how the property (vi) (eq.(3.1.9)) is derived from (4). Recalling (1) and (2), and using the formula

$$\int_{0-}^{\infty} \delta(t)f(t)dt = \frac{1}{2}\{ f(0+)+f(0-) \},$$

we have

$$\langle C(s),Q(s)\rangle = \frac{1}{2} \Pi \{Q(d/dt)h(0+)\}'.$$

Owing to a fundamental property of the Laplace transformation, it follows that the right-hand side of the above equation is equal to that of (3.1.9).

Remark 3.2.1 Replacing $\int_{0-}^\infty$ in (4) with $\int_{0+}^\infty$, we can define another inner product

$$\lhd P(s),Q(s)\rhd \triangleq \int_{0+}^{\infty} \{P(d/dt)h(t)\} \Pi \{Q(d/dt)h(t)\}' dt. \qquad (5)$$

The frequency domain representation of this inner product is

$$\lhd P(s),Q(s)\rhd$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [P(i\omega)C^{-1}(i\omega)]_{sp} \Pi [C_*^{-1}(i\omega)Q_*(i\omega)]_{sp} d\omega. \qquad (6)$$

It is obvious that if $P(s) \dashv C(s)$ and $Q(s) \dashv C(s)$ then $\lhd P(s),Q(s)\rhd = \langle P(s),Q(s)\rangle$. However, the property (v) under the weakened assumption (3.1.12) is not valid for the inner product (5). Furthermore, the property (vi) is replaced with the following: If $Q(s) \dashv C(s)$, then

$$\lhd C(s), Q(s) \rhd = 0. \tag{7}$$

<The stochastic representation>

Let $u(t)$ be the p-dimensional Gaussian white noise process with variance parameter Π; i.e.,

$$E[u(t_1)u'(t_2)] = \delta(t_1 - t_2)\,\Pi , \tag{8}$$

where $E[\ldots]$ denotes the expectation of a random variable. The process $u(t)$ can be regarded as the formal derivative $dw(t)/dt$ of Wiener's Brownian motion process $w(t)$ such that

$$E[w(t_1)w'(t_2)] = \min(t_1, t_2)\,\Pi . \tag{9}$$

Suppose that $P(s) \dashv C(s)$ and $Q(s) \dashv C(s)$, and let $p(t)$ and $q(t)$ be the responses of $P(s)C^{-1}(s)$ and $Q(s)C^{-1}(s)$, respectively, to the input $u(t)$; i.e.,

$$\begin{cases} p(t) \triangleq P(d/dt)C^{-1}(d/dt)u(t) & (10) \\ q(t) \triangleq Q(d/dt)C^{-1}(d/dt)u(t). & (11) \end{cases}$$

Since $C(s)$ is assumed to be a stable PM, $p(t)$ and $q(t)$ can be made staionary with a suitable choice of initial conditions. Then $p(t)$ and $q(t)$ are of finite variance, and according to the spectral theory of stochastic porocesses we have

$$R_{pq}(t_1-t_2) \triangleq E[p(t_1)q'(t_2)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(i\omega)C^{-1}(i\omega)\,\Pi\, C_*^{-1}(i\omega)Q_*(i\omega)e^{i\omega(t_1-t_2)}d\omega. \qquad (12)$$

In particular, we have

$$\langle P(s),Q(s)\rangle = E[p(t)q'(t)]. \qquad (13)$$

This is the stochastic representation of the inner product for the case where $P(s) \dashv C(s)$ and $Q(s) \dashv C(s)$. In this representation, the property (v) (eq.(3.1.3)) of the inner product is written as

$$E[\frac{d}{dt}p(t)\cdot q'(t)] + E[p(t)\cdot\frac{d}{dt}q'(t)] = 0,$$

which is regarded as a consequence of the stationarity

$$d\ E[p(t)q'(t)] = 0 \qquad (14)$$

and of the Leibniz's formula

$$d\left\{p(t)q'(t)\right\} = dp(t)\cdot q'(t) + p(t)\cdot dq'(t). \qquad (15)$$

When $P(s) \sim C(s)$ but $P(s) \not\dashv C(s)$, however, $p(t)$ is not a stochastic process in the ordinary sense, and we need some new notion in order to extend the representation (13) to this case. One way to deal with such a situation is to use the concept of random distributions ([14]), which is the stochastic version of L. Schwartz's distributions. In this framework, $p(t)$ and $q(t)$ are regarded as random distributions, and the formula (12), where R_{pq} is a distribution in general, is always justified. Hence, the representation (13) is also justified when $P(s) \sim C(s)$ and

$Q(s) \prec C(s)$. However, this framework is too formal, and it is difficult to see the meaning of the property (vi) (eq.(3.1.9)) in this framework. In the sequel, we take another approach which is more elementary.

When $P(s) \sim C(s)$, $p(t)$ is expressed formally as

$$p(t) = d\tilde{p}(t)/dt, \tag{16}$$

where $\tilde{p}(t)$ is a stochastic process in the ordinary sense. Using this expression, eq.(13) is written as

$$\langle P(s), Q(s) \rangle = E[d\tilde{p}(t) \cdot q'(t)]/dt. \tag{17}$$

The problem is to provide $d\tilde{p}(t) \cdot q'(t)$ with a definite meaning. To this problem, we can apply one of the two well-known methods of stochastic calculus: the Itô calculus and the Stratonovich calculus ([15]). Roughly speaking, the Itô calculus treats

$$d\tilde{p}(t) * q'(t) \triangleq \{ \tilde{p}(t+dt) - \tilde{p}(t) \} q'(t) \quad (dt > 0), \tag{18}$$

while the Stratonovich calculus treats

$$d\tilde{p}(t) \circ q'(t) \triangleq \{ \tilde{p}(t+dt) - \tilde{p}(t) \} \{ q'(t+dt) + q'(t) \} /2 \quad (dt > 0). \tag{19}$$

Note that the restriction $(dt > 0)$ cannot be deleted in the above definitions. These two notions are related to each other by

$$d\tilde{p}(t) \circ q'(t) = d\tilde{p}(t) * q'(t) + \frac{1}{2} d\tilde{p}(t) dq'(t). \tag{20}$$

Now, we claim that if $P(s) \sim C(s)$ and if $Q(s) \prec C(s)$ then

$$\langle P(s),Q(s)\rangle = E[d\tilde{p}(t)\circ q'(t)]/dt \tag{21}$$

(Proof) It is sufficient to show the above equation for $P(s)=C(s)$. In this case, it follows from (10) that $p(t)=u(t)$ $=dw(t)/dt$ and that $\tilde{p}(t)=w(t)$. Hence, recalling the property (vi) (eq.(3.1.9)), the equation (21) is written as

$$E[dw(t)\circ q'(t)]/dt = \frac{1}{2}\Pi K', \tag{22}$$

where $K \triangleq [Q(s)C^{-1}(s)]_{-1}$. Let us prove this. First, we have

$$E[dw(t)*q'(t)] = 0, \tag{23}$$

because $dw(t)$ ($= w(t+dt)-w(t)$) and $q'(t)$ are independent when $dt>0$ (see (18)). Therefore, from (20) we get

$$E[dw(t)\circ q'(t)] = \frac{1}{2}E[dw(t)dq'(t)]. \tag{24}$$

Next, we note that $sQ(s)$ is uniquely represented as

$$sQ(s) = K\,C(s) + R(s)$$

with a PM $R(s)$ such that $R(s)\dashv C(s)$. This means that

$$dq(t) = K\,dw(t) + r(t)dt, \tag{25}$$

where $r(t) \triangleq R(d/dt)C^{-1}(d/dt)u(t)$. It can be shown from (8) (or (9)) that

$$E[dw(t)dw'(t)] = \Pi\,dt,$$

and therefore from (25) we obtain

$$E[dw(t)dq'(t)] = \Pi K'dt + o(dt).$$

This equation, combined with (24), leads to the desired equation (22). Thus, the representation (21) has been proved. (QED)

It is known that the Stratonovich calculus preserves all the formal rules of the usual differential calculus. For instance, the following Leibniz's formula is valid:

$$d\{p(t)q'(t)\} = dp(t)\circ q'(t) + p(t)\circ dq'(t). \tag{26}$$

This is a generalization of (15) and proves the property (v) under the weakened assumption (3.1.12). It is noted that the Itô calculus does not preserve the Leibniz's formula.

Remark 3.2.2 Comparing (23) with (7), we can see that

$$\langle\!\langle P(s),Q(s)\rangle\!\rangle = E[d\tilde{p}(t)*q'(t)]/dt. \tag{27}$$

3.3. Defining the inner product without the stability assumption

So far, we have investigated the inner product defined by the integral on the imaginary axis (3.1.2) from given $(C(s),\Pi)$ under the assumption that $C(s)$ is stable. In this section, we will extend the definition of the inner product to the general case without the stability assumption, for the purpose of studying the stability of $C(s)$ in terms of the defined inner product. The definition by the integral (3.1.2) is not suitable for our purpose, because the integral does not have information about whether $C(s)$ is stable or not. Alternatively, as the definition we adopt the equations (3.1.19) and (3.1.22) which was appeared in the second method of calculating the inner product; i.e., given a $p \times p$ nonsingular PM $C(s)$ and a $p \times p$ positive-definite matrix Π, the inner product between two arbitrary PM's with p columns, say $P(s)$ and $Q(s)$, is defined by

$$\langle P(s),Q(s)\rangle \triangleq \frac{1}{2}[P(s)\{C^{-1}(s)W(s)-W_{\star}(s)C_{\star}^{-1}(s)\}Q_{\star}(s)]_{-1}, \tag{1}$$

where $W(s)$ is a $p \times p$ PM satisfying

$$C(s)W_{\star}(s) + W(s)C_{\star}(s) = \Pi. \tag{2}$$

About the equation (2), the following result is known ([1]): Under the assumption that

$$\det C(s) \text{ and } \det C_{\star}(s) \text{ have no common zeros,} \tag{3}$$

there exists a PM solution $W(s)$ of (2), and if $W(s)$ is constrained to be such that $C^{-1}(s)W(s)$ is strictly proper, then

$W(s)$ is unique. From now on, we assume (3). Owing to the uniqueness result mentioned above, the definition (1) does not depend on a choice of a solution $W(s)$ of (2), because if both $W_1(s)$ and $W_2(s)$ are solutions then $C^{-1}(s)W_1(s)-C^{-1}(s)W_2(s)$ turns out to be a PM.

Let us examine the validity of the fundamental properties (i)-(vi) in Sec.3.1. The validity of (i) and (ii) is obvious. It is also clear from (1) that (v) is valid under the weakened assumption (3.1.12). The property (iii) is easily verified by using the formula $[R_*(s)]_{-1}=-[R(s)]'_{-1}$. Modifying (2) as

$$C^{-1}(s)W(s)-W_*(s)C_*^{-1}(s) = 2C^{-1}(s)W(s)-C^{-1}(s)\,\Pi\,C_*^{-1}(s),$$

we can see that (vi) follows immediately from (1). Thus the validity of all the fundamental properties but (iv) has been verified. On the other hand, the positivity (iv) is not valid in general. (The inner product is said to be <u>indefinite</u> in this sense.) Actually, we have the following theorem.

<u>Theorem 3.3.1</u> $C(s)$ is stable if and only if the inner product defined from $(C(s),\Pi)$ is positive, where Π is an arbitrary positive-definite matrix.

In proving the above theorem, we use the fact that if $P(s)\dashv C(s)$ and if $Q(s)\dashv C(s)$ then the definition of $\langle P(s),Q(s)\rangle$ by (1) and (2) is equivalent to the definition by the equations (3.1.13)-(3.1.18) which was appeared in the first method of calculating $\langle P(s),Q(s)\rangle$; i.e., given $(F,G,S(s))$ such that

$$
\begin{cases}
(F,G) \text{ is controllable} & (4) \\
C(s) \text{ and } S(s) \text{ are right coprime} & (5) \\
(sI-F)^{-1}G = S(s)C^{-1}(s), & (6)
\end{cases}
$$

then $X \triangleq \langle S(s),S(s)\rangle$ satisfies the Lyapunov equation

$$FX + XF' + G\Pi G' = 0. \qquad (7)$$

This fact is shown by noting that

$$
\begin{aligned}
&\langle sS(s),S(s)\rangle \\
&= FX + G\langle C(s),S(s)\rangle \qquad \text{(by (6))} \\
&= FX + \frac{1}{2}G\Pi[S(s)C^{-1}(s)]'_{-1} \qquad \text{(by the property (vi))} \\
&= FX + \frac{1}{2}G\Pi G' \qquad \text{(by (6))}
\end{aligned}
$$

and by invoking the property (v). Now we can see from (4)-(7) that the stability of $C(s)$ is equivalent to the positive-definiteness of X, which is also equivalent to the positivity of the inner product. Thus, Th.3.3.1 has been proved.

Remark 3.3.2 Owing to (4) and (7), $X=\langle S(s),S(s)\rangle$ is a nonsingular matrix in general. This property will be referred to as the nonsingularity of the inner product in later sections.

4. THE CONTINUOUS-TIME THEORY OF THE STABILITY AND ORTHOGONAL POLYNOMIAL MATRICES

4.1. The stability and orthogonal polynomial matrices

Suppose that we are given a $p \times p$ positive-definite matrix Π and a $p \times p$ nonsingular PM $C(s)$ satisfying (3.3.3), i.e.,

$$\det C(s) \text{ and } \det C_*(s) \text{ have no common zeros.}$$

Then, in the same way as in Sec.3.3 we can define from $(C(s),\Pi)$ the inner product $\langle P(s),Q(s)\rangle$ for PM's $P(s)$ and $Q(s)$ satisfying (3.1.5). In this section we further assume that $C(s)$ is column reduced with descending column degrees $n_1 \geq \cdots \geq n_p$, which allows us to use the notation and the results in Chap.2. In this situation, the condition (3.1.5) is equivalent to

$$\deg P(s) \leq n \quad \text{and} \quad \deg Q(s) \leq n-1. \tag{1}$$

This means that if $P(s)$ and $Q(s)$ are written as

$$\begin{cases} P(s) = \sum_{j=0}^{n} P_j \, T_j(s) \\ Q(s) = \sum_{j=0}^{n-1} Q_j \, T_j(s) \end{cases} \tag{2}$$

then the following inner product is defined:

$$\langle P(s),Q(s)\rangle = \sum_{i=0}^{n} \sum_{j=0}^{n-1} P_i \, X_{ij} \, Q_j', \tag{3}$$

where

$$X_{ij} \triangleq \langle T_i(s), T_j(s) \rangle \quad : r(i) \times r(j). \tag{4}$$

In particular, if both deg P(s) and deg Q(s) are less than n, then eq.(3) is written as

$$\langle P(s), Q(s) \rangle = [P_0, \ldots, P_{n-1}] \; X \; [Q_0, \ldots, Q_{n-1}]', \tag{5}$$

where

$$X \triangleq \langle T(s), T(s) \rangle$$

$$= \begin{bmatrix} X_{0,0} & \cdots & X_{0,n-1} \\ \vdots & & \vdots \\ X_{n-1,0} & \cdots & X_{n-1,n-1} \end{bmatrix} \quad : N \times N.$$

Recalling that the companion pair (Γ, B) defined from C(s) satisfies (2.1.25), and applying the argument about (F,G,S(s)) in (3.3.4)-(3.3.7) to $(\Gamma, B, T(s))$, we can see that X is the unique solution of the Lyapunov equation

$$\Gamma X + X\Gamma' + B \Pi B' = 0. \tag{6}$$

When a monic PM of degree j ($0 \leqq j \leqq n$), say

$$R_j(s) = T_j(s) + R_{j,j-1}T_{j-1}(s) + \ldots + R_{j,0}T_0(s) \quad : r(j) \times p, \tag{7}$$

satisfies the orthogonal condition

$$\langle R_j(s), T_i(s) \rangle = 0 \quad \text{for } \forall i = 0,1,\ldots,j-1, \tag{8}$$

we say that $R_j(s)$ is an <u>orthogonal PM of degree j</u> defined from $(C(s), \Pi)$. Let X_j ($0 \leqq j \leqq n-1$) be a submatrix of X defined as

$$X_j \triangleq \begin{bmatrix} X_{0,0} & \cdots & X_{0,j} \\ \vdots & & \vdots \\ X_{j,0} & \cdots & X_{j,j} \end{bmatrix} . \tag{9}$$

Then (8) is written as

$$[R_{j,0},\ldots,R_{j,j-1}]\ X_{j-1} + [X_{j,0},\ldots,X_{j,j-1}] = 0. \tag{10}$$

Therefore the condition that

$$X_j \text{ is nonsingular for } \forall j = 0,1,\ldots,n-1 \tag{11}$$

ensures the existence and the uniqueness of $R_j(s)$ for every $j=0,1,\ldots,n-1$. From now on, we assume (11). Note that if C(s) is stable then X is positive-definite and the assumption is satisfied.

The orthogonal PM's $\{R_j(s)\ ;\ 0 \leq j \leq n\}$ constitute an orthogonal system; i.e.,

$$\langle R_i(s), R_j(s) \rangle = 0 \quad \text{if } i \neq j.$$

Hence, defining a $r(j) \times r(j)$ symmetric matrix

$$\varepsilon_j \triangleq \langle R_j(s), R_j(s) \rangle \tag{12}$$

$$(\;= \langle R_j(s), T_j(s) \rangle \quad \text{by (8)} \;)$$

and letting

$$R(s) \triangleq [R_0'(s),\ \ldots\ , R_{n-1}'(s)]' \quad : N \times p, \tag{13}$$

we have

$$\langle R(s), R(s) \rangle = (E \triangleq) \text{ block diag } \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}\}. \quad (14)$$

Using the expression

$$R(s) = R\, T(s)$$

where

$$R \triangleq \begin{bmatrix} I_{r(0)} & & & \bigcirc \\ R_{1\,0} & I_{r(1)} & & \\ \vdots & \vdots & \ddots & \\ R_{n-1\,0} & R_{n-1\,1} & \cdots & I_{r(n-1)} \end{bmatrix} \quad : N \times N \quad (15)$$

eq.(14) is written as

$$R\, X\, R' = E. \quad (16)$$

We can see from (16) that the assumption (11) implies the nonsingularity of ε_j for $\forall j=0,1,\ldots,n-1$. Moreover, since the stability of C(s) is equivalent to the positive-definiteness of X (see (6)), we obtain the following theorem.

<u>Theorem 4.1.1</u> C(s) is stable if and only if ε_j is positive-definite for $\forall j=0,1,\ldots,n-1$.

The above theorem itself might be a trivial result. Nevertheless, it will be shown later that combination of the theorem and a recursive formula for $\{R_j(s)\}$ yields several important results.

<u>Remark 4.1.2</u> According to the fact that in (6) the inertia of X completely determines the location of the eigen-

values of Γ relative to the imaginary axis (see Theorem 3.3 in [9]), we can generalize Th.4.1.1 as follows. Let k_j be the number of positive eigenvalues of ε_j. Then the number of zeros of det C(s) in the open left half-plane is $\sum_{j=0}^{n-1} k_j$.

The orthogonal PM's $\{R_j(s)\}$ can be used as a basis for representing PM's as well as $\{T_j(s)\}$. The representation of C(s) w.r.t. the basis $\{R_j(s)\}$ is as follows.

<u>Theorem 4.1.3</u> C(s) is written as

$$C(s) = C_n \{ R_n(s) + \frac{1}{2} \bar{\Pi} \Lambda'_{n-1} \varepsilon_{n-1}^{-1} R_{n-1}(s) \}, \tag{17}$$

where

$$\bar{\Pi} \triangleq C_n^{-1} \Pi C_n'^{-1}. \tag{18}$$

(Proof) We note that the property (vi) in Sec.3.1 (eq.(3.1.9)) is equivalent to

$$\langle C(s), T_j(s) \rangle = \begin{cases} 0 & \text{if } 0 \leqq j \leqq n-2 \\ \frac{1}{2} \Pi C_n'^{-1} \Lambda'_{n-1} & \text{if } j = n-1 \end{cases} \tag{19}$$

and that C(s) is characterized as a PM of degree n having the highest degree coefficient C_n and obeying (19). Hence the equation (17) is readily proved by verifying that its right-hand side satisfies (19). (QED)

As the next step, we will investigate the mutual relation among $\{R_j(s)\}$. For this purpose we introduce new quantities:

$$\theta_j \triangleq \langle sR_j(s), R_j(s)\rangle \quad : r(j) \times r(j) \tag{20}$$

for j=0,1,...,n-1. It then turns out from the property (v) in Sec.3.1 (eq.(3.1.3)) that θ_j is skew-symmetric; i.e., $\theta_j' = -\theta_j$.

First, we consider the strictly regular case where $n=n_i$ for $\forall i$ (see Ex.2.1.1). Then we have the following three term recurrence for j=1,2,...,n-1:

$$R_{j+1}(s) = (sI-\theta_j\varepsilon_j^{-1})\, R_j(s) + \varepsilon_j\varepsilon_{j-1}^{-1}\, R_{j-1}(s). \tag{21}$$

Starting from the initial condition

$$R_0(s) = I \qquad R_1(s) = sI - \theta_0\varepsilon_0^{-1}, \tag{22}$$

$\{R_j(s)\}$ can be produced recursively by (21).

(Proof of (21)) It is sufficient to verify that the right-hand side of (21) is orthogonal to $R_i(s)$ for $\forall i=0,1,\ldots,j$. Owing to the property (v) in Sec.3.1 (eq.(3.1.3)), the inner product between the right-hand side and $R_i(s)$ is written as

$$- \langle R_j(s), sR_i(s)\rangle - \theta_j\varepsilon_j^{-1}\langle R_j(s), R_i(s)\rangle$$
$$+ \varepsilon_j\varepsilon_{j-1}^{-1}\langle R_{j-1}(s), R_i(s)\rangle.$$

For i=0,1,...,j-2, it is obvious from the orthogonality of $\{R_j(s)\}$ that the above expression is equal to 0. It is also clear from the definitions of $\{\varepsilon_j\}$ and of $\{\theta_j\}$ that the above expression vanishes for i=j-1 and for i=j. (QED)

Example 4.1.4 (The scalar case : the Routh-Hurwitz test)
Suppose that C(s) is a monic scalar polynomial of degree n written as

$$C(s) = s^n + C_{n-1}s^{n-1} + \dots + C_0. \tag{23}$$

Then, since the skew-symmetric matrices $\{\theta_j\}$ vanish, the equations (21) and (22) are reduced to

$$\begin{cases} R_0(s) = 1 \qquad R_1(s) = s & (24) \\ R_{j+1}(s) = s\,R_j(s) + e_j R_{j-1}(s), & (25) \end{cases}$$

where $e_j \triangleq \varepsilon_j/\varepsilon_{j-1}$ $(1 \leqq j \leqq n-1)$. It follows from the above equations that $R_j(s)$ is of the form

$$R_j(s) = s^j + R_{j,j-2}s^{j-2} + R_{j,j-4}s^{j-4} + \dots . \tag{26}$$

On the other hand, Th.4.1.3 shows that

$$C(s) = R_n(s) + e_n R_{n-1}(s), \tag{27}$$

where $e_n \triangleq \Pi/2\varepsilon_{n-1}$. Invoking (26) and comparing (27) with (23), we can see that $R_n(s)$, $R_{n-1}(s)$ and e_n are obtained directly from C(s) as

$$\begin{cases} R_n(s) = s^n + C_{n-2}s^{n-2} + C_{n-4}s^{n-4} + \dots \\ R_{n-1}(s) = s^{n-1} + (C_{n-3}/C_{n-1})s^{n-3} + (C_{n-5}/C_{n-1})s^{n-5} + \dots \\ e_n = C_{n-1}. \end{cases} \tag{28}$$

Furthermore, eq.(25) can be interpreted as the Euclidean algorithm producing $\{e_j, R_{j-1}(s)\}$ from $\{R_{j+1}(s), R_j(s)\}$. Thus it is seen that the equations (24)-(27) are equivalent to the procedure of the Routh-Hurwitz stability test (see, for instance, [8]) except for the order of computation. It is obvious that the stability criterion of the Routh-Hurwitz test

$$e_j > 0 \quad \text{for} \quad \forall j = 1,2,\ldots,n$$

is equivalent to the criterion of Th.4.1.1.

Remark 4.1.5 The above example shows that, in the scalar case, $R_n(s)$ and $R_{n-1}(s)$ are obtained directly by decomposing $C(s)$ into the even power part and the odd power part as in (28) and that we can produce $R_{n-2}(s)$, $R_{n-3}(s)$, ..., $R_0(s)$ in that order from $R_n(s)$ and $R_{n-1}(s)$ by the Euclidean algorithm (25). This is a special feature of the scalar case. In the matrix case, we cannot compute $R_n(s)$ and $R_{n-1}(s)$ directly from $C(s)$, and instead we must compute $\{R_j(s)\}$ in reverse order by (12), (20) and (21), for which we need to solve the PM equation (3.3.2) or the Lyapunov equation (6). It is noted that a similar situation arises in the discrete-time case concerning the Schur-Cohn test and the Levinson algorithm. See [5], [10] and [20] for this problem.

Example 4.1.6 (The first order case : the Lyapunov test) Suppose that $C(s)$ is written as

$$C(s) = sI - F. \tag{29}$$

It then follows from (17) and (22) that

$$C(s) = R_1(s) + \frac{1}{2}\Pi\,\varepsilon_0^{-1}R_0(s)$$

$$= sI - \theta_0\varepsilon_0^{-1} + \frac{1}{2}\Pi\,\varepsilon_0^{-1},$$

which leads to

$$F = (\theta_0 - \frac{1}{2}\Pi)\,\varepsilon_0^{-1}. \tag{30}$$

It is immediate from the above equation (or from (6)) that ε_0 satisfies the Lyapunov equation

$$F\varepsilon_0 + \varepsilon_0 F' + \Pi = 0. \tag{31}$$

The skew-symmetric matrix θ_0 is given by

$$\theta_0 = F\varepsilon_0 + \frac{1}{2}\Pi.$$

The stability criterion $\varepsilon_0 > 0$ of Th.4.1.1 is nothing but the criterion of the Lyapunov test.

<u>Example 4.1.7</u> (The scond order case) Let us consider the second order case:

$$C(s) = s^2 I + s\,C_1 + C_0. \tag{32}$$

From (17), (21) and (22), we have

$$\begin{cases} R_1(s) = s\,I - \theta_0\varepsilon_0^{-1} \\ R_2(s) = s^2\,I - s\,(\theta_1\varepsilon_1^{-1} + \theta_0\varepsilon_0^{-1}) + \theta_1\varepsilon_1^{-1}\theta_0\varepsilon_0^{-1} + \varepsilon_1\varepsilon_0^{-1} \end{cases} \tag{33}$$

$$C(s) = R_2(s) + \frac{1}{2}\Pi\,\varepsilon_1^{-1}R_1(s), \tag{34}$$

which yields

$$\begin{cases} C_1 = (\frac{1}{2}\Pi - \theta_1)\varepsilon_1^{-1} - \theta_0\varepsilon_0^{-1} \\ C_0 = \varepsilon_1\varepsilon_0^{-1} - (\frac{1}{2}\Pi - \theta_1)\varepsilon_1^{-1}\theta_0\varepsilon_0^{-1}. \end{cases} \tag{35}$$

This means that, for an arbitrary positive-definite matrix Π, a PM C(s) of the form (32) is always expressed as (35) with symmetric matrices $\{\varepsilon_0, \varepsilon_1\}$ and skew-symmetric matrices $\{\theta_0, \theta_1\}$. In this expression, the stability of C(s) is equivalent to the positive-definiteness of ε_0 and of ε_1. Considering the special case where $\theta_0 = 0$, we obtain the following result: if there exists a symmetric matrix ε_1 such that

$$\begin{aligned} &\varepsilon_1^{-1}C_0 = C_0'\varepsilon_1^{-1} \ (= \varepsilon_0^{-1}) \\ &C_1\varepsilon_1 + \varepsilon_1 C_1' \ (= \Pi) > 0, \end{aligned} \tag{36}$$

then the stability of C(s) is equivalent to the condition

$$\varepsilon_1 > 0 \quad \text{and} \quad \varepsilon_1^{-1}C_0 > 0.$$

Application of the result to the case where $\varepsilon_1 = I$ and to the case where $\varepsilon_1 = C_0$ leads to the following two well-known sufficient conditions for the stability of C(s):

(i) $C_0 = C_0' > 0$ and $C_1 + C_1' > 0$

(ii) $C_0 = C_0' > 0$ and $C_1C_0 + C_0C_1' > 0$.

We now return to the general case where C(s) is column reduced with the column degrees $n_1 \geqq \cdots \geqq n_p$. Recalling (2.1.16), we can generalize (21) and (22) as follows.

Theorem 4.1.8 The orthogonal PM's $\{R_j(s)\}$ satisfy the following recurrence relationship:

$$\begin{cases} R_0(s) = T_0(s), \qquad \Lambda_0 R_1(s) = (sI - \theta_0 \varepsilon_0^{-1})R_0(s), \\ \Lambda_j R_{j+1}(s) = (sI - \theta_j \varepsilon_j^{-1})R_j(s) + \varepsilon_j \Lambda'_{j-1} \varepsilon_{j-1}^{-1} R_{j-1}(s). \end{cases} \tag{37}$$

$$(1 \leqq j \leqq n-1)$$

In the above recursion, however, the r(j+1)-r(j) lowest rows of $R_{j+1}(s)$ are not determined from $\{R_j(s), R_{j-1}(s)\}$, and therefore we cannot use (37) for producing $\{R_j(s)\}$ in general. In order to derive a recursive algorithm for computing $\{R_j(s)\}$, we define auxiliary PM's $\{Y_j(s); j=0,1,\ldots,n\}$ by the following conditions.

(i) $Y_j(s)$ is a $\{p-r(j)\}\times p$ PM written as

$$Y_j(s) = \Omega_j + \sum_{i=0}^{j-1} Y_{j,i}\, T_i(s), \tag{38}$$

where

$$\Omega_j = [\ O \mid I_{p-r(j)}\ ] \ \ : \{p-r(j)\}\times p.$$

(ii) $Y_j(s)$ satisfies the orthogonal condition

$$\langle Y_j(s), T_i(s)\rangle = 0 \quad \text{for } \forall i=0,1,\ldots,j-1. \tag{39}$$

(In the case where $r(j) = p$, $Y_j(s)$ is regarded as a $0 \times p$ matrix formally.) Note that the assumption (11) ensures the existence and the uniqueness of $Y_j(s)$. We further define

$$\xi_j \triangleq \langle Y_j(s), R_j(s) \rangle \quad : \{p-r(j)\}\times r(j) \tag{40}$$

$$(= \langle Y_j(s), T_j(s) \rangle \quad \text{by (39) })$$

$$(= \langle \Omega_j, R_j(s) \rangle \quad \text{by (8) })$$

for $j=0,1,\ldots,n-1$. Now, we have the following theorem.

<u>Theorem 4.1.9</u> $\{R_j(s)\}$ and $\{Y_j(s)\}$ are computed by the recurrence formula:

$$\left[\begin{array}{c} R_{j+1}(s) \\ \hline Y_{j+1}(s) \end{array}\right] \begin{array}{l} \}\, r(j+1) \\ \}\, p-r(j+1) \end{array}$$

$$= \left[\begin{array}{c} (sI - \theta_j\varepsilon_j^{-1})R_j(s) + \varepsilon_j\Lambda'_{j-1}\varepsilon_{j-1}^{-1} R_{j-1}(s) \\ \hline Y_j(s) - \xi_j\varepsilon_j^{-1} R_j(s) \end{array}\right] \begin{array}{l} \}\, r(j) \\ \}\, p-r(j) \end{array}$$

$$(1 \leqq j \leqq n-1) \tag{41}$$

with the initial condition:

$$\left[\begin{array}{c} R_0(s) \\ \hline Y_0(s) \end{array}\right] \begin{array}{l} \}\, r(0) \\ \}\, p-r(0) \end{array} = I_p$$

$$\left[\begin{array}{c} R_1(s) \\ \hline Y_1(s) \end{array}\right] \begin{array}{l} \}\, r(1) \\ \}\, p-r(1) \end{array} = \left[\begin{array}{c} (sI - \theta_0\varepsilon_0^{-1})\; T_0(s) \\ \hline \Omega_0 - \xi_0\varepsilon_0^{-1}\; T_0(s) \end{array}\right] \begin{array}{l} \}\, r(0) \\ \}\, p-r(0) \end{array} . \tag{42}$$

(Proof) It is sufficient to show that the right-hand side of (41), say $Z_{j+1}(s)$, is expressed as the sum of $[T'_{j+1}(s), \Omega'_{j+1}]'$ and a PM of degree less than j+1, and that $Z_{j+1}(s)$ is orthogonal to every PM of degree less than j+1. The first statement on $Z_{j+1}(s)$ is readily verified by noting that

$$\begin{bmatrix} s\ T_j(s) \\ \hline \Omega_j \end{bmatrix} = \begin{bmatrix} T_{j+1}(s) \\ \hline \Omega_{j+1} \end{bmatrix},$$

and the second statement is clear from (8), (37), (39) and (40).

(QED)

<u>Remark 4.1.10</u> In the argument of the present section, the quantities $\{R_j(s)\}$, $\{Y_j(s)\}$, $\{\varepsilon_j\}$, $\{\theta_j\}$ and $\{\xi_j\}$ have been defined from $(C(s),\Pi)$ via the inner product, and have been shown to satisfy (17), (41) and (42). Conversely, we can prove that if quantities $\{R_j(s)\}$,$\{Y_j(s)\}$, etc. satisfy (17), (41) and (42) then they coincide with those defined via the inner product. Therefore, the equations (17), (41) and (42) can also be adopted as the definition of $\{R_j(s)\}$, $\{Y_j(s)\}$, etc.

4.2. Schwarz matrices and the Routh approximation

In the previous section, the quantities $\bar{\Pi}$, $\{\varepsilon_j\}$ and $\{\theta_j\}$ were defined from given $(C(s),\Pi)$. Using these quantities, let

$$e_j \triangleq \begin{cases} \varepsilon_j \, \Lambda'_{j-1} \, \varepsilon_{j-1}^{-1} & : r(j) \times r(j-1) \qquad (1 \leqq j \leqq n-1) \\ \frac{1}{2}\bar{\Pi} \, \Lambda'_{n-1} \, \varepsilon_{n-1}^{-1} & : r(n) \times r(n-1) \qquad (j = n) \end{cases}$$

$$f_j \triangleq \theta_j \, \varepsilon_j^{-1} \quad : r(j) \times r(j) \qquad (0 \leqq j \leqq n-1)$$

$$\Theta \triangleq \begin{bmatrix} f_0 & \Lambda_0 & & & \bigcirc \\ -e_1 & f_1 & \Lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ \bigcirc & & -e_{n-2} & f_{n-2} & \Lambda_{n-2} \\ & & & -e_{n-1} & -e_n + f_{n-1} \end{bmatrix} : N \times N$$

We call Θ the <u>block-Schwarz matrix</u> defined from $(C(s),\Pi)$. Note that Θ is composed of a positive-definite matrix $\bar{\Pi}$, nonsingular symmetric matrices $\{\varepsilon_j\}$ and skew-symmetric matrices $\{\theta_j\}$.

In the case where $C(s)$ is a scalar polynomial, it turns out that $\Lambda_j = 1$ and that $f_j = 0$. Moreover, the quantities $\{e_j\}$ are obtained by the procedure of the Routh-Hurwitz stability test for $C(s)$ as shown in Ex.4.1.4. Thus we can see that Θ is the Schwarz matrix defined from $C(s)$ in the usual definition (see, for instance, [2]).

Owing to Th.4.1.3 and Th.4.1.8, we have

$$(sI - \Theta)\, R(s) = B\, C(s), \tag{1}$$

where $R(s)$ and B are defined by (4.1.13) and (2.1.23) respectively. It is noted that eq.(1) characterizes Θ. This characterization leads to the following properties of Θ. First, Θ is similar to the block-companion matrix Γ defined from C(s). Indeed, comparison of (1) with (2.1.25) yields

$$\begin{cases} \Theta = R\ \Gamma\ R^{-1} \\ B = R\ B, \end{cases} \tag{2}$$

where R is a matrix defined by (4.1.15). Next, it follows from (2) and from (4.1.16) that the Lyapunov equation (4.1.6) is transformed into

$$\Theta E + E\Theta' + B\Pi B' = 0. \tag{3}$$

This means that the similarity transformation (2) from the companion pair (Γ,B) into the Schwarz pair (Θ,B) block-diagonalizes the solution of the Lyapunov equation (4.1.6). Note that eq.(3) can also be verified by a direct calculation.

Suppose that H(s) is a $q \times p$ strictly proper rational matrix such that

$$H(s) = D(s)\ C^{-1}(s), \tag{4}$$

where D(s) is a $q \times p$ PM. Then the companion pair (Γ,B), together with the coefficient matrix D of D(s) defined by (2.1.26), provides the block-companion type controller form realization (Γ,B,D) of the MFD (4) as shown in (2.1.27). Owing to the similarity relation (2), a realization (Θ,B,V) of H(s) is obtained by defining additionally

$$V \triangleq D\,R^{-1} \quad : q \times N. \tag{5}$$

Note that

$$D(s) = V\,R(s) = \sum_{j=0}^{n-1} v_j\,R_j(s),$$

where

$$V = [v_0, v_1, \ldots, v_{n-1}], \qquad v_j : q \times r(j).$$

We call (Θ, B, V) the controllable Schwarz form realization of $H(s)$ defined from the right MFD (4) and Π. We note that the observable Schawarz form realization of a left MFD can also be defined in a similar way.

As an application of the above results, we will construct the Routh approximation for MIMO (multiple-input, multiple-output) systems in the sequel. The Routh approximation method for reducing order of SISO (single-input, single-output) systems was introduced by Hutton and Friedland in [11] and was shown to have some nice properties there. This method consists of two steps: first, derive the Schwarz form realization of a given higher order system, and next, construct the approximant by truncating a part of the state variables of the realization. The extension of the method to the MIMO systems is a quite easy poroblem now.

Suppose that a $q \times p$ strictly proper rational matrix $H(s)$ is written in a right irreducible MFD (4), and let (Θ, B, V) be the controllable Schwarz form realization defined from the MFD and Π.

We further assume that H(s) is stable, which means that

$$\varepsilon_j > 0 \quad \text{for } \forall j=0,1,\ldots,n-1. \tag{6}$$

For each m=1,2,...,n, let

$$N_m \triangleq \sum_{j=n-m}^{n-1} r(j),$$

$$\Theta_m \triangleq$$

$$\begin{bmatrix} f_{n-m} & \Lambda_{n-m} & & & \bigcirc \\ -e_{n-m+1} & f_{n-m+1} & \Lambda_{n-m+1} & & \\ & \ddots & \ddots & \ddots & \\ & & -e_{n-2} & f_{n-2} & \Lambda_{n-2} \\ \bigcirc & & & -e_{n-1} & -e_n+f_{n-1} \end{bmatrix} : N_m \times N_m$$

$$B_m \triangleq [\ O \mid C_n'^{-1} \Lambda_{n-1}'\ ]' \quad : N_m \times p$$

$$V_m \triangleq [v_{n-m},\ldots,v_{n-1}] \quad : q \times N_m.$$

It should be noted that Θ_m is the block-Schwarz matrix composed of $\{\varepsilon_j\ ;\ n-m \leqq j \leqq n-1\}$, $\{\theta_j\ ;\ n-m \leqq j \leqq n-1\}$ and $\overline{\Pi}$. This fact proves the following:

$$\Theta_m E_m + E_m \Theta_m' + B_m \Pi B_m' = 0, \tag{7}$$

where

$$E_m \triangleq \text{block diag}\{\varepsilon_{n-m},\ldots,\varepsilon_{n-1}\} \quad : N_m \times N_m.$$

Now, we define the <u>m-th order Routh approximant</u> of H(s) by

$$H_m(s) \triangleq V_m\,(sI - \Theta_m)^{-1}\,B_m. \tag{8}$$

In the SISO case, this definition is shown to coincide with that of [11].

Remark 4.2.1 Since the controllable Schwarz form realization (Θ, B, V) of $H(s)$ is defined from the right irreducible MFD (4) and Π, the definition of the Routh approximants seems to depend on a choice of a right irreducible MFD of $H(s)$. However, it will be shown in Sec.4.4 that the approximants are uniquely determined by $(H(s), \Pi)$. On the other hand, the definition depends actually on a choice of Π. Furthermore, the use of a left irreducible MFD and of the observable Schwarz form realization provides another definition. In the SISO case, all the ambiguity of definition vanishes.

The fundamental properties of the SISO approximation ([11]) are generalized to the MIMO case as follows. First, since the stability condition (6) implies the positive-definiteness of E_m, it follows from (7) that Θ_m is a stable matrix. Thus we have:

(i) The approximation preserves stability; i.e., $H_m(s)$ is stable for $\forall m$.

The second property of the approximation is concerned with Markov parameters. Expand $H(s)$ and $H_m(s)$ as

$$\begin{cases} H(s) = h_0 \, s^{-1} + h_1 \, s^{-2} + \dots \\ H_m(s) = h_{m,0} \, s^{-1} + h_{m,1} \, s^{-2} + \dots \; . \end{cases}$$

The matrices $\{h_j\}$ and $\{h_{m,j}\}$ are called the Markov parameters of the systems, and are represented as

$$\begin{cases} h_j = V\,\Theta^j\,B \\ h_{m,j} = V_m\,\Theta_m^j\,B_m. \end{cases} \qquad (j=0,1,\ldots)$$

Recalling that Θ_m, B_m and V_m are right lower submatrices of Θ, B and V respectively, and noting the sparseness of the matrices, we can prove the following:

(ii) The m-th order approximation preserves the first m Markov parameters; i.e.,

$$h_{m,j} = h_j \quad \text{for } \forall j=0,1,\ldots,m-1.$$

The last property shown below is quite peculiar to the Routh approximation. Let $h_m(t)$ $(t \geqq 0)$ be the impulse response of $H_m(s)$, which is represented as

$$h_m(t) = V_m \exp(t\Theta_m)\,B_m. \tag{9}$$

We define the <u>impulse response energy</u> of $H_m(s)$ w.r.t. Π as

$$\Delta_m \triangleq \int_0^\infty h_m(t)\,\Pi\,h_m'(t)\,dt \quad : q \times q.$$

Noting that the solution E_m of the Lyapunov equation (7) is represented as

$$E_m = \int_0^\infty \exp(t\Theta_m)B_m\,\Pi\,B_m'\exp(t\Theta_m')\,dt,$$

we obtain from (9)

$$\Delta_m = V_m E_m V_m' = \sum_{j=n-m}^{n-1} v_j \varepsilon_j v_j' .$$

This leads to the following:

(iii) The impulse response energy of the m-th order approximant decreases monotonically as m is lowered; i.e.,

$$\Delta_{m+1} \geqq \Delta_m \quad (1 \leqq \forall m \leqq n-1).$$

<u>Remark 4.2.2</u> In the definition of the m-th order approximant (8), modify the Schwarz matrix Θ_m, which is composed of $\{\varepsilon_j, \theta_j \; ; \; n-m \leqq j \leqq n-1\}$ and of $\bar{\Pi}$, into $\bar{\Theta}_m$ by replacing θ_{n-m} with an arbitrary $r(n-m) \times r(n-m)$ skew-symmetric matrix $\bar{\theta}_{n-m}$. Then we can see that $\bar{H}_m(s) \triangleq V_m(sI - \bar{\Theta}_m)^{-1} B_m$ satisfies the properties (i)-(iii). This means that $\bar{H}_m(s)$ can also be regarded as a MIMO version of the Routh approximation.

4.3. The duality of polynomial matrices

The concept of dual PM's was first introduced by Anderson and Bitmead [1] for studying the stability of PM's. The concept was also applied by Anderson and Kailath [3] to the problem of deriving the backward Markovian model of a stochastic process from a given forward Markovian model. In this section, we will discuss the duality in the framework of inner product and of orthogonal PM's.

Suppose that $C(s)$ and $D(s)$ are $p \times p$ nonsingular PM's and that Π is a $p \times p$ positive-definite matrix. When the following two conditions are satisfied, it is said that $D(s)$ is a <u>dual PM</u> of $C(s)$ w.r.t. Π.

(i) $C_{\star}(s)\Pi^{-1}C(s) = D_{\star}(s)\Pi^{-1}D(s).$ (1)

(ii) $\det C(s) = \pm \det D(-s).$ (2)

The above definition is a slight generalization of the definition in [1] where Π was restricted to I. The fundamental result on the existence and the uniqueness of dual PM's was shown in [1] and [3] for $\Pi=I$, and is immediately extended to our case as follows. Every nonsingular PM $C(s)$ has a dual PM $D(s)$ w.r.t. Π. Furthermore, if $C(s)$ satisfies the condition (3.3.3), then a dual PM is unique up to a constant Π-orthogonal left factor; i.e., if $D(s)$ is a dual PM, then $\bar{D}(s)$ is also a dual PM if and only if there exists a matrix K such that

$$K \Pi K' = \Pi \qquad (3)$$

$$\tilde{D}(s) = K\, D(s). \tag{4}$$

From now on, we assume (3.3.3).

When $C(s)$ and $D(s)$ are mutually dual w.r.t. Π, $D(s)C^{-1}(s)$ is Π-unitary on the imaginary axis $s = i\omega$. Since the Π-unitary group is compact, each element of $D(s)C^{-1}(s)$ is bounded on the imaginary axis, which implies that $D(s)C^{-1}(s)$ is proper and that $\lim_{s\to\infty} D(s)C^{-1}(s)$ is Π-orthogonal. Hence there exists a unique dual PM $D(s)$ such that

$$\text{(iii)} \qquad \lim_{s\to\infty} D(s)C^{-1}(s) = I. \tag{5}$$

A PM $D(s)$ satisfying (i)-(iii) is simply called <u>the dual PM</u> of $C(s)$ w.r.t. Π.

<u>Example 4.3.1</u> When $C(s)$ is a scalar monic polynomial of degree n, the dual polynomial of $C(s)$ is given by

$$D(s) = (-1)^n C(-s), \tag{6}$$

being independent of Π.

<u>Example 4.3.2</u> (Anderson and Bitmead [1]) When $C(s)$ is written as $C(s) = sI - F$, the dual PM w.r.t. Π is given by

$$D(s) = sI + \varepsilon_0 F' \varepsilon_0^{-1}, \tag{7}$$

where ε_0 is the solution of (4.1.31) in Ex.4.1.6.

The duality is related to the stability as follows ([1]).

Factorize Π^{-1} as

$$\Pi^{-1} = L'L \tag{8}$$

where L is a square nonsingular matrix, and define

$$S(s) \triangleq L\, D(s)C^{-1}(s)\, L^{-1}. \tag{9}$$

Then (1) leads to

$$S(s)\, S_{*}(s) = I. \tag{10}$$

Generally a rational matrix S(s) obeying (10) is said to be lossless bounded real (LBR) when it is stable. In our case, since C(s) and D(s) are right coprime under the assumption (3.3.3), it follows that

$$C(s) \text{ is stable iff } S(s) \text{ is LBR.} \tag{11}$$

It is well known (see, for instance, [6]) that the LBR property completely characterizes the scattering matrices of lossless networks, which are composed of lossless circuit elements including capacitors, inductors, transformers and gyrators. When S(s) is the scattering matrix of a lossless network, the immittance (i.e. impedance or admittance) matrix of the network is given by

$$\begin{aligned} Z(s) &\triangleq \{I - S(s)\}\{I + S(s)\}^{-1} \\ &= L\{C(s)-D(s)\}\{C(s)+D(s)\}^{-1}L^{-1} \end{aligned} \tag{12}$$

or its inverse. We can see that the LBR property of S(s) is transformed into the lossless positive real (LPR) property of $Z(s)$; i.e.,

$$\begin{cases} Z(s) + Z_*(s) = 0 & (13) \\ Z(s) + Z'(\bar{s}) \geqq 0 \quad \text{in } \mathrm{Re}[s] > 0, & (14) \end{cases}$$

where $\bar{s}$ denotes the complex cojugate of s. It is obvious that

$$C(s) \text{ is stable iff } Z(s) \text{ is LPR.} \tag{15}$$

The above argument shows that, once a dual PM of $C(s)$ is obtained, the stability test for $C(s)$ is reduced to the LBR (LPR) test for $S(s)$ (for $Z(s)$). It should be noted that the LBR (LPR) test, for which various efficient methods are known ([1]), is much easier than the stability test in general.

Now, let us investigate the mutual relationship between the duality and the notion of inner product. Owing to the assumption (3.3.3), there exists a PM $W(s)$ satisfying (3.3.2); i.e.,

$$C(s)W_*(s) + W(s)C_*(s) = \Pi. \tag{16}$$

Similarly, since the dual PM $D(s)$ also satisfies the same condition as (3.3.3) (see (2)), there exists a PM $Y(s)$ such that

$$D(s)Y_*(s) + Y(s)D_*(s) = \Pi. \tag{17}$$

We can see from (1) that

$$C^{-1}(s)W(s) - Y_*(s)D_*^{-1}(s) = D^{-1}(s)Y(s) - W_*(s)C_*^{-1}(s).$$

Coimparison of the poles of both sides of the above equation shows that $C^{-1}(s)W(s)-Y_{*}(s)D_{*}^{-1}(s)$ must be a PM. Thus we have

$$C^{-1}(s)W(s)-W_{*}(s)C_{*}^{-1}(s) \equiv -\{D^{-1}(s)Y(s)-Y_{*}(s)D_{*}^{-1}(s)\} \quad \text{modulo PM.} \tag{18}$$

Hence, recalling the expression (3.3.1), we see that, for arbitrary PM's $P(s)$ and $Q(s)$, the inner product $\langle P(s),Q(s)\rangle_C$ defined from $(C(s),\Pi)$ and the inner product $\langle P(s),Q(s)\rangle_D$ defined from $(D(s),\Pi)$ are related to each other as

$$\langle P(s),Q(s)\rangle_C = - \langle P(s),Q(s)\rangle_D . \tag{19}$$

The above equation allows us to characterize $D(s)$ in terms of $\langle\ \rangle_C$ as follows.

<u>Theorem 4.3.3</u> A PM $D(s)$ obeying (5) is the dual of $C(s)$ w.r.t. Π if and only if the following holds for $^{\forall}Q(s) \dashv C(s)$:

$$\langle D(s),Q(s)\rangle_C = -\langle C(s),Q(s)\rangle_C . \tag{20}$$

(Proof) As in (3.1.9), $D(s)$ satisfies

$$\langle D(s),Q(s)\rangle_D = \frac{1}{2}\Pi\,[Q(s)D^{-1}(s)]'_{-1} \tag{21}$$

for $^{\forall}Q(s) \dashv D(s)$, or equivalently for $^{\forall}Q(s) \dashv C(s)$. Under the constraint (5), the right-hand side of (21) coincides with that of (3.1.9), and therefore (19) leads to (20). Since $D(s)$ satisfying (5) and (20) is unique because of the nonsingularity of $\langle\ \rangle_C$ (see Remark 3.3.2), the dual PM $D(s)$ is characterized by these equations. (QED)

If D(s) is the dual of C(s) w.r.t. Π, and if U(s) is a unimodular PM, then the dual of C(s)U(s) w.r.t. Π is D(s)U(s), obviously. Therefore, we can assume with no loss of generality that C(s) is column reduced with descending column degrees. Using the notation of Sec.4.1, we have the following.

<u>Theorem 4.3.4</u> The dual PM of C(s) w.r.t. Π is given by

$$D(s) = C_n \{ R_n(s) - \tfrac{1}{2}\bar{\Pi}\,\Lambda'_{n-1}\varepsilon_{n-1}^{-1}R_{n-1}(s) \}. \tag{22}$$

(Proof) Immediate from Th.4.3.3. (See the proof of Th.4.1.3) (QED)

<u>Example 4.3.5</u> (The scalar case) Application of the above theorem to Ex.4.1.4 shows that

$$D(s) = R_n(s) - e_n R_{n-1}(s) = (-1)^n\, C(-s),$$

which coincides with the result of Ex.4.3.1.

<u>Example 4.3.6</u> (The first order case) Application of the above theorem to Ex.4.1.6 shows that

$$\begin{aligned} D(s) &= R_1(s) - \tfrac{1}{2}\Pi\,\varepsilon_0^{-1}R_0(s) \\ &= sI - (\theta_0 + \tfrac{1}{2}\Pi)\varepsilon_0^{-1}. \end{aligned} \tag{23}$$

Since eq.(4.1.30) leads to

$$\varepsilon_0 F' = -(\theta_0+\frac{1}{2}\Pi),$$

the result of Ex.4.3.2 (eq.(7)) follows from (23).

Example 4.3.7 (The second order case) In the situation of Ex.4.1.7, assume that there exists a symmetric matrix ε_1 satisfying (4.1.36), and let $\Pi \triangleq C_1\varepsilon_1 + \varepsilon_1 C_1'$. This assumption means that $\theta_0 = 0$ in (4.1.33), and we have

$$\begin{cases} R_1(s) = sI \\ R_2(s) = s^2 I - s\theta_1\varepsilon_1^{-1} + \varepsilon_1\varepsilon_0^{-1}. \end{cases}$$

Thus, it follows from Th.4.1.3 and from Th.4.3.4 that

$$C(s) = s^2 I + s(\frac{1}{2}\Pi-\theta_1)\varepsilon_1^{-1} + \varepsilon_1\varepsilon_0^{-1}$$

$$D(s) = s^2 I - s(\frac{1}{2}\Pi+\theta_1)\varepsilon_1^{-1} + \varepsilon_1\varepsilon_0^{-1}.$$

We can see from the above equations that the dual of $C(s)$ w.r.t. Π is written as

$$D(s) = \varepsilon_1 C_*(s) \varepsilon_1^{-1}.$$

In particular, we have:

(i) If $C_0 = C_0'$ and if $C_1+C_1' > 0$, then the dual of $C(s)$ w.r.t. C_1+C_1' is $C_*(s)$.

(ii) If $C_0 = C_0'$ and if $C_1C_0+C_0C_1' > 0$, then the dual of $C(s)$ w.r.t. $C_1C_0+C_0C_1'$ is $C_0C_*(s)C_0^{-1}$.

According to Th.4.1.3 and Th.4.3.4, Z(s) in (12) is written as

$$Z(s) = \frac{1}{2} LC_n \bar{\Pi} \Lambda'_{n-1} \varepsilon_{n-1}^{-1} R_{n-1}(s) R_n^{-1}(s) C_n^{-1} L^{-1} \tag{24}$$

In the sequel, we will investigate the internal structure of Z(s) in the light of the recursive formula (4.1.37) in Th.4.1.8, in order to show that the stability criterion of Th.4.1.1 is actually interpreted as a criterion of the LPR-ness of Z(s).

First, we make some preliminary definitions. Since $R_j(s)$ is a $r(j) \times p$ PM of degree j, the last p-r(j) columns of $R_j(s)$ constitute a zero matrix, and we denote the remaining nonzero $r(j) \times r(j)$ PM by $\hat{R}_j(s)$; i.e.,

$$R_j(s) = [\hat{R}_j(s) \mid O]. \tag{25}$$

Moreover, since $R_j(s)$ is monic and is written as

$$R_j(s) = T_j(s) + Q(s)$$

where Q(s) is a PM of degree less than j, it follows from (2.1.16) that

$$\Lambda_{j-1} R_j(s) = s\, T_{j-1}(s) + \Lambda_{j-1} Q(s),$$

which implies that the last p-r(j-1) columns of $\Lambda_{j-1} R_j(s)$ constitute a zero matrix. Hence, denoting the remaining nonzero $r(j-1) \times r(j-1)$ PM by $\hat{\hat{R}}_{j-1}(s)$, we have

$$\Lambda_{j-1} R_j(s) = [\hat{\hat{R}}_j(s) \mid O]. \tag{26}$$

The following is obvious:

$$\Lambda_{j-1}\hat{R}_j(s) = \hat{\hat{R}}_j(s)\Lambda_{j-1}. \tag{27}$$

Taking the $r(j) \times r(j)$ nonzero parts of the both sides of the recursive formula (4.1.37), we obtain

$$\begin{cases} \hat{R}_0(s) = I \qquad \hat{\hat{R}}_1(s) = sI - \theta_0\varepsilon_0^{-1} \\ \hat{\hat{R}}_{j+1}(s) = (sI - \theta_j\varepsilon_j^{-1})\hat{R}_j(s) + \varepsilon_j\Lambda'_{j-1}\varepsilon_{j-1}^{-1}\hat{R}_{j-1}(s)\Lambda_{j-1}. \end{cases} \tag{28}$$

For $j=1,2,\ldots,n$, let

$$Z_j(s) \triangleq \hat{\hat{R}}_j(s)\hat{R}_{j-1}^{-1}(s)\varepsilon_{j-1} \quad : r(j-1) \times r(j-1). \tag{29}$$

Then it follows from (27) and (28) that

$$\begin{cases} Z_1(s) = s\varepsilon_0 - \theta_0 \\ Z_{j+1}(s) = s\varepsilon_j - \theta_j + K'_jZ_j^{-1}(s)K_j \qquad (1 \leqq j \leqq n-1) \end{cases} \tag{30}$$

where

$$K_j \triangleq \Lambda_{j-1}\varepsilon_j.$$

On the other hand, recalling (4.1.18) and (8), we obtain from (24)

$$Z(s) = \frac{1}{2} K'_nZ_n^{-1}(s)K_n \tag{31}$$

where

$$K_n \triangleq \Lambda_{n-1}C_n^{-1}L^{-1}.$$

Now, referring to the following fundamental results on the LPR property, we can see from (30) and (31) that $\{Z_j(s)\}$ and $Z(s)$ are LPR if and only if $\varepsilon_j > 0$ for $\forall j=0,1,\ldots,n-1$.

(i) For an arbitrary positive-definite matrix ε, $s\varepsilon$ is LPR. ($s\varepsilon$ is interpreted as the impedance matrix of a multiport inductor or the admittance matrix of a multiport capacitor.)

(ii) Every skew-symmetric matrix θ is LPR. (θ is interpreted as the immittance matrix of a multiport gyrator.)

(iii) If Z(s) is LPR and if K is a constant matrix with the number of columns same as Z(s), then $K'Z(s)K$ is LPR. ($K'Z(s)K$ is interpreted as the immittance matrix of the network obtained from a network whose immittance matrix is Z(s), with a cascade-loaded multiport transformer.)

(iv) If Y(s) and Z(s) are LPR and of the same size, then Y(s)+Z(s) is LPR. (Y(s)+Z(s) is interpreted as the impedance (admittance) matrix of the network obtained by the series (parallel) connection of two networks whose impedance (admittance) matrix are Y(s) and Z(s).)

(v) If Z(s) is LPR and is nonsingular, then $Z^{-1}(s)$ is LPR. (This fact corresponds to the impedance-admittance transformation of a network.)

(vi) A square rational matrix Z(s) is LPR if and only if its being so is deducible from the above (i)-(v).

Remark 4.3.8 The equations (30) and (31) can be regarded as the procedure of the Cauer synthesis ([6]) applied to Z(s).

Remark 4.3.9 We can see from (30) and (31) that $Z'(s) = Z(s)$ if and only if $\theta_j = 0$ for $\forall j=0,1,\ldots,n-1$. This fact is an example of the general principle that a passive network is reciprocal if and only if it cantains no gyrators ([6]).

Remark 4.3.10 Using (30), we can readily prove the following by induction on j:

$$Z_j(s) + Z_{j*}(s) = 0. \tag{32}$$

This equation is equivalent to the following:

$$\hat{\hat{R}}_{j*}(s)\varepsilon_{j-1}^{-1}\hat{R}_{j-1}(s) + \hat{R}_{j-1*}(s)\varepsilon_{j-1}^{-1}\hat{\hat{R}}_j(s) = 0, \tag{33}$$

which is a new property of the orthogonal PM's.

4.4. Further study of orthogonal polynomial matrices

Throughout this section, we work with a fixed $\Pi > 0$ and fixed descending column degrees $n_1 \geqq \ldots \geqq n_p$. We denote by $\mathcal{C}$ the totality of column reduced PM's with the prescribed column degrees as in Chap.2 (see (2.1.20)), and by $\hat{\mathcal{C}}$ the totality of $C(s) \in \mathcal{C}$ satisfying (3.3.3) and (4.1.11).

In Sec.2.2, it has been shown that the group $\mathcal{U}$ (see (2.2.1)) is acting on $\mathcal{C}$, and the behavior of the action has been investigated in terms of the coefficient matrices $\{C_j\ ;\ j=0, 1,\ldots,n\}$, which constitute a coordinate system of $\mathcal{C}$. We can see that the subset $\hat{\mathcal{C}}$ of $\mathcal{C}$ is also parametrized by $\{C_j\}$, with some restricting condition, and is closed under the action of $\mathcal{U}$. On the other hand, the arguments in Sec.4.1 (see Th.4.1.9 and Remark 4.1.10) show that $\hat{\mathcal{C}}$ is parametrized by another coordinate system $\Phi \triangleq (C_n, \{\varepsilon_j, \theta_j, \xi_j\ ;\ j=0,1,\ldots,n-1\})$, where C_n is a $p \times p$ nonsingular matrix, ε_j is a $r(j) \times r(j)$ nonsingular symmetric matrix, θ_j is a $r(j) \times r(j)$ skew-symmetric matrix, and ξ_j is a $(p-r(j)) \times r(j)$ matrix. Indeed, we can verify that the dimension of Φ, which is the sum of $\dim C_n = p^2$, $\dim \varepsilon_j = r(j)(r(j)+1)/2$, $\dim \theta_j = r(j)(r(j)-1)/2$ and $\dim \xi_j = (p-r(j))r(j)$ for $j=0,1,\ldots,n-1$, is equal to $\dim \hat{\mathcal{C}} = p(p+N)$. In the present section, we will investigate the behavior of the quantities in Φ under the action of $\mathcal{U}$, and will elucidate the respective roles of the quantities thereby.

Suppose that

$$\bar{C}(s) = C(s)U(s), \tag{1}$$

where $C(s), \bar{C}(s) \in \hat{\mathcal{C}}$ and $U(s) \in \mathcal{U}$. (In the sequel, we use for $C(s)$ the notation of Sec.4.1 as it is, and for $\bar{C}(s)$ the notation with $\bar{\ }$.) Let $\langle\ \rangle_C$ and $\langle\ \rangle_{\bar{C}}$ be the inner products defined from $(C(s),\Pi)$ and $(\bar{C}(s),\Pi)$, respectively. Then, recalling the definition of the inner products (see (3.3.1) and (3.3.2)), we have

$$\langle P(s),Q(s)\rangle_C = \langle P(s)U(s),Q(s)U(s)\rangle_{\bar{C}}. \tag{2}$$

Owing to the fact that deg $P(s)U(s)$ = deg $P(s)$ for $^{\forall}P(s)$ (see Remark 2.2.4), it follows from (2) and from the definition of the orthogonal PM $R_j(s)$ of degree j that $R_j(s)U(s)$ is also of degree j and is orthogonal to every PM of degree less than j, with respect to the inner product $\langle\ \rangle_{\bar{C}}$. Furthermore, we can see from (2.1.10) that the highest dgree coefficient matrix of $R_j(s)U(s)$ is $F_n^{(j)}$. Thus, it is concluded that the orthogonal PM of degree j defined from $(\bar{C}(s),\Pi)$ is given by

$$\bar{R}_j(s) = (F_n^{(j)})^{-1}R_j(s)U(s). \tag{3}$$

It follows immediately from (2) and (3) that

$$\begin{cases} \bar{\varepsilon}_j = (F_n^{(j)})^{-1}\varepsilon_j(F_n^{(j)})'^{-1} \\ \bar{\theta}_j = (F_n^{(j)})^{-1}\theta_j(F_n^{(j)})'^{-1}. \end{cases} \tag{4}$$

Let (Θ,B) and $(\bar{\Theta},\bar{B})$ be the Schwarz pairs (see Sec.4.2) defined from $(C(s),\Pi)$ and from $(\bar{C}(s),\Pi)$, respectively. Then (4) yields

$$\begin{cases} \bar{\Theta} = F^{-1}\,\Theta\,F \\ \bar{B} = F^{-1}B, \end{cases} \tag{5}$$

where

$$F \triangleq \text{block diag } \{F_n^{(0)}, \ldots, F_n^{(n-1)}\} \quad : N \times N.$$

(We can also prove (5) from (1) and (3) by invoking (4.2.1).) It should be noted that eq.(5), together with $\bar{V} = VF$, represents the arbitrariness of the Schwarz form realizations defined from irreducible right MFD's of a given system. Using this fact, and noting the block-diagonal structure of F, we can show that the definition of the Routh approximation (4.2.8) does not depend on a choice of an irreducible right MFD of a given system (see Remark 4.2.1).

Now, suppose that $U(s) \in \mathcal{U}_n$, or equivalently that $F_n = I$, in (1). Then, from (4) and (2.2.12) we have

$$\bar{C}_n = C_n, \quad \bar{\varepsilon}_j = \varepsilon_j, \quad \bar{\theta}_j = \theta_j \quad (0 \leqq {}^{\forall}j \leqq n-1). \tag{6}$$

This means that the quantities $(C_n, \{\varepsilon_j\}, \{\theta_j\})$ constitute an <u>invariant</u> with respect to the action of $\mathcal{U}_n$. Moreover, the invariant is <u>complete</u> in the sense that, for arbitrary $C(s), \bar{C}(s) \in \hat{\mathcal{C}}$, the condition (6) is equivalent to the existence of $U(s) \in \mathcal{U}_n$ satisfying (1). Let us prove this. Assume that C(s) and $\bar{C}(s)$ satisfy (6). Then they have a same Schwarz pair, and therefore we obtain from (4.2.1)

$$\bar{R}(s)\bar{C}^{-1}(s) = R(s)C^{-1}(s).$$

The above equation proves the existence of a unimodular PM U(s) satisfying (1), because the both sides are irreducible MFD's.

Furthermore, since $\bar{C}_n = C_n$ by assumption, it turns out that U(s) $\in \mathcal{U}_n$.

Next, we investigate the behavior of the quantities $\{\xi_j\}$ under the action of $\mathcal{U}_n$. Suppose that $C(s) \in \hat{\mathcal{C}}$ has the coordinate $(C_n, \{\varepsilon_j\}, \{\theta_j\}, \{\xi_j\})$. Then, for arbitrary $\{\bar{\xi}_j\}$, there exists a unimodular PM $U(s) \in \mathcal{U}_n$ such that C(s)U(s) has the coordinate $(C_n, \{\varepsilon_j\}, \{\theta_j\}, \{\bar{\xi}_j\})$. Indeed, using the recursion in Th.4.1.9, we can construct $\bar{C}(s)$ having the coordinate $(C_n, \{\varepsilon_j\}, \{\theta_j\}, \{\bar{\xi}_j\})$, for which there exists $U(s) \in \mathcal{U}_n$ such that $\bar{C}(s) = C(s)U(s)$ as mentioned above.

The above arguments are summarized in the following theorem.

<u>Theorem 4.4.1</u> Among the quantities in the coordinate system $(C_n, \{\varepsilon_j\}, \{\theta_j\}, \{\xi_j\})$ of $\hat{\mathcal{C}}$, $(C_n, \{\varepsilon_j\}, \{\theta_j\})$ constitute a complete invariant with respect to the action of $\mathcal{U}_n$. On the other hand, the values of $\{\xi_j\}$ can arbitrarily varied by the action of $\mathcal{U}_n$.

We have seen that the quantities $\{\varepsilon_j\}$ determine whether C(s) is stable or not (Th.4.1.1), and that $\{\xi_j\}$ correspond to the degree of freedom of C(s)U(s) for $U(s) \in \mathcal{U}_n$ (Th.4.4.1). Now, we elucidate the role of $\{\theta_j\}$ by the following two theorems.

<u>Theorem 4.4.2</u> For $^{\forall}C(s) \in \hat{\mathcal{C}}$, the following conditions are mutually equivalent.

(i) $C_*(s)\Pi^{-1}C(s)T_n(-1)$ is symmetric.

(ii) The dual of C(s) w.r.t. Π is written as

$$D(s) = C(-s)T_n(-1).$$

(iii) The orthogonal PM's $\{R_j(s)\}$ satisfy

$$R_j(s) = \hat{R}_j(-s)T_j(-1) \quad \text{for } \forall j=0,1,\ldots,n,$$

where $\hat{R}_j(s)$ is defined by (4.3.25).

(iv) The inner product defined from $(C(s),\Pi)$ satisfies

$$\langle P(s),Q(s)\rangle = \langle P(-s)T_n(-1),Q(-s)T_n(-1)\rangle$$

for $\forall P(s)$, $\forall Q(s)$.

<u>Theorem 4.4.3</u> For $\forall C(s) \in \hat{\mathcal{C}}$, the following conditions are mutually equivalent.

(i) $\theta_j = 0$ for $\forall j=0,1,\ldots,n-1$.

(ii) There exists a unimodular PM $U(s) \in \mathcal{U}$ (or $\mathcal{U}_n$) such that C(s)U(s) satisfies the conditions (i)-(iv) in Th.4.4.2.

(iii) S(s) defined by (4.3.9) is symmetric.

(iv) Z(s) defined by (4.3.12) is symmetric.

(v) $Z_j(s)$ defined by (4.3.29) is symmetric for $\forall j=0,1,\ldots,n-1$.

<u>Remark 4.4.4</u> If C(s) satisfies the conditions in Th.4.4.2, then $R_n(s)$ and $R_{n-1}(s)$ can be obtained from $(C(s),\Pi)$

directly as in the scalar case (see Remark 4.1.5). Therefore, we can use the straightforward extension of the Routh-Hurwitz method for testing the stability of C(s). It should be noted that we can readily examine whether the condition (i) in the theorem is satisfied or not.

<u>Remark 4.4.5</u> In the strictly regular case (see Ex.2.1.1), the conditions in Th.4.4.2 are written as follows.

(i) $C_*(s)\Pi^{-1}C(s)$ is symmetric.

(ii) $D(s) = (-1)^n C(-s)$.

(iii) $R_j(s) = (-1)^j R_j(-s)$ for $\forall j=0,1,\ldots,n$.

(iv) $\langle P(s),Q(s)\rangle = \langle P(-s),Q(-s)\rangle$ for $\forall P(s)$, $\forall Q(s)$.

Furthermore, since $\mathcal{U}_n = \{I\}$ in this case (Ex.2.2.6), all the conditions in Th.4.4.2 and in Th.4.4.3 are mutually equivalent. In particular, the conditions are always satisfied in the scalar case. On the other hand, in the general column reduced case, the conditions in Th.4.4.2 are stronger than those in Th.4.4.3. We note that the conditions in Th.4.4.3 are invariant under the action of $\mathcal{U}$, while those in Th.4.4.2 are not so.

(Proof of Th.4.4.2) (i)$\Leftrightarrow$(ii) : Obvious from the definition of D(s) ((4.3.1),(4.3.2),(4.3.5)). (ii)$\Rightarrow$(iv) : Assume (ii). Then a solution Y(s) of (4.3.17) is obtained from a solution W(s) of (4.3.16) by

$$Y(s) = W(-s)T_n(-1).$$

Hence, it follows from (4.3.18) that

$$C^{-1}(s)W(s)-W_*(s)C_*^{-1}(s)$$

$$\equiv -T_n(-1)\{C^{-1}(-s)W(-s)-W_*(-s)C_*^{-1}(-s)\}T_n(-1) \quad \text{modulo PM},$$

which proves (iv) by using (3.3.1). (iv)$\Longrightarrow$(iii) : Assume (iv). Then noting that

$$R_j(-s)T_j(-1) = (-1)^{n-j}R_j(-s)T_n(-1),$$

we can see that $R_j(-s)T_j(-1)$ is orthogonal to every PM of degree less than j. Moreover, since $R_j(-s)T_j(-1)$ is of degree j and is monic, the condition (iii) is concluded. (iii)$\Longrightarrow$(ii) : Obvious from Th.4.1.3 and Th.4.3.4. (QED)

(Proof of Th.4.4.3) (i)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(v) : Obvious from (4.3.30-31) (see Remark 4.3.9). (iii)$\Longleftrightarrow$(iv) : Obvious from (4.3.12). (i)$\Longrightarrow$(ii) : Assume (i). Then, according to Th.4.4.1, there exists a unimodular PM $U(s) \in \mathcal{U}_n$ such that $\bar{C}(s) \triangleq C(s)U(s)$ satisfies $\bar{\theta}_j=0$ and $\bar{\xi}_j=0$ for $\forall j$. Using (4.1.41-42), we can prove by induction on j that $\bar{C}(s)$ satisfies (iii) in Th.4.4.2. (ii)$\Longrightarrow$(i) : Assume (ii). Then $\bar{C}(s) = C(s)U(s)$ satisfies (iii) in Th.4.4.2, which implies that $\bar{\theta}_j=0$ for $\forall j$ (see (4.1.37)). Thus, recalling (4), the condition (i) is derived.

(QED)

5. A SURVEY OF THE DISCRETE-TIME STABILITY THEORY FOR POLYNOMIAL MATRICES

In this chapter, we will present the discrete-time (d.t.) results which correspond to the continuous-time (c.t.) results in Chap.3 and Chap.4. In addition to a survey of known results such as the LWR algorithm, some new results will be derived from a comparative viewpoint on the d.t. case and the c.t. case.

Suppose that we are given a $p \times p$ nonsingular PM $A(z)$ and a $p \times p$ positive-definite matrix Σ. Let us define from $(A(z),\Sigma)$ the inner product $\langle P(z),Q(z)\rangle$, where $P(z)$ and $Q(z)$ are arbitrary PM's with p columns such that

$$P(z) \sim A(z) \quad \text{and} \quad Q(z) \sim A(z),$$

which means that $P(z)A^{-1}(z)$ and $Q(z)A^{-1}(z)$ are proper. To begin with, we consider the case where $A(z)$ is d.t. stable, in the sense that all the zeros of $\det A(z)$ lie in the open unit disk. In this case, we define the inner product as

$$\langle P(z),Q(z)\rangle \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\omega})A^{-1}(e^{i\omega})\,\Sigma\, A'^{-1}(e^{-i\omega})Q'(e^{-i\omega})d\omega.$$

It is obvious that the inner product satisfies the same fundamental properties as (i)-(iv) in Sec.3.1 for the c.t. case. Furthermore, it holds that

$$\begin{cases} \langle zP(z),zQ(z)\rangle = \langle P(z),Q(z)\rangle & (1) \\ \langle A(z),Q(z)\rangle = \Sigma\{\lim_{z\to\infty} Q(z)A^{-1}(z)\}', & (2) \end{cases}$$

which can be regarded as the counterparts of (v) and (vi) in Sec.3.1. Note that $A(z)$ is orthogonal to every $Q(z)$ such that $Q(z) \dashv A(z)$ (i.e., $Q(z)A^{-1}(z)$ is strictly proper), while in the c.t. case $C(s)$ is not orthogonal to $Q(s)$ if $sQ(s) \not\dashv C(s)$. Note also that

$$\langle A(z), A(z) \rangle = \Sigma,$$

while in the c.t. case $\langle C(s), C(s) \rangle$ is not defined.

As in the c.t. case, there are two methods to compute the inner product. The first method is as follows. Suppose that matrices $\{F, G\}$ and a PM $S(z)$ satisfy the following.

$$\begin{cases} (F,G) \text{ is controllable.} & (3) \\ A(z) \text{ and } S(z) \text{ are right coprime.} & (4) \\ (zI - F)^{-1}G = S(z)A^{-1}(z). & (5) \end{cases}$$

If $P(z) \dashv A(z)$ and if $Q(z) \dashv A(z)$, then there exist constant matrices P and Q such that

$$\begin{cases} P(z) = P\,S(z) \\ Q(z) = Q\,S(z), \end{cases} \tag{6}$$

and we have

$$\langle P(z), Q(z) \rangle = P\,X\,Q' \tag{7}$$

where X is the unique solution of the Lyapunov equation

$$X = FXF' + G\Sigma G'. \tag{8}$$

If $P(z) \sim A(z)$ and if $Q(z) \sim A(z)$, then we have

$$\begin{cases} P(z) = K\,A(z) + \bar{P}(z) \\ Q(z) = L\,A(z) + \bar{Q}(z), \end{cases} \tag{9}$$

where K and L are constant matrices, and $\bar{P}(z)$ and $\bar{Q}(z)$ are PM's satisfying $\bar{P}(z) \dashv A(z)$ and $\bar{Q}(z) \dashv A(z)$. Hence we can compute $\langle P(z), Q(z) \rangle$ by

$$\langle P(z), Q(z) \rangle = K \Sigma L' + \langle \bar{P}(z), \bar{Q}(z) \rangle. \tag{10}$$

The second method to compute the inner product is based on the equation

$$A(z)W'(z^{-1}) + W(z)A'(z^{-1}) = \Sigma, \tag{11}$$

where $W(z)$ is a $p \times p$ unknown PM. The above equation has a solution $W(z)$ under the assumption that $A(z)$ is d.t. stable (see [5]), and we can show that if $P(z) \sim A(z)$ and if $Q(z) \sim A(z)$ then $\langle P(z), Q(z) \rangle$ is given by

$$\begin{aligned} \langle P(z), Q(z) \rangle = {} & [P(z)A^{-1}(z)W(z)Q'(z^{-1})]_0 \\ & + [P(z^{-1})W'(z)A'^{-1}(z)Q'(z)]_0, \end{aligned} \tag{12}$$

where $[\ ]_0$ denotes the constant term of a rational matrix defined by

$$[R(z)]_0 \triangleq [R(z)]_{pol} \Big|_{z=0}. \qquad \text{(See Def.3.1.2.)} \tag{13}$$

It is noted that although the solutions of (11) are not unique the right-hand side of (12) is uniquely determined, being independent of a choice of a solution $W(z)$.

In the general case where $A(z)$ is not necessarily stable, we define the inner product $\langle P(z),Q(z)\rangle$ by (3)-(10), or equivalently by (11)-(12). It is shown that the definition is valid if

$$\det A(z) \text{ and } \det A(z^{-1}) \text{ have no common zeros.} \tag{14}$$

As the next step, we introduce the notion of orthogonal PM's. We assume that $A(z)$ is column reduced with descending column degrees $(n=)n_1 \geqq \dots \geqq n_p$, and use the notation of Sec.2.1. Then $A(z)$ is written as

$$A(z) = A_n T_n(z) + A_{n-1} T_{n-1}(z) + \dots + A_0 T_0(z), \tag{15}$$

where $A_n \in GL(p)$ and $A_j \in \mathbb{R}^{p\times r(j)}$ for $j=0,1,\dots,n-1$. If a PM, say $A_j(z)$, is of degree j and monic, and if $A_j(z)$ satisfies the orthogonal condition

$$\langle A_j(z),T_i(z)\rangle = 0 \quad \text{for} \quad \forall i=0,1,\dots,j-1, \tag{16}$$

we say that $A_j(z)$ is a <u>j-th degree forward orthogonal PM</u> defined from $(A(z),\Sigma)$. If a PM $B_j(z)$ is of degree at most j and satisfies

$$B_j(0) = [I_{r(j)} \,|\, O\,] \quad : r(j)\times p \tag{17}$$

and

$$\langle B_j(z),zT_i(z)\rangle = 0 \quad \text{for} \quad \forall i=0,1,\dots,j-1, \tag{18}$$

we say that $B_j(z)$ is a <u>j-th degree backward orthogonal PM</u> defined from $(A(z),\Sigma)$. $B_j(z)$ is also defined as a PM such that

$B_j(z^{-1})\tilde{T}_j(z)$ is of degree j and monic, and that

$$\langle B_j(z), T_i(z^{-1})\tilde{T}_j(z)\rangle = 0 \quad \text{for} \quad {}^{\forall}i=0,1,\ldots,j-1, \tag{19}$$

where $\tilde{T}_j(z)$ is a $p\times p$ diagonal PM defined as

$$\tilde{T}_j(z) \triangleq \mathrm{diag}\{s^j, s^{j-n+n_2}, \ldots, s^{j-n+n_{r(j)}}, 0, \ldots, 0\}.$$

Under a condition similar to (4.1.11), the existence and the uniqueness of forward and backward orthogonal PM's are ensured. We note that (2) leads to

$$A(z) = A_n \cdot A_n(z), \tag{20}$$

which corresponds to Th.4.1.3 in the c.t. case.

Let

$$\begin{cases} \delta_j^A \triangleq \langle A_j(z), A_j(z)\rangle \\ \delta_j^B \triangleq \langle B_j(z), B_j(z)\rangle \end{cases} \tag{21}$$

for j=0,1,...,n. Then, corresponding to Th.4.1.1, the following is obtained.

$$\begin{aligned} A(z) \text{ is stable } &\Longleftrightarrow \delta_j^A > 0 \quad \text{for } {}^{\forall}j=0,1,\ldots,n-1 \\ &\Longleftrightarrow \delta_j^B > 0 \quad \text{for } {}^{\forall}j=0,1,\ldots,n-1 \end{aligned} \tag{22}$$

Note that

$$\Sigma = A_n \delta_n^A A_n', \tag{23}$$

which follows from (20).

In the strictly regular case where $T_j(z) = z^j I$ for every j, the orthogonal PM's are generated by the well-known LWR algorithm in the following.

(The LWR algorithm)

Set

$$\begin{cases} A_0(z) = B_0(z) = I & (24) \\ \delta_0^A = \delta_0^B = \langle I, I \rangle, & (25) \end{cases}$$

and calculate for $j=0,1,\ldots,n-1$

$$\pi_j \triangleq \langle zA_j(z), B_j(z) \rangle \qquad (26)$$

$$\begin{cases} \gamma_{j+1}^A \triangleq -\pi_j (\delta_j^B)^{-1} & (27) \\ \gamma_{j+1}^B \triangleq -\pi_j' (\delta_j^A)^{-1} & (28) \end{cases}$$

$$\begin{cases} A_{j+1}(z) = zA_j(z) + \gamma_{j+1}^A B_j(z) & (29) \\ B_{j+1}(z) = B_j(z) + \gamma_{j+1}^B zA_j(z) & (30) \end{cases}$$

$$\begin{cases} \delta_{j+1}^A = (I - \gamma_{j+1}^A \gamma_{j+1}^B) \delta_j^A & (31) \\ \delta_{j+1}^B = (I - \gamma_{j+1}^B \gamma_{j+1}^A) \delta_j^B. & (32) \end{cases}$$

In Sec.4.1 we derived a recursive algorithm producing the orthogonal PM's for an arbitrary column reduced PM for the c.t. case (Th.4.1.9). Similarly, we can generalize the LWR algorithm to the general column reduced case. The generalized LWR algorithm produces recursively $\{A_j(z)\}$ and $\{B_j(z)\}$ together with auxiliary PM's $\{V_j(z)\}$ and $\{W_j(z)\}$ as follows.

(The generalized LWR algorithm)

$$\left[\begin{array}{c} A_0(z) \\ \hline V_0(z) \end{array}\right]\begin{array}{l}\}\ r(0) \\ \}\ p-r(0)\end{array} = \left[\begin{array}{c} B_0(z) \\ \hline W_0(z) \end{array}\right]\begin{array}{l}\}\ r(0) \\ \}\ p-r(0)\end{array} = I_p$$

$$\left[\begin{array}{c} A_{j+1}(z) \\ \hline V_{j+1}(z) \end{array}\right]\begin{array}{l}\}\ r(j+1) \\ \}\ p-r(j+1)\end{array} = \left[\begin{array}{c} zA_j(z) + \gamma^A_{j+1}B_j(z) \\ \hline V_j(z) - K_{j+1}A_j(z) \end{array}\right]\begin{array}{l}\}\ r(j) \\ \}\ p-r(j)\end{array}$$

$$\left[\begin{array}{c} B_{j+1}(z) \\ \hline W_{j+1}(z) \end{array}\right]\begin{array}{l}\}\ r(j+1) \\ \}\ p-r(j+1)\end{array} = \left[\begin{array}{c} B_j(z) + \gamma^B_{j+1}zA_j(z) \\ \hline W_j(z) - L_{j+1}zA_j(z) \end{array}\right]\begin{array}{l}\}\ r(j) \\ \}\ p-r(j)\end{array}$$

$$(j=0,1,\ldots,n-1)$$

where γ^A_{j+1} and γ^B_{j+1} are defined by (26)-(28), and

$$\begin{cases} K_{j+1} \triangleq \langle V_j(z), A_j(z)\rangle(\delta^A_j)^{-1} \\ L_{j+1} \triangleq \langle W_j(z), zA_j(z)\rangle(\delta^A_j)^{-1}. \end{cases}$$

The equations (31) and (32) should be modified into

$$\begin{cases} \Lambda_j\delta^A_{j+1}\Lambda'_j = (I - \gamma^A_{j+1}\gamma^B_{j+1})\ \delta^A_j \\ \Lambda_j\delta^B_{j+1}\Lambda'_j = (I - \gamma^B_{j+1}\gamma^A_{j+1})\ \delta^B_j. \end{cases}$$

$$(\Lambda_j \triangleq [I_{r(j)} \mid 0\ ] : r(j)\times r(j+1))$$

Note that in the above equation some elements of δ^A_{j+1} and δ^B_{j+1} are not determined from δ^A_j and δ^B_j if $r(j)<r(j+1)$.

Let us compare the d.t. results mentioned above with the corresponding c.t. results in Chap.4. For simplicity we restrict ourselves to the monic strictly regular case; i.e., we assume that

$$\begin{cases} A(z) = z^n I + z^{n-1} A_{n-1} + \dots + A_0 & (33) \\ C(s) = s^n I + s^{n-1} C_{n-1} + \dots + C_0 . & (34) \end{cases}$$

We have seen in Sec.4.1 and in Sec.4.4 that the symmetric matrices $\{\varepsilon_j\}$ and the skew-symmetric matrices $\{\theta_j\}$ constitute jointly a 'coordinate system' for expressing C(s), and that these matrices provide us with important information about C(s) (see Th.4.1.1, Th.4.4.2, Th.4.4.3, Remark 4.4.4 and Remark 4.4.5). On the other hand, the situation seems much more complicated in the d.t. case. The LWR algorithm has a dualistic structure which is not observed in the corresponding c.t. algorithm (4.1.21). Furthermore, the correspondence between the quantities appeared in these two algorithms is not clear. Indeed, we cannot see what quantities are the d.t. counterparts of $\{\theta_j\}$, while it might be evident that the counterparts of $\{\varepsilon_j\}$ are $\{\delta_j^A\}$. The matrices $\{\gamma_j^A\}$ ($\{\pi_j\}$) cannot be regarded as the counterparts of $\{\theta_j\}$, because $\{\delta_j^A\}$ and $\{\gamma_j^A\}$ ($\{\delta_j^A\}$ and $\{\pi_j\}$) are not independent each other. In the sequel, we will derive a new version of the LWR algorithm including two sets of matrices $\{\delta_j\}$ and $\{u_j\}$ which can be regarded as the counterparts of $\{\varepsilon_j\}$ and $\{\theta_j\}$, respectively.

To begin with, we make some definitions. Suppose that Δ is a $p \times p$ nonsingular symmetric matix. A $p \times p$ matrix H is said to be <u>Δ-symmetric</u> if

$$H \Delta = \Delta H', \tag{35}$$

and a $p \times p$ matrix U is said to be _Δ-orthogonal_ if

$$U \Delta U' = \Delta. \tag{36}$$

(The above definition is more convenient for the later use than the popular definition where H′ and U′ are said to be Δ-symmetric and Δ-orthogonal in (35) and (36).)

We consider the problem of representing a given $p \times p$ matrix A as

$$A = H U \tag{37}$$

by a Δ-symmetric matrix H and a Δ-orthogonal matrix U. Such a pair (H,U) is said to be a _quasi Δ-polar decomposition_ of A. In the case where Δ is positive-definite, it is known that every $p \times p$ matrix A has such a decomposition. Moreover, if A is non-singular then there exists a unique decomposition such that H is _Δ-positive_, i.e.,

$$H \Delta = \Delta H' > 0. \tag{38}$$

The pair (H,U) satisfying (36)-(38) is called _the Δ-polar decomposition_ of A.

On the other hand, if Δ is not positive-definite, there may possibly be a matrix A having no quasi Δ-polar decompositions. (For example, if

$$\Delta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and if

$$|a-d| < |b-c| \quad \text{and} \quad |a+d| < |b+c| ,$$

then there is no (H,U) satisfying (35)-(37)). Furthermore, even if A has quasi Δ-polar decompositions, we need some other condition than the Δ-positivity of H in order to specify one of the decompositions. Nevertheless, we can generalize the notion of Δ-polar decomposition to an arbitrary nonsingular symmetric matrix Δ as follows. Let $\mathcal{A}$ be the totality of $p \times p$ matrices, and let

$$\mathcal{H} \triangleq \{ H \in \mathcal{A} \mid H\Delta = \Delta H' \}$$

$$\mathcal{U} \triangleq \{ U \in \mathcal{A} \mid U\Delta U' = \Delta \}.$$

We define a mapping ϕ as

$$\phi : \mathcal{H} \times \mathcal{U} \longrightarrow \mathcal{A}$$
$$(H,U) \longmapsto (H,U) = HU.$$

Then the differential of ϕ at (I,I) is given by

$$[d\phi]_{(I,I)} : \mathcal{H} \times \mathcal{L} \longrightarrow \mathcal{A}$$
$$(H,L) \longmapsto H+L,$$

where

$$\mathcal{L} \triangleq \{ L \in \mathcal{A} \mid L\Delta + \Delta L' = 0 \}.$$

A matrix L belonging to $\mathcal{L}$ is said to be <u>Δ-skew-symmetric</u>. Evidently $[d\phi]_{(I,I)}$ is a linear isomorphism, and hence there exists a neighborhood $\mathcal{H}_0 \times \mathcal{U}_0$ of (I,I) in $\mathcal{H} \times \mathcal{U}$ such that the restriction of ϕ to $\mathcal{H}_0 \times \mathcal{U}_0$, say ϕ_0, is a diffeomorphism onto $\mathcal{A}_0 \triangleq \phi(\mathcal{H}_0 \times \mathcal{U}_0)$. This means that for an arbitrary $A \in \mathcal{A}_0$ there exists a unique $(H,U) \in \mathcal{H}_0 \times \mathcal{U}_0$ satisfying A = HU. We call the pair (H,U) <u>the Δ-polar decomposition</u> of A. It is noted that $\mathcal{A}_0$ is a neighborhood of I in $\mathcal{A}$ and therefore the Δ-polar decomposition of A is defined when A is sufficiently near to I. It is clear from the definition that if a one parameter family of $p \times p$ matrices $\{A(t)\ ;\ t>0\}$ satisfies

$$A(t) \longrightarrow I \quad \text{as} \quad t \downarrow 0 \tag{39}$$

then the Δ-polar decomposition (H(t),U(t)) of A(t) also satisfies

$$\begin{cases} H(t) \longrightarrow I \\ U(t) \longrightarrow I \end{cases} \quad \text{as} \quad t \downarrow 0. \tag{40}$$

Moreover, if $\{A(t)\}$ has the one-sided differential at $t = 0$

$$\dot{A}(0) \triangleq \lim_{t \downarrow 0} (A(t) - I)/t, \tag{41}$$

then $\{H(t)\}$ and $\{U(t)\}$ also have

$$\begin{cases} \dot{H}(0) \triangleq \lim\limits_{t \downarrow 0} (H(t) - I)/t \\ \dot{U}(0) \triangleq \lim\limits_{t \downarrow 0} (U(t) - I)/t, \end{cases} \tag{42}$$

and it holds that

$$\begin{cases} \dot{H}(0) \in \mathcal{H}, \quad \dot{U}(0) \in \mathcal{L}, \\ \dot{A}(0) = \dot{H}(0) + \dot{U}(0). \end{cases} \tag{43}$$

Now, let us return to the task of deriving a version of the LWR algorithm possessing the desired structure. We consider the following recursive algorithm, called the (quasi-)polar-type LWR algorithm, which produces PM's $\{A_j(z)\}$ and $\{\bar{B}_j(z)\}$ together with matrices $\{\delta_j\}$, $\{\alpha_j\}$, $\{h_j\}$ and $\{u_j\}$.

(The polar-type LWR algorithm)

Set

$$\begin{cases} A_0(z) = \bar{B}_0(z) = I & (44) \\ \delta_0 = \langle I, I\rangle, & (45) \end{cases}$$

and calculate for j=0,1,...,n-1

$$\alpha_j = \langle zA_j(z), \bar{B}_j(z)\rangle \delta_j^{-1} \quad (46)$$

$$(h_j, u_j) = \text{a quasi } \delta_j\text{-polar decomposition of } \alpha_j \quad (47)$$

$$\begin{cases} A_{j+1}(z) = zA_j(z) - \alpha_j \bar{B}_j(z) & (48) \\ \bar{B}_{j+1}(z) = h_j zA_j(z) - u_j \bar{B}_j(z) & (49) \end{cases}$$

$$\delta_{j+1} = \delta_j - \alpha_j \delta_j \alpha_j' \; (= (I - h_j^2)\delta_j). \quad (50)$$

If (47) is replaced with

$$(h_j, u_j) = \text{the } \delta_j\text{-polar decomposition of } \alpha_j, \quad (47)'$$

the algorithm is called the canonical polar-type LWR algorithm.

In the (quasi-)polar-type algorithm, we can easily verify the following by induction on j:

$$\begin{cases} \langle A_j(z), z^i\rangle = 0 \quad \text{for } \forall i=0,1,\ldots,j-1 & (51) \\ \langle \bar{B}_j(z), z^i\rangle = 0 \quad \text{for } \forall i=1,2,\ldots,j & (52) \end{cases}$$

$$\delta_j = \langle A_j(z), A_j(z) \rangle = \langle \bar{B}_j(z), \bar{B}_j(z) \rangle. \tag{53}$$

It is also shwon that $A_j(z)$ is a monic PM of degree j. Consequently, $\{A_j(z)\}$ are the forward orthogonal PM's defined from $(A(z), \Sigma)$, and $\delta_j = \delta_j^A$. Similarly, we can show that

$$\bar{B}_j(z) = (-1)^j u_{j-1} \cdot \cdots \cdot u_0 B_j(z),$$

where $B_j(z)$ is the j-th degree backward orthogonal PM defined fron $(A(z), \Sigma)$.

We should examine the definability of quasi polar decompositions in the algorithm. In the case where A(z) is d.t. stable, it turns out that $\{\delta_j\}$ are all positive-definite, and hence the algorithm is always valid. In the general case without the stability assumption on A(z), it may possibly occur that the algorithm does not work. In particular, the validity of the canonical algorithm requires the condition that every α_j is sufficiently near to I, or equivalently that A(z) is sufficiently near to $(z-1)^n I$. As we shall see in Sec.8.2, this condition means that A(z) can be regarded as an approximation of a c.t. PM C(s) of the form (34).

Let us investigate the mutual relation among the quantities appeared in the canonical algorithm. In the situation where Σ (= δ_n) is given, it is obvious that A(z) (= $A_n(z)$) is determined by $\{\delta_j \ ; \ 0 \leqq j \leqq n-1\}$ and $\{\alpha_j \ ; \ 0 \leqq j \leqq n-1\}$. However, $\{\delta_j\}$ and $\{\alpha_j\}$ are not independent each other, but constrained by (50). The constraint completely determines the factor h_j of α_j from (δ_j, δ_{j+1}), while the factor u_j is free of the constraint. Consequently,

$A(z)$ is determined by $\{\delta_j\}$ and $\{u_j\}$, which are independent each other. This means that $\{\delta_j \; ; \; u_j\}$ constitute a coordinate system for expressing $A(z)$. We can regard these quantities as the counterparts of $\{\varepsilon_j \; ; \; \theta_j\}$. (More precisely, δ_j corresponds to ε_j, and u_j corresponds to $\theta_j \varepsilon_j^{-1}$. See Prop.8.2.8.)

In the quasi-polar type algorithm, the matrices $\{\delta_j \; ; \; u_j\}$ cannot be regarded as a coordinate system because of the arbitrariness of quasi polar decompositions in (47). However, we can show the following important theorem corresponding to the c.t. results in Th.4.4.2, Th.4.4.3 and Remark 4.4.5.

Theorem 5.1 For an arbitrary $p \times p$ PM $A(z)$ of the form (33) and for an arbitrary $p \times p$ positive-definite matrix Σ, the following conditions are mutually equivalent.

(i) $A'(z^{-1})\Sigma^{-1}A(z)$ is symmetric.

(ii) For $\forall j=0,1,\ldots,n$,

$$\begin{cases} B_j(z) = z^j A_j(z^{-1}) \\ \delta_j^B = \delta_j^A. \end{cases}$$

(iii) Quasi polar decompositions $\{(h_j, u_j)\}$ in the polar-type LWR algorithm can be chosen so that $u_j = I$ for $\forall j=0,1,\ldots,n-1$.

We note that the above theorem can be extended to the general column reduced case, as well as in Th.4.4.2 and Th.4.4.3.

PART II

THE MULLIS-ROBERTS TYPE APPROXIMATIONS

6. THE MULLIS-ROBERTS TYPE APPROXIMATIONS FOR DISCRETE-TIME SYSTEMS

Mullis and Roberts ([21]) proposed two methods for approximating discrete-time SISO (single-input-single-output) systems —— the modified least squares approximation and the interpolatory approximation ——, and elucidated the properties of the approximations by investigating their mutual relationship. The extension of the methods to the MIMO (multiple-input-multiple-output) case was studied by Inouye ([12]). In this chapter we will present the fundamental theory for the two methods, most of which is based on [21] and [12].

6.1. The modified least squares approximation (MLSA)

Suppose that we are given a discrete-time stable and strictly proper rational matrix $G(z) : p \times q$, and consider the problem of approximating $G(z)$ by a rational matrix $\hat{G}(z)$ of the form

$$\hat{G}(z) = \hat{A}^{-1}(z)\, \hat{B}(z) \tag{1}$$

where

$$\begin{cases} \hat{A}(z) = z^n I + z^{n-1} \hat{A}_{n-1} + \dots + \hat{A}_0 \quad : p \times p & (2) \\ \hat{B}(z) = z^{n-1} \hat{B}_{n-1} + \dots + \hat{B}_0 \quad : p \times q. & (3) \end{cases}$$

The above equations mean that all the observability indices of $\hat{G}(z)$ are less than or equal to n. From now on, we assume the original system $G(z)$ to be such that

$$\left\{\begin{array}{l}\text{all the observability indices of } G(z) \text{ are} \\ \text{greater than or equal to } n,\end{array}\right. \tag{4}$$

which will simplify the later arguments.

Now, we formulate <u>the n-th order modified least squares approximation (MLSA) problem</u> as follows: find a rational matrix $\hat{G}(z)$ of the form (1) minimizing the criterion

$$D \triangleq \|\hat{A}(z)G(z) - \hat{B}(z)\|^2, \tag{5}$$

where the norm is defined by

$$\|R(z)\|^2 \triangleq \text{trace } \frac{1}{2\pi}\int_{-\pi}^{\pi} R(e^{i\omega})R'(e^{-i\omega})d\omega. \tag{6}$$

The reason for the adjective 'modified' is that the criterion D can be regarded as a modification of the L^2 criterion

$$C \triangleq \|G(z) - \hat{G}(z)\|^2.$$

First, invoking that a polynomial matrix and a d.t. stable strictly proper rational matrix are always orthogonal on the unit circle, we have

$$D = \|[\hat{A}(z)G(z)]_{sp}\|^2 + \|[\hat{A}(z)G(z)]_{pol} - \hat{B}(z)\|^2, \tag{7}$$

where $[\ ]_{sp}$ and $[\ ]_{pol}$ denote the strictly proper part and the polynomial part of a rational matrix as defined in Def.3.1.2. It follows from (7) that $\hat{B}(z)$ minimizing D for a fixed $\hat{A}(z)$ is given by

$$\hat{B}(z) = [\hat{A}(z)G(z)]_{pol} \tag{8}$$

Note that the right-hand side of the above is of degree at most n-1 and is written as (3). We can see that

$$(8) \Longleftrightarrow \hat{A}(z)G(z) - \hat{B}(z) \text{ is strictly proper.}$$

$$\Longleftrightarrow \hat{A}(z)G(z) - \hat{B}(z) = O(z^{-1})$$

$$\Longleftrightarrow G(z) - \hat{G}(z) = O(z^{-n-1}), \tag{9}$$

where $O(\cdot)$ denotes Landau's symbol expressing the order of a function as $z \to \infty$. Using the impulse response sequence $\{g_j ; j=0,1,\ldots\}$ of $G(z)$ defined by

$$G(z) = g_0 z^{-1} + g_1 z^{-2} + \ldots. , \tag{10}$$

the condition (9) is written as

$$\hat{G}(z) = g_0 z^{-1} + \ldots. + g_{n-1} z^{-n} + O(z^{-n-1}). \tag{11}$$

This means that the first n elements of the impulse response sequence of $\hat{G}(z)$ are equal to those of $G(z)$.

Under the mutually equivalent conditions (8),(9) and (11), the criterion D depends on $\hat{A}(z)$ alone, and is written as

$$D = \| [\hat{A}(z)G(z)]_{sp} \|^2$$

$$= \| G^{(n)}(z) + \hat{A}_{n-1} G^{(n-1)}(z) + \ldots + \hat{A}_0 G^{(0)}(z) \|^2$$

where

$$G^{(k)}(z) \triangleq [z^k G(z)]_{sp} . \tag{12}$$

Hence, defining for m=1,2,...

$$K_m \triangleq \begin{bmatrix} K_{0,0} & \cdots & K_{0,m-1} \\ \vdots & & \vdots \\ K_{m-1,0} & \cdots & K_{m-1,m-1} \end{bmatrix} : \quad mp \times mp \tag{13}$$

where

$$K_{jk} \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} G^{(j)}(e^{i\omega}) G^{(k)\prime}(e^{-i\omega}) d\omega \quad : \quad p \times p, \tag{14}$$

we have

$$D = \text{trace } [\hat{A}_0, \ldots, \hat{A}_{n-1}, I] K_{n+1} [\hat{A}_0, \ldots, \hat{A}_{n-1}, I]'. \tag{15}$$

Note that K_m is positive-semidefinite for every m.

The minimization of D in (15) is a simple least squares problem, and a necessary and sufficient condition for $\hat{A}(z)$ to minimize D is that $\hat{A}(z)$ satisfies the normal equation:

$$[\hat{A}_0, \ldots, \hat{A}_{n-1}, I] K_{n+1} = [0, \ldots, 0, \delta], \tag{16}$$

where δ is the indeterminate coefficient. Note that δ is a $p \times p$ positive-semidefinite matrix. The minimum of D is given by

$$\min D = \text{trace } \delta.$$

We call δ <u>the approximation error matrix</u> of the n-th order MLSA.

The solutions of the n-th order MLSA problem are completely characterized by (11) and (16). In particular, if

$$K_n > 0 \tag{17}$$

then the solutions are unique. It will be shown later that (17) is equivalent to the assumption (4).

Remark 6.1.1 Inouye ([12]) has shown that even if K_n is singular the approximant $\hat{G}(z)$ is uniquely determined, in spite of the non-uniqueness of $\hat{A}(z)$.

In order to derive some properties of the MLSA, we investigate the structure of $\{\kappa_{ij}\}$. First, since (10) implies that

$$z^k G(z) = (g_0 z^{k-1} + \dots + g_{k-1}) + (g_k z^{-1} + g_{k+1} z^{-2} + \dots)$$

we have

$$G^{(k)}(z) = g_k z^{-1} + g_{k+1} z^{-2} + \dots \,. \tag{18}$$

Hence, the definition (14) leads to

$$\kappa_{ij} = \sum_{k=0}^{\infty} g_{i+k} g'_{j+k} = r_{|i-j|} - \sum_{k=1}^{\min(i,j)} g_{i-k} g'_{j-k} \tag{19}$$

where

$$r_j \triangleq \sum_{k=0}^{\infty} g_{j+k} g'_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\omega} G(e^{i\omega}) G'(e^{-i\omega}) d\omega. \tag{20}$$

We call $\{r_j; j = 0,1,2,\dots\}$ <u>the auto-covariance sequence</u> of $G(z)$. It follows from (19) that the matrix K_m can be written in terms of the 2n+1 matrices $\{g_0,\dots,g_{m-2}; r_0,\dots,r_{m-1}\}$ as

$$K_m = R_m - G_m G'_m \tag{21}$$

where

$$R_m \triangleq \begin{bmatrix} r_0 & r'_1 & \cdots & r'_{m-1} \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & r'_1 \\ r_{m-1} & \cdots & r_1 & r_0 \end{bmatrix} \qquad G_m \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ g_0 & 0 & & \vdots \\ g_1 & g_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g_{m-2} & \cdots & g_1 & g_0 \end{bmatrix} . \tag{22}$$

Thus we can see from (11) and (16) that the solution of the n-th order MLSA problem is determined by the quantities $\{g_0,\ldots,g_{n-1};\ r_0,\ldots,r_n\}$.

Another representation of κ_{ij} than (19) can be obtained via a state space description. Suppose that a state space realization of G(z)

$$G(z) = W(sI - U)^{-1}V \tag{23}$$

is minimal. Then, using the equation

$$z^k(zI-U)^{-1} = z^{k-1}I + z^{k-2}U + \ldots + U^{k-1} + U^k(zI-U)^{-1},$$

we have

$$G^{(k)}(z) = WU^k(zI - U)^{-1}V. \tag{24}$$

Since U is a discrete-time stable matrix, the Lyapunov equation

$$X = UXU' + VV' \tag{25}$$

has a unique solution X, which is represented as

$$X = \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{i\omega}I - U)^{-1}V\,V'(e^{-i\omega}I - U)'^{-1}d\omega.$$

Hence, we obtain from (14) and (24)

$$\kappa_{ij} = WU^iXU'^jW'. \tag{26}$$

Noting that X is positive-definite because of the controllability of (U,V), we can see from (26) that the condition (17) for the unique existence of solutions of the n-th order MLSA problem is equivalent to

$$\text{rank } [W', U'W', \ldots, U'^{n-1}W']' = np,$$

which is nothing but the assumption (4).

As the next step, we examine the stability of the approximant $\hat{G}(z)$. Owing to the special structure (21) of the matrices $\{K_m\}$, the normal equation (16) can be modified into the following Lyapunov equation (see [21] and [12]) :

$$K_n = \underset{\sim}{\hat{A}}\, K_n\, \underset{\sim}{\hat{A}}' + \underset{\sim}{g}_n \underset{\sim}{g}_n' + \underset{\sim}{b}\, \delta\, \underset{\sim}{b}', \tag{27}$$

where

$$\underset{\sim}{\hat{A}} \triangleq \text{comp}\,\{\hat{A}(z)\}$$

$$= \begin{bmatrix} 0 & I & & \bigcirc \\ & \bigcirc & \ddots & \\ & & & I \\ -\hat{A}_0 & -\hat{A}_1 & \cdots & -\hat{A}_{n-1} \end{bmatrix} \quad : \quad np \times np$$

$$\underset{\sim}{g}_n \triangleq [g_0', \ldots, g_{n-1}']' \quad : \ np \times q$$

$$\underset{\sim}{b} \triangleq [0, \ldots, 0, I]' \quad : \ np \times p.$$

Since K_n is positive-definite under the assumption (4), we can see from the above equation that $\underset{\sim}{\hat{A}}$ is a discrete-time stable matrix if and only if

$$(\underset{\sim}{\hat{A}}, [\underset{\sim}{g}_n, \underset{\sim}{b}\gamma]) \text{ is controllable,} \tag{28}$$

where γ is a $p \times p$ matrix such that $\gamma\gamma' = \delta$. In Appendix 6.1.4 at the end of the present section, we will show that (28) is derived from the assumption (4). Thus, the stability of $\underset{\sim}{\hat{A}}$, or equivalently of $\hat{A}(z)$, is guaranteed by (4). We note that (28) is easily verified in the following two special cases.

(i) All the obsevability indices of $G(z)$ are greater than n.

(ii) All the obsevability indices of $G(z)$ are equal to n.

Indeed, (i) implies that δ is positive-definite, which leads to the controllability of $(\hat{\underset{\sim}{A}}, \underset{\sim}{b}\gamma)$, and (ii) implies that $\hat{G}(z) = G(z)$, which leads to the controllability of $(\hat{\underset{\sim}{A}}, \underset{\sim}{g}_n)$ (see Lemma 6.2.1).

<u>Remark 6.1.2</u> Without the assumption (4), $\hat{A}(z)$ may possibly have unstable zeros. Nevertheless, it can be shown ([12]) that $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ is always a stable system. This means that the unstable zeros of $\hat{A}(z)$ are cancelled out by those of $\hat{B}(z)$.

The results of the present section are summarized in the following theorem.

<u>Theorem 6.1.3</u> For a given stable system $G(z)$ satisfying (4), the solution $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ of the n-th order MLSA problem satisfies the following.

(i) $(\hat{A}(z), \hat{B}(z))$ is uniquely determined by the 2n+1 matrix data $\{g_0, \ldots, g_{n-1}; r_0, \ldots, r_n\}$ taken from $G(z)$.

(ii) $\hat{G}(z)$ preserves $\{g_0, \ldots, g_{n-1}\}$.

(iii) $\hat{A}(z)$ is a stable PM.

Appendix 6.1.4 (Proof of (28)) We first note that the controllability of $(\hat{\underset{\sim}{A}}, [\underset{\sim}{g}_n, \underset{\sim}{b}\gamma])$ is equivalent to the left coprimeness of $\hat{A}(z)$ and $[\hat{B}(z), \gamma]$. (The proof of this fact is quite similar to that of Lemma 6.2.1 in the next section, and is omitted.) Hence, it suffices to show that $x'\hat{A}(z)$ and $[x'\hat{B}(z), x'\gamma]$ are left coprime for any nonzero p-vector x. If $x'\gamma \neq 0$, the coprimeness is obvious. So, we assume that $x'\gamma = 0$ and that $x \neq 0$. It then follows that

$$\|x'(\hat{A}(z)G(z) - \hat{B}(z))\|^2 = x'\,\delta\,x = (x'\gamma)^2 = 0,$$

from which we have

$$x'\hat{A}(z)G(z) = x'\hat{B}(z). \tag{29}$$

Now, suppose that $x'\hat{A}(z)$ and $x'\hat{B}(z)$ are not left coprime. Then a greatest common left devisor of $(x'\hat{A}(z), x'\hat{B}(z))$, say d(z), is not a constant, and therefore the polynomial row vector

$$p'(z) \triangleq d^{-1}(z)x'\hat{A}(z) \quad : 1 \times p$$

can be written as

$$p'(z) = z^{n-1}\, p'_{n-1} + \cdots + z\, p'_1 + p'_0.$$

On the other hand, we can see from (29) that $p'(z)G(z)$ (= $d^{-1}(z)x'\hat{B}(z)$) is also a polynomial row vector. This fact is represented, in terns of a minimal realization (U,V,W) of G(z), as

$$p'_0W + p'_1WU + \cdots + p'_{n-1}WU^{n-1} = 0.$$

This contradicts the assumption (4). (QED)

6.2. The interpolatory approximation (IA)

To begin with, we consider the special case of the MLSA problem where all the observability indices of G(z) are n. It then turns out that $\hat{G}(z) = G(z)$ and that $\delta = 0$ in the normal equation (6.1.16). Hence the Lyapunov equation (6.1.27) is reduced to

$$K_n = \hat{\underset{\sim}{A}} K_n \hat{\underset{\sim}{A}}' + \underset{\sim}{g}_n \underset{\sim}{g}_n'. \tag{1}$$

Since K_n is composed of the 2n-1 matrix data $\{g_0, \ldots, g_{n-2}; r_0, \ldots, r_{n-1}\}$ taken from $\hat{G}(z) = G(z)$, the equation (1) expresses a condition which the 2n data $\{g_0, \ldots, g_{n-1}; r_0, \ldots, r_{n-1}\}$ should obey, together with (6.1.11):

$$\hat{G}(z) = g_0 z^{-1} + \ldots. + g_{n-1} z^{-n} + O(z^{-n-1}). \tag{2}$$

The equation (1) is also derived in the following way. Let

$$\hat{G}_k(z) \triangleq [z^k \hat{G}(z)]_{sp} .$$

Then it follows from (2) that

$$z\hat{G}_k(z) = \hat{G}_{k+1}(z) + g_k . \tag{3}$$

On the other hand, we have

$$\hat{G}_n(z) + \hat{A}_{n-1}\hat{G}_{n-1}(z) + \ldots. + \hat{A}_0 \hat{G}_0(z)$$
$$= [\hat{A}(z)\hat{G}(z)]_{sp} = [\hat{B}(z)]_{sp} = 0. \tag{4}$$

These equations yield

$$[\hat{G}_0'(z),\ldots,\hat{G}_{n-1}'(z)]' = (zI - \hat{\underset{\sim}{A}})^{-1}\underset{\sim}{g}_n \ . \tag{5}$$

Recalling the definition of K_n (6.1.13-14), we can see that (5) leads to (1).

As a by-product of (5), we obtain the following lemma, which will be used in later arguments.

Lemma 6.2.1 If $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ of the form (6.1.1-3) satisfies (2), the controllability of $(\hat{\underset{\sim}{A}},\underset{\sim}{g}_n)$ is equivalent to the left coprimeness of $\hat{A}(z)$ and $\hat{B}(z)$.

(Proof) From (5) we have

$$\hat{G}(z) = \underset{\sim}{e}'(zI - \hat{\underset{\sim}{A}})^{-1}\underset{\sim}{g}_n \tag{6}$$

where

$$\underset{\sim}{e}' \triangleq [I,0,\ldots,0] \quad : p \times np. \tag{7}$$

The triplet $(\hat{\underset{\sim}{A}},\underset{\sim}{g}_n,\underset{\sim}{e}')$ is called the observability realization of $\hat{G}(z)$ ([17]). Noting that $(\hat{\underset{\sim}{A}},\underset{\sim}{e}')$ is always observable, and considering the McMillan degree of $\hat{G}(z)$, we can see that $(\hat{\underset{\sim}{A}},\underset{\sim}{g}_n)$ is controllable if and only if $\hat{A}(z)$ and $\hat{B}(z)$ are left coprime.

(QED)

Now, return to the general situation where a stable system $G(z)$ satisfying (6.1.4) is given, and consider the problem of characterizing and constructing systems of the form (6.1.1-3) which preserve (or match, or interpolate) the 2n matrix data $\{g_0,\ldots,g_{n-1};r_0,\ldots,r_{n-1}\}$ taken from $G(z)$. This problem is called the n-th order interpolatory approximation (IA) problem.

Suppose that a system $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ preserves the data.

Then $\hat{G}(z)$ satisfies (1) and (2), evidently. Furthermore, recalling that the positive-definiteness of K_n is equivalent to the assumption (6.1.4), we can see that the observability indices of $\hat{G}(z)$ are not less than n, also. This means that $\hat{A}(z)$ and $\hat{B}(z)$ are left coprime. Conversely, suppose that $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ satisfies (1) and (2), and that $\hat{A}(z)$ and $\hat{B}(z)$ are left coprime. Then, since $(\hat{\underset{\sim}{A}}, \underset{\sim}{g}_n)$ is controllable (Lemma 6.2.1), and since K_n is positive-definite, it follows from (1) that $\hat{\underset{\sim}{A}}$ is a stable matrix. It is known that the stability of $\hat{\underset{\sim}{A}}$ guarantees the uniqueness of solutions of the Lyapunov equation (1). Hence, it is concluded from (1) and (2) that $\hat{G}(z)$ preserves the data. Thus, we obtain the following theorem.

<u>Theorem 6.2.2</u> Let $\{g_0, \ldots, g_{n-1}; r_0, \ldots, r_{n-1}\}$ be the data taken from a stable system G(z) satisfying (6.1.4). Then, a system $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ of the form (6.1.1-3) is stable and preserves the data, if and only if $\hat{A}(z)$ and $\hat{B}(z)$ are left coprime and $\hat{G}(z)$ satisfies (1) and (2).

<u>Remark 6.2.3</u> If $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ satisfies (1) and (2) but does not satisfy the coprimeness, then $\hat{G}(z)$ does not preserve $\{r_0, \ldots, r_{n-1}\}$. Furthermore $\hat{A}(z)$ is not a stable PM in this case. Nevertheless, it can be shwon from (1) that $\hat{G}(z)$ is stable (cf. Remark 6.1.2).

As the next step, we proceed to the problem of how to construct $\hat{G}(z) = \hat{A}^{-1}(z)\hat{B}(z)$ satisfying (1) and (2). Since $\hat{B}(z)$

is determined from $\hat{A}(z)$ by (2), we have only to construct $\hat{A}(z)$ satisfying (1). Our approach is based on the following result : Modify the matrix K_{n+1}, which is composed of the 2n+1 matrix data $\{g_0,\ldots,g_{n-1};r_0,\ldots,r_n\}$, into $\bar{K}(\bar{r}_n)$ by replacing r_n with a free matrix parameter $\bar{r}_n$. ($K_{n+1} = \bar{K}_{n+1}(r_n)$, in particular.) Then it is shown (see Inouye [12]) that the equation

$$K_n = \underset{\sim}{\hat{A}} K_n \underset{\sim}{\hat{A}}' + \underset{\sim}{g}_n \underset{\sim}{g}_n' + \underset{\sim}{b}\, \bar{\delta}\, \underset{\sim}{b}' \tag{8}$$

is equivalent to the existence of a matrix $\bar{r}_n$ such that

$$[\hat{A}_0,\ldots,\hat{A}_{n-1},I]\ K_{n+1}(\bar{r}_n) = [0,\ldots,0,\bar{\delta}], \tag{9}$$

where $\bar{\delta}$ is an arbitrary symmetric matrix. Therefore, the Lyapunov equation (1) is equivalent to the following condition:

$$\exists\, \bar{r}_n \quad \text{s.t.} \quad [\hat{A}_0,\ldots,\hat{A}_{n-1},I]\bar{K}_{n+1}(\bar{r}_n) = [0,\ldots,0,0] \tag{10}$$

Let us construct $\hat{A}(z)$ satisfying (10). First, for arbitrary $p \times p$ PM's such that

$$\begin{cases} P(z) = z^n P_n + \ldots\ldots + z P_1 + P_0 \\ Q(z) = z^n Q_n + \ldots\ldots + z Q_1 + Q_0 \end{cases}$$

we define the inner product $\langle P(z),Q(z)\rangle$ as

$$\begin{aligned} \langle P(z),Q(z)\rangle &\overset{\Delta}{=} [P_0,\ldots,P_n]K_{n+1}[Q_0,\ldots,Q_n]' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G_P(e^{i\omega})G_Q'(e^{-i\omega})d\omega, \end{aligned} \tag{11}$$

where

$$G_P(z) \triangleq [P(z)G(z)]_{sp}\ , \qquad G_Q(z) \triangleq [Q(z)G(z)]_{sp}\ .$$

Then, we can see that the existence of $(\bar{r}_n, \bar{\delta})$ satisfying (9) is equivalent to the condition

$$\langle \hat{A}(z), z^j I \rangle = 0 \quad \text{for } \forall j=1,2,\ldots,n-1. \tag{12}$$

The solutions of the above equation are parametrized as follows:

$$\hat{A}(z) = z\, E(z) + \hat{A}_0 F(z), \tag{13}$$

where $\hat{A}_0$ $(= \hat{A}(0))$ is a free matrix parameter, and $E(z)$ and $F(z)$ are PM's such that

$$\begin{cases} E(z) = z^{n-1}\, I + z^{n-2}\, E_{n-2} + \cdots\cdot + E_0 \\ \langle zE(z), z^j I \rangle = 0 \quad \text{for } \forall j=1,2,\ldots,n-1 \end{cases} \tag{14}$$

$$\begin{cases} F(z) = z^{n-1}\, F_{n-1} + \cdots\cdot + z\, F_1 + I \\ \langle F(z), z^j I \rangle = 0 \quad \text{for } \forall j=1,2,\ldots,n-1. \end{cases} \tag{15}$$

It should be noted that $E(z)$ and $F(z)$ are determined from the data $\{g_0, \ldots, g_{n-1}; r_0, \ldots, r_{n-1}\}$ and are independent of r_n. We can show that if $\hat{A}(z)$ is written as (13) then $\bar{\delta}$ in (9) is given by

$$\bar{\delta} = \alpha - \hat{A}_0\, \beta\, \hat{A}_0' \tag{16}$$

where

$$\alpha \triangleq \langle zE(z), zE(z) \rangle \quad : p \times p \tag{17}$$

$$\beta \triangleq \langle F(z), F(z) \rangle \quad : p \times p. \tag{18}$$

Note that α and β are determined from the data $\{g_0, \ldots, g_{n-1};$

$r_0,\ldots,r_{n-1}\}$. Consequently, we see that $\hat{A}(z)$ satisfies (10) if and only if it is written as (13) with $\hat{A}_0$ satisfying

$$\hat{A}_0 \beta \hat{A}_0' = \alpha. \tag{19}$$

The solutions of (19) are constructed as follows. Since $\alpha \geqq 0$ and $\beta > 0$, they have positive-semidefinite square root matrices $\alpha^{1/2}$ and $\beta^{1/2}$. Let

$$T \triangleq \beta^{1/2}(\beta^{-1/2}\alpha\beta^{-1/2})^{1/2}\beta^{-1/2}. \tag{20}$$

Then it is easy to see that $\hat{A}_0 = \pm T$ are solutions of (19). We note that T is β-positive-semidefinite, i.e.,

$$T\beta = \beta T' \geqq 0. \tag{21}$$

In general, a matrix $\hat{A}_0$ is a solution of (19) if and only if there exists a β-orthogonal matrix U :

$$U \beta U' = \beta \tag{22}$$

such that

$$(-1)^n \hat{A}_0 = T\,U. \tag{23}$$

(The significance of the factor $(-1)^n$ in (23) lies only in its convenience for the arguments in Se.8.3.) It should be noted that if $\hat{A}_0$ is nonsingular then T and U are uniquely determined from $\hat{A}_0$ by (21)-(23). Indeed, (T,U) is the β-polar decomposition of $\hat{A}_0$ defined in Chap.5.

From the above arguments, we obtain the following theorem.

Theorem 6.2.4 A PM $\hat{A}(z)$ satisfies (1), if and only if it is written as (13) and there exists a β-orthogonal matrix U satisfying (23).

We denote by $\hat{A}(z;U)$ the solution of (1) satisfying (23), and by $\hat{G}(z;U) = \hat{A}^{-1}(z;U)\hat{B}(z;U)$ the corresponding solution of (1) and (2). It is noted that the mapping : $U \longmapsto \hat{G}(z;U)$ is one-to-one if α is nonsingular.

Remark 6.2.5 Consider the case where p = 1, which includes the SISO case. It then turns out that the β-orthogonal group is $\{1,-1\}$. Hence, the equation (1) has (at most) two solutions, i.e., $\hat{A}(z;1)$ satisfying $(-1)^n\hat{A}_0 \geqq 0$ and $\hat{A}(z;-1)$ satisfying $(-1)^n\hat{A}_0 \leqq 0$.

7. THE MULLIS-ROBERTS TYPE APPROXIMATIONS FOR CONTINUOUS-TIME SYSTEMS

7.1. MLSA and IA for continuous-time systems

The purpose of this section is to develop the theory of the MLSA and the IA for continuous-time (c.t.) systems, corresponding to the discrete-time (d.t.) theory in the previous chapter.

Suppose that we are given a $p \times q$ c.t. stable and strictly proper rational matrix $H(s)$, and consider the problem of approximating $H(s)$ by a rational matrix $\hat{H}(s)$ of the form

$$\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s) \tag{1}$$

where

$$\begin{cases} \hat{C}(s) = s^n I + s^{n-1} \hat{C}_{n-1} + \dots + \hat{C}_0 \quad : p \times p & (2) \\ \hat{D}(s) = s^{n-1} \hat{D}_{n-1} + \dots + \hat{D}_0 \quad : p \times q. & (3) \end{cases}$$

As in the previous chapter, we assume that

$$\begin{cases} \text{all the observability indices of } H(s) \text{ are} \\ \text{greater than or equal to } n. \end{cases} \tag{4}$$

To begin with, we formulate the c.t. analogue of the d.t. MLSA problem in Sec.6.1. It seems evident that the criterion D in (6.1.5) corresponds to

$$E \triangleq \| \hat{C}(s)H(s) - \hat{D}(s) \|^2, \tag{5}$$

where the norm is defined by

$$\|R(s)\|^2 \triangleq \text{trace}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} R(i\omega)R'(-i\omega)d\omega. \tag{6}$$

However, there is a remarkable difference between D and E that E is not finite in general, while D is always finite. So, we investigate the condition for the finiteness of E. Since H(s) is assumed to be c.t. stable, $\hat{C}(s)H(s)-\hat{D}(s)$ has no poles on the imaginary axis. Therefore, a necessary and sufficient condition for E to be finite is that $\hat{C}(s)H(s)-\hat{D}(s)$ is strictly proper. Using Landau's symbol $O(\cdot)$, the finiteness condition is represented as

$$\hat{C}(s)H(s) - \hat{D}(s) = O(s^{-1}), \tag{7}$$

which is equivalent to

$$H(s) - \hat{H}(s) = O(s^{-n-1}). \tag{8}$$

Define the Markov parameters $\{h_j\}$ of H(s) by

$$H(s) = h_0 s^{-1} + h_1 s^{-2} + \ldots. \tag{9}$$

Then the condition (8) is written as

$$\hat{H}(s) = h_0 s^{-1} + \ldots + h_{n-1}s^{-n} + O(s^{-n-1}). \tag{10}$$

That is, E is finite if and only if $\hat{H}(s)$ preserves the first n Markov parameters $\{h_0,\ldots,h_{n-1}\}$ of the original system H(s). In the sequel, we consider the problem of finding a system $\hat{H}(s)$ of the form (1)-(3) which satisfies (10) and minimizes E. This is called <u>the n-th order continuous-time modified least squares approximation (c.t. MLSA) problem</u>.

The finiteness condition (7) yields

$$\hat{D}(s) = [\hat{C}(s)H(s)]_{pol} \tag{11}$$

and

$$\hat{C}(s)H(s) - \hat{D}(s) = [\hat{C}(s)H(s)]_{sp}$$

$$= H^{(n)}(s) + \hat{C}_{n-1}H^{(n-1)}(s) + \ldots + \hat{C}_0 H^{(0)}(s), \tag{12}$$

where $[\ ]_{pol}$ and $[\ ]_{sp}$ denote the polynomial part and the strictly proper part of a rational matrix as in Def.3.1.2, and

$$H^{(k)}(s) \triangleq [s^k H(s)]_{sp} . \tag{13}$$

Hence, defining for k=1,2,...

$$M_k \triangleq \begin{bmatrix} \mu_{0,0} & \cdots & \mu_{0,k-1} \\ \vdots & & \vdots \\ \mu_{k-1,0} & \cdots & \mu_{k-1,k-1} \end{bmatrix} \quad : kp \times kp \tag{14}$$

where

$$\mu_{jk} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} H^{(j)}(i\omega) H^{(k)\prime}(-i\omega) d\omega \quad : p \times p, \tag{15}$$

we have

$$E = \text{trace } [\hat{C}_0, \ldots, \hat{C}_{n-1}, I] M_{n+1} [\hat{C}_0, \ldots, \hat{C}_{n-1}, I]'. \tag{16}$$

Note that M_k is positive-semidefinite for every k.

It is clear from (16) that a PM $\hat{C}(s)$ minimizes E if and only if it satisfies the normal equation:

$$[\hat{C}_0, \ldots, \hat{C}_{n-1}, I]\ M_{n+1} = [0, \ldots, 0, \varepsilon], \tag{17}$$

where ε : $p \times p$ is the indeterminate coefficient which turns out positive-semidefinite. The minimum of E is given by

$$\min E = \text{trace } \varepsilon.$$

We call ε the approximation error matrix of the n-th order MLSA. Corresponding to the fact that (6.1.4) is equivalent to (6.1.17) it can be shown that the assumption (4) is equivalent to

$$M_n > 0, \tag{18}$$

which guarantees the existence and the uniqueness of $\hat{C}(s)$ satisfying (17). Thus, the two conditions (10) and (17), which correspond to (6.1.11) and (6.1.16), determine the unique solution $\hat{H}(s)$ of the n-th order MLSA problem.

Let us investigate the structure of the matrices $\{\mu_{jk}\}$. Noting that (cf.(6.1.18))

$$H^{(k)}(s) = h_k s^{-1} + h_{k+1} s^{-2} + \dots \tag{19}$$

and that (cf.(6.2.3))

$$s\, H^{(k)}(s) = H^{(k+1)}(s) + h_k , \tag{20}$$

and recalling Lemma 3.1.1, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)H^{(j)}(i\omega)H^{(k)\prime}(-i\omega)d\omega &= \mu_{j+1,k} + \tfrac{1}{2}h_j h_k' \\ &= -\mu_{j,k+1} - \tfrac{1}{2}h_j h_k' , \end{aligned} \tag{21}$$

which yields the following important equation:

$$\mu_{j+1,k} + \mu_{j,k+1} + h_j h_k' = 0. \tag{22}$$

For $j=0,1,...,$ let

$$\mu_j \triangleq \mu_{j,j} \quad : p \times p \tag{23}$$

$$\begin{aligned}\lambda_j &\triangleq \mu_{j+1,j} + \tfrac{1}{2}h_j h_j' \quad : p \times p \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)\, H^{(j)}(i\omega)\, H^{(j)\prime}(-i\omega)\, d\omega.\end{aligned} \tag{24}$$

It should be noted that μ_j is symmetric and that λ_j is skew-symmetric. We can see from (22) that the matrix M_k $(k=1,2,...)$ is composed of $\{h_0,...,h_{k-2};\mu_0,...,\mu_{k-1};\lambda_0,...,\lambda_{k-2}\}$.

Remark 7.1.1 A time domain representation of μ_{jk} is derived as follows. Let $\{h(t)\ ;\ t\geqq 0\}$ be the impulse response of $H(s)$, and let $h^{(k)}(t) = d^k h(t)/dt^k$ for $k=0,1,...$. Then, we can show that

$$H^{(k)}(s) = \int_{0+}^{\infty} h^{(k)}(t)\, e^{-st}\, dt. \tag{25}$$

(Since $h^{(k)}(t)$ has a singularity at $t=0$, $\int_{0+}^{\infty}$ cannot be replaced with $\int_0^{\infty}$ in the above equation.) It follows from (25) that

$$\mu_{jk} = \int_{0+}^{\infty} h^{(j)}(t)\, h^{(k)\prime}(t)\, dt. \tag{26}$$

We note that eq.(22) can be obtained by application of integration by parts to (26) together with use of the formula

$$h_k = h^{(k)}(0+) = \lim_{t\downarrow 0} h^{(k)}(t). \tag{27}$$

It has been shown above that the n-th order approximant $\hat{H}(s)$ is determined from $\{h_0,\ldots,h_{n-1}\}$ and M_{n+1} by (11) and (17). However, partitioning (17) into

$$\begin{cases} [\hat{C}_0,\ldots,\hat{C}_{n-1},I]\ M_a = [0,\ldots,0] & (28) \\ [\hat{C}_0,\ldots,\hat{C}_{n-1},I]\ M_b = \varepsilon & (29) \end{cases}$$

where

$$M_{n+1} = [M_a,M_b], \quad M_a : (n+1)p \times np, \quad M_b : (n+1)p \times p,$$

we can see that (28) completely determines $\hat{C}(s)$ and that (29) is not necessary if ε is not required. Since M_a consists of the data $\{h_0,\ldots,h_{n-1};\mu_0,\ldots,\mu_{n-1};\lambda_0,\ldots,\lambda_{n-1}\}$, $\hat{H}(s)$ is determined from the same data. Note that the number of independent elements of the data is np(p+q) and is equal to the number of free parameters of $\hat{H}(s)$.

Remark 7.1.2 It is interesting to compare the above situation with the corresponding situation in the discrete-time case. We have seen in Sec.6.1 that, in order to determine the solution $\hat{G}(z)$ of the n-th order d.t. MLSA problem, the data $\{g_0,\ldots,g_{n-1};r_0,\ldots,r_n\}$ taken from the original system G(z) is necessary. The number of independent elements of the data is np(p+q)+p(p+1)/2, which is greater than the number of free parameters of $\hat{G}(z)$. This is an essential difference between the c.t. MLSA and the d.t. MLSA, and is closely related to the results shown below.

It is shown that (28) is equivalent to the equation

$$\underset{\sim}{\hat{C}}\, M_n = \begin{bmatrix} \mu_{1,0} & \cdots & \mu_{1,n-1} \\ \vdots & & \vdots \\ \mu_{n,0} & \cdots & \mu_{n,n-1} \end{bmatrix}$$

where

$$\underset{\sim}{\hat{C}} \triangleq \text{comp}\,\{\, \hat{C}(s) \,\}$$

$$= \begin{bmatrix} 0 & I & & 0 \\ \vdots & & \ddots & \\ & 0 & & I \\ -\hat{C}_0 & -\hat{C}_1 & \cdots & -\hat{C}_{n-1} \end{bmatrix} \quad : \quad np \times np$$

Hence, we can see from (22) that

$$\underset{\sim}{\hat{C}} M_n + M_n \underset{\sim}{\hat{C}}' + \underset{\sim}{h}_n \underset{\sim}{h}_n' = 0, \tag{30}$$

where

$$\underset{\sim}{h}_n \triangleq [h_0', \ldots, h_{n-1}']' \quad : \; np \times q.$$

The above Lyapunov equation seems to correspond to the equation (6.2.1) for IA rather than (6.1.27) for MLSA. Indeed, we have the following theorem corresponding to Th.6.2.2.

<u>Theorem 7.1.3</u> Let $\{h_0, \ldots, h_{n-1}; \mu_0, \ldots, \mu_{n-1}; \lambda_0, \ldots, \lambda_{n-2}\}$ be the data taken from a stable system H(s) satisfying (4). Then, a system $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the form (1)-(3) is stable and preserves the data, if and only if $\hat{C}(s)$ and $\hat{D}(s)$ are left coprime and $\hat{H}(s)$ satisfies (10) and (30).

The problem of finding systems of the form (1)-(3)

preserving the data $\{h_0,\ldots,h_{n-1};\mu_0,\ldots,\mu_{n-1};\lambda_0,\ldots,\lambda_{n-2}\}$ is called the n-th order continuous-time interpolatory approximation (c.t. IA) problem. The above argument shows that the solution $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the MLSA problem turns out a solution of the IA problem if $\hat{C}(s)$ and $\hat{D}(s)$ are left coprime. Furthermore, noting that eq.(28) is valid even if M_a defined from H(s) is replaced with $\hat{M}_a$ defined from $\hat{H}(s)$, and invoking that M_a contains the data λ_{n-1}, we can see that $\hat{H}(s)$ also preserves λ_{n-1}. It should be noted that the left coprimeness is a 'generic' property in the sense that $\hat{C}(s)$ and $\hat{D}(s)$ almost always turn out to be left coprime.

Next, consider the case where $\hat{C}(s)$ and $\hat{D}(s)$ are not left coprime. In this case, $\hat{H}(s)$ does not preserve the data. Moreover det $\hat{C}(s)$ has zeros on the imaginary axis. Nevertheless, we can see from (30) that $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ turns out stable by pole-zero cancellation.

The above results are summarized in the following theorem.

Theorem 7.1.4 For a given stable system H(s) satisfying (4), the solution $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the n-th order MLSA problem satisfies the following.

(i) $(\hat{C}(s),\hat{D}(s))$ is uniquely determined by the data $\{h_0,\ldots h_{n-1};\mu_0,\ldots,\mu_{n-1};\lambda_0,\ldots,\lambda_{n-1}\}$ taken from H(s).

(ii) $\hat{H}(s)$ preserves $\{h_0,\ldots,h_{n-1}\}$.

(iii) $\hat{H}(s)$ is stable.

(iv) If $\hat{C}(s)$ and $\hat{D}(s)$ are left coprime, then $\hat{C}(s)$ is a stable PM and $\hat{H}(s)$ preserves $\{\mu_0,\ldots,\mu_{n-1};\lambda_0,\ldots,\lambda_{n-1}\}$.

Remark 7.1.5 It is obvious from the above theorem that the c.t. MLSA is generically transitive in the following sense: Let ℓ, m, n be integers such that $n > m > \ell$, and let $H_n(s)$ be an n-th order stable system. We denote by $\hat{H}_m(s)$ and $\hat{H}_\ell(s)$ the m-th order approximant and the ℓ-th order approximant for $H_n(s)$, respectively, and by $\hat{\hat{H}}_\ell(s)$ the ℓ-th order approximant for $\hat{H}_m(s)$. Then it almost always holds that $\hat{\hat{H}}_\ell(s) = \hat{H}_\ell(s)$ (Fig.1). It should be noted that the d.t. MLSA does not satisfy the transitivity.

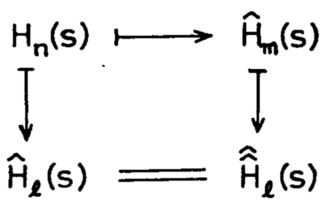

Fig.1 The transitivity of the c.t. MLSA

Remark 7.1.6 There is another difference between the c.t. MLSA and the d.t. MLSA. Let ε_n and δ_n be the approximation error matrices of the n-th order c.t. MLSA problem and the n-th order d.t. MLSA problem, respectively. Noting that the transformation : $(\hat{A}(z),\hat{B}(z)) \longmapsto (z\hat{A}(z),z\hat{B}(z))$ preserves the criterion D in (6.1.5), we can see that the matrices $\{\delta_n\}$ satisfy the monotonicity $\delta_n \geqq \delta_{n+1}$ ([12],[21]). In the c.t. case, the matrices $\{\varepsilon_n\}$ do not satisfy the monotonicity.

We have seen that in the c.t. case the solution of the MLSA problem is also a solution of the IA problem if the solution satisfies the coprimeness. As the next step, we proceed to parametrize all the solutions of the IA problem. Since the numerator $\hat{D}(s)$ is determined from the denominator $\hat{C}(s)$ by (10), we have only to parametrize $\hat{C}(s)$ satisfying (30).

Modify the matrix M_a in (28), which is composed of the data $\{h_0,\ldots,h_{n-1};\mu_0,\ldots,\mu_{n-1};\lambda_0,\ldots,\lambda_{n-1}\}$, into $\bar{M}_a(\bar{\lambda}_{n-1})$ by replacing λ_{n-1} with a free matrix parameter $\bar{\lambda}_{n-1}$. Then it can be proved that the equation (30) is equivalent to the existence of a matrix $\bar{\lambda}_{n-1}$ such that

$$\begin{cases} \bar{\lambda}_{n-1}' = -\bar{\lambda}_{n-1} & (31) \\ [\hat{C}_0,\ldots,\hat{C}_{n-1},I]\ \bar{M}_a(\bar{\lambda}_{n-1}) = [0,\ldots,0]. & (32) \end{cases}$$

Eq.(32) can be partitioned into

$$\begin{cases} [\hat{C}_0,\ldots,\hat{C}_{n-1},I]\ M_c = [0,\ldots,0] & (33) \\ [\hat{C}_0,\ldots,\hat{C}_{n-1},I]\ \bar{M}_d(\bar{\lambda}_{n-1}) = [0,\ldots,0], & (34) \end{cases}$$

where

$$M_c \triangleq \begin{bmatrix} \mu_{0,0} & \cdots & \mu_{0,n-2} \\ \vdots & & \vdots \\ \mu_{n,0} & \cdots & \mu_{n,n-2} \end{bmatrix} \quad : \quad (n+1)p \times (n-1)p$$

$$\bar{M}_d(\bar{\lambda}_{n-1}) \triangleq \begin{bmatrix} \mu_{0,n-1} \\ \vdots \\ \mu_{n-1,n-1} \\ \bar{\lambda}_{n-1}-\frac{1}{2}h_{n-1}h'_{n-1} \end{bmatrix} \quad : \quad (n+1)p \times p.$$

$$(\bar{M}_a(\bar{\lambda}_{n-1}) = [M_c, \bar{M}_d(\bar{\lambda}_{n-1})])$$

Note that M_c is determined by the given data and is independent of the parameter $\bar{\lambda}_{n-1}$. Now, we define the inner product $\langle P(s),Q(s)\rangle$ for arbitrary $p \times p$ PM's $P(s)$ and $Q(s)$ of the form

$$\begin{cases} P(s) = s^n P_n + \dots + s P_1 + P_0 \\ Q(s) = s^n Q_n + \dots + s Q_1 + Q_0 \end{cases}$$

by

$$\langle P(s),Q(s)\rangle \triangleq [P_0,\dots,P_n]M_{n+1}[Q_0,\dots,Q_n]' \qquad (35)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_P(i\omega)H_Q'(-i\omega)d\omega,$$

where

$$H_P(s) \triangleq [P(s)H(s)]_{sp} \qquad H_Q(s) \triangleq [Q(s)H(s)]_{sp} .$$

We note that the inner product satisfies the following special property (cf.(22)):

$$\langle sP(s),Q(s)\rangle + \langle P(s),sQ(s)\rangle + P\{h\}Q'\{h\} = 0, \qquad (36)$$

where

$$\begin{cases} P\{h\} \triangleq P_n h_n + \dots + P_0 h_0 \\ Q\{h\} \triangleq Q_n h_n + \dots + Q_0 h_0. \end{cases}$$

(See Remark 3.2.1.) Using the inner product, eq.(33) is represented as

$$\langle \hat{C}(s), s^j I \rangle = 0 \quad \text{for } \forall j=0,1,\dots,n-2. \tag{37}$$

The solutions of (37) are parametrized as follows:

$$\hat{C}(s) = s^n I + K(s) + \hat{C}_{n-1} L(s), \tag{38}$$

where $\hat{C}_{n-1}$ is a free matrix parameter, and K(s) and L(s) are PM's such that

$$\begin{cases} K(s) = s^{n-2} K_{n-2} + \dots + K_0 \\ \langle s^n I + K(s), s^j I \rangle = 0 \quad \text{for } \forall j=0,1,\dots,n-2 \end{cases} \tag{39}$$

$$\begin{cases} L(s) = s^{n-1} I + s^{n-2} L_{n-2} + \dots + L_0 \\ \langle L(s), s^j I \rangle = 0 \quad \text{for } \forall j=0,1,\dots,n-2. \end{cases} \tag{40}$$

We can see that $\hat{C}(s)$ in (38) satisfies (34) if and only if

$$\xi + \hat{C}_{n-1}\eta + \bar{\lambda}_{n-1} - \frac{1}{2} h_{n-1} h'_{n-1} = 0, \tag{41}$$

where

$$\begin{cases} \xi \triangleq \langle K(s), s^{n-1} I \rangle \quad : p \times p & (42) \\ \eta \triangleq \langle L(s), L(s) \rangle \quad : p \times p. & (43) \end{cases}$$

Obviously, the existence of a skew-symmetric matrix $\bar{\lambda}_{n-1}$ satisfying (41) is equivalent to the equation

$$\hat{C}_{n-1}\eta + \eta \hat{C}'_{n-1} + \xi + \xi' - h_{n-1} h'_{n-1} = 0. \tag{44}$$

Therefore it is concluded that $\hat{C}(s)$ satisfies (30) if and only if

$\hat{C}(s)$ is written as (38) with $\hat{C}_{n-1}$ obeying (44).

The equation (44) is easily solved as

$$- \hat{C}_{n-1} = R + W, \tag{45}$$

where R is a matrix defined as

$$R = \frac{1}{2}(\xi + \xi' - h_{n-1}h'_{n-1})\eta^{-1}, \tag{46}$$

and W is a matrix parameter which is free within the η-skew-symmetry

$$W\eta + \eta W' = 0. \tag{47}$$

Note that R is η-symmetric in the sense that

$$R\eta = \eta R'. \tag{48}$$

The above results are summarized in the following theorem.

Theorem 7.1.7 A PM $\hat{C}(s)$ is a solution of (30), if and only if it is written as (38) and there exists a η-skew-symmetric matrix W satisfying (45).

We denote by $\hat{C}(s;W)$ the solution of (30) satisfying (45), and by $\hat{H}(s;W) = \hat{C}^{-1}(s;W)\hat{D}(s;W)$ the corresponding solution of (10) and (30). Obviously, the mapping : $W \longmapsto \hat{H}(s;W)$ is one-to-one.

Remark 7.1.8 Consider the case where p = 1, which includes the SISO case. Then, the solutions of (10) and (30) are unique. Note that the unique solution coincides with the solution of the MLSA problem.

7.2. The recursive structure of MLSA

In the previous section, the properties of approximants have been investigated for a fixed order n. We will study in the present section the mutual relation among approximants of several orders. We restrict ourselves to the SISO case where p=q=1 for simplicity.

Assume that the original system H(s) is of order N, and for n=1,2,...,N, denote by $H_n(s) = C_n^{-1}(s)D_n(s)$ the solution of the n-th order MLSA (= IA) problem for H(s). Note that $H_N(s) = H(s)$ in particular. Using the inner product defined by (7.1.35), the normal equation (7.1.17) is represented as

$$\langle C_n(s), s^j \rangle = \begin{cases} 0 & \text{if } j=0,1,\dots,n-1 \\ \varepsilon_n & \text{if } j=n. \end{cases} \quad (1)$$

Owing to the special property (7.1.36) of the inner product, the normal equation (1) can be solved recursively as follows:

$$\begin{cases} C_0(s) = 1, \qquad C_1(s) = s + \nu_0^2/2\varepsilon_0 & (2) \\ V_0(s) = \nu_0/\varepsilon_0 & (3) \end{cases}$$

$$\begin{cases} C_{k+1}(s) = (s+\nu_k^2/2\varepsilon_k)C_k(s) \\ \qquad\qquad + (\varepsilon_k/\varepsilon_{k-1})C_{k-1}(s) + \nu_k V_{k-1}(s) & (4) \\ V_k(s) = V_{k-1}(s) + (\nu_k/\varepsilon_k)C_k(s) & (5) \end{cases}$$

where

$$\nu_k \triangleq C_k\{h\}. \quad (6)$$

The polynomial $V_k(s)$ is written as

$$V_k(s) = \sum_{j=0}^{k} (\nu_j/\varepsilon_j)C_j(s)$$

and is characterized by the property

$$\langle V_k(s), P(s) \rangle = P\{h\} \quad \text{if} \quad \deg P(s) \leqq k.$$

The proof of (2)-(5) is similar to that of (4.1.21), and is omitted.

The above recursion yields a state space realization of $H_n(s)$ as follows. Fix an integer n ($1 \leqq n \leqq N$), and let for $k=0,1,\ldots,n$

$$x_k(s) \triangleq [C_k(s)H_n(s)]_{sp} \ .$$

Then the following is obvious.

$$\begin{cases} x_0(s) = H_n(s) & (7) \\ x_n(s) = 0. & (8) \end{cases}$$

Noting that (6) leads to

$$sx_k(s) = [sC_k(s)H_n(s)]_{sp} + \nu_k \ ,$$

we obtain from (2)-(5)

$$sx_0(s) = x_1(s) - (\nu_0^2/2\varepsilon_0)x_0(s) + \nu_0 \qquad (9)$$

$$sx_k(s) = x_{k+1}(s) - (\nu_k^2/2\varepsilon_k)x_k(s) - (\varepsilon_k/\varepsilon_{k-1})x_{k-1}(s)$$

$$- \nu_k \sum_{j=0}^{k-1} (\nu_j/\varepsilon_j)x_j(s) + \nu_k \; . \qquad (10)$$

$(1 \leqq \forall k \leqq n-1)$

Now, define

$$\begin{cases} a_{j,k} \triangleq - \nu_j \nu_k/\varepsilon_k \\ b_k \triangleq - \varepsilon_k/\varepsilon_{k-1} \end{cases}$$

$$\underset{\sim}{\Theta}_n \triangleq \begin{bmatrix} a_{00}/2 & 1 & & & \bigcirc \\ a_{10}+b_1 & a_{11}/2 & 1 & \ddots & \\ \vdots & \vdots & & & 1 \\ a_{n-1\ 0} & a_{n-1\ 1} & \cdots & & a_{n-1\ n-1}/2 \end{bmatrix} \quad : n \times n$$

$$\underset{\sim}{\nu}_n \triangleq [\nu_0, \ldots, \nu_{n-1}]' \quad : n \times 1,$$

Then, from (8)-(10) we have

$$(sI - \underset{\sim}{\Theta}_n)[x_0(s), \ldots, x_{n-1}(s)]' = \underset{\sim}{\nu}_n \; .$$

Hence it follows from (7) that

$$H_n(s) = \underset{\sim}{e}_n'(sI - \underset{\sim}{\Theta}_n)^{-1}\underset{\sim}{\nu}_n$$

where

$$\underset{\sim}{e}_n' = [1, 0, \ldots, 0].$$

That is, $(\underset{\sim}{\Theta}_n, \underset{\sim}{\nu}_n, \underset{\sim}{e}_n')$ is a realization of $H_n(s)$. This realization is similar to the observability realization (see (6.2.6))

$$H_n(s) = \underset{\sim}{e}_n'(sI - \underset{\sim}{C}_n)^{-1}\underset{\sim}{h}_n$$

where

$$\underset{\sim}{C}_n \triangleq \text{comp}\ \{C_n(s)\} \ : \ n \times n$$

$$\underset{\sim}{h}_n \triangleq [h_0, \ldots, h_{n-1}]' \quad : \ n \times 1.$$

Indeed, defining a $n \times n$ matrix T_n by

$$[C_0(s), \ldots, C_{n-1}(s)]' = T_n[1, s, \ldots, s^{n-1}]',$$

we can show that

$$\begin{cases} \underset{\sim}{\Theta}_n = T_n \ \underset{\sim}{C}_n \ T_n^{-1} \\ \underset{\sim}{\nu}_n = T_n \ \underset{\sim}{h}_n \\ \underset{\sim}{e}_n = \underset{\sim}{e}_n \ T_n^{-1}. \end{cases} \tag{11}$$

Furthermore, eq.(1) yields

$$T_n M_n T_n' = (E_n \triangleq) \ \text{diag} \ \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}\},$$

and therefore it follows from (11) that eq.(7.1.30) is transformed into

$$\underset{\sim}{\Theta}_n \ E_n + E_n \ \underset{\sim}{\Theta}_n' + \underset{\sim}{\nu}_n \ \underset{\sim}{\nu}_n' = 0. \tag{12}$$

This means that the similarity transformation (11) diagonalizes the solution of the Lyapunov equation (7.1.30).

Remark 7.2.1 Consider the special case where $h_0 = \ldots = h_{N-2} = 0$ ($\Longleftrightarrow \nu_0 = \ldots = \nu_{N-2} = 0$) in the above arguments. It then turns out that the recursion (2)-(5) coincides with (4.1.24-27), which is nothing but the reversed procedure of the Routh-Hurwitz test, and that $\{C_0(s),\ldots,C_{N-1}(s)\}$ become the orthogonal polynomials defined from $C_N(s)$. In addition, $\underset{\sim}{\Theta}_N$ becomes the Schwarz matrix defined from $C_N(s)$. In this respect, the results in the present section can be regarded as a generalization of some results in Chap.4.

We can see that $\underset{\sim}{\Theta}_{n-1}$, $\underset{\sim}{\nu}_{n-1}$ and $\underset{\sim}{e}'_{n-1}$ are (left-upper) sub-matrices of $\underset{\sim}{\Theta}_n$, $\underset{\sim}{\nu}_n$ and $\underset{\sim}{e}'_n$, respectively. This means that $H_{n-1}(s)$ is obtained from $H_n(s)$ by cutting off the effect of x_{n-1} in the n state variables $\{x_0,x_1,\ldots,x_{n-1}\}$ of the realization $(\underset{\sim}{\Theta}_n,\underset{\sim}{\nu}_n,\underset{\sim}{e}'_n)$. Indeed, the equations (8)-(10), which represent $H_n(s)$, turn out to represent $H_{n-1}(s)$ by putting 0 in the place of $x_{n-1}(s)$. Let us illustrate the situation using a block diagram. For k= 1,2,...,n-1, let

$$w_k(s) \triangleq 1 - \sum_{j=0}^{k-1}(\nu_j/\varepsilon_j)x_j(s).$$

Then the equations (9) and (10) are written as

$$\begin{cases} x_1(s) = -\nu_0 + (s + \nu_0^2/2\varepsilon_0)x_0(s) \\ w_1(s) = 1 - (\nu_0/\varepsilon_0)x_0(s) \end{cases}$$

$$\begin{cases} x_{k+1}(s) - (\varepsilon_k/\varepsilon_{k-1})x_{k-1}(s) \\ \qquad = -\nu_k w_k(s) + (s + \nu_k^2/2\varepsilon_k)x_k(s) \\ w_{k+1}(s) = w_k(s) - (\nu_k/\varepsilon_k)x_k(s). \\ \qquad (1 \leqq {}^{\forall}k \leqq n-1) \end{cases}$$

It can be seen from the above equations that $H_n(s)$ can be realized by connecting some block elements of the form in Fig.1. Consider the case where N = 4 for instance. Then the original system $H(s) = H_4(s)$ is realized as in Fig.2, and the n-th order approximant $H_n(s)$ (n=1,2,3) is obtained by cutting the connections on the corresponding dotted line.

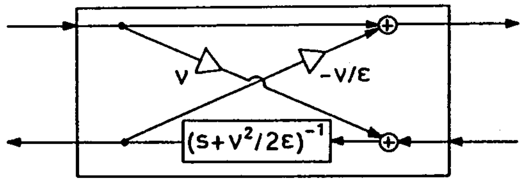

Fig.1 The block element specified by (ν,ε)

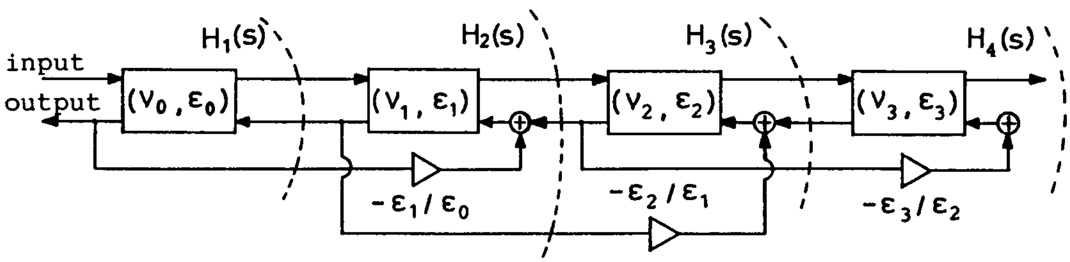

Fig.2 A block diagram representation of the c.t. MLSA

7.3. The weighted MLSA problem

In this section, we will study the following problem: Given a $p \times q$ strictly proper and c.t. stable rational matrix $H(s)$ satisfying (7.1.4) as in Sec.7.1, and given a nonsingular $q \times q$ PM $W(s)$ such that

$$\text{(i)} \quad W_*(s) \ (\triangleq W'(-s)) = W(s) \tag{1}$$

$$\text{(ii)} \quad W(i\omega) \geqq 0 \ \text{ for } \forall\omega : \text{real} \qquad (i^2 = -1) \tag{2}$$

$$\text{(iii)} \quad \deg W(s) \leqq 2n \tag{3}$$

$$(\text{i.e., } \deg [W(s)]_{ij} \leqq 2n \ \text{ for } \forall i,\ \forall j),$$

find a rational matrix $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the form (7.1.1-3) minimizing the criterion

$$E_W \triangleq \|\hat{C}(s)H(s) - \hat{D}(s)\|_W^2 , \tag{4}$$

where the norm is defined by

$$\|R(s)\|_W^2 \triangleq \text{trace}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} R(i\omega)W^{-1}(i\omega)R'(-i\omega)d\omega. \tag{5}$$

More precisely, denoting by $\mathcal{C}$ the totality of $p \times p$ monic PM's of degree n (i.e. of the form (7.1.2)), and letting for each $\hat{C}(s) \in \mathcal{C}$

$$\mathcal{D}_{\hat{C}} \triangleq \{\hat{D}(s) : \text{a } p \times q \text{ PM} \mid \deg \hat{D}(s) \leqq n-1$$

$$\text{and } \|\hat{C}(s)H(s) - \hat{D}(s)\|_W^2 < \infty \}, \tag{6}$$

the problem is formulated as

$$\min_{\hat{C}\in\mathcal{C}} \ \min_{\hat{D}\in\mathcal{D}_{\hat{C}}} \|\hat{C}(s)H(s) - \hat{D}(s)\|_W^2 . \tag{7}$$

This is called <u>the n-th order weighted MLSA problem with the weighting PM W(s)</u>. We will present below two apparently different results providing the solution of the problem, both of which reduce the problem to a weightless MLSA problem studied in Sec.7.1.

In order to describe the results, we need to make some definitions. First, owing to the assumptions (1) and (2), the weight W(s) can be factorized as

$$W(s) = X_*(s)\ X(s) \tag{8}$$

where X(s) is a $q \times q$ nonsingular PM. We can assume without loss of generality that X(s) is semi-stable in the sense that

$$\det X(s) \neq 0 \quad \text{in} \quad \text{Re } s > 0. \tag{9}$$

Such a PM X(s) is said to be a <u>semi-stable factor</u> of W(s). Similarly, there always exists a PM Y(s) such that

$$\begin{cases} W(s) = Y_*(s)\ Y(s) & (10) \\ \det Y(s) \neq 0 \quad \text{in} \quad \text{Re } s < 0. & (11) \end{cases}$$

Such a PM Y(s) is said to be an <u>anti-stable factor</u> of W(s). It is noted that X(s) and Y(s) are mutually dual w.r.t. I in the sense of Sec.4.3. We also note that the assumption (3) yields

$$\deg X(s) \leqq n \tag{12}$$

$$\deg Y(s) \leqq n. \tag{13}$$

Next, we define the 'stable parts' of rational matrices. An arbitrary rational matrix R(s) is represented uniquely as

$$R(s) = P(s) + R_+(s) + R_-(s)$$

where P(s) is a PM, $R_+(s)$ is a strictly proper and c.t. stable rational matrix, and $R_-(s)$ is a strictly proper rational matrix having no poles in the open left half-plane. In this representation, we call $R_+(s)$ the <u>stable part</u> of R(s) and write it as

$$R_+(s) = [R(s)]_+ .$$

The following lemma will play a fundamental role in later arguments.

<u>Lemma 7.3.1</u> Let H(s) be a $p \times q$ strictly proper and c.t. stable rational matrix, let P(s) be a $p \times p$ PM, and let Q(s) be a $q \times q$ PM. Then there exists uniquely a $p \times q$ PM R(s) such that

$$\{P(s)H(s) - R(s)\}Q^{-1}(s) = [P(s)H(s)Q^{-1}(s)]_+ .$$

(Proof) If such an R(s) exists then it is written as

$$R(s) = P(s)H(s) - [P(s)H(s)Q^{-1}(s)]_+Q(s),$$

and therefore the uniqueness of R(s) is obvious. Conversely, suppose that a rational matrix R(s) is written as above. Then it is clear that R(s) has no poles in Re $s \geqq 0$. On the other hand, noting that $[R(s)Q^{-1}(s)]_+ = 0$ by definition, we can see that R(s) has no poles in Re $s < 0$, also. This means that R(s) is a PM, which completes the proof. (QED)

Now we present the main theorems, the proofs of which will be shown at the end of this section. The first theorem gives the solution of the weighted MLSA problem with the weighting PM W(s) by the use of an anti-stable factor Y(s) of W(s).

Theorem 7.3.2 Let $\hat{\tilde{H}}(s)$ be the solution of the n-th order weightless MLSA problem for

$$\tilde{H}(s) \triangleq [H(s)Y^{-1}(s)]_{+}. \tag{14}$$

Then the solution $\hat{H}(s)$ of the n-th order weighted MLSA problem for H(s) is given by

$$\hat{H}(s) = [\hat{\tilde{H}}(s)Y(s)]_{+}. \tag{15}$$

The above result is illustrated as follows. Let us denote by $\mathcal{H}$ the totality of $p \times q$ strictly proper stable rational matrix, by $\mathcal{H}_{\geq n}$ the totality of elements of $\mathcal{H}$ satisfying (7.1.4), and by $\mathcal{H}_{\leq n}$ the totality of elements of $\mathcal{H}$ of the form (7.1.1-3). Then the n-th order weighted MLSA problem with the weighting PM W(s) defines the mapping

$$\Phi_W^{(n)} \ : \ \begin{array}{ccc} \mathcal{H}_{\geq n} & \longrightarrow & \mathcal{H}_{\leq n} \\ \Psi & & \Psi \\ H(s) & \longmapsto & \hat{H}(s). \end{array}$$

We also define the mappings

$$\Pi_Y : \mathcal{H} \longrightarrow \mathcal{H}$$
$$\begin{array}{ccc} \cup & & \cup \\ H(s) & \longmapsto & [H(s)Y(s)]_+ \end{array}$$

$$\Pi_{Y^{-1}} : \mathcal{H} \longrightarrow \mathcal{H}$$
$$\begin{array}{ccc} \cup & & \cup \\ H(s) & \longmapsto & [H(s)Y^{-1}(s)]_+ \end{array} .$$

Using these mappings, the result in Th.7.3.1 is represented as

$$\Phi_W^{(n)} = \Pi_Y \circ \Phi_I^{(n)} \circ \Pi_{Y^{-1}} .$$

Since Π_Y and $\Pi_{Y^{-1}}$ turn out to be each other's inverse mapping by the following lemma, we have the commutative diagram in Fig.1

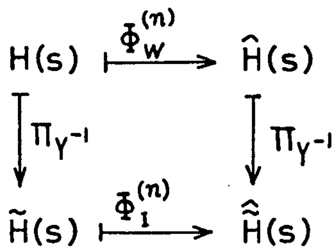

Fig.1 The reduction of the weighted MLSA to the weightless MLSA

<u>Lemma 7.3.3</u> If $H(s) \in \mathcal{H}$ and if $Y(s)$ is an anti-stable PM, then

$$[[H(s)Y(s)]_+ Y^{-1}(s)]_+ = [[H(s)Y^{-1}(s)]_+ Y(s)]_+ = H(s).$$

(Proof) Putting $(H'(s), Y'(s), I)$ in the place of

$(H(s),P(s),Q(s))$ in Lemma 7.3.1, it is shown that there exists a PM $R(s)$ such that

$$[H(s)Y(s)]_+Y^{-1}(s) = H(s) - R(s)Y^{-1}(s).$$

Now, operate with $[\ ]_+$ on the both sides of the above. Then, owing to the anti-stability of $Y(s)$, the second term of the right-hand side vanishes, and we obtain

$$[[H(s)Y(s)]_+Y^{-1}(s)]_+ = H(s).$$

Similarly, putting $(H(s),I,Y(s))$ in the place of $(H(s),P(s),Q(s))$ in Lemma 7.3.1, it is shown that there exists a PM $\bar{R}(s)$ such that

$$[H(s)Y^{-1}(s)]_+Y(s) = H(s) - \bar{R}(s),$$

which yields

$$[[H(s)Y^{-1}(s)]_+Y(s)]_+ = H(s).$$

(QED)

The second theorem described below uses a semi-stable factor $X(s)$ of $W(s)$ for the same purpose as of the first theorem. Factorize $X(s)$ as

$$X(s) = X_+(s)\ Z(s) \tag{16}$$

where $X_+(s)$ and $Z(s)$ are PM's such that

$$\begin{cases} \det X_+(s) \neq 0 \quad \text{in} \quad \operatorname{Re} s \geqq 0 & (17) \\ \det Z(s) \neq 0 \quad \text{in} \quad \operatorname{Re} s \neq 0, & (18) \end{cases}$$

and consider a triplet (S(s),F,G), where S(s) is a PM and F and G are constant matrices, such that

$$\begin{cases} S(s)X_+^{-1}(s) = (sI-F)^{-1}G, & (19) \\ (F,G) \text{ is controllable,} & (20) \\ X_+(s) \text{ and } S(s) \text{ are right coprime.} & (21) \end{cases}$$

In this situation, the second theorem gives the solution of the weighted MLSA problem as follows.

Theorem 7.3.4 Let

$$\Xi(s) \triangleq \begin{bmatrix} S(s)X_+^{-1}(s) \\ [T(s)H(s)X^{-1}(s)]_+ \end{bmatrix} \tag{22}$$

where

$$T(s) \triangleq [I, sI, \ldots, s^{n-1}I]' \quad : np \times p, \tag{23}$$

and let $\hat{\Xi}(s)$ be the solution of the 1st order weightless MLSA problem for $\Xi(s)$. Then $\hat{\Xi}(s)$ is written as

$$\hat{\Xi}(s) = \begin{bmatrix} S(s)X_+^{-1}(s) \\ [T(s)\hat{H}(s)X^{-1}(s)]_+ \end{bmatrix} \tag{24}$$

where $\hat{H}(s)$ is the solution of the n-th order weighted MLSA problem for H(s).

The properties of the weighted MLSA are drived by the above theorems from those of the weightless MLSA. In the situations of the theorems, let

$\{\tilde{h}_j\} \triangleq$ the Markov parameters of $\tilde{H}(s)$

$$\text{i.e.,} \quad \tilde{H}(s) = \tilde{h}_0 s^{-1} + \tilde{h}_1 s^{-2} + \ldots. \tag{25}$$

$$\tilde{H}^{(j)}(s) \triangleq [s^j \tilde{H}(s)]_{sp} \tag{26}$$

$$\tilde{\mu}_{jk} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{H}^{(j)}(i\omega)\, \tilde{H}^{(k)\prime}(-i\omega)\, d\omega \tag{27}$$

$$\tilde{\mu}_j \triangleq \tilde{\mu}_{jj} \tag{28}$$

$$\begin{aligned} \tilde{\lambda}_j &\triangleq \tilde{\mu}_{j+1,j} + \frac{1}{2} \tilde{h}_j \tilde{h}_j' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)\, \tilde{H}^{(j)}(i\omega)\, \tilde{H}^{(j)\prime}(-i\omega)\, d\omega. \end{aligned} \tag{29}$$

$$\xi \triangleq \lim_{s\to\infty} s\, \Xi(s) \tag{30}$$

$$\Delta \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \Xi(i\omega)\, \Xi'(-i\omega)\, d\omega \tag{31}$$

$$\begin{aligned} \Gamma &\triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} [(i\omega)\Xi(i\omega)]_{sp}\, \Xi'(-i\omega)\, d\omega + \frac{1}{2} \xi \xi' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)\, \Xi(i\omega)\, \Xi'(-i\omega)\, d\omega. \end{aligned} \tag{32}$$

We regard these quantities as 'data' about $H(s)$. Then the following is immediate from the above theorems and Th.7.1.4.

Corollary 7.3.5 For a given $H(s) \in \mathcal{H}$, the solution $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the n-th order weighted MLSA problem satisfies the following.

(i) $(\hat{C}(s),\hat{D}(s))$ is uniquely determined by the data $\{\tilde{h}_0,\ldots \tilde{h}_{n-1};\tilde{\mu}_0,\ldots,\tilde{\mu}_{n-1};\tilde{\lambda}_0,\ldots,\tilde{\lambda}_{n-1}\}$ taken from H(s), as well as by the data $\{\xi;\Delta,\Gamma\}$ taken from H(s).

(ii) $\hat{H}(s)$ preserves $\{\tilde{h}_0,\ldots,\tilde{h}_{n-1}\}$ and ξ.

(iii) $\hat{H}(s)$ is stable.

(iv) If $\hat{C}(s)$ and $\hat{D}(s)$ are left coprime, then $\hat{C}(s)$ is a stable PM, and $\hat{H}(s)$ preserves $\{\tilde{\mu}_0,\ldots,\tilde{\mu}_{n-1};\tilde{\lambda}_0,\ldots,\tilde{\lambda}_{n-1}\}$ and $\{\Delta,\Gamma\}$.

Let us investigate the meaning of the above quantities for some special cases. First, we consider the case where W(s) satisfies the following conditions.

(a) $\det W(i\omega) \neq 0$ for $\forall\omega$: real. (33)

(b) $\lim_{s\to\infty} s^{-2n} W(s)$ is a nonsingular matrix. (34)

These are equivalent to the condition that a semi-stable factor X(s) of W(s) is strictly stable and written as

$$X(s) = s^n X_n + s^{n-1} X_{n-1} + \ldots + X_0$$

with X_n nonsingular. In this case, it turns out that $X_+(s) = X(s)$ in (16)-(18), and we can choose T(s) as S(s) in (19)-(21). Then $\Xi(s)$ in (22) is written as

$$\Xi(s) = \begin{bmatrix} T(s) \\ T(s)H(s) \end{bmatrix} X^{-1}(s).$$

Now ξ in (30) is reduced to

$$\xi = [0,\ldots,X_n'^{-1},0,\ldots,0]'$$

and has no information about H(s). Thus the solution $\hat{H}(s) = \Phi_W^{(n)}(H(s))$ is determined by Δ and Γ in (31)-(32), and preserves them generically. We can see that Δ and Γ are composed of the following submatrices:

$$P_{jk} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^j\ W^{-1}(i\omega)\ (-i\omega)^k\ d\omega$$

$$Q_{jk} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^j\ H(i\omega)\ W^{-1}(i\omega)\ (-i\omega)^k\ d\omega$$

$$R_{jk} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^j\ H(i\omega)\ W^{-1}(i\omega)\ H'(-i\omega)\ (-i\omega)^k\ d\omega$$

$(j,k = 0,1,\ldots,n,$ except for $(j,k) = (n,n))$.

It is noted that $\{P_{jk}\}$ has no information about H(s). Hence, the solution $\hat{H}(s)$ is determined by $\{Q_{jk};R_{jk}\}$. The meaning of these matrices will be clarified by considering the following stochastic situation. Let u(·) be a stationary process with the power spectrum $W^{-1}(s)$, which is obtained as the output of the system $X^{-1}(s)$ to a normalized white noise input, and let y(·) be the

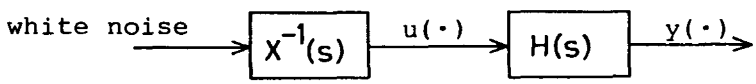

Fig.2 The definition of u and y

output of the system H(s) to the input $u(\cdot)$ (Fig.2). Then we have

$$\begin{cases} P_{jk} = E[u^{(j)}(t)\cdot u^{(k)'}(t)] \\ Q_{ij} = E[y^{(j)}(t)\cdot u^{(k)'}(t)] \\ R_{jk} = E[y^{(j)}(t)\cdot y^{(k)'}(t)] \end{cases}$$

where

$$\begin{cases} u^{(j)}(t) \triangleq d^j u(t)/dt^j \\ y^{(j)}(t) \triangleq d^j y(t)/dt^j . \end{cases}$$

We summarize the above results as follows.

Corollary 7.3.6 If W(s) satisfies (33) and (34), then $\hat{H}(s) \triangleq \Phi_W^{(n)}(H(s))$ is determined by the data $\{Q_{jk};R_{jk}\}$ taken from H(s), and preserves the data generically.

Next, we consider the case where

$$W(s) = s^r(-s)^r I \qquad (r=0,1,\ldots,n).$$

In this case, a semi-stable factor and an anti-stable factor of W(s) can be chosen as

$$X(s) = Y(s) = s^r I.$$

We will investigate the structure of the quantities $\{\tilde{h}_j;\tilde{\mu}_j;\tilde{\lambda}_j\}$ defined in (25)-(29). For an arbitrary integer k, let

$$H^{(k)}(s) \triangleq [s^k H(s)]_+ \,. \tag{35}$$

Then $\tilde{H}(s)$ in (14) is written as

$$\tilde{H}(s) = H^{(-r)}(s). \tag{36}$$

We see that (35) is a generalization of the previous definition of $H^{(k)}(s)$ (7.1.13) where k was restricted to nonnegative integers. It should be noted that the following is valid for arbitrary integers j and k:

$$[s^j H^{(k)}(s)]_+ = H^{(j+k)}(s). \tag{37}$$

Denoting the impulse response of $H^{(k)}(s)$ by $h^{(k)}(t)$, we have

$$h^{(k-1)}(t) = -\int_t^\infty h^{(k)}(t_1)\, dt_1 \qquad (t>0).$$

We define further for arbitrary integers j and k

$$h_k \triangleq \lim_{s\to\infty} sH^{(k)}(s) = h^{(k)}(0+)$$

$$\mu_{j,k} \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} H^{(j)}(i\omega)\, H^{(k)\prime}(-i\omega)\, d\omega$$

$$= \int_{0+}^{\infty} h^{(j)}(t)\, h^{(k)\prime}(t)\, dt$$

$$\mu_k \triangleq \mu_{k,k}$$

$$\lambda_k \triangleq \mu_{k+1,k} + \frac{1}{2} h_k h_k' \,.$$

It is clear that these definitions include the previous definitions of the quantities in Sec 7.1 as a special case. Noting that

$$s\, H^{(k)}(s) = h_k + H^{(k+1)}(s),$$

we can see the interesting fact that $H^{(k)}(s)$ is expanded in two ways as

$$H^{(k)}(s) = h_k s^{-1} + h_{k+1} s^{-2} + \dots$$
$$= - h_{k-1} - h_{k-2} s - h_{k-3} s^2 - \dots .$$

In particular, we have

$$H(s) = - h_{-1} - h_{-2} s - h_{-3} s^2 - \dots ,$$

which means that the quantities $\{h_{-1}, h_{-2}, \dots\}$ are equivalent to the so-called 'time moments' of $H(s)$.

It is seen from (36) and (37) that

$$\begin{cases} \tilde{h}_k = h_{k-r} \\ \tilde{\mu}_{j,k} = \mu_{j-r,k-r} \\ \tilde{\mu}_k = \mu_{k-r} \\ \tilde{\lambda}_k = \lambda_{k-r} . \end{cases}$$

Thus we have the following.

<u>Corollary 7.3.7</u> If $W(s) = s^r(-s)^r I$ $(0 \leqq r \leqq n)$, then $\hat{H}(s) \triangleq \Phi_W^{(n)}(H(s))$ is determined by the data $\{h_{-r}, \dots, h_{n-r-1}; \mu_{-r}, \dots, \mu_{n-r-1}; \lambda_{-r}, \dots, \lambda_{n-r-1}\}$ taken from $H(s)$, and preserves the data generically.

Let us proceed to prove the theorems. We begin with investigation of the structure of the set $\mathcal{D}_{\hat{C}}$ in (6) for an arbitrarily fixed $\hat{C}(s) \in \mathcal{C}$. Let X(s) and Y(s) be a semi-stable factor and an anti-stable factor of W(s), respectively. Then we have

$$\begin{aligned} E_W &= \| \hat{C}(s)H(s) - \hat{D}(s) \|_W^2 \\ &= \| \{ \hat{C}(s)H(s) - \hat{D}(s) \} X^{-1}(s) \|^2 \\ &= \| \{ \hat{C}(s)H(s) - \hat{D}(s) \} Y^{-1}(s) \|^2. \end{aligned} \tag{38}$$

Hence the finiteness of E_W is equivalent to the condition that $\{\hat{C}(s)H(s)-\hat{D}(s)\}X^{-1}(s)$ (or $\{\hat{C}(s)H(s) - \hat{D}(s)\}Y^{-1}(s)$) is strictly proper and has no poles on the imaginary axis. We can write the condition as

$$\{\hat{C}(s)H(s) - \hat{D}(s)\} X^{-1}(s) \in \mathcal{H}. \tag{39}$$

Owing to (12), the above condition implies that deg D(s) $\leqq$ n-1. Therefore we obtain

$$\mathcal{D}_{\hat{C}} = \{ \hat{D}(s) : p \times q \text{ PM} \mid (39) \} . \tag{40}$$

Now we define a PM $\hat{D}_1(s)$ by

$$\{ \hat{C}(s)H(s) - \hat{D}_1(s) \} X^{-1}(s) = [\hat{C}(s)H(s)X^{-1}(s)]_+, \tag{41}$$

whose existence and uniqueness are guaranteed by Lemma 7.3.1. Then it is clear that

$$\hat{D}_1(s) \in \mathcal{D}_{\hat{C}} . \tag{42}$$

Noting that $\mathcal{H}$ is a linear space, we see from (40) and (42) that a PM $\hat{D}(s)$ belongs to $\mathcal{D}_{\hat{C}}$ if and only if

$$\{\hat{D}(s) - \hat{D}_1(s)\}\ X^{-1}(s) \in \mathcal{H}. \tag{43}$$

Using the factorization in (16)-(18), the above condition is represented as

$$\exists P(s) \in \mathcal{P} \quad \text{s.t.} \quad \hat{D}(s) - \hat{D}_1(s) = P(s)Z(s) \tag{44}$$

where

$$\mathcal{P} \triangleq \{P(s) : p \times q \text{ PM} \mid P(s)X_+^{-1}(s) \text{ is strictly proper}\}. \tag{45}$$

Thus we have

$$\mathcal{D}_{\hat{C}} = \hat{D}_1(s) + \mathcal{P} \cdot Z(s). \tag{46}$$

Similarly, defining a PM $\hat{D}_2(s)$ by

$$\{\hat{C}(s)H(s) - \hat{D}_2(s)\}\ Y^{-1}(s) = [\hat{C}(s)H(s)Y^{-1}(s)]_+ , \tag{47}$$

we have

$$\mathcal{D}_{\hat{C}} = \hat{D}_2(s) + \mathcal{P} \cdot Z(s). \tag{48}$$

These representations show that $\mathcal{D}_{\hat{C}}$ is an affine space such that

$$\dim \mathcal{D}_{\hat{C}} = \dim \mathcal{P} = p\ N_+$$

where $N_+ \triangleq \deg \det X_+(s)$, i.e., N_+ is the number of stable zeros of $X(s)$. We note that $Y(s)$ can also be factorized as

$$Y(s) = Y_-(s)\ Z(s) \tag{49}$$

where $Y_-(s)$ is a PM such that

$$\det Y_{-}(s) \neq 0 \quad \text{in} \quad \text{Re } s \leqq 0, \tag{50}$$

and we have

$$\mathcal{P} = \{ P(s) : p \times q \text{ PM} \mid P(s) Y_{-}^{-1}(s) \text{ is strictly proper} \}. \tag{51}$$

Now we present the proof of Th.7.3.2. We first claim that for $\forall \hat{D}(s) \in \mathcal{D}_{\hat{C}}$

$$\| \hat{C}(s)H(s) - \hat{D}(s) \|_W^2$$

$$= \| \hat{C}(s)H(s) - \hat{D}_2(s) \|_W^2 + \| \hat{D}(s) - \hat{D}_2(s) \|_W^2 . \tag{52}$$

(Proof) Define for arbitrary rational matrices F(s) and G(s) the inner products $\langle F(s),G(s) \rangle$ and $\langle F(s),G(s) \rangle_W$ by

$$\langle F(s),G(s) \rangle \triangleq \int_{-\infty}^{\infty} F(i\omega)\ G'(-i\omega)\ d\omega$$

$$\langle F(s),G(s) \rangle_W \triangleq \int_{-\infty}^{\infty} F(i\omega)\ W^{-1}(i\omega)\ G'(-i\omega)\ d\omega,$$

and let

$$J \triangleq \langle\ \hat{C}(s)H(s)-\hat{D}_2(s)\ ,\ \hat{D}(s)-\hat{D}_2(s)\ \rangle_W$$

$$= \langle\ \{\hat{C}(s)H(s)-\hat{D}_2(s)\}Y^{-1}(s)\ ,\ \{\hat{D}(s)-\hat{D}_2(s)\}Y^{-1}(s)\ \rangle.$$

Using (47)-(49), we have

$$J = \langle\ [\hat{C}(s)H(s)Y^{-1}(s)]_+\ ,\ P(s)Y_{-}^{-1}(s)\ \rangle,$$

where P(s) is an element of $\mathcal{P}$. This means that J is obtained by integrating $[\hat{C}(s)H(s)Y^{-1}(s)]_+ Y_{-*}^{-1}(s)P_*(s)$ on the imaginary axis. Therefore, noting that both $[\hat{C}(s)H(s)Y^{-1}(s)]_+$ and $Y_{-*}^{-1}(s)P_*(s)$ are strictly proper and stable, we obtain J=0 (see Lemma 3.1.1), which leads to (52). (QED)

It is immediate from (52) that

$$\|\hat{C}(s)H(s) - \hat{D}_2(s)\|_W^2 = \min_{\hat{D}(s)\in \mathcal{D}_{\hat{C}}} \|\hat{C}(s)H(s) - \hat{D}(s)\|_W^2 ;$$

i.e., given a denominator $\hat{C}(s)$, the numerator minimizing the criterion E_W is $\hat{D}_2(s)$ in (47), and the minimum of E_W is

$$E_W = \| [\hat{C}(s)H(s)Y^{-1}(s)]_+\|^2. \tag{53}$$

Therefore, determining first the denominator $\hat{C}(s)$ by minimization of E_W in (53) and next the numerator $\hat{D}_2(s)$ by (47), we obtain the solution of the weighted MLSA problem for H(s) as

$$\hat{H}(s) = \hat{C}^{-1}(s)\ \hat{D}_2(s). \tag{54}$$

Define $\tilde{H}(s)$ by (14). Then it can be shown that for $^{\forall}\hat{C}(s)\in \mathcal{C}$

$$[\hat{C}(s)H(s)Y^{-1}(s)]_+ = [\hat{C}(s)\tilde{H}(s)]_{sp} . \tag{55}$$

Hence, recalling the arguments in Sec.7.1, we see that the solution $\hat{\tilde{H}}(s)$ of the n-th order weightless MLSA problem for $\tilde{H}(s)$ has the same denominator as $\hat{H}(s)$ and is written as

$$\hat{\tilde{H}}(s) = \hat{C}^{-1}(s)\ \hat{\tilde{D}}(s) \tag{56}$$

where $\hat{\tilde{D}}(s)$ is defined by (see (7.1.12))

$$\hat{C}(s)\tilde{H}(s) - \hat{\tilde{D}}(s) = [\hat{C}(s)\tilde{H}(s)]_{sp} . \tag{57}$$

It follows from (47), (55) and (57) that

$$\hat{C}(s)\tilde{H}(s) - \hat{\tilde{D}}(s) = \{\hat{C}(s)H(s) - \hat{D}_2(s)\}\, Y^{-1}(s),$$

which leads to

$$H(s) - \hat{H}(s) = \{\tilde{H}(s) - \hat{\tilde{H}}(s)\}\, Y(s).$$

Operating with $[\ ]_+$ on the both sides of the above and apealing to Lemma 7.3.3, we obtain the desired equation (15). Thus Th.7.3.2 has been proved.

Next, let us prove Th.7.3.4. Suppose that $\hat{C}(s) \in \mathcal{C}$ and that $\hat{D}(s) \in \mathcal{D}_{\hat{C}}$. Then, according to (46), $\hat{D}(s)$ is uniquely represented as

$$\hat{D}(s) = \hat{D}_1(s) + P(s)Z(s), \tag{58}$$

where $\hat{D}_1(s)$ is a PM defined by (41), and $P(s)$ is a PM such that $P(s)X_+^{-1}(s)$ is strictly proper. Owing to (19)-(21), $P(s)$ is uniquely represented as

$$P(s) = K\, S(s) \tag{59}$$

where K is a $p \times N_+$ constant matrix ($N_+ = \deg \det X_+(s)$). Let

$$\Omega \triangleq \begin{bmatrix} F & O \\ BK & \Gamma \end{bmatrix} \ : (N_+ + np) \times (N_+ + np) \tag{60}$$

where (Γ, B) is the companion pair defined from $\hat{C}(s)$ (see (2.1.22-23)). Using (2.1.25), (16), (19), (41), (58) and (59), it can be shown that

$$(sI - \Omega)\, \Xi(s) - \xi = \begin{bmatrix} O \\ \{\hat{C}(s)H(s) - \hat{D}(s)\}X^{-1}(s) \end{bmatrix}$$

where ξ is a constant matrix defined by (30). From this result, we can see the following fact: defining K and Γ from the solution $\hat{H}(s) = \hat{C}^{-1}(s)\hat{D}(s)$ of the n-th order weighted MLSA problem for H(s), and constructing Ω from these matrices as (60), the solution $\hat{\Xi}(s)$ of the 1st order weightless MLSA problem for $\Xi(s)$ is given by

$$\hat{\Xi}(s) = (sI - \Omega)^{-1}\xi . \qquad (61)$$

It is easy to verify that (61) leads to (24). Thus Th.7.3.4 has been proved.

PART III

THE CONTINUOUS-TIME LIMITS OF THE DISCRETE-TIME RESULTS

8. THE CONTINUOUS-TIME LIMITS OF THE DISCRETE-TIME RESULTS

8.1. The continuous-time limits of discrete-time systems

In this section, we will present the fundamental framework for treating the continuous-time (c.t.) limits of discrete-time (d.t.) systems as preliminaries for the succeeding two sections.

Suppose that we are given a one-parameter family of PM's in z, say $\{P^{(t)}(z)\ ;\ 0<t\leqq t_0\}$, and a PM in s, say P(s), such that the degrees of the elements of $P^{(t)}(z)$ and P(s) are at most n. Then they are written as

$$\begin{cases} P^{(t)}(z) = (\frac{z-1}{t})^n P_n^{(t)} + \dots + (\frac{z-1}{t}) P_1^{(t)} + P_0^{(t)} & (1) \\ P(s) = s^n P_n + \dots + s P_1 + P_0, & (2) \end{cases}$$

where $\{P_j^{(t)}\}$ and $\{P_j\}$ are constant matrices. Now, we interpret t as the 'unit time length' or the 'sampling period', and (z-1)/t as the difference operator approximating the differential operator s. In this interpretation, $P^{(t)}(z)$ can be regarded as a d.t. approximation of P(s) if $P_j^{(t)}$ is near to P_j for every j. When $P_j^{(t)}$ converges to P_j as $t \downarrow 0$ for every j, we say that <u>$P^{(t)}(z)$ converges to P(s) as $t \downarrow 0$</u>, and write the convergence as

$$P^{(t)}(z) \longrightarrow P(s) \qquad (t \downarrow 0). \qquad (3)$$

In (3) P(s) is called the <u>c.t. limit</u> of $\{P^{(t)}(s)\}$.

Since (1) is equivalent to

$$P^{(t)}(1+ts) = s^n P_n^{(t)} + \dots + s P_1^{(t)} + P_0^{(t)}$$

the convergence (3) can be represented as

$$P^{(t)}(1+ts) \longrightarrow P(s) \qquad (t \downarrow 0)$$

which is a usual convergence of PM's in s. This is the representation of (3) using the s-z transformation

a) $z = 1+ts \qquad (s = (z-1)/t).$

In general, if an s-z transformation $z = \zeta^{(t)}(s)$ satisfies

$$\{\zeta^{(t)}(s) - 1\}/t \longrightarrow s \qquad (t \downarrow 0) \tag{4}$$

for $\forall s$, then (3) is equivalent to the pointwise convergence

$$P^{(t)}(\zeta^{(t)}(s)) \longrightarrow P(s) \qquad (t \downarrow 0).$$

We present below some examples of such s-z transformations other than a).

b) $z = 1/(1-ts) \qquad (s = (1-z^{-1})/t)$

c) $z = \frac{2+ts}{2-ts} \qquad (s = \frac{2(z-1)}{t(z+1)})$

d) $z = e^{ts}$

Next, suppose that we are given a one-parameter family of rational matrices in z, say $\{R^{(t)}(z);0<t\leqq t_0\}$, and a rational matrix in s, say R(s). If there exist scalar polynomials $\{p^{(t)}(z)\}$ and p(s) together with PM's $\{Q^{(t)}(z)\}$ and Q(s) such that

$$\deg p^{(t)}(z) = \deg p(s) \quad \text{for } \forall t \tag{5}$$

$$\begin{cases} R^{(t)}(z) = Q^{(t)}(z)/p^{(t)}(z) & (6) \\ R(s) = Q(s)/p(s) & (7) \end{cases}$$

$$\begin{cases} p^{(t)}(z) \longrightarrow p(s) \quad (t \downarrow 0) & (8) \\ Q^{(t)}(z) \longrightarrow Q(s) \quad (t \downarrow 0), & (9) \end{cases}$$

we say that <u>$R^{(t)}(z)$ converges to $R(s)$ as $t \downarrow 0$</u>, and write the convergence as

$$R^{(t)}(z) \longrightarrow R(s) \quad (t \downarrow 0). \tag{10}$$

In (10) $R(s)$ is called the <u>c.t. limit</u> of $\{R^{(t)}(z)\}$.

It is clear from the definition that (10) implies the pointwise convergence

$$R^{(t)}(\zeta^{(t)}(s)) \longrightarrow R(s) \quad (t \downarrow 0), \tag{11}$$

where $\zeta^{(t)}(s)$ is an arbitrary s-z transformation satisfying (4). On the other hand, the converse is not necessarily true. For instance, the rational functions

$$\begin{cases} R^{(t)}(z) = 1/z \quad \text{(being independent of t)} \\ R(s) = 1 \end{cases} \tag{12}$$

satisfy (11), but they conflict with the condition (5) and hence do not satisfy (10).

Obviously, the convergence (10) is equivalent to the existence of PM's $\{P^{(t)}(z)\}$, $\{Q^{(t)}(z)\}$, $P(s)$ and $Q(s)$ such that

$$\deg \det P^{(t)}(z) = \deg \det P(s) \quad \text{for } \forall t$$

$$\begin{cases} R^{(t)}(z) = P^{(t)-1}(z)\ Q^{(t)}(z) \\ R(s) = P^{-1}(s)\ Q(s) \end{cases}$$

$$\left(\text{or} \quad \begin{cases} R^{(t)}(z) = Q^{(t)}(z)\ P^{(t)-1}(z) \\ R(s) = Q(s)\ P^{-1}(s) \end{cases} \right)$$

$$\begin{cases} P^{(t)}(z) \longrightarrow P(s) \qquad (t \downarrow 0) \\ Q^{(t)}(z) \longrightarrow Q(s) \qquad (t \downarrow 0). \end{cases}$$

In particular, if $R^{(t)}(z)$ and $R(s)$ are written as

$$R^{(t)}(z) = A^{(t)-1}(z)\ B^{(t)}(z) \tag{13}$$

$$R(s) = C^{-1}(s)\ D(s) \tag{14}$$

where

$$A^{(t)}(z) = z^n\ I + z^{n-1}\ A_{n-1}^{(t)} + \dots + A_0^{(t)} \tag{15}$$

$$B^{(t)}(z) = z^{n-1}\ B_{n-1}^{(t)} + \dots + B_0^{(t)} \tag{16}$$

$$C(s) = s^n\ I + s^{n-1}\ C_{n-1} + \dots + C_0 \tag{17}$$

$$D(s) = s^{n-1}\ D_{n-1} + \dots + D_0\ , \tag{18}$$

and if

$$t^{-n}\ A^{(t)}(z) \longrightarrow C(s) \qquad (t \downarrow 0) \tag{19}$$

$$t^{-n}\ B^{(t)}(z) \longrightarrow D(s) \qquad (t \downarrow 0), \tag{20}$$

then $R^{(t)}(z)$ converges to $R(s)$ as $t \downarrow 0$. It is noted that the PM

$$C^{(t)}(s) \triangleq t^{-n} A^{(t)}(1+ts) \tag{21}$$

can be written as

$$C^{(t)}(s) = s^n I + s^{n-1} C_{n-1}^{(t)} + \dots + C_0^{(t)} \tag{22}$$

and that (19) is represented as

$$C_j^{(t)} \longrightarrow C_j \qquad (t \downarrow 0) \quad \text{for } \forall j=0,1,\dots,n-1. \tag{23}$$

Rewriting (21) and (22) as

$$\begin{aligned} A^{(t)}(z) &= t^n C^{(t)}\left(\frac{z-1}{t}\right) \\ &= (z-1)^n I + t(z-1)^{n-1} C_{n-1}^{(t)} + \dots + t^n C_0^{(t)}, \end{aligned} \tag{24}$$

we can see that (19) implies the convergence

$$A^{(t)}(z) \longrightarrow (z-1)^n I \qquad (t \downarrow 0). \tag{25}$$

When $R^{(t)}(z)$ and $R(s)$ are both strictly proper, the convergence (10) is also equivalent to the existence of $(A^{(t)}, B^{(t)}, C^{(t)})$ and (F,G,H) such that

$$\begin{cases} R^{(t)}(z) = C^{(t)} (zI - A^{(t)})^{-1} B^{(t)} \\ R(s) = H (sI - F)^{-1} G \end{cases}$$

$$\begin{cases} (A^{(t)}-I)/t \longrightarrow F \\ B^{(t)}/t \longrightarrow G \qquad\qquad (t \downarrow 0) \\ C^{(t)} \longrightarrow H. \end{cases}$$

We present below two elementary lemmas which will be used in the succeeding sections.

Lemma 8.1.1 If rational matrices $\{R^{(t)}(z); 0<t\leqq t_0\}$ and $R(s)$ satisfy (10), then their polynomial parts and strictly proper parts (see Def.3.1.2) also satisfy

$$\begin{cases} [R^{(t)}(z)]_{pol} \longrightarrow [R(s)]_{pol} \\ [R^{(t)}(z)]_{sp} \longrightarrow [R(s)]_{sp} \, . \end{cases} \qquad (t \downarrow 0) \qquad (26)$$

Lemma 8.1.2 Suppose that rational matrices $R_1^{(t)}(z)$ and $R_2^{(t)}(z)$ are both strictly proper and d.t. stable for $0<\forall t\leqq t_0$, and that rational matrices $R_1(s)$ and $R_2(s)$ are both strictly proper and c.t. stable. If

$$R_j^{(t)}(z) \longrightarrow R_j(s) \qquad (t \downarrow 0) \qquad \text{for } j=1,2 \qquad (27)$$

then

$$t^{-1} \int_{-\pi}^{\pi} R_1^{(t)}(e^{i\omega}) \, R_2^{(t)\prime}(e^{-i\omega}) \, d\omega$$

$$\longrightarrow \int_{-\infty}^{\infty} R_1(i\omega) \, R_2'(-i\omega) \, d\omega \qquad (t \downarrow 0). \qquad (28)$$

The first lemma is immediate from the definition of the convergence. It is worth noting that the lemma is due to the condition (5). Indeed, (26) does not hold with the example in (12), which does not satisfy (5).

The second lemma is proved as follows. Let for $j=1,2$

$$r_j^{(t)}(\omega) \triangleq \begin{cases} R_j^{(t)}(e^{ti\omega}) & \text{if } -\pi/t < \omega < \pi/t \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$t^{-1} \int_{-\pi}^{\pi} R_1^{(t)}(e^{i\omega})\, R_2^{(t)\prime}(e^{-i\omega})\, d\omega$$

$$= \int_{-\infty}^{\infty} r_1^{(t)}(\omega)\, r_2^{(t)\prime}(-\omega)\, d\omega.$$

As an example of (11), the following pointwise convergence is obtained for j=1,2:

$$r_j^{(t)}(\omega) \longrightarrow R_j(i\omega) \qquad (t \downarrow 0). \tag{29}$$

Furthermore, it follows from the assumption of the lemma that $R_1(s)R_2'(-s)$ is absolutely integrable on the imaginary axis, and that there exists an absolutely integrable function f such that

$$|\, r_1^{(t)}(\omega)\, r_2^{(t)}(-\omega) \,| \leqq f(\omega) \qquad \text{for } \forall t, \forall\omega.$$

Therefore, Lebesgue's convergence theorem guarantees that (29) implies (28).

<u>Remark 8.1.3</u> The strictly proper condition in Lemma 8.1.2 is indispensable. For instance, suppose that $R_1^{(t)}(z)$ and R(s) are strictly proper and stable as in the lemma, and that $R_2^{(t)}(z) = R_2(s) = I$. Then we have

$$\begin{cases} \int_{-\pi}^{\pi} R_1^{(t)}(e^{i\omega})\, R_2^{(t)\prime}(e^{-i\omega})\, d\omega = 0 \\ \int_{-\infty}^{\infty} R_1(i\omega)\, R_2'(-i\omega)\, d\omega = \pi \lim_{s\to\infty} sR_1(s). \end{cases}$$

Thus (28) does not hold in general. In this case, $R_1(s)R_2'(-s)$ is not absolutely integrable on the imaginary axis, and hence we cannot apply Lebesgue's theorem.

8.2. The continuous-time limits of the discrete-time orthogonal polynomial matrices

In Chap.4 we constructed the c.t. theory of orthogonal PM's, and in Chap.5 surveyed the corresponding d.t. theory. In this section, we will investigate how the d.t. theory 'converges' to the c.t. theory as the 'unit time length' t tends to 0, where only the strictly regular case will be treated for simplicily.

Suppose that we are given a one-parameter family of $p \times p$ PM's $\{A^{(t)}(z); 0<t\leqq t_0\}$ and a $p \times p$ PM $C(s)$ of the form

$$\begin{cases} A^{(t)}(z) = z^n I + z^{n-1} A_{n-1}^{(t)} + \dots + A_0^{(t)} & (1) \\ C(s) = s^n I + s^{n-1} C_{n-1} + \dots + C_0 , & (2) \end{cases}$$

and also given a one-parameter family of $p \times p$ positive-definite matrices $\{\Sigma^{(t)}; 0<t\leqq t_0\}$ and a $p \times p$ positive-definite matrix Π. We asuume that (see (8.1.19))

$$t^{-n} A^{(t)}(z) \longrightarrow C(s) \qquad (t \downarrow 0) \tag{3}$$

and that

$$t^{-2n+1} \Sigma^{(t)} \longrightarrow \Pi \qquad (t \downarrow 0). \tag{4}$$

Furthermore, we make the additional assumption that $C(s)$ satisfies (3.3.3), which allows us to define from $(C(s), \Pi)$ the inner product $\langle P(s), Q(s) \rangle$ as in Sec.3.3, where $P(s)$ and $Q(s)$ are arbitrary $p \times p$ PM's in s. This assumption, together with the convergence (3), implies that $A^{(t)}(z)$ satisfies (5.14) for sufficiently small $t>0$, and thus we can define from $(A^{(t)}(z), \Sigma^{(t)})$ the inner product $\langle P(z),Q(z)\rangle_t$ as in Chap.5, where $P(z)$ and $Q(z)$ are arbitrary $p\times p$ PM's in z.

Under these assumptions we have the following proposition.

Proposition 8.2.1 Let $\{P^{(t)}(z)\}$ and $\{Q^{(t)}(z)\}$ be one-parameter families of $p\times p$ PM's in z of degree at most n-1, and let $P(s)$ and $Q(s)$ be $p\times p$ PM's in s of degree at most n-1. If

$$\begin{cases} P^{(t)}(z) \longrightarrow P(s) \\ Q^{(t)}(z) \longrightarrow Q(s) \end{cases} \qquad (t \downarrow 0) \tag{5}$$

then

$$\langle P^{(t)}(z),Q^{(t)}(z)\rangle_t \longrightarrow \langle P(s),Q(s)\rangle \qquad (t \downarrow 0). \tag{6}$$

(Proof) Since $P^{(t)}(1+ts)$, $Q^{(t)}(1+ts)$, $P(s)$ and $Q(s)$ are of degree at most n-1 as PM's in s, they are written as

$$\begin{cases} P^{(t)}(1+ts) = P^{(t)}\cdot T(s) \\ Q^{(t)}(1+ts) = Q^{(t)}\cdot T(s) \end{cases} \tag{7}$$

$$\begin{cases} P(s) = P\cdot T(s) \\ Q(s) = Q\cdot T(s), \end{cases} \tag{8}$$

where $P^{(t)}$, $Q^{(t)}$, P and Q are $p\times np$ constant matrices, and

$$T(s) \triangleq [I,sI,\ldots,s^{n-1}I]' \quad : np\times p.$$

Using these expressions, the convergences in (5) are written as

$$\begin{cases} P^{(t)} \longrightarrow P \\ Q^{(t)} \longrightarrow Q. \end{cases} \tag{9}$$

Let (Γ,B) and $(\Gamma^{(t)},B)$ be the companion pairs defined from $C(s)$ and $C^{(t)}(s) \triangleq t^{-n} A^{(t)}(1+ts)$, respectively, which are characterized by the equations

$$(sI - \Gamma)^{-1}B = T(s)\ C^{-1}(s) \tag{10}$$

$$(sI - \Gamma^{(t)})^{-1}B = T(s)\ C^{(t)-1}(s). \tag{11}$$

(See (2.1.22-25).) Then the convergence (3) is written as

$$\Gamma^{(t)} \longrightarrow \Gamma \qquad (t \downarrow 0). \tag{12}$$

According to the definition of the inner product $\langle\ \rangle$ (Sec.3.3), it follows from (7) and (10) that

$$\langle P(s),Q(s)\rangle = P\ X\ Q' \tag{13}$$

where X is the unique solution of the c.t. Lyapunov equation

$$\Gamma\ X + X\ \Gamma' + B\,\Pi\,B' = 0. \tag{14}$$

Similarly, rewriting (7) and (11) as

$$\begin{cases} P^{(t)}(z) = P^{(t)}\cdot T((z-1)/t) \\ Q^{(t)}(z) = Q^{(t)}\cdot T((z-1)/t) \end{cases}$$

$$(zI - F^{(t)})^{-1}(t^{-n+1}B) = T((z-1)/t)\ A^{(t)-1}(z)$$

where

$$F^{(t)} \triangleq I + t\Gamma^{(t)},$$

we see that

$$\langle P^{(t)}(z), Q^{(t)}(z)\rangle_t = P^{(t)} X^{(t)} Q^{(t)\prime}$$

where $X^{(t)}$ is the unique solution of the d.t. Lyapunov equation

$$X^{(t)} = F^{(t)} X^{(t)} F^{(t)\prime} + t^{-2n+2} B \Sigma^{(t)} B', \tag{15}$$

which is also written as

$$\Gamma^{(t)} X^{(t)} + X^{(t)} \Gamma^{(t)\prime} + t\, \Gamma^{(t)} X^{(t)} \Gamma^{(t)\prime} + B(t^{-2n+1}\Sigma^{(t)})B' = 0. \tag{16}$$

Comparison of (16) with (14) shows that the convergences (4) and (12) yield

$$X^{(t)} \longrightarrow X \qquad (t \downarrow 0). \tag{17}$$

Thus, (6) is proved from (9), (13), (15) and (17). (QED)

It should be noted that the above proposition is not straightforwardly extended to the case where deg $P^{(t)}(z)$ = deg $P(s) = n$. For instance, the PM's

$$\begin{cases} P^{(t)}(z) \triangleq t^{-n} A^{(t)}(z) \\ P(s) \triangleq C(s) \end{cases}$$

$$\begin{cases} Q^{(t)}(z) \triangleq (\frac{z-1}{t})^{n-1} I \\ Q(s) \triangleq s^{n-1} I \end{cases}$$

satisfy (5), but it follows from (3.1.9) and (5.2) that

$$\begin{cases} \langle P^{(t)}(z), Q^{(t)}(z)\rangle_t = 0 \\ \langle P(s), Q(s)\rangle = \frac{1}{2}\Pi, \end{cases}$$

and hence (6) does not hold (cf. Remark 8.1.3). Instead, we have the following proposition.

Proposition 8.2.2 Suppose that the degrees of $P^{(t)}(z)$ and $P(s)$ are at most n, and that the degrees of $Q^{(t)}(z)$ and $Q(s)$ are at most n-1. If they satisfy the convergences in (5), then as $t \downarrow 0$

(i) $$\langle P^{(t)}(z), Q^{(t)}(z)\rangle_t \longrightarrow \langle P(s), Q(s)\rangle - \frac{1}{2} P_n \Pi Q'_{n-1} \tag{18}$$

$$\left(P(s) = \sum_j s^j P_j , \quad Q(s) = \sum_j s^j Q_j \right)$$

(ii) $$\langle P^{(t)}(z), zQ^{(t)}(z)\rangle_t \longrightarrow \langle P(s), Q(s)\rangle + \frac{1}{2} P_n \Pi Q'_{n-1} \tag{19}$$

(iii) $$\langle P^{(t)}(z), (\tfrac{z+1}{2})Q^{(t)}(z)\rangle_t \longrightarrow \langle P(s), Q(s)\rangle. \tag{20}$$

(Proof) (i) : We can write $P^{(t)}(z)$ and $P(s)$ as

$$P^{(t)}(z) = t^{-n} P_n^{(t)} A^{(t)}(z) + \bar{P}^{(t)}(z) \tag{21}$$

$$P(s) = P_n C(s) + \bar{P}(s) \tag{22}$$

where $P_n^{(t)}$ is a constant matrices, and $\bar{P}^{(t)}(z)$ and $\bar{P}(s)$ are PM's of degree at most n-1. Using these expressions, we obtain from (3) and (5)

$$\begin{cases} P_n^{(t)} \longrightarrow P_n & (t \downarrow 0) & (23) \\ \bar{P}^{(t)}(z) \longrightarrow \bar{P}(s) & (t \downarrow 0). & (24) \end{cases}$$

Since $A^{(t)}(z)$ is orthogonal to $Q^{(t)}(z)$, we have from (21)

$$\langle P^{(t)}(z), Q^{(t)}(z)\rangle_t = \langle \bar{P}^{(t)}(z), Q^{(t)}(z)\rangle_t ,$$

and hence Prop.8.2.1 shows that

$$\langle P^{(t)}(z), Q^{(t)}(z)\rangle_t \longrightarrow \langle \bar{P}(s), Q(s)\rangle.$$

Recalling (22) and (3.1.9), we see that the limit in the above convergence is written as

$$\langle \bar{P}(s), Q(s)\rangle = \langle P(s), Q(s)\rangle - \frac{1}{2} P_n \Pi Q'_{n-1} . \tag{25}$$

Thus (18) has been proved. (ii) : We can write $zQ^{(t)}(z)$ as

$$z\, Q^{(t)}(z) = t^{-n+1} Q_{n-1}^{(t)} A^{(t)}(z) + \bar{Q}^{(t)}(z) \tag{26}$$

where $Q_{n-1}^{(t)}$ is a constant matrix, and $\bar{Q}^{(t)}(z)$ is a PM of degree at most n-1. It is obvious that $Q_{n-1}^{(t)}$ is the (n-1)-th coefficient matrix of the PM $Q^{(t)}(1+ts)$, from which we have

$$Q_{n-1}^{(t)} \longrightarrow Q_{n-1} . \tag{27}$$

Noting that (5) leads to

$$z\, Q^{(t)}(z) \longrightarrow Q(s),$$

we can see from (26) and (27) that

$$\bar{Q}^{(t)}(z) \longrightarrow Q(s). \tag{28}$$

Now, it follows from (21) and (26) that

$$\langle P^{(t)}(z), zQ^{(t)}(z)\rangle_t = t^{-2n+1} P_n^{(t)} \Sigma^{(t)} Q_{n-1}^{(t)\prime} + \langle \bar{P}^{(t)}(z), \bar{Q}^{(t)}(z)\rangle_t . \qquad (29)$$

The first term of the right-hand side in (29) converges to $P_n \Pi Q'_{n-1}$ because of (4), (23) and (27), while the second term converges to $\langle \bar{P}(s), Q(s)\rangle$ because of (24) and (28). Thus, we obtain (19) by recalling (25). (iii) : Immediate from (i) and (ii). (QED)

Remark 8.2.3 Using the stochastic interpretation of the inner product $\langle\ \rangle$ in Sec.3.2, the results in Prop.8.2.2 are represented as

$$\langle P^{(t)}(z), Q^{(t)}(z)\rangle_t \longrightarrow E[d\tilde{p}(t)*q'(t)]/dt$$

$$(= \langle P(s), Q(s)\rangle, \text{ see Rem.3.2.2})$$

$$\langle P^{(t)}(z), (z-1)Q^{(t)}(z)\rangle_t \longrightarrow E[d\tilde{p}(t)dq'(t)]/dt$$

$$\langle P^{(t)}(z), (\tfrac{z+1}{2})Q^{(t)}(z)\rangle_t \longrightarrow E[d\tilde{p}(t)\circ q'(t)]/dt.$$

Thus we see that the equation

$$(\tfrac{z+1}{2})Q^{(t)}(z) = Q^{(t)}(z) + \tfrac{1}{2}(z-1)Q^{(t)}(z)$$

corresponds to (3.2.20), which represents the relation between the Itô calculus and the Stratonovich calculus.

Now let us investigate the limiting behavior of the quantities

$$A_j^{(t)}(z),\ B_j^{(t)}(z),\ (\delta_j^A)^{(t)},\ (\delta_j^B)^{(t)},\ (\gamma_j^A)^{(t)},\ (\gamma_j^B)^{(t)}$$

$$(j=0,1,\dots,n)$$

generated from $(A^{(t)}(z),\Sigma^{(t)})$ by the LWR algorithm (5.24-32), relating them with the quantities

$$R_j(s),\ \varepsilon_j,\ \theta_j \qquad (j=0,1,\dots,n)$$

generated from $(C(s),\Pi)$ by the algorithm (4.1.21).

To begin with, we present the result on the forward orthogonal PM's $\{A_j^{(t)}(z)\}$.

Proposition 8.2.4 As $t \downarrow 0$,

$$\begin{cases} t^{-j}\ A_j^{(t)}(z) \longrightarrow R_j(s) & \qquad (30) \\ & (0 \leqq j \leqq n-1) \\ t^{-2j}(\delta_j^A)^{(t)} \longrightarrow \varepsilon_j & \qquad (31) \end{cases}$$

$$\begin{cases} t^{-n}\ A_n^{(t)}(z) \longrightarrow C(s) & \qquad (32) \\ t^{-2n+1}(\delta_n^A)^{(t)} \longrightarrow \Pi\ . & \qquad (33) \end{cases}$$

(Proof) Note that (32) and (33) are nothing but the assumptions (3) and (4) (see (5.20) and (5.23), and note that $A_n=I$ here), and that (31) is immediately derived from (30) (see Prop.8.2.1, (4.1.12) and (5.21)). Hence we have only to prove (30). For $j=0,1,\dots,n-1$ let

$$R_j^{(t)}(s) \triangleq t^{-j}\ A_j^{(t)}(1+ts),$$

which can be written as

$$R_j^{(t)}(s) = s^j I + s^{j-1} R_{j,j-1}^{(t)} + \dots + R_{j,0}^{(t)} .$$

Then we have

$$t^{-j} A_j^{(t)}(z) = \left(\frac{z-1}{t}\right)^j I + \left(\frac{z-1}{t}\right)^{j-1} R_{j,j-1}^{(t)} + \dots + R_{j,0}^{(t)} .$$

Since $t^{-j}A_j^{(t)}(z)$ is characterized as a PM of the above form such that (see (5.16))

$$\langle t^{-j}A_j^{(t)}(z) , \left(\frac{z-1}{t}\right)^k I \rangle_t = 0 \quad \text{for } \forall k=0,1,\dots,j-1,$$

we can see from Prop.8.2.1 that $t^{-j}A_j^{(t)}(z)$ converges to $R_j(s)$, which is characterized as a monic PM of degree j being orthogonal to $s^k I$ for $\forall k=0,1,\dots,j-1$. Thus (30) has been proved. (QED)

Next, we investigate the limiting behavior of the backward PM's $\{B_j^{(t)}(z)\}$.

<u>Proposition 8.2.5</u> As $t \downarrow 0$,

$$\begin{cases} t^{-j} B_j^{(t)}(z) \longrightarrow (-1)^j R_j(s) & (34) \\ t^{-2j}(\delta_j^B)^{(t)} \longrightarrow \varepsilon_j & (35) \end{cases} \qquad (0 \leqq j \leqq n-1)$$

$$\begin{cases} t^{-n} B_n^{(t)}(z) \longrightarrow (-1)^n D(s) & (36) \\ t^{-2n+1}(\delta_n^B)^{(t)} \longrightarrow \Pi , & (37) \end{cases}$$

where D(s) is the dual PM of C(s) w.r.t. Π defined by (4.3.1-2) and (4.3.5).

(Proof) For $j=0,1,\ldots,n$, let

$$R_j^{(t)}(s) \triangleq (-t)^{-j} (1-ts)^j B_j^{(t)}(1/(1-ts)).$$

Then, since $z^j B_j^{(t)}(z^{-1})$ is a monic PM of degree j in z, $R_j^{(t)}(s)$ turns out to be a monic PM of degree j in s, and can be written as

$$R_j^{(t)}(s) = s^j I + s^{j-1} R_{j,j-1}^{(t)} + \ldots + R_{j,0}^{(t)} ,$$

which yields

$$(-t)^{-j} B_j^{(t)}(z)$$

$$= \sigma_{j,j}^{(t)}(z) I + \sigma_{j,j-1}^{(t)}(z) R_{j,j-1}^{(t)} + \ldots + \sigma_{j,0}^{(t)}(z) R_{j,0}^{(t)} \tag{38}$$

where

$$\sigma_{j,k}^{(t)}(z) \triangleq z^j ((1-z^{-1})/t)^k \qquad (0 \leqq k \leqq j).$$

It is noted that $\sigma_{j,k}^{(t)}(z)$ is a polynomial of degree j in z, and that

$$\sigma_{j,k}^{(t)}(z) \longrightarrow s^k \qquad (t \downarrow 0). \tag{39}$$

We can see that the orthogonal condition (5.18) for $B_j^{(t)}(z)$ is equivalent to

$$\langle (-t)^{-j} B_j^{(t)}(z) , \sigma_{j,k}^{(t)}(z) I \rangle_t = 0$$

$$\text{for } \forall k=0,1,\ldots,j-1. \tag{40}$$

For $j=0,1,\ldots,n-1$, it follows from (39) and Prop.8.2.1 that

$$\langle \sigma_{j,k}^{(t)}(z) I , \sigma_{j,\ell}^{(t)}(z) I \rangle_t \longrightarrow \langle s^k I, s^\ell I \rangle, \tag{41}$$

$$(0 \leqq k, \ell \leqq j)$$

and therefore (34) is obtained by comparing (38) and (40) with (4.1.7) and (4.1.8) (where $T_i(s) = s^i I$ in our case). The convergence (35) is immediate from (34). For j=n, however, (41) does not hold. Instead, noting that

$$\sigma_{n,\ell}^{(t)}(z) = z\,\sigma_{n-1,\ell}^{(t)}(z) \qquad (0 \leqq \ell \leqq n-1),$$

it is shown from (ii) of Prop.8.2.2 that

$$\langle\, \sigma_{n,k}^{(t)}(z)I\ ,\ \sigma_{n,\ell}^{(t)}(z)I\ \rangle_t$$

$$\longrightarrow \begin{cases} \langle s^n I, s^{n-1} I\rangle + \frac{1}{2}\Pi & \text{if } (k,\ell) = (n,n-1) \\ \langle s^k I, s^\ell I\rangle & \text{otherwise.} \end{cases} \tag{42}$$

Invoking that D(s) is characterized as a monic PM of degree n satisfying

$$\langle D(s), s^k I\rangle = \begin{cases} 0 & \text{if } 0 \leqq k \leqq n-2 \\ -\frac{1}{2}\Pi & \text{if } k=n-1, \end{cases} \tag{43}$$

(see (4.3.20) and (4.1.19))

we obtain the convergence (36) from (38)-(40) and (42). Applying (iii) of Prop.8.2.2 to the equation

$$\langle\, (-t)^{-n} B_n^{(t)}(z)\ ,\ (\tfrac{z+1}{2})(\tfrac{z-1}{t})^{n-1}\ \rangle_t = -\tfrac{1}{2}\, t^{-2n+1} (\delta_n^B)^{(t)},$$

we can see that (37) is derived from (36) and (43). (QED)

The result on the reflection coefficient matrices $\{(\gamma_j^A)^{(t)}\}$

and $\{(\gamma_j^B)^{(t)}\}$ is as follows.

Proposition 8.2.6 As $t \downarrow 0$,

$$\begin{cases} (\gamma_j^A)^{(t)} \longrightarrow (-1)^j I & (44) \\ & (1 \leqq j \leqq n) \\ (\gamma_j^B)^{(t)} \longrightarrow (-1)^j I & (45) \end{cases}$$

$$\begin{cases} t^{-2}\{I - (\gamma_j^A)^{(t)}\ (\gamma_j^B)^{(t)}\} \longrightarrow \varepsilon_j \varepsilon_{j-1}^{-1} & (46) \\ & (1 \leqq j \leqq n-1) \\ t^{-2}\{I - (\gamma_j^B)^{(t)}\ (\gamma_j^A)^{(t)}\} \longrightarrow \varepsilon_j \varepsilon_{j-1}^{-1} & (47) \end{cases}$$

$$\begin{cases} t^{-1}\{I - (\gamma_n^A)^{(t)}\ (\gamma_n^B)^{(t)}\} \longrightarrow \Pi\, \varepsilon_{n-1}^{-1} & (48) \\ t^{-1}\{I - (\gamma_n^B)^{(t)}\ (\gamma_n^A)^{(t)}\} \longrightarrow \Pi\, \varepsilon_{n-1}^{-1} & (49) \end{cases}$$

$$\begin{cases} t^{-1}\{(\gamma_j^A)^{(t)} + (\gamma_{j+1}^A)^{(t)}\} \longrightarrow (-1)^{j+1}\ \theta_j \varepsilon_j^{-1} & (50) \\ t^{-1}\{(\gamma_j^B)^{(t)} + (\gamma_{j+1}^B)^{(t)}\} \longrightarrow (-1)^{j}\ \theta_j \varepsilon_j^{-1} & (51) \end{cases}$$

$$(0 \leqq j \leqq n-2)$$

$$\begin{cases} t^{-1}\{(\gamma_{n-1}^A)^{(t)} + (\gamma_n^A)^{(t)}\} \longrightarrow (-1)^n (\theta_{n-1} - \tfrac{1}{2}\Pi)\varepsilon_{n-1}^{-1} & (52) \\ t^{-1}\{(\gamma_{n-1}^B)^{(t)} + (\gamma_n^B)^{(t)}\} \longrightarrow (-1)^{n-1} (\theta_{n-1} + \tfrac{1}{2}\Pi)\varepsilon_{n-1}^{-1} & (53) \end{cases}$$

(Proof) We will give here only the proofs of (44), (46), (48), (50) and (52). First, (44) is obtained from the fact that

$$(\gamma_j^A)^{(t)} = A_j^{(t)}(0)$$

$$A_j^{(t)}(z) \longrightarrow (z-1)^j\ I \qquad \text{(see (8.1.25))}.$$

Next, (46) and (48) are verified by recalling (31), (33) and (5.31). Finally, (50) and (52) are proved by comparing (4.1.20) with the equation

$$\langle (z-1)A_j^{(t)}(z), B_j^{(t)}(z)\rangle_t = -\{(\gamma_j^A)^{(t)} + (\gamma_{j+1}^A)^{(t)}\}(\delta_j^B)^{(t)}$$

and by appealing to the previous propositions in the present section. (QED)

Remark 8.2.7 It can be shown that the LWR algorithm is equivalent to the following recursion:

$$A_{j+1}(z) = (z-1)\ A_j(z) + (\gamma_j^A + \gamma_{j+1}^A)\ B_j(z)$$

$$+ (I - \gamma_j^A \gamma_j^B)\ z\ A_{j-1}(z)$$

$$B_{j+1}(z) = -(z-1)\ B_j(z) + (\gamma_j^B + \gamma_{j+1}^B)\ z\ A_j(z)$$

$$+ (I - \gamma_j^B \gamma_j^A)\ z\ B_{j-1}(z)$$

We can see from Prop.8.2.4-6 that the above recursion 'converges' to the recursion (4.1.21) together with (4.1.17). Conversely, comparison of these recursions provides us with alternative proofs of Prop.8.2.4-6. In addition, we note that the three term recurrence version of the LWR algorithm ([18])

$$A_{j+1}(z) = (z-1)\ A_j(z) + \{ I + \gamma_{j+1}^A (\gamma_j^A)^{-1} \} A_j(z)$$

$$\gamma_{j+1}^A \{ \gamma_j^B - (\gamma_j^A)^{-1} \}\ z\ A_{j-1}(z)$$

also 'converges' to (4.1.21) and (4.1.17).

Finally, we investigate the limiting behavior of the quantities

$$\delta_j^{(t)},\ \alpha_j^{(t)},\ h_j^{(t)},\ u_j^{(t)}$$

generated from $(A^{(t)}(z), \Sigma^{(t)})$ by the canonical polar-type LWR algorithm (5.44-50) in which (5.47) is replaced with (5.47)'.

<u>Proposition 8.2.8</u> As $t \downarrow 0$,

$$t^{-2j}\ \delta_j^{(t)} \longrightarrow \varepsilon_j \tag{54}$$

$$t^{-1}\{\alpha_j^{(t)} - I\} \longrightarrow \theta_j\ \varepsilon_j^{-1} \tag{55}$$

$$t^{-1}\{h_j^{(t)} - I\} \longrightarrow 0 \tag{56}$$

$$t^{-1}\{u_j^{(t)} - I\} \longrightarrow \theta_j\ \varepsilon_j^{-1} \tag{57}$$

for $\forall j=0,1,\ldots,n-1$.

(Proof) Note that (54) is nothing but (31) (see (5.53)). It can be shown from (5.47-48) and (5.51) that

$$\langle (z-1)A_j^{(t)}(z)\ ,\ A_j^{(t)}(z) \rangle_t = \{\alpha_j^{(t)}\ h_{j-1}^{(t)} - I\}\delta_j^{(t)}.$$

$$(h_{-1}^{(t)} \triangleq I)$$

Comparing this equation with (4.1.20), we obtain

$$t^{-1}\{\alpha_j^{(t)}\ h_{j-1}^{(t)} - I\} \longrightarrow \theta_j\ \varepsilon_j^{-1}.$$

Using this convergence, and noting that (55) implies (56)-(57) (see (5.41-43)), we can prove (55)-(57) by induction on j. (QED)

8.3. The continuous-time limits of the discrete-time Mullis-Roberts type approximations

Suppose that we are given a one-parameter family of $p \times q$ strictly proper and d.t. stable rational matrices $\{G^{(t)}(z); 0<t\leqq t_0\}$ which converges to a strictly proper and c.t. stable rational matrix $H(s)$; i.e.,

$$G^{(t)}(z) \longrightarrow H(s) \qquad (t \downarrow 0). \tag{1}$$

In addition, we assume that $G^{(t)}(z)$ and $H(s)$ satisfy (6.1.4) and (7.1.4), respectively. Let us denote by

$$\begin{cases} \hat{G}^{(t)}(z) = \hat{A}^{(t)-1}(z)\ \hat{B}^{(t)}(z) \\ \hat{H}(s) = \hat{C}^{-1}(s)\ \hat{D}(s) \end{cases}$$

the solutions of

$$\begin{cases} \text{the n-th order d.t. MLSA problem for } G^{(t)}(z) \\ \text{the n-th order c.t. MLSA problem for } H(s) \end{cases}$$

respectively, and by $\delta^{(t)}$ and ε the associated approximation error matrices (see (6.1.16) and (7.1.17)). We also denote by

$$\begin{cases} \hat{G}^{(t)}(z;U) = \hat{A}^{(t)-1}(z;U)\ \hat{B}^{(t)}(z;U) \\ \hat{H}(s;W) = \hat{C}^{-1}(s;W)\ \hat{D}(s;W) \end{cases}$$

the solutions of

$$\begin{cases} \text{the n-th order d.t. IA problem for } G^{(t)}(z) \\ \text{the n-th order c.t. IA problem for } H(s) \end{cases}$$

specified by the conditions (6.2.23) and (7.1.45), respectively, where

$$\begin{cases} U \text{ is an arbitrary } \beta^{(t)}\text{-orthogonal matrix (see (6.2.22))} \\ W \text{ is an arbitrary } \eta\text{-skew-symmetric matrix (see (7.1.47))} \end{cases}$$

and $\beta^{(t)}$ and η are matrices defined from $G^{(t)}(z)$ and $H(s)$ by (6.2.18) and (7.1.43).

The aim of the present section is to show the following theorem.

<u>Theorem 8.3.1</u>

(i) As $t \downarrow 0$,

$$\begin{cases} \hat{G}^{(t)}(z) \longrightarrow \hat{H}(s) & (2) \\ t^{-2n-1}\ \delta^{(t)} \longrightarrow \varepsilon\,. & (3) \end{cases}$$

(ii) $\hat{G}^{(t)}(z;U^{(t)})$ converges to a continuous-time system as $t \downarrow 0$ if and only if there exists a matrix W such that as $t \downarrow 0$

$$\{U^{(t)} - I\}/t \longrightarrow W. \quad (4)$$

The convergence (4) implies that W is η-skew-symmetric and that as $t \downarrow 0$

$$\hat{G}^{(t)}(z;U^{(t)}) \longrightarrow \hat{H}(s;W). \quad (5)$$

Corollary 8.3.2 In the case where p=1 (including the scalar case), both $\hat{G}^{(t)}(z)$ and $\hat{G}^{(t)}(z;1)$ converge to $\hat{H}(s)$, while $\hat{G}^{(t)}(z;-1)$ has no continuous-time limit.

The above corollary is immediately derived from Th.8.3.1 by recalling Rem.6.2.5 and Rem.7.1.8.

Let us proceed to prove the theorem. First, we note that (2) and (5) are equivalent to

$$\begin{cases} t^{-n}\,\hat{A}^{(t)}(z) \longrightarrow \hat{C}(s) & (6) \\ & (t \downarrow 0) \\ t^{-n}\,\hat{B}^{(t)}(z) \longrightarrow \hat{D}(s) & (7) \end{cases}$$

and

$$\begin{cases} t^{-n}\,\hat{A}^{(t)}(z;U^{(t)}) \longrightarrow \hat{C}(s;W) & (8) \\ & (t \downarrow 0) \\ t^{-n}\,\hat{B}^{(t)}(z;U^{(t)}) \longrightarrow \hat{D}(s;W) & (9) \end{cases}$$

respectively (see (8.1.19-20)). Since the numerators

$$\hat{B}^{(t)}(z), \quad \hat{B}^{(t)}(z;U^{(t)})$$

$$\hat{D}(s), \quad \hat{D}(s;W)$$

are the polynomial parts of

$$\hat{A}^{(t)}(z)G^{(t)}(z), \quad \hat{A}^{(t)}(z;U^{(t)})G^{(t)}(z)$$

$$\hat{C}(s)H(s), \quad \hat{C}(s;W)H(s)$$

respectively (see (6.1.8) and (7.1.11)), we see from Lemma 8.1.1 that (6) and (8) imply (7) and (9) respectively. Hence it suffices to verify (6) and (8).

For arbitrary $p \times p$ PM's $P(z)$ and $Q(z)$ we define

$$\langle P(z),Q(z)\rangle_t \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} G_P^{(t)}(e^{i\omega})\ G_Q^{(t)\prime}(e^{-i\omega})\ d\omega$$

where

$$G_P^{(t)}(z) \triangleq [P(z)G^{(t)}(z)]_{sp}\ , \qquad G_Q^{(t)}(z) \triangleq [Q(z)G^{(t)}(z)]_{sp}$$

as in (6.2.11). We also define as in (7.1.35)

$$\langle P(s),Q(s)\rangle \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} H_P(i\omega)\ H_Q'(-i\omega)\ d\omega$$

where

$$H_P(s) \triangleq [P(s)H(s)]_{sp}\ , \qquad H_Q(s) \triangleq [Q(s)H(s)]_{sp}\ .$$

According to Lemma 8.1.1 and Lemma 8.1.2, we have the following.

<u>Lemma 8.3.3</u> If

$$\begin{cases} P^{(t)}(z) \longrightarrow P(s) \\ Q^{(t)}(z) \longrightarrow Q(s) \end{cases} \qquad (t \downarrow 0)$$

then

$$t^{-1}\ \langle P^{(t)}(z),Q^{(t)}(z)\rangle_t \longrightarrow \langle P(s),Q(s)\rangle.$$

Now, noting that the normal equations (6.1.16) and (7.1.17), which determine $\hat{A}^{(t)}(z)$, $\delta^{(t)}$, $\hat{C}(s)$, and ε, are written as

$$\langle \hat{A}^{(t)}(z),z^jI\rangle_t = \begin{cases} 0 & \text{for } {}^\forall j=0,1,\ldots,n-1 \\ \delta^{(t)} & \text{for } j=n \end{cases} \tag{10}$$

$$\langle \hat{C}(s),s^jI\rangle = \begin{cases} 0 & \text{for } {}^\forall j=0,1,\ldots,n-1 \\ \varepsilon & \text{for } j=n, \end{cases} \tag{11}$$

we can verify (6) and (3) by an argument similar to the proof of Prop.8.2.4. The item (i) in the theorem is thus proved.

In order to prove (ii), it is necessary to investigate the relation between $\{E^{(t)}(z), F^{(t)}(z), \alpha^{(t)}, \beta^{(t)}, T^{(t)}\}$ defined by (6.2.14-20) and $\{K(s), L(s), \xi, \eta, R\}$ defined by (7.1.39-46). We claim that as $t \downarrow 0$

$$t^{-n+1}\ zE^{(t)}(z) \longrightarrow L(s) \tag{12}$$

$$t^{-n+1}\ F^{(t)}(z) \longrightarrow (-1)^{n-1}\ L(s) \tag{13}$$

$$t^{-n}\{\ zE^{(t)}(z) + (-1)^{n}F^{(t)}(z)\ \} \longrightarrow s^{n}I + K(s) \tag{14}$$

$$t^{-2n+1}\ \alpha^{(t)} \longrightarrow \eta \tag{15}$$

$$t^{-2n+1}\ \beta^{(t)} \longrightarrow \eta \tag{16}$$

$$t^{-2n}\{\alpha^{(t)} - \beta^{(t)}\} \longrightarrow \xi + \xi' - h_{n-1}h'_{n-1} \tag{17}$$

$$\{T^{(t)} - I\}\ /t \longrightarrow R. \tag{18}$$

The first two convergences (12) and (13) are readily proved by arguments similar to the proofs of Prop.8.2.4 and Prop.8.2.5, and (15) and (16) are immediate from (10) and (11). The convergence (14), which is equivalent to

$$\tilde{K}^{(t)}(s) \triangleq t^{-n}\{\ (1+ts)E^{(t)}(1+ts) + (-1)^{n}F^{(t)}(1+ts)\ \}$$

$$\longrightarrow s^{n}\ I + K(s),$$

is also proved similarly by noting that $\tilde{K}^{(t)}(s)$ is of the form

$$\tilde{K}^{(t)}(s) = s^{n}\ I + s^{n-1}\ \tilde{K}^{(t)}_{n-1} + \dots + \tilde{K}^{(t)}_{0}$$

and that

$$\tilde{K}^{(t)}_{n-1} - t\,\tilde{K}^{(t)}_{n-2} + \dots + (-t)^{n-1}\,\tilde{K}^{(t)}_{0} = 0.$$

(The last equation is derived from calculation of $\tilde{K}^{(t)}(-t^{-1})$.) We will prove (17) with the aid of the convergence (6). First, invoking that

$$\hat{C}^{(t)}(s) \triangleq t^{-n}\,\hat{A}^{(t)}(1+ts)$$

$$= s^n\,I + s^{n-1}\,\hat{C}^{(t)}_{n-1} + \dots + \hat{C}^{(t)}_{0}$$

$$\longrightarrow \hat{C}(s) = s^n\,I + s^{n-1}\,\hat{C}_{n-1} + \dots + \hat{C}_0$$

and that

$$\hat{A}^{(t)}_0 \triangleq \hat{A}^{(t)}(0) = t^n\,\hat{C}^{(t)}(-t^{-1})$$

$$= (-1)^n\,(I - t\,\hat{C}^{(t)}_{n-1} + t^2\,\hat{C}^{(t)}_{n-2} - \dots),$$

we have

$$\{(-1)^n\hat{A}^{(t)}_0 - I\}/t \longrightarrow -\hat{C}_{n-1}\,. \tag{19}$$

Next, we note that (cf.(6.2.16))

$$\delta^{(t)} = \alpha^{(t)} - \hat{A}^{(t)}_0\,\beta^{(t)}\,\hat{A}^{(t)\prime}_0, \tag{20}$$

which is a consequence of the fact that the solution $\hat{A}^{(t)}(z)$ of the equation (10) is also a solution of (6.2.12) and hence is written as (6.2.13). Rewriting (20) as

$$t^{-2n+1}\delta^{(t)} = t^{-2n+1}\alpha^{(t)} - (-1)^n\hat{A}^{(t)}_0\,t^{-2n+1}\beta^{(t)}\,(-1)^n\hat{A}^{(t)\prime}_0,$$

and calculating the right differential coefficients of the both sides at t=0 by the use of (3), (15), (16) and (19), we obtain

$$0 = \lim_{t \downarrow 0} \{ t^{-2n+1} \alpha^{(t)} - \eta \}/t - \lim_{t \downarrow 0} \{ t^{-2n+1} \beta^{(t)} - \eta \}/t$$

$$+ \hat{C}_{n-1} \eta + \eta \hat{C}'_{n-1}$$

or equivalently

$$t^{-2n} \{ \alpha^{(t)} - \beta^{(t)} \} \longrightarrow - (\hat{C}_{n-1} \eta + \eta \hat{C}'_{n-1}). \qquad (21)$$

On the other hand, noting that the solution $\hat{C}(s)$ of the equation (11) is also a solution of (7.1.37) and hence is written as (7.1.38), we have

$$0 = \langle \hat{C}(s), s^{n-1} I \rangle$$

$$= \langle s^n I, s^{n-1} I \rangle + \xi + \hat{C}_{n-1} \eta .$$

Therefore, recalling the property (7.1.36) of the inner product, we can see that (21) leads to (17). Finally, the remaining convergence (18) is derived from (15)-(17) by noting that $T^{(t)}$ is the unique $\beta^{(t)}$-positive-semidefinite matrix satisfying

$$T^{(t)} \beta^{(t)} T^{(t)\prime} = \alpha^{(t)} \qquad (22)$$

and that R is the unique η-symmetric matrix satisfying

$$R\eta + \eta R' = \xi + \xi' - h_{n-1} h'_{n-1} .$$

(Calculate the right differential coefficients of the both sides of $t^{-2n+1} \times (22)$ at $t=0$.)

Now let us prove the item (ii) of the theorem. Rewriting (6.2.13) as

$$\hat{A}^{(t)}(z;U^{(t)}) = \{zE^{(t)}(z) + (-1)^n F^{(t)}(z)\}$$

$$- \{(-1)^n \hat{A}_0^{(t)}(U^{(t)}) - I\}(-1)^{n-1} F^{(t)}(z)$$

where

$$\hat{A}_0^{(t)}(U^{(t)}) \triangleq \hat{A}^{(t)}(0;U^{(t)}),$$

and invoking (13) and (14), we see that the existence of the c.t. limit of $t^{-n}\hat{A}^{(t)}(z;U^{(t)})$ is equivalent to the existence of the limit of $\{(-1)^n\hat{A}_0^{(t)}(U^{(t)})-I\}/t$ as $t \downarrow 0$, which is also equivalent to the existence of W in (4) because of the equation

$$(-1)^n \hat{A}_0^{(t)}(U^{(t)}) = T^{(t)} U^{(t)} \qquad \text{(see (6.2.23))}$$

in which $T^{(t)}$ always satisfies (18). If W in (4) exists, then as $t \downarrow 0$

$$\{(-1)^n\hat{A}_0^{(t)}(U^{(t)}) - I\}/t \longrightarrow R + W \tag{23}$$

$$t^{-n}\hat{A}^{(t)}(z;U^{(t)}) \longrightarrow s^n I + K(s) - (R+W)L(s). \tag{24}$$

Furthermore, owing to the $\beta^{(t)}$-orthogonality (6.2.22) of $U^{(t)}$ and to the convergences (15) and (16), it turns out that W is η-skew-symmetric. This means that -(R+W) is the (n-1)-th degree coefficient matrix $\hat{C}_{n-1}(W)$ of the PM $\hat{C}(s;W)$ (see (7.1.45)). Hence, it follows from (7.1.38) that the limit in (24) is $\hat{C}(s;W)$, which yields (5). The item (ii) has thus been proved.

REFERENCES

[1] B.D.O. Anderson and R.R. Bitmead, "Stability of matrix polynomials," Int. J. Contr., vol.26, no.2, pp.235-247, 1977.

[2] B.D.O. Anderson, E.I. Jury and M. Mansour, "Schwarz matrix properties for continuous and discrete time systems," Int. J. Contr., vol.23, no.1, pp.1-16, 1976.

[3] B.D.O. Anderson and T. Kailath, "Forwards, backwards, and dynamically reversible Markovian models of second-order processes," IEEE Trans. Circuits and Syst., vol.CAS-26, no.11, pp.956-965, 1979.

[4] B.D.O. Anderson and R.E. Skelton, "The generation of all q-Markov Covers," to appear.

[5] B.D.O. Anderson and A.C. Tsoi, "Connecting forward and backward autoregressive models," IEEE Trans. Automat. Contr., vol.AC-29, no.10, 1984.

[6] B.D.O. Anderson and S. Vongpanitlerd, "Network analysis and synthesis," Prentice-Hall, 1973.

[7] E. Eitelberg, "Model reduction by minimizing the weighted equation error," Int. J. Contr., vol.34, no.6, pp.1113-1123, 1981.

[8] F.R. Gantmakher, "Theory of matrices," Chelsea Publishing Co. New York, 1959.

[9] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^{∞}-error bounds," Int. J. Contr., vol.39, no.6, pp.1115-1193, 1984.

[10] M.T. Hadidi, M. Morf and B. Porat, "Efficient construction of canonical ladder forms for vector autoregressive processes," IEEE Trans. Automat. Contr., vol.AC-27, no.6, pp.1222-1233, 1982.

[11] M.F. Hutton and B. F. Friedland, "Routh approximations for reducing order of linear, time-invariant systems," IEEE Trans. Automat. Contr., vol.AC-20, no.3, pp.329-337, 1975.

[12] Y. Inouye, "Approximation of multivariable linear systems with impulse response and autocorrelation sequences," Automatica, vol.19, no.3, pp.265-277, 1983.

[13] Y. Inouye, "Some notes on the second-order interpolation problem of digital filters," IEEE Trans. Acoust. Speech and Signal Process., vol.ASSP-31, no.1, pp.209-212, 1983.

[14] K. Ito, "Stationary random distributions," Mem. Coll. Sci. Univ. Kyoto, Ser.A, vol.28, pp.209-223, 1954.

[15] K. Ito, "Stochatic differentials," Appl. Math. Optim., vol.1, pp.347-381, 1975.

[16] T. Kailath, "A view of three decades of linear filtering theory," IEEE Trans. Inform. Theory, vol.IT-20, no.2, pp.146-181, 1974.

[17] T. Kailath, "Linear systems," Prentice-Hall, 1980.

[18] T. Kailath, "Linear estimations for stationary and near stationary processes," in Modern Signal Processing ed. by T. Kailath, Hemisphere Publishing Co., pp.59-128, 1985.

[19] R.E. Kalman, "Design of a self-optimizing control system," Trans. ASME, vol.80, pp.468-478, 1958.

[20] Y. Monden and S. Arimoto, "Generalized Rouche's theorem and its application to multivariable autoregressions," IEEE Trans. Accoust. Speech and Signal Process., vol.ASSP-28, no.6, pp.733-738, 1980.

[21] C.T. Mullis and R.A. Roberts, "The use of second-order information in the approximation of discrete-time linear systems," IEEE Trans. Acoust. Speech and Signal Process., vol.ASSP-24, no.3, pp.226-238, 1976.

[22] H. Nagaoka, "The Mullis-Roberts type approximation for continuous-time linear systems," (in Japanese,) Trans. IECE of Japan, vol.J69-A, no.6, pp.694-702, 1986.

[23] H. Nagaoka, Y. Monden and S. Arimoto, "An approach to the continuous-time stability criterion of polynomial matrices via orthogonal polynomial matrices," (in Japanese,)

Trans. IECE of Japan, vol.J68-A, no.6, pp.549-556, 1985.

[24] H. Nagaoka, Y. Monden and S. Arimoto, "The continuous-time limit of the discrete-time stability theory," Proc. ICASSP 86, pp.269-272, 1986.

[25] G. Obinata and H. Inooka, "A method for reducing the order of multivariable stochastic systems," IEEE Trans. Automat. Contr., vol.AC-22, no.4, pp.676-677, 1977.

[26] V. M. Popov, "Some properties of control systems with irreducible matrix transfer functions," in Lecture Notes in Mathematics, vol.144, Springer, Berlin, pp.169-180, 1969.

[27] K. Steiglitz and L.E. McBride, "A technique for the identification of linear systems," IEEE Trans. Automat. Contr., vol.AC-10, pp.461-464, 1965.