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Tests for Exponentiality under Random Censorship Model

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Abstract

The aim of this thesis is to present classes of test statistics for testing exponentiality against some alternatives with an aging property under a random censorship model. The exponential distribution is widely used in the fields of reliability, survival analysis and life testing because of its simple nature. The feature of the exponential distribution can be described as constant failure rate function or constant mean residual life function. This property presents a good description of the life length of a unit which does not age with time. But there are some situations that the occurrences of the initial failures and the wearout failures cause the changes of the failure rate function and the mean residual life function. Such aging properties give rise to the correspondings for the life distribution. By using the concepts of six nonparametric models for life distributions with the aging property, we consider six testing problems under the random censorship model.

Censored data arise naturally in many fields. The underlying test may be a destructive one so that units on test can not be re-used or, because of time and or cost constraint, we can not afford to wait indefinitely for all the units to fail. And as in a clinical trial, patients may enter the study at different times and leave, or die from a cause different from the one under investigation. Depending on the nature of the underlying tests, some types of censored data may be found, and we deal with the censored data observed under the random censorship model.

In Chapter 2 we give these notions on life distributions and types of censoring and we review in Chapter 3 some basic results from the theory of counting processes, martingale limit theorem and von Mises statistical functionals. Based on these mathematical foundations, we give an asymptotic theory of all proposed statistics presented in Chapters 4-6 with unified approach.

Each of the sections of these chapters discusses different testing problems under the random censorship model and proposes some classes of test statistics based on the Kaplan-Meier estimator. The asymptotic distribution of all proposals is derived under the null hypothesis and fixed distributions. And a consistent estimator of the asymptotic variance of each statistic under the null hypothesis is constructed from the theory of counting processes. The comparison of the tests on the basis of the Pitman asymptotic efficacy is also given for some alternatives under the proportional censoring model and we recommend one test from this result for each testing problem.

Contents

Chapter 1

Introduction

The exponential distribution is as widely used in reliability and life testing as the normal distribution is in other areas of statistics. One of the reasons is that the mathematics associated with the exponential distribution is relatively simple. In this respect the “lack-of-memory” property that the remaining life of a used unit whose life time is represented by exponential variable is independent of its initial age plays a central role.

This property can be expressed by a simple functional equation of the life distribution function (*df*) and presents a good description of the life length of a unit which does not age with time. But there are some situations that the occurrences of the initial failures and the wearout failures cause the changes of the failure rate function and the mean residual function, and such aging properties give rise to correspondings for the life *df*.

In Section 2.1 of Chapter 2 we define a variety of life distributions according to aging properties represented in nonparametric forms. These are: the *increasing failure rate* (IFR), the *increasing failure rate average* (IFRA), the *new better than used* (NBU), the *new better than used in expectation* (NBUE), the *decreasing mean residual life* (DMRL) and the *harmonic new better than used in expectation* (HNBUE) classes. Each of the notions of aging has a simple statistical interpretation and has a dual property by reversing the inequality or the direction of monotonicity. These are named DFR, DFRA, NWU, NWUE, IMRL and HNWUE, respectively. The IFR distribution has an increasing failure rate function and the IFRA distribution has an increasing failure rate average. The NBU property states that the conditional survival probability of a used but unfailed unit at any age is less than or equal to the corresponding probability of a new unit and the IFRA class is contained in the NBU class. The NBUE property is a weaker version than the NBU property, and says that the expected life length of a new unit is greater than or equal to the expected remaining life of a used but unfailed unit at any age. The DMRL distribution has a decreasing mean residual life function and the corresponding class contains the IFRA class, but is contained in the NBUE class. The notion of the HNBUE property may be interpreted as stating that the integral harmonic mean value of the mean residual life function at any age is less than or equal to the integral harmonic mean value of a new unit. The class of the HNBUE life distributions contains the above five classes and may be considered as a more natural class of the life *df*’s. In this thesis we consider these life distribution classes as the alternatives for testing exponentiality.

Section 2.2 of Chapter 2 defines three types of censoring: Type I, Type II and random censoring. Censoring is often occurred in survival analysis and life testing, and the censored sample contains only partial information about a population of interest. In Chapters 4-6 the problem of testing exponentiality is considered under the random censorship model.

In Section 3.1 of Chapter 3 we present the theory of counting process and martingale limit theorem in connection with the treatment of the censored data observed under the random censorship model. In the censored case we make statistical inferences about the population distribution $F(t)$ or about some functionals of $F(t)$ by the use of the Kaplan-Meier estimator $\widehat{F_n}(t)$. This Kaplan-Meier estimator may be considered as a generalization of the usual empirical *df* in the uncensored case. This fact together with the theory of counting processes and martingale limit theorem helps us to discuss the behavior of statistics based on the Kaplan-Meier estimator in the uncensored case, as well as in the censored case, with a unified approach. The normalized process of this estimator is known to be expressed as the stochastic integral with respect to the martingale generated from the censored data and this fact plays an important role in deriving the asymptotics of all the test statistics proposed in this thesis.

In order to study the asymptotic behavior of statistics that are functionals of the empirical *df*, von Mises [55] proposed a technique based on a form of Taylor expansion involving the derivatives of the functionals. In Section 3.2 of Chapter 3 we review the approach presented in Fernholz [16] based on the Hadamard differentiability of the functionals. Kumazawa [36], [38], [41], [42] used her method to derive the asymptotic behavior of the statistics represented as a functional of the Kaplan-Meier estimator $\widehat{F_n}(t)$.

In Section 4.1 of Chapter 4 the problem of testing exponentiality against the IFR alternatives is considered. The test statistic is constructed by using the property of the scaled total time on test (TTT-) transforms $H_F^{-1}(t)$ and was discussed in Kumazawa [45]. The concept of the scaled TTT-transforms introduced by Barlow and Campo [4] has proven to be very useful in the statistical analysis of reliability and life testing. The asymptotic distributions of the suitably normalized version of the statistic under the null hypothesis and fixed alternatives are given. The efficacy consideration of the test for some IFR alternatives under the proportional censoring model is also presented and it is shown that the efficacy decreases with the value of the expected proportion of observing the censored data.

We present in Section 4.2 of Chapter 4 two classes of test statistics for testing against the IFRA alternatives. The first class was proposed by Kumazawa [36] and includes the class of the statistics given by Deshpande [14] in the uncensored case. The second one is considered as a generalization of the statistic introduced in Kumazawa [45] that utilized the property of the TTT-transforms. The asymptotic distribution of the proposed statistics is derived and the comparison of the tests is made on the basis of the Pitman asymptotic efficacy.

Section 5.1 of Chapter 5 deals with the testing problem against the NBU alternatives. In Kumazawa [42] the Kaplan-Meier estimator was used to generalize the class of the statistics proposed in Koul [32]. The asymptotic distribution of the statistic with a weight function is shown and the efficacies of the statistics for some alternatives are computed. From the numerical evaluation of these efficacies one test is recommended.

For testing against the NBUE alternatives we present in Section 5.2 of Chapter 5 three statistics N_1, N_2 and N_3. The statistics N_1 and N_2 are constructed by the same method as given in the previous and were proposed by Kumazawa [37], [41]. In De Souza Borges, Proschan and Rodrigues [13] the N_1-statistic with constant weight function was considered in the uncensored case. And the N_2-statistic with constant weight function was considered by Hollander and Proschan

[20] in the uncensored case and by Koul and Susarla [34] in the censored case with a modified Kaplan-Meier estimator. The N_3-statistic is represented as a Kolmogorov-Smirnov type and was introduced in Kumazawa [47]. For the first two statistics N_1 and N_2 the asymptotic distribution is found to be normal, but the asymptotic distribution of the third one is shown to be not normal. So the efficacy comparison of the tests is made between the N_1- and N_2-tests, and we recommend the use of the test based on the one member of the class of the N_2-statistics.

In Section 6.1 of Chapter 6 two tests for exponentiality against the DMRL alternatives are given. The proposed test statistics P_1 and P_2 are constructed from the two measures of exponentiality towards DMRL-ness. The first measure is based on the property of the scaled TTT-transforms and the second one uses the notion of the definition. The P_1-statistic is a generalized version of the one introduced by Kumazawa [45] and the class of the P_2-statistics contains the one proposed in Bergman and Klefsjö [9], in which a modified Kaplan-Meier estimator was used and the proof given in there seems complicated. The asymptotic distributions of the statistics are given and the efficacies of the tests against some alternatives are presented to select the optimal test for the testing problem.

Finally, we consider in Section 6.2 of Chapter 6 the problem of testing exponentiality against the HNBUE alternatives. Bergman and Klefsjö [8] introduced the test statistics Q_1 and Q_2 based on the modified Kaplan-Meier estimator and Kumazawa [46] proposed the Q_3-statistics for this testing problem. We present proofs on the asymptotic distributions of these statistics by the theory of counting processes and martingale limit theorem. The efficacy consideration shows the use of the one member of the Q_2-statistics.

Chapter 2

Life Distributions and Types of Censoring

2.1 Life Distributions

We formulate a variety of life distributions based on notions of aging, which afford nonparametric statisticians an opportunity to consider inferences according to their probabilistic and geometrical properties.

Definition 2.1.1. *A life distribution* $F(t)$ *is a probability distribution satisfying* $F(t) = 0$ *for* $t < 0$. *The corresponding survival function is given by* $S(t) := \overline{F}(t) := 1 - F(t)$. *The function*

$$\Lambda(t) := \int_0^t \frac{dF(s)}{1 - F(s-)} \tag{2.1.1}$$

is called the hazard function associated with $F(t)$.

Note that when $F(t)$ has a density $f(t)$ and $S(t) > 0$,

$$\frac{d\Lambda(t)}{dt} = \frac{f(t)}{1 - F(t-)} := \lambda(t)$$

is referred to as the failure rate function. Here we may interpret $\lambda(t)dt$ as the probability that a unit alive at time t will fail in $[t, t + dt)$, where dt is small.

For a discussion of life distribution classes, we need the following notations:

$$\mu_F := \int_0^\infty S(t)dt;$$
$$\tau_F := \sup\{t : F(t) < 1\};$$
$$F_t(s) := \{F(t + s) - F(t)\}/S(t).$$

Definition 2.1.2. *(a)* $F(t)$ *is increasing failure rate (IFR) if* $\overline{F}_t(s)$ *is decreasing in* $t \in [0, \tau_F)$ *for each* $s \geq 0$.

(b) $F(t)$ *is increasing failure rate average (IFRA) if* $\Lambda(t)/t$ *is increasing in* $t \in [0, \tau_F)$.

(c) *$F(t)$ is new better than used (NBU) if $\overline{F}_t(s) \le \overline{F}(s)$ for all t and $s \in [0, \tau_F)$.*

(d) *$F(t)$ is new better than used in expectation (NBUE) if $F(t)$ has a finite mean μ_F and $\mu_F \ge e_F(t)$ for all $t \in [0, \tau_F)$, where*

$$e_F(t) := \begin{cases} \int_t^\infty S(s)ds/S(t) & \textit{if } S(t) > 0, \\ 0 & \textit{otherwise,} \end{cases} \tag{2.1.2}$$

and is called the mean residual life at age t.

(e) *$F(t)$ is decreasing mean residual life (DMRL) if $F(t)$ has a finite mean μ_F and $e_F(t)$ is decreasing in all $t \in [0, \tau_F)$.*

(f) *$F(t)$ is harmonic new better than used in expectation (HNBUE) if $F(t)$ has a finite mean μ_F and*

$$\int_t^\infty S(s)ds \le \mu_F \exp(-t/\mu_F) \quad \textit{for all } t \in [0, \tau_F).$$

By reversing the inequalities and the directions of monotonicity we get the six classes DFR, DFRA, NWU, NWUE, IMRL and HNWUE, respectively. Here D=*decreasing* , I=*increasing* and W=*worse* .

2.2 LIFE DISTRIBUTIONS

Different properties of the five classes IFR, IFRA, NBU, DMRL and NBUE and their duals were considered by authors such as Marshall and Proschan [50], Barlow and Proschan [7], Langberg, León and Proschan [48] and Hollander and Proschan [21]. The classes of the HNBUE and HNWUE life distributions were first introduced by Rolski [52] and investigated by Klefsjö [28], [30]. Here the chain of implications holds among these life distribution classes:

$$\begin{tikzcd} IFR \arrow[r, Rightarrow] \arrow[d, Rightarrow] & IFRA \arrow[r, Rightarrow] & NBU \arrow[d, Rightarrow] & \\ DMRL \arrow[rr, Rightarrow] & & NBUE \arrow[r, Rightarrow] & HNBUE \end{tikzcd}$$

In this thesis we consider these life distribution classes as the alternatives for testing exponentiality under the random censorship model defined in the next section.

2.3 Types of Censoring

Censored data arise naturally in a number of fields, particularly in problems of reliability and survival analysis, and contain only partial information about the population distribution of interest. We discuss three types of right censoring. To this end, let $X_1^\circ, X_2^\circ, \cdots, X_n^\circ$ be independently, identically distributed (*iid*) with life *df* $F(t)$.

(a) *Type I Censoring.* We assume that n units are put on test and we terminate our test at a predetemined time T, so that complete information on the first k order statistics

$$X_{(1)}^\circ \le X_{(2)}^\circ \le \cdots \le X_{(k)}^\circ$$

is available. Here the number k is an integer-valued random variable (*rv*) with

$$X^\circ_{(k)} \leq T \leq X^\circ_{(k+1)}.$$

Each of the remaining unobserved life times is known to be greater than the time T.

(b) *Type II Censoring.* As in Type I Censoring, n units are simultaneously put on test and we terminate our test after a predetermined number (or fraction) of failures are obtained. In this case we have complete information on the first r observations

$$X^\circ_{(1)} \leq X^\circ_{(2)} \leq \cdots \leq X^\circ_{(r)}$$

and the remaining observations are known to be greater than $X^\circ_{(r)}$. Here the number r (or r/n) is a fixed constant.

(c) *Random Censoring.* Let $U_1, U_2, \cdots, U_n$ be *iid* with possibly discontinuous and defective *df* $G(t)$. U_i is considered as the censoring times associated with X°_i and $G(t)$ is referred to as the censoring *df*. Then we can only observe (X_i, δ_i), $1 \leq i \leq n$, where

$$X_i := \min(X^\circ_i, U_i) \quad \text{and} \quad \delta_i := \mathbf{1}_{\{X^\circ_i \leq U_i\}}$$

with $\mathbf{1}_A$ the indicator of the set A. Here we assume that X°_i and U_i are stochastically independent. And in Chapters 4–6 we assume that the population *df* $F(t)$ is continuous. This random censorship model arises in medical applications with animal studies or clinical trials. In a clinical trial, patients may enter the study at different times: then each is treated with one of several possible therapies. We want to observe their life times, but censoring occurs according to loss to follow-up, drop-out and termination of the study.

Chapter 3

Counting Processes and von Mises Functionals

3.1 Counting Processes

It was demonstrated by Aalen [1] how the theory of multivariate counting processes gives a general framework in which both censored survival data and inhomogenuous Markov processes may be analyzed, and how by means of martingale central limit theorem the asymptotic behavior for the one- and the two-sample statistics and generalizations to censored data may be derived. Here we give the results from the theory of multivariate counting processes in connection with the treatment of the random censorship model.

Let $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t : t \in [0, \infty)\}$ be a fixed stochastic basis. A multivariate stochastic process $\mathbf{N}(t) = (N_1(t), N_2(t), \cdots, N_k(t))$ defined on the time interval $[0, \infty)$ is called a multivariate counting process if each of the k component processes $N_i(t)$ has a sample function which is a right-continuous step function with zero at time zero and with a finite number of jumps, each of size $+1$, and if furthermore two different component processes can not jump at the same time. Then by Theorem I.9 of Meyer [51], there exist right continuous, nondecreasing, predictable processes $A_i(t)$ with zero at time zero such that

$$M_i(t) := N_i(t) - A_i(t)$$

are local martingales for $i = 1, 2, \cdots, k$. The process $A_i(t)$ is called the compensator of $N_i(t)$.

Under the random censorship model described in Section 2.2, we can observe the possibly right censored data (X_i, δ_i), $1 \le i \le n$. Define stochastic process $N(t)$ on $[0, \infty)$ by

$$N(t) := \sum_{i=1}^{n} \mathbf{1}_{\{X_i \le t, \delta_i = 1\}} :$$

$N(t)$ represents the number of the uncensored units observed to failure at time t or earlier. By Lemma 2.3 of Gill [18],

$$M(t) := N(t) - \int_0^t Y(s) d\Lambda(s) \tag{3.1.1}$$

is a square integrable martingale on $[0, \infty)$, where

$$Y(t) := \sum_{i=1}^{n} \mathbf{1}_{\{X_i \geq t\}}. \tag{3.1.2}$$

Here the process $Y(t)$ represents the number of the units at risk at time t and the function $\Lambda(t)$ denotes the hazard function associated with the *df* $F(t)$ defined in the equation (2.1.1) of Section 2.1.

Under the random censorship model, we make statistical inferences about the population *df* $F(t)$ or about some functionals of $F(t)$ by the use of the Kaplan-Meier estimator $\widehat{F}_n(t)$ or the functionals of $\widehat{F}_n(t)$. The estimator $\widehat{F}_n(t)$ was first introduced in Kaplan and Meier [26] and is defined by

$$\widehat{F}_n(t) := 1 - \widehat{S}_n(t) := 1 - \prod_{s=0}^{t} \left\{ 1 - \frac{dN(s)}{Y(s)} \right\} \tag{3.1.3}$$

on the basis of the censored data (X_i, δ_i), $1 \leq i \leq n$. Note that when we get a complete sample the Kaplan-Meier estimator $\widehat{F}_n(t)$ reduces to the usual empirical *df*. The asymptotic behavior of $\widehat{F}_n(t)$ on the whole line is discussed by Gill [17], [18] using the theory of counting processes and martingale central limit theorem.

We present some results necessary to discuss the asymptotic distribution of the test statistics based on the Kaplan-Meier estimator $\widehat{F}_n(t)$ in the later chapters. For any process $W(t)$ we define the stopped process $W^T(t)$ by

$$W^T(t) := W(T \wedge t)$$

with $T = \max_{1 \leq i \leq n} X_i$.

Lemma 3.1.1. (Gill [18]) *For all t we have*

$$Z_n(t) : = n^{1/2} \frac{\widehat{F}_n(t) - F(t)}{S(t)} \tag{3.1.4}$$

$$= \int_0^t H_n(s) dM(s), \tag{3.1.5}$$

where

$$H_n(s) : = n^{1/2} \frac{\widehat{S}_n(s-) J(s)}{S(s) Y(s)} \tag{3.1.6}$$

and $J(s) := \mathbf{1}_{\{Y(s) > 0\}}$.

We denote by $D[0, t]$ the space of right continuous functions defined on the interval $[0, t]$ with left limits, with the Skorokhod metric topology.

Lemma 3.1.2. (Gill [18]) *Let* $h(t)$ *be a nonnegative continuous and nonincreasing function on the interval* $[0, \tau_H]$ *such that*

$$\int_0^{\tau_H} h^2(t) dC(t) < \infty,$$

where

$$C(t) := \int_0^t \frac{d\Lambda(s)}{1 - H(s-)}, \tag{3.1.7}$$

$$\tau_H := \sup\{t : H(t) < 1\}, \tag{3.1.8}$$

and

$$H(t) := 1 - S(t)\overline{G}(t).$$

Then the stochastic processes

$$(h(\cdot)Z_n(\cdot))^T, \quad (\int_0^{\cdot} h(t)dZ_n(t))^T, \quad and \quad (\int_0^{\cdot} Z_n(t)dh(t))^T$$

converge jointly in $D[0, \tau_H]$ *weakly as* $n \to \infty$ *to processes*

$$h(\cdot)Z(\cdot), \quad \int_0^{\cdot} h(t)dZ(t), \quad and \quad \int_0^{\cdot} Z(t)dh(t),$$

respectively, where $Z(t)$ *is a Gaussian process with zero mean and covariance function* $E\{Z(s)Z(t)\} = C(s \wedge t)$.

3.2 Von Mises Functionals

In order to study the asymptotic behavior of statistics that are functionals of the empirical *df*, von Mises [55] proposed a technique based on a form of Taylor expansion involving the derivatives of the functionals. The approach presented in Fernholz [16] was constructed on the basis of the Hadamard differentiability of the functionals, and Kumazawa [36], [38], [41], [42] used her method to derive the asymptotic behavior of the statistics represented as functionals of the Kaplan-Meier estimator $\widehat{F}_n(t)$ defined by the equation (3.1.3).

Let $T(F)$ be a functional based on $F \in \mathcal{D}$, a class of *df*'s. And let $\mathcal{V}$ and $\mathcal{W}$ be topological vector spaces and $\mathcal{L}(\mathcal{V}, \mathcal{W})$ the set of continuous linear transformations from $\mathcal{V}$ to $\mathcal{W}$. Let $\mathcal{S}$ be a class of compact subsets of $\mathcal{V}$ such that $\mathcal{S}$ contains all singletons, and let $\mathcal{A}$ be an open set of $\mathcal{V}$.

Definition 3.2.1. *A functional* $T : \mathcal{A} \to \mathcal{W}$ *is Hadamard differentiable at* $F \in \mathcal{A}$ *if there exists* $T'_F(\cdot) \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ *such that for any* $K \in \mathcal{S}$

$$\lim_{t \to 0} \frac{T(F + tH) - T(F) - T'_F(tH)}{t} = 0$$

uniformly for $H \in K$. *The linear transformation* $T'_F(\cdot)$ *is called the Hadamard derivative of* $T(\cdot)$ *at* F.

Then the following result follows from Theorem 4.4.2 of Fernholz [16].

Theorem 3.2.1. (Kumazawa [38]) *Suppose that* $\widetilde{F}_n(t)$ *is an estimator of population df* $F(t)$ *such that the stochastic process* $\{n^{1/2}[\widetilde{F}_n\{F^{-1}(t)\} - \widetilde{F}\{F^{-1}(t)\}] : 0 \le t \le 1\}$ *converges in* $D[0,1]$ *weakly as* $n \to \infty$ *to a continuous Gaussian process* $W(t)$ *with zero mean and continuous covariance function, where* $\widetilde{F}(t)$ *is a version (possibly stochastic) of* $F(t)$*. And suppose that the induced functional* $\tau(g) := T(g \circ F)$ *for* $g \in D[0,1]$ *is Hadamard differentiable at the identity function* $I(t) := t$ *with derivative* $\tau'_I(\cdot)$ *and that*

$$n^{1/2}\{T(\widetilde{F}) - T(F)\} \to 0 \textit{ in probability as } n \to \infty.$$

Then we have as $n \to \infty$

$$n^{1/2}\{T(\widetilde{F}_n) - T(F)\} \to_d N(0, \sigma^2)$$

provided $\sigma^2 = Var\{\tau'_I(W)\} > 0$.

Since most statistics such as L-, M- and R-statistics can be expressed as Hadamard differentiable functionals as shown in Fernholz [16], the above result would help us to derive the asymptotic normality of the statistics. The other forms of statistics based on the Kaplan-Meier estimator were discussed in Kumazawa [38].

Chapter 4

Tests for IFR and IFRA

4.1 The IFR Alternative

We consider to test the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

against the alternative

$$\mathcal{H}_1 : F(t) \text{ is IFR, but not exponential,}$$

on the basis of the possibly right censored data (X_i, δ_i), $1 \leq i \leq n$, defined in Section 2.2.

In analyzing the life distribution classes with the aging properties, Barlow and Campo [4] and Klefsjö [29] proved that the different forms of the aging properties can be expressed by their corresponding properties of the scaled total time on test (TTT-) transforms $H_F^{-1}(t)$, where

$$H_F^{-1}(t) := \frac{\int_0^{F^{-1}(t)} S(u)du}{\mu_F} \tag{4.1.1}$$

for $0 \leq t \leq 1$.

From the result of Barlow and Campo [4] we have that $H_F^{-1}(t) \equiv t$ for exponential distribution $F(t)$ and that $F(t)$ is IFR if and only if $H_F^{-1}(t)$ is concave on the unit interval $[0, 1]$. Using this property, Kumazawa [45] considered the measure

$$\begin{aligned}
&\int_0^1 \int_0^{1-t} \int_0^s su\{\frac{H_F^{-1}(t+u) - H_F^{-1}(t)}{u} - \frac{H_F^{-1}(t+s) - H_F^{-1}(t)}{s}\}dudsdt \\
&= \frac{\int_0^\infty F(t)S^2(t)\{1 - 2F(t)\}dt}{\mu_F}
\end{aligned}$$

of discrepancy between exponentiality and continuous IFR distribution, and proposed the test statistic

$$K_1 := \frac{\int_0^T \widehat{F}_n(t)\widehat{S}_n^2(t)\{1 - 2\widehat{F}_n(t)\}dt}{\widehat{\mu}_n}, \tag{4.1.2}$$

where

$$\widehat{\mu}_n := \int_0^T \widehat{S}_n(t)dt. \tag{4.1.3}$$

The statistic K_1 may be considered as a generalization of the test statistic

$$\sum_{j=0}^{n-2}\sum_{k=2}^{n-j}\sum_{\nu=1}^{k-1}\{k(D_{j+\nu} - D_j) - \nu(D_{j+k} - D_j)\}$$

introduced in Klefsjö [31] under the uncensored model which we have a complete sample Y_1, Y_2, $\cdots$, Y_n from $F(t)$, where

$$D_j := \frac{\int_0^{F_n^{-1}(j/n)} \overline{F}_n(t)dt}{\sum_{k=1}^n Y_k/n}$$

and

$$F_n(t) := \frac{\sum_{i=1}^n \mathbf{1}_{\{Y_i \le t\}}}{n}.$$

Since $\widehat{F}_n(T) < 1$ almost surely if the largest observation T of the X_i's is censored, the integral region in defining the statistic K_1 becomes to the finite random interval $[0, T]$. Note that we reject the null hypothesis $\mathcal{H}_0$ in favor of the alternative $\mathcal{H}_1$ for large values of the statistic K_1.

Theorem 4.1.1. (Kumazawa [45]) *Suppose that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} h_1^2(t)dC(t) < \infty \tag{4.1.4}$$

and

$$n^{1/2}h_1(T) \to 0 \text{ in probability as } n \to \infty, \tag{4.1.5}$$

where $C(t)$ and τ_H are defined in the equations (3.1.7) and (3.1.8) in Lemma 3.1.2 of Section 3.1, respectively, and

$$h_1(t) := \int_t^\infty S(u)du.$$

Let the function $g(t)$ be of form $\prod_{i=1}^{\ell}(t-\alpha_i)$ with $\alpha_1 = 0$ and any α_i, $2 \le i \le \ell$. Then the sequence of the rv's

$$n^{1/2}\left\{\frac{\int_0^T g\{\widehat{S}_n(t)\}dt}{\widehat{\mu}_n} - \frac{\mu_2}{\mu_F}\right\}$$

converges in distribution as $n \to \infty$ to a normal rv with zero mean and variance

$$\frac{\int_0^{\tau_H}\{\mu_2 h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4},$$

where

$$\mu_2 := \int_0^\infty g\{S(t)\}dt$$

and

$$h_2(t) := \int_t^\infty S(u)g'\{S(u)\}du.$$

Proof: We prove the convergence in distribution of the sequence of the *rv*'s

$$W_n := sn^{1/2}\Big[\int_0^T g\{\widehat{S}_n(u)\}du - \int_0^\infty g\{S(u)\}du\Big] + tn^{1/2}\Big\{\int_0^T \widehat{S}_n(u)du - \int_0^\infty S(u)du\Big\}$$

to an appropriate normal *rv* for any real numbers s and t according to the Cramér-Wold Device. We set

$$V_n := sn^{1/2}\int_0^T \Big[g\{\widehat{S}_n(u)\} - g\{S(u)\}\Big]du + tn^{1/2}\int_0^T \{\widehat{S}_n(u) - S(u)\}du.$$

Then we obtain for some constant $M > 0$ by the condition (4.1.5)

$$\begin{aligned} n^{1/2}|W_n - V_n| &= n^{1/2}|s\int_T^\infty g\{S(u)\}du + t\int_T^\infty S(u)du| \\ &\leq (M|s| + |t|)n^{1/2}h_1(T) = o_p(n^0) \quad \text{as } n \to \infty. \end{aligned}$$

By applying the formula

$$\prod_{i=1}^{\ell} a_i - \prod_{i=1}^{\ell} b_i = \sum_{k=1}^{\ell}(a_k - b_k)\prod_{i=k+1}^{\ell} a_i \prod_{j=1}^{k-1} b_j$$

to the *rv* V_n, we have

$$V_n = sn^{1/2}\sum_{k=1}^{\ell}\int_0^T \{F(u) - \widehat{F}_n(u)\}\prod_{i=k+1}^{\ell}\{\widehat{S}_n(u) - \alpha_i\}\prod_{j=1}^{k-1}\{S(u) - \alpha_j\}du + tn^{1/2}\int_0^T \{F(u) - \widehat{F}_n(u)\}du.$$

Since the Kaplan-Meier estimator $\widehat{F}_n(t)$ is uniformly consistent on the interval $[0, T]$ from the main result of Wang [56], the Slutsky's Theorem implies that V_n is asymptotically equivalent to

$$\begin{aligned} &sn^{1/2}\sum_{k=1}^{\ell}\int_0^T \{F(u) - \widehat{F}_n(u)\}\prod_{i=k+1}^{\ell}\{S(u) - \alpha_i\}\prod_{j=1}^{k-1}\{S(u) - \alpha_j\}du + tn^{1/2}\int_0^T \{F(u) - \widehat{F}_n(u)\}du \\ &= \int_0^T Z_n(u)dh_{s,t}(u), \end{aligned}$$

where $Z_n(u) = n^{1/2}\{\widehat{F}_n(u) - F(u)\}/S(u)$ and $h_{s,t}(u) := sh_2(u) + th_1(u)$. Hence Lemma 3.1.2 of Section 3.1 together with the condition (4.1.4) yields that

$$V_n \to_d N(0, \int_0^{\tau_H} h_{s,t}^2(u)dC(u)) \quad \text{as } n \to \infty.$$

Therefore the desired result follows from Corollary 3.1 of Serfling [53]. □

Corollary 4.1.1. (Kumazawa [45]) *Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ satisfies the conditions (4.1.4) and (4.1.5) of Theorem 4.1.1. Then we have as $n \to \infty$*

$$n^{1/2}\left\{\frac{\int_0^T g\{\widehat{S}_n(t)\}dt}{\widehat{\mu}_n} - \frac{\mu_2}{\mu_F}\right\} \to_d N(0, \sigma^2),$$

where

$$\sigma^2 := \int_0^\infty \Big[\mu_2 S(t) - \mu_F g\{S(t)\}\Big]^2 dC(t)/\mu_F^2.$$

This corollary shows that under the null hypothesis $\mathcal{H}_0$ the asymptotic variance of the suitably normalized version of the test statistic K_1 defined in the equation (4.1.2) is given by

$$\int_0^\infty g^2\{S(t)\}dC(t)$$

with $g(t) := t^2(1-t)(2t-1)$. Since this quantity depends on the unknown parameter μ and the censoring *df* $G(t)$, we must construct a consistent estimator from the censored observations (X_i, δ_i), $1 \le i \le n$. The same method as given in the proof of Lemma 2.4 of Kumazawa [42] shows that

$$\widehat{\sigma}_n^2 := \int_0^T g^2\{\widehat{S}_n(t-)\}d\widehat{C}(t)$$

is a consistent estimator of σ^2, where

$$\widehat{C}(t) := n\int_0^t \frac{J(u)}{Y(u)\{Y(u)-1\}}dN(u)$$

and $J(u)$ is given in Lemma 3.1.1 of Section 3.1. Hence the asymptotically exact test based on the statistic K_1 can be constructed by using this estimator $\widehat{\sigma}_n^2$.

Next we compute the efficacies of the test statistic K_1 against some alternatives under the proportional censoring model where the censoring *df* $G(t)$ is given by $\overline{G}(t) = S^\lambda(t)$ with censoring parameter λ: the value of λ has relation with the probability of obtaining uncensored observation, *i.e.*, $P(X_1 \le U_1) = \frac{1}{1+\lambda}$. In this situation Corollary 4.1.2 implies $\mu = 1$ without loss of generality and $0 < \lambda < 1$. And the asymptotic variance of the suitably normalized version of K_1 under $\mathcal{H}_0$ is given by

$$\sigma_\lambda^2 := \frac{4}{7-\lambda} - \frac{12}{6-\lambda} + \frac{13}{5-\lambda} - \frac{6}{4-\lambda} + \frac{1}{3-\lambda}.$$

Table. 4.1 : Efficacies of the statistic K_1 when the censoring parameter $\lambda = 0, \frac{1}{10}, \frac{1}{2}, \frac{3}{4}$

	Alternative			
λ	*(i)*	*(ii)*	*(iii)*	*(iv)*
0	.72834	.36459	.05833	.19632
$\frac{1}{10}$	.66199	.33138	.05302	.17843
$\frac{1}{2}$	.42453	.21251	.03400	.11443
$\frac{3}{4}$	.30424	.15229	.02437	.08200

The following IFR life *df*'s are considered as the alternative for testing exponentiality.

(i) $1 - \exp(-t^{\theta+1})$ (Weibull),

(ii) $1 - \exp\{-(t + \theta t^2/2)\}$ (Linear failure rate),

(iii) $1 - \exp\left[-\{t + \theta(t + e^{-t} - 1)\}\right]$ (Makeham),

and

(iv) $\int_0^t s^\theta e^{-s} ds/\Gamma(1+\theta)$ (Gamma),

where $t \geq 0$, $\theta \geq 0$ and the null distribution $\mathcal{H}_0$ with $\mu = 1$ is obtained when $\theta = 0$. Since each of the families $\{F_\theta(t)\}$ of the alternatives (i)-(iv) listed in the above satisfies the conditions (A.1)-(A.4) given by Kumazawa [40], it is seen that the sequence $\{F_{\theta_n}(t)\}$ with $\theta_n = cn^{-1/2}$ and $c > 0$ is contiguous to the null *df* and that the efficacy of the test statistic K_1 is equal to

$$eff(K_1) := \lim_{n\to\infty} \left\{ \frac{dE_\theta[K_1]}{d\theta}\bigg|_{\theta=0} \right\}^2 /(nVar_0[K_1]),$$

where $E_\theta[\cdot]$ denotes the expectation under the *df* F_θ and $Var_0[\cdot]$ the variance under the null hypothesis $\mathcal{H}_0$.

After some calculations we obtain that for the alternative (i)

$$eff(K_1) = (-\frac{3}{2}\ln 2 + \ln 3)^2/\sigma_\lambda^2,$$

for (ii)

$$eff(K_1) = (\frac{1}{24})^2/\sigma_\lambda^2,$$

for (iii)

$$eff(K_1) = (\frac{1}{60})^2/\sigma_\lambda^2$$

and for (iv)

$$eff(K_1) = (-\frac{7}{3}\ln 2 + \frac{3}{2}\ln 3)^2/\sigma_\lambda^2.$$

Table 4.1 shows the efficacies of the test statistic K_1 for the alternatives (i)-(iv) and some values of the censoring parameter λ. The above results reveal that the efficacy decreases with the value of λ.

4.2 The IFRA Alternative

For the problem of testing the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

versus the alternative

$$\mathcal{H}_2 : F(t) \text{ is IFRA, but not exponential,}$$

we may consider the following two measures of exponentiality against the IFRA *df*'s: for nondecreasing function $\psi_1(t) \geq 0$ and constant $\beta > 1$,

$$\Delta_1 := \int_0^\infty \psi_1\{S(\beta t)\}dF(t)$$

and for nonnegative function $\psi_2(t)$,

$$\Delta_2 := \frac{\int_0^\infty \Psi_2\{S(t)\}\int_0^t S(u)dudF(t)}{\mu_F},$$

where

$$\Psi_2(t) := \psi_2(1-t)\{\int_0^1 s\psi_2(s)ds - 2\int_0^{1-t} s\psi_2(s)ds\}.$$

The first measure Δ_1 relies on the fact that $F(t)$ is IFRA if and only if for all $\beta > 1$

$$S^\beta(t) \geq S(\beta t) \quad \text{for any } t \geq 0.$$

Since the scaled TTT-transforms $H_F^{-1}(t)$ defined in the equation (4.1.1) of Section 4.1 share the property that $H_F^{-1}(t)/t$ is decreasing in $t \in [0,1]$ for the IFRA *df*'s from Theorem 2.1 of Barlow and Campo [4], we may consider the measure

$$\int_0^1 \int_s^1 st\psi_2(s)\psi_2(t)\{\frac{H_F^{-1}(s)}{s} - \frac{H_F^{-1}(t)}{t}\}dtds$$
$$= \frac{\int_0^\infty \Psi_2\{S(t)\}\int_0^t S(s)dsdF(t)}{\mu_F} = \Delta_2$$

for continuous IFRA *df*'s with positive weight function $\psi_2(t)$.

Now from these measures we obtain two classes of test statistics

$$L_1(\psi_1,\beta) := \int_0^T \psi_1\{\widehat{S}_n(\beta t)\}d\widehat{F}_n(t) \tag{4.2.1}$$

and

$$L_2(\psi_2) := \frac{\int_0^T \Psi_2\{\widehat{S}_n(t)\}\int_0^t \widehat{S}_n(u)dud\widehat{F}_n(t)}{\widehat{\mu}_n} \tag{4.2.2}$$

by substituting the Kaplan-Meier estimator $\widehat{F}_n(t)$ for $F(t)$.

Under the uncensored model Deshpande [14] studied the statistic $L_1(\psi_1,\beta)$ with $\psi_1(t) = t$ for the above testing problem. And the statistic $L_1(\psi_1,\beta)$ was introduced by Kumazawa [36] and is a version of the test statistic proposed by Kumazawa [35] for testing exponentiality against the NBU alternatives in the uncensored case with β integer ≥ 2. We reject the null hypothesis $\mathcal{H}_0$ in favor of the alternative $\mathcal{H}_2$ for small values of $L_1(\psi_1,\beta)$.

And the statistic $L_2(\psi_2)$ is an extended version of the test statistic $L_2(\psi_2)$ with $\psi_2(t) \equiv constant$, proposed in Kumazawa [45], by introducing the weight function $\psi_2(t)$ in the measure Δ_2. In the uncensored data Klefsjö [31] investigated the testing problem on the basis of the same property of the scaled TTT-transforms and proposed the test statistic, which is seen to be asymptotically equivalent to $L_2(\psi_2)$ with $\psi_2(t) \equiv constant$ in the uncensored case. Here we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_2$ for large values of $L_2(\psi_2)$.

For testing against the IFRA alternatives, Barlow and Proschan [6] proposed the test statistics based on the normalized spacings which are generalized to treat the censored observations under the Type II censoring model, and proved unbiasedness of the test against the alternatives.

Theorem 4.2.1. (Kumazawa [36]) *Suppose that the weight function $\psi_1(t)$ is continuous and piecewise differentiable with bounded derivatives. And suppose that the df $F(t)$ is absolutely continuous and that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} S^2(t)dC(t) < \infty \tag{4.2.3}$$

and

$$n^{1/2}\psi_1\{S(T)\}S(T/\beta) \to 0 \textit{ in probability as } n \to \infty. \tag{4.2.4}$$

Then the sequence of the rv's $n^{1/2}\{L_1(\psi_1,\beta) - W(F)\}$ converges in distribution as $n \to \infty$ to a normal rv B with zero mean and variance $E[B^2]$, where

$$W(F) := \int_0^\infty \psi_1\{S(\beta t)\}dF(t),$$

$$B := -\int_0^\infty Z(\beta t)S(\beta t)\psi_1'\{S(\beta t)\}dF(t) + \int_0^\infty Z(t/\beta)S(t/\beta)\psi_1'\{S(t)\}dF(t),$$

and $Z(t)$ is the limiting process of the normalized Kaplan-Meier process $Z_n(t)$ given in Lemma 3.1.2 of Section 3.1.

Proof: Following Theorem 3.2.2 of Section 3.2, we first show that the induced functional $\tau(g) := W(g \circ F)$ for $g \in D[0,1]$ can be expressed as a composition of Hadamard differentiable transformations. For fixed $F(t)$, $\psi_1(t)$ and β, we define

$$\gamma_1(g_1)(s) := \beta F^{-1} \circ g_1^{\star}(s),$$
$$\gamma_2(g_1, g_2)(s) := \psi_1[1 - g_1 \circ F\{g_2(s)\}]$$

and

$$\gamma_3(g_1) := \int_0^1 g_1(u)du,$$

where $g_1 \in D[0,1]$, $g_2 \in L^1[0,1]$, $0 \leq s \leq 1$, $F^{-1}(s) := \inf\{t : F(t) \geq s\}$ and $g_1^{\star}(s) := \inf\{t, 1 : g(t) \geq s\}$. Then from Propositions 6.1.1, 6.1.2 and 6.1.6 of Fernholz [16] the above transformations $\gamma_1(\cdot) - \gamma_3(\cdot)$ are all Hadamard differentiable at $I(t)$. Therefore $\tau(g) = \gamma_3 \circ \gamma_2\{g, \gamma_1(g)\}$ is Hadamard differentiable at $I(t)$ by the chain rule of Proposition 3.1.2 of Fernholz [16].

Next we note that

$$\begin{aligned}
&n^{1/2}|\{L_1(\psi_1,\beta) - W(F)\} - \{W(\widehat{F}_n^T) - W(F^T)\}| \\
&= n^{1/2}|\int_{T/\beta}^{\infty} \psi_1\{S(\beta x)\}dF(x) - \int_{T/\beta}^{T} \psi_1\{S(T)\}dF(x)| \\
&\leq 2n^{1/2}\psi_1\{S(T)\}S(T/\beta) \\
&= o_p(n^0) \quad \text{as } n \to \infty.
\end{aligned}$$

Hence Theorem 3.2.2 of Section 3.2 together with some calculations yields the desired result. □

We consider the weight function $\psi_1(u) = u^{\alpha}$ as a special case.

Corollary 4.2.1. (Kumazawa [36]) *Let $\psi_1(u) = u^{\alpha}$, $\alpha \geq 1$. Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ satisfies the conditions (4.2.3) and (4.2.4) of Theorem 4.2.1. Then $n^{1/2}\{L_1(u^{\alpha},\beta) - \upsilon^{-1}\}$ converges in distribution as $n \to \infty$ to a normal distribution with mean zero and variance*

$$\sigma^2_{\alpha,\beta} := \int_0^{\infty} f_{\alpha,\beta}\{S(t)\}dC(t),$$

where

$$f_{\alpha,\beta}(t) := (\alpha/\upsilon)^2[\beta^2 t^{2\upsilon} - 2\beta t^{(\beta+1)\upsilon/\beta} + t^{2\upsilon/\beta}]$$

and $\upsilon := \alpha\beta + 1$.

Because of the dependency of the statistic $L_1(u^{\alpha},\beta)$, $\alpha \geq 1$, $\beta > 1$, on the unknowns μ and $G(t)$, we must estimate the asymptotic variance $\sigma^2_{\alpha,\beta}$ from the observations (X_i, δ_i), $1 \leq i \leq n$. To this end, we set

$$\widehat{\sigma}^2_{\alpha,\beta} := \int_0^{T} f_{\alpha,\beta}\{\widehat{S}_n(t-)\}d\widehat{C}(t).$$

Then it is seen that $\widehat{\sigma}^2_{\alpha,\beta}$ is a consistent estimator of $\sigma^2_{\alpha,\beta}$ by the same method as given in Section 4.1. Hence the test rejecting $\mathcal{H}_0$ in favor of $\mathcal{H}_2$ for

$n^{1/2}\{L_1(u^\alpha, \beta) - \upsilon^{-1}\}/\widehat{\sigma}_{\alpha,\beta} < z_\eta$ is consistent against all continuous IFRA alternatives, where z_η is the η -percentile of a standard normal distribution.

Next in order to derive the asymptotic distribution of the statistic $L_2(\psi_2)$, we discuss the asymptotic distribution of the statistic in the form

$$T_n(\psi) := \frac{\int_0^T \psi\{\widehat{S}_n(t)\} \int_0^t \widehat{S}_n(u)du d\widehat{F}_n(t)}{\widehat{\mu}_n}, \tag{4.2.5}$$

where $\widehat{\mu}_n$ is defined in the equation (4.1.3). Some test statistics proposed in this thesis can be expressed as this form and we apply the following result to investigate their asymptotics.

Theorem 4.2.2. (Kumazawa [41]) *Suppose that the df $F(t)$ is absolutely continuous and that the df's $F(t)$, $G(t)$ and the weight function $\psi(t)$ satisfy the conditions*

$$\int_0^\infty \Big[\psi\{S(s)\}S(s) + \Psi\{S(s)\}\Big]ds < \infty, \tag{4.2.6}$$

$$n^{1/2}\Psi\{S(T)\} \to 0 \quad \textit{in probability as } n \to \infty, \tag{4.2.7}$$

$$\int_0^{\tau_H} h_i^2(t)dC(t) < \infty, \tag{4.2.8}$$

and

$$n^{1/2}h_i(T) \to 0 \quad \textit{in probability as } n \to \infty \tag{4.2.9}$$

for $i = 1$ and 2, where

$$\Psi(t) := \int_0^t \psi(u)du,$$

$$h_1(t) := \int_t^\infty S(u)du,$$

and

$$h_2(t) := \int_t^\infty \Big[\psi\{S(s)\}S(s) + \Psi\{S(s)\}\Big]S(s)ds.$$

Then we have as $n \to \infty$

$$n^{1/2}\left\{T_n(\psi) - \frac{\mu_\psi}{\mu_F}\right\} \to_d N(0, \sigma_\psi^2),$$

where

$$\mu_\psi := \int_0^\infty \Psi\{S(s)\}S(s)ds \tag{4.2.10}$$

and

$$\sigma_\psi^2 := \frac{\int_0^{\tau_H}\{\mu_\psi h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4}. \tag{4.2.11}$$

Proof: We first show that the random vector

$$A_n := n^{1/2}\Big(\int_0^T \widehat{S}_n(t)dt - \mu_F, \int_0^T \psi\{\widehat{S}_n(s)\}\int_0^s \widehat{S}_n(u)dud\widehat{F}_n(s) - \mu_2\Big)$$

is asymptotically equivalent to

$$B_n := -\Big(\int_0^T h_1(t)H_n(t)dM(t), \int_0^T h_2(t)H_n(t)dM(t)\Big),$$

where $M(t)$ and $H_n(t)$ are defined in the equations (3.1.1) and (3.1.6) in Section 3.1, respectively.

We set

$$W_n := \int_0^T \psi\{\widehat{S}_n(s)\}\int_0^s \widehat{S}_n(u)dud\widehat{F}_n(s)$$

and

$$W(F) := \int_0^\infty \psi\{S(s)\}\int_0^s S(u)dudF(s)$$

to investigate the asymptotic behavior of the second component of the random vector A_n. For fixed $F(t)$ and $\psi(t)$, we define

$$\gamma_1(g)(s) := F^{-1} \circ g^\star(s)$$

and

$$\gamma_2(g) := \int_0^1 \Psi(1-t)(1-t)dg(t)$$

for $s \in [0,1]$ and $g \in D[0,1]$, where $g^\star(s) = \inf\{t, 1 : g(t) \geq s\}$. Since the transformations $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are Hadamard differentiable at $I(t)$ from Proposition 6.1.1 of Fernholz [16], the functional $\tau(\cdot)$ induced on $D[0,1]$ by $\tau(g) := W(g \circ F)$ for $g \in D[0,1]$ is Hadamard differentiable at $I(t)$ by the chain rule and the expression that $\tau(g) = \gamma_2\{\gamma_1(g)\}$. Note that the derivative $\tau_I'(g)$ of $\tau(g)$ at $I(t)$ is given by

$$\tau_I'(g) = -\int_0^\infty [\psi\{S(s)\}S(s) + \Psi\{S(s)\}]g \circ F(s)ds.$$

We obtain from the conditions (4.2.7) and (4.2.9)

$$\begin{aligned}
&n^{1/2}|\{W_n - W(F)\} - \{W(\widehat{F}_n^T) - W\{T(F^T)\}|\\
&= n^{1/2}\int_T^\infty \psi\{S(s)\}\int_0^s S(u)dudF(s)\\
&= n^{1/2}\Psi\{S(T)\}\int_0^T S(s)ds + n^{1/2}\int_T^\infty \Psi\{S(s)\}S(s)ds\\
&\leq n^{1/2}\mu_F\Psi\{S(T)\} + n^{1/2}\int_T^\infty \Psi\{S(s)\}S(s)ds\\
&= o_p(n^0) \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore the Hadamard differentiability of $\tau(\cdot)$ implies that as $n \to \infty$

$$\begin{aligned} n^{1/2}\{W_n - W(F)\} &= \tau_I'(\phi_n) + o_p(n^0) \\ &= h_2(T)Z_n(T) - \int_0^T h_2(t)H_n(t)dM(t) + o_p(n^0), \end{aligned}$$

where

$$\phi_n(t) := n^{1/2}\{\widehat{F}_n^T \circ F^{-1}(t) - F^T \circ F^{-1}(t)\} \quad \text{for } 0 \le t \le 1.$$

Hence Remark 2.2 of Gill [18] yields that

$$n^{1/2}\{W_n - W(F)\} = -\int_0^T h_2(t)H_n(t)dM(t) + o_p(n^0) \quad \text{as } n \to \infty.$$

By applying Lemma 3.1.1 of Section 3.1 to the first component of the random vector A_n, it is seen that

$$n^{1/2}\{\int_0^T \widehat{S}_n(s)ds - \mu_F\} = -\int_0^T h_1(t)H_n(t)dM(t) + o_p(n^0) \quad \text{as } n \to \infty.$$

Therefore the random vector A_n is asymptotically equivalent to B_n from Theorem 4.4 of Billingsley [10].

Since each component of the random vector B_n is represented as the stochastic integral with respect to the square integrable martingale $M(t)$, and the functions $h_i(t)$'s and the process $H_n(t)$ are predictable, Theorem 2.1 of Andersen *et al.* [2] together with the condition (4.2.8) implies that B_n converges in distribution as $n \to \infty$ to a normal distribution with zero mean vector and dispersion matrix $\{\sigma_{i,j}\}_{1 \le i,j \le 2}$, where

$$\sigma_{i,j} := \int_0^{\tau_H} h_i(t)h_j(t)dC(t).$$

Hence we can conclude the proof from Corollary 3.3 of Serfling [53] and some calculations. □

Corollary 4.2.2. *Suppose that the df's $F(t)$, $G(t)$ and the function $\Psi_2(t)$ satisfy the conditions of Theorem 4.2.3. Then we have as $n \to \infty$*

$$n^{1/2}\left\{L_2(\psi_2) - \frac{\mu_{\Psi_2}}{\mu_F}\right\} \to_d N(0, \sigma^2_{\Psi_2}),$$

where μ_{Ψ_2} and $\sigma^2_{\Psi_2}$ denote the correspondings to those given in the equations (4.2.10) and (4.2.11) of Theorem 4.2.3 for the function $\Psi_2(t)$, respectively.

Corollary 4.2.3. *Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ and the function $\Psi_2(t)$ satisfy the conditions (4.2.6)-(4.2.9) of Theorem 4.2.3. Then $n^{1/2}L_2(\psi_2)$ converges in distribution as $n \to \infty$ to a normal distribution with mean zero and variance*

$$\sigma^2_{\psi_2} : = \int_0^\infty \varphi^2\{S(t)\}S^2(t)dC(t),$$

where

$$\varphi(t) : = \int_0^t \Psi_2(s)ds.$$

This corollary shows that under the null hypothesis the asymptotic variance $\sigma^2_{\psi_2}$ of the statistic $L_2(\psi_2)$ defined in the equation (4.2.2) depends on the unknowns μ and $G(t)$. Similarly as in the case of the statistic $L_1(\psi_1,\beta)$ we can find a consistent estimator

$$\widehat{\sigma}^2_n := \int_0^T \varphi^2\{\widehat{S}_n(t-)\}\widehat{S}^2_n(t-)d\widehat{C}(t).$$

This helps us to construct the asymptotically exact test based on $L_2(\psi_2)$.

Now we compare the efficacy of the test statistics $L_1(\psi_1,\beta)$, $\alpha \geq 1$, $\beta > 1$, and $L_2(\psi_2)$ for the alternatives (i)-(iv) listed in Section 4.1 under the proportional censoring model with the censoring parameter λ. In this situation from Corollaries 4.2.2 and 4.2.5 we may take $\mu = 1$ and $0 < \lambda < 1$. And the asymptotic variances of the suitably normalized versions of the statistics $L_1(u^\alpha,\beta)$ and $L_2(\psi_2)$ under $\mathcal{H}_0$ are given by

$$\sigma_1^2 := \beta\left(\frac{\alpha}{\upsilon}\right)^2\left[\frac{\beta}{2\alpha\beta+1-\lambda} + \frac{1}{2\upsilon-\beta(\lambda+1)} - \frac{2\beta}{\upsilon(\beta+1)-\beta(\lambda+1)}\right]$$

and

$$\sigma_2^2 := \int_0^1 \varphi^2(t)t^{-\lambda}dt,$$

respectively. Then we have the following efficacies of the statistics $L_1(u^\alpha,\beta)$ against the alternatives: for (i)

$$eff\{L_1(u^\alpha,\beta)\} = (\alpha\beta\ln\beta)^2/(\upsilon^4\sigma_1^2),$$

for (ii)

$$eff\{L_1(u^\alpha,\beta)\} = \{\alpha\beta(\beta-1)\}^2/(\upsilon^4\sigma_1^2),$$

for (iii)

$$eff\{L_1(u^\alpha,\beta)\} = \{\frac{\alpha+1}{\upsilon} - \frac{2}{\upsilon+1} - \frac{\alpha}{\upsilon+\beta}\}^2/\sigma_1^2$$

and for (iv)

$$eff\{L_1(u^\alpha,\beta)\} = \frac{\{(\beta-1)\ln\upsilon - \alpha\beta\ln\beta\}^2}{\upsilon^2(\upsilon-\beta)^2\sigma_1^2}.$$

Table. 4.2 : Efficacies of the IFRA-test statistics when the censoring parameter $\lambda = \frac{1}{10}$ and $\frac{3}{4}$

	Alternative			
Statistic	*(i)*	*(ii)*	*(iii)*	*(iv)*
	$\lambda = 1/10$			
$L_1(u^1, 1.03)$	1.2677	0.3168	0.0626	0.4775
$L_1(u^1, 1.95)$	1.2841	0.2985	0.0606	0.4967
$L_1(u^{1.5}, 3.42)$	1.0506	0.1083	0.0329	0.4981
$L_2(u^0)$	1.1945	0.2404	0.0554	0.4729
$L_2(u^1)$	1.0810	0.3594	0.0643	0.5756
	$\lambda = 3/4$			
$L_1(u^1, 1.03)$	0.7012	0.1752	0.0346	0.2641
$L_1(u^1, 1.95)$	0.7421	0.1725	0.0348	0.2870
$L_1(u^{1.5}, 3.42)$	0.8561	0.1600	0.0268	0.4058
$L_2(u^0)$	0.7545	0.1518	0.0349	0.2987
$L_2(u^1)$	0.5184	0.1728	0.0308	0.2760

For the test statistic $L_2(\psi_2)$ we consider the weight function to be of form $\psi_2(u) = u^\rho, \rho > -1$. Corollary 4.2.5 implies that

$$\begin{aligned}\sigma^2_{2,\rho} :=& \frac{1}{(\rho+2)^2}\Bigg(\frac{1}{(\rho+1)^2(2\rho+3)^2(1-\lambda)} + \frac{B(1-\lambda, 2\rho+3)}{(\rho+1)^2} + \\ &+ \frac{4B(1-\lambda, 4\rho+7)}{(2\rho+3)^2} - \frac{2B(1-\lambda, \rho+2)}{(\rho+1)^2(2\rho+3)} + \\ &+ \frac{4B(1-\lambda, 2\rho+4)}{(\rho+1)(2\rho+3)^2} - \frac{4B(1-\lambda, 3\rho+5)}{(\rho+1)(2\rho+3)}\Bigg),\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the Beta function. Here we consider the two test statistics $L_2(u^0)$ and $L_2(u^1)$. Kumazawa [45] discussed the statistic A_2^c equivalent to $L_2(u^0)$. Then the asymptotic variances of the suitably normalized versions of $L_2(u^0)$ and $L_2(u^1)$ under $\mathcal{H}_0$ are given by

$$\sigma^2_{2,0} = \frac{1}{9(7-\lambda)} - \frac{2}{3(6-\lambda)} + \frac{4}{3(5-\lambda)} - \frac{1}{4-\lambda} + \frac{1}{4(3-\lambda)}$$

and

$$\begin{aligned}\sigma^2_{2,1} =& \frac{1}{9}\Bigg(\frac{1}{100(1-\lambda)} + \frac{B(1-\lambda, 5)}{4} + + \frac{4B(1-\lambda, 11)}{25} - \\ &- \frac{B(1-\lambda, 3)}{10} + \frac{2B(1-\lambda, 6)}{25} - \frac{2B(1-\lambda, 8)}{5}\Bigg),\end{aligned}$$

respectively.

And the efficacies of $L_2(u^0)$ and $L_2(u^1)$ are given: for the alternative (i)

$$eff\{L_2(u^0)\} = (-\frac{5}{2}\ln 2 + 2\ln 3)^2/\sigma^2_{2,0},$$
$$eff\{L_2(u^1)\} = (-\frac{77}{90}\ln 2 + \frac{33}{90}\ln 3 + \frac{2}{15}\ln 5)^2/\sigma^2_{2,1},$$

for (ii)

$$eff\{L_2(u^0)\} = (\frac{5}{24})^2/\sigma^2_{2,0},$$
$$eff\{L_2(u^1)\} = (\frac{19}{1350})^2/\sigma^2_{2,1},$$

for (iii)

$$eff\{L_2(u^0)\} = (\frac{1}{10})^2/\sigma^2_{2,0},$$
$$eff\{L_2(u^1)\} = (\frac{15}{2520})^2/\sigma^2_{2,1},$$

and for (iv)

$$eff\{L_2(u^0)\} = (-\frac{10}{3}\ln 2 + 3\ln 3)^2/\sigma^2_{2,0},$$
$$eff\{L_2(u^1)\} = (-\frac{281}{225}\ln 2 + \frac{14}{25}\ln 3 + \frac{1}{6}\ln 5)^2/\sigma^2_{2,1},$$

respectively.

Table 4.2 shows the efficacies of the IFRA-test statistics $L_1(u^1, 1.03)$, $L_1(u^1, 1.95)$, $L(u^{1.5}, 3.42)$ and $L_2(u^0)$ for the alternatives listed in Section 4.1 and some values of the censoring parameter λ. For the statistic $L_1(u^\alpha, \beta)$ we choose the values of α and β so as to maximize its efficacy against a particular alternative. Since $\overline{\delta}_n := \sum_{i=1}^n \delta_i/n$ is an estimate of $P(X_1 \leq U_1) = \frac{1}{1+\lambda}$, we recommend the $L_1(u^1, 1.95)$-test statistic for small values of $\overline{\delta}_n$ and the $L_1(u^1, 1.03)$-test statistic for large values of $\overline{\delta}_n$ in the sense of the Pitman asymptotic relative efficiency.

Chapter 5

Tests for NBU and NBUE

5.1 The NBU Alternative

In this section we are interested in testing the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

versus the alternative

$$\mathcal{H}_3 : F(t) \text{ is NBU, but not exponential,}$$

under the random censorship model.

Koul [32] considered the parameter

$$\begin{aligned} &\int_0^\infty \int_0^\infty [\psi\{S(s)\}\psi\{S(t)\} - \psi\{S(s+t)\}]dF(s)dF(t) \\ &= \{\int_0^1 \psi(s)ds\}^2 - \int_0^\infty \int_0^\infty \psi\{S(s+t)\}dF(s)dF(t) \end{aligned}$$

as a measure of the deviation of $F(t)$ from exponentiality towards the NBU alternatives and developed the class of the test statistics in the uncensored case. Here the weight function $\psi(\cdot)$ is assumed to be nondecreasing. This parameter with $\psi(t) = t$ was first investigated in Hollander and Proschan [19].

For the testing problem based on the censored observations (X_i, δ_i), $1 \leq i \leq n$, Kumazawa [42] proposed the class of the statistics

$$M_1(\psi) := \int_0^T \int_0^T \psi\{\widehat{S}_n(s+t)\}d\widehat{F}_n(s)d\widehat{F}_n(t), \tag{5.1.1}$$

which corresponds to the the Koul's [32] NBU statistic in the uncensored case. The statistic $M_1(\psi)$ with $\psi(t) = t$ was considered in Chen, Hollander and Langberg [11] using a modified Kaplan-Meier estimator of $F(t)$.

Theorem 5.1.1. (Kumazawa [39], [42]) *Suppose that the weight function $\psi(t)$ is continuous and piecewise differentiable with bounded derivatives. And suppose that the df $F(t)$ is absolutely continuous and that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} S^2(t)dC(t) < \infty \tag{5.1.2}$$

and

$$n^{1/2}\psi\{S(T)\} \to 0 \text{ in probability as } n \to \infty. \tag{5.1.3}$$

Then the sequence of the rv's $n^{1/2}\{M_1(\psi)-W(F)\}$ converges in distribution as $n \to \infty$ to a normal rv B with zero mean and variance $E[B^2]$, where

$$\begin{aligned} W(F) &:= \int_0^\infty \int_0^\infty \psi\{S(s+t)\}dF(s)dF(t), \\ B &:= -\int_0^\infty \int_0^\infty Z(s+t)S(s+t)\psi'\{S(s+t)\}dF(s)dF(t)+ \\ &\quad + 2\int_0^\infty \int_0^t Z(t-s)S(t-s)dF(s)\psi'\{S(t)\}dF(t) \end{aligned}$$

and $Z(t)$ is the limiting process of $Z_n(t)$ given in Lemma 3.1.2 of Section 3.1.

Proof: To apply Theorem 3.2.2 of Section 3.2, we first note that the induced functional $\tau(g) := W(g \circ F)$ for $g \in D[0,1]$ can be expressed as a composition of Hadamard differentiable transformations. For fixed $F(t)$ and $\psi(t)$, we define

$$\begin{aligned} \gamma_1(g_1)(s) &:= F^{-1} \circ g_1^\star(s), \\ \gamma_2(g_2)(s,t) &:= g_2(s) + g_2(t), \\ \gamma_3(g_3, g_1)(s,t) &:= \psi[1 - g_1 \circ F\{g_3(s,t)\}], \end{aligned}$$

and

$$\gamma_4(g_3) := \int_0^1 \int_0^1 g_3(s,t)dsdt,$$

where $g_1 \in D[0,1]$, $g_2 \in L^1[0,1]$, $g_3 \in L^1[0,1] \times [0,1]$, $0 \le s, t \le 1$ and $g_1^\star(s) = \inf\{t, 1 : g(t) \ge s\}$. Then from Propositions 6.1.1, 6.1.2 and 6.1.6 of Fernholz [16] the above transformations $\gamma_1(\cdot) - \gamma_4(\cdot)$ are all Hadamard differentiable at $I(t)$. Therefore $\tau(g) = \gamma_4 \circ \gamma_3\{\gamma_2 \circ \gamma_1(g), g\}$ is Hadamard differentiable at $I(t)$ by the chain rule of Proposition 3.1.2 of Fernholz [16].

Next we have

$$
\begin{aligned}
& n^{1/2}|\{M_1(\psi) - W(F)\} - \{W(\widehat{F}_n^T) - W(F^T)\}| \\
& \leq n^{1/2}|\int_0^T \int_{T-t}^{\infty} \psi\{S(s+t)\}dF(s)dF(t)|+ \\
& \quad + n^{1/2}|\int_T^{\infty} \int_0^{\infty} \psi\{S(s+t)\}dF(s)dF(t)|+ \\
& \quad + n^{1/2}|\int_0^T \int_{T-t}^{T} \psi\{S(T)\}dF(s)dF(t)| \\
& \leq 3n^{1/2}\psi\{S(T)\} \\
& = o_p(n^0) \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence the desired result follows from Theorem 3.2.2 of Section 3.2 and some calculations. □

We consider the weight function $\psi(u) = u^\alpha$ as a special case.

Corollary 5.1.1. (Kumazawa [42]) *Let* $\psi(u) = u^\alpha$, $\alpha \geq 1$. *Suppose that under the null hypothesis* $\mathcal{H}_0$ *the censoring df* $G(t)$ *satisfies the conditions (5.1.2) and (5.1.3) of Theorem 5.1.1. Then* $n^{1/2}\{M_1(u^\alpha) - (\alpha+1)^{-2}\}$ *converges in distribution as* $n \to \infty$ *to a normal distribution with mean zero and variance*

$$\int_0^{\infty} f_\alpha\{S(t)\}dC(t),$$

where

$$f_\alpha(t) := \alpha^2(\alpha+1)^{-4}\{(\alpha+1)\ln t + 1\}^2 t^{2\alpha+2}.$$

From Lemma 2.4 of Kumazawa [42], the asymptotic variance of $M_1(u^\alpha)$, $\alpha \geq 1$, under the null hypothesis may be consistently estimated by

$$\widehat{\sigma}_\alpha^2 := \int_0^T f_\alpha\{\widehat{S}_n(t-)\}d\widehat{C}(t).$$

Using this estimator, we can construct the asymptotically exact test based on the statistic $M_1(u^\alpha), \alpha \geq 1$.

Now we compute the efficacies of the test statistics $M_1(u^\alpha)$, $\alpha \geq 1$, against the alternatives (i)-(iv) listed in Section 4.1 under the proportional censoring model. From Corollary 5.1.2 we may take $\mu = 1$ and $0 < \lambda < 1$, and the asymptotic variance under $\mathcal{H}_0$ is found to be equal to

$$\sigma_\alpha^2 := \frac{\alpha^2\{(\alpha+1)^2 + (\alpha-\lambda)^2\}}{(2\alpha-\lambda+1)^3(\alpha+1)^4}.$$

Some calculations yield that for the alternative (i)

$$eff\{M_1(u^\alpha)\} = \frac{\alpha^2}{(\alpha+1)^6\sigma_\alpha^2},$$

Table. 5.1 : Efficacies of the NBU-test statistics when the censoring parameter $\lambda = \frac{1}{10}$ and $\frac{3}{4}$

	Alternative			
λ	*(i)*	*(ii)*	*(iii)*	*(iv)*
		$\lambda = 1/10$		
$M_1(u^1)$	1.2676	0.3169	0.0625	0.4774
$M_1(u^{1.5})$	1.1560	0.1849	0.0481	0.4862
$M_1(u^2)$	1.0366	0.1151	0.0364	0.4737
		$\lambda = 3/4$		
$M_1(u^1)$	0.7009	0.1752	0.0346	0.2640
$M_1(u^{1.5})$	0.8062	0.1289	0.0335	0.3391
$M_1(u^2)$	0.8075	0.0897	0.0283	0.3690

for (ii)

$$eff\{M_1(u^\alpha)\} = \frac{\alpha^2}{(\alpha+1)^8\sigma_\alpha^2},$$

for (iii)

$$eff\{M_1(u^\alpha)\} = \frac{\alpha^2}{(\alpha+1)^4(\alpha+2)^4\sigma_\alpha^2},$$

and for (iv)

$$eff\{M_1(u^\alpha)\} = \frac{\{\ln(\alpha+1)-\alpha\}^2}{\alpha^2(\alpha+1)^4\sigma_\alpha^2}.$$

Table 5.1 shows the efficacies of the test statistics $M_1(u^1)$, $M_1(u^{1.5})$ and $M_1(u^2)$ for the alternatives (i)-(iv) and some values of the censoring parameter λ. We recommend the $M_1(u^1)$-test statistic for the testing problem in the sense of Pitman asymptotic relative efficiency.

Remark. Joe and Proschan [24], [25] obtained some results on the *decreasing 100α-percentile* ($0 < \alpha < 1$) *residual* and the *new better than used with respect to the 100α-percentile* aging properties and developed the statistic for testing exponentiality against these life distributions in the uncensored case. And Hollander, Park and Proschan [22], [23] introduced the *new better than used at time* t_0 aging property and considered the problem of testing exponentiality versus this aging property in the uncensored and the censored case. Under the random censorship model Kumazawa [44] proposed the classes of the test statistics generalizing their statistics to accommodate the censored data, and derived the asymptotic distributions of the statistics under some milder conditions.

5.2 The NBUE Alternative

We develop a test of the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

against the alternative

$$\mathcal{H}_4 : F(t) \text{ is NBUE, but not exponential,}$$

on the basis of the possibly right censored data (X_i, δ_i), $1 \leq i \leq n$, defined in Section 2.2.

Kumazawa [37] introduced the class of the test statistics

$$N_1(\psi_1) := \frac{\int_0^T \Psi_1(t)\widehat{S}_n(t)dt}{\widehat{\mu}_n \int_0^T \psi_1(s)\widehat{S}_n(s)ds} \tag{5.2.1}$$

based on the measure

$$\int_0^\infty \psi_1(t)\{\mu_F S(t) - \int_t^\infty S(s)ds\}dt$$

of exponentiality against the NBUE life *df*'s using weight function $\psi_1(t)$, where

$$\Psi_1(t) := \int_0^t \psi_1(s)ds.$$

The measure with $\psi_1(t) \equiv$ *constant* was considered in De Souza Borges, Proschan and Rodrigues [13] for the above testing problem in the uncensored case. Note that we reject the null hypothesis $\mathcal{H}_0$ in favor of $\mathcal{H}_4$ for small values of the statistic $N_1(\psi_1)$.

The parameter

$$\int_0^\infty \psi_2\{S(t)\}\{\mu_F F(t) - \int_0^t S(s)ds\}dF(t)$$

as a measure of the deviation of $F(t)$ towards the NBUE alternatives with weight function $\psi_2(t)$ was considered in Kumazawa [41], and the class of the test statistics

$$N_2(\psi_2) := \frac{\int_0^T \psi_2\{\widehat{S}_n(t)\} \int_0^t \widehat{S}_n(s)dsd\widehat{F}_n(t)}{\widehat{\mu}_n} \tag{5.2.2}$$

was discussed. Hollander and Proschan [20] used this measure with $\psi_2(t) \equiv$ *constant* and proposed the resulting statistic in the uncensored case. In the censored case Koul and Susarla [34] generalized the statistic given in Hollander and Proschan [20] based on a modified Kaplan-Meier estimator, and gave the asymptotics of their statistic under some strong regularity conditions.

Noting that a life *df* $F(t)$ is NBUE if and only if the scaled TTT-transforms $H_F^{-1}(t)$ defined by the equation (4.1.1) of Section 4.1 satisfy $H_F^{-1}(t) \geq t$ for all $t \in [0, 1]$ from Theorem 2.4 of Klefsjö [29], we may consider

$$\Delta_2(\psi) := \int_0^1 \psi(1-t)\{H_F^{-1}(t) - t\}dt$$

as a measure of NBUE-ness with weight function $\psi(t)$. Then change of variable formula in multiple integral shows that

$$\Delta_2(\psi) = \frac{\int_0^\infty \psi\{S(t)\} \int_0^t S(s)dsdF(t)}{\mu_F} - \int_0^1 t\psi(1-t)dt,$$

which also yields the $N_2(\psi_2)$-statistic. The measure $\Delta_2(\psi)$ with $\psi(t) \equiv$ *constant* was used by Kumazawa [45] and investigated the resulting statistic A_3^c under the random censorship model. Based on the same property, Klefsjö [31] proposed the statistic A_3, which is known to be the cumulative TTT-statistic discussed in Barlow *et al.* [3], Chapter 6, on testing against the IFR alternatives and which is equivalent to the Hollander and Proschan's [20] statistic. Note that we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_4$ for large values of $N_2(\psi_2)$.

On the basis of the fact that the NBUE property is expressed by means of the mean residual life $e_F(t)$ defined in the equation (2.1.2) of Section 2.1, the test statistic

$$N_3 := \sup_{0 \leq t \leq T} \widehat{S}_n(t)\left(1 - \frac{\widehat{e}_t}{\widehat{\mu}_n}\right) \tag{5.2.3}$$

was introduced in Kumazawa [47], where

$$\widehat{e}_t := \frac{\int_t^T \widehat{S}_n(s)ds}{\widehat{S}_n(t)} \quad \text{for } 0 \leq t \leq T.$$

Kumazawa [43] discussed the asymptotic behavior of the suitably normalized version of $\widehat{e}_t$ on the fixed interval $[0, u]$, $0 < u < \tau_H$, under the random censorship. The statistic N_3 may be considered as a natural extension of the statistic given by Barlow and Doksum [5] and Koul [33] in the uncensored case, and we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_4$ for large values of N_3.

In order to derive the asymptotic distribution of the statistic $N_1(\psi_1)$, we first assume that $\psi_1(t)$ is not constant on the unit interval $[0, 1]$.

Theorem 5.2.1. (Kumazawa [37]) *Suppose that the weight function $\psi_1(t)$ is nonnegative and right continuous. And suppose that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} h_i^2(t)dC(t) < \infty \tag{5.2.4}$$

and

$$n^{1/2}h_i(T) \to 0 \quad \textit{in probability as } n \to \infty \tag{5.2.5}$$

for $i = 1, 2$ *and* 3*, where* T *is the largest observation of the* X_i*'s,*

$$h_1(t) := \int_t^\infty S(s)ds,$$
$$h_2(t) := \int_t^\infty \psi_1(s)S(s)ds$$

and

$$h_3(t) := \int_t^\infty \Psi_1(s)S(s)ds.$$

Then we have as $n \to \infty$

$$n^{1/2}\Big\{N_1(\psi_1) - \frac{\mu_3}{\mu_F\mu_2}\Big\} \to_d N(0, \sigma^2),$$

where $\mu_2 = h_2(0), \mu_3 = h_3(0)$ *and*

$$\sigma^2 := \Big(\frac{\mu_3}{\mu_F\mu_2}\Big)^2 \int_0^{\tau_H} \Big\{\frac{h_1(t)}{\mu_F} + \frac{h_2(t)}{\mu_2} - \frac{h_3(t)}{\mu_3}\Big\}^2 dC(t).$$

Proof: By applying the same method as given in the proof of Theorem 4.2.3 of Section 4.2, it is seen that the random vector

$$A_n := n^{1/2}\Big(\int_0^T \widehat{S}_n(t)dt - \mu_F, \int_0^T \psi_1(t)\widehat{S}_n(t)dt - \mu_2, \int_0^T \Psi_1(t)\widehat{S}_n(t)dt - \mu_3\Big)$$

is asymptotically equivalent to

$$B_n := \Big(\int_0^T Z_n(t)dh_1(t), \int_0^T Z_n(t)dh_2(t), \int_0^T Z_n(t)dh_3(t)\Big).$$

Since the random vector B_n converges in distribution as $n \to \infty$ to a normal distribution with zero mean vector and dispersion matrix $\{\sigma_{i,j}\}_{1\le i,j\le 3}$ with

$$\sigma_{i,j} := \int_0^{\tau_H} h_i(t)h_j(t)dC(t),$$

the desired result follows from Corollary 3.3 of Serfling [53]. □

Next we consider the test statistic given by

$$\frac{\int_0^T t\widehat{S}_n(t)dt}{\widehat{\mu}_n^2}$$

in the case of $\psi_1(t) \equiv$ *constant*. This statistic is also considered as a test statistic for testing against the HNBUE alternatives in Section 6.2 and treated in a more general framework: we have the following result from Theorem 6.2.5 of Section 6.2.

Corollary 5.2.1. (Kumazawa [37]) *Suppose that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} h_i^2(t)dC(t) < \infty$$

and

$$n^{1/2}h_i(T) \to 0 \quad \textit{in probability as } n \to \infty,$$

for $i = 1$ and 2, where

$$h_1(t) := \int_t^\infty S(s)ds$$

and

$$h_2(t) := \int_t^\infty sS(s)ds.$$

Then we have as $n \to \infty$

$$n^{1/2}\left\{\frac{\int_0^T s\widehat{S}_n(s)ds}{\widehat{\mu}_n^2} - \frac{\mu_2}{\mu_F^2}\right\} \to_d N(0, \sigma^2),$$

where $\mu_2 = h_2(0)$ and

$$\sigma^2 := \frac{\int_0^{\tau_H}\{2\mu_2 h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^6}.$$

The asymptotic behavior of the statistic $N_1(\psi_1)$ under the null hypothesis can be summarized as follows.

Corollary 5.2.2. (Kumazawa [37]) *Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ and the weight function $\psi_1(t)$ satisfy the conditions (5.2.4) and (5.2.5) of Theorem 5.2.1. Then $n^{1/2}\{N_1(\psi_1) - 1\}$ converges in distribution as $n \to \infty$ to a normal distribution with mean zero and variance*

$$\sigma^2 := \int_0^\infty \left\{1 - \frac{\Psi_1(t)}{\mu_2}\right\}^2 S^2(t)dC(t).$$

The similar methods as given in earliers show that

$$\widehat{\sigma}_n^2 := \int_0^T \left\{1 - \frac{\Psi_1(t)}{\widehat{\mu}_2}\right\}^2 \widehat{S}_n^2(t-)d\widehat{C}(t)$$

is a consistent estimator of σ^2 given in Corollary 5.2.3, where

$$\widehat{\mu}_2 := \int_0^T \psi_1(s)\widehat{S}_n(s)ds.$$

In order to derive the asymptotic distribution of $N_2(\psi_2)$ from Theorem 4.2.3 of Section 4.2, we set

$$\Psi_2(t) := \int_0^t \psi_2(s)ds,$$
$$h_1(t) := \int_t^\infty S(s)ds,$$

and

$$h_2(t) := \int_0^\infty [\psi_2\{S(s)\}S(s) + \Psi_2\{S(s)\}]S(s)ds.$$

Corollary 5.2.3. (Kumazawa [41]) *Suppose that the df's $F(t)$, $G(t)$ and the weight function $\psi_2(t)$ satisfy the conditions of Theorem 4.2.3 of Section 4.2. Then we have as $n \to \infty$*

$$n^{1/2}\left\{N_2(\psi_2) - \frac{\mu_{\psi_2}}{\mu_F}\right\} \to_d N(0, \sigma^2_{\psi_2}),$$

where

$$\mu_{\psi_2} := \int_0^\infty \Psi_2\{S(s)\}S(s)ds$$

and

$$\sigma^2_{\psi_2} := \frac{\int_0^{\tau_H} \{\mu_{\psi_2} h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4}.$$

Corollary 5.2.4. (Kumazawa [41]) *Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ and the weight function $\psi_2(t)$ satisfy the conditions (4.2.6)-(4.2.9) of Theorem 4.2.3 of Section 4.2. Then $n^{1/2}\{N_2(\psi_2) - \nu\}$ converges in distribution as $n \to \infty$ to a normal distribution with mean zero and variance*

$$\sigma^2_{\psi_2} := \int_0^\infty [\nu - \Psi_2\{S(t)\}]^2 S^2(t)dC(t),$$

where

$$\nu := \int_0^1 \Psi_2(s)ds.$$

A consistent estimator

$$\widehat{\sigma}_n^2 := \int_0^T [\nu - \Psi_2\{\widehat{S}_n(t-)\}]^2 \widehat{S}_n^2(t-)d\widehat{C}(t)$$

can be constructed from the previous discussions and we can obtain an asymptotically exact test based on the test statistic $N_2(\psi_2)$ defined in the equation (5.2.2) by using this estimator.

We need the following lemma to give the asymptotic distribution of the test statistic N_3 defined in the equation (5.2.3).

Lemma 5.2.1. (Kumazawa [47]) *Suppose that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} S^2(t)dC(t) < \infty, \tag{5.2.6}$$

$$\int_0^{\tau_H} h^2(t)dC(t) < \infty, \tag{5.2.7}$$

and

$$n^{1/2}h(T) \to 0 \quad \text{in probability as } n \to \infty, \tag{5.2.8}$$

where

$$h(t) := \int_t^{\infty} S(s)ds.$$

Then the stochastic process

$$B_n(t) := n^{1/2}\left\{\widehat{S}_n(t)\left(1-\frac{\widehat{e}_t}{\widehat{\mu}_n}\right) - S(t)\left(1-\frac{e_F(t)}{\mu_F}\right)\right\}$$

for $0 \le t \le T$ converges weakly in $D[0, \tau_H]$ as $n \to \infty$ to a Gaussian process $B(t)$ with zero mean and covariance function

$$E\{B(s)B(t)\} = \int_0^{\tau_H} g_s(u)g_t(u)dC(u),$$

where

$$g_s(u) := \mathbf{1}_{\{u\le s\}}\left\{\frac{h(s)}{\mu_F} - S(s)\right\} - \mathbf{1}_{\{u\ge s\}}\frac{h(u)}{\mu_F} + \frac{h(s)h(u)}{\mu_F^2}.$$

Proof: We have for $0 \le t \le T$

$$B_n(t) = -S(t)Z_n(t) - \frac{\int_t^T Z_n(u)dh(u)}{\widehat{\mu}_n} + \frac{h(t)\int_0^T Z_n(u)dh(u)}{\mu_F\widehat{\mu}_n} - \frac{n^{1/2}h(T)h(t)}{\mu_F\widehat{\mu}_n} + \frac{n^{1/2}h(T)}{\widehat{\mu}_n} \tag{5.2.9}$$

with $Z_n(t) = n^{1/2}\{\widehat{F}_n(t) - F(t)\}/S(t)$. Hence Lemma 3.1.2 of Section 3.1 together with the Crámer-Wold Device and the Slutsky's Theorem implies that the limiting process of $B_n(t)$ can be expressed as

$$\begin{aligned} &-S(t)Z(t) - \frac{\int_t^{\tau_H} Z(u)dh(u)}{\mu_F} + \frac{h(t)\int_0^{\tau_H} Z(u)dh(u)}{\mu_F^2} \\ &= \int_0^{\tau_H} g_t(u)dZ(u) = B(t), \end{aligned}$$

where $Z(t)$ is the limiting process of $Z_n(t)$. Therefore some calculations yield the desired result. □

Theorem 5.2.2. (Kumazawa [47]) *Suppose that under the null hypothesis $\mathcal{H}_0$ the censoring df $G(t)$ satisfies the conditions (5.2.6)-(5.2.8) of Lemma 5.2.6. Then we have as $n \to \infty$*

$$\frac{n^{1/2}N_3}{\widehat{\sigma}_n} \to_d \sup_{0\le t<\infty}\Big[W\{\frac{\sigma(t)}{\sigma(\infty)}\} - F(t)W(1)\Big],$$

where $W(t)$ denotes a standard Gaussian process with zero mean and covariance function $E\{W(s)W(t)\} = s \wedge t$,

$$\sigma^2(t) := \int_0^t S^2(s)dC(s)$$

and

$$\widehat{\sigma}_n^2 := \int_0^T \widehat{S}_n^2(s-)d\widehat{C}(s).$$

Proof: Note that we are in the situation $\tau_H = \tau_F = \infty$. Lemma 5.2.6 shows that the limiting process of $B_n(t)$ given in the equation (5.2.9) under the null hypothesis $\mathcal{H}_0$ is given by

$$B(t) = \int_0^\infty \{\mathbf{1}_{\{u<t\}} - F(t)\}S(u)dZ(u).$$

Then it is seen that the stochastic process $\{B(t) : 0 \le t < \infty\}$ has the same distribution as the process $\{W\{\sigma(t)\} - F(t)W\{\sigma(\infty)\} : 0 \le t < \infty\}$. Hence we can conclude the proof from the Continuous Mapping Theorem and the fact that $\widehat{\sigma}_n^2$ is a consistent estimator of $\sigma^2(\infty)$. □

In the uncensored case the variance $\sigma^2(t)$ becomes to $F(t)$, so we have

$$\begin{aligned}\lim_{n\to\infty} P\Big(\frac{n^{1/2}N_3}{\widehat{\sigma}_n} \le x\Big) &= P\Big(\sup_{0\le t<\infty}[W\{F(t)\} - F(t)W(1)] \le x\Big)\\ &= P\Big(\sup_{0\le t\le 1}\{W(t) - tW(1)\} \le x\Big)\\ &= 1 - \exp(-2x^2) \quad \text{for all } x \ge 0,\end{aligned}$$

which can be also derived by the result of Barlow and Doksum [5] since in this situation $\widehat{\sigma}_n^2 = \sum_{i=1}^n \frac{n-i}{n^2(n-i+1)}$ has the limiting value 1.

Here we assume that under the null hypothesis $F(t)$ the censoring $df\, G(t)$ satisfies $F\circ\varphi^{-1}(s) \ge s$ for all $s \ge 0$ with $\varphi(s) := \sigma(s)/\sigma(\infty)$: this condition holds for the proportional censoring model given by $\overline{G}(t) = S^{\lambda}(t)$ with $0 < \lambda < 1$. Then we have for all $x \ge 0$

$$\begin{aligned}\lim_{n\to\infty} P\Big(\frac{n^{1/2}N_3}{\widehat{\sigma}_n} \le x\Big) &= P\Big(\sup_{0\le t<\infty}[W\{\varphi(t)\} - F(t)W(1)] \le x\Big)\\ &= P\Big(\sup_{0\le t\le 1}\{W(t) - F\circ\varphi^{-1}(t)W(1)\} \le x\Big)\\ &\ge P\Big(\sup_{0\le t\le 1}\{W(t) - tW(1)\} \le x\Big) = 1 - \exp(-2x^2).\end{aligned}$$

Table. 5.2 : Efficacies of the NBUE-test statistics when the censoring parameter $\lambda = \frac{1}{10}$ and $\frac{3}{4}$

		Alternative		
Statistic	*(i)*	*(ii)*	*(iii)*	*(iv)*
		$\lambda = 1/10$		
$N_1(u^0)$	0.7218	0.7217	0.0451	0.1804
$N_1(u^{0.5})$	0.3785	0.5195	0.0241	0.0831
$N_1(u^1)$	0.2307	0.4101	0.0144	0.0455
$N_2(u^0)$	1.2474	0.6490	0.0721	0.3874
$N_2(u^{0.5})$	1.3319	0.5711	0.0728	0.4408
$N_2(u^1)$	1.3732	0.5056	0.0711	0.4776
		$\lambda = 3/4$		
$N_1(u^0)$	0.0100	0.01	0.0006	0.0025
$N_1(u^{0.5})$	0.0019	0.0026	0.0001	0.0004
$N_1(u^1)$	0.0003	0.0006	0.0000	0.0000
$N_2(u^0)$	0.1863	0.0969	0.0107	0.0578
$N_2(u^{0.5})$	0.2738	0.1174	0.0149	0.0906
$N_2(u^1)$	0.3497	0.1287	0.0181	0.1216

The asymptotic distribution of the suitably normalized version of N_3 under the null hypothesis for arbitrary $G(t)$ can not be evaluated and the above expression would be useful to determine the critical point of the N_3-test.

Now we shall compare the efficacies of the test statistics $N_1(\psi_1)$ and $N_2(\psi_2)$ for the alternatives (i)-(iv) given in Section 4.1 under the proportional censoring model. For the selection of the weight function we take $\psi_1(t) = \psi_2(t) = t^\alpha$. Then Corollaries 5.2.3 and 5.2.5 imply that $\mu = 1$ and the censoring parameter λ satisfies $0 < \lambda < 1$: here we assume $\alpha > -1/2$ for the statistic $N_1(u^\alpha)$ and $\alpha > -1$ for $N_2(u^\alpha)$. And the asymptotic variances under $\mathcal{H}_0$ are given by

$$\sigma_1^2 := \frac{\Gamma(2\alpha+3) - 2(1-\lambda)^{\alpha+1} + (1-\lambda)^{2\alpha+2}}{(1-\lambda)^{2\alpha+3}\Gamma^2(\alpha+2)}$$

and

$$\sigma_2^2 := \{\frac{1}{(1-\lambda)(\alpha+2)^2} - \frac{2}{(\alpha+2)(\alpha+2-\lambda)} + \frac{1}{(2\alpha+3-\lambda)}\}/(\alpha+1)^2,$$

respectively. Then some calculations yield that: for the alternative (i)

$$eff\{N_1(u^\alpha)\} = \{\gamma + \frac{\Gamma'(\alpha+2)}{\Gamma(\alpha+2)}\}^2/\sigma_1^2,$$

$$eff\{N_2(u^\alpha)\} = \{\frac{\ln(\alpha+2)}{(\alpha+1)(\alpha+2)\sigma_2}\}^2,$$

for (ii)

$$eff\{N_1(u^{\alpha})\} = \frac{(\alpha+1)^2}{\sigma_1^2},$$
$$eff\{N_2(u^{\alpha})\} = \frac{1}{(\alpha+2)^4\sigma_2^2},$$

for (iii)

$$eff\{N_1(u^{\alpha})\} = \frac{(2^{\alpha+1}-1)^2}{2^{2\alpha+4}\sigma_1^2},$$
$$eff\{N_2(u^{\alpha})\} = \left\{\frac{1}{2(\alpha+2)(\alpha+3)\sigma_2}\right\}^2,$$

and for (iv)

$$eff\{N_1(u^{\alpha})\} = \frac{(\alpha+1)^2}{(\alpha+2)^2\sigma_1^2},$$
$$eff\{N_2(u^{\alpha})\} = \left\{\frac{\ln(\alpha+2)}{(\alpha+1)^2} - \frac{1}{(\alpha+1)(\alpha+2)}\right\}^2/\sigma_2^2,$$

respectively, where γ is the Euler's constant.

Table 5.2 gives the efficacies of the test statistics $N_1(u^0)$, $N_1(u^{0.5})$, $N_1(u^1)$, $N_1(u^2)$, $N_2(u^0)$, $N_2(u^{0.5})$, $N_2(u^1)$ and $N_2(u^2)$ for the alternatives and some values of the censoring parameter λ. Here we recommend the use of the $N_2(u^1)$-test for the testing problem in the sense of the Pitman asymptotic relative efficiency.

Chapter 6

Tests for DMRL and HNBUE

6.1 The DMRL Alternative

Kumazawa [45] considered the measure of dispersion from exponentiality to DMRL life *df*'s $F(t)$ given by

$$\int_0^1 \int_s^1 (1-s)(1-t)\{\frac{1-H_F^{-1}(s)}{1-s} - \frac{1-H_F^{-1}(t)}{1-t}\}dtds,$$

and proposed the test statistic A_4^c for testing the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

against the alternative

$$\mathcal{H}_5 : F(t) \text{ is DMRL, but not exponential}$$

based on the Kaplan-Meier estimator $\widehat{F}_n(t)$. Here the property that the scaled TTT-transforms $H_F^{-1}(t)$ defined in the equation (4.1.1) of Section 4.1 satisfy that $\{1 - H_F^{-1}(t)\}/(1-t)$ is nonincreasing in $t \in [0,1]$ for DMRL life *df*'s from Theorem 2.5 of Klefsjö [29] is used in defining the above measure. By using weight function $\psi_1(t)$ this measure can be generalized as

$$\begin{aligned}\Delta_1 : &= \int_0^1 \int_s^1 (1-s)(1-t)\psi_1(1-s)\psi_1(1-t)\{\frac{1-H_F^{-1}(s)}{1-s} - \frac{1-H_F^{-1}(t)}{1-t}\}dtds \\ &= \frac{\int_0^1 \Psi_1(t)dt - \int_0^\infty \Psi_1\{S(t)\}\int_0^t S(s)dsdF(t)}{\mu_F},\end{aligned}$$

where

$$\Psi_1(t) := \psi_1(t)\{\int_0^1 s\psi_1(s)ds - 2\int_t^1 s\psi_1(s)ds\}.$$

The method of replacing $F(t)$ by the Kaplan-Meier estimator $\widehat{F}_n(t)$ suggests to construct the test statistic

$$P_1(\psi_1) := \frac{\int_0^T \Psi_1\{\widehat{S}_n(t)\}\int_0^t \widehat{S}_n(s)dsd\widehat{F}_n(t)}{\widehat{\mu}_n} \tag{6.1.1}$$

for the above testing problem. Klefsjö [31] developed the test statistic A_4 in the uncensored case by using the same property of the scaled TTT-transforms $H_F^{-1}(t)$ and the A_4-statistic may be considered as the corresponding to $P_1(\psi_1)$ with $\psi_1(t) \equiv \textit{constant}$ in the uncensored case. Note that we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_5$ for small values of $P_1(\psi_1)$.

And we may use

$$\Delta_2(\alpha,\beta) := \int_0^\infty \int_0^t S^{\alpha+1}(s) S^{\beta+1}(t) \{e_F(s) - e_F(t)\} dF(s) dF(t)$$

as a measure of DMRL-ness of life *df* $F(t)$. The measure $\Delta_2(\alpha,\beta)$ with $\alpha = \beta = 0$ was first considered by Hollander and Proschan [20] and Chen, Hollander and Langberg [12]. Bergman and Klefsjö [9] discussed $\Delta_2(\alpha,\beta)$ with α and β nonnegative integers. It is seen that the measure $\Delta_2(\alpha,\beta)$ with $\alpha = \beta$ is equal to $\Delta_1(\psi_1)$ with $\psi_1(u) = u^\alpha$, so in this case the both measures lead us to construct the equivalent test. Then some simple calculations yield that

$$\begin{aligned}\Delta_2(\alpha,\beta) &= \int_0^\infty \{a_1 + a_2 S^{\beta+1}(t) + a_3 S^{\alpha+\beta+3}(t)\} S(t) dt \\ &=: \int_0^\infty g\{S(t)\} dt, \qquad (6.1.2)\end{aligned}$$

where

$$\begin{aligned} a_1 &: = -\frac{1}{(\beta+1)(\beta+2)(\alpha+\beta+3)}, \\ a_2 &: = \frac{1}{(\alpha+2)(\beta+1)}, \end{aligned}$$

and

$$a_3 : = -\frac{\alpha+\beta+4}{(\alpha+2)(\beta+2)(\alpha+\beta+3)}.$$

The test statistic

$$P_2(\alpha,\beta) := \frac{\int_0^T g\{\widehat{S}_n(t)\} dt}{\widehat{\mu}_n} \qquad (6.1.3)$$

may be constructed by the use of the Kaplan-Meier estimator $\widehat{F}_n(t)$ and we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_5$ for large values of $P_2(\alpha,\beta)$. Chen, Hollander and Langberg [12] and Bergman and Klefsjö [9] considered the test statistic based on a modified Kaplan-Meier estimator, and proved the asymptotic normality of the normalized version under some stronger conditions than those given in the below.

In order to derive the asymptotic distribution of the $P_1(\psi_1)$-statistic, we apply Theorem 4.2.3 of Section 4.2. To this end we set

$$\varphi(t) := \int_0^t \Psi_1(s) ds. \qquad (6.1.4)$$

Corollary 6.1.1. *Suppose that the df's $F(t)$, $G(t)$ and the function $\Psi_1(t)$ satisfy the conditions of Theorem 4.2.3 of Section 4.2. Then we have as $n \to \infty$*

$$n^{1/2}\left\{P_1(\psi_1) - \frac{\mu_{\psi_1}}{\mu_F}\right\} \to_d N(0, \sigma^2_{\psi_1}),$$

where

$$\mu_{\psi_1} : = \int_0^\infty \varphi\{S(s)\}S(s)ds,$$
$$\sigma^2_{\psi_1} : = \frac{\int_0^{\tau_H}\{\mu_{\psi_1}h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4},$$

and

$$h_2(t) : = \int_t^\infty \Big[\Psi_1\{S(s)\}S(s) + \varphi\{S(s)\}\Big]S(s)ds.$$

Corollary 6.1.2. *Suppose that under the null hypothesis $\mathcal{H}_0$ the df $G(t)$ and the function $\Psi_1(t)$ satisfy the conditions (4.2.6)-(4.2.9) of Theorem 4.2.3 of Section 4.2. Then we have as $n \to \infty$*

$$n^{1/2}\{P_1(\psi_1) - \upsilon\} \to_d N(0, \sigma^2_{\psi_1}),$$

where

$$\upsilon : = \int_0^1 \varphi(s)ds \qquad (6.1.5)$$

and

$$\sigma^2_{\psi_1} : = \int_0^\infty \Big[\upsilon - \varphi\{S(t)\}\Big]^2 S^2(t)dC(t).$$

From this corollary the asymptotic variance $\sigma^2_{\psi_1}$ under the null hypothesis depends on the unknown parameters μ and $G(t)$. By the similar method as given in the previous sections we can construct a consistent estimator

$$\widehat{\sigma}^2_{\psi_1} := \int_0^T \Big[\upsilon - \varphi\{\widehat{S}_n(t-)\}\Big]^2 \widehat{S}^2_n(t-)d\widehat{C}(t)$$

by the theory of counting processes.

Next we consider the asymptotic behavior of the test statistic $P_2(\alpha,\beta)$ defined in the equation (6.1.6). This result can be proved by Theorem 4.1.1 of Section 4.1 and stated as follows.

Corollary 6.1.3. *Suppose that the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} h_1^2(t)dC(t) < \infty \qquad (6.1.6)$$

and

$$n^{1/2}h_1(T) \to 0 \text{ in probability as } n \to \infty, \tag{6.1.7}$$

where

$$h_1(t) := \int_t^{\infty} S(u)du.$$

Then the sequence of the rv's

$$n^{1/2}\left\{P_2(\alpha,\beta) - \frac{\mu_2}{\mu_F}\right\}$$

converges in distribution as $n \to \infty$ *to a normal rv with zero mean and variance*

$$\frac{\int_0^{\tau_H}\{\mu_2 h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4},$$

where

$$\mu_2 := \int_0^{\infty} g\{S(t)\}dt,$$

$$h_2(t) := \int_t^{\infty} S(u)g'\{S(u)\}du,$$

and the function $g(t)$ *is defined in the equation (6.1.2).*

Corollary 6.1.4. *Suppose that under the null hypothesis* $\mathcal{H}_0$ *the df* $G(t)$ *satisfies the conditions (6.1.9) and (6.1.10) of Corollary 6.1.3. Then we have as* $n \to \infty$

$$n^{1/2}P_2(\alpha,\beta) \to_d N(0, \sigma^2_{\alpha,\beta}),$$

where

$$\sigma^2_{\alpha,\beta} := \int_0^{\infty} g^2\{S(t)\}dC(t).$$

Because of the dependency of the asymptotic variance of the statistic $P_2(\alpha,\beta)$ under the null hypothesis on the unknowns μ and $G(t)$, we may consider an estimator

$$\widehat{\sigma}^2_{\alpha,\beta} := \int_0^{T} g^2\{\widehat{S}_n(t-)\}d\widehat{C}(t).$$

The consistency can be proved by the same technique as used in the proof of Lemma 2.4 of Kumazawa [42].

Now we compare the efficacies of the test statistics $P_1(\psi_1)$ and $P_2(\alpha,\beta)$ for the alternatives (i)-(iv) presented in Section 4.1 under the proportional censoring model with the censoring parameter λ. From Corollaries 6.1.2 and 6.1.4 we may take $\mu = 1$ and $0 < \lambda < 1$. And the asymptotic variances of the suitably normalized versions of $P_1(\psi_1)$ and $P_2(\alpha,\beta)$ under $\mathcal{H}_0$ are given by

$$\sigma_1^2 := \int_0^1 \{\upsilon - \varphi(t)\}^2 t^{-\lambda} dt$$

Table. 6.1 : Efficacies of the DMRL-test statistics when the censoring parameter $\lambda = \frac{1}{10}$ and $\frac{3}{4}$

	Alternative			
λ	*(i)*	*(ii)*	*(iii)*	*(iv)*
		$\lambda = 1/10$		
$P_2(0,0)$	0.5638	0.6609	0.0469	0.1131
$P_2(0,.5)$	0.6575	0.6334	0.0540	0.1435
$P_2(0,1)$	0.7284	0.5997	0.0585	0.1701
$P_2(.5,0)$	0.5942	0.6608	0.0491	0.1222
$P_2(.5,.5)$	0.6840	0.6293	0.0557	0.1526
$P_2(.5,1)$	0.7509	0.5931	0.0597	0.1787
$P_2(1,0)$	0.6211	0.6611	0.0510	0.1306
$P_2(1,.5)$	0.7076	0.6256	0.0571	0.1608
$P_2(1,1)$	0.7711	0.5874	0.0606	0.1866
		$\lambda = 3/4$		
$P_2(0,0)$	0.0534	0.0625	0.0044	0.0107
$P_2(0,.5)$	0.0857	0.0826	0.0070	0.0187
$P_2(0,1)$	0.1182	0.0973	0.0095	0.0276
$P_2(.5,0)$	0.0594	0.0660	0.0049	0.0122
$P_2(.5,.5)$	0.0934	0.0859	0.0076	0.0208
$P_2(.5,1)$	0.1269	0.1002	0.0100	0.0302
$P_2(1,0)$	0.0647	0.0689	0.0053	0.0136
$P_2(1,.5)$	0.1003	0.0887	0.0081	0.0228
$P_2(1,1)$	0.1348	0.1026	0.0106	0.0326

and

$$\sigma_2^2 := \frac{a_1^2}{1-\lambda} + \frac{a_2^2}{2\beta+3-\lambda} + \frac{a_3^2}{2\alpha+2\beta+7-\lambda} + \\ + \frac{2a_1a_2}{\beta+2-\lambda} + \frac{2a_1a_3}{\alpha+\beta+4-\lambda} + \frac{2a_2a_3}{\alpha+2\beta+5-\lambda},$$

respectively, where $\varphi(t)$, υ and a_i's are given in the equations (6.1.7), (6.1.8) and (6.1.3)-(6.1.5), respectively. Here for the statistic $P_1(\psi_1)$ we consider the weight function $\psi_1(t)$ to be of form t^α. So the resulting statistic $P_1(\psi_1)$ is equivalent to the statistic $P_2(\alpha, \alpha)$.

Then the efficacies of the $P_2(\alpha,\beta)$-test are given as follows: for the alternative (i)

$$eff\{P_2(\alpha,\beta)\} = \left\{\frac{\ln(\beta+2)}{(\beta+1)(\beta+2)} - \frac{\ln(\alpha+\beta+4)}{(\beta+2)(\alpha+\beta+3)}\right\}^2 / \{(\alpha+2)^2\sigma_2^2\},$$

for (ii)

$$eff\{P_2(\alpha,\beta)\} = \left\{\frac{1}{(\beta+2)^2(\alpha+\beta+4)\sigma_2}\right\}^2,$$

for (iii)

$$eff\{P_2(\alpha,\beta)\} = \left\{\frac{a_1}{2} + \frac{a_2}{\beta+3} + \frac{a_3}{\alpha+\beta+5}\right\}^2 / \sigma_2^2,$$

and for (iv)

$$eff\{P_2(\alpha,\beta)\} = \left\{a_1 + \frac{a_2 \ln(\beta+2)}{\beta+1} + \frac{a_3 \ln(\alpha+\beta+4)}{\alpha+\beta+3}\right\}^2 / \sigma_2^2.$$

Some numerical evaluations of the above expressions with some values of α and β yield the entries of Table 6.1. Here we recommend the test based on the $P_2(1,1)$-statistic for this testing problem.

6.2 The HNBUE Alternative

For testing the null hypothesis

$$\mathcal{H}_0 : F(t) = 1 - \exp(-t/\mu) \quad \text{for } t \geq 0 \quad (\mu \text{ unspecified})$$

versus the alternative

$$\mathcal{H}_6 : F(t) \text{ is HNBUE, but not exponential,}$$

under the random censorship model, Bergman and Klefsjö [8] proposed the class of the test statistics $Q_1(k)$ and $Q_2(k)$ with k integer ≥ 2 based on the property that if $F(t)$ is HNBUE then for $k = 2, 3, \cdots$,

$$\int_0^\infty S^k(t)dt \geq \frac{\mu_F}{k}$$

and

$$\int_0^\infty \{1 - F^k(t)\}dt \leq \mu_F \nu_k$$

with $\nu_k := \sum_{j=1}^k \frac{1}{j}$. In Bergman and Klefsjö [8] a modified Kaplan-Meier estimator was used to define the statistics $Q_1(k)$ and $Q_2(k)$ and the asymptotic normality of the suitably normalized versions of the statistics was derived under some strong conditions. Then these statistics can be represented as

$$Q_1(k) := \frac{\int_0^T \widehat{S}_n^k(t)dt}{\widehat{\mu}_n} \tag{6.2.1}$$

and

$$Q_2(k) := \frac{\int_0^T \{1 - \widehat{F}_n^k(t)\}dt}{\widehat{\mu}_n} \tag{6.2.2}$$

by using the Kaplan-Meier estimator $\widehat{F}_n(t)$ defined in the equation (4.1.1) of Section 4.1. Here we reject the null hypothesis $\mathcal{H}_0$ in favor of $\mathcal{H}_6$ for large values of $Q_1(k)$ and reject $\mathcal{H}_0$ for small values of $Q_2(k)$. It is seen that the test statistic $N_2(\psi)$ with $\psi(t) = t^\alpha$ and α nonnegative integer, introduced in Section 5.2 for testing against the NBUE alternatives, is asymptotically equivalent to $Q_1(\alpha + 2)$.

Kumazawa [46] introduced a measure of exponentiality against the HNBUE life *df*'s given by

$$\begin{aligned}\Delta_1 :&= \int_0^\infty \psi(t)\Big\{\mu_F \exp(-t/\mu_F) - \int_t^\infty S(u)du\Big\}dt \\ &= \mu_F^2 \int_0^\infty \psi(\mu_F t)e^{-t}dt - \int_0^\infty S(u)\int_0^u \psi(t)dtdu\end{aligned}$$

with nonnegative weight function $\psi(t)$. Note that $\Delta_1 = 0$ when $F \in \mathcal{H}_0$ and $\Delta_1 > 0$ when $F \in \mathcal{H}_6$. If we select the weight function $\psi(t) := t^\alpha$ with $\alpha > 0$, the above measure Δ_1 becomes

$$\Delta_1 = \mu^{\alpha+1}\Gamma(\alpha) - \frac{\int_0^\infty u^\alpha S(u)du}{\alpha}$$

and a class of the test statistics

$$Q_3(\alpha) := \frac{\int_0^T u^\alpha \widehat{S}_n(u)du}{\widehat{\mu}_n^{\alpha+1}} \tag{6.2.3}$$

for $\alpha > 0$ may be constructed by using the Kaplan-Meier estimator $\widehat{F}_n(t)$. This statistic in the uncensored case closely relates to the class of the statistics T_α introduced in Kimball [27] and may be considered as a natural extension of T_α for the censored observations. Some properties of T_α are discussed in Lee *et al.* [49] in detail. Under the uncensored model Singh and Kochar [54] considered the above testing problem by using the weight function $\psi(t) := \exp(-t/\mu_F)/\mu_F$ in the measure Δ_1 and discussed some properties of the resulting test statistic.

Here Theorem 9.4 of Dharmadhikari and Joag-dev [15] states that $F(t)$ is an HNBUE life *df* if, and only if,

$$\int_0^\infty g(t)dF(t) \le \frac{\int_0^\infty g(t)\exp(-t/\mu_F)dt}{\mu_F}$$

for all nondecreasing, convex function $g(t)$ on $[0, \infty)$. Hence the measure Δ_1 may be also derived from this characterization of the HNBUE life distributions. Note that we reject $\mathcal{H}_0$ in favor of $\mathcal{H}_6$ for small values of $Q_3(\alpha)$.

Now the asymptotic distribution of the test statistic $Q_1(k)$ can be derived from Theorem 4.1.1 of Section 4.1.

Corollary 6.2.1. *Suppose that for fixed integer* $k \geq 2$ *the df's* $F(t)$ *and* $G(t)$ *satisfy the conditions*

$$\int_0^{\tau_H} h_1^2(t)dC(t) < \infty \tag{6.2.4}$$

and

$$n^{1/2}h_1(T) \to 0 \quad \text{in probability as } n \to \infty, \tag{6.2.5}$$

where

$$h_1(t) := \int_t^{\infty} S(u)du.$$

Then we have as $n \to \infty$

$$n^{1/2}\left\{Q_1(k) - \frac{\mu_2}{\mu_F}\right\} \to_d N(0, \sigma_k^2),$$

where

$$\sigma_k^2 : = \frac{\int_0^{\tau_H}\{\mu_2 h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4},$$

$$\mu_2 : = \int_0^{\infty} S^k(t)dt,$$

and

$$h_2(t) : = k\int_t^{\infty} S^k(u)du.$$

Corollary 6.2.2. *Suppose that for fixed integer* $k \geq 2$ *and the null hypothesis* $F(t)$*, the censoring df* $G(t)$ *satisfies the conditions (6.2.4) and (6.2.5) of Corollary 6.2.1. Then we have as* $n \to \infty$

$$n^{1/2}\left\{Q_1(k) - \frac{1}{k}\right\} \to_d N(0, \sigma_{1,k}^2),$$

where

$$\sigma_{1,k}^2 := \frac{\int_0^{\infty}\{S(t) - kS^k(t)\}^2 dC(t)}{k^2}.$$

Again we can prove the asymptotic normality of the suitably normalized version of the statistic $Q_2(k)$ from Theorem 4.1.1 of Section 4.1.

Corollary 6.2.3. *Suppose that for fixed integer* $k \geq 2$ *the df* $F(t)$ *and the censoring df* $G(t)$ *satisfy the conditions (6.2.4) and (6.2.5) of Corollary 6.2.1. Then we have as* $n \to \infty$

$$n^{1/2}\left\{Q_2(k) - \frac{\mu_2}{\mu_F}\right\} \to_d N(0, \sigma_k^2),$$

where

$$\sigma_k^2 := \frac{\int_0^{\tau_H}\{\mu_2 h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^4},$$

$$\mu_2 := \int_0^{\infty}\{1 - F^k(t)\}dt,$$

$$h_1(t) := \int_t^{\infty} S(u)du,$$

and

$$h_2(t) := k\int_t^{\infty} F^{k-1}(u)S(u)du.$$

Corollary 6.2.4. *Suppose that for fixed integer $k \geq 2$ and the null hypothesis $F(t)$, the censoring df $G(t)$ satisfies the conditions (6.2.4) and (6.2.5) of Corollary 6.2.1. Then we have as $n \to \infty$*

$$n^{1/2}\{Q_2(k) - \nu_k\} \to_d N(0, \sigma_{2,k}^2),$$

where

$$\sigma_{2,k}^2 := \int_0^{\infty}\{\nu_k S(t) + F^k(t) - 1\}^2 dC(t).$$

From Corollaries 6.2.2 and 6.2.4 the asymptotic variances of the suitably normalized versions of $Q_1(k)$ and $Q_2(k)$ under $\mathcal{H}_0$ hypothesis are found to depend on the unknowns μ and $G(t)$, and may be estimated by

$$\widehat{\sigma}_{1,k}^2 := \frac{\int_0^T\{\widehat{S}_n(t-) - k\widehat{S}_n^k(t-)\}^2 d\widehat{C}(t)}{k^2}$$

and

$$\widehat{\sigma}_{2,k}^2 := \int_0^T\{\nu_k\widehat{S}_n(t-) + \widehat{F}_n^k(t-) - 1\}^2 d\widehat{C}(t),$$

respectively. The consistency of these estimators can be proved by the same method as given in Section 4.1.

Next we consider the asymptotic distribution of the test statistic $Q_3(\alpha)$ defined in the equation (6.2.3).

Theorem 6.2.1. (Kumazawa (1989a)) *Suppose that for fixed constant $\alpha > 0$ the df's $F(t)$ and $G(t)$ satisfy the conditions*

$$\int_0^{\tau_H} h_i^2(t)dC(t) < \infty \tag{6.2.6}$$

and

$$n^{1/2}h_i(T) \to 0 \quad \textit{in probability as } n \to \infty \tag{6.2.7}$$

for $i = 1$ *and* 2, *where*

$$h_1(t) := \int_t^\infty S(u)du$$

and

$$h_2(t) := \int_t^\infty u^\alpha S(u)du.$$

Then we have as $n \to \infty$

$$n^{1/2}\Big\{Q_3(\alpha) - \frac{\mu_2}{\mu_F^{\alpha+1}}\Big\} \to_d N(0, \sigma_\alpha^2),$$

where $\mu_2 := h_2(0)$ *and*

$$\sigma_\alpha^2 := \frac{\int_0^{\tau_H} \{\mu_2(\alpha + 1)h_1(t) - \mu_F h_2(t)\}^2 dC(t)}{\mu_F^{2\alpha+4}}.$$

Proof: We have

$$\begin{aligned} W_n :&= n^{1/2}\Big\{Q_3(\alpha) - \frac{\mu_2}{\mu_F^{\alpha+1}}\Big\} \\ &= \frac{\int_0^T Z_n(s)dh_2(s)}{\widehat{\mu}_n^{\alpha+1}} - \mu_2 \frac{n^{1/2}(\widehat{\mu}_n^{\alpha+1} - \mu_F^{\alpha+1})}{\widehat{\mu}_n^{\alpha+1}\mu_F^{\alpha+1}} - \frac{n^{1/2}h_2(T)}{\widehat{\mu}_n^{\alpha+1}} \end{aligned}$$

with $Z_n(t) = n^{1/2}\{\widehat{F}_n(t) - F(t)\}/S(s)$. It is seen from Corollary 3.3 of Serfling (1980) and the fact that $\widehat{\mu}_n$ is a consistent estimator of μ_F that the second term of the right hand side is asymptotically equivalent to

$$-\frac{\mu_2(\alpha + 1)\int_0^T Z_n(s)dh_1(s)}{\mu_F^{\alpha+2}}$$

from the conditions (6.2.6) and (6.2.7).

Now Lemma 3.1.2 of Section 3.1 together with the Cramér-Wold Device implies that the random vector

$$\Big(\int_0^T Z_n(t)dh_1(t), \int_0^T Z_n(t)dh_2(t)\Big)$$

converges in distribution as $n \to \infty$ to

$$\Big(\int_0^{\tau_H} Z(t)dh_1(t), \int_0^{\tau_H} Z(t)dh_2(t)\Big)$$

with the limiting process $Z(t)$ of $Z_n(t)$. Hence the limiting *rv* of W_n as $n \to \infty$ can be expressed as

$$\frac{\int_0^{\tau_H} Z(s)dh_2(s)}{\mu^{\alpha+1}} - \frac{\mu_2(\alpha+1)\int_0^{\tau_H} Z(s)dh_1(s)}{\mu_F^{\alpha+2}}$$
$$= \frac{\int_0^{\tau_H}\{\mu_2(\alpha+1)h_1(t) - \mu_F h_2(t)\}dZ(t)}{\mu_F^{\alpha+2}}.$$

The desired result follows from the Fubini's Theorem and some calculations. □

Corollary 6.2.5. *Suppose that for fixed constant $\alpha > 0$ and the null hypothesis $F(t)$, the censoring df $G(t)$ satisfies the conditions (6.2.6) and (6.2.7) of Theorem 6.2.5. Then we have as $n \to \infty$*

$$n^{1/2}\{Q_3(\alpha) - \Gamma(\alpha+1)\} \to_d N(0, \sigma^2_{3,\alpha}),$$

where

$$\sigma^2_{3,\alpha} := \int_0^\infty \left\{\Gamma(\alpha+2)S(u) - \frac{\int_u^\infty t^\alpha S(t)dt}{\mu^{\alpha+1}}\right\}^2 dC(u).$$

Because of the dependency of the asymptotic variance $\sigma^2_{3,\alpha}$ under $\mathcal{H}_0$ on the unknown parameters μ and $G(t)$, we may use a consistent estimator

$$\widehat{\sigma}^2_{3,\alpha} := \int_0^T \left\{\Gamma(\alpha+2)\widehat{S}_n(u-) - \frac{\int_u^T t^\alpha \widehat{S}_n(t)dt}{\widehat{\mu}_n^{\alpha+1}}\right\}^2 d\widehat{C}(u)$$

by the theory of counting processes.

We shall consider the efficacies of the tests based on the statistics $Q_1(k)$, $Q_2(k)$ and $Q_3(\alpha)$ against the alternatives (i)-(iv) listed in Section 4.1 under the proportional censoring model with $\overline{G}(t) = S^\lambda(t)$. Then Corollaries 6.2.2, 6.2.4 and 6.2.6 imply that $\mu = 1$ and $0 < \lambda < 1$. And the asymptotic variances of their suitably normalized versions under $\mathcal{H}_0$ are given by

$$\sigma^2_{1,k} : = \frac{1}{k^2(1-\lambda)} - \frac{2}{k(k-\lambda)} + \frac{1}{2k-\lambda-1},$$
$$\sigma^2_{2,k} : = \sum_{i=2}^{2k} \frac{a_{i,k}}{i-1-\lambda},$$

and

$$\sigma^2_{3,k} : = (k!)^2 \sum_{i=0}^{2k} \frac{i!\, b_{i,k}}{(1-\lambda)^{i+1}},$$

respectively, where

$$a_{i,k} : = \sum_{\ell+m=i} c_{\ell,k}c_{m,k},$$
$$b_{i,k} : = \sum_{\ell+m=i} d_{\ell,k}d_{m,k},$$

Table. 6.2 : Efficacies of the HNBUE-test statistics when the censoring parameter $\lambda = \frac{1}{10}$ and $\frac{3}{4}$

	Alternative			
λ	*(i)*	*(ii)*	*(iii)*	*(iv)*
		$\lambda = 1/10$		
$Q_1(2)$	1.2474	0.6490	0.0721	0.3874
$Q_1(3)$	1.3732	0.5056	0.0711	0.4776
$Q_2(2)$	1.2474	0.6490	0.0721	0.3874
$Q_2(3)$	1.1289	0.6932	0.0691	0.3287
$Q_3(1)$	0.7217	0.7217	0.0451	0.1804
$Q_3(2)$	0.3889	0.5600	0.0243	0.0847
		$\lambda = 3/4$		
$Q_1(2)$	0.1863	0.0969	0.0107	0.0578
$Q_1(3)$	0.3497	0.1287	0.0181	0.1216
$Q_2(2)$	0.1863	0.0969	0.0107	0.0578
$Q_2(3)$	0.1386	0.0851	0.0084	0.0403
$Q_3(1)$	0.01	0.01	0.0006	0.0025
$Q_3(2)$	0.0008	0.0011	0.0001	0.0002

$$c_{i,k} := \begin{cases} \nu_k - k & \text{for } i = 1 \\ (-1)^i \binom{k}{i} & \text{for } i = 2, 3, \cdots, k, \end{cases}$$

and

$$d_{i,k} := \begin{cases} -k & \text{for } i = 0 \\ \frac{1}{i!} & \text{for } i = 1, 2, \cdots, k. \end{cases}$$

Here we assume for the statistic $Q_3(\alpha)$ that $\alpha = k$ is positive integer.

As stated in the beginning of this section the $Q_1(k)$-statistic is equivalent to the $N_2(\psi)$-statistic with $\psi(t) = t^{k-2}$, we do not give the expressions for the efficacies of the test based on $Q_1(k)$. Then some calculations show that for the alternative (i)

$$eff\{Q_2(k)\} = \left\{k \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i} \frac{\ln(i+1)}{(i+1)^2}\right\}^2 / \sigma^2_{2,k},$$

$$eff\{Q_3(k)\} = \Big(\Gamma(k+2)\{1 - \nu_{k+1}\}\Big)^2 / \sigma^2_{3,k},$$

for (ii)

$$eff\{Q_2(k)\} = \left\{\nu_k - k \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \frac{1}{(i+1)^3}\right\}^2 / \sigma^2_{2,k},$$

$$eff\{Q_3(k)\} = \left\{\frac{k\Gamma(k+2)}{2}\right\}^2 / \sigma^2_{3,k},$$

for (iii)

$$eff\{Q_2(k)\} = \left\{\frac{k}{2}\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}\frac{i}{(i+1)^2(i+2)}\right\}^2/\sigma^2_{2,k},$$
$$eff\{Q_3(k)\} = \Gamma^2(k+1)\left(\frac{k-1}{2}+\frac{1}{2^{k+1}}\right)^2/\sigma^2_{3,k},$$

and for (iv)

$$eff\{Q_2(k)\} = \left\{\nu_k - k\sum_{i=1}^{k-1}(-1)^i\binom{k-1}{i}\frac{\ln(i+1)}{i(i+1)}\right\}^2/\sigma^2_{2,k},$$
$$eff\{Q_3(k)\} = \Gamma^2(k+1)\{\nu_{k+1}-k-1\}^2/\sigma^2_{3,k},$$

respectively, where γ denotes the Euler's constant.

Table 6.2 shows the efficacies of the tests based on the statistics $Q_1(2)$, $Q_1(3)$, $Q_2(2)$, $Q_2(3)$, $Q_3(2)$ and $Q_3(2)$ against the alternatives (i)-(iv) and some values of the censoring parameter λ. The poorness of the performance of the test based on the T_α-statistic, equivalent to the $Q_3(\alpha)$-statistic in the uncensored case, was pointed out in Lee *et al.*[49] and it seems that the $Q_3(\alpha)$-statistic in the censored case inherits the characteristic of the T_α-statistic. Here we recommend the $Q_1(3)$-test for testing exponentiality against the HNBUE alternatives under the censored model based on the concept of the Pitman asymptotic relative efficiency.

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