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DESIGN OF STRATEGY-PROOF MECHANISMS

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Design of Strategy-Proof Mechanisms

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Abstract

We consider the problem of choosing a single allocation from the set of feasible allocations in economic environments. We may use mechanisms which determine the final allocation on the basis of preferences of agents. However, selfish agents may manipulate their preferences in order to achieve an allocation in their favor. To overcome this difficulty, we must design strategy-proof mechanisms in which truthful revelation of preferences is a dominant strategy for each agent. In this thesis, we study the possibility of designing strategy-proof mechanisms in several economic environments.

In Chapters 2 and 3, we consider the mechanism design problem for the allocation of an indivisible good when monetary compensation is possible. In Chapter 2, we consider the possibility of designing strategy-proof and Pareto efficient mechanisms on finitely restricted preference domains. First, we show that there is no strategy-proof and Pareto efficient mechanism on some preference domains consisting of a sufficiently large but finite number of quasi-linear preferences. Next, we prove that there is no strategy-proof, Pareto efficient, and equally compensatory mechanism on arbitrary preference domains consisting of more than three quasi-linear preferences. We conclude that the impossibility of strategy-proof and Pareto efficient mechanisms is very strong. In Chapter 3, we give up Pareto efficiency and try to understand the structure of strategy-proof mechanisms. We characterize the set of strategy-proof, individually rational, equally compensatory, demand monotonic mechanisms. Those mechanisms have the following properties: (i) they determine the allocation of monetary compensation depending only on who receives the indivisible good; (ii) they allocate the indivisible good to one of the pre-specified one or two agent(s); and (iii) they disregard preferences of agents other than the pre-specified agent(s). This characterization enables us to understand that those mechanisms are very inefficient and asymmetric.

In Chapters 4 - 6, we consider the mechanism design problem for the provision of public goods. In Chapter 4, we consider the effect of partial exclusion on the design of

strategy-proof mechanisms for the provision of a fixed sized public project, that is, one indivisible unit of a non-rivalrous good. For the case of a non-excludable public project, we characterize the unanimous mechanisms by strategy-proofness, individual rationality, and citizen sovereignty. For the case of an excludable public project, we characterize the largest unanimous mechanisms by strategy-proofness, individual rationality, demand monotonicity, and access independence. We conclude that partial exclusion always improves the efficiency of strategy-proof mechanisms. In Chapter 5, we consider the effect of fixed costs on the design of strategy-proof mechanisms for the provision of public goods. First, we consider the case of a cost function without fixed costs. We show that the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property. Next, we consider the case of a cost function with positive fixed costs. We show that the restriction of the range of mechanisms is necessary for designing strategy-proof and individually rational mechanisms. This result implies that the existence of fixed costs limits the variety of our choices, and it is less desirable in terms of efficiency. In Chapter 6, we reconsider Serizawa's (1996) characterization of strategy-proof, individually rational, no exploitative, and non-bossy mechanisms for the provision of public goods. He leaves an open question whether or not non-bossiness is necessary for his characterization. We show that strategy-proofness, individual rationality, and no exploitation imply non-bossiness. As a corollary, we provide a new characterization of strategy-proof, individually rational, and no exploitative mechanisms.

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Chapter 1

Introduction

We consider the problem of choosing a single allocation in economic environments. When a society consisting of several agents has to choose from the set of feasible allocations, it may rely on a certain mechanism which determines the final allocation on the basis of preferences of agents. Formally, a *mechanism* is a function which associates a feasible allocation with each combination of preferences of agents. We would like to design mechanisms which choose a desirable allocation (e.g. an efficient allocation, an equitable allocation, etc.) for each combination of preferences. Since preferences of agents are private information, agents are required to report their preferences in order to implement mechanisms. However, selfish agents may manipulate their preferences in order to achieve an allocation in their favor. As a result, the allocation chosen by the mechanism is desirable on the basis of reported preferences of agents, but may be far from desirable on the basis of true preferences of agents. This is the problem of manipulation. To overcome this problem, we must design mechanisms in which truthful revelation of preferences is a dominant strategy for each agent. We call such mechanisms *strategy-proof* (Gibbard, 1973; Satterthwaite, 1975). Agents have no incentive to manipulate their preferences in strategy-proof mechanisms. Strategy-proofness is an attractive requirement from the viewpoint of decentralization. An advantage of strategy-proof mechanisms is the weakness of the assumption on informational requirements. Each agent is assumed to know his own preference, but not assumed to know other agents' preferences (in contrast to Nash-type implementation) or prior distribution of preferences (in contrast to Bayesian implementation). In this thesis, we study the possibility of designing strategy-proof mechanisms which satisfy the other normative criteria in several economic environments.

In Chapter 2, we consider economies with a single indivisible good and a transferable good. The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. As examples, consider the division of an estate consisting of a house and cash in a bereaved family, and the allocation of a single task and bonuses in a firm. We consider mechanisms which determine who consumes the indivisible good and how much compensation the other agents receive on the basis of preferences of agents. We regard the following two axioms as desiderata for mechanisms. The first axiom is strategy-proofness. The second axiom is Pareto efficiency (the allocation chosen by the mechanism is always Pareto efficient). We consider the possibility of designing strategy-proof and Pareto efficient mechanisms in economies with an indivisible good and money. A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences. However, it is well known that the possibility of designing strategy-proof mechanisms depends on the size of the preference domain of the mechanisms. Therefore, we tackle the question whether or not strategy-proof and Pareto efficient mechanisms exist given some restrictions of the preference domain of the mechanisms.

First, we consider some finite restrictions of the preference domain in order to understand how strong the impossibility result is. We show that there is no strategy-proof and Pareto efficient mechanism on some preference domains consisting of a sufficiently large but finite number of quasi-linear preferences. The impossibility result holds true even on finitely restricted preference domains. A possible drawback of this result is that the preference domains consist of a very large number of preferences when the number of agents is large. Next, we impose an auxiliary axiom named equal compensation (the non-consumers of the indivisible good receive the same amount of monetary compensation) and consider the possibility of such mechanisms on small preference domains. We show that there is no strategy-proof, Pareto efficient, and equally compensatory mechanism on arbitrary preference domains consisting of more than three quasi-linear preferences. Finally, we describe the structure of strategy-proof and Pareto efficient mechanisms on very small preference domains consisting of two or three quasi-linear preferences. We conclude that the impossibility of strategy-proof and Pareto efficient mechanisms is inevitable since such small preference domains are very unrealistic.

In Chapter 3, we consider economies with a single indivisible good and a transferable good. By the results of Chapter 2, we must give up Pareto efficiency in order to design reasonable strategy-proof mechanisms. We think of the following four axioms as desiderata for mechanisms. The first axiom is strategy-proofness. The next two axioms are individual rationality (all agents end up no worse off than at the status quo) and equal compensation, which are related to equity. The last axiom is demand monotonicity (the consumer of the indivisible good is unchanged when the consumer increases his demand for the indivisible good and no other agents increase their demand), which is a weakening of Pareto efficiency. In this chapter we characterize the set of mechanisms which satisfy these four axioms on the set of all quasi-linear preferences. As a result, we answer the questions (i) how inefficient strategy-proof mechanisms are, and (ii) how asymmetry strategy-proof mechanisms are.

First, we show that if a mechanism satisfies strategy-proofness, equal compensation, and demand monotonicity, then it satisfies the constant transfer property (the allocation of monetary compensation depends only on who receives the indivisible good). Second, we prove that any mechanism which satisfies four axioms allocates the indivisible good to one of the pre-specified one or two agent(s), and disregards preferences of agents other than the pre-specified agent(s). When the set of potential consumers of the indivisible good consists of two agents (without loss of generality, we call them agents 1 and 2), we define two types of mechanisms. The decisive mechanisms require that agent 1 (agent 2 respectively) get the indivisible good if he wants it under a pre-specified monetary compensation, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. The unilaterally unanimous mechanisms require that agent 1 (agent 2 respectively) get the indivisible good if both agents want agent 1 (agent 2 respectively) to get it under a pre-specified monetary compensation, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. When the set of potential consumers consists of only one agent, we define the dictatorial mechanisms (one of the agents always consumes the indivisible good without compensation). Finally, we provide the following characterization: a mechanism satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity if and only if it is decisive, unilaterally unanimous, or dictatorial. This characterization enables us to understand that those mechanisms are very inefficient and asymmetric.

In Chapter 4, we consider the provision of a fixed sized public project, that is, one indivisible unit of a non-rivalrous good. We first consider the case that the public project is non-excludable. As examples, consider the provision of national defense, pollution-control devices, fireworks displays, and street lighting. Here we consider mechanisms which determine whether to provide the project and how to divide the costs among agents. We next consider the case that the public project is excludable. As examples, consider the provision of cable TV, computer networks, airports, and highways. Here we consider mechanisms which determine whether to provide the project, how to divide the costs among agents, and who is allowed to consume the project.

Moulin (1994) characterizes "the conservative equal-costs mechanism" by coalitional strategy-proofness, individual rationality, and symmetry for the provision of non-excludable public projects. Moreover, he proposes "the serial mechanism" for the provision of excludable public projects, and shows that the serial mechanism Pareto dominates the conservative equal-costs mechanisms.

In this chapter we provide some characterizations by strategy-proofness instead of coalitional strategy-proofness for the provision of excludable versus non-excludable public projects. These characterizations enable us to understand the effect of partial exclusion on the design of strategy-proof mechanisms. We regard the following axioms as desiderata for mechanisms. The first axiom is strategy-proofness. The next four auxiliary axioms are individual rationality, demand monotonicity ((i) the set of consumers of the project does not shrink when the demand of no agent decreases; and (ii) the set of consumers of the project is unchanged when the demand of no current consumer decreases and the demand of no current non-consumer increases), citizen sovereignty (society has access to either level of the project), and access independence (each agent has access to either level of the project regardless of other agents' preferences).

First, we consider the case of a non-excludable public project. Constant cost sharing pre-specifies a cost sharing pattern for society, and requires that if the project is provided, then agents should share its cost according to the cost sharing pattern. Serizawa (1996) shows that constant cost sharing is a necessary condition for strategy-proofness in the two-agent case. We prove that constant cost sharing is a necessary condition for strategy-proofness and individual rationality in the n-agent case. The unanimous mechanisms are defined as follows: (i) they are constant cost sharing; and (ii) they provide the project if each agent's willingness to pay is larger than or equal to his cost share. We characterize the unanimous mechanisms as the set of strategy-proof, individually rational, and citizen sovereign mechanisms.

Second, we consider the case of an excludable public project. Semiconstant cost sharing pre-specifies a cost sharing pattern for each coalition, and requires that if the project is provided for agents in some coalition, then those agents should share its cost according to the cost sharing pattern for the coalition. We prove that semiconstant cost sharing is a necessary condition for strategy-proofness in the two-agent case, and it is a necessary condition for strategy-proofness, individual rationality, and demand monotonicity in the n-agent case. The largest unanimous mechanisms are defined as follows: (i) they are semiconstant cost sharing; and (ii) they provide the project for the largest coalition such that the willingness to pay of each member of the coalition is larger than or equal to his cost share. We characterize the largest unanimous mechanisms as the set of strategy-proof, individually rational, demand monotonic, and access independent mechanisms.

Comparing the two classes of mechanisms, we conclude that partial exclusion always improves efficiency, that is, it is always possible to design some largest unanimous mechanism (for an excludable public project) which Pareto dominates a given unanimous mechanism (for a non-excludable public project).

In Chapter 5, we consider the provision of public goods. Moulin (1994) characterizes "the conservative equal-costs mechanism" by coalitional strategy-proofness, individual rationality, and symmetry in economies with one private good and one public good. His result relies on the assumption that public goods can be produced without fixed costs. It is more natural, however, to assume that we need positive fixed costs to produce public goods. In this chapter we incorporate the consideration of fixed costs, and provide several characterizations by strategy-proofness instead of coalitional strategy-proofness.

We introduce the notion of a cost sharing rule, which associates a cost sharing pattern with each level of public goods. Assuming that a cost sharing rule is exogenously given, we consider mechanisms which determine only the level of public goods. One interpretation of this model is that the revision of tax rules is less frequent than public decisions. We think of the following two axioms as desiderata for mechanisms. The first axiom is strategy-proofness. The second axiom is individual rationality. We characterize the set of strategy-proof and individually rational mechanisms.

First, for the sake of comparison, we consider the case of a cost function without fixed costs. In economies with one private good and one public good, we show that the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property (any feasible level of the public good is attainable by the mechanism) for a certain class of cost sharing rules. In economies with one private good and several public goods, it follows from a general result of Zhou (1991a) that there is no strategy-proof and individually rational mechanism.

Next, we consider the case of a cost function with positive fixed costs. Since the cost function has fixed costs, it has the non-convexity. Thus, any cost sharing rule must have the non-convexity. We present the set of strategy-proof and individually rational mechanisms by restricting the range of mechanisms to recover the convexity of the cost sharing rule. Those mechanisms are the variants of the minimal provision mechanism. Conversely, if the restriction of the range of mechanisms is not sufficient to recover the convexity of the cost sharing rule, the non-convexity prevents us from designing strategy-proof and individually rational mechanisms. These results imply that we must restrict the range of mechanisms if we want to design strategy-proof and individually rational mechanisms. In other words, the non-convexity of cost sharing rules limits the variety of our choices, and therefore it is less desirable in terms of efficiency.

In Chapter 6, we consider the provision of public goods. In contrast to Chapter 5, we consider mechanisms which determine both the level of public goods and how to divide the costs among agents. Serizawa (1996) characterizes the set of "semiconvex cost sharing schemes determined by the minimum demand principle" by strategy-proofness, individual rationality, no exploitation, and non-bossiness in economies with one private good and one public good. However, there is a criticism on the non-bossiness axiom since the economic interpretation of non-bossiness is not so clear. Moreover, he leaves an open question whether or not non-bossiness is necessary for his characterization. Therefore, it is an interesting question what class of mechanisms is characterized without non-bossiness. We show that any strategy-proof, individually rational, and no exploitative mechanism must satisfy non-bossiness in economies with one private good and one public good. As a corollary, we characterize the set of semiconvex cost sharing schemes determined by the minimum demand principle by strategy-proofness, individual rationality, and no exploitation.

Chapter 2

Strategy-Proof and Efficient Allocation of an Indivisible Good on Finitely Restricted Preference Domains [*]

2.1. Introduction

We consider economies with a single indivisible good and a transferable good.[1] The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. We consider allocation mechanisms which determine who consumes the indivisible good and how much compensation the other agents receive on the basis of preferences of agents. We regard the following axioms as desiderata for mechanisms. The first axiom is strategy-proofness. A mechanism satisfies strategy-proofness if truthful revelation of preferences is a dominant strategy for each agent. The second one is Pareto efficiency. A mechanism satisfies Pareto efficiency if it always chooses a Pareto efficient allocation. We study the possibility of designing strategy-proof and Pareto efficient mechanisms.

The possibility of designing strategy-proof mechanisms depends on the size of the preference domain of the mechanisms. In a social choice framework, Gibbard (1973) and Satterthwaite (1975) establish the impossibility of strategy-proof mechanisms when the preference domain is "unrestricted", whereas Moulin (1980) and Barbera and Jackson (1994) characterize a rich class of strategy-proof, Pareto efficient, and anonymous mechanisms when the preference domain is restricted to "single peaked" preferences.

In two-agent pure exchange economies, Zhou (1991b) shows that there is no strategy-proof, Pareto efficient, and non-dictatorial mechanism on the usual economic preference domain, and Schummer (1997) proves the same impossibility result even when the

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[*] This chapter is based on Ohseto (1999c).

[1] This type of economies has been studied in much of the literature. For details, see Tadenuma and Thomson (1993, 1995).

preference domain is restricted to (i) "homothetic" preferences, or (ii) more than three "linear" preferences. Therefore, the impossibility of strategy-proof and Pareto efficient mechanisms is well established in the two-agent case.

However, when we consider economies with private goods, there is a crucial difference between the two-agent case and the case of more than two agents. Satterthwaite and Sonnenschein (1981) point out that there exist strategy-proof, Pareto efficient, and non-dictatorial mechanisms in the case of more than two agents. However, it is very difficult to characterize such strategy-proof mechanisms because of the concept of strategy-proofness and the presence of private goods. When some agent (e.g. agent 1) changes his preference and others remain unchanged, strategy-proofness puts constraint on agent 1's consumption bundle directly, but on other agents' consumption bundles indirectly (e.g. through budget balance). Satterthwaite and Sonnenschein (1981) introduce non-bossiness to overcome this difficulty. Barbera and Jackson (1995) also use non-bossiness in order to characterize the set of strategy-proof and anonymous mechanisms in the case of more than two agents. However, we do not invoke non-bossiness since the economic interpretation of non-bossiness is not so clear.

We consider the possibility of strategy-proof and Pareto efficient mechanisms in economies with an indivisible good and money. A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences.[2] First, we consider some finite restrictions of the preference domain in order to understand how strong the impossibility result is. We show that there is no strategy-proof and Pareto efficient mechanism on some preference domains consisting of a sufficiently large but finite number of quasi-linear preferences. The impossibility result holds true even on finitely restricted preference domains. A possible drawback of this result is that the preference domains consist of a very large number of preferences when the number of agents is large. Next, we impose an auxiliary axiom

[2] To escape the impossibility result, one may weaken the incentive criterion from strategy-proofness to Bayesian incentive compatibility (d'Aspremont and Gerard-Varet, 1979; Myerson and Satterthwaite, 1983).

"equal compensation" and consider the possibility of such mechanisms on small preference domains. We show that there is no strategy-proof, Pareto efficient, and equally compensatory mechanism on arbitrary preference domains consisting of more than three quasi-linear preferences. Finally, we describe the structure of strategy-proof and Pareto efficient mechanisms on very small preference domains consisting of two or three quasi-linear preferences. We conclude that the impossibility of strategy-proof and Pareto efficient mechanisms is inevitable since such small preference domains are very unrealistic.

This chapter is organized as follows. In Section 2.2, we introduce notation and definitions. In Section 2.3, we show the main impossibility result on sufficiently large but finite preference domains. In Section 2.4, we show some impossibility results on small preference domains. In Section 2.5, we summarize the results.

2.2. Notation and Definitions

We consider economies with a single indivisible good and a transferable good. The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. Let $N=\{1,...,n\}$ $(n\geq 2)$ be the set of agents. For each $i\in N$, the consumption space of agent i is the set of pairs $(t_i, x_i)\in R\times\{0,1\}$, where t_i denotes money he receives and x_i denotes his consumption of the indivisible good. The amount of money each agent receives may be negative. Each agent $i\in N$ has a quasi-linear preference on his consumption space. Let U_A be the set of all quasi-linear preferences on $R\times\{0,1\}$ which can be represented by a quasi-linear utility function $u_i(t_i, x_i)=t_i+v_i(x_i)$. For each $u_i\in U_A$, let $\lambda(u_i)$ denote agent i's willingness to pay for the indivisible good, that is, $u_i(t_i,0)=u_i(t_i-\lambda(u_i),1)$ for all $t_i\in R$. We consider an arbitrary preference domain U which is a finite subset of U_A. Let #U denote the number of preferences in U. A preference profile is a list $u=(u_1,\dots,u_n)\in U^n$. Let M be the amount of money which is allocated to agents. We assume that M is known and fixed. The set of feasible allocations is $Z=\{z=(t_1,\dots,t_n;x_1,\dots,x_n)\in R^n\times\{0,1\}^n | \sum_{i\in N} t_i = M \text{ and } \sum_{i\in N} x_i = 1\}$. The set of feasible transfer allocations is $Z_T=\{t=(t_1,\dots,t_n)\in R^n | \sum_{i\in N} t_i = M\}$. A *mechanism (defined on U^n)* is a function f: $U^n \rightarrow Z$, which associates a feasible allocation with each preference profile. Let $F(U^n)$ be the *set of mechanisms (defined on U^n)*. Given $f \in F(U^n)$ and $u \in U^n$, we write as $f(u)=(t_1(u),\ldots,t_n(u);x_1(u),\ldots,x_n(u))$, $f_i(u)=(t_i(u),x_i(u))$, and $f_t(u)=(t_1(u),\ldots,t_n(u))$. Given $f \in F(U^n)$, let $C_f(u)=\{i \in N| \, x_i(u)=1\}$ denote the consumer of the indivisible good at $u \in U^n$. Given $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, the notation $(\bar{u}_i, u_{-i})$ represents the preference profile obtained from u after the replacement of u_i by $\bar{u}_i$. We introduce the main axioms.

Definition 2.1. A mechanism $f \in F(U^n)$ satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(\bar{u}_i, u_{-i}))$.

Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. If a mechanism $f \in F(U^n)$ does not satisfy strategy-proofness, then there exist $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$ such that $u_i(f_i(\bar{u}_i, u_{-i})) > u_i(f_i(u))$. We then say that agent i can manipulate f at u via $\bar{u}_i$.

Definition 2.2. A mechanism $f \in F(U^n)$ satisfies *Pareto efficiency* iff for all $u \in U^n$, there is no $z \in Z$ such that [for all $i \in N$, $u_i(t_i, x_i) \geq u_i(f_i(u))$] and [for some $i \in N$, $u_i(t_i, x_i) > u_i(f_i(u))$].

Definition 2.3. A mechanism $f \in F(U^n)$ satisfies *equal compensation* iff for all $u \in U^n$ and $i, j \notin C_f(u)$, $t_i(u)=t_j(u)$.

Equal compensation requires that the non-consumers of the indivisible good should receive the same amount of money.

The following lemma is a well known result, and the proof will be omitted (see e.g. Mas-Colell, Whinston, and Green (1995), p. 862).

Lemma 2.1. *A mechanism $f \in F(U^n)$ satisfies Pareto efficiency if and only if for all $u \in U^n$, $C_f(u) \subset \underset{i \in N}{Argmax} \{ \lambda(u_i) \}$.*

2.3. Sufficiently Large but Finite Preference Domains

A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences.

Theorem 2.1. (Holmström, 1979). *There is no strategy-proof and Pareto efficient mechanism $f \in F(U_A^n)$.*[3]

Notice that the preference domain considered in Holmström (1979) contains an infinite number of preferences. We consider the possibility of strategy-proof and Pareto efficient mechanisms on some finite subsets of quasi-linear preferences. Given two integers a, b arbitrarily, let [a,...,b] denote the set of integers between a and b inclusive. Let $U_{[a,b]}=\{u_i \in U_A | \lambda(u_i) \in [a,\ldots,b]\}$.

The following lemma presents a necessary condition of strategy-proof and Pareto efficient mechanisms when the preference domain is restricted to $U_{[a,b]}$. It states that if agent j consumes the indivisible good at preference profile u, and if the other agent i can consume it by changing his preference, then the amount of money agent i receives decreases by $\lambda(u_j)-1$ at least and $\lambda(u_j)+1$ at most. In other words, agent i must pay $\lambda(u_j)-1$ at least and $\lambda(u_j)+1$ at most in order to consume the indivisible good.

Lemma 2.2. *Assume that a mechanism $f \in F(U_{[a,b]}^n)$ satisfies strategy-proofness and Pareto efficiency. For all $u \in U_{[a+1,b-1]}^n$, $i \in N$, and $\bar{u}_i \in U_{[a,b]}$, if $C_f(u)=\{j\} \neq \{i\}=C_f(\bar{u}_i, u_{-i})$, then $\lambda(u_j)-1 \leq t_i(u)-t_i(\bar{u}_i, u_{-i}) \leq \lambda(u_j)+1$.*

Proof. Suppose first that for some $u \in U_{[a+1,b-1]}^n$, $i \in N$, and $\bar{u}_i \in U_{[a,b]}$, $C_f(u)=\{j\} \neq \{i\}=C_f(\bar{u}_i, u_{-i})$ and $t_i(u)-t_i(\bar{u}_i, u_{-i})<\lambda(u_j)-1$. Let $\hat{u}_i \in U_{[a,b]}$ be such that $\lambda(\hat{u}_i)=\lambda(u_j)-1$. It follows from Lemma 2.1 that $C_f(\hat{u}_i, u_{-i}) \neq \{i\}$. By strategy-proofness,

[3] By using a general result of Holmström (1979), Schummer (1998) proves that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences in economies with multiple indivisible goods and money.

$f_i(\widehat{u}_i,u_{-i})=(t_i(u),0)$. Since $\widehat{u}_i(t_i(\overline{u}_i,u_{-i}),1)=\widehat{u}_i(t_i(\overline{u}_i,u_{-i})+\lambda(\widehat{u}_i),0)>\widehat{u}_i(t_i(u)-\lambda(u_j)+1+\lambda(\widehat{u}_i),0)=\widehat{u}_i(t_i(u),0)$, agent i can manipulate f at $(\widehat{u}_i,u_{-i})$ via $\overline{u}_i$.

Suppose next that for some $u\in U^n_{[a+1,b-1]}$, $i\in N$, and $\overline{u}_i\in U_{[a,b]}$, $C_f(u)=\{j\}\neq\{i\}=C_f(\overline{u}_i,u_{-i})$ and $t_i(u)-t_i(\overline{u}_i,u_{-i})>\lambda(u_j)+1$. Let $\widehat{u}_i\in U_{[a,b]}$ be such that $\lambda(\widehat{u}_i)=\lambda(u_j)+1$. It follows from Lemma 2.1 that $C_f(\widehat{u}_i,u_{-i})=\{i\}$. By strategy-proofness, $f_i(\widehat{u}_i,u_{-i})=(t_i(\overline{u}_i,u_{-i}),1)$. Since $\widehat{u}_i(t_i(u),0)=\widehat{u}_i(t_i(u)-\lambda(\widehat{u}_i),1)>\widehat{u}_i(t_i(\overline{u}_i,u_{-i})+\lambda(u_j)+1-\lambda(\widehat{u}_i),1)=\widehat{u}_i(t_i(\overline{u}_i,u_{-i}),1)$, agent i can manipulate f at $(\widehat{u}_i,u_{-i})$ via u_i. **Q.E.D.**

We show the non-existence of strategy-proof and Pareto efficient mechanisms on a sufficiently large but finite number of quasi-linear preferences.

Theorem 2.2. *Let $U_{[a,b]}$ be such that $b\text{-}a>\frac{2^n+2n^2-2n-2}{n-1}$. There is no strategy-proof and Pareto efficient mechanism $f\in F(U^n_{[a,b]})$.*

Proof. Suppose that there exists a strategy-proof and Pareto efficient mechanism $f\in F(U^n_{[a,b]})$. For each $i\in N$, let $u_i^-,u_i^+\in U_{[a,b]}$ be such that $\lambda(u_i^-)=a+i$ and $\lambda(u_i^+)=b-i$. Then, $\lambda(u_1^-)<\cdots<\lambda(u_n^-)<\lambda(u_n^+)<\cdots<\lambda(u_1^+)$ and $\lambda(u_n^+)-\lambda(u_n^-)>\frac{2^n-2}{n-1}$. Let $U^*=\times_{i\in N}\{u_i^-,u_i^+\}$. Let $\overline{U}^*=\{u\in U^*|$ there are an even number of agents who reveal u_i^+ at $u\}$ and $\widehat{U}^*=\{u\in U^*|$ there are an odd number of agents who reveal u_i^+ at $u\}$. By budget balance, we have that

$$\sum_{i\in N} t_i(u)=M \quad \text{for all } u\in\overline{U}^*, \text{ and} \tag{2.1}$$

$$\sum_{i\in N} t_i(u)=M \quad \text{for all } u\in\widehat{U}^*. \tag{2.2}$$

We provide necessary conditions on $t_i(\cdot)$ for any pair of preference profiles where only agent i reveals different preferences, that is, (u_i^-,u_{-i}), $(u_i^+,u_{-i})\in U^*$. We consider the following six cases.

Case 1. Let $(u_i^-,u_{-i})\in\widehat{U}^*$ and $(u_i^+,u_{-i})\in\overline{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $j\leq i$. It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{j\}$. By strategy-proofness,

$$t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})=0. \tag{2.3}$$

Case 2. Let $(u_i^-,u_{-i})\in\overline{U}^*$ and $(u_i^+,u_{-i})\in\widehat{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $j\leq i$. It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{j\}$. By strategy-proofness,

$$t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})=0. \tag{2.4}$$

Case 3. Let $(u_i^-,u_{-i})\in\widehat{U}^*$ and $(u_i^+,u_{-i})\in\overline{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $i<j<n$. Suppose that agent j reveals u_j^- at (u_i^-,u_{-i}). Since $\lambda(u_j^-)<\lambda(u_n^-)<\lambda(u_n^+)$, it follows from Lemma 2.1 that $C_f(u_i^-,u_{-i})\neq\{j\}$. This is a contradiction. Hence, agent j reveals u_j^+ at (u_i^-,u_{-i}). It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{i\}$. It follows from Lemma 2.2 that

$$\lambda(u_j^+)-1\leq t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})\leq\lambda(u_j^+)+1. \tag{2.5}$$

Case 4. Let $(u_i^-,u_{-i})\in\overline{U}^*$ and $(u_i^+,u_{-i})\in\widehat{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $i<j<n$. Suppose that agent j reveals u_j^- at (u_i^-,u_{-i}). Since $\lambda(u_j^-)<\lambda(u_n^-)<\lambda(u_n^+)$, it follows from Lemma 2.1 that $C_f(u_i^-,u_{-i})\neq\{j\}$. This is a contradiction. Hence, agent j reveals u_j^+ at (u_i^-,u_{-i}). It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{i\}$. It follows from Lemma 2.2 that

$$\lambda(u_j^+)-1\leq t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})\leq\lambda(u_j^+)+1. \tag{2.6}$$

Case 5. Let $(u_i^-,u_{-i})\in\widehat{U}^*$ and $(u_i^+,u_{-i})\in\overline{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $i<j=n$. That is, $(u_i^-,u_{-i})=(u_1^-,\ldots,u_{n-1}^-,u_n^+)$. It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{i\}$. It follows from Lemma 2.2 that

$$\lambda(u_n^+)-1\leq t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})\leq\lambda(u_n^+)+1. \tag{2.7}$$

Case 6. Let $(u_i^-,u_{-i})\in\overline{U}^*$ and $(u_i^+,u_{-i})\in\widehat{U}^*$ be such that $C_f(u_i^-,u_{-i})=\{j\}$ with $i<j=n$. That is, $(u_i^-,u_{-i})=(u_1^-,\ldots,u_n^-)$. It follows from Lemma 2.1 that $C_f(u_i^+,u_{-i})=\{i\}$. It follows from Lemma 2.2 that

$$\lambda(u_n^-)-1\leq t_i(u_i^-,u_{-i})-t_i(u_i^+,u_{-i})\leq\lambda(u_n^-)+1. \tag{2.8}$$

We count the number of inequalities derived in Cases 3 and 4. Fix any j ($\neq$1,n). The condition $C_f(u_i^-,u_{-i})=\{j\}$ requires that $u_1=u_1^-,\ldots,u_{j-1}=u_{j-1}^-,u_j=u_j^+$. Each agent k=1,..., j-1 is a possible candidate for agent i. Each agent l=j+1,...,n reveals either u_l^- or u_l^+. Thus, for any j ($\neq$1,n), we derive $(j-1)\cdot2^{n-j}$ inequalities. By the summation through j, we have that $\sum_{j=2}^{n-1}(j-1)\cdot2^{n-j}=2\{\sum_{j=2}^{n-1}(j-1)\cdot2^{n-j}\}-\sum_{j=2}^{n-1}(j-1)\cdot2^{n-j}=2^{n-1}+2^{n-2}+\cdots+2^2-2(n-2)=\sum_{j=2}^{n-1}2^j-2(n-2)=2\{\sum_{j=2}^{n-1}2^j\}-\sum_{j=2}^{n-1}2^j-2(n-2)=2^n-2^2-2(n-2)=2^n-2n$. Notice that each case provides the same number of inequalities for any j ($\neq$1,n). Therefore, each case provides $2^{n-1}-n$ inequalities.

We count the number of inequalities derived in Cases 5 and 6. The condition $C_f(u_i^-,u_{-i})=\{n\}$ requires that $u_1=u_1^-,\ldots,u_{n-1}=u_{n-1}^-$. Each agent k=1,...,n-1 is a possible candidate for agent i. Agent n reveals either u_n^- or u_n^+. Thus, we derive 2(n-1) inequalities. Notice that each case provides the same number of inequalities. Therefore, each case provides n-1 inequalities.

We consider the summation of all the equations (or inequalities) in (2.1), (2.3), (2.5), (2.7) and all the equations (or inequalities) multiplied -1 in (2.2), (2.4), (2.6), (2.8). For each $u \in \overline{U}^*$ and $i \in N$, the term $t_i(u)$ appears once in (2.1) and once in one of (2.3) - (2.8). For each $u \in \widehat{U}^*$ and $i \in N$, the term $t_i(u)$ appears once in (2.2) and once in one of (2.3) - (2.8). Notice that the terms $t_i(u)$ cancel out each other in the summation process. Since (2.1) and (2.2) provide the same number of equations, the terms M cancel out each other in the summation process. Since (2.5) and (2.6) provide the same number of inequalities for any j ($\neq$1,n), the terms $\lambda(u_j^+)$ cancel out each other in the summation process. Therefore, the summation provides the inequality $-2^n+2+(n-1)\{\lambda(u_n^+)-\lambda(u_n^-)\}\leq 0 \leq 2^n-2+(n-1)\{\lambda(u_n^+)-\lambda(u_n^-)\}$. Since $\lambda(u_n^+)-\lambda(u_n^-) > \frac{2^n-2}{n-1}$, the left-hand inequality is a contradiction. **Q.E.D.**

2.4. Small Preference Domains

In this section we consider the possibility of strategy-proof, Pareto efficient, and equally compensatory mechanisms on small preference domains. That is, we tackle the question whether or not, given any restriction of the preference domain, such mechanisms exist.

We describe a fundamental structure of strategy-proof, Pareto efficient, and equally compensatory mechanisms. We show that those mechanisms almost satisfy the constant transfer property: transfer allocations depend only on who consumes the indivisible good. We introduce some formal notation and definitions. A *transfer allocation function* is a function $\pi: N \to Z_T$, which associates a feasible transfer allocation with each consumer of the indivisible good. For each $i \in N$, we let $\pi(i)=(\pi_1(i),\dots,\pi_j(i),\dots,\pi_n(i))$, where $\pi_j(i)$ represents the amount of money agent j receives when agent i consumes the indivisible good. Let Π denote the *set of transfer allocation functions*. A mechanism $f \in F(U^n)$ satisfies the *constant transfer property on* $\mathcal{U}$ *(*$\mathcal{U} \subset U^n$*) relative to* $\pi \in \Pi$ iff for all $u \in \mathcal{U}$, $[C_f(u)=\{i\} \Rightarrow f_t(u)=\pi(i)]$.

Given an arbitrary preference domain U, we let u_i^h, $u_i^l \in U$ be such that $\lambda(u_i^h) \geq \lambda(u_i) \geq \lambda(u_i^l)$ for all $u_i \in U$. Such u_i^h and u_i^l exist uniquely since U is a finite subset of quasi-linear preferences. Given the Cartesian product of the preference domain U^n, we let $\Gamma(U^n)=\{u\in U^n|$ there is at most one agent who reveals u_i^h at $u\}$.

Ohseto (1999b) shows that any strategy-proof, Pareto efficient, and equally compensatory mechanism $f\in F(U_A^n)$ satisfies the constant transfer property on U_A^n relative to some $\pi\in\Pi$. When the preference domain is finitely restricted, we can show the following limited version of that result.[4]

Lemma 2.3. *If a mechanism $f\in F(U^n)$ satisfies strategy-proofness, Pareto efficiency, and equal compensation, then f satisfies the constant transfer property on $\Gamma(U^n)$ relative to some $\pi\in\Pi$.*

Proof. It is sufficient to show that $C_f(u)=C_f(\bar{u})$ implies $f(u)=f(\bar{u})$ for all $u, \bar{u}\in\Gamma(U^n)$. Without loss of generality, we assume that $C_f(u)=C_f(\bar{u})=\{1\}$. It follows from Lemma 2.1 that only agent 1 may reveal u_1^h at $u, \bar{u}$. It follows from Lemma 2.1 that $C_f(u_1^h,u_{-1})=C_f(u_1^h,\bar{u}_{-1})=\{1\}$. By strategy-proofness, $f_1(u)=f_1(u_1^h,u_{-1})$ and $f_1(\bar{u})=f_1(u_1^h,\bar{u}_{-1})$. By equal compensation, $f(u)=f(u_1^h,u_{-1})$ and $f(\bar{u})=f(u_1^h,\bar{u}_{-1})$. It follows from Lemma 2.1 that $C_f(u_1^h,u_2^l,u_3,\ldots,u_n)=C_f(u_1^h,u_2^l,\bar{u}_3,\ldots,\bar{u}_n)=\{1\}$. By strategy-proofness, $f_2(u_1^h,u_{-1})=f_2(u_1^h,u_2^l,u_3,\ldots,u_n)$ and $f_2(u_1^h,\bar{u}_{-1})=f_2(u_1^h,u_2^l,\bar{u}_3,\ldots,\bar{u}_n)$. By equal compensation, $f(u_1^h,u_{-1})=f(u_1^h,u_2^l,u_3,\ldots,u_n)$ and $f(u_1^h,\bar{u}_{-1})=f(u_1^h,u_2^l,\bar{u}_3,\ldots,\bar{u}_n)$. Repeatedly applying the same argument to the remaining agents, we have that $f(u)=f(u_1^h,u_{-1}^l)$ and $f(\bar{u})=f(u_1^h,u_{-1}^l)$. Therefore, $f(u)=f(\bar{u})$. **Q.E.D.**

We show the non-existence of strategy-proof, Pareto efficient, and equally compensatory mechanisms on arbitrary preference domains which consist of more than three quasi-linear preferences.

[4] Let $U=\{u_i^a,u_i^b\}$, where $\lambda(u_i^a)=1$ and $\lambda(u_i^b)=2$. Let n=3 and a mechanism $f\in F(U^n)$ be such that $f(u_1^a,u_2^a,u_3^a)=f(u_1^b,u_2^a,u_3^a)=(-1,1/2,1/2;1,0,0)$, $f(u_1^a,u_2^b,u_3^a)=f(u_1^b,u_2^b,u_3^a)=(2/3,-4/3,2/3;0,1,0)$, $f(u_1^a,u_2^a,u_3^b)=f(u_1^b,u_2^a,u_3^b)=f(u_1^a,u_2^b,u_3^b)=(2/3,2/3,-4/3;0,0,1)$, and $f(u_1^b,u_2^b,u_3^b)=(-4/3,2/3,2/3;1,0,0)$. Then, f satisfies strategy-proofness, Pareto efficiency, and equal compensation, but does not satisfy the constant transfer property on U^n.

Theorem 2.3. *Let $\#U \geq 4$. There is no strategy-proof, Pareto efficient, and equally compensatory mechanism $f \in F(U^n)$.*

Proof. Let u_i^a, u_i^b, u_i^c, u_i^d be preferences in U such that $\lambda(u_i^a)<\lambda(u_i^b)<\lambda(u_i^c)<\lambda(u_i^d)$. Suppose that there exists a strategy-proof, Pareto efficient, and equally compensatory mechanism $f \in F(U^n)$. It follows from Lemma 2.3 that f satisfies the constant transfer property on $\Gamma(U^n)$ relative to some $\pi \in \Pi$. Notice that $(u_1^a,\ldots,u_n^a)$, $(u_1^c,\ldots,u_n^c) \in \Gamma(U^n)$, and for all $i \in N$, (u_i^b,u_{-i}^a), $(u_i^d,u_{-i}^c) \in \Gamma(U^n)$. First, we assume that $C_f(u_1^a,\ldots,u_n^a)=\{j\}$. It follows from Lemma 2.1 that for all $i \neq j$, $C_f(u_i^b,u_{-i}^a)=\{i\}$. By strategy-proofness, $u_i^a(\pi_i(j),0) \geq u_i^a(\pi_i(i),1)$ and $u_i^b(\pi_i(i),1) \geq u_i^b(\pi_i(j),0)$. Hence, $\pi_i(i)+\lambda(u_i^a) \leq \pi_i(j) \leq \pi_i(i)+\lambda(u_i^b)$ for all $i \neq j$. Adding up these inequalities for all $i \neq j$, we have that $\sum_{i \neq j} \pi_i(i)+(n-1)\lambda(u_i^a) \leq \sum_{i \neq j} \pi_i(j) \leq \sum_{i \neq j} \pi_i(i)+(n-1)\lambda(u_i^b)$. By budget balance $\sum_{i \in N} \pi_i(j)=M$, we have that $(n-1)\lambda(u_i^a) \leq M-\sum_{i \in N} \pi_i(i) \leq (n-1)\lambda(u_i^b)$. Next, we assume that $C_f(u_1^c,\ldots,u_n^c)=\{k\}$. It follows from Lemma 2.1 that for all $i \neq k$, $C_f(u_i^d,u_{-i}^c)=\{i\}$. By strategy-proofness, $u_i^c(\pi_i(k),0) \geq u_i^c(\pi_i(i),1)$ and $u_i^d(\pi_i(i),1) \geq u_i^d(\pi_i(k),0)$. Hence, $\pi_i(i)+\lambda(u_i^c) \leq \pi_i(k) \leq \pi_i(i)+\lambda(u_i^d)$ for all $i \neq k$. Adding up these inequalities for all $i \neq k$, we have that $\sum_{i \neq k} \pi_i(i)+(n-1)\lambda(u_i^c) \leq \sum_{i \neq k} \pi_i(k) \leq \sum_{i \neq k} \pi_i(i)+(n-1)\lambda(u_i^d)$. By budget balance $\sum_{i \in N} \pi_i(k)=M$, we have that $(n-1)\lambda(u_i^c) \leq M-\sum_{i \in N} \pi_i(i) \leq (n-1)\lambda(u_i^d)$. Since $\lambda(u_i^b)<\lambda(u_i^c)$, this is a contradiction. **Q.E.D.**

We have two corollaries to Theorem 2.3. The first one follows from the fact that equal compensation is vacuously true when n=2. The second one follows from the fact that envy-freeness (Foley, 1967) implies Pareto efficiency and equal compensation in our model (Svensson, 1983).

Corollary 2.1. *Let $n=2$ and $\#U \geq 4$. There is no strategy-proof and Pareto efficient mechanism $f \in F(U^n)$.*[5]

[5] Schummer (1998) shows a similar result in economies with multiple indivisible goods and money.

Corollary 2.2. *Let $\#U \geq 4$. There is no strategy-proof and envy-free mechanism $f \in F(U^n)$.*[6]

Next, we characterize the set of strategy-proof and Pareto efficient mechanisms on very small preference domains in the two-agent case. The following two theorems show that strategy-proofness puts some constraint on transfer allocation functions. It turns out that there is a trade-off between the restriction of the preference domain and the constraint on transfer allocation functions. The arguments are much the same as Lemma 2.3 and Theorem 2.3, and the proofs will be omitted.

Lemma 2.4. *Let $n=2$. If a mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency, then f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$.*

Theorem 2.4. *Let $n=2$ and $\#U=3$. Assume that $U=\{u_i^a, u_i^b, u_i^c\}$, where $\lambda(u_i^a) < \lambda(u_i^b) < \lambda(u_i^c)$. A mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency if and only if (i) for all $u \in U^n$, $C_f(u) \subset \underset{i \in N}{Argmax}\{\lambda(u_i)\}$, and (ii) f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$, where $\sum_{i \in N} \pi_i(i) = M - \lambda(u_i^b)$.*

Theorem 2.5. *Let $n=2$ and $\#U=2$. Assume that $U=\{u_i^a, u_i^b\}$, where $\lambda(u_i^a) < \lambda(u_i^b)$. A mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency if and only if (i) for all $u \in U^n$, $C_f(u) \subset \underset{i \in N}{Argmax}\{\lambda(u_i)\}$, and (ii) f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$, where $M - \lambda(u_i^b) \leq \sum_{i \in N} \pi_i(i) \leq M - \lambda(u_i^a)$.*

As in the two-agent case, we can design strategy-proof, Pareto efficient, equally compensatory mechanisms on very small preference domains in the n-agent case.

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[6] It also follows from a general result of Tadenuma and Thomson (1995) that there is no strategy-proof and envy-free mechanism $f \in F(U_A^n)$.

Example 2.1. Let #U=3. Assume that $U=\{u_i^a,u_i^b,u_i^c\}$, where $\lambda(u_i^a)<\lambda(u_i^b)<\lambda(u_i^c)$. Consider mechanisms $f\in F(U^n)$ such that for all $u\in U^n$, (i) $C_f(u)\subset \underset{i\in N}{\mathrm{Argmax}}\ \{\lambda(u_i)\}$, (ii) $t_i(u)=\frac{M-(n-1)\lambda(u_i^b)}{n}$ for $i\in C_f(u)$, and (iii) $t_j(u)=\frac{M+\lambda(u_i^b)}{n}$ for all $j\notin C_f(u)$. Then, these mechanisms satisfy strategy-proofness, Pareto efficiency, and equal compensation.

Example 2.2. Let #U=2. Assume that $U=\{u_i^a,u_i^b\}$, where $\lambda(u_i^a)<\lambda(u_i^b)$. Consider mechanisms $f\in F(U^n)$ such that for all $u\in U^n$, (i) $C_f(u)\subset \underset{i\in N}{\mathrm{Argmax}}\ \{\lambda(u_i)\}$, (ii) $t_i(u)=\frac{M-(n-1)\lambda'}{n}$ for $i\in C_f(u)$, and (iii) $t_j(u)=\frac{M+\lambda'}{n}$ for all $j\notin C_f(u)$, where $\lambda(u_i^a)\leq\lambda'\leq\lambda(u_i^b)$. Then, these mechanisms satisfy strategy-proofness, Pareto efficiency, and equal compensation.

2.5. Conclusion

We studied the problem of allocating a single indivisible good when monetary compensation is possible. A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences. We considered some finite restrictions of the preference domain in order to understand how strong the impossibility result is. We proved that there is no strategy-proof and Pareto efficient mechanism when (i) the preference domain consists of a sufficiently large but finite number of quasi-linear preferences (Theorem 2.2), or (ii) the preference domain consists of more than three quasi-linear preferences and equal compensation is imposed on mechanisms (Theorem 2.3). We conclude that the impossibility result is very strong since such drastic restrictions of the preference domain are very unrealistic.

Chapter 3

Strategy-Proof Allocation Mechanisms for Economies with an Indivisible Good **

3.1. Introduction

We consider economies with a single indivisible good and a transferable good. The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. This chapter looks for desirable allocation mechanisms which determine who consumes the indivisible good and how much compensation the other agents receive from the consumer. We think of the following four axioms as desiderata for mechanisms. The first axiom is strategy-proofness. Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. It is an attractive requirement from the viewpoint of decentralization. The next two axioms are related to equity. They are individual rationality (all agents end up no worse off than at the status quo) and equal compensation (the non-consumers of the indivisible good receive the same amount of monetary compensation). The last axiom is demand monotonicity, which requires that the consumer of the indivisible good remain unchanged when the consumer increases his demand for the indivisible good and no other agents increase their demand. This requirement is necessary for Pareto efficiency, but rather weaker than Pareto efficiency. We attempt to design mechanisms that satisfy these four axioms.

There is a huge literature on strategy-proofness. It is well known that strategy-proofness is a strong requirement in a social choice framework. The Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) states that any strategy-proof mechanism whose range contains more than two outcomes must be dictatorial. Under the requirement of Pareto efficiency, the parallel impossibility results can be established in

** This chapter is based on Ohseto (1999b).

economic environments. Zhou (1991b), improving upon Hurwicz (1972) and Dasgupta, Hammond, and Maskin (1979), shows that strategy-proofness and Pareto efficiency imply dictatorship in two-agent pure exchange economies. Hurwicz and Walker (1990) prove that any strategy-proof mechanism is generically Pareto inefficient in a model that includes pure exchange economies with a transferable good. These results suggest that we should give up Pareto efficiency in order to construct reasonable strategy-proof mechanisms. Barbera and Jackson (1995) characterize the set of strategy-proof, anonymous, and non-bossy mechanisms in pure exchange economies. The class of such mechanisms is rather rich; moreover, those mechanisms fulfill satisfactory properties of coalitional strategy-proofness, envy-freeness (Foley, 1967), and individual rationality. Serizawa (1996, 1999) presents similar characterizations in economies with one private good and one public good. Their characterizations enable us to understand how inefficient strategy-proof mechanisms are.

In economies with an indivisible good and money, it follows from a general result of Tadenuma and Thomson (1995) that there is no strategy-proof and envy-free mechanism.[7] Although envy-freeness is a concept of equity, it implies Pareto efficiency in these economies (Svensson, 1983). It also follows from a general result of Holmström (1979) that there is no strategy-proof and Pareto efficient mechanism in these economies. In this chapter we adopt individual rationality and equal compensation as mild requirements of equity, and demand monotonicity as a minimum requirement of efficiency. We will check in the next section that each axiom is strictly weaker than envy-freeness, and these axioms together do not imply Pareto efficiency in these economies.

First, we show that if a mechanism satisfies strategy-proofness, equal compensation, and demand monotonicity, then it satisfies the constant transfer property (the allocation of monetary transfer depends only on who receives the indivisible good). Second, we prove that any mechanism that satisfies our four axioms allocates the indivisible good to one of the pre-specified one or two agent(s), and disregards preferences of agents other than the

[7] They establish a more general result that any subcorrespondence of the envy-free correspondence is manipulable in the sense of Hurwicz (1972).

pre-specified agent(s). When the set of potential consumers of the indivisible good consists of two agents (without loss of generality, we call them agents 1 and 2), we construct two types of mechanisms: the decisive mechanisms and the unilaterally unanimous mechanisms. Decisiveness requires that agent 1 (agent 2 respectively) get the indivisible good if he wants it at the cost of a pre-specified level of compensation, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. Unilateral unanimity requires that agent 1 (agent 2 respectively) get the indivisible good if both agents want agent 1 (agent 2 respectively) to get it under a pre-specified monetary transfer, and agent 2 (agent 1 respectively) get the indivisible good without compensation otherwise. When the set of potential consumers consists of only one agent, we construct the dictatorial mechanisms: one of the agents always consumes the indivisible good without compensation. Finally, we provide the following characterization: a mechanism satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity if and only if it is decisive, unilaterally unanimous, or dictatorial. This characterization enables us to understand that those mechanisms are very inefficient. Moreover, those mechanisms have serious asymmetry (e.g. (i) the decisive mechanisms determine allocations on the basis of only one agent's preferences; and (ii) the unilaterally unanimous mechanisms and the dictatorial mechanisms always guarantee one of the agents at least the utility level of having the indivisible good without compensation). In contrast to Barbera and Jackson (1995), the presence of an indivisible good induces serious asymmetry in mechanisms.

This chapter is organized as follows. In Section 3.2, we introduce notation and definitions. In Section 3.3, we describe a fundamental structure of strategy-proof, individually rational, equally compensatory, and demand monotonic mechanisms. In Section 3.4, we provide a full characterization of those mechanisms. In Section 3.5, we summarize the results and state some remarks.

3.2. Notation and Definitions

Let $N=\{1,...,n\}$ $(n\geq 2)$ be the set of agents. Consider a single indivisible good and a transferable good. The indivisible good can be assigned to only one agent. The transferable good, regarded as money, is used for compensation. The society must decide who consumes the indivisible good and how much compensation the other agents receive. A consumption bundle of agent i is a pair $(t_i, x_i) \in R \times \{0,1\}$, where $t_i \in R$ represents the net monetary transfer which agent i receives (if $t_i > 0$) or agent i pays (if $t_i < 0$), and $x_i \in \{0,1\}$ denotes agent i's consumption of the indivisible good. The set of feasible allocations is $Z = \{(t_1, \ldots, t_n; x_1, \ldots, x_n) \in R^n \times \{0,1\}^n \mid \sum_{i \in N} t_i = 0 \text{ and } \sum_{i \in N} x_i = 1\}$. The set of feasible transfer allocations is $Z_T = \{(t_1, \ldots, t_n) \in R^n \mid \sum_{i \in N} t_i = 0\}$.

Each agent $i \in N$ has a preference on his consumption space $R \times \{0,1\}$. Let U be the set of all quasi-linear preferences which can be represented by a quasi-linear utility function $u_i(t_i, x_i) = t_i + v_i(x_i)$, where $0 = v_i(0) < v_i(1) < +\infty$.[8] For each $u_i \in U$, let $\lambda(u_i) = v_i(1) - v_i(0)$. We can interpret $\lambda(u_i)$ as agent i's willingness to pay for the indivisible good, that is, $u_i(t_i, 0) = u_i(t_i - \lambda(u_i), 1)$ for all $t_i \in R$. Notice that $u_i \in U$ and $\overline{u}_i \in U$ are identical preferences if and only if $\lambda(u_i) = \lambda(\overline{u}_i)$. A list $u = (u_1, \ldots, u_n) \in U^n$ is called a preference profile.

For each coalition C in N, let -C represent coalition N\C. Let $(\overline{u}_C, u_{-C})$ denote the preference profile whose i-th component is $\overline{u}_i$ if $i \in C$ and u_i if $i \notin C$. When $C = \{i\}$, we simply denote $(\overline{u}_{\{i\}}, u_{-\{i\}})$ by $(\overline{u}_i, u_{-i})$.

A *mechanism* is a function $f: U^n \to Z$, which associates a feasible allocation with each preference profile. For each $u \in U^n$, we let $f(u) = (t_1(u), \ldots, t_n(u); x_1(u), \ldots, x_n(u))$. Let f_t and f_i be functions such that for each $u \in U^n$, $f_t(u) = (t_1(u), \ldots, t_n(u))$ and $f_i(u) = (t_i(u), x_i(u))$, respectively. For each $u \in U^n$, let $C_f(u) = \{i \in N \mid x_i(u) = 1\}$ represent the consumer of the indivisible good, and $NC_f(u) = \{i \in N \mid x_i(u) = 0\}$ represent the non-consumers of the indivisible good. Notice that $\#C_f(u) = 1$ and $\#NC_f(u) = n-1$ for each $u \in U^n$. Let $R_f = \{i \in N \mid$ there exists some $u \in U^n$ such that $C_f(u) = \{i\}\}$ denote the set of agents who have an opportunity to receive the indivisible good through the mechanism f.

We think of the following four axioms as desiderata for mechanisms.

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[8] This implies that the indivisible good is a "good" for all agent. This is not a restrictive assumption since we present impossibility results.

Definition 3.1. A mechanism f satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\overline{u}_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(\overline{u}_i, u_{-i}))$.

If a mechanism f does not satisfy strategy-proofness, then there exist some $u \in U^n$, $i \in N$, and $\overline{u}_i \in U$ such that $u_i(f_i(\overline{u}_i, u_{-i})) > u_i(f_i(u))$; thus we say that agent i can manipulate f at u via $\overline{u}_i$.

Definition 3.2. A mechanism f satisfies *individual rationality* iff for all $u \in U^n$ and $i \in N$, $u_i(f_i(u)) \geq u_i(0,0)$.

Definition 3.3. A mechanism f satisfies *equal compensation* iff for all $u \in U^n$ and i, $j \in NC_f(u)$, $t_i(u) = t_j(u)$.

Definition 3.4. A mechanism f satisfies *demand monotonicity* iff for all u, $\overline{u} \in U^n$ such that $\lambda(\overline{u}_i) > \lambda(u_i)$ for $i \in C_f(u)$ and $\lambda(\overline{u}_j) \leq \lambda(u_j)$ for all $j \in NC_f(u)$, $C_f(u) = C_f(\overline{u})$.

Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. Individual rationality requires that all agents should end up no worse off than at the status quo. Equal compensation requires that the amount of monetary transfer should be the same for all non-consumers of the indivisible good. Demand monotonicity requires that an increase of the consumer's demand and non-increase of the non-consumers' demand should not change the consumer of the indivisible good.

These four axioms are independent as shown in Examples 4.2 - 4.5. Here we present simple examples which draw a clear distinction between strategy-proofness and demand monotonicity.

Example 3.1. Let n=3 and a mechanism f be such that for all $u \in U^3$, if $\lambda(u_3) \geq 1$, then $C_f(u) = \{1\}$ and $f_t(u) = (0,0,0)$, and if $\lambda(u_3) < 1$, then $C_f(u) = \{2\}$ and $f_t(u) = (0,0,0)$. Then, f satisfies strategy-proofness, individual rationality, and equal compensation, but does not satisfy demand monotonicity.

Example 3.2. Let n=2 and a mechanism f be such that for all $u \in U^2$, if $\lambda(u_1) \geq \lambda(u_2)$, then $C_f(u)=\{1\}$ and $f_t(u)=(0,0)$, and if $\lambda(u_1)<\lambda(u_2)$, then $C_f(u)=\{2\}$ and $f_t(u)=(0,0)$. Then, f satisfies demand monotonicity, individual rationality, and equal compensation, but does not satisfy strategy-proofness.

Example 3.1 shows that strategy-proofness does not imply demand monotonicity under the requirements of individual rationality and equal compensation: to check that f violates demand monotonicity, it is sufficient to see that $C_f(u)=\{1\}$ when $\lambda(u_1)=\lambda(u_2)=\lambda(u_3)=1$, and $C_f(\overline{u})=\{2\}$ when $\lambda(\overline{u}_1)=2$, $\lambda(\overline{u}_2)=\lambda(\overline{u}_3)=\frac{1}{2}$. This mechanism depends only on agent 3's preferences and never allocates the indivisible good to agent 3. However, it is possible to construct a mechanism which satisfies our axioms except demand monotonicity, which incorporates preferences of all agents, and which potentially allocates the indivisible good to any agent (see Example 3.4). Example 3.2 proves that demand monotonicity does not imply strategy-proofness under the requirements of individual rationality and equal compensation: to check that f violates strategy-proofness, it is sufficient to see that $C_f(u_1,u_2)=\{1\}$ and $C_f(u_1,\overline{u}_2)=\{2\}$ when $\lambda(u_1)=\lambda(u_2)=1$, $\lambda(\overline{u}_2)=2$. This example also satisfies Pareto efficiency defined below. A similar example that contains any number of agents will be constructed in Example 3.7.

We then discuss the relationships between our axioms (especially demand monotonicity and strategy-proofness) and Pareto efficiency. In this model, Pareto efficiency can be represented as follows (see e.g. Mas-Colell, Whinston, and Green (1995), p. 862). A mechanism f satisfies *Pareto efficiency* iff for all $u \in U^n$, $C_f(u) \subset \operatorname{Argmax}_{i \in N} \{\lambda(u_i)\}$. We prove that Pareto efficiency implies demand monotonicity, but not vice versa.

Lemma 3.1. *If a mechanism f satisfies Pareto efficiency, then f satisfies demand* *monotonicity.*

Proof. Consider any $u \in U^n$. It follows from Pareto efficiency that $C_f(u)=\{i\}$ implies $\lambda(u_i) \geq \lambda(u_j)$ for all $j \neq i$. Consider any $\bar{u} \in U^n$ such that $\lambda(\bar{u}_i) > \lambda(u_i)$ for $i \in C_f(u)$ and $\lambda(\bar{u}_j) \leq \lambda(u_j)$ for all $j \in NC_f(u)$. It is clear that $\lambda(\bar{u}_i) > \lambda(\bar{u}_j)$ for all $j \neq i$, and thus it follows from Pareto efficiency that $C_f(\bar{u})=\{i\}$. **Q.E.D.**

Example 3.3. Let n=2 and a mechanism f be such that for all $u \in U^2$, if $\lambda(u_1) \geq 1$, then $C_f(u)=\{1\}$ and $f_t(u)=(-1,1)$, and if $\lambda(u_1)<1$, then $C_f(u)=\{2\}$ and $f_t(u)=(0,0)$. Then, f satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity, but does not satisfy Pareto efficiency.

Example 3.3 shows that demand monotonicity does not imply Pareto efficiency: to check that f violates Pareto efficiency, it is sufficient to see that $C_f(u)=\{1\}$ when $\lambda(u_1)=1$, $\lambda(u_2)=2$. This example also proves the existence of the mechanism which satisfies our four axioms (it is a member of the decisive mechanisms which we will define in Section 3.4).

It follows from a general result of Holmström (1979) that there is no strategy-proof and Pareto efficient mechanism. Example 3.2 proves that Pareto efficiency does not imply strategy-proofness, and Example 3.3 proves that strategy-proofness does not imply Pareto efficiency. Therefore, strategy-proofness and Pareto efficiency are independent.

We finally discuss the relationships between our axioms and envy-freeness. A mechanism f satisfies *envy-freeness* iff for all $u \in U$ and $i, j \in N$, $u_i(f_i(u)) \geq u_i(f_j(u))$. A general result of Tadenuma and Thomson (1995) implies the non-existence of strategy-proof and envy-free mechanisms. It follows from Lemmas 1 and 2 in Tadenuma and Thomson (1995) that envy-freeness implies individual rationality. It is evident from the definitions that envy-freeness implies equal compensation. It follows from the fact that envy-freeness implies Pareto efficiency (Svensson, 1983) and Lemma 3.1 that envy-freeness implies demand monotonicity. Therefore, our axioms except strategy-proofness are strictly weaker than envy-freeness. The relationships among our axioms, Pareto efficiency, and envy-freeness are illustrated in Figure 3.1.

3.3. Preliminary Results

In this section we describe a fundamental structure of strategy-proof, individually rational, equally compensatory, and demand monotonic mechanisms. First, we prove that those mechanisms have some constancy relative to transfer allocations, that is, those mechanisms specify the same pattern of transfer allocations whenever they allocate the indivisible good to the same agent.

A *transfer allocation function* is a function π: $N \to Z_T$, which associates a feasible transfer allocation with each consumer of the indivisible good. For each $i \in N$, we let $\pi(i)=(\pi_1(i),\ldots,\pi_j(i),\ldots,\pi_n(i))$, where $\pi_j(i)$ represents the amount of money which agent j receives when agent i consumes the indivisible good. Let Π denote the *set of transfer allocation functions*. A mechanism f satisfies the *constant transfer property relative to* $\pi \in \Pi$ iff for all $u \in U^n$, $[C_f(u)=\{i\} \Rightarrow f_t(u)=\pi(i)]$. A mechanism f satisfies the *constant transfer property* iff for some $\pi \in \Pi$, f satisfies the constant transfer property relative to π.

Theorem 3.1. *If a mechanism f satisfies strategy-proofness, equal compensation, and demand monotonicity, then f satisfies the constant transfer property.*

To prove this theorem, we have prepared the following useful lemmas.

Lemma 3.2. *For any mechanism f, $u=(u_i, u_{-i}) \in U^n$, and $\overline{u}_i \in U$, if f satisfies strategy-proofness, $f_i(u)=(t_i(u),1)$, and $\lambda(\overline{u}_i)>\lambda(u_i)$, then $f_i(\overline{u}_i, u_{-i})=(t_i(u),1)$.*

Proof. Suppose toward contradiction that $f_i(\overline{u}_i,u_{-i})=(t_i(\overline{u}_i,u_{-i}),x_i(\overline{u}_i,u_{-i}))\neq(t_i(u),1)$. If $x_i(\overline{u}_i,u_{-i})=0$ and $t_i(\overline{u}_i,u_{-i})>t_i(u)+\lambda(u_i)$, then since $u_i(t_i(u),1)=u_i(t_i(u)+\lambda(u_i),0)$, agent i can manipulate f at u via $\overline{u}_i$. If $x_i(\overline{u}_i,u_{-i})=0$ and $t_i(\overline{u}_i,u_{-i})<t_i(u)+\lambda(\overline{u}_i)$, then since $\overline{u}_i(t_i(u),1)=\overline{u}_i(t_i(u)+\lambda(\overline{u}_i),0)$, agent i can manipulate f at $(\overline{u}_i,u_{-i})$ via u_i. If $x_i(\overline{u}_i,u_{-i})=0$, then it must hold that $t_i(u)+\lambda(u_i)\geq t_i(\overline{u}_i,u_{-i})\geq t_i(u)+\lambda(\overline{u}_i)$, which contradicts $\lambda(\overline{u}_i)>\lambda(u_i)$. Hence, $x_i(\overline{u}_i,u_{-i})=1$. It is clear that $x_i(\overline{u}_i,u_{-i})=1$ and $t_i(\overline{u}_i,u_{-i})\neq t_i(u)$ contradict strategy-proofness. **Q.E.D.**

Lemma 3.3. *For any mechanism f, $u=(u_i,u_{-i})\in U^n$, and $\bar{u}_i\in U$, if f satisfies strategy-proofness, $f_i(u)=(t_i(u),0)$, and $\lambda(u_i)>\lambda(\bar{u}_i)$, then $f_i(\bar{u}_i,u_{-i})=(t_i(u),0)$.*

Proof. Suppose toward contradiction that $f_i(\bar{u}_i,u_{-i})=(t_i(\bar{u}_i,u_{-i}),x_i(\bar{u}_i,u_{-i}))\neq(t_i(u),0)$. If $x_i(\bar{u}_i,u_{-i})=1$ and $t_i(\bar{u}_i,u_{-i})>t_i(u)-\lambda(u_i)$, then since $u_i(t_i(u),0)=u_i(t_i(u)-\lambda(u_i),1)$, agent i can manipulate f at u via $\bar{u}_i$. If $x_i(\bar{u}_i,u_{-i})=1$ and $t_i(\bar{u}_i,u_{-i})<t_i(u)-\lambda(\bar{u}_i)$, then since $\bar{u}_i(t_i(u),0)=\bar{u}_i(t_i(u)-\lambda(\bar{u}_i),1)$, agent i can manipulate f at $(\bar{u}_i,u_{-i})$ via u_i. If $x_i(\bar{u}_i,u_{-i})=1$, then it must hold that $t_i(u)-\lambda(u_i)\geq t_i(\bar{u}_i,u_{-i})\geq t_i(u)-\lambda(\bar{u}_i)$, which contradicts $\lambda(u_i)>\lambda(\bar{u}_i)$. Hence, $x_i(\bar{u}_i,u_{-i})=0$. It is clear that $x_i(\bar{u}_i,u_{-i})=0$ and $t_i(\bar{u}_i,u_{-i})\neq t_i(u)$ contradict strategy-proofness.
Q.E.D.

Lemma 3.4. *Assume that a mechanism f satisfies strategy-proofness, equal compensation, and demand monotonicity. For all u, $\bar{u}\in U^n$ such that $\lambda(\bar{u}_i)>\lambda(u_i)$ for $i\in C_f(u)$ and $\lambda(\bar{u}_j)<\lambda(u_j)$ for all $j\in NC_f(u)$, it holds that $f(u)=f(\bar{u})$.*

Proof. It follows from Lemma 3.2 that $f_i(u)=f_i(\bar{u}_i,u_{-i})$. By equal compensation, it holds that $f(u)=f(\bar{u}_i,u_{-i})$. Choose arbitrarily $j\in NC_f(u)$. By demand monotonicity, it holds that $C_f(\bar{u}_{\{i,j\}},u_{-\{i,j\}})=C_f(u)=\{i\}$. It follows from Lemma 3.3 that $f_j(\bar{u}_i,u_{-i})=f_j(\bar{u}_{\{i,j\}},u_{-\{i,j\}})$. By equal compensation, it holds that $f(\bar{u}_i,u_{-i})=f(\bar{u}_{\{i,j\}},u_{-\{i,j\}})$. Repeat this argument successively to all $k\in NC_f(u)$ with $k\neq j$. Then, we have $f(u)=f(\bar{u})$. **Q.E.D.**

Proof of Theorem 3.1. Choose any u, $\bar{u}\in U^n$ such that $C_f(u)=C_f(\bar{u})$. Consider $\hat{u}\in U^n$ such that $\lambda(\hat{u}_i)>\max\{\lambda(u_i),\lambda(\bar{u}_i)\}$ for $i\in C_f(u)=C_f(\bar{u})$ and $\lambda(\hat{u}_j)<\min\{\lambda(u_j),\lambda(\bar{u}_j)\}$ for all $j\in NC_f(u)=NC_f(\bar{u})$. It follows from Lemma 3.4 that $f(u)=f(\hat{u})$ and $f(\bar{u})=f(\hat{u})$. Hence, it holds that $f_t(u)=f_t(\bar{u})$. This implies that f satisfies the constant transfer property.
Q.E.D.

Theorem 3.1 puts a strong restriction on the structure of mechanisms, but no restriction on the choice of transfer allocation functions. The following lemmas provide some necessary conditions on transfer allocation functions.

Lemma 3.5. *Assume that a mechanism f satisfies the constant transfer property relative to* $\pi\in\Pi$. *If f satisfies individual rationality, then there exists* $i\in R_f$ *such that* $\pi(i)=(0,\ldots,0)$.

Proof. For each $i\in R_f$, there exists $u\in U^n$ such that $C_f(u)=\{i\}$. By individual rationality, it must hold that $t_j(u)=\pi_j(i)\geq 0$ for all $j\in NC_f(u)$. By budget balance, $t_i(u)=\pi_i(i)\leq 0$. Therefore, $\pi_i(i)\leq 0$ for all $i\in R_f$. Suppose toward contradiction that there is no agent $k\in R_f$ such that $\pi_k(k)=0$, that is, $\pi_i(i)<0$ for all $i\in R_f$. Consider $\overline{u}\in U^n$ such that $-\lambda(\overline{u}_i)>\pi_i(i)$ for all $i\in R_f$. It follows from individual rationality that $t_i(\overline{u})=\pi_i(i)\geq-\lambda(\overline{u}_i)$ for $i\in C_f(\overline{u})\subset R_f$, which contradicts the construction of $\overline{u}_i$. **Q.E.D.**

Lemma 3.6. *Assume that a mechanism f satisfies the constant transfer property relative to* $\pi\in\Pi$. *If f satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity, then there exist no two agents* $i, j\in R_f$ *such that* $\pi(i)=\pi(j)=(0,\ldots,0)$.

Proof. Let $\overline{R}_f=\{i\in R_f | \pi(i)=(0,\ldots,0)\}$. Assume, on the contrary, that there exist two agents $i, j\in\overline{R}_f$. Since $i, j\in R_f$, there exist $u, \overline{u}\in U^n$ such that $C_f(u)=\{i\}$ and $C_f(\overline{u})=\{j\}$. Consider $\widehat{u}\in U^n$ such that $-\lambda(\widehat{u}_k)>\pi_k(k)$ for all $k\in R_f\backslash\overline{R}_f$ and $\lambda(\widehat{u}_l)<\min\{\lambda(u_l),\lambda(\overline{u}_l)\}$ for all $l\in N$. Consider $\widetilde{u}_i, \widetilde{u}_j\in U$ such that $\lambda(\widetilde{u}_i)>\lambda(u_i)$ and $\lambda(\widetilde{u}_j)>\lambda(\overline{u}_j)$. It follows from Lemma 3.4 that $f(u)=f(\widetilde{u}_i,\widehat{u}_{-i})$ and $f(\overline{u})=f(\widetilde{u}_j,\widehat{u}_{-j})$. Hence, $f_i(\widetilde{u}_i,\widehat{u}_{-i})=(0,1)$ and $f_j(\widetilde{u}_j,\widehat{u}_{-j})=(0,1)$. We show that $C_f(\widehat{u})$ is indeterminable. If $C_f(\widehat{u})=\{k\}$ for some $k\in R_f\backslash\overline{R}_f$, then $t_k(\widehat{u})=\pi_k(k)\geq-\lambda(\widehat{u}_k)$ by individual rationality, which contradicts the construction of $\widehat{u}_k$. If $C_f(\widehat{u})=\{l\}$ for some $l\in\overline{R}_f\backslash\{i\}$, then since $f_i(\widehat{u})=(0,0)$, agent i can manipulate f at $\widehat{u}$ via $\widetilde{u}_i$. If $C_f(\widehat{u})=\{i\}$, then since $f_j(\widehat{u})=(0,0)$, agent j can manipulate f at $\widehat{u}$ via $\widetilde{u}_j$.
Q.E.D.

These lemmas show that there must be asymmetry in mechanisms, that is, there is only one agent who can consume the indivisible good without compensating the other agents.

3.4. Main Results

In this section we provide a full characterization of strategy-proof, individually rational, equally compensatory, and demand monotonic mechanisms. First, we show that those mechanisms have serious asymmetry, that is, the set of potential consumers of the indivisible good through the mechanisms consists of at most two agents.

Theorem 3.2. *If a mechanism f satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity, then $\#R_f \leq 2$.*

Proof. Assume, on the contrary, that $\#R_f \geq 3$. Without loss of generality, we assume that $R_f \supset \{1,2,n\}$. It follows from Theorem 3.1 that f satisfies the constant transfer property relative to some $\pi \in \Pi$. It follows from Lemmas 3.5 and 3.6 that there exists only one agent $i \in R_f$ such that $\pi(i)=(0,...,0)$. Without loss of generality, we assume that $\pi(n)=(0,...,0)$. Hence, $\pi(i) \neq (0,...,0)$ for all $i \in R_f \backslash \{n\}$. By individual rationality and budget balance, $\pi_i(i)<0$ for all $i \in R_f \backslash \{n\}$. By equal compensation, $\pi_j(i)>0$ for all $i \in R_f \backslash \{n\}$ and all $j \neq i$. Since $R_f \supset \{1,2\}$, there exist u, $u' \in U^n$ such that $C_f(u)=\{1\}$ and $C_f(u')=\{2\}$. For all $i \in R_f \backslash \{n\}$, choose some $\overline{u}_i \in U$ such that $\lambda(\overline{u}_i)<\min\{\lambda(u_i),\lambda(u_i')\}$ and $-\lambda(\overline{u}_i)>\pi_i(i)$. For all $j \in -R_f \cup \{n\}$, choose some $\overline{u}_j \in U$ such that $\lambda(\overline{u}_j)<\min\{\lambda(u_j),\lambda(u_j')\}$. Choose some $\widehat{u}_1$, $\widehat{u}_2 \in U$ such that $\lambda(\widehat{u}_1)>\lambda(u_1)$ and $\lambda(\widehat{u}_2)>\lambda(u_2')$. It follows from Lemma 3.4 that $f(u)=f(\widehat{u}_1,\overline{u}_{-1})$ and $f(u')=f(\widehat{u}_2,\overline{u}_{-2})$. Hence, $C_f(\widehat{u}_1,\overline{u}_{-1})=\{1\}$ and $C_f(\widehat{u}_2,\overline{u}_{-2})=\{2\}$. Choose some $\widetilde{u}_1$, $\widetilde{u}_2 \in U$ such that $\pi_1(2)-\pi_1(1)>\lambda(\widetilde{u}_1)>-\pi_1(1)$ and $\pi_2(1)-\pi_2(2)>\lambda(\widetilde{u}_2)>-\pi_2(2)$. The following steps lead to a contradiction (see Figure 3.2).

Step 1. $C_f(\widetilde{u}_1,\overline{u}_{-1})=\{1\}$.

By individual rationality, if $C_f(\widetilde{u}_1,\overline{u}_{-1})=\{k\}$ for any $k \in R_f \backslash \{1,n\}$, then $t_k(\widetilde{u}_1,\overline{u}_{-1})=\pi_k(k) \geq -\lambda(\overline{u}_k)$, which contradicts the construction of $\overline{u}_k$. Notice that $\widetilde{u}_1(\pi_1(1),1)=\widetilde{u}_1(\pi_1(1)+\lambda(\widetilde{u}_1),0)>\widetilde{u}_1(0,0)=\widetilde{u}_1(\pi_1(n),0)$. If $C_f(\widetilde{u}_1,\overline{u}_{-1})=\{n\}$, then agent 1 can manipulate f at $(\widetilde{u}_1,\overline{u}_{-1})$ via $\widehat{u}_1$. Hence, we have that $C_f(\widetilde{u}_1,\overline{u}_{-1})=\{1\}$.

Step 2. $C_f(\widetilde{u}_2,\overline{u}_{-2})=\{2\}$.

By individual rationality, if $C_f(\widetilde{u}_2,\overline{u}_{-2})=\{k\}$ for any $k \in R_f \backslash \{2,n\}$, then

$t_k(\tilde{u}_2,\bar{u}_{-2})=\pi_k(k)\geq-\lambda(\bar{u}_k)$, which contradicts the construction of $\bar{u}_k$. Notice that $\tilde{u}_2(\pi_2(2),1)=\tilde{u}_2(\pi_2(2)+\lambda(\tilde{u}_2),0)>\tilde{u}_2(0,0)=\tilde{u}_2(\pi_2(n),0)$. If $C_f(\tilde{u}_2,\bar{u}_{-2})=\{n\}$, then agent 2 can manipulate f at $(\tilde{u}_2,\bar{u}_{-2})$ via $\hat{u}_2$. Hence, we have that $C_f(\tilde{u}_2,\bar{u}_{-2})=\{2\}$.

Step 3. $f(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})$ *is indeterminable.*

By individual rationality, if $C_f(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})=\{k\}$ for any $k\in R_f\backslash\{1,2,n\}$, then $t_k(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})=\pi_k(k)\geq-\lambda(\bar{u}_k)$, which contradicts the construction of $\bar{u}_k$. Notice that $\tilde{u}_1(\pi_1(2),0)=\tilde{u}_1(\pi_1(2)-\lambda(\tilde{u}_1),1)>\tilde{u}_1(\pi_1(1),1)$ and $\tilde{u}_2(\pi_2(1),0)=\tilde{u}_2(\pi_2(1)-\lambda(\tilde{u}_2),1)>\tilde{u}_2(\pi_2(2),1)$. If $C_f(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})=\{1\}$ or $\{n\}$, then agent 1 can manipulate f at $(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})$ via $\bar{u}_1$. If $C_f(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})=\{2\}$, then agent 2 can manipulate f at $(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})$ via $\bar{u}_2$. Therefore, $C_f(\tilde{u}_{\{1,2\}},\bar{u}_{-\{1,2\}})$ is indeterminable.

Q.E.D.

Theorem 3.2 is a tight result. We present mechanisms which satisfy any three axioms and the condition $\#R_f>2$.

Example 3.4. Let $\pi\in\Pi$ be such that $\pi_j(i)=0$ for all i, $j\in N$. A mechanism f satisfies the constant transfer property relative to π, and for all $u\in U^4$, $C_f(u)$ is defined as follows.

		$\lambda(u_4)\geq1$		$\lambda(u_4)<1$	
		$\lambda(u_2)\geq1$	$\lambda(u_2)<1$	$\lambda(u_2)\geq1$	$\lambda(u_2)<1$
$\lambda(u_3)\geq1$	$\lambda(u_1)\geq1$	{4}	{3}	{4}	{3}
	$\lambda(u_1)<1$	{2}	{2}	{2}	{2}
$\lambda(u_3)<1$	$\lambda(u_1)\geq1$	{1}	{3}	{1}	{3}
	$\lambda(u_1)<1$	{1}	{4}	{1}	{4}

Then, f satisfies strategy-proofness, individual rationality, equal compensation, and $\#R_f=n=4$, but does not satisfy demand monotonicity.[9]

[9] This example is suggested by Miki Kato. It is also possible to construct this type of mechanisms with more than four agents.

Example 3.5. Let $\pi\in\Pi$ be such that $\pi_i(i)=-(n-i)$ for all $i\in N$, $\pi_j(i)=1$ for all $i, j\in N$ with $i<j$, and $\pi_j(i)=0$ for all $i, j\in N$ with $i>j$. A mechanism f satisfies the constant transfer property relative to π, and for all $u\in U^n$, if $\lambda(u_i)>n-i$ for some $i\in N\backslash\{n\}$ and $\lambda(u_j)\leq n-j$ for all $j\in N$ with $i>j$, then $C_f(u)=\{i\}$, and if $\lambda(u_j)\leq n-j$ for all $j\in N\backslash\{n\}$, then $C_f(u)=\{n\}$. Then, f satisfies strategy-proofness, individual rationality, demand monotonicity, and $\#R_f=n$, but does not satisfy equal compensation.

Example 3.6. Let $\pi\in\Pi$ be such that $\pi_i(i)=-(n-1)$ for all $i\in N$ and $\pi_j(i)=1$ for all $i, j\in N$ with $i\neq j$. A mechanism f satisfies the constant transfer property relative to π, and for all $u\in U^n$, if $\lambda(u_i)>n$ for some $i\in N\backslash\{n\}$ and $\lambda(u_j)\leq n$ for all $j\in N$ with $i>j$, then $C_f(u)=\{i\}$, and if $\lambda(u_j)\leq n$ for all $j\in N\backslash\{n\}$, then $C_f(u)=\{n\}$. Then, f satisfies strategy-proofness, equal compensation, demand monotonicity, and $\#R_f=n$, but does not satisfy individual rationality.

Example 3.7. Let a mechanism f be such that for all $u\in U^n$, $C_f(u)=\min_{i\in N}\{\operatorname{Argmax}_{i\in N}\{\lambda(u_i)\}\}$, $t_i(u)=-\lambda(u_i)$ for $i\in C_f(u)$, and $t_j(u)=\frac{1}{n-1}\lambda(u_i)$ for all $j\in NC_f(u)$. Then, f satisfies individual rationality, equal compensation, demand monotonicity, and $\#R_f=n$, but does not satisfy strategy-proofness.

The first three examples do not use information of preferences effectively. In Example 3.4, each agent's preferences have no influence on whether or not he gets the indivisible good, and the configuration of transfer allocations. In Examples 3.5 and 3.6, the mechanisms determine allocations without incorporating agent n's preferences. In contrast, the last example uses preferences effectively and satisfies Pareto efficiency at the cost of strategy-proofness.

Next, we characterize the set of mechanisms that satisfy strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f=2$. We find again asymmetry in those mechanisms, that is, they determine allocations only on the basis of preferences of agents in R_f.

Lemma 3.7. *If a mechanism f satisfies strategy-proofness, individual rationality, equal compensation, demand monotonicity, and* $\#R_f=2$, *then* $f(u)=f(u_{R_f},\bar{u}_{-R_f})$ *for all* $u, \bar{u}\in U^n$.

Proof. For simplicity of arguments, we assume $R_f=\{1,2\}$. Suppose toward contradiction that $f(u)\neq f(u_{\{1,2\}},\bar{u}_{-\{1,2\}})$. It follows from Theorem 3.1 that f satisfies the constant transfer property relative to some $\pi\in\Pi$. Thus, $C_f(u)\neq C_f(u_{\{1,2\}},\bar{u}_{-\{1,2\}})$. Without loss of generality, we assume that $C_f(u)=\{1\}$ and $C_f(u_{\{1,2\}},\bar{u}_{-\{1,2\}})=\{2\}$. There exists some k ($3\leq k\leq n$) such that $C_f(u_{\{1,...,k\}},\bar{u}_{-\{1,...,k\}})=\{1\}$ and $C_f(u_{\{1,...,k-1\}},\bar{u}_{-\{1,...,k-1\}})=\{2\}$. It follows from individual rationality and Lemmas 3.5 and 3.6 that either $\pi_1(1)<\pi_2(2)=0$ or $\pi_2(2)<\pi_1(1)=0$. Consider the case of $\pi_1(1)<\pi_2(2)=0$. By equal compensation, $\pi_k(1)>\pi_k(2)$. Hence, agent k can manipulate f at $(u_{\{1,...,k-1\}},\bar{u}_{-\{1,...,k-1\}})$ via u_k, which contradicts strategy-proofness. The other case is similar. **Q.E.D.**

We define two classes of mechanisms which depend only on preferences of potential consumers.

Definition 3.5. A mechanism f is *decisive* iff (A1) $R_f=\{i,j\}$ for some $i, j\in N$, (A2) f satisfies the constant transfer property relative to some $\pi\in\Pi$ such that $\pi_i(i)=-(n-1)\rho<0$, $\pi_k(i)=\rho>0$ for all $k\neq i$, and $\pi_l(j)=0$ for all $l\in N$, and (A3) for all $u\in U^n$, $[\lambda(u_i)>(n-1)\rho \Rightarrow C_f(u)=\{i\}]$ and $[\lambda(u_i)<(n-1)\rho \Rightarrow C_f(u)=\{j\}]$, where ρ is a positive real number.

Definition 3.6. A mechanism f is *unilaterally unanimous* iff (B1) $R_f=\{i,j\}$ for some $i, j\in N$, (B2) f satisfies the constant transfer property relative to some $\pi\in\Pi$ such that $\pi_i(i)=-(n-1)\rho<0$, $\pi_k(i)=\rho>0$ for all $k\neq i$, and $\pi_l(j)=0$ for all $l\in N$, and (B3) for all $u\in U^n$, $[\lambda(u_i)>(n-1)\rho$ and $\lambda(u_j)<\rho \Rightarrow C_f(u)=\{i\}]$ and $[\lambda(u_i)<(n-1)\rho$ or $\lambda(u_j)>\rho \Rightarrow C_f(u)=\{j\}]$, where ρ is a positive real number.

Here, ρ represents the amount of the transfer from agent i to each of the other agents when agent i receives the indivisible good. (A1) and (B1) say that the set of potential consumers consists of two agents indexed by i and j. (A2) and (B2) say that the mechanism satisfies the constant transfer property relative to some transfer allocation function in which agent i pays the equal amount of money to the other agents when he gets the indivisible good and agent j pays nothing when he gets it. (A3) says that agent i gets the indivisible good if agent i wants it under a given transfer allocation, and agent j gets the indivisible good otherwise. (B3) says that agent i gets the indivisible good if agent i wants it and agent j does not want it (hence, both agents want agent i to get it) under a given transfer allocation, and agent j gets the indivisible good otherwise. Figures 3.3 and 3.4 illustrate the structure of the decisive mechanisms and the unilaterally unanimous mechanisms respectively.

Lemma 3.8. *If a mechanism f satisfies strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f=2$, then f is decisive or unilaterally unanimous.*

Proof. Assuming that f is not decisive, we show that f is unilaterally unanimous. (B1) is trivial. (B2) is straightforward from Theorem 3.1, Lemmas 3.5 and 3.6, and equal compensation. We prove (B3). Let $\rho=\frac{-1}{n-1}\pi_i(i)$. Notice that for all u_i, $u_j\in U$, $[\lambda(u_i)>(n-1)\rho \Leftrightarrow u_i(\pi_i(i),1)=u_i(\pi_i(i)+\lambda(u_i),0)=u_i(-(n-1)\rho+\lambda(u_i),0)>u_i(0,0)=u_i(\pi_i(j),0)]$, and $[\lambda(u_j)>\rho \Leftrightarrow u_j(\pi_j(j),1)=u_j(\pi_j(j)+\lambda(u_j),0)=u_j(\lambda(u_j),0)>u_j(\rho,0)=u_j(\pi_j(i),0)]$.

Step 1. For all $u\in U^n$ such that $\lambda(u_i)<(n-1)\rho$, $C_f(u)=\{j\}$.

Assume, on the contrary, that $C_f(u)=\{i\}$. By individual rationality, it must hold that $t_i(u)=\pi_i(i)=-(n-1)\rho\geq-\lambda(u_i)$. It contradicts $\lambda(u_i)<(n-1)\rho$.

Step 2. For all $u\in U^n$ such that $\lambda(u_i)>(n-1)\rho$ and $\lambda(u_j)<\rho$, $C_f(u)=\{i\}$.

Since $i\in R_f$, there exists $\overline{u}\in U^n$ such that $C_f(\overline{u})=\{i\}$. For all $u_i\in U$ such that $\lambda(u_i)>(n-1)\rho$, it must hold that $C_f(u_i,\overline{u}_{-i})=\{i\}$; otherwise agent i can manipulate f at $(u_i,\overline{u}_{-i})$ via $\overline{u}_i$. For all u_i, $u_j\in U$ such that $\lambda(u_i)>(n-1)\rho$ and $\lambda(u_j)<\rho$, it must hold that $C_f(u_{\{i,j\}},\overline{u}_{-\{i,j\}})=\{i\}$; otherwise agent j can manipulate f at $(u_{\{i,j\}},\overline{u}_{-\{i,j\}})$ via $\overline{u}_j$. By Lemma

3.7, we obtain a desired conclusion.

Step 3. For all $u \in U^n$ such that $\lambda(u_i) \geq (n-1)\rho$ and $\lambda(u_j) > \rho$, $C_f(u) = \{j\}$.

Since f is not decisive, there exists $\hat{u} \in U^n$, where $\lambda(\hat{u}_i) > (n-1)\rho$, such that $C_f(\hat{u}) = \{j\}$. For all $u_i \in U$ such that $\lambda(u_i) \geq (n-1)\rho$, it must hold that $C_f(u_i, \hat{u}_{-i}) = \{j\}$; otherwise agent i can manipulate f at $\hat{u}$ via u_i. For all u_i, $u_j \in U$ such that $\lambda(u_i) \geq (n-1)\rho$ and $\lambda(u_j) > \rho$, it must hold that $C_f(u_{\{i,j\}}, \hat{u}_{-\{i,j\}}) = \{j\}$; otherwise agent j can manipulate f at $(u_{\{i,j\}}, \hat{u}_{-\{i,j\}})$ via $\hat{u}_j$. By Lemma 3.7, we have a desired conclusion. **Q.E.D.**

The definition of the decisive mechanisms does not specify an allocation for all $u \in U^n$ such that $\lambda(u_i) = (n-1)\rho$. Let $U_A = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) < \rho\}$, $U_B = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) = \rho\}$, and $U_C = \{u \in U^n | \lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) > \rho\}$. We use the notation $C_f(\overline{U}) = \{k\}$ for some $\overline{U} \subset U^n$ and $k \in N$ when $C_f(u) = \{k\}$ for all $u \in \overline{U}$. We consider necessary conditions on allocations for preference profiles in U_A, U_B, and U_C. If there exist u, $\overline{u} \in U_A$ such that $C_f(u) = \{i\}$ and $C_f(\overline{u}) = \{j\}$, then by $u_i = \overline{u}_i$ and Lemma 3.7, agent j can manipulate f at $\overline{u}$ via u_j. Hence, it must hold that either $C_f(U_A) = \{i\}$ or $C_f(U_A) = \{j\}$. Similarly, it must hold that either $C_f(U_C) = \{i\}$ or $C_f(U_C) = \{j\}$. It follows from Lemma 3.7 that either $C_f(U_B) = \{i\}$ or $C_f(U_B) = \{j\}$. We can find the following eight patterns for the specification of allocations for U_A, U_B, and U_C.

[α1] $C_f(U_A) = \{i\}$, $C_f(U_B) = \{i\}$, $C_f(U_C) = \{i\}$.

[α2] $C_f(U_A) = \{i\}$, $C_f(U_B) = \{i\}$, $C_f(U_C) = \{j\}$.

[α3] $C_f(U_A) = \{i\}$, $C_f(U_B) = \{j\}$, $C_f(U_C) = \{j\}$.

[α4] $C_f(U_A) = \{j\}$, $C_f(U_B) = \{j\}$, $C_f(U_C) = \{j\}$.

[α5] $C_f(U_A) = \{i\}$, $C_f(U_B) = \{j\}$, $C_f(U_C) = \{i\}$.

[α6] $C_f(U_A) = \{j\}$, $C_f(U_B) = \{i\}$, $C_f(U_C) = \{i\}$.

[α7] $C_f(U_A) = \{j\}$, $C_f(U_B) = \{i\}$, $C_f(U_C) = \{j\}$.

[α8] $C_f(U_A) = \{j\}$, $C_f(U_B) = \{j\}$, $C_f(U_C) = \{i\}$.

Similarly, the definition of the unilaterally unanimous mechanisms does not specify an allocation for all $u \in U^n$ such that [$\lambda(u_i) = (n-1)\rho$ and $\lambda(u_j) \leq \rho$], and [$\lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) = \rho$]. Let $U_D = \{u \in U^n | \lambda(u_i) > (n-1)\rho$ and $\lambda(u_j) = \rho\}$. We can find the following eight patterns for the specification of allocations for U_A, U_B, and U_D.

[β1] $C_f(U_A)=\{i\}$, $C_f(U_B)=\{i\}$, $C_f(U_D)=\{i\}$.

[β2] $C_f(U_A)=\{i\}$, $C_f(U_B)=\{j\}$, $C_f(U_D)=\{i\}$.

[β3] $C_f(U_A)=\{i\}$, $C_f(U_B)=\{j\}$, $C_f(U_D)=\{j\}$.

[β4] $C_f(U_A)=\{j\}$, $C_f(U_B)=\{j\}$, $C_f(U_D)=\{i\}$.

[β5] $C_f(U_A)=\{j\}$, $C_f(U_B)=\{j\}$, $C_f(U_D)=\{j\}$.

[β6] $C_f(U_A)=\{i\}$, $C_f(U_B)=\{i\}$, $C_f(U_D)=\{j\}$.

[β7] $C_f(U_A)=\{j\}$, $C_f(U_B)=\{i\}$, $C_f(U_D)=\{i\}$.

[β8] $C_f(U_A)=\{j\}$, $C_f(U_B)=\{i\}$, $C_f(U_D)=\{j\}$.

Theorem 3.3. *A mechanism f satisfies strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f=2$ if and only if (i) f is decisive with one of [α1] - [α4], or (ii) f is unilaterally unanimous with one of [β1] - [β5].*

Proof. It follows from Lemma 3.8 that if f satisfies strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f=2$, then f is decisive or unilaterally unanimous. It is easy to show that if f is decisive with one of [α5] - [α8], then f violates strategy-proofness. Similarly, it is easy to show that if f is unilaterally unanimous with one of [β6] - [β8], then f violates strategy-proofness. This proves necessity. Straightforward proofs of sufficiency are omitted. **Q.E.D.**

Finally, we characterize the set of mechanisms that satisfy strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f=1$. We introduce the dictatorial mechanisms: there is an agent who always consumes the indivisible good without compensation to the other agents.

Definition 3.7. A mechanism f is *dictatorial* iff there is an agent $i\in N$ such that for all $u\in U^n$, $f_i(u)=(0,1)$ and $f_j(u)=(0,0)$ for all $j\neq i$.

The following theorem is straightforward, and the proof will be omitted.

Theorem 3.4. *A mechanism f satisfies strategy-proofness, individual rationality, equal compensation, demand monotonicity, and $\#R_f = 1$ if and only if f is dictatorial.*

3.5. Conclusion

In the previous section we divided the set of mechanisms into three classes based on the number of potential consumers, and we characterized the set of strategy-proof, individually rational, equally compensatory, and demand monotonic mechanisms for each of the classes (Theorems 3.2, 3.3, and 3.4). It may be convenient to sum up those results as the following theorem.

Theorem 3.5. *A mechanism f satisfies strategy-proofness, individual rationality, equal compensation, and demand monotonicity if and only if (i) f is decisive with one of [$\alpha 1$] - [$\alpha 4$], (ii) f is unilaterally unanimous with one of [$\beta 1$] - [$\beta 5$], or (iii) f is dictatorial.*

These three types of mechanisms have the following common properties: (i) they determine the allocation of monetary transfer depending on who receives the indivisible good; (ii) they allocate the indivisible good to one of the pre-specified (one or two) agent(s); and (iii) they disregard preferences of agents other than the pre-specified agent(s).

It follows from a general result of Tadenuma and Thomson (1995) that there is no strategy-proof and envy-free mechanism. Although our axioms of individual rationality, equal compensation, and demand monotonicity are strictly weaker than envy-freeness, it is impossible to construct attractive mechanisms which satisfy strategy-proofness, individual rationality, equal compensation, and demand monotonicity. This characterization shows that the presence of an indivisible good yields serious asymmetry in mechanisms.

It follows from a general result of Holmström (1979) that there is no strategy-proof and Pareto efficient mechanism. However, it was not yet clear how inefficient strategy-proof mechanisms are. This characterization enables us to understand that those mechanisms are very inefficient. It is easy to see that any decisive mechanism, unilaterally unanimous mechanism, or dictatorial mechanism fails to achieve a Pareto efficient allocation for many preference profiles.

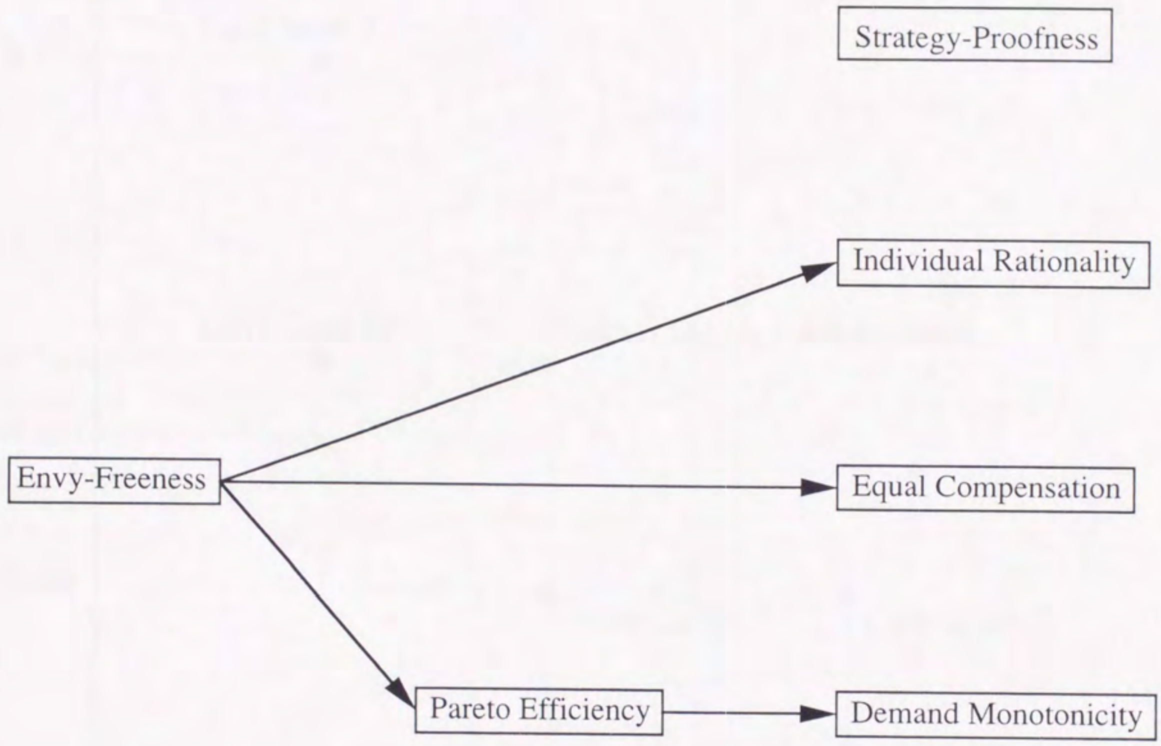

Figure 3.1. The relationships among six axioms.

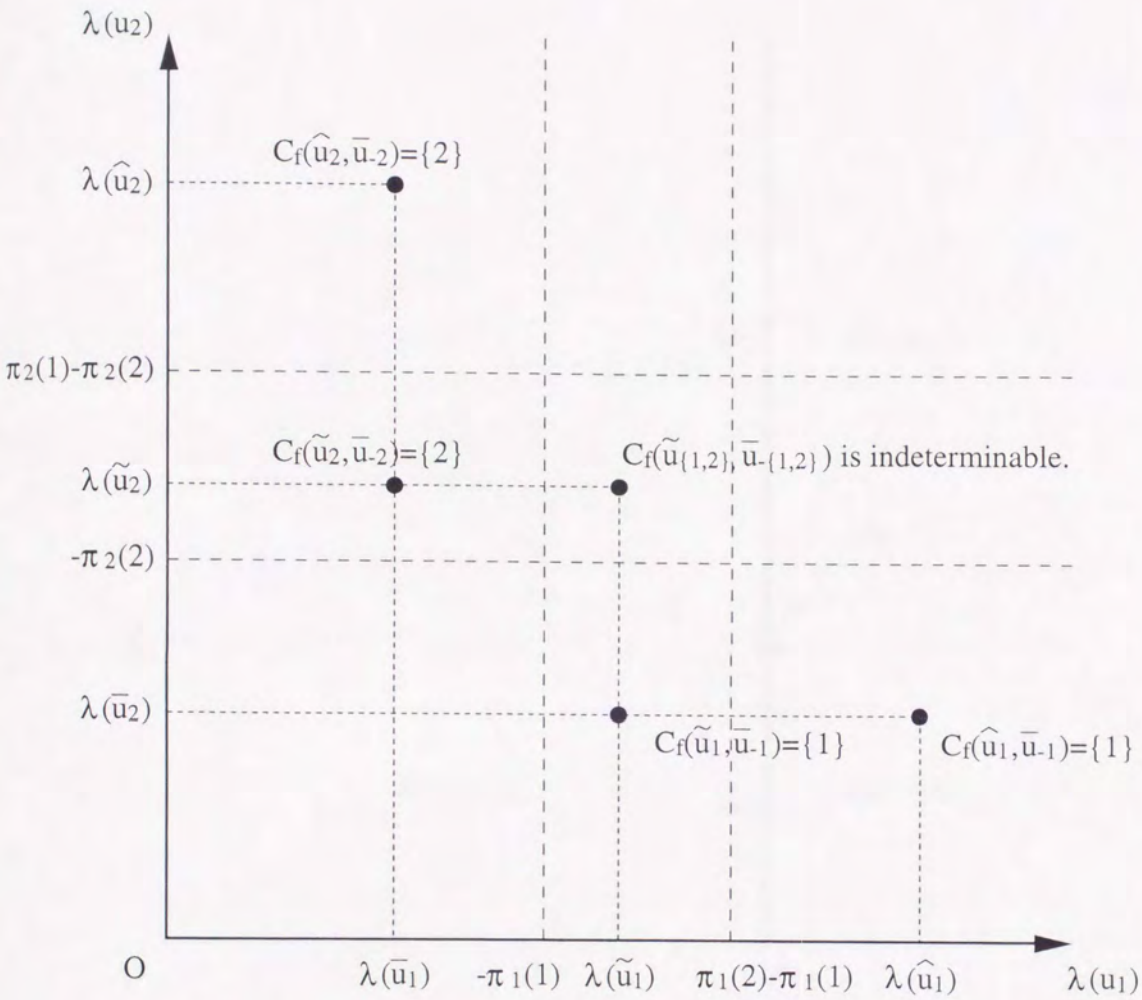

Figure 3.2. An illustration of the proof of Theorem 3.2: the set of preference profiles where $(\bar{u}_3, \ldots, \bar{u}_n)$ is fixed.

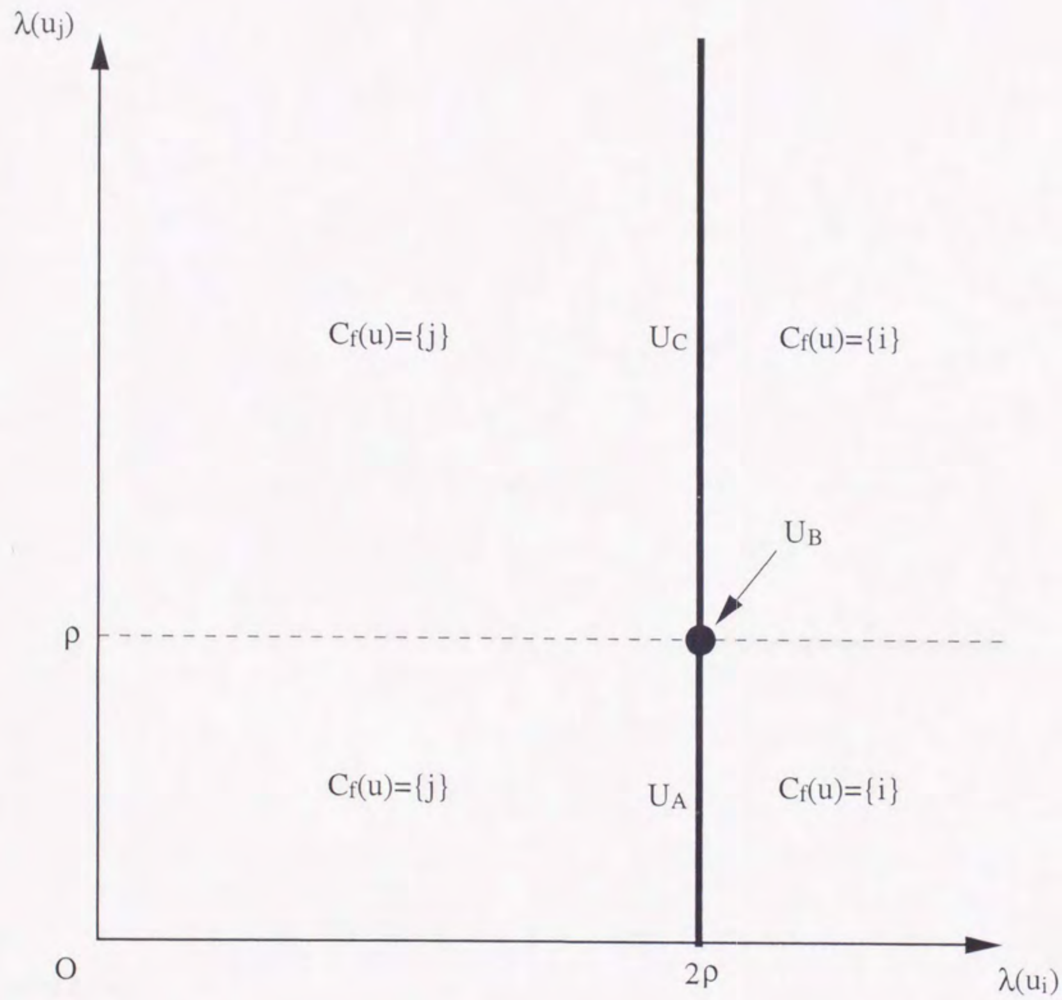

Figure 3.3. An illustration of a decisive mechanism when n=3.

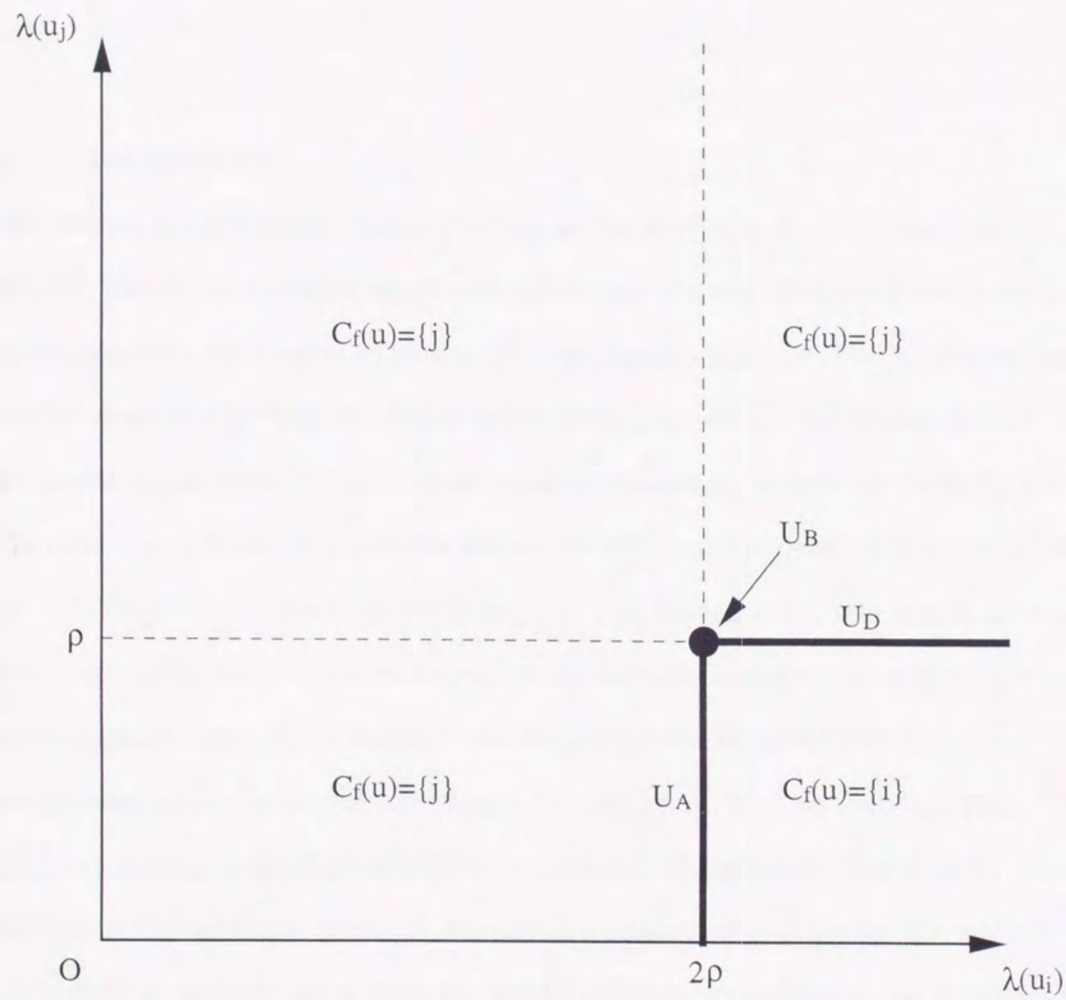

Figure 3.4. An illustration of a unilaterally unanimous mechanism when n=3.

Chapter 4

Characterizations of Strategy-Proof Mechanisms for Excludable versus Non-Excludable Public Projects***

4.1. Introduction

We address the mechanism design problem for the provision of a fixed sized public project, that is, the provision of one indivisible unit of a non-rivalrous good. Some public projects are non-excludable by nature. We first consider that case. The issue there is to decide whether to provide the project and how to divide the cost of producing it, if produced. Some public projects can be made excludable by appropriate methods. Cable TV is an example. We next consider this possibility. Here the question is still whether to provide the project and how to divide the cost of producing it, but also who is allowed to consume it. The main axiom we impose on mechanisms is strategy-proofness (Gibbard, 1973; Satterthwaite, 1975). Strategy-proofness requires that truthful revelation of preferences should be a dominant strategy for each agent. We also introduce four auxiliary axioms. Individual rationality requires that all agents should end up no worse off than at the status quo. Demand monotonicity requires that (i) the set of consumers of the project should not shrink when the demand of no agent decreases, and that (ii) the set of consumers of the project should be unchanged when the demand of no current consumer decreases and the demand of no current non-consumer increases. Citizen sovereignty requires that society should have access to either level of the project. Access independence requires that each agent should have access to either level of the project regardless of other agents' preferences.

For the provision of non-excludable public goods, Serizawa (1996) characterizes the set of strategy-proof, individually rational, and "non-bossy" (Satterthwaite and

*** This chapter is based on Ohseto(1999a).

Sonnenschein, 1981) mechanisms, which he calls "semiconvex cost sharing schemes determined by the minimum demand principle".[10] Those mechanisms are very far from being Pareto efficient since they divide the cost of producing the public good according to a fixed cost sharing rule. Non-bossiness substantially narrows down the class of strategy-proof mechanisms, and thus plays an important role in his characterization. However, this property stands on weak normative ground. Therefore, a characterization without non-bossiness is an important extension of his result.

For the provision of non-excludable public goods, Moulin (1994) characterizes "the conservative equal-costs mechanism" by imposing coalitional strategy-proofness, individual rationality, and symmetry. For the provision of excludable public goods, he proves that "the serial mechanism" satisfies coalitional strategy-proofness, the stand alone test, and symmetry, and Pareto dominates the conservative equal-costs mechanism.[11] Coalitional strategy-proofness is very meaningful, but it is a stronger requirement than strategy-proofness and non-bossiness. It is an interesting question what class of mechanisms is characterized by strategy-proofness instead of coalitional strategy-proofness.[12] Moreover, we would like to know how the class of strategy-proof mechanisms enlarges if we drop symmetry.

First, we consider the case of a non-excludable public project. We introduce the notions of "constant cost sharing" and "the unanimous mechanisms". Constant cost sharing pre-specifies a cost sharing pattern for society, and requires that if the project is provided, then agents should share its cost according to the cost sharing pattern. Serizawa

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[10] Serizawa (1996) considers the case of the continuous provision, whereas we consider the case of the discrete provision. However, his result applies to the case of discrete provision as well. The same comment applies to Moulin (1994), Saijo (1991), and Serizawa (1999) discussed later.

[11] The stand alone test is stronger than individual rationality. Saijo (1991) proves that there is no mechanism which satisfies strategy-proofness and the stand alone test for the provision of non-excludable public goods.

[12] Serizawa (1999) proves that the conservative equal-costs mechanism is the unique mechanism which satisfies strategy-proofness, individual rationality, and symmetry for the provision of non-excludable public goods.

(1996) shows that constant cost sharing is a necessary condition for strategy-proofness in the two-agent case. We prove that constant cost sharing is a necessary condition for strategy-proofness and individual rationality in the n-agent case. The unanimous mechanisms are defined as follows: (i) they are constant cost sharing; and (ii) they provide the project if and only if each agent's willingness to pay is larger than or equal to his cost share. We characterize the unanimous mechanisms as the set of strategy-proof, individually rational, and citizen sovereign mechanisms.

Second, we consider the case of an excludable public project. We introduce the notions of "semiconstant cost sharing" and "the largest unanimous mechanisms". Semiconstant cost sharing pre-specifies a cost sharing pattern for each coalition, and requires that if the project is provided for agents in some coalition, then those agents should share its cost according to the cost sharing pattern for the coalition. We prove that semiconstant cost sharing is a necessary condition for strategy-proofness in the two-agent case, and it is a necessary condition for strategy-proofness, individual rationality, and demand monotonicity in the n-agent case. The largest unanimous mechanisms are defined as follows: (i) they are semiconstant cost sharing; and (ii) they provide the project for the largest coalition such that the willingness to pay of each member of the coalition is larger than or equal to his cost share. We characterize the largest unanimous mechanisms as the set of strategy-proof, individually rational, demand monotonic, and access independent mechanisms.

Comparing the two classes of mechanisms, we conclude that admitting partial exclusion always improves efficiency, that is, it is always possible to construct some largest unanimous mechanism (for an excludable public project) which Pareto dominates a given unanimous mechanism (for a non-excludable public project). Moreover, the largest unanimous mechanisms are very attractive for their simplicity: (i) each agent has only to report his willingness to pay; and (ii) there exists a simple algorithm to calculate the final allocation.

Before closing this section, we discuss the relationship between our results and the work of Deb and Razzolini (1999a, 1999b). For the provision of an excludable public project, they focus on a particular member of the largest unanimous mechanisms (which they call "the auction like mechanism" or "the serial cost sharing"), which divides the cost of the project among the consumers of the project equally. They characterize this mechanism by strategy-proofness, individual rationality, symmetry, and some auxiliary axioms ("directional non-bossiness" and "free entry" (1999a); "upper semicontinuity" and "voluntariness" (1999b)). The comparison of their and our characterization results makes it clear that symmetry substantially narrows down the class of strategy-proof mechanisms.

This chapter is organized as follows. In Section 4.2, we introduce notation and definitions. In Section 4.3, we characterize the set of strategy-proof, individually rational, and citizen sovereign mechanisms for a non-excludable public project. In Section 4.4, we characterize the set of strategy-proof, individually rational, demand monotonic, and access independent mechanisms for an excludable public project. In Section 4.5, we discuss the validity of admitting partial exclusion.

4.2. Notation and Definitions

Let $N=\{1,\ldots,n\}$ ($n\geq 2$) be the set of agents. We consider the provision of a fixed sized public project, that is, there is one indivisible unit of a non-rivalrous good $y\in\{0,1\}$. The cost function $c(y)$ is normalized in such a way that $c(0)=0$ and $c(1)=1$. We assume that the public project is non-excludable in Section 4.3 and excludable in Section 4.4.

The consumption space of agent $i\in N$ is the set of pairs $(x_i,y_i)\in R\times\{0,1\}$, where x_i denotes his cost share and y_i denotes his consumption of the project. The set of feasible allocations is $Z=\{z=(x_1,\ldots,x_n;y_1,\ldots,y_n)\in R^n\times\{0,1\}^n \mid \sum_{i\in N} x_i=c(\max_{i\in N} y_i)$ and $x_i\geq 0$ for all $i\in N\}$. The set of feasible cost shares is $\Delta=\{s=(s_1,\ldots,s_n)\in R^n \mid \sum_{i\in N} s_i=1$ and $s_i\geq 0$ for all $i\in N\}$.

Each agent $i\in N$ has a preference on his consumption space. We assume that preferences can be represented by a utility function $u_i(x_i,y_i)$. Each preference u_i is continuous and strictly decreasing in x_i, strictly increasing in y_i, and satisfies the following property: for all $x_i\in R$, there exist $\overline{x}_i\in R$ and $\widehat{x}_i\in R$ such that $u_i(\overline{x}_i,1)=u_i(x_i,0)$

and $u_i(\widehat{x}_i,0)=u_i(x_i,1)$. Let U be the set of all such preferences. For each $u_i \in U$, let $\lambda(u_i)$ denote agent i's willingness to pay at the status quo, that is, $u_i(\lambda(u_i),1)=u_i(0,0)$.[13] A list $u=(u_1,\ldots,u_n) \in U^n$ is called a preference profile.

Let 2^N be the set of all coalitions in N, where $\varnothing \in 2^N$. For each coalition $C \in 2^N \setminus \{\varnothing, N\}$, let -C represent coalition N\C. Let $(\overline{u}_C, u_{-C})$ denote the preference profile whose i-th component is $\overline{u}_i$ if $i \in C$ and u_i if $i \notin C$. For simplicity of notation, we write $(\overline{u}_{\{i\}}, u_{-\{i\}})$ as $(\overline{u}_i, u_{-i})$ and $(\overline{u}_{\{i,j\}}, u_{-\{i,j\}})$ as $(\overline{u}_{i,j}, u_{-i,-j})$.

A *mechanism* is a function f: $U^n \to Z$, which associates a feasible allocation with each preference profile. Given a mechanism f and a preference profile u, we write $f(u)=(x_1(u),\ldots,x_n(u);y_1(u),\ldots,y_n(u))$. We use the notation $x(u)=(x_1(u),\ldots,x_n(u))$, $y(u)=(y_1(u),\ldots,y_n(u))$, and $f_i(u)=(x_i(u),y_i(u))$. For simplicity of notation, we abbreviate $y(u)=(0,...,0)$ as $y(u)=0$ and $y(u)=(1,...,1)$ as $y(u)=1$

For each $u \in U^n$, let $I_f(u)=\{i \in N \mid y_i(u)=1\}$ represent the set of users of the project. We call agents in $I_f(u)$ the included agents, and agents in $N \setminus I_f(u)$ the excluded agents.

We introduce three central axioms on mechanisms.

Definition 4.1. A mechanism f satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\overline{u}_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(\overline{u}_i, u_{-i}))$.

Definition 4.2. A mechanism f satisfies *individual rationality* iff for all $u \in U^n$ and $i \in N$, $u_i(f_i(u)) \geq u_i(0,0)$.

Definition 4.3. A mechanism f satisfies *demand monotonicity* iff for all u, $\overline{u} \in U^n$,
(i) $[\lambda(\overline{u}_i) \geq \lambda(u_i)$ for all $i \in N \Rightarrow I_f(\overline{u}) \supset I_f(u)]$, and
(ii) $[\lambda(\overline{u}_i) \geq \lambda(u_i)$ for all $i \in I_f(u)$ and $\lambda(\overline{u}_j) \leq \lambda(u_j)$ for all $j \notin I_f(u) \Rightarrow I_f(u)=I_f(\overline{u})]$.

Strategy-proofness requires that truthful revelation of preferences should be a

[13] The assumption that u_i is strictly increasing in y_i implies that $\lambda(u_i)>0$.

dominant strategy for each agent. Individual rationality requires that all agents should end up no worse off than at the status quo. Demand monotonicity requires that (i) the set of included agents should not shrink when the demand of no agent decreases, and that (ii) the set of included agents should be unchanged when the demand of no included agent decreases and the demand of no excluded agent increases.[14]

We introduce two axioms concerning access to the project.

Definition 4.4. A mechanism f satisfies *citizen sovereignty* iff there exist u, $\bar{u} \in U^n$ such that y(u)=0 and y($\bar{u}$)=1.

Definition 4.5. A mechanism f satisfies *access independence* iff for each $i \in N$, there exist u_i, $\bar{u}_i \in U$ such that for all $u_{-i} \in U^{n-1}$, $y_i(u_i, u_{-i})=0$ and $y_i(\bar{u}_i, u_{-i})=1$.

Citizen sovereignty requires that society should have access to either level of the project. Access independence requires that each agent should have access to either level of the project regardless of other agents' preferences. It is clear that access independence is stronger than citizen sovereignty.

We use Pareto dominance for welfare comparisons between two mechanisms. A mechanism f *Pareto dominates* another mechanism $\bar{f}$ iff (i) for all $u \in U^n$ and $i \in N$, $u_i(f_i(u)) \geq u_i(\bar{f}_i(u))$, and (ii) for some $u \in U^n$ and $i \in N$, $u_i(f_i(u)) > u_i(\bar{f}_i(u))$. A mechanism f *weakly Pareto dominates* another mechanism $\bar{f}$ iff (i) holds.

4.3. Strategy-Proof Mechanisms for a Non-Excludable Public Project

In this section we consider mechanisms for the provision of a non-excludable public project satisfying the following non-excludability assumption.

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[14] When we only consider the set of quasi-linear preferences, the premise of the second part of demand monotonicity is the same as that of Maskin monotonicity (Maskin, 1999), but the conclusion of the second part of demand monotonicity is strictly weaker than that of Maskin monotonicity.

Assumption 4.1. (Non-Excludability). Given a mechanism f, either $y(u)=0$ or $y(u)=1$ for each $u \in U^n$.

Definition 4.6. Given $s \in \Delta$, a mechanism f is *constant cost sharing relative to s* iff for all $u \in U^n$, $[y(u)=0 \Rightarrow x(u)=(0,...,0)]$ and $[y(u)=1 \Rightarrow x(u)=s]$.

A mechanism is *constant cost sharing* iff for some $s \in \Delta$, f is constant cost sharing relative to s. Constant cost sharing pre-specifies a feasible cost share for society. If a mechanism is constant cost sharing, then it is very far from being Pareto efficient. However, Serizawa (1996) proves that constant cost sharing is a necessary condition for strategy-proofness in the two-agent case.

Theorem 4.1. (Serizawa, 1996). *If $N=\{1,2\}$, and a mechanism f satisfies strategy-proofness, then f is constant cost sharing.*

We prove that constant cost sharing is a necessary condition for strategy-proofness and individual rationality in the n-agent case.

Theorem 4.2. *If a mechanism f satisfies strategy-proofness and individual rationality, then f is constant cost sharing.*

Proof. The argument consists of two steps.

Step 1. For all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, if $y(u)=1$ and $\lambda(u_i)<\lambda(\bar{u}_i)$, then $f(u)=f(\bar{u}_i, u_{-i})$.

Suppose that $f_i(u) \neq f_i(\bar{u}_i, u_{-i})$. Since $y(\bar{u}_i, u_{-i})=1$ contradicts strategy-proofness, $f_i(\bar{u}_i, u_{-i})=(0,0)$. By individual rationality, $x_i(u) \leq \lambda(u_i)$. Since $x_i(u)<\lambda(\bar{u}_i)$, $\bar{u}_i(x_i(u),1)>\bar{u}_i(0,0)$, and thus $\bar{u}_i(f_i(u))>\bar{u}_i(f_i(\bar{u}_i, u_{-i}))$, which contradicts strategy-proofness. Therefore, $f_i(u)=f_i(\bar{u}_i, u_{-i})$. Suppose that $x(u) \neq x(\bar{u}_i, u_{-i})$. Then, there exist $j,k \in N$ $(j,k \neq i)$ such that $x_j(u)>x_j(\bar{u}_i, u_{-i})$ and $x_k(u)<x_k(\bar{u}_i, u_{-i})$. Let $\bar{u}_j$, $\bar{u}_k \in U$ be such that $x_j(u)>\lambda(\bar{u}_j)>x_j(\bar{u}_i, u_{-i})$ and $x_k(u)<\lambda(\bar{u}_k)<x_k(\bar{u}_i, u_{-i})$. By strategy-proofness and individual rationality, $y(\bar{u}_j, u_{-j})=0$ and $y(\bar{u}_{i,k}, u_{-i,-k})=0$, and thus $x(\bar{u}_j, u_{-j})=(0,...,0)$ and

$x(\overline{u}_{i,k},u_{-i,-k})=(0,...,0)$. By strategy-proofness, $y(\overline{u}_k,u_{-k})=1$ and $y(\overline{u}_{i,j},u_{-i,-j})=1$. Let $s_i=x_i(\overline{u}_k,u_{-k})$ and $\overline{s}_i=x_i(\overline{u}_{i,j},u_{-i,-j})$. By strategy-proofness, $u_i(f_i(\overline{u}_k,u_{-k}))\geq u_i(f_i(\overline{u}_{i,k},u_{-i,-k}))$ and $u_i(f_i(\overline{u}_j,u_{-j}))\geq u_i(f_i(\overline{u}_{i,j},u_{-i,-j}))$, and thus $u_i(s_i,1)\geq u_i(0,0)\geq u_i(\overline{s}_i,1)$. By strategy-proofness, $\overline{u}_i(f_i(\overline{u}_{i,j},u_{-i,-j}))\geq\overline{u}_i(f_i(\overline{u}_j,u_{-j}))$ and $\overline{u}_i(f_i(\overline{u}_{i,k},u_{-i,-k}))\geq\overline{u}_i(f_i(\overline{u}_k,u_{-k}))$, and thus $\overline{u}_i(\overline{s}_i,1)\geq\overline{u}_i(0,0)\geq\overline{u}_i(s_i,1)$. Since u_i and $\overline{u}_i$ are strictly decreasing in agent i's cost share, it follows that $s_i=\overline{s}_i$. This implies that $\lambda(u_i)=\lambda(\overline{u}_i)$, which is a contradiction.

Step 2. f is constant cost sharing.

Suppose that there exist u, $\overline{u}\in U^n$ such that $y(u)=y(\overline{u})=1$ and $x(u)\neq x(\overline{u})$. Let $\widehat{u}_1\in U$ be such that $\lambda(\widehat{u}_1)>\lambda(u_1)$ and $\lambda(\widehat{u}_1)>\lambda(\overline{u}_1)$. By Step 1, $f(u)=f(\widehat{u}_1,u_{-1})$ and $f(\overline{u})=f(\widehat{u}_1,\overline{u}_{-1})$. Applying this argument to the remaining agents successively, we have that $f(\widehat{u})=f(u)\neq f(\overline{u})=f(\widehat{u})$ for some $\widehat{u}\in U^n$, which is a contradiction. **Q.E.D.**

These theorems are tight. Example 4.1 shows that strategy-proofness is necessary for Theorems 4.1 and 4.2. Example 4.2 shows that individual rationality is indispensable for Theorem 4.2.

Example 4.1. Let n=2 and a mechanism f be defined by $f(u)=(\frac{\lambda(u_1)}{\lambda(u_1)+\lambda(u_2)},\frac{\lambda(u_2)}{\lambda(u_1)+\lambda(u_2)};1)$ for all $u\in U^n$ such that $\lambda(u_1)+\lambda(u_2)\geq 1$, and $f(R)=(0,0;0)$ otherwise. Then, f satisfies individual rationality, but is not constant cost sharing.

Example 4.2. Let n=3 and a mechanism f be defined by $f(u)=(\frac{1}{3},\frac{2}{3},0;1)$ for all $u\in U^n$ such that $\lambda(u_3)\geq 1$, and $f(u)=(\frac{2}{3},\frac{1}{3},0;1)$ otherwise. Then, f satisfies strategy-proofness, but is not constant cost sharing.

Next, we examine when the project is provided. Given $u\in U^n$ and $s\in\Delta$, let $A(u,s,\geq)=\{i\in N|\ \lambda(u_i)\geq s_i\}$ denote the *set of agents whose willingness to pay is larger than or equal to their component of s*, and $A(u,s,>)=\{i\in N|\ \lambda(u_i)>s_i\}$ denote the *set of agents whose willingness to pay is strictly larger than their component of s*.

Definition 4.7. Given $s \in \Delta$, a mechanism f respects the *unanimity relative to s* iff (i) f is constant cost sharing relative to s, (ii) for all $u \in U^n$, $[A(u,s,\geq) \neq N \Rightarrow y(u)=0]$, and (iii) for all $u \in U^n$, $[A(u,s,\geq)=N \Rightarrow y(u)=1]$. Given $s \in \Delta$, a mechanism f respects the *weak unanimity relative to s* iff (i) and (ii) hold, and (iii)′ for all $u \in U^n$, $[A(u,s,>)=N \Rightarrow y(u)=1]$.

Unanimity relative to s says that the project is provided if and only if all agents approve the provision of the project coupled with the cost share s. We prove that the unanimous mechanisms are on the Pareto frontier of the set of strategy-proof, individually rational, and citizen sovereign mechanisms.

Theorem 4.3. (i) *If a mechanism f respects the unanimity relative to some $s \in \Delta$, then f satisfies strategy-proofness, individual rationality, and citizen sovereignty.*
(ii) If a mechanism f satisfies strategy-proofness, individual rationality, and citizen sovereignty, then f is weakly Pareto dominated by a unanimous mechanism $\overline{f}$ relative to some $s \in \Delta$.

Proof. (i) We show that f satisfies strategy-proofness. Let $u \in U^n$, $i \in N$, and $\overline{u}_i \in U$. We consider the following three cases. If $i \notin A(u,s,\geq)$, then $f_i(u)=(0,0)$ and $u_i(0,0)>u_i(s_i,1)$. If $i \in A(u,s,\geq)=N$, then $f_i(u)=(s_i,1)$ and $u_i(s_i,1) \geq u_i(0,0)$. If $i \in A(u,s,\geq) \neq N$, then there exists $j \in N$ $(j \neq i)$ such that $j \notin A(u,s,\geq)$, and thus $f_i(u)=f_i(\overline{u}_i,u_{-i})=(0,0)$. It follows that $u_i(f_i(u)) \geq u_i(f_i(\overline{u}_i,u_{-i}))$ in all cases. It is clear that f satisfies individual rationality and citizen sovereignty.
(ii) The argument consists of two steps.
Step 1. f respects the weak unanimity relative to some $s \in \Delta$.

By Theorem 4.2, f is constant cost sharing relative to some $s \in \Delta$. Suppose first that for some $u \in U^n$, $A(u,s,\geq) \neq N$ and $y(u)=1$. Let $i \notin A(u,s,\geq)$. It is clear that $f_i(u)=(s_i,1)$ and $u_i(0,0)>u_i(s_i,1)$, which contradicts individual rationality. Suppose next that for some $u \in U^n$, $A(u,s,>)=N$ and $y(u)=0$. By citizen sovereignty, there exists $\overline{u} \in U^n$ such that $y(\overline{u})=1$. Since f is constant cost sharing relative to s, $f(\overline{u})=(s;1)$. By strategy-proofness,

$f_1(u_1, \bar{u}_{-1})=(s_1, 1)$. Since f is constant cost sharing relative to s, $f(u_1, \bar{u}_{-1})=(s;1)$. If the remaining agents change preferences from $\bar{u}_i$ to u_i successively, we have that $f(u)=(s;1)$, which is a contradiction.

Step 2. *f is weakly Pareto dominated by the unanimous mechanism $\bar{f}$ relative to s.*

By Step 1, $\bar{f}(u)=f(u)$ for all $u \in U^n$ such that $A(u,s,\geq)\neq N$ or $A(u,s,>)=N$. Let $u \in U^n$ be such that $A(u,s,\geq)=N$ and $A(u,s,>)\neq N$. It follows that either $\bar{f}(u)=f(u)=(s;1)$ or $[\bar{f}(u)=(s;1)$ and $f(u)=(0,...,0;0)]$. Since $u_i(s_i,1)\geq u_i(0,0)$ for all $i \in N$, $\bar{f}$ weakly Pareto dominates f. **Q.E.D.**

4.4. Strategy-Proof Mechanisms for an Excludable Public Project

In this section we consider mechanisms for the provision of an excludable public project. We introduce some definitions. A *cost sharing rule* is a function π: $2^N \backslash \{\varnothing\} \rightarrow \Delta$, which associates a feasible cost share with each possible set of users of the project. For each $C \in 2^N \backslash \{\varnothing\}$, let $\pi(C)=(\pi_1(C),\ldots,\pi_n(C))$. Let Π denote the *set of cost sharing rules*.

Definition 4.8. Given $\pi \in \Pi$, a mechanism f is *semiconstant cost sharing relative to π* iff for all $u \in U^n$, $[I_f(u)=\varnothing \Rightarrow x(u)=(0,...,0)]$ and $[I_f(u)=C\ (\neq\varnothing) \Rightarrow x(u)=\pi(C)]$.

A mechanism f is *semiconstant cost sharing* iff for some $\pi \in \Pi$, f is semiconstant cost sharing relative to π. Semiconstant cost sharing pre-specifies a feasible cost share for each possible set of users of the project. The number of possible feasible cost shares is equal to the number of possible coalitions. If a mechanism is semiconstant cost sharing, then it divides the cost of the project according to a finite set of feasible cost shares, and thus it fails to achieve Pareto efficiency. This structure is quite similar to that of mechanisms in Barbera and Jackson (1995).

We prove that semiconstant cost sharing is a necessary condition for strategy-proofness in the two-agent case.

Theorem 4.4. *If N={1,2}, and a mechanism f satisfies strategy-proofness, then f is* *semiconstant cost sharing.*

Proof. Suppose that there exist u, $\bar{u} \in U^n$ such that $I_f(u)=I_f(\bar{u})$ and $x(u) \neq x(\bar{u})$. Consider the following two cases.

Case 1. $I_f(u)=I_f(\bar{u})=N$.

Let $f(u)=(s_1,s_2;1,1)$ and $f(\bar{u})=(\bar{s}_1,\bar{s}_2;1,1)$. Without loss of generality, assume that $s_1>\bar{s}_1$ and $s_2<\bar{s}_2$. Let $\hat{u}_1$, $\hat{u}_2 \in U$ be such that $\lambda(\hat{u}_1)>s_1$ and $\lambda(\hat{u}_2)>\bar{s}_2$. By strategy-proofness, $f_1(\hat{u}_1,u_2)=(s_1,1)$ and $f_1(\hat{u}_1,\bar{u}_2)=(\bar{s}_1,1)$, and thus by strategy-proofness, $f(\hat{u}_1,u_2)=(s_1,s_2;1,0)$ and $f(\hat{u}_1,\bar{u}_2)=(\bar{s}_1,\bar{s}_2;1,1)$. By strategy-proofness, $f_2(u_1,\hat{u}_2)=(s_2,1)$ and $f_2(\bar{u}_1,\hat{u}_2)=(\bar{s}_2,1)$, and thus by strategy-proofness, $f(u_1,\hat{u}_2)=(s_1,s_2;1,1)$ and $f(\bar{u}_1,\hat{u}_2)=(\bar{s}_1,\bar{s}_2;0,1)$. Since $f(u_1,\hat{u}_2)=(s_1,s_2;1,1)$ and $\lambda(\hat{u}_1)>s_1$, by strategy-proofness, $f_1(\hat{u}_1,\hat{u}_2)=(s_1,1)$. Since $f(\hat{u}_1,\bar{u}_2)=(\bar{s}_1,\bar{s}_2;1,1)$ and $\lambda(\hat{u}_2)>\bar{s}_2$, by strategy-proofness, $f_2(\hat{u}_1,\hat{u}_2)=(\bar{s}_2,1)$. It follows from $s_1>\bar{s}_1$ that $s_1+\bar{s}_2>\bar{s}_1+\bar{s}_2=1$, which is a contradiction.

Case 2. $I_f(u)=I_f(\bar{u})=\{i\}$.

Without loss of generality, let i=1. Let $f(u)=(s_1,s_2;1,0)$ and $f(\bar{u})=(\bar{s}_1,\bar{s}_2;1,0)$. Without loss of generality, assume that $s_1>\bar{s}_1$ and $s_2<\bar{s}_2$. Let $\hat{u}_1 \in U$ be such that $\lambda(\hat{u}_1)>s_1$. By Case 1, $f(\hat{u}_1,u_2)=(s_1,s_2;1,0)$. Let $\hat{s}_2$, $\tilde{s}_2 \in R$ be such that $u_2(\hat{s}_2,1)=u_2(s_2,0)$ and $\bar{u}_2(\tilde{s}_2,1)=\bar{u}_2(\bar{s}_2,0)$. Let $\hat{u}_2 \in U$ be such that $\hat{u}_2(\hat{s}_2\text{-}\varepsilon,1)=\hat{u}_2(s_2,0)$ and $\hat{u}_2(\tilde{s}_2\text{-}\delta,1)=\hat{u}_2(\bar{s}_2,0)$ for some ε, $\delta>0$. By strategy-proofness, $f_2(\bar{u}_1,\hat{u}_2)=(\bar{s}_2,0)$. Since $\bar{s}_2>s_2\geq 0$, $f(\bar{u}_1,\hat{u}_2)=(\bar{s}_1,\bar{s}_2;1,0)$. By strategy-proofness, $f_1(\hat{u}_1,\hat{u}_2)=(\bar{s}_1,1)$ and $f_2(\hat{u}_1,\hat{u}_2)=(s_2,0)$. It follows from $s_1>\bar{s}_1$ that $\bar{s}_1+s_2<s_1+s_2=1$, which is a contradiction. **Q.E.D.**

We prove that semiconstant cost sharing is a necessary condition for strategy-proofness, individual rationality, and demand monotonicity in the n-agent case.

Theorem 4.5. *If a mechanism f satisfies strategy-proofness, individual rationality, and demand monotonicity, then f is semiconstant cost sharing.*[15]

[15] It follows immediately from this theorem that *if a mechanism f satisfies strategy-proofness, individual rationality, demand monotonicity, and symmetry, then f is semiconstant cost sharing relative to* $\pi^e \in \Pi$, *where for each* $C \in 2^N \backslash \{\emptyset\}$, $\pi_i^e(C)=\frac{1}{|C|}$ *for all* $i \in C$, *and* $\pi_j^e(C)=0$ *for all* $j \notin C$.

Proof. The argument consists of three steps.

Step 1. *For all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, if $y_i(u)=1$ and $\lambda(u_i)<\lambda(\bar{u}_i)$, then $f(u)=f(\bar{u}_i, u_{-i})$.*

By demand monotonicity, $I_f(u)=I_f(\bar{u}_i,u_{-i})$. Suppose that $x(u)\neq x(\bar{u}_i,u_{-i})$. By strategy-proofness, $x_i(u)=x_i(\bar{u}_i,u_{-i})$. Then, there exists $j\in N$ ($j\neq i$) such that $0\leq x_j(u)<x_j(\bar{u}_i,u_{-i})$. This fact and individual rationality imply that $j\in I_f(\bar{u}_i,u_{-i})$, and thus $j\in I_f(u)$. Let $\bar{u}_j\in U$ be such that $x_j(u)<\lambda(\bar{u}_j)<x_j(\bar{u}_i,u_{-i})$. By strategy-proofness, $f_j(\bar{u}_j,u_{-j})=(x_j(u),1)$. By strategy-proofness and individual rationality, $f_j(\bar{u}_{i,j},u_{-i,-j})=(0,0)$. Since $\lambda(u_i)<\lambda(\bar{u}_i)$, this contradicts demand monotonicity.

Step 2. *For all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, if $y_i(u)=0$ and $\lambda(u_i)>\lambda(\bar{u}_i)$, then $f(u)=f(\bar{u}_i, u_{-i})$.*

The argument is very similar to that of Step 1.

Step 3. *f is semiconstant cost sharing.*

Suppose that there exist u, $\bar{u}\in U^n$ such that $I_f(u)=I_f(\bar{u})$ and $x(u)\neq x(\bar{u})$. Let $\hat{u}_1\in U$ be such that $\lambda(\hat{u}_1)>\lambda(u_1)$ and $\lambda(\hat{u}_1)>\lambda(\bar{u}_1)$ if $1\in I_f(u)=I_f(\bar{u})$, and $\lambda(\hat{u}_1)<\lambda(u_1)$ and $\lambda(\hat{u}_1)<\lambda(\bar{u}_1)$ otherwise. By Steps 1 and 2, $f(u)=f(\hat{u}_1,u_{-1})$ and $f(\bar{u})=f(\hat{u}_1,\bar{u}_{-1})$. Applying this argument to the remaining agents successively, we have that $f(\hat{u})=f(u)\neq f(\bar{u})=f(\hat{u})$ for some $\hat{u}\in U^n$, which is a contradiction. **Q.E.D.**

These theorems are tight. Example 4.1 shows that strategy-proofness is necessary for Theorems 4.4 and 4.5. Example 4.2 shows that individual rationality is necessary for Theorem 4.5. The next example shows that demand monotonicity is indispensable for Theorem 4.5.

Example 4.3. Let $n=3$ and a mechanism f be defined by

(i) $f(u)=(\frac{1}{3},\frac{2}{3},0;1,1,0)$ for all $u\in U^n$ such that $[\lambda(u_1)\geq\frac{1}{3}$ and $\lambda(u_2)\geq\frac{2}{3}]$ and $\lambda(u_3)\geq 1$,

(ii) $f(u)=(0,0,0;0,0,0)$ for all $u\in U^n$ such that $[\lambda(u_1)<\frac{1}{3}$ or $\lambda(u_2)<\frac{2}{3}]$ and $\lambda(u_3)\geq 1$,

(iii) $f(u)=(\frac{2}{3},\frac{1}{3},0;1,1,0)$ for all $u\in U^n$ such that $[\lambda(u_1)\geq\frac{2}{3}$ and $\lambda(u_2)\geq\frac{1}{3}]$ and $\lambda(u_3)<1$, and

(iv) $f(u)=(0,0,0;0,0,0)$ for all $u\in U^n$ such that $[\lambda(u_1)<\frac{2}{3}$ or $\lambda(u_2)<\frac{1}{3}]$ and $\lambda(u_3)<1$.

Then, f satisfies strategy-proofness and individual rationality, but is not semiconstant cost sharing.

Next, we examine when the project is provided and who are the users of the project. Given $u \in U^n$, $\pi \in \Pi$, and $C \in 2^N \setminus \{\varnothing\}$, let $A(u,\pi(C),\geq)=\{i \in C \mid \lambda(u_i) \geq \pi_i(C)\}$ denote the *set of agents in coalition C whose willingness to pay is larger than or equal to their cost share specified by* π, and $A(u,\pi(C),>)=\{i \in C \mid \lambda(u_i) > \pi_i(C)\}$ denote the *set of agents in coalition C whose willingness to pay is strictly larger than their cost share specified by* π.

Lemma 4.1. *If a mechanism f satisfies strategy-proofness, individual rationality, demand monotonicity, and access independence, then f is semiconstant cost sharing relative to some* $\pi \in \Pi$*, and for all* $u \in U^n$ *and* $C \in 2^N \setminus \{\varnothing\}$*,*

(i) $[A(u,\pi(C),\geq) \neq C \Rightarrow I_f(u) \neq C]$*, and*

(ii) $[A(u,\pi(C),>) = C \Rightarrow I_f(u) \supset C]$*.*

Proof. By Theorem 4.5, f is semiconstant cost sharing relative to some $\pi \in \Pi$.
(i) Suppose that there exist $u \in U^n$ and $C \in 2^N \setminus \{\varnothing\}$ such that $A(u,\pi(C),\geq) \neq C$ and $I_f(u)=C$. There exists some $i \in C$ such that $i \notin A(u,\pi(C),\geq)$ and $f_i(u)=(\pi_i(C),1)$, which contradicts individual rationality.
(ii) By access independence, there exists $\bar{u} \in U^n$ such that $I_f(\bar{u})=C$. Let $i \in C$. Suppose first that $\lambda(u_i) \geq \lambda(\bar{u}_i)$. By demand monotonicity, $I_f(\bar{u})=I_f(u_i,\bar{u}_{-i})$. Suppose next that $\lambda(u_i) < \lambda(\bar{u}_i)$. Since $f_i(\bar{u})=(\pi_i(C),1)$ and $\lambda(u_i) > \pi_i(C)$, by strategy-proofness, $f_i(u_i,\bar{u}_{-i})=(\pi_i(C),1)$. Since $i \in I_f(u_i,\bar{u}_{-i})$ and $\lambda(u_i) < \lambda(\bar{u}_i)$, by demand monotonicity, $I_f(\bar{u})=I_f(u_i,\bar{u}_{-i})$. Hence, $I_f(u_i,\bar{u}_{-i})=C$ in both cases. Applying this argument to all $i \in C$ successively, we have that $I_f(u_C,\bar{u}_{-C})=C$. Let $C'=C \cup \{i \notin C \mid \lambda(u_i) < \lambda(\bar{u}_i)\}$. By demand monotonicity, $I_f(u_{C'},\bar{u}_{-C'})=C$. Since $\lambda(u_i) \geq \lambda(\bar{u}_i)$ for all $i \notin C'$, by demand monotonicity, $I_f(u) \supset C$. **Q.E.D.**

We now introduce two additional conditions on cost sharing rules. A cost sharing rule π is *user monotonic* iff for all $C, \bar{C} \in 2^N \setminus \{\varnothing\}$ such that $C \supset \bar{C}$, $\pi_i(\bar{C}) \geq \pi_i(C)$ for all $i \in \bar{C}$. A cost sharing rule π is *user liable* iff for all $C \in 2^N \setminus \{\varnothing\}$, $\pi_i(C) > 0$ for all $i \in C$ and $\pi_j(C)=0$ for all $j \notin C$. User monotonicity requires that the share of the cost of the included agents should not increase when the set of included agents expands.[16] User liability requires that the included agents should pay something and the excluded agents should pay nothing.[17] Let Π^* denote the *set of user monotonic and user liable cost sharing rules*. We show that user monotonicity and user liability are necessary conditions for strategy-proof, individually rational, demand monotonic, and access independent mechanisms.

Lemma 4.2. *If a mechanism f satisfies strategy-proofness, individual rationality, demand monotonicity, and access independence, then f is semiconstant cost sharing relative to some user monotonic and user liable cost sharing rule $\pi\in\Pi^*$.*

Proof. By Theorem 4.5, f is semiconstant cost sharing relative to some $\pi\in\Pi$. First, we show that π is user monotonic. Suppose that there exist $C, \overline{C}\in 2^N\backslash\{\varnothing\}$ such that $C\supset\overline{C}$ and $\pi_j(C)>\pi_j(\overline{C})$ for some $j\in\overline{C}$. By access independence, there exists $u\in U^n$ such that $I_f(u)=C$. For all $i\in C$, let $\overline{u}_i\in U$ be such that $\lambda(\overline{u}_i)\geq\lambda(u_i)$ and $\lambda(\overline{u}_i)>\pi_i(\overline{C})$. Let $\overline{u}=(\overline{u}_C,u_{-C})$. By demand monotonicity, $I_f(\overline{u})=C$. Hence, $f_j(\overline{u})=(\pi_j(C),1)$. Let $\widehat{u}_j\in U$ be such that $\pi_j(C)>\lambda(\widehat{u}_j)>\pi_j(\overline{C})$. Since $A((\widehat{u}_j,\overline{u}_{-j}),\pi(\overline{C}),>)=\overline{C}$, by Lemma 4.1, $I_f(\widehat{u}_j,\overline{u}_{-j})\supset\overline{C}$. Hence, $j\in I_f(\widehat{u}_j,\overline{u}_{-j})$. By strategy-proofness, $f_j(\widehat{u}_j,\overline{u}_{-j})=(\pi_j(C),1)$, which contradicts individual rationality.

Next, we show that π is user liable. Let $C\in 2^N\backslash\{\varnothing\}$. By access independence, there exists $u\in U^n$ such that $I_f(u)=C$. By individual rationality, $\pi_i(C)=x_i(u)=0$ for all $i\notin C$. Suppose that $\pi_j(N)=0$ for some $j\in N$. By access independence, there exists $u\in U^n$ such that $I_f(u)=N\backslash\{j\}$. For all $i\in N\backslash\{j\}$, let $\overline{u}_i\in U$ be such that $\lambda(\overline{u}_i)\geq\lambda(u_i)$ and $\lambda(\overline{u}_i)>\pi_i(N)$. By demand monotonicity, $I_f(u_j,\overline{u}_{-j})=N\backslash\{j\}$. Since $\lambda(u_j)>\pi_j(N)=0$, $A((u_j,\overline{u}_{-j}),\pi(N),>)=N$. By Lemma 4.1, $I_f(u_j,\overline{u}_{-j})=N$, which is a contradiction. Hence, $\pi_i(N)>0$ for all $i\in N$. Since π is user monotonic, $\pi_i(C)\geq\pi_i(N)>0$ for all $C\in 2^N\backslash\{\varnothing\}$ and $i\in C$. **Q.E.D.**

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[16] The idea of user monotonicity is much the same as that of "population monotonicity" introduced under a different name by Thomson (1983a, 1983b) in the context of bargaining.

[17] The notion of user liability is technically the same as that of "strong individual rationality" introduced by Roth (1977) in the context of bargaining. However, the meaning is somewhat different since we consider here the cost sharing problem.

Next, we identify the largest coalition whose members at a given preference profile approve the provision of the project coupled with the pre-specified cost sharing rule. A coalition $C \in 2^N \setminus \{\emptyset\}$ is the *largest unanimous coalition at $u \in U^n$ relative to $\pi \in \Pi^*$* iff (i) $A(u,\pi(C),\geq)=C$, and (ii) $A(u,\pi(\overline{C}),\geq) \neq \overline{C}$ for all $\overline{C} \in 2^N \setminus \{\emptyset\}$ such that $\overline{C} \supset C$ and $\overline{C} \neq C$. *The largest unanimous coalition is unique if it exists.* To check this, suppose that there exist two largest unanimous coalitions C and $\overline{C}$ at $u \in U^n$ relative to $\pi \in \Pi^*$. Notice that $C \not\subset \overline{C}$ and $\overline{C} \not\subset C$. Since π is user monotonic, $\pi_i(C) \geq \pi_i(C \cup \overline{C})$ for all $i \in C$ and $\pi_i(\overline{C}) \geq \pi_i(C \cup \overline{C})$ for all $i \in \overline{C}$. Since $A(u,\pi(C),\geq)=C$ and $A(u,\pi(\overline{C}),\geq)=\overline{C}$, $\lambda(u_i) \geq \pi_i(C \cup \overline{C})$ for all $i \in C \cup \overline{C}$. Hence, $A(u,\pi(C \cup \overline{C}),\geq)=C \cup \overline{C}$, which is a contradiction.

Definition 4.9. Given $\pi \in \Pi^*$, a mechanism f respects the *largest unanimity relative to π* iff (i) f is semiconstant cost sharing relative to π, (ii) for all $u \in U^n$, if there is no largest unanimous coalition at u relative to π, then $I_f(u)=\emptyset$, and (iii) for all $u \in U^n$, if C is the largest unanimous coalition at u relative to π, then $I_f(u)=C$. Given $\pi \in \Pi^*$, a mechanism f respects the *weakly largest unanimity relative to π* iff (i) and (ii) hold, and (iii)′ for all $u \in U^n$, if C is the largest unanimous coalition at u relative to π, then $I_f(u) \subset C$ and $[A(u,\pi(C),>)=C \Rightarrow I_f(u)=C]$.

Largest unanimity relative to π says that the project is provided for the largest coalition whose members approve the provision of the project coupled with the cost share specified by π, and the project is not provided if no such coalition exists. The largest unanimous mechanisms are simple in two respects: (i) they need little information, that is, each agent has only to report a positive real number $\lambda(u_i)$; and (ii) there exists a simple algorithm to implement them. We prove that the largest unanimous mechanisms are on the Pareto frontier of the set of strategy-proof, individually rational, demand monotonic, and access independent mechanisms.[18]

[18] Deb and Razzolini (1999a, 1999b) focus on the largest unanimous mechanism relative to $\pi^e \in \Pi^*$. Notice that Theorem 4.6 and Footnote 15 lead to an alternative characterization of that mechanism.

Theorem 4.6. (i) *If a mechanism f respects the largest unanimity relative to some $\pi \in \Pi^*$, then f satisfies strategy-proofness, individual rationality, demand monotonicity, and access independence.*

(ii) *If a mechanism f satisfies strategy-proofness, individual rationality, demand monotonicity, and access independence, then f is weakly Pareto dominated by a largest unanimous mechanism $\overline{f}$ relative to some $\pi \in \Pi^*$.*

Proof. (i) First, we show that f satisfies strategy-proofness. Suppose that there exist $u \in U^n$, $i \in N$, and $\overline{u}_i \in U$ such that $u_i(f_i(\overline{u}_i, u_{-i})) > u_i(f_i(u))$. Let $C = I_f(u)$ and $\overline{C} = I_f(\overline{u}_i, u_{-i})$. Let $i \in C$ and $i \in \overline{C}$. Since π is user monotonic, $\overline{C} \not\subset C$. Since π is user monotonic, $A(u, \pi(C), \geq) = C$ and $A((\overline{u}_i, u_{-i}), \pi(\overline{C}), \geq) = \overline{C}$ imply that $A(u, \pi(C \cup \overline{C}), \geq) = C \cup \overline{C}$. This contradicts the fact that C is the largest unanimous coalition at u relative to π. Similarly, the case of $i \notin C$ and $i \in \overline{C}$ leads to a contradiction. Let $i \in C$ and $i \notin \overline{C}$. Since $A(u, \pi(C), \geq) = C$, $u_i(\pi_i(C), 1) \geq u_i(0,0)$. Hence, $u_i(f_i(u)) \geq u_i(f_i(\overline{u}_i, u_{-i}))$, which is a contradiction. Let $i \notin C$ and $i \notin \overline{C}$. Hence, $f_i(u) = f_i(\overline{u}_i, u_{-i}) = (0,0)$, which is a contradiction.

Next, we show that f satisfies demand monotonicity. Let $u, \overline{u} \in U^n$ be such that $\lambda(\overline{u}_i) \geq \lambda(u_i)$ for all $i \in N$. Let $C = I_f(u)$ and $\overline{C} = I_f(\overline{u})$. Since $A(u, \pi(C), \geq) = C$ implies that $A(\overline{u}, \pi(C), \geq) = C$, it follows that $C \subset \overline{C}$. Let $u, \overline{u} \in U^n$ be such that $\lambda(\overline{u}_i) \geq \lambda(u_i)$ for all $i \in I_f(u)$ and $\lambda(\overline{u}_j) \leq \lambda(u_j)$ for all $j \notin I_f(u)$. Let $C = I_f(u)$ and $\overline{C} = I_f(\overline{u})$. Since $A(u, \pi(C), \geq) = C$ implies that $A(\overline{u}, \pi(C), \geq) = C$, it follows that $C \subset \overline{C}$. Suppose that $C \neq \overline{C}$. Since π is user monotonic, $A(u, \pi(C), \geq) = C$ and $A(\overline{u}, \pi(\overline{C}), \geq) = \overline{C}$ imply that $A(u, \pi(\overline{C}), \geq) = \overline{C}$. Therefore, $I_f(u) \supset \overline{C}$, which is a contradiction.

It is clear that f satisfies individual rationality and access independence.

(ii) The argument consists of two steps.

Step 1. *f respects the weakly largest unanimity relative to some $\pi \in \Pi^*$.*

By Lemma 4.2, f is semiconstant cost sharing relative to some $\pi \in \Pi^*$. Suppose that there is no largest unanimous coalition at u relative to π. By Lemma 4.1, $I_f(u) \neq C$ for all $C \in 2^N \setminus \{\varnothing\}$. Hence, $I_f(u) = \varnothing$. Suppose that C is the largest unanimous coalition at u relative to π. Since π is user monotonic, $A(u, \pi(\overline{C}), \geq) \neq \overline{C}$ for all $\overline{C} \in 2^N \setminus \{\varnothing\}$ such that

$\overline{C}\not\subset C$. By Lemma 4.1, $I_f(u)\neq\overline{C}$ for all $\overline{C}\in 2^N\backslash\{\varnothing\}$ such that $\overline{C}\not\subset C$. Hence, $I_f(u)\subset C$. Suppose that C is the largest unanimous coalition at u relative to π, and $A(u,\pi(C),>)=C$. By Lemma 4.1, $I_f(u)\supset C$. Hence, $I_f(u)=C$.

Step 2. *f is weakly Pareto dominated by the largest unanimous mechanism $\bar{f}$ relative to π.*

Let $u\in U^n$ be such that C is the largest unanimous coalition at u relative to π, and $A(u,\pi(C),>)\neq C$. Therefore, $I_{\bar{f}}(u)=C\supset I_f(u)$. Since π is user monotonic, $u_i(\bar{f}_i(u))\geq u_i(f_i(u))$ for all $i\in I_f(u)$. Then, $\bar{f}_i(u)=(\pi_i(C),1)$ and $f_i(u)=(0,0)$ for all $i\in C\backslash I_f(u)$. Since $A(u,\pi(C),\geq)=C$, $u_i(\bar{f}_i(u))\geq u_i(f_i(u))$ for all $i\in C\backslash I_f(u)$. It is clear that $\bar{f}_i(u)=f_i(u)=(0,0)$ for all $i\notin C$. Hence, $u_i(\bar{f}_i(u))\geq u_i(f_i(u))$ for all $i\in N$. It follows from the definitions of the mechanisms that $\bar{f}(u)=f(u)$ for all other $u\in U^n$. Therefore, $\bar{f}$ weakly Pareto dominates f.

Q.E.D.

Finally, we explain how to find the largest unanimous coalition at $u\in U^n$ relative to $\pi\in\Pi^*$. Suppose that $A(u,\pi(C),\geq)\neq C$ for some $C\in 2^N\backslash\{\varnothing\}$. Let $\overline{C}\in 2^N\backslash\{\varnothing\}$ be such that $\overline{C}\subset C$. Since π is user monotonic, if $\pi(C)$ is rejected by some $i\in\overline{C}$, then $\pi(\overline{C})$ is also rejected by $i\in\overline{C}$. Hence, $A(u,\pi(\overline{C}),\geq)=\overline{C}$ implies that $\overline{C}\subset A(u,\pi(C),\geq)$. With this observation, we present the following algorithm which implements the largest unanimous mechanisms.

Algorithm 4.1. *The following algorithm implements the largest unanimous mechanism f relative to $\pi\in\Pi^*$.*

Step 0: Collect $u\in U^n$.

Step 1: Let C=N.

Step 2: If $A(u,\pi(C),\geq)=C$, then go to Step 5.

Step 3: If $A(u,\pi(C),\geq)=\varnothing$, then go to Step 6.

Step 4: Let $C=A(u,\pi(C),\geq)$ and go to Step 2.

Step 5: $I_f(u)=C$ and $x(u)=\pi(C)$; end.

Step 6: $I_f(u)=\varnothing$ and $x(u)=(0,...,0)$; end.

4.5. Conclusion

We characterized two classes of strategy-proof mechanisms for the provision of a fixed sized public project. Partial excludability of the project led us to the set of largest unanimous mechanisms, whereas the non-excludability of the project led us to only the set of unanimous mechanisms. We compare these two classes of mechanisms, and justify partial exclusion for the design of strategy-proof mechanisms. The following remark formally states that the largest unanimous mechanisms perform better than the unanimous mechanisms from the point of view of efficiency.

Remark 4.1. *The largest unanimous mechanism f relative to $\pi \in \Pi^*$ Pareto dominates the unanimous mechanism $\overline{f}$ relative to $s \in \Delta$, where $\pi(N)=s$.*

Chapter 5

Strategy-Proof Mechanisms in Public Good Economies[****]

5.1. Introduction

When a society provides public goods, it has to determine the level of public goods to produce and how to divide the costs among agents. A mechanism is a function that describes the decision-making based on preferences of agents. Moulin (1994) characterizes "the conservative equal-costs mechanism" by coalitional strategy-proofness, individual rationality, and symmetry in economies with one private good and one public good. His result relies on the assumption that public goods can be produced without fixed costs. It is more natural, however, to assume that we need positive fixed costs to produce public goods. In this chapter we incorporate the consideration of fixed costs, and present not only positive results but also negative results. These results shed light on the boundary between possibility results and impossibility results.

We study mechanisms that satisfy two basic axioms. The first axiom is strategy-proofness. A mechanism satisfies strategy-proofness if it is a dominant strategy for each agent to reveal preferences truthfully. Moulin's result is quite appealing, because it is well known that strategy-proofness is a strong requirement in general environments. The Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) establish that, under minor conditions, any strategy-proof mechanism must be dictatorial. Recently, Barbera and Peleg (1990) and Zhou (1991a) prove similar powerful impossibility results. The second axiom is individual rationality. A mechanism satisfies individual rationality if all agents end up no worse off than at the status quo. No agent lacks an incentive to participate in individually rational mechanisms. We characterize the set of strategy-proof and individually rational mechanisms in more natural economic environments.

[****] This chapter is based on Ohseto (1997).

We introduce the notion of a cost sharing rule, which associates a cost sharing pattern with each level of public goods. Assuming that cost sharing rules are exogenously given, we consider mechanisms that determine the level of public goods. One interpretation of this setting is that the revision of tax rules is less frequent than public decisions. Moreover, the set of cost sharing rules we deal with is restricted to a reasonable one. That is, we require that cost sharing rules have the same properties (continuity, convexity, etc.) as the cost function. The equal cost sharing rule (the costs are divided among agents equally) and proportional cost sharing rules (the costs are divided among agents according to a given proportion vector) are examples of this set.

First, for the sake of comparison, we consider the case of a cost function with no fixed costs. In economies with one private good and one public good, we show that the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property for any cost sharing rule. The full-range property is the condition that any feasible level of the public good is attainable by the mechanism. If we turn our attention to the case of one private good and several public goods, the result drastically changes. That is, it follows from a general result of Zhou (1991a) that there is no strategy-proof and individually rational mechanism.

Next, we consider the case of a cost function with positive fixed costs. Since the cost function has fixed costs, the cost function has the non-convexity. Thus, any cost sharing rule must have the non-convexity. We present the set of strategy-proof and individually rational mechanisms by restricting the range of mechanisms to recover the convexity of the cost sharing rule. Those mechanisms are the variants of the minimal provision mechanism. Conversely, if the restriction of the range is not sufficient to recover the convexity of the cost sharing rule, the non-convexity prevents us from constructing strategy-proof and individually rational mechanisms. These results imply that we must restrict the range of mechanisms if we want to design strategy-proof and individually rational mechanisms. In other words, the non-convexity of cost sharing rules limits the variety of our choices, and therefore it is less desirable in terms of efficiency. These results also describe the boundary between possibility results and impossibility results.

To conclude this section, we relate our results to recent work on the characterization of strategy-proof mechanisms in public good economies. Barbera and Jackson (1994) present a full characterization of strategy-proof mechanisms. Their work does not specify the cost function explicitly, and does not apply to the case of a cost function with positive fixed costs. Serizawa (1996) characterizes the set of strategy-proof, individually rational, and non-bossy mechanisms in economies with one private good and one public good. Since the economic interpretation of non-bossiness is not so clear, we do not invoke on this condition. Serizawa (1999) characterizes the set of strategy-proof, individually rational, and symmetric mechanisms in economies with one private good and one public good. His characterization is a refinement of Moulin (1994) since it identifies the conservative equal-costs mechanism by using strategy-proofness instead of coalitional strategy-proofness. Applying his results to the case of non-convex cost functions leads to impossibility results, which are closely related to our impossibility results.

This chapter is organized as follows. In Section 5.2, we introduce notation and definitions. In Section 5.3, we consider the case without fixed costs. In Section 5.4, we study the case with fixed costs. In Section 5.5, we summarize the results.

5.2. Notation and Definitions

Let $N=\{1,\dots,n\}$ $(n\geq 2)$ be the set of agents. There are two types of goods x and y, where x is a (one-dimensional) private good and $y=(y_1,\dots,y_m)$ is an m-dimensional vector of public goods. Public good i can be produced at any level y_i in $Y_i=[0,y_{i\ max}]$. The capacity $y_{i\ max}$ is finite for all i. Let $Y=\prod_{i=1}^{m} Y_i$.[19] A cost function of public goods is given by $c(y_1,\dots,y_m)$. We assume that $c(0,\dots,0)=0$, $c(y_1,\dots,y_m)$ is continuous and convex on Y except at the origin $(0,\dots,0)$, and $c(y_1,\dots,y_m)$ is strictly increasing in each y_i. Let $X=R_+$ denote the possible range of the costs.

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[19] The assumption that the space of public goods is a Cartesian product is made for simplicity. We only use the convexity of that space in Theorems 5.2 and 5.6. When agents have a limit of their cost share, the set of possible combinations of public goods has the convexity in this model, and thus theorems still hold in this case.

A consumption bundle of agent $i \in N$ is $(y_1,\ldots,y_m;x_i)$, where x_i is agent i's share of the costs of producing $y=(y_1,\ldots,y_m)$ units of public goods. Each agent $i \in N$ has a preference on $Y \times X$, which can be represented by a utility function u_i. For each agent, let U denote the set of possible preferences, which consists of all continuous, strictly convex and monotonic (non-decreasing in $y_1,\ldots,y_m$ and non-increasing in x_i) preferences on $Y \times X$. Given any $u_i \in U$ and any set $B \subset Y \times X$, $\text{Argmax}(u_i;B)$ denotes the set of maximal consumption bundles of u_i on B. If $\text{Argmax}(u_i;B)$ consists of a single consumption bundle, we use $\text{argmax}(u_i;B)$ to represent the unique member of this set. A list $u=(u_1,\ldots,u_n) \in U^n$ is called a preference profile. Given $u \in U^n$, $i, j \in N$, $\bar{u}_i \in U$, and $\widehat{u}_j \in U$, we denote by $(\bar{u}_i, u_{-i})$ the preference profile obtained from u after the replacement of u_i by $\bar{u}_i$, and by $(\bar{u}_i, \widehat{u}_j, u_{-i,-j})$ the preference profile obtained from u after the replacement of u_i and u_j by $\bar{u}_i$ and $\widehat{u}_j$.

A *cost sharing function for agent i* is a function π_i: $Y \to X$, which associates agent i's share of the costs with each level of public goods. A list $\pi=(\pi_1,\ldots,\pi_n)$ is called a *cost sharing rule*.

Definition 5.1. For any cost function c, a cost sharing rule $\pi=(\pi_i)_{i \in N}$ is *feasible* iff for all $y \in Y$, $\sum_{i \in N} \pi_i(y)=c(y)$.

For any cost function c, let Π_c denote the *set of feasible cost sharing rules*.

Example 5.1. (1) For any cost function c, the *equal cost sharing rule*, $\pi_i(y)=\frac{c(y)}{n}$ for all $i \in N$, is feasible.

(2) For any cost function c, any *proportional cost sharing rule*, $\pi_i(y)=p_i c(y)$ where $p_i>0$ for all $i \in N$ and $\sum_{i \in N} p_i=1$, is feasible.

Given any $\pi \in \Pi_c$, the set of feasible allocations is $Z^\pi=\{(y;x_1,\ldots,x_n) \mid y \in Y \text{ and } x_i=\pi_i(y) \text{ for all } i \in N\}$, and the set of feasible consumption bundles of agent i (that is, agent i's consumption space) is $Z_i^\pi=\{(y;x_i) \mid y \in Y \text{ and } x_i=\pi_i(y)\}$.

Given any $\pi \in \Pi_c$, a *mechanism* is a function f^π: $U^n \rightarrow Z^\pi$, which associates a feasible allocation with each preference profile. The range of f^π is denoted by $R(f^\pi)$. Let $\#R(f^\pi)$ and $\dim(R(f^\pi))$ denote the cardinality and the dimension of the projection of $R(f^\pi)$ on Y, respectively. Let f_i^π be a function corresponding to f^π that associates a feasible consumption bundle of agent i with each preference profile. The range of f_i^π is denoted by $R(f_i^\pi)$.

The notion of cost sharing rules is found in Mas-Colell (1980) and Mas-Colell and Silvestre (1989). They define the concept of cost share equilibrium as a unanimously preferred allocation supported by some cost sharing rule. Although any cost share equilibrium is Pareto efficient, there is no cost share equilibrium on some fixed Z^π for most preference profiles. Hence, any mechanism f^π usually fails to achieve Pareto efficiency.

We introduce two main axioms.

Definition 5.2. A mechanism f^π satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, $u_i(f_i^\pi(u)) \geq u_i(f_i^\pi(\bar{u}_i, u_{-i}))$.

Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. If a mechanism f^π does not satisfy strategy-proofness, then there exist some $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$ such that $u_i(f_i^\pi(\bar{u}_i, u_{-i})) > u_i(f_i^\pi(u))$, and therefore we say that agent i can manipulate f^π at u via $\bar{u}_i$.

Definition 5.3. A mechanism f^π satisfies *individual rationality* iff for all $u \in U^n$ and $i \in N$, $u_i(f_i^\pi(u)) \geq u_i(0,\ldots,0;0)$.

Individual rationality requires that all agents should end up no worse off than at the status quo.

The following lemma is useful in the subsequent sections.

Lemma 5.1. (Barbera and Peleg, 1990; Zhou, 1991a). *Given any cost sharing rule* $\pi \in \Pi_c$*, if a mechanism* f^{π} *satisfies strategy-proofness,* $(y;\pi(y)) \in R(f^{\pi})$ *and* $(y;\pi_i(y)) = \arg\max(u_i;R(f_i^{\pi}))$ *for all* $i \in N$*, then* $f^{\pi}(u)=(y;\pi(y))$*.*

Proof. Since $(y;\pi(y)) \in R(f^{\pi})$, we can choose $\bar{u} \in U^n$ such that $f^{\pi}(\bar{u})=(y;\pi(y))$. Suppose toward contradiction that $f^{\pi}(u) \neq (y;\pi(y))$. Let $z^i = f^{\pi}(u_1,\ldots,u_i,\bar{u}_{i+1},\ldots,\bar{u}_n)$ for $i=0,\ldots,n$. Then, $z^0=(y;\pi(y))$ and $z^n \neq (y;\pi(y))$. Hence, there exists j ($1 \leq j \leq n$) such that $z^{j-1}=(y;\pi(y))$ and $z^j \neq (y;\pi(y))$. Therefore, agent j can manipulate f^{π} at $(u_1,\ldots,u_j,\bar{u}_{j+1},\ldots,\bar{u}_n)$ via $\bar{u}_j$.

Q.E.D.

5.3. The Minimal Provision Mechanism: The Case without Fixed Costs

In this section we consider the case where the cost function is continuous, convex, strictly increasing, and $c(0,\ldots,0)=0$. That is, the cost function has no fixed costs. We impose the following assumption on cost sharing rules, which requires that cost sharing rules should have the same properties as the cost function.

Assumption 5.1. Each cost sharing rule $\pi=(\pi_i)_{i \in N}$ satisfies the following properties: each π_i is continuous, convex, strictly increasing, and $c(0,\ldots,0)=0$.

Let $\Pi_c^{'}$ be the *set of feasible cost sharing rules satisfying Assumption 5.1.* For any cost function c, the equal cost sharing rule and all other proportional cost sharing rules belong to $\Pi_c^{'}$.

5.3.1. The Case with One Public Good

We consider economies with one private good x and one public good y. Since each cost sharing function π_i is convex and each preference u_i is strictly convex, the maximal consumption bundle of u_i on Z_i^{π} is uniquely determined. Notice that given any cost sharing rule π, the useful information about each preference u_i reduces to its restriction on the consumption space Z_i^{π}. Moreover, since there is a one to one and onto projection of Z_i^{π} on Y, we can regard the restricted preferences on Z_i^{π} as preferences on Y.

For any $\pi\in\Pi_c'$ and $u_i\in U$, let $(y_i^*(u_i);\pi_i(y_i^*(u_i)))=\mathrm{argmax}(u_i;Z_i^\pi)$. For any $u\in U^n$, let $y^*(u)=\min_{i\in N} y_i^*(u_i)$.

Definition 5.4. The *minimal provision mechanism* $f^{\pi*}$ associates $(y^*(u);\pi(y^*(u)))$ with each $u\in U^n$.[20]

Although we define the minimal provision mechanism in the form of direct revelation mechanisms, it works in a simple manner as follows. Each agent $i\in N$ has only to reveal his maximal consumption bundle $(y_i^*(u_i);\pi_i(y_i^*(u_i)))$ of $u_i\in U$ on Z_i^π. Then, the society chooses the minimum $y_i^*(u_i)$ as the level of the public good and divides the costs among agents according to π.

Lemma 5.2. *For any cost sharing rule $\pi\in\Pi_c'$, the minimal provision mechanism $f^{\pi*}$ satisfies strategy-proofness and individual rationality.*

Proof. Consider the restriction of $u_i\in U$ on Z_i^π. Let $\overline{U}_i$ denote the set of all preferences on Z_i^π obtained by such restriction. Since each cost sharing function π_i is convex and preference u_i is strictly convex, each preference in $\overline{U}_i$ is single peaked on Z_i^π.[21] Then, the usual argument on single peakedness proves that $f^{\pi*}$ satisfies strategy-proofness (see Black (1948), Moulin (1980), and Barbera and Jackson (1994), for details). Further, since $0\leq y^*(u)\leq y_i^*(u_i)$ for all $u\in U^n$ and $i\in N$, single peakedness implies that $u_i(0;0)\leq u_i(y^*(u);\pi_i(y^*(u)))\leq u_i(y_i^*(u_i);\pi_i(y_i^*(u_i)))$. This shows that $f^{\pi*}$ satisfies individual rationality. **Q.E.D.**

We introduce the definition of the full-range property and characterize the set of

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[20] The minimal provision mechanism chooses a cost share equilibrium for any $u\in U^n$ such that $y^*(u)=y_i^*(u_i)$ for all $i\in N$.

[21] A preference u_i is *single peaked* on Z_i^π iff $y'<y''<y_i^*(u_i)$ implies $u_i(y';\pi_i(y'))<u_i(y'';\pi_i(y''))<u_i(y_i^*(u_i);\pi_i(y_i^*(u_i)))$ and $y_i^*(u_i)<y''<y'$ implies $u_i(y_i^*(u_i);\pi_i(y_i^*(u_i)))>u_i(y'';\pi_i(y''))>u_i(y';\pi_i(y'))$.

strategy-proof, individually rational mechanisms with the full-range property.

Definition 5.5. A mechanism f^{π} satisfies the *full-range property* iff for all $y \in [0, y_{max}]$, there exists $u \in U^n$ such that $f^{\pi}(u)=(y;\pi(y))$.

Theorem 5.1. *For any cost sharing rule $\pi \in \Pi_c^{'}$, the minimal provision mechanism $f^{\pi *}$ is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property.*

Proof. It follows from Lemma 5.2 that $f^{\pi *}$ satisfies strategy-proofness and individual rationality. It is clear that $f^{\pi *}$ satisfies the full-range property. We prove uniqueness.

Suppose toward contradiction that there exists a mechanism f^{π} other than $f^{\pi *}$ that satisfies the premises of the theorem. The allocations of f^{π} and $f^{\pi *}$ are different at some $u \in U^n$. We consider two possible cases.

First, suppose that there exists $u \in U^n$ such that $f^{\pi}(u)=(\bar{y}^1;\pi(\bar{y}^1))$, where $\bar{y}^1 < y^*(u)$ (see Figure 5.1). Without loss of generality, we can assume $y^*(u)=y_1^*(u_1) \leq y_2^*(u_2) \leq \cdots \leq y_n^*(u_n)$ by permuting indexes of agents. Then, it holds that $y^*(u)=y_1^*(u_1) \neq y_n^*(u_n)$, otherwise it follows from Lemma 5.1 that $f^{\pi}(u)=(y^*(u);\pi(y^*(u)))$. Let j be the smallest index such that $y^*(u) \neq y_j^*(u_j)$. For $k=j,...,n$, let $\bar{u}_k \in U$ be such that $\mathrm{argmax}(\bar{u}_k;Z_k^{\pi})=(y^*(u);\pi_k(y^*(u)))$ and $\bar{u}_k(y_k^*(u_k);\pi_k(y_k^*(u_k)))=\bar{u}_k(0;0)$. Then, by the construction of $\bar{u}_j$, $f^{\pi}(\bar{u}_j,u_{-j})=(\bar{y}^2;\pi(\bar{y}^2))$, where $\bar{y}^2 \in [0,y_j^*(u_j)]$, otherwise individual rationality is not satisfied for agent j. Hence, it holds that $\bar{y}^2 \leq \bar{y}^1$, otherwise agent j can manipulate f^{π} at u via $\bar{u}_j$. Similarly, $f^{\pi}(\bar{u}_j,\bar{u}_{j+1},u_{-j,-[j+1]})=(\bar{y}^3;\pi(\bar{y}^3))$, where $\bar{y}^3 \in [0,y_{j+1}^*(u_{j+1})]$, otherwise individual rationality is not satisfied for agent j+1. Hence, it holds that $\bar{y}^3 \leq \bar{y}^2$, otherwise agent j+1 can manipulate f^{π} at $(\bar{u}_j,u_{-j})$ via $\bar{u}_{j+1}$. Applying this argument to each agent k such that $y^*(u) \neq y_k^*(u_k)$ successively, we obtain $f^{\pi}(u_1,\ldots,u_{j-1},\bar{u}_j,\ldots,\bar{u}_n)=(\bar{y};\pi(\bar{y}))$, where $\bar{y} \leq \bar{y}^1 < y^*(u)$. However, it follows from Lemma 5.1 that $f^{\pi}(u_1,\ldots,u_{j-1},\bar{u}_j,\ldots,\bar{u}_n)=(y^*(u);\pi(y^*(u)))$. This is a contradiction.

Next, suppose that there exists $u \in U^n$ such that $f^{\pi}(u)=(\tilde{y}^1;\pi(\tilde{y}^1))$, where $y^*(u) < \tilde{y}^1$ (see Figure 5.2). Without loss of generality, we assume $y^*(u)=y_1^*(u_1) \leq y_2^*(u_2) \leq \cdots \leq y_n^*(u_n)$. Let

$\widetilde{u}_1 \in U$ be such that $\operatorname{argmax}(\widetilde{u}_1;Z_1^\pi)=(y^*(u);\pi_1(y^*(u)))$, and for some $\widetilde{y}^2 \in (y^*(u),\widetilde{y}^1)$, $\widetilde{u}_1(\widetilde{y}^2;\pi_1(\widetilde{y}^2))=\widetilde{u}_1(0;0)$. It follows from individual rationality for agent 1 that $f^\pi(\widetilde{u}_1,u_{-1})=(\widetilde{y}^3;\pi(\widetilde{y}^3))$, where $\widetilde{y}^3 \in [0,\widetilde{y}^2]$. Choose $\widetilde{y}^4$ such that $u_1(\widetilde{y}^4;\pi_1(\widetilde{y}^4))=u_1(\widetilde{y}^1;\pi_1(\widetilde{y}^1))$, if any. If there is no such $\widetilde{y}^4$, then any $\widetilde{y}^3 \in [0,\widetilde{y}^2]$ is strictly preferred to $\widetilde{y}^1$ at u_1, and thus agent 1 can manipulate f^π at u via $\widetilde{u}_1$. Hence, there exists such $\widetilde{y}^4$. Notice that for all $y \in (\widetilde{y}^4,\widetilde{y}^1)$, $u_1(y;\pi_1(y))>u_1(\widetilde{y}^4;\pi_1(\widetilde{y}^4))=u_1(\widetilde{y}^1;\pi_1(\widetilde{y}^1))$. If $\widetilde{y}^3 \in (\widetilde{y}^4,\widetilde{y}^2]$, then agent 1 can manipulate f^π at u via $\widetilde{u}_1$. Then, $\widetilde{y}^3$ must be in $[0,\widetilde{y}^4]$. Hence, it holds that $\widetilde{y}^3 \leq \widetilde{y}^4 < y^*(\widetilde{u}_1,u_{-1})$. Applying the first argument to $(\widetilde{u}_1,u_{-1}) \in U^n$ leads to a contradiction. **Q.E.D.**

Moulin (1994) studies mechanisms that determine both the level of public goods and the cost share. He characterizes the mechanism satisfying coalitional strategy-proofness, individual rationality, symmetry, and the full-range property. The proof of his theorem consists of two steps. First, he shows, from a result of Moulin (1993), that any coalitional strategy-proof and symmetric mechanism divides the costs equally. Next, he shows, from a result of Barbera and Jackson (1994) improving upon Moulin (1980), that any mechanism satisfying his four axioms provides minimal public goods.

On the other hand, Theorem 5.1 applies to mechanisms that determine only the level of public goods, assuming that the cost sharing rule is exogenously given. However, we use only strategy-proofness, individual rationality, and the full-range property. All the results in Sections 5.3 and 5.4 correspond to Moulin's second step. They can be extended to his type of results, because our class of cost sharing rules includes the equal cost sharing rule.

In addition to strategy-proofness and individual rationality, the full-range property plays an important role in Theorem 5.1. Even if we drop the full-range property, a similar result still holds when we consider the set of mechanisms whose range contains the origin, and is closed and connected. However, the existence or the uniqueness of strategy-proof and individually rational mechanisms is strongly dependent on the range of mechanisms. Several range conditions are discussed in Section 5.4.

5.3.2. The Case with Several Public Goods

We consider economies with one private good x and two or more public goods $y_1,\ldots,y_m$ ($m\geq2$). In contrast to the case of one private good and one public good, we can derive the following negative result from a general result of Zhou (1991a).

Theorem 5.2. *For any cost sharing rule $\pi\in\Pi_c'$, there is no mechanism f^π satisfying strategy-proofness, individual rationality, and $\dim(R(f^\pi))\geq2$.*[22]

Proof. Consider the restriction of $u_i\in U$ on Z_i^π. Let $\overline{U}_i$ denote the set of all preferences on Z_i^π obtained by such restriction. Let $\overline{U}$ denote the set of all continuous and strictly convex preferences on Y. For each $v\in\overline{U}$, we can find some $u_i\in\overline{U}_i$ such that u_i and v are the same preferences with respect to the public goods components since each cost sharing function π_i is continuous, convex, and strictly increasing, and each preference u_i is continuous, strictly convex, and monotonic. Hence, $\overline{U}_i$ includes the set of all continuous and strictly convex preferences with respect to the public goods components. In such an environment, a general result of Zhou (1991a) proves that strategy-proofness and $\dim(R(f^\pi))\geq2$ imply dictatorship.[23] We show that dictatorship is inconsistent with individual rationality. Suppose that agent $i\in N$ is the dictator. Let $u\in U^n$ be such that $\mathrm{argmax}(u_i;R(f_i^\pi))=(y_1,\ldots,y_m;x_i)\neq(0,\ldots,0;0)$ and $\mathrm{argmax}(u_j;Z_j^\pi)=(0,\ldots,0;0)$ for some $j\neq i$. A dictatorial mechanism f^π associates $(y;\pi(y))$, where $y=(y_1,\ldots,y_m)$, with $u\in U^n$. Individual rationality is not satisfied for agent j because $u_j(0,\ldots,0;0)>u_j(f_j^\pi(u))=u_j(y;\pi_j(y))$. **Q.E.D.**

5.4. Impossibility Results: The Case with Fixed Costs

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[22] If the range of mechanisms is only one-dimension (that is, the range is a straight line through the origin with respect to Y), the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property along the one-dimensional line.

[23] A mechanism f^π is *dictatorial* iff there exists agent $i\in N$, who is called the dictator, such that for all $u\in U^n$ and $z_i\in R(f_i^\pi)$, $u_i(f_i^\pi(u))\geq u_i(z_i)$.

In this section, we consider the case where the cost function has positive fixed costs. Each cost function c is represented by the form $c=c^f+c^v$, where c^f is the fixed cost function and c^v is the variable cost function. We assume that $c^f(0,\dots,0)=0$, $c^f(y_1,\dots,y_m)=\bar{c}>0$ if $(y_1,\dots,y_m)\neq(0,\dots,0)$, and c^v is continuous, convex, strictly increasing, and $c^v(0,\dots,0)=0$. No cost sharing rule considered in Section 5.3 is feasible in such a situation since it is not feasible sufficiently near the origin. We deal with cost sharing rules in which each cost sharing function π_i is represented by $\pi_i=\pi_i^f+\pi_i^v$, where π_i^f is a fixed cost sharing function and π_i^v is a variable cost sharing function. We impose the following assumption on cost sharing rules.

Assumption 5.2. Each cost sharing rule $\pi=(\pi_i)_{i\in N}=(\pi_i^f+\pi_i^v)_{i\in N}$ satisfies the following properties: for each $i\in N$, $\pi_i^f(0,\dots,0)=0$, $\pi_i^f(y_1,\dots,y_m)=\bar{c}_i>0$ if $(y_1,\dots,y_m)\neq(0,\dots,0)$, where $\sum_{i\in N}\bar{c}_i=\bar{c}$, and π_i^v is continuous, convex, strictly increasing, and $\pi_i^v(0,\dots,0)=0$.

Let Π_c'' be the *set of feasible cost sharing rules satisfying Assumption 5.2*. For any cost function c, the equal cost sharing rule and all other proportional cost sharing rules belong to Π_c''. Notice that any cost sharing rule $\pi\in\Pi_c''$ divides fixed costs among every agent, namely, every agent has to pay the share of fixed costs $\bar{c}_i$ if public goods are provided. Thus, each cost sharing function π_i is not convex near the origin, and π_i is not continuous at the origin.

5.4.1. The Case with One Public Good

We consider economies with one private good x and one public good y. Since each agent has positive fixed cost share, each cost sharing function π_i is neither continuous nor convex on the whole domain. The restriction of any $u_i\in U$ on Z_i^π is no longer single peaked. Hence, the minimal provision mechanism does not work as in the case without fixed costs. However, if we restrict the range of mechanisms properly, the minimal provision mechanism may satisfy strategy-proofness and individual rationality. The following two lemmas indicate how to restrict the range of mechanisms.

Lemma 5.3. *Given any cost sharing rule $\pi \in \Pi_c''$, if a mechanism f^π satisfies strategy-proofness, then $R(f^\pi)$ is closed.*[24]

Proof. Choose any $(y;\pi(y)) \in \text{Closure}(R(f^\pi))$. If $y=0$, $(y;\pi(y))$ is an isolated point of Z^π, and thus $(y;\pi(y)) \in R(f^\pi)$. We consider the case of $y \in (0,y_{max})$. Choose $u \in U^n$ such that $\text{argmax}(u_i;Z_i^\pi)=(y;\pi_i(y))$ for all $i \in N$. Suppose toward contradiction that $f^\pi(u)=(\hat{y};\pi(\hat{y}))$, where $\hat{y} \neq y$. We can choose y' and y'' such that $y \in (y',y'')$, $u_i(y';\pi_i(y'))>u_i(0;0)$, $u_i(y';\pi_i(y'))>u_i(\hat{y};\pi_i(\hat{y}))$, and $u_i(y'';\pi_i(y''))>u_i(\hat{y};\pi_i(\hat{y}))$ for all $i \in N$. Since $(y;\pi(y)) \in \text{Closure}(R(f^\pi))$, there exists some $(\bar{y};\pi(\bar{y})) \in R(f^\pi)$ such that $\bar{y} \in (y',y'')$. For all $i \in N$, choose $\bar{u}_i \in U$ such that $\text{argmax}(\bar{u}_i;Z_i^\pi)=(\bar{y};\pi_i(\bar{y}))$. Then, $f^\pi(\bar{u}_1,u_{-1})=(\bar{y}^1;\pi(\bar{y}^1))$, where $\bar{y}^1 \in Y\backslash(y',y'')$, otherwise agent 1 can manipulate f^π at u via $\bar{u}_1$. Similarly, $f^\pi(\bar{u}_1,\bar{u}_2,u_{-1,-2})=(\bar{y}^2;\pi(\bar{y}^2))$, where $\bar{y}^2 \in Y\backslash(y',y'')$, otherwise agent 2 can manipulate f^π at $(\bar{u}_1,u_{-1})$ via $\bar{u}_2$. Applying this argument successively, we obtain $f^\pi(\bar{u})=(\bar{y}^n;\pi(\bar{y}^n))$, where $\bar{y}^n \in Y\backslash(y',y'')$. It contradicts the fact that $f^\pi(\bar{u})=(\bar{y};\pi(\bar{y}))$ by Lemma 5.1. Therefore, $f^\pi(u)=(y;\pi(y))$, and thus $(y;\pi(y)) \in R(f^\pi)$. The remaining case of $y=y_{max}$ is similar.
Q.E.D.

Lemma 5.4. *Given any cost sharing rule $\pi \in \Pi_c''$, if a mechanism f^π satisfies individual rationality, then $(0;0,...,0) \in R(f^\pi)$.*

Proof. It is straightforward from the definition of individual rationality and the existence of $u_i \in U$ such that $\text{argmax}(u_i;Z_i^\pi)=(0;0)$. **Q.E.D.**

These range conditions are necessary and sufficient for the existence of strategy-proof and individually rational mechanisms in the case without fixed costs. However, they are not sufficient in the case with fixed costs.

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[24] A closed range of mechanisms is a necessary condition for strategy-proofness in several environments. See Barbera and Peleg (1990), Zhou (1991a), and Barbera and Jackson (1994).

Definition 5.6. A cost sharing rule $\pi \in \Pi_c''$ is *essentially convex on $W \subset Y$* iff each π_i is convex on W.[25]

In the above definition it is not required that W be a convex set in Y. For any $\pi \in \Pi_c''$ and any $W \subset Y$, let $Z^\pi(W)=\{(y;\pi(y)) | y \in W\}$ and $Z_i^\pi(W)=\{(y;\pi_i(y)) | y \in W\}$. Figures 5.3 and 5.4 give some examples of $Z_i^\pi(W)$, where π_i is convex on a closed set W. The maximal consumption bundles of $u_i \in U$ on $Z_i^\pi(W)$ consist of a single consumption bundle or two consumption bundles, when W is closed and f is essentially convex on W. Denote such consumption bundles by $(y_i^+(u_i)|_w;\pi_i(y_i^+(u_i)|_w))$ and $(y_i^*(u_i)|_w;\pi_i(y_i^*(u_i)|_w))$, where $y_i^+(u_i)|_w \leq y_i^*(u_i)|_w$. Notice that if $y_i^+(u_i)|_w \neq y_i^*(u_i)|_w$ for some $u_i \in U$, then $(y;\pi(y)) \notin Z^\pi(W)$ for all $y \in (y_i^+(u_i)|_w, y_i^*(u_i)|_w)$. For any $u \in U^n$, let $y^+(u)|_w=\min_{i \in N} y_i^+(u_i)|_w$ and $y^*(u)|_w=\min_{i \in N} y_i^*(u_i)|_w$. For any $u \in U^n$, let $T(u)=\{i \in N | y^+(u)|_w=y_i^+(u_i)|_w \text{ and } y^*(u)|_w=y_i^*(u_i)|_w\}$ be the set of agents with minimal demand for the public good. We define a class of mechanisms similar to the minimal provision mechanism defined in Section 5.3.

Definition 5.7. Given a cost sharing rule $\pi \in \Pi_c''$ and a closed set $W \subset Y$ such that π is essentially convex on W and $0 \in W$, a mechanism f^π into $Z^\pi(W)$ is in the *class of minimal provision mechanisms* iff it associates either $(y^+(u)|_w;\pi(y^+(u)|_w))$ or $(y^*(u)|_w;\pi(y^*(u)|_w))$ with each $u \in U^n$.

Notice that this definition allows that a mechanism f^π into $Z^\pi(W)$ associates $(y^+(u)|_w;\pi(y^+(u)|_w))$ for some $u \in U^n$, and $(y^*(\bar{u})|_w;\pi(y^*(\bar{u})|_w))$ for another $\bar{u} \in U^n$.

We characterize the set of strategy-proof and individually rational mechanisms with the range condition $R(f^\pi)=Z^\pi(W)$. The condition $R(f^\pi)=Z^\pi(W)$ means that f^π satisfies the full-range property on W.

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[25] Notice that any proportional cost sharing rule π is essentially convex on $W \subset Y$ if and only if the cost function c is convex on W.

Theorem 5.3. *For any cost sharing rule $\pi\in\Pi_c''$ and any closed set $W\subset Y$ such that π is essentially convex on W and $0\in W$, a mechanism f^π into $Z^\pi(W)$ satisfies strategy-proofness, individual rationality, and $R(f^\pi)=Z^\pi(W)$ if and only if it is in the class of minimal provision mechanisms and it satisfies the following Condition (α):*

for any $u, \bar{u}\in U^n$ such that $y^+(u)|_w=y^+(\bar{u})|_w\neq y^*(u)|_w=y^*(\bar{u})|_w$, $T(u)\supset T(\bar{u})$, and $u_i=\bar{u}_i$ for all $i\in T(\bar{u})$, it holds that either $f^\pi(u)=(y^+(u)|_w;\pi(y^+(u)|_w))$ or $f^\pi(\bar{u})=(y^*(\bar{u})|_w;\pi(y^*(\bar{u})|_w))$.

Proof. *Necessity.* It follows immediately from Theorem 5.1 that if a mechanism f^π into $Z^\pi(W)$ satisfies strategy-proofness, individual rationality, and $R(f^\pi)=Z^\pi(W)$, then it is in the class of minimal provision mechanisms. Here we show that it satisfies Condition (α). Suppose toward contradiction that a mechanism f^π into $Z^\pi(W)$ satisfies strategy-proofness, individual rationality, $R(f^\pi)=Z^\pi(W)$, and that for some u, $\bar{u}\in U^n$ such that $y^+(u)|_w=y^+(\bar{u})|_w\neq y^*(u)|_w=y^*(\bar{u})|_w$, $T(u)\supset T(\bar{u})$, and $u_i=\bar{u}_i$ for all $i\in T(\bar{u})$, $f^\pi(u)=(y^*(u)|_w;\pi(y^*(u)|_w))$ and $f^\pi(\bar{u})=(y^+(\bar{u})|_w;\pi(y^+(\bar{u})|_w))$.

Consider some $i\notin T(\bar{u})$, if any. Notice that, by the definition of $T(\bar{u})$, $\bar{u}_i(y^*(u)|_w;\pi_i(y^*(u)|_w))>\bar{u}_i(y^+(u)|_w;\pi_i(y^+(u)|_w))$ since π_i is convex on W and $\bar{u}_i$ is strictly convex. It follows from strategy-proofness and the fact that f^π into $Z^\pi(W)$ is in the class of minimal provision mechanisms that $f^\pi(u_i,\bar{u}_{-i})=(y^+(\bar{u})|_w;\pi(y^+(\bar{u})|_w))$. Repeat this argument for all $i\notin T(\bar{u})$ and notice that $u_i=\bar{u}_i$ for all $i\in T(\bar{u})$. Hence, we obtain $f^\pi(u)=(y^+(\bar{u})|_w;\pi(y^+(\bar{u})|_w))$, which is a contradiction.

Sufficiency. Suppose that a mechanism f^π into $Z^\pi(W)$ is in the class of minimal provision mechanisms and it satisfies Condition (α). Choose any $\bar{u}\in U^n$. If $y^+(\bar{u})|_w=y^*(\bar{u})|_w$, then it follows from a similar argument to Lemma 5.2 that no agent can manipulate f^π at $\bar{u}$. Consider the case of $y^+(\bar{u})|_w\neq y^*(\bar{u})|_w$. For each $i\in T(\bar{u})$, one of the maximal consumption bundles of $\bar{u}_i$ on $Z_i^\pi(W)$ is given by $f^\pi(\bar{u})$. For each $i\notin T(\bar{u})$, agent i may be able to change the allocation of f^π in his favor by changing his preference into some $u_i\in U$.

If $f^\pi(\bar{u})=(y^+(\bar{u})|_w;\pi(y^+(\bar{u})|_w))$, we consider the following six cases.

Case 1. $y^*(\bar{u})|_w<y_i^+(u_i)|_w$

Case 2. $y^*(\overline{u})|_w = y_i^+(u_i)|_w$

Case 3. $y^+(\overline{u})|_w = y_i^+(u_i)|_w$ and $y^*(\overline{u})|_w = y_i^*(u_i)|_w$

Case 4. $y^+(\overline{u})|_w = y_i^+(u_i)|_w = y_i^*(u_i)|_w$

Case 5. $y_i^+(u_i)|_w < y_i^*(u_i)|_w = y^+(\overline{u})|_w$

Case 6. $y_i^*(u_i)|_w < y^+(\overline{u})|_w$

In Cases 1, 2, and 3, preference profile $\overline{u}$ and $(u_i, \overline{u}_{-i})$ satisfy the premises of Condition (α). Therefore, it holds that $f^\pi(u_i, \overline{u}_{-i}) = (y^+(\overline{u})|_w; \pi(y^+(\overline{u})|_w))$ since $f^\pi(\overline{u}) \neq (y^*(\overline{u})|_w; \pi(y^*(\overline{u})|_w))$. In Case 4, since f^π is in the class of minimal provision mechanisms, it holds that $f^\pi(u_i, \overline{u}_{-i}) = (y^+(\overline{u})|_w; \pi(y^+(\overline{u})|_w))$. In Cases 5 and 6, it holds that either $f^\pi(u_i, \overline{u}_{-i}) = (y_i^+(u_i)|_w; \pi(y_i^+(u_i)|_w))$ or $f^\pi(u_i, \overline{u}_{-i}) = (y_i^*(u_i)|_w; \pi(y_i^*(u_i)|_w))$. Since π_i is convex on W, $\overline{u}_i$ is strictly convex, and $y^+(\overline{u})|_w < y_i^+(\overline{u}_i)|_w$, it is clear that $f_i^\pi(u_i, \overline{u}_{-i})$ is not any better than $f_i^\pi(\overline{u})$ for $i \notin T(\overline{u})$ at $\overline{u}_i$. It is easy to check the case of $f^\pi(\overline{u}) = (y^*(\overline{u})|_w; \pi(y^*(\overline{u})|_w))$. Therefore, for each $i \notin T(\overline{u})$, agent i can not manipulate f^π at $\overline{u} \in U^n$. Then, f^π into $Z^\pi(W)$ satisfies strategy-proofness. Notice that $0 \leq y^+(u)|_w \leq y^*(u)|_w \leq y_i^*(u_i)|_w$ for all $u \in U^n$ and $i \in N$. Since π_i is convex on W and any $u_i \in U$ is strictly convex, it holds that $u_i(0;0) \leq u_i(y^+(u)|_w; \pi_i(y^+(u)|_w)) \leq u_i(y^*(u)|_w; \pi_i(y^*(u)|_w)) \leq u_i(y_i^*(u_i)|_w; \pi_i(y_i^*(u_i)|_w))$. This implies that f^π into $Z^\pi(W)$ satisfies individual rationality. It is clear that $R(f^\pi) = Z^\pi(W)$.

Q.E.D.

In the class of minimal provision mechanisms, we choose the following mechanism and call it the minimal provision mechanism henceforth. We will show that the minimal provision mechanism is on the Pareto frontier of the class of the minimal provision mechanisms. We say that a mechanism f^π *weakly Pareto dominates* $\overline{f}^\pi$ iff for all $u \in U^n$ and $i \in N$, $u_i(f_i^\pi(u)) \geq u_i(\overline{f}_i^\pi(u))$.

Definition 5.8. Given a cost sharing rule $\pi \in \Pi_c''$ and a closed set $W \subset Y$ such that π is essentially convex on W and $0 \in W$, the *minimal provision mechanism* $f^{\pi *}$ *into* $Z^\pi(W)$ associates $(y^*(u)|_w; \pi(y^*(u)|_w))$ with each $u \in U^n$.

Theorem 5.4. *For any cost sharing rule $\pi \in \Pi_c^{''}$ and any closed set $W \subset Y$ such that π is essentially convex on W and $0 \in W$, any mechanism f^{π} into $Z^{\pi}(W)$ satisfying strategy-proofness, individual rationality, and $R(f^{\pi})=Z^{\pi}(W)$ is weakly Pareto dominated by the minimal provision mechanism $f^{\pi *}$ into $Z^{\pi}(W)$.*

Proof. It follows from Theorem 5.3 that any mechanism f^{π} into $Z^{\pi}(W)$ satisfying strategy-proofness, individual rationality, and $R(f^{\pi})=Z^{\pi}(W)$ is in the class of minimal provision mechanisms. Clearly, $y^{+}(u)|_W \leq y^{*}(u)|_W \leq y_i^{*}(u_i)|_W$ for all $u \in U^n$ and $i \in N$. Since π_i is convex on W and u_i is strictly convex, it holds that $u_i(y^{+}(u)|_W;\pi_i(y^{+}(u)|_W)) \leq u_i(y^{*}(u)|_W;\pi_i(y^{*}(u)|_W)) \leq u_i(y_i^{*}(u_i)|_W;\pi_i(y_i^{*}(u_i)|_W))$ for all $u \in U^n$ and $i \in N$. Since $f^{\pi *}$ into $Z^{\pi}(W)$ always chooses $(y^{*}(u)|_W;\pi(y^{*}(u)|_W))$, any mechanism f^{π} into $Z^{\pi}(W)$ satisfying the premises of the theorem is weakly Pareto dominated by $f^{\pi *}$ into $Z^{\pi}(W)$. **Q.E.D.**

We consider the opposite case of Theorem 5.3. We define the non-convexity of cost sharing rules on a triplet T, three distinct points in Y, and show that it is impossible to construct strategy-proof and individually rational mechanisms if the range of mechanisms includes three allocations which induce the non-convexity of cost sharing rules.

Definition 5.9. A cost sharing rule $\pi \in \Pi_c^{''}$ is *essentially non-convex on a triplet $T \subset Y$* iff each π_i is non-convex on T, that is, $\pi_i(y^2) > \frac{y^3-y^2}{y^3-y^1}\pi_i(y^1) + \frac{y^2-y^1}{y^3-y^1}\pi_i(y^3)$ for $T=\{y^1,y^2,y^3 \in Y \mid y^1<y^2<y^3\}$.[26]

Theorem 5.5. *For any cost sharing rule $\pi \in \Pi_c^{''}$, there is no mechanism f^{π} satisfying strategy-proofness, individual rationality, and such that $R(f^{\pi})$ contains $Z^{\pi}(T)$ for some triplet $T \subset Y$ on which f is essentially non-convex.*

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[26] Notice that any proportional cost sharing rule π is essentially non-convex on some triplet $T \subset Y$ if and only if the cost function c is non-convex on T. For any subset $W \subset Y$, any proportional cost sharing rule π is either essentially convex on W or essentially non-convex on some triplet $T \subset W$.

Proof. Suppose toward contradiction that there exist a mechanism f^{π} and some triplet $T \subset Y$ which satisfy the premises of the theorem. Since each variable cost sharing function π_i^v is convex on Y, $0 \in Y$ must be included in this triplet T. Let 0, y', and y" (0<y'<y") be the elements of T. Let S(y') and S(y") be some closed segments in Y including y', y", respectively. Let S([y',y"]) be some closed segment in Y including S(y') and S(y"). Denote the boundary points of S by Bd(S). If $y'' \neq y_{max}$, choose S(y'), S(y"), and S([y',y"]) such that $y' \notin Bd(S(y'))$, $y'' \notin Bd(S(y''))$, $S(y') \cap S(y'') = \varnothing$, $Bd(S(y')) \cap Bd(S([y',y''])) = \varnothing$, $Bd(S(y'')) \cap Bd(S([y',y''])) = \varnothing$, $0 \notin Bd(S([y',y'']))$, and that f is essentially non-convex on $\{0\} \cup Bd(S(y''))$. If $y'' = y_{max}$, choose S(y'), S(y"), and S([y',y"]) such that $y' \notin Bd(S(y'))$, $\{y''\} \neq S(y'')$, $S(y') \cap S(y'') = \varnothing$, $Bd(S(y')) \cap Bd(S([y',y''])) = \varnothing$, $0 \notin Bd(S([y',y'']))$, and that f is essentially non-convex on $\{0\} \cup Bd(S(y''))$. We can find the following four types of preferences for each agent (see Figures 5.5 - 5.8).

(i) u_i: $\mathrm{argmax}(u_i;Z_i^{\pi}) = (y';\pi_i(y'))$ and $u_i(y;\pi_i(y)) = u_i(0;0)$ for all $y \in Bd(S(y'))$.

(ii) $\tilde{u}_i$: $\mathrm{argmax}(\tilde{u}_i;Z_i^{\pi}) = (y'';\pi_i(y''))$, $\tilde{u}_i(y;\pi_i(y)) = \tilde{u}_i(0;0)$ for all $y \in Bd(S([y',y'']))\backslash\{y''\}$, and $\tilde{u}_i(\tilde{y}^1;\pi_i(\tilde{y}^1)) = \tilde{u}_i(\tilde{y}^2;\pi_i(\tilde{y}^2))$ for $\tilde{y}^1, \tilde{y}^2 \in Bd(S(y''))\backslash\{y''\}$.

(iii) $\hat{u}_i$: $\mathrm{argmax}(\hat{u}_i;Z_i^{\pi}) = (y';\pi_i(y'))$, $\hat{u}_i(y;\pi_i(y)) = \hat{u}_i(0;0)$ for all $y \in Bd(S([y',y'']))\backslash\{y''\}$, and $\hat{u}_i(\hat{y}^1;\pi_i(\hat{y}^1)) = \hat{u}_i(\hat{y}^2;\pi_i(\hat{y}^2))$ for $\hat{y}^1, \hat{y}^2 \in Bd(S(y'))$.

(iv) $\bar{u}_i$: $\mathrm{argmax}(\bar{u}_i;Z_i^{\pi}) = (y'';\pi_i(y''))$ and $\bar{u}_i(y;\pi_i(y)) = \bar{u}_i(0;0)$ for all $y \in Bd(S(y''))\backslash\{y''\}$.

Notice that $\tilde{u}_i(y;\pi_i(y)) > \tilde{u}_i(0;0)$ for all $y \in S(y')$ and $\hat{u}_i(y;\pi_i(y)) > \hat{u}_i(0;0)$ for all $y \in S(y'')$.

By Lemma 5.1, $f^{\pi}(u) = (y';\pi(y'))$. Even if agent n changes his preference into $\tilde{u}_n$, individual rationality for agent 1 requires $f^{\pi}(\tilde{u}_n,u_{-n}) = (y;\pi(y))$, where $y \in \{0\} \cup S(y')$. If y=0, agent n can manipulate f^{π} at $(\tilde{u}_n,u_{-n})$ via u_n since $\tilde{u}_n(y';\pi_n(y')) > \tilde{u}_n(0;0)$. Hence, $y \in S(y')$. Again, individual rationality for agent 1 requires $f^{\pi}(\tilde{u}_{n-1},\tilde{u}_n,u_{-[n-1],-n}) = (\bar{y};\pi(\bar{y}))$, where $\bar{y} \in \{0\} \cup S(y')$. If $\bar{y}=0$, agent n-1 can manipulate f^{π} at $(\tilde{u}_{n-1},\tilde{u}_n,u_{-[n-1],-n})$ via u_{n-1} since $\tilde{u}_{n-1}(y;\pi_{n-1}(y)) > \tilde{u}_{n-1}(0;0)$ for all $y \in S(y')$. Hence, $\bar{y} \in S(y')$. Applying this argument successively, we obtain

$$f^{\pi}(u_1,\tilde{u}_{-1}) = (y;\pi(y)), \text{ where } y \in S(y'). \tag{5.1}$$

By Lemma 5.1, $f^{\pi}(\bar{u}_1,\bar{u}_2,\tilde{u}_{-1,-2}) = (y'';\pi(y''))$. Individual rationality for agent 2 requires

$f^\pi(\widehat{u}_1,\overline{u}_2,\widetilde{u}_{-1,-2})=(y;\pi(y))$, where $y\in\{0\}\cup S(y'')$. If $y=0$, agent 1 can manipulate f^π at $(\widehat{u}_1,\overline{u}_2,\widetilde{u}_{-1,-2})$ via $\overline{u}_1$ since $\widehat{u}_1(y'';\pi_1(y''))>\widehat{u}_1(0;0)$. Hence, we obtain

$$f^\pi(\widehat{u}_1,\overline{u}_2,\widetilde{u}_{-1,-2})=(y;\pi(y)),\text{ where } y\in S(y''). \qquad (5.2)$$

By Lemma 5.1, $f^\pi(\widetilde{u})=(y'';\pi(y''))$. Individual rationality for agent 1 requires $f^\pi(\widehat{u}_1,\widetilde{u}_{-1})=(\widehat{y};\pi(\widehat{y}))$, where $\widehat{y}\in\{0\}\cup S([y',y''])$. If $\widehat{y}=0$, agent 1 can manipulate f^π at $(\widehat{u}_1,\widetilde{u}_{-1})$ via $\widetilde{u}_1$ since $\widehat{u}_1(y'';\pi_1(y''))>\widehat{u}_1(0;0)$. Hence, we obtain

$$f^\pi(\widehat{u}_1,\widetilde{u}_{-1})=(\widehat{y};\pi(\widehat{y})),\text{ where } \widehat{y}\in S([y',y'']). \qquad (5.3)$$

If $\widehat{y}\notin S(y')$, it follows from (5.1) and (5.3) that agent 1 can manipulate f^π at $(\widehat{u}_1,\widetilde{u}_{-1})$ via u_1 since $\widehat{u}_1(y;\pi_1(y))>\widehat{u}_1(\overline{y};\pi_1(\overline{y}))$ for all $y\in S(y')$ and $\overline{y}\in S([y',y''])\backslash S(y')$. If $\widehat{y}\notin S(y'')$, it follows from (5.2) and (5.3) that agent 2 can manipulate f^π at $(\widehat{u}_1,\widetilde{u}_{-1})$ via $\overline{u}_2$ since $\widetilde{u}_2(y;\pi_2(y))>\widetilde{u}_2(\overline{y};\pi_2(\overline{y}))$ for all $y\in S(y'')$ and $\overline{y}\in S([y',y''])\backslash S(y'')$. Therefore, $\widehat{y}$ must be in both $S(y')$ and $S(y'')$, which contradicts the assumption that $S(y')\cap S(y'')=\emptyset$. **Q.E.D.**

5.4.2. The Case with Several Public Goods

We consider economies with one private good x and several public goods $y_1,\ldots,y_m$ ($m\geq 2$). We present the same negative result as in the case of no fixed costs.

Theorem 5.6. *For any cost sharing rule $\pi\in\Pi_c''$, there is no mechanism f^π satisfying strategy-proofness, individual rationality, and $\dim(R(f^\pi))\geq 2$.*

Proof. Suppose toward contradiction that there exists a mechanism f^π which satisfies the premises of the theorem. It follows from the same reason as Lemma 5.4 that individual rationality requires $(0,...,0;0,...,0)\in R(f^\pi)$. Since $\dim(R(f^\pi))\geq 2$, we can choose y', $y''\in Y$ such that $(y';\pi(y'))\in R(f^\pi)$, $(y'';\pi(y''))\in R(f^\pi)$, and 0, y', y'' are not on a straight line. Consider the case of $y', y''\notin Bd(Y)$. Let $S(y')$ and $S(y'')$ be some closed sets in Y including y', y'', respectively. Let $S(\{y',y''\})$ be some closed set in Y including $S(y')$ and $S(y'')$. We can choose sufficiently large and strictly convex sets $S(y')$, $S(y'')$, and $S(\{y',y''\})$ such that $y'\notin Bd(S(y'))$, $y''\notin Bd(S(y''))$, $S(y')\cap S(y'')=\emptyset$, $Bd(S(y'))\cap Bd(S(\{y',y''\}))=\emptyset$, $Bd(S(y''))\cap Bd(S(\{y',y''\}))=\emptyset$ and $0\notin Bd(S(\{y',y''\}))$. Here, a sufficiently large set means a set including points near the origin. Then, we can construct the preferences used in Theorem 5.5. All the other cases are similar. The rest of the proof is the same as Theorem 5.5.   **Q.E.D.**

5.4.3. The Case of Linear Cost Sharing Rules

As a corollary of Theorems 5.5 and 5.6, we present a simple negative result. Let $\Pi_c^{'''}$ be the *subset of $\Pi_c^{''}$ satisfying the following Assumption 5.3.*

Assumption 5.3. Each variable cost sharing function π_i^V is linear.

Corollary 5.1. *For any cost sharing rule $\pi \in \Pi_c^{'''}$, there is no mechanism f^π satisfying strategy-proofness, individual rationality, and $\#R(f^\pi) \geq 3$.*

Proof. It follows immediately from Theorem 5.5 when the range of mechanisms is one-dimension, and from Theorem 5.6 when the range of mechanisms is greater than one-dimension.   **Q.E.D.**

5.5. Conclusion

We showed that the minimal provision mechanism is the unique mechanism satisfying strategy-proofness, individual rationality, and the full-range property in economies with one private good and one public good when the cost sharing rule has the convex property. Even if the cost sharing rule has positive fixed costs and thus it includes a non-convex portion, the proper restriction of the range of mechanisms guarantees that the minimal provision mechanism satisfies strategy-proofness and individual rationality. Conversely, if the restriction is not sufficient, the implied non-convexity of the cost sharing rule leads to an impossibility result. Moreover, we proved that there is no strategy-proof and individually rational mechanism in economies with one private good and several public goods.

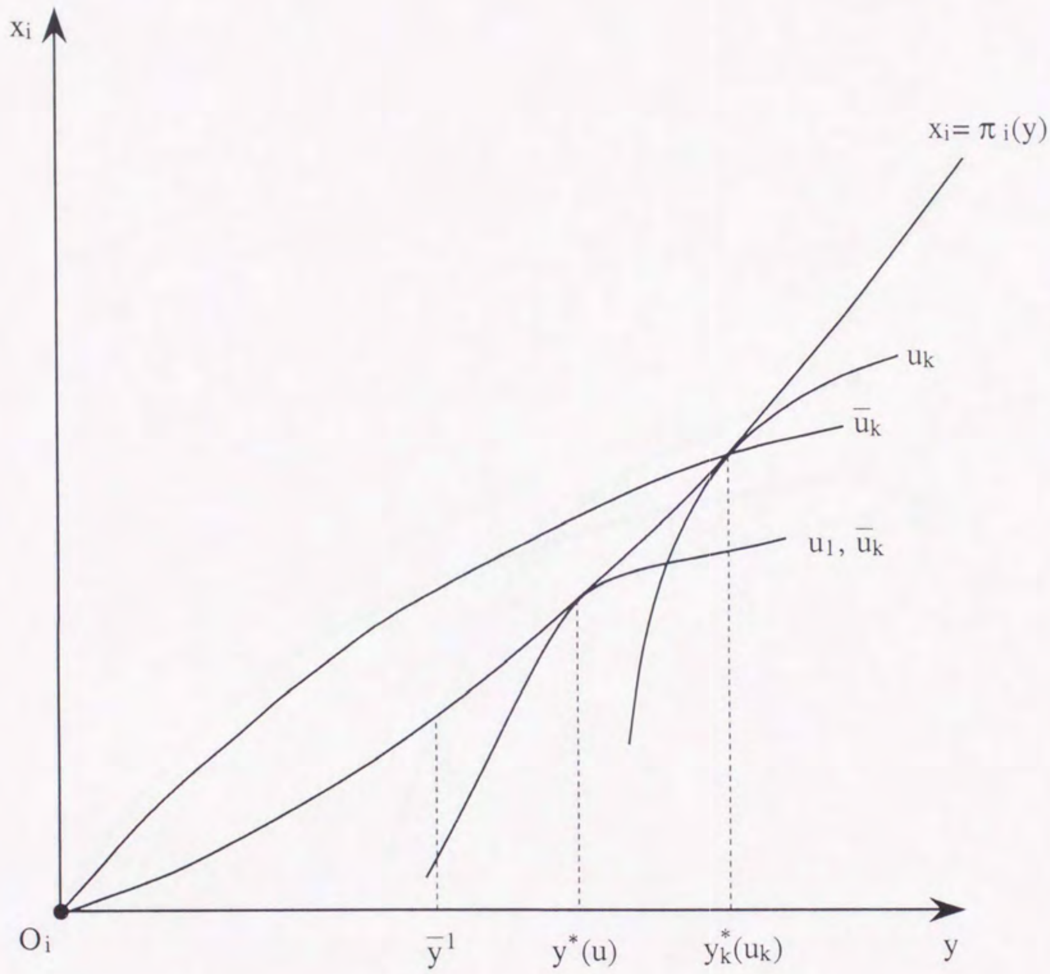

Figure 5.1. An illustration of the proof of Theorem 5.1.

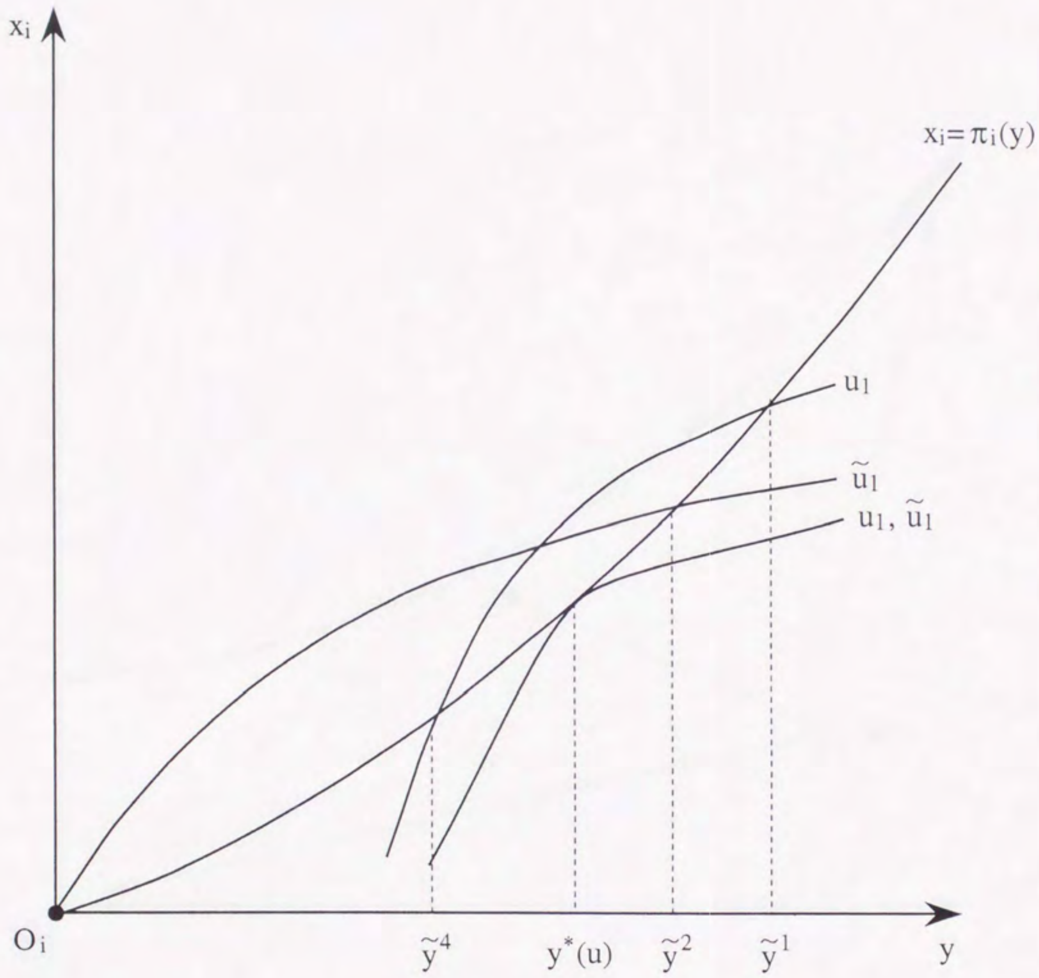

Figure 5.2. An illustration of the proof of Theorem 5.1.

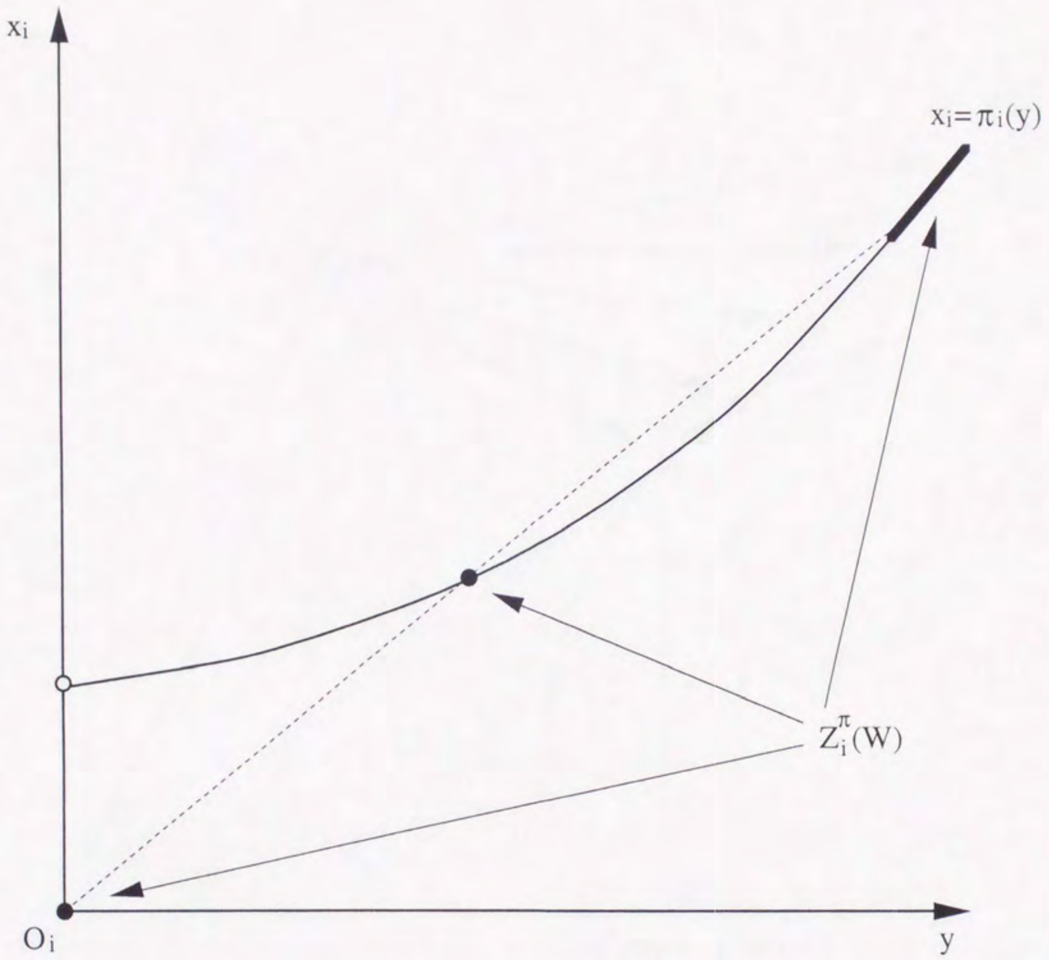

Figure 5.3. An example of $Z_i^\pi(W)$.

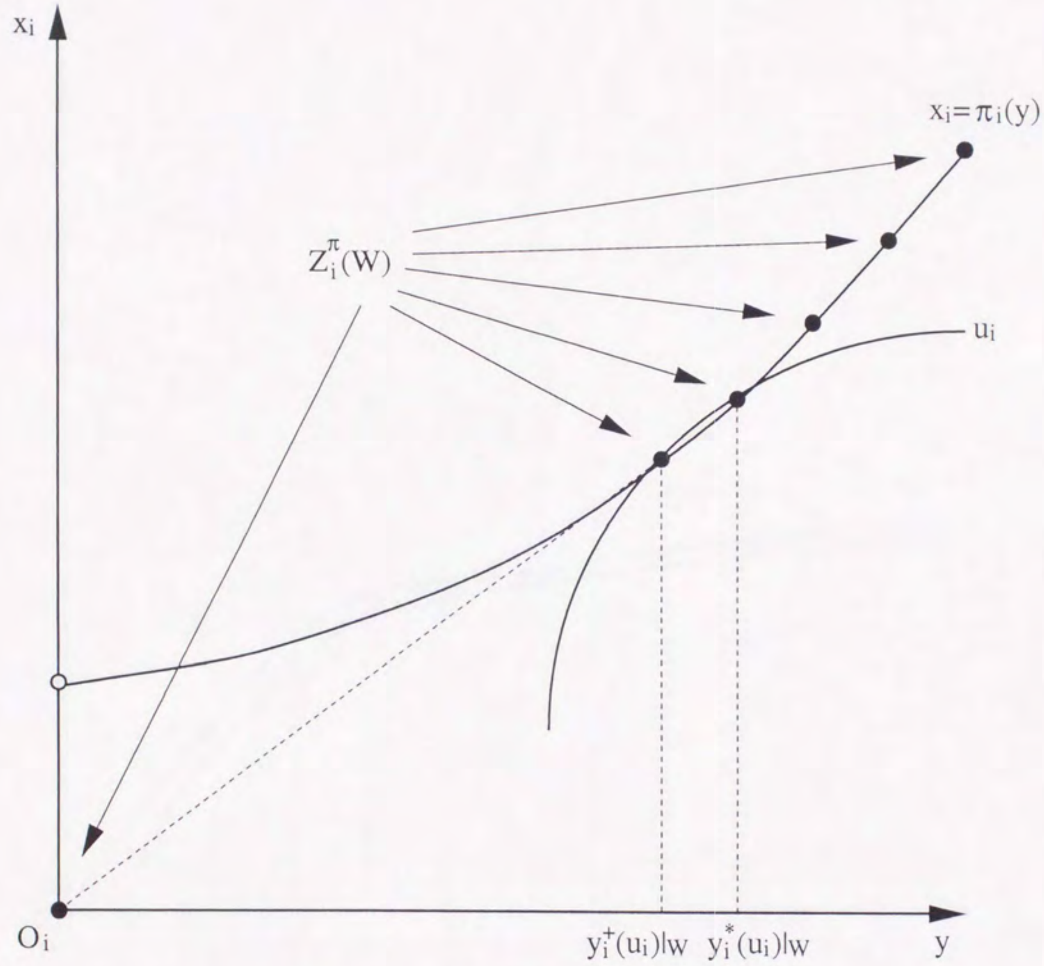

Figure 5.4. An example of $Z_i^{\pi}(W)$.

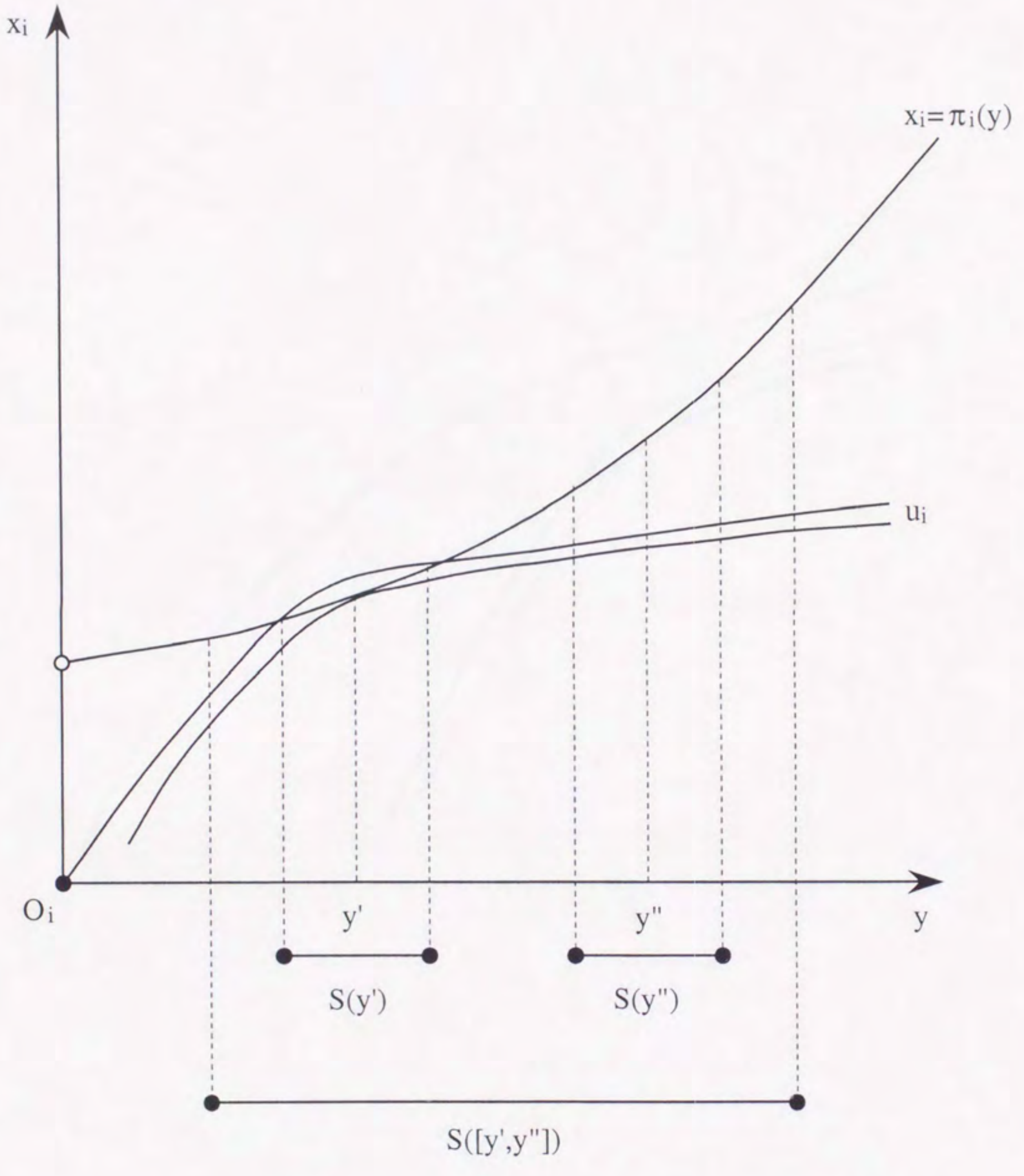

Figure 5.5. An example of preference u_i.

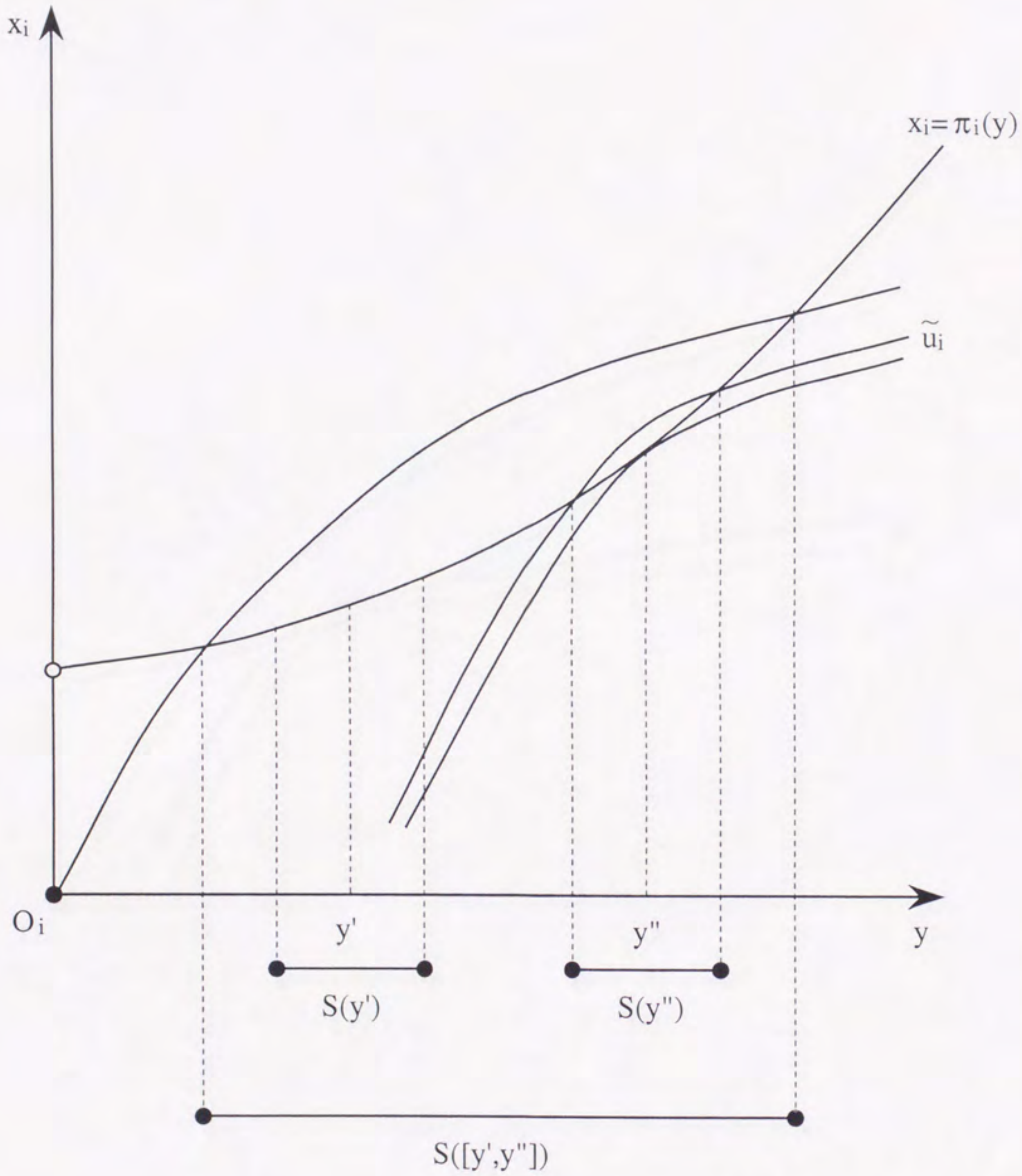

Figure 5.6. An example of preference $\widetilde{u}_i$.

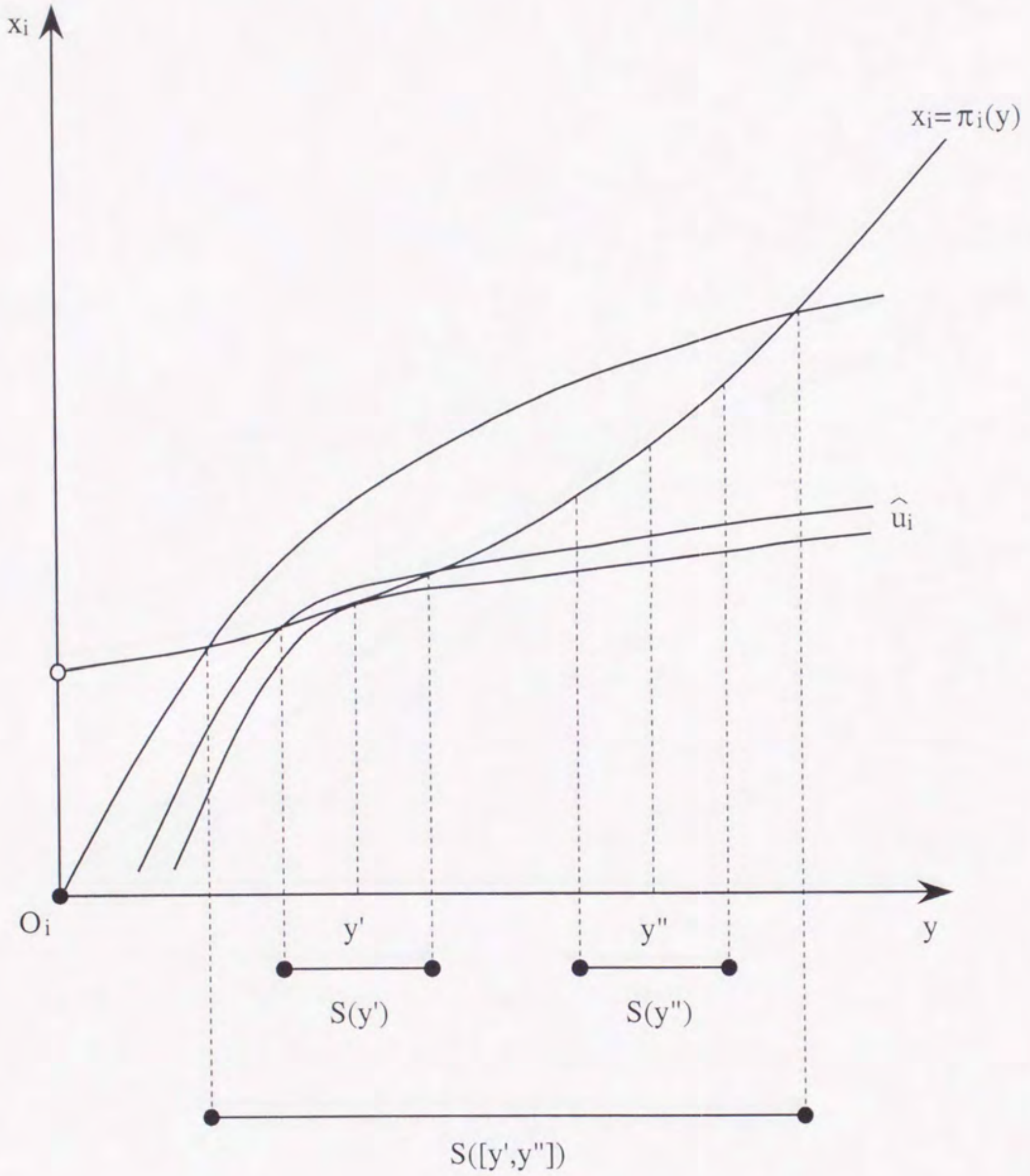

Figure 5.7. An example of preference $\widehat{u}_i$.

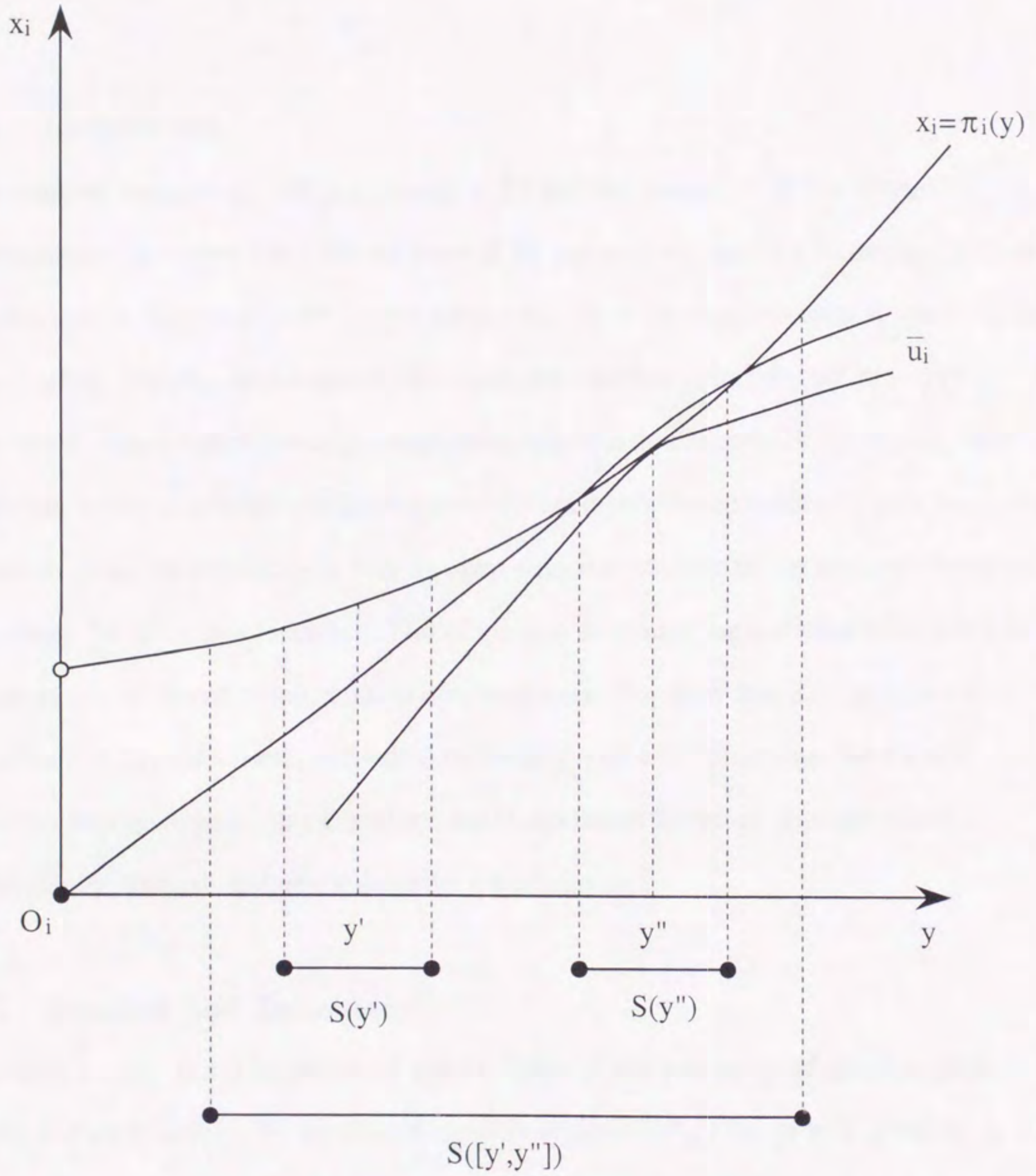

Figure 5.8. An example of preference $\bar{u}_i$.

Chapter 6

Strategy-Proof and Individually Rational Mechanisms for Public Good Economies: A Note[*****]

6.1. Introduction

We consider economies with one private good and one public good. We consider mechanisms that determine both the level of the public good and how to divide the costs among agents. Serizawa (1996) characterizes the set of mechanisms named "semiconvex cost sharing schemes determined by the minimum demand principle" by strategy-proofness, individual rationality, no exploitation, and non-bossiness. However, there is a criticism on the non-bossiness axiom since the economic interpretation of non-bossiness is not so clear. Moreover, he leaves an open question whether or not non-bossiness is necessary for his characterization. Therefore, it is an interesting question what class of mechanisms is characterized without non-bossiness. We show that if a mechanism satisfies strategy-proofness, individual rationality, and no exploitation, then it also satisfies non-bossiness. As a corollary, we characterize the set of strategy-proof, individually rational, and no exploitative mechanisms.

6.2. Notation and Definitions[27]

Let $N=\{1,...,n\}$ $(n\geq 2)$ be the set of agents. There is one private good and one public good. For each agent $i \in N$, we denote agent i's endowment of the private good by e_i. The initial amount of the public good is assumed to be zero. The public good can be produced using the private good which is regarded as money. For each agent $i \in N$, we denote agent i's consumption of the private good by x_i. The amount of the public good is denoted by y. The cost function c(y) of the public good is a continuous and increasing function from

[*****] This chapter is based on Deb and Ohseto (1999).

[27] We follow the model in Serizawa (1996).

R_+ to R_+ such that $c(0)=0$ and for all $y \in R_+$, $\liminf_{\varepsilon \to +0} \frac{c(y+\varepsilon) - c(y)}{\varepsilon} > 0$, $\limsup_{\varepsilon \to +0} \frac{c(y+\varepsilon) - c(y)}{\varepsilon} < \infty$, $\liminf_{\varepsilon \to +0} \frac{c(y) - c(y-\varepsilon)}{\varepsilon} > 0$, $\limsup_{\varepsilon \to +0} \frac{c(y) - c(y-\varepsilon)}{\varepsilon} < \infty$. The set of feasible allocations is $Z=\{(x_1,\ldots,x_n;y) \in R_+^{n+1} \mid c(y)=\sum_{i \in N} (e_i-x_i)\}$.

Each agent $i \in N$ has a preference on R_+^2 which can be represented by a utility function $u_i(x_i,y)$. Let U be the set of continuous, strictly convex, and strictly increasing preferences on R_+^2. A list $u=(u_1,\ldots,u_n) \in U^n$ is called a preference profile. We denote generic elements of U by u_i, $\bar{u}_i$, $\hat{u}_i$,..., and generic elements of U^n by $u, \bar{u},\ldots$, respectively. Given $u \in U^n$, $i, j \in N$, $\bar{u}_i \in U$, and $\hat{u}_j \in U$, we denote by $(\bar{u}_i,u_{-i})$ the preference profile obtained from u after the replacement of u_i by $\bar{u}_i$, and by $(\bar{u}_i,\hat{u}_j,u_{-i,-j})$ the preference profile obtained from u after the replacement of u_i and u_j by $\bar{u}_i$ and $\hat{u}_j$. The upper contour set and the lower contour set of $u_i \in U$ at $(\bar{x}_i,\bar{y}) \in R_+^2$ are defined by $UC(u_i; (\bar{x}_i,\bar{y}))=\{(x_i,y) \mid u_i(x_i,y) \geq u_i(\bar{x}_i,\bar{y})\}$ and $LC(u_i; (\bar{x}_i,\bar{y}))=\{(x_i,y) \mid u_i(\bar{x}_i,\bar{y}) \geq u_i(x_i,y)\}$, respectively. A *mechanism* is a function $f: U^n \to Z$, which associates a feasible allocation with each preference profile. Given a mechanism f and $u \in U^n$, we will write $f(u)=(x_1(u),\ldots,x_n(u);y(u))$ and $f_i(u)=(x_i(u),y(u))$.

Definition 6.1. A mechanism f satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(\bar{u}_i,u_{-i}))$.

Definition 6.2. A mechanism f satisfies *individual rationality* iff for all $u \in U^n$ and $i \in N$, $u_i(f_i(u)) \geq u_i(e_i,0)$.

Definition 6.3. A mechanism f satisfies *no exploitation* iff for all $u \in U^n$ and $i \in N$, $x_i(u) \leq e_i$.

Definition 6.4. A mechanism f satisfies *non-bossiness* iff for all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, $[f_i(u)=f_i(\bar{u}_i,u_{-i}) \Rightarrow f(u)=f(\bar{u}_i,u_{-i})]$.

Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. Individual rationality requires that all agents end up no worse off than at the status quo. No exploitation requires that no agent receive the private good in addition to his endowment of the private good. Non-bossiness requires that by changing his preferences, no agent can change the allocation without changing his consumption bundle.

6.3. Results

Theorem 6.1. *If a mechanism f satisfies strategy-proofness, individual rationality, and no exploitation, then f satisfies non-bossiness.*[28]

Proof. Suppose that a mechanism f satisfies strategy-proofness, individual rationality, and no exploitation. The proof is divided into three steps.

Step 1. *Suppose that $y(u)=y(\overline{u})$ and $f(u)\neq f(\overline{u})$ for some $u, \overline{u}\in U^n$. Then, for any $i\in N$, there exists some $\widetilde{u}_i\in U$ such that $y(\widetilde{u}_i, u_{-i})=y(\widetilde{u}_i, \overline{u}_{-i})$ and $f(\widetilde{u}_i, u_{-i})\neq f(\widetilde{u}_i, \overline{u}_{-i})$.*

Case 1. $x_i(u)=x_i(\overline{u})$.

We can choose some $\widetilde{u}_i\in U$ such that for all $x_i\in R_+$,

$$UC(\widetilde{u}_i; (x_i,y(u))) \cap LC(u_i; (x_i,y(u))) = \{(x_i,y(u))\}, \text{ and} \tag{6.1}$$

$$UC(\widetilde{u}_i; (x_i,y(u))) \cap LC(\overline{u}_i; (x_i,y(u))) = \{(x_i,y(u))\}. \tag{6.2}$$

By (6.1), strategy-proofness implies $f_i(\widetilde{u}_i,u_{-i})=f_i(u)$. By (6.2), strategy-proofness implies $f_i(\widetilde{u}_i,\overline{u}_{-i})=f_i(\overline{u})$. Hence, $y(\widetilde{u}_i,u_{-i})=y(\widetilde{u}_i,\overline{u}_{-i})$. The proof of this case is complete if $f(\widetilde{u}_i,u_{-i})\neq f(\widetilde{u}_i,\overline{u}_{-i})$. Assume to the contrary that $f(\widetilde{u}_i,u_{-i})=f(\widetilde{u}_i,\overline{u}_{-i})$. Then, it holds that either $f(u)\neq f(\widetilde{u}_i,u_{-i})$ or $f(\overline{u})\neq f(\widetilde{u}_i,\overline{u}_{-i})$. Without loss of generality, we assume that $f(u)\neq f(\widetilde{u}_i,u_{-i})$. Since the allocation is balanced and $f_i(u)=f_i(\widetilde{u}_i,u_{-i})$, we can choose some agent $j\neq i$ such that $x_j(u)>x_j(\widetilde{u}_i,u_{-i})$. We can choose some $\widehat{u}_j\in U$ such that

$$UC(\widehat{u}_j; f_j(u)) \cap LC(u_j; f_j(u)) = \{f_j(u)\}, \text{ and} \tag{6.3}$$

[28] It follows from Deb, Razzolini, and Seo (1995) and Ohseto (1999a) that strategy-proofness, individual rationality, and no exploitation imply non-bossiness for the case of the binary provision of the public good (namely, y=0 or y=1). This theorem generalizes the result to the case of an arbitrary set of the levels of the public good. The case of the binary provision or the continuous provision follows as a corollary of this theorem.

$$UC(\widehat{u}_j; (e_j,0)) \cap LC(u_j; f_j(\widetilde{u}_i,u_{-i})) \cap \{(x_j,y) | \, y \geq y(u)\} = \varnothing. \tag{6.4}$$

This construction is possible since f satisfies no exploitation (see Figure 6.1). By (6.3), strategy-proofness implies $f_j(\widehat{u}_j,u_{-j})=f_j(u)$. By (6.4), strategy-proofness and individual rationality imply $y(\widetilde{u}_i,\widehat{u}_j,u_{-i,-j})<y(\widetilde{u}_i,u_{-i})$. By strategy-proofness, $u_i(f_i(\widehat{u}_j,u_{-j}))\geq u_i(f_i(\widetilde{u}_i,\widehat{u}_j,u_{-i,-j}))$ and $\widetilde{u}_i(f_i(\widetilde{u}_i,\widehat{u}_j,u_{-i,-j}))\geq \widetilde{u}_i(f_i(\widehat{u}_j,u_{-j}))$. Hence, by (6.1), it must hold that $f_i(\widetilde{u}_i,\widehat{u}_j,u_{-i,-j})=f_i(\widehat{u}_j,u_{-j})$. This contradicts the fact that $y(\widetilde{u}_i,\widehat{u}_j,u_{-i,-j})<y(\widetilde{u}_i,u_{-i})=y(u)=y(\widehat{u}_j,u_{-j})$.

Case 2. $x_i(u)\neq x_i(\overline{u})$.

We can choose some $\widetilde{u}_i\in U$ which satisfies (6.1) and (6.2) for all $x_i\in R_+$. By strategy-proofness, $f_i(\widetilde{u}_i,u_{-i})=f_i(u)$ and $f_i(\widetilde{u}_i,\overline{u}_{-i})=f_i(\overline{u})$. They imply $y(\widetilde{u}_i,u_{-i})=y(\widetilde{u}_i,\overline{u}_{-i})$ and $f(\widetilde{u}_i,u_{-i})\neq f(\widetilde{u}_i,\overline{u}_{-i})$.

Step 2. It holds that $f(u)=f(\overline{u})$ for all u, $\overline{u}\in U^n$ such that $y(u)=y(\overline{u})$.

Assume to the contrary that $y(u)=y(\overline{u})$ and $f(u)\neq f(\overline{u})$ for some u, $\overline{u}\in U^n$. By Step 1, there exists some $\widetilde{u}_1\in U$ such that $y(\widetilde{u}_1,u_{-1})=y(\widetilde{u}_1,\overline{u}_{-1})$ and $f(\widetilde{u}_1,u_{-1})\neq f(\widetilde{u}_1,\overline{u}_{-1})$. Repeatedly applying Step 1 to the remaining agents, we can find $\widetilde{u}_2\in U$, $\widetilde{u}_3\in U$,..., $\widetilde{u}_{n-1}\in U$ such that $y(u_n,\widetilde{u}_{-n})=y(\overline{u}_n,\widetilde{u}_{-n})$ and $f(u_n,\widetilde{u}_{-n})\neq f(\overline{u}_n,\widetilde{u}_{-n})$. Therefore, applying Step 1 to agent n leads to a contradiction.

Step 3. f satisfies non-bossiness.

It is obvious that for all $u\in U^n$, $i\in N$, and $\overline{u}_i\in U$, $[f_i(u)=f_i(\overline{u}_i,u_{-i}) \Rightarrow y(u)=y(\overline{u}_i,u_{-i})]$. It follows from Step 2 that for all $u\in U^n$, $i\in N$, and $\overline{u}_i\in U$, $[y(u)=y(\overline{u}_i,u_{-i}) \Rightarrow f(u)=f(\overline{u}_i,u_{-i})]$. Therefore, f satisfies non-bossiness. **Q.E.D.**

We will explain the non-redundancy of the three axioms of strategy-proofness, individual rationality, and no exploitation for this theorem. Notice that any mechanism satisfies non-bossiness in the two-agent case. Each example below satisfies two axioms out of three, but does not satisfy the axiom of non-bossiness.

Example 6.1. Let $n=3$, $e_1=e_2=e_3=1$, and $c(1)=1$. Let a mechanism f be such that for all $u\in U^n$, (i) if $u_1(0,1)\geq u_1(1,0)$, $u_2(0,1)\geq u_2(1,0)$, and $u_3(0,1)\geq u_3(1,0)$, then

$f(u)=(0,1,1;1)$, (ii) if $u_1(0,1)\geq u_1(1,0)$, $u_2(0,1)\geq u_2(1,0)$, and $u_3(0,1)<u_3(1,0)$, then $f(u)=(1,0,1;1)$, and (iii) if $u_1(0,1)<u_1(1,0)$ or $u_2(0,1)<u_2(1,0)$, then $f(u)=(1,1,1;0)$. Then, f satisfies individual rationality and no exploitation, but does not satisfy strategy-proofness or non-bossiness.

Example 6.2. Let $n=3$, $e_1=e_2=e_3=1$, and $c(1)=1$. Let a mechanism f be such that for all $u\in U^n$, (i) if $u_3(0,1)\geq u_3(1,0)$, then $f(u)=(0,1,1;1)$, and (ii) if $u_3(0,1)<u_3(1,0)$, then $f(u)=(1,0,1;1)$. Then, f satisfies strategy-proofness and no exploitation, but does not satisfy individual rationality or non-bossiness.

Example 6.3. Let $n=3$, $e_1=e_2=e_3=1$, and $c(1)=\frac{1}{2}$. Let a mechanism f be such that for all $u\in U^n$, (i) if $u_3(0,1)\geq u_3(2,0)$, then $f(u)=(\frac{3}{2},1,0;1)$, (ii) if $u_3(2,0)>u_3(0,1)\geq u_3(1,0)$, then $f(u)=(1,\frac{3}{2},0;1)$, and (iii) if $u_3(1,0)>u_3(0,1)$, then $f(u)=(1,1,1;0)$. Then, f satisfies strategy-proofness and individual rationality, but does not satisfy no exploitation or non-bossiness.

6.4. Conclusion

Serizawa (1996) defined the set of mechanisms called "semiconvex cost sharing schemes determined by the minimum demand principle".[29] He characterized it by the four axioms of strategy-proofness, individual rationality, no exploitation, and non-bossiness. Now we establish the following new characterization as a corollary of his and our results.[30]

Corollary 6.1. *A mechanism f satisfies strategy-proofness, individual rationality, and no exploitation if and only if f is a semiconvex cost sharing scheme determined by the minimum demand principle.*

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[29] Refer to Serizawa (1996) for the precise definition.

[30] Using Theorem 6.1, most of the results in Serizawa (1996) can be reestablished without the non-bossiness axiom. For example, it can be shown that strategy-proofness, individual rationality, and no exploitation imply coalitional strategy-proofness.

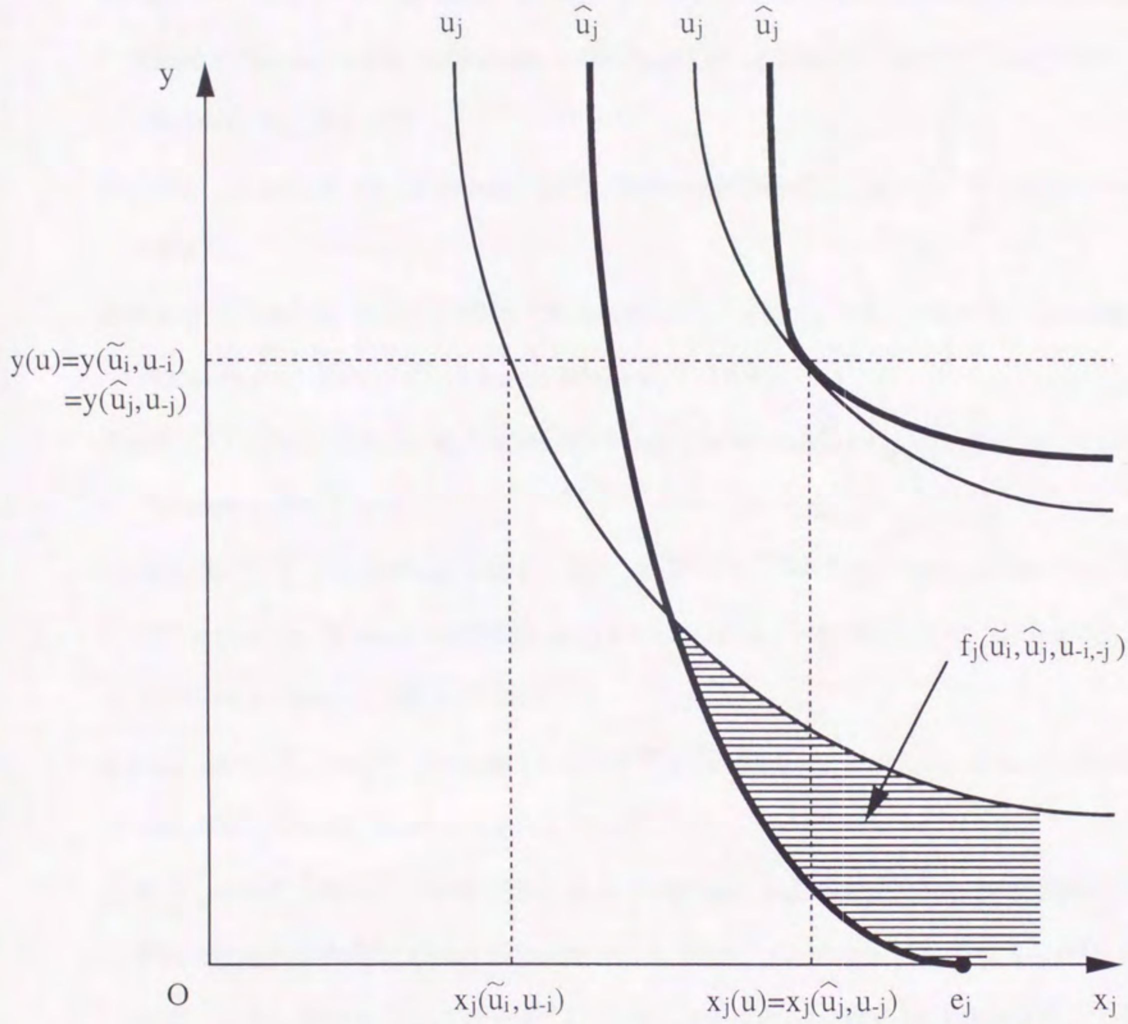


Figure 6.1. An illustration of the proof of Theorem 6.1.

References

Barbera, S., and M. O. Jackson (1994): "A Characterization of Strategy-Proof Social Choice Functions for Economies with Pure Public Goods," *Social Choice and Welfare*, 11, 241-252.

Barbera, S., and M. O. Jackson (1995): "Strategy-Proof Exchange," *Econometrica*, 63, 51-87.

Barbera, S., and B. Peleg (1990): "Strategy-Proof Voting Schemes with Continuous Preferences," *Social Choice and Welfare*, 7, 31-38.

Black, D. (1948): "On the Rationale of Group Decision-Making," *Journal of Political Economy*, 56, 23-34.

Dasgupta, P., P. Hammond, and E. Maskin (1979): "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility," *Review of Economic Studies*, 44, 185-216.

d'Aspremont, C., and L. Gerard-Varet (1979): "Incentives and Incomplete Information," *Journal of Public Economics*, 11, 25-45.

Deb, R., and S. Ohseto (1999): "Strategy-Proof and Individually Rational Social Choice Functions for Public Good Economies: A Note," *Economic Theory*, 14, 685-689.

Deb, R., and L. Razzolini (1999a): "Auction Like Mechanisms for Pricing Excludable Public Goods," *Journal of Economic Theory*, forthcoming.

Deb, R., and L. Razzolini (1999b): "Voluntary Cost Sharing for an Excludable Public Project," *Mathematical Social Sciences*, 37, 123-138.

Deb, R., L. Razzolini, and T. K. Seo (1995): "Strategy-Proof Cost Sharing, Ability to Pay and Free Provision of an Indivisible Public Good," mimeo, Southern Methodist University.

Foley, D. (1967): "Resource Allocation and the Public Sector," *Yale Economic Essays*, 7, 45-98.

Gibbard, A. (1973): "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41, 587-601.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 47, 1137-1144.

Hurwicz, L. (1972): "On Informationally Decentralized Systems," in *Decision and Organization*, ed. by C. B. McGuire and R. Radner. Amsterdam: North-Holland, pp. 297-336.

Hurwicz, L., and M. Walker (1990): "On the Generic Nonoptimality of Dominant-Strategy Allocation Mechanisms: A General Theorem that Includes Pure Exchange Economies," *Econometrica*, 58, 683-704.

Mas-Colell, A. (1980): "Efficiency and Decentralization in the Pure Theory of Public Goods," *Quarterly Journal of Economics*, 94, 625-641.

Mas-Colell, A., and J. Silvestre (1989): "Cost Share Equilibria: A Lindahlian Approach," *Journal of Economic Theory*, 47, 239-256.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory.* New York: Oxford University Press.

Maskin, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66, 23-38.

Moulin, H. (1980): "On Strategy-Proofness and Single Peakedness," *Public Choice*, 35, 437-455.

Moulin, H. (1993): "On the Fair and Coalition-Strategyproof Allocation of Private Goods," in *Frontiers in Game Theory*, ed. by K. Binmore, A. Kirman, and P. Tani. Cambridge: MIT Press, pp. 151-163.

Moulin, H. (1994): "Serial Cost-Sharing of Excludable Public Goods," *Review of Economic Studies*, 61, 305-325.

Myerson, R. B., and M. A. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29, 265-281.

Ohseto, S. (1997): "Strategy-Proof Mechanisms in Public Good Economies," *Mathematical Social Sciences*, 33, 157-183.

Ohseto, S. (1999a): "Characterizations of Strategy-Proof Mechanisms for Excludable versus Non-Excludable Public Projects," *Games and Economic Behavior*, forthcoming.

Ohseto, S. (1999b): "Strategy-Proof Allocation Mechanisms for Economies with an Indivisible Good," *Social Choice and Welfare*, 16, 121-136.

Ohseto, S. (1999c): "Strategy-Proof and Efficient Allocation of an Indivisible Good on Finitely Restricted Preference Domains," mimeo, Tokyo Metropolitan University.

Roth, A. E. (1977): "Individual Rationality and Nash's Solution to the Bargaining Problem," *Mathematics of Operations Research*, 2, 64-65.

Saijo, T. (1991): "Incentive Compatibility and Individual Rationality in Public Good Economies," *Journal of Economic Theory*, 55, 203-212.

Satterthwaite, M. A. (1975): "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory*, 10, 187-217.

Satterthwaite, M. A., and H. Sonnenschein (1981): "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies*, 48, 587-597.

Schummer, J. (1997): "Strategy-Proofness versus Efficiency on Restricted Domains of Exchange Economies," *Social Choice and Welfare*, 14, 47-56.

Schummer, J. (1998): "Eliciting Preferences to Assign Positions and Compensation," *Games and Economic Behavior*, forthcoming.

Serizawa, S. (1996): "Strategy-Proof and Individually Rational Social Choice Functions for Public Good Economies," *Economic Theory*, 7, 501-512.

Serizawa, S. (1999): "Strategy-Proof and Symmetric Social Choice Functions for Public Good Economies," *Econometrica*, 67, 121-145.

Svensson, L. G. (1983): "Large Indivisibles: An Analysis with Respect to Price Equilibrium and Fairness," *Econometrica*, 51, 939-954.

Tadenuma, K., and W. Thomson (1993): "The Fair Allocation of an Indivisible Good When Monetary Compensations Are Possible," *Mathematical Social Sciences*, 25, 117-132.

Tadenuma, K., and W. Thomson (1995): "Games of Fair Division," *Games and*

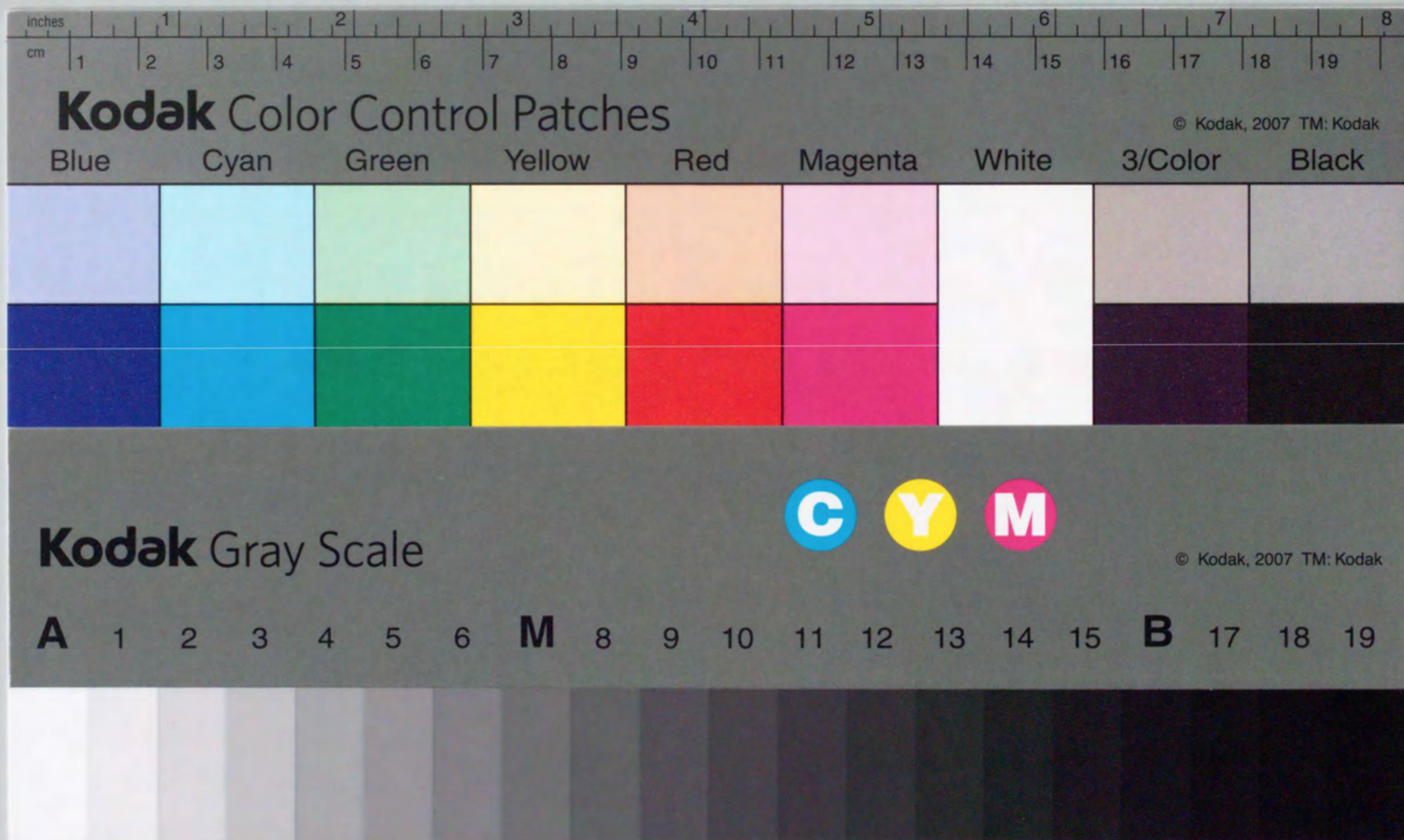
Economic Behavior, 9, 191-204.

Thomson, W. (1983a): "The Fair Division of a Fixed Supply among a Growing Population," *Mathematics of Operations Research*, 8, 319-326.

Thomson, W. (1983b): "Problems of Fair Division and the Egalitarian Principle," *Journal of Economic Theory*, 31, 211-226.

Zhou, L. (1991a): "Impossibility of Strategy-Proof Mechanisms in Economies with Pure Public Goods," *Review of Economic Studies*, 58, 107-119.

Zhou, L. (1991b): "Inefficiency of Strategy-Proof Allocation Mechanisms in Pure Exchange Economies," *Social Choice and Welfare*, 8, 247-254.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