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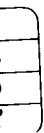
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**HOLOMORPHIC VERTICAL LINE BUNDLE OF THE TWISTOR
SPACE OVER A QUATERNIONIC MANIFOLD**

(四元数多様体上のツイスター空間の正則垂直直線束)

Toshimasa KOBAYASHI



HOLOMORPHIC VERTICAL LINE BUNDLE OF THE TWISTOR SPACE OVER A QUATERNIONIC MANIFOLD

TOSHIMASA KOBAYASHI

ABSTRACT. The vertical bundle of the twistor fibration over a 4-dimensional self-dual manifold is a holomorphic line bundle and plays an important role in a study of the twistor space. On the other hand, the vertical bundle of the twistor space over a quaternionic manifold is not a holomorphic line bundle, in general. We shall give the condition for a vertical bundle to be a holomorphic line bundle.

1. INTRODUCTION

We are concerned with holomorphic structures on the vertical bundle of the twistor fibration over a quaternionic manifold.

For an oriented m -dimensional conformal manifold M , we may consider a Weyl structure D on M , which is a symmetric linear connection preserving the conformal structure of M . Over M , there is a line bundle L associated to the $CO(m)$ -principal bundle of M and the representation $A \mapsto |\det A|^{\frac{1}{m}}$ of the linear group. Thus a Weyl structure D on M induces a linear connection D^L on L . In the case of $m = 4$, if the curvature of D^L is a self-dual 2-form, then D is called a self-dual Weyl structure. While it is known that if M is a 4-dimensional self-dual manifold, then there is a complex 3-manifold Z fibered over M by a family of projective lines. Z is called the twistor space of M . The vertical bundle Θ of Z is considered as a complex line bundle over Z and has a natural Hermitian metric. We choose a Weyl structure D on M , then a linear connection ∇ on Θ is induced by D . If the curvature of ∇ is of type (1,1) relative to the complex structure on Z , then we call ∇ a *Chern connection*. A Chern connection on Θ induces a holomorphic structure that renders Θ a holomorphic line bundle over Z . In particular, if D is the Levi-Civita connection of a self-dual metric on M , then the induced connection ∇ on Θ is a Chern connection, and $\otimes^2\Theta$ is isomorphic to the dual bundle of the canonical bundle of Z as a holomorphic bundle.

Gauduchon showed that for a 4-dimensional self-dual manifold, a linear connection ∇ on Θ is a Chern connection if and only if a Weyl structure D that induces ∇ is self-dual. Furthermore, if M is compact, he classified the types of the conformal structures

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admitting holomorphic sections on $\otimes^p \Theta$. Using these results and a vanishing theorem, he proved that if the conformal class of M contains a metric with negative scalar curvature then the twistor space of M does not contain any nontrivial divisor.

A $4n$ -dimensional manifold ($n \geq 2$) is called *quaternionic* if it has a $GL(n, \mathbb{H})Sp(1)$ -structure preserved by a torsion-free connection. We note that if $n = 1$ then $GL(1, \mathbb{H})Sp(1) = CO(4)$. Salamon showed that there is a twistor space Z over a quaternionic manifold M . The fiber Z_x over each point $x \in M$ is a real 2-sphere, which parametrizes almost complex structures on $T_x M$, and the total space of Z admits a complex structure. Therefore, we regard the notion of quaternionic manifold as a generalization of that of self-dual manifold and examine quaternionic manifolds and their twistor spaces.

In the next section, we recall the twistor space of a quaternionic manifold. We express a twistor space and its vertical bundle as associated bundles with the $GL(n, \mathbb{H})Sp(1)$ -principal bundle and representations of $GL(n, \mathbb{H})Sp(1)$. Thus we see that a connection D on a quaternionic manifold induces a connection ∇ on a vertical bundle. Further, we may describe the curvature R^∇ of ∇ explicitly, and see the relation between the curvatures R^∇ and R^D . In Section 3, we recall representations of the structure group $GL(n, \mathbb{H})Sp(1)$ and the first prolongation of its Lie algebra. Combining the Clebsch-Gordan formula and the formulas of irreducible decompositions of $GL(n, \mathbb{H})$ -modules, we describe the first prolongation as a $GL(n, \mathbb{H})Sp(1)$ -module. In Section 4, we shall study a curvature of a quaternionic manifold by means of representation theory. We consider R^D as a 2-form with values in the Lie algebra $\mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)$ of $GL(n, \mathbb{H})Sp(1)$. From the first Bianchi identity, we see that R^D determines an element of a Spencer cohomology. By using some irreducible decompositions of $GL(n, \mathbb{H})Sp(1)$ -modules, we have the irreducible decomposition of a curvature of a quaternionic manifold. In Section 5, we have the main theorem. From the results in Sections 3 and 4, we may describe a curvature of a quaternionic manifold explicitly. We shall obtain the condition for the vertical bundle of the twistor space of a quaternionic manifold to have a Chern connection. We also find that this condition corresponds to the condition for a Weyl structure to be self-dual in the case of a 4-dimensional self-dual manifold. In Section 6, we deal with hypercomplex manifolds. A $4n$ -dimensional manifold that has a $GL(n, \mathbb{H})$ -structure with a torsion-free connection is called a *hypercomplex manifold*. We note that the class of hypercomplex manifolds is included in that of quaternionic manifolds. It is known that a hypercomplex manifold has a unique torsion-free connection. It is called the *Obata connection*. Applying the theorem in Section 5 to the case of a hypercomplex manifold, we see that an Obata connection induces a Chern connection on a vertical line bundle.

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2. TWISTOR SPACES

Let M be a quaternionic manifold, which is a real $4n$ -dimensional manifold, $n \geq 2$, with a $GL(n, \mathbb{H})Sp(1)$ -structure admitting a torsion-free connection. We choose a connection D out of such connections. We denote by E, H the standard complex representations of $GL(n, \mathbb{H}), Sp(1)$ on $\mathbb{C}^{2n}, \mathbb{C}^2$ respectively. The complex vector spaces E and H possess antilinear structure maps $v \mapsto \tilde{v}$ commuting with the action of the respective groups and satisfying $\tilde{\tilde{v}} = -v$. Such representations are called quaternionic. Then the complexified cotangent bundle of M has the form

$$(2.1) \quad (T^*M)^{\mathbb{C}} \cong \mathbf{E} \otimes_{\mathbb{C}} \mathbf{H},$$

where \mathbf{E}, \mathbf{H} are vector bundles associated to representations E, H respectively. The symmetric powers $S^k H (k \geq 0)$ are the irreducible complex representations of $Sp(1)$. If k is even, then $S^k H$ has a real structure induced from the structure map of H , so we regard it as a real vector space. In particular, $S^2 H$ is the adjoint representation of $Sp(1)$. There is an $Sp(1)$ -invariant skew form $\omega_H \in \Lambda^2 H^*$ which induces an isomorphism $H \cong H^*$. Using the inclusion $S^2 H \hookrightarrow H \otimes H \cong_{\omega_H} H \otimes H^* = \text{End} H$, we may identify $\mathfrak{sp}(1)$ with $S^2 H$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $S^2 H \subset H \otimes H$ induced by ω_H . If $J, K \in S^2 H$, then as endomorphisms of TM ,

$$(2.2) \quad J \circ K + K \circ J = -\langle J, K \rangle 1.$$

We consider the bundle

$$Z = \{J \in S^2 \mathbf{H} \mid \langle J, J \rangle^{1/2} = \sqrt{2}\}$$

whose fiber Z_x over a point $x \in M$ is a real 2-sphere. From (2.2), an element $J \in Z_x$ defines an almost complex structure on $T_x M$. The bundle Z is called the twistor space of M . Let π be the natural projection from Z to M and Θ the vertical tangent bundle on Z . For any point $J \in Z_x$, we have a natural identification

$$\Theta_J = \{A \in S^2 \mathbf{H} \mid J \circ A = -A \circ J\},$$

where $\Theta_J = T_J Z_x$ is the fiber of Θ at J . The bundle Θ admits a complex structure determined by

$$\mathcal{J}A = J \circ A, \quad A \in \Theta_J.$$

An inner product $\langle \cdot, \cdot \rangle$ on Θ_J is induced by the embedding of Θ_J in $S^2 \mathbf{H}$. \mathcal{J} is compatible with $\langle \cdot, \cdot \rangle$, so Θ has a canonical hermitian structure. We denote by $\Omega^{(x)}$ the Kähler form on $\Theta_J (J \in Z_x)$ induced by $\langle \cdot, \cdot \rangle$. Let v^D denote the vertical projection from TZ to Θ with respect to D . Any vector U on Z , at a point J , is represented by

$$U = (v^D(U), X),$$

where $X = \pi_*(U)$ is the projection of U in $T_x M$. Thus we obtain an almost complex structure \mathcal{J} on Z defined by

$$\mathcal{J}U = (J \circ v^D(U), JX).$$

Salamon showed that \mathcal{J} is integrable when M is a quaternionic manifold. We define Π the orthogonal projection of $\pi^*S^2\mathbf{H}$ onto Θ such that for any point J of Z_x ,

$$\Pi^J(A) = A - \frac{1}{2}\langle A, J \rangle J, \quad A \in S^2\mathbf{H}.$$

A connection D on M induces a connection D^{Ad} on $S^2\mathbf{H}$ via the adjoint representation of $Sp(1)$. We denote by π^*D^{Ad} the pull back connection on $\pi^*S^2\mathbf{H}$. We may define a hermitian connection ∇ on Θ as follows :

$$\nabla = \Pi \circ \pi^*D^{Ad},$$

more explicitly,

$$\nabla_U \tilde{A} = \widetilde{D_X^{Ad} A} - \frac{1}{2}\langle A, J \rangle v^D(U), \quad U \in T_J Z,$$

where \tilde{A} is a vertical vector field on Z defined by

$$\tilde{A}(J) = \Pi^J(A), \quad A \in S^2\mathbf{H}, \quad J \in Z_x.$$

We may compute the curvature of ∇ as follows.

Lemma 2.1 ([3]). *Let R^∇ denote the curvature of the hermitian connection ∇ on Θ induced by a connection D of M . Then we have*

- (1) $R_{B,C}^\nabla A = \frac{1}{2}\Omega^{(x)}(C, B)\mathcal{J}A,$
- (2) $R_{B,\tilde{X}}^\nabla A = 0,$
- (3) $R_{\tilde{X},\tilde{Y}}^\nabla A = \Pi^J[R^D(X, Y), A],$

where $A, B, C \in \Theta_J$, $X, Y \in T_x M$, \tilde{X}, \tilde{Y} is the horizontal lift of X, Y respectively, and R^D is the curvature of D .

Proof. (1) We note that $[B, C] = \Omega^{(x)}(B, C)J$ and (2.2), we have

$$\begin{aligned} R_{B,C}^\nabla A &= \nabla_B \nabla_{\tilde{C}} \tilde{A} - \nabla_C \nabla_{\tilde{B}} \tilde{A} - \nabla_{[\tilde{B}, \tilde{C}]} \tilde{A} \\ &= \nabla_B \left(-\frac{1}{2}\langle A, J \rangle \tilde{C}\right) - \nabla_C \left(-\frac{1}{2}\langle A, J \rangle \tilde{B}\right) \\ &= -\frac{1}{2} \{ (\langle A, \nabla_B J \rangle) C - (\langle A, \nabla_C J \rangle) B \} \\ &= \frac{1}{2} (\langle A, C \rangle B - \langle A, B \rangle C) \\ &= \frac{1}{2} \Omega^{(x)}(C, B) \mathcal{J}A. \end{aligned}$$

(2) We note that $[\tilde{X}, \tilde{B}]$ is vertical, we have

$$\begin{aligned}
R_{B, \tilde{X}}^\nabla A &= \nabla_B \nabla_{\tilde{X}} \tilde{A} - \nabla_{\tilde{X}} \nabla_B \tilde{A} - \nabla_{[\tilde{B}, \tilde{X}]} \tilde{A} \\
&= \nabla_B (\widetilde{D_X^{Ad} A}) - \nabla_{\tilde{X}} (-\frac{1}{2} \langle A, J \rangle \tilde{B}) \\
&= -\frac{1}{2} \langle D_X^{Ad} A, J \rangle B + \frac{1}{2} D_X^{Ad} (\langle A, J \rangle B) \\
&= -\frac{1}{2} \langle D_X^{Ad} A, J \rangle B + \frac{1}{2} (\langle D_X^{Ad} A, J \rangle + \langle A, D_X^{Ad} J \rangle) B \\
&= 0.
\end{aligned}$$

(3) We note that $R^{D^{Ad}} = d(Ad)(R^D) = ad(R^D)$, we have

$$\begin{aligned}
R_{\tilde{X}, \tilde{Y}}^\nabla A &= \nabla_{\tilde{X}} \nabla_{\tilde{Y}} \tilde{A} - \nabla_{\tilde{Y}} \nabla_{\tilde{X}} \tilde{A} - \nabla_{[\tilde{X}, \tilde{Y}]} \tilde{A} \\
&= \nabla_{\tilde{X}} (\widetilde{D_Y^{Ad} A}) - \nabla_{\tilde{Y}} (\widetilde{D_X^{Ad} A}) - \widetilde{D_{[X, Y]}^{Ad} A} \\
&= \widetilde{D_X^{Ad} D_Y^{Ad} A} - \widetilde{D_Y^{Ad} D_X^{Ad} A} - \widetilde{D_{[X, Y]}^{Ad} A} \\
&= \Pi^J(R^{D^{Ad}}(X, Y)A) \\
&= \Pi^J[R^D(X, Y), A]. \quad \square
\end{aligned}$$

From this lemma, we see that R^∇ is \mathcal{J} -invariant in cases of (1) and (2). In (3), $[\ , \]$ is the bracket of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)$ of the structure group $GL(n, \mathbb{H})Sp(1)$. R^D is a 2-form with values in \mathfrak{g} and A is in $\Theta_J \subset S^2H \cong \mathfrak{sp}(1)$, so we take notice of the component on $\mathfrak{sp}(1)$ of R^D in Section 5. By virtue of representation theory, we examine the curvature of a connection on a quaternionic manifold.

3. REPRESENTATIONS OF $GL(n, \mathbb{H})Sp(1)$

We denote by G the structure group $GL(n, \mathbb{H})Sp(1)$ of M . Let $\mathfrak{g}^{(1)}$ be the first prolongation of the Lie algebra \mathfrak{g} of G and T the representation of G corresponding to the tangent bundle. We have

$$\mathfrak{g} \subset \text{End}T = T \otimes T^*,$$

then $\mathfrak{g}^{(1)}$ is defined to be the kernel of the skewing mapping

$$\partial : \mathfrak{g} \otimes T^* \rightarrow T \otimes \Lambda^2 T^*.$$

We shall determine the above homomorphism for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1) \cong E^*E \oplus S^2H$. Tensor products are indicated either in the usual way or simply by juxtaposition. From (2.1), we have

$$\mathfrak{g} \otimes T^* \cong (E^*E \oplus S^2H) \otimes EH,$$

and

$$\begin{aligned} T \otimes \Lambda^2 T^* &\cong E^* H \otimes \Lambda^2(EH) \\ &\cong E^* H \otimes (S^2 E \oplus \Lambda^2 E S^2 H). \end{aligned}$$

There is a contraction $\varphi : E^* \otimes S^2 E \rightarrow E$, so by Schur's lemma, E appears in $E^* \otimes S^2 E$, and we have

$$(3.1) \quad E^* \otimes S^2 E \cong E \oplus C,$$

where $C = \ker \varphi$. In a similar fashion, we see

$$(3.2) \quad E^* \otimes \Lambda^2 E \cong E \oplus D.$$

C and D are both irreducible. Combining the above isomorphisms and the Clebsch-Gordan formula

$$(3.3) \quad S^j H \otimes S^k H \cong \bigoplus_{r=0}^{\min(j,k)} S^{j+k-2r} H,$$

we have

Lemma 3.1 ([8]).

$$\begin{aligned} \mathfrak{g} \otimes T^* &\cong 3EH \oplus CH \oplus DH \oplus ES^3 H, \\ T \otimes \Lambda^2 T^* &\cong 2EH \oplus CH \oplus DH \oplus ES^3 H \oplus DS^3 H, \end{aligned}$$

where nEH denotes an isotypic component isomorphic to the direct sum of n copies of EH .

From this lemma, we obtain

Proposition 3.1 ([8]).

$$\mathfrak{g}^{(1)} = \ker \partial \cong EH.$$

We represent the isomorphism in Proposition 3.1 more precisely. There is one copy of EH in each of the three terms on the right-hand side of

$$\mathfrak{g} \otimes T^* \cong (\mathbb{C} \oplus \mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1)) \otimes EH.$$

We take a basis $\{e_i\}_{i=1}^{2n}$ of E , such that $\tilde{e}_j = e_{j+n}, \widetilde{e_{j+n}} = -e_j$ ($j = 1, \dots, n$), and an $SU(2)$ -basis $\{h, \tilde{h}\}$ of H ($\omega_H(h, \tilde{h}) = 1$), where $v \mapsto \tilde{v}$ are antilinear structure maps

commuting with the action of $GL(n, \mathbb{H})$ or $Sp(1)$ and satisfying $\tilde{v} = -v$. Let $\{e^i\}_{i=1}^{2n}$ denote the dual basis of E^* , then

$$\begin{aligned}\alpha_1 &= \sum_{i=1}^{2n} (e^i h e_i \tilde{h} - e^i \tilde{h} e_i h) e_1 h \in \mathbb{C} \otimes EH, \\ \alpha_2 &= \sum_{i=1}^{2n} (e^i h e_1 \tilde{h} - e^i \tilde{h} e_1 h) e_i h - \frac{1}{2n} \alpha_1 \in \mathfrak{sl}(n, \mathbb{H}) \otimes EH, \\ \alpha_3 &= \sum_{i=1}^{2n} \{2e^i h e_i h e_1 \tilde{h} - (e^i \tilde{h} e_i h + e^i h e_i \tilde{h}) e_1 h\} \in \mathfrak{sp}(1) \otimes EH,\end{aligned}$$

are representatives of the element $e_1 h$ in each of the three copies of EH , and $\ker \partial$ is spanned by the element

$$\begin{aligned}(3.4) \quad \alpha &= \frac{n+1}{n} \alpha_1 + 2\alpha_2 + \alpha_3 \\ &= \sum_{i=1}^{2n} \{(e^i h e_i \tilde{h} - e^i \tilde{h} e_i h) e_1 h + 2(e^i h e_1 \tilde{h} - e^i \tilde{h} e_1 h) e_i h \\ &\quad + 2e^i h e_i h e_1 \tilde{h} - (e^i \tilde{h} e_i h + e^i h e_i \tilde{h}) e_1 h\}.\end{aligned}$$

By using (3.4), in Section 5, we may describe a curvature of a quaternionic manifold concretely.

4. CURVATURE OF A QUATERNIONIC MANIFOLD

We consider the Spencer complex

$$\dots \rightarrow \mathfrak{g}^{(r)} \otimes \Lambda^{s-1} T^* \rightarrow \mathfrak{g}^{(r-1)} \otimes \Lambda^s T^* \rightarrow \mathfrak{g}^{(r-2)} \otimes \Lambda^{s+1} T^* \rightarrow \dots,$$

where $\mathfrak{g}^{(r)}$ denotes the r -th prolongation of \mathfrak{g} , where $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = T$. The cohomology at the point $\mathfrak{g}^{(r-1)} \otimes \Lambda^s T^*$ is denoted by $H^{r,s}(\mathfrak{g})$.

For a quaternionic manifold M with a torsion-free connection D , the curvature R^D of D lies in $\mathfrak{g} \otimes \Lambda^2 T^*$. The first Bianchi identity implies that $\partial R = 0$, and hence R^D represents the cohomology class in $H^{1,2}(\mathfrak{g})$ of the sequence

$$\mathfrak{g}^{(1)} \otimes T^* \rightarrow \mathfrak{g} \otimes \Lambda^2 T^* \rightarrow T \otimes \Lambda^3 T^*.$$

In order to decompose these spaces, we introduce some irreducible decompositions of $GL(n, \mathbb{H})$ -modules. First,

$$(4.1) \quad \begin{cases} E \otimes S^2 E \cong S^3 E \oplus F, \\ E \otimes \Lambda^2 E \cong \Lambda^3 E \oplus F', \end{cases}$$

where modules F and F' are irreducible, and $F \cong F'$ via Schur's lemma. Secondly,

$$(4.2) \begin{cases} E^* \otimes S^3 E \cong S^2 E \oplus U, \\ E^* \otimes \Lambda^3 E \cong \Lambda^2 E \oplus V, \end{cases}$$

with U, V irreducible, and from (4.1) and (4.2),

$$(4.3) \begin{cases} E^* \otimes E \otimes S^2 E \cong S^2 E \oplus U \oplus E^* F, \\ E^* \otimes E \otimes \Lambda^2 E \cong \Lambda^2 E \oplus V \oplus E^* F. \end{cases}$$

We see that both left-hand members in (4.3) contain $E \otimes E$ from (3.1) and (3.2), thus we have that

$$E^* F \cong S^2 E \oplus \Lambda^2 E \oplus W,$$

for some irreducible module W . Thirdly,

$$\Lambda^3(EH) \cong \Lambda^3 E S^3 H \oplus FH.$$

Combining the above decompositions and the Clebsch-Gordan formula (3.3), we have

Lemma 4.1 ([8]).

$$\begin{aligned} \mathfrak{g} \otimes \Lambda^2 T^* &\cong 2S^2 E \oplus 2\Lambda^2 E \oplus U \oplus W \oplus (2S^2 E \oplus 3\Lambda^2 E \oplus V \oplus W)S^2 H \oplus \Lambda^2 E S^4 H, \\ T \otimes \Lambda^3 T^* &\cong S^2 E \oplus \Lambda^2 E \oplus W \oplus (S^2 E \oplus 2\Lambda^2 E \oplus V \oplus W)S^2 H \oplus (\Lambda^2 E \oplus V)S^4 H. \end{aligned}$$

On the other hand, from (2.1) and Proposition 3.1, we have

$$(4.4) \quad \mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH \cong S^2 E \oplus \Lambda^2 E \oplus (S^2 E \oplus \Lambda^2 E)S^2 H.$$

Thus we see that the components of $\mathfrak{g} \otimes \Lambda^2 T^*$ minus those of $\partial(\mathfrak{g}^{(1)} \otimes T^*)$ all occur in $T \otimes \Lambda^3 T^*$ with the exception of U . Using Schur's lemma, we may check that $\partial : \mathfrak{g} \otimes \Lambda^2 T^* \rightarrow T \otimes \Lambda^3 T^*$ has full rank. Hence we obtain

Proposition 4.1 ([8]).

$$H^{1,2}(\mathfrak{g}) \cong U.$$

Therefore, the curvature R^D has the form

$$(4.5) \quad R^D = \partial\left(\sum_i v_i \otimes t^i\right) + R_U,$$

where $v_i \in \mathfrak{g}^{(1)}$, $t^i \in T^*$, and $R_U \in U$, i.e., R^D decomposes into irreducible $GL(n, \mathbb{H})Sp(1)$ -components in $S^2 E, \Lambda^2 E, S^2 E S^2 H, \Lambda^2 E S^2 H$, and U .

Remark. In the case of a 4-dimensional conformal manifold, we see that $\mathfrak{g}^{(1)} \otimes T^* \cong S^2E \oplus \mathbb{C} \oplus S^2ES^2H \oplus S^2H$ and $H^{1,2}(\mathfrak{g}) \cong U \oplus S^4H$. Thus a curvature has its components in S^2E , \mathbb{C} , S^2ES^2H , S^2H , U and S^4H . If M is self-dual, then the S^4H -component vanishes. The components lying in \mathbb{C} , S^2ES^2H , and U correspond to the parts of the scalar curvature, the traceless Ricci curvature, and the self-dual Weyl tensor, respectively. And the S^2E -component and the S^2H -component correspond to the self-dual part and the anti-self-dual part of the curvature of D^L respectively.

5. CHERN CONNECTIONS

Let X be a complex manifold and \mathcal{L} a Hermitian line bundle over X . A Hermitian connection on \mathcal{L} is called a *Chern connection*, if its curvature is of type (1,1) with respect to the complex structure on X . It is well-known that for any fixed Hermitian structure on \mathcal{L} , there is a natural bijection between Chern connections and holomorphic structures on \mathcal{L} , obtained by identifying a Chern connection with its (0,1)-part. In Section 2, we have seen that the twistor space of a quaternionic manifold is a complex manifold and its vertical bundle is a Hermitian line bundle. In this section, we shall obtain the condition for a Hermitian connection on the vertical bundle to be a Chern connection.

We extend the curvature R of a torsion-free connection on a quaternionic manifold to a complex bilinear form, also denote it by R , on $TM^{\mathbb{C}}$. We see that the U -component R_U of R is $\mathfrak{gl}(n, \mathbb{H})$ -valued. So from (4.5), we also see that the $\mathfrak{sp}(1)$ -component of R is constructed by the vectors $e_p h e_q h$, $e_p h e_q \tilde{h}$, $e_p \tilde{h} e_q h$, and $e_p \tilde{h} e_q \tilde{h}$ in $\mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH$. We denote the coefficients of these vectors by α_{pq} , $\alpha_{p\bar{q}}$, $\alpha_{\bar{p}q}$, and $\alpha_{\bar{p}\bar{q}}$ respectively. On the other hand, from (3.4), we may express the component on $\mathfrak{sp}(1)$ of R as follows :

$$\begin{aligned} R(e^p h, e^q h)_{S^2H} &= a_{pq} h \cdot h + b_{pq} \tilde{h} \cdot h, \\ R(e^p h, e^{\bar{q}} \tilde{h})_{S^2H} &= a_{p\bar{q}} h \cdot h + b_{p\bar{q}} \tilde{h} \cdot h + c_{p\bar{q}} \tilde{h} \cdot \tilde{h}, \\ R(e^p \tilde{h}, e^q h)_{S^2H} &= a_{\bar{p}q} h \cdot h + b_{\bar{p}q} \tilde{h} \cdot h + c_{\bar{p}q} \tilde{h} \cdot \tilde{h}, \\ R(e^p \tilde{h}, e^{\bar{q}} \tilde{h})_{S^2H} &= b_{\bar{p}\bar{q}} \tilde{h} \cdot h + c_{\bar{p}\bar{q}} \tilde{h} \cdot \tilde{h}, \end{aligned}$$

where $a \cdot b$ means the symmetric product of a and b . We note that coefficients a_{pq} , $a_{p\bar{q}}$, $a_{\bar{p}q}$, b_{pq} , $b_{p\bar{q}}$, $b_{\bar{p}q}$, $b_{\bar{p}\bar{q}}$, $c_{p\bar{q}}$, $c_{\bar{p}q}$, $c_{\bar{p}\bar{q}}$ and α_{pq} , $\alpha_{p\bar{q}}$, $\alpha_{\bar{p}q}$, $\alpha_{\bar{p}\bar{q}}$ satisfy the following relations :

$$(5.1) \quad \begin{cases} a_{pq} = \alpha_{p\bar{q}} - \alpha_{q\bar{p}}, & a_{p\bar{q}} = -\alpha_{pq}, & a_{\bar{p}q} = \alpha_{qp}, \\ b_{pq} = \alpha_{\bar{p}\bar{q}} - \alpha_{\bar{q}\bar{p}}, & b_{p\bar{q}} = -\alpha_{\bar{p}q} - \alpha_{q\bar{p}}, & b_{\bar{p}q} = \alpha_{p\bar{q}} + \alpha_{\bar{q}p}, & b_{\bar{p}\bar{q}} = \alpha_{qp} - \alpha_{pq}, \\ c_{p\bar{q}} = -\alpha_{\bar{q}\bar{p}}, & c_{\bar{p}q} = \alpha_{\bar{p}\bar{q}}, & c_{\bar{p}\bar{q}} = \alpha_{\bar{q}p} - \alpha_{\bar{p}q} \end{cases} \quad (p, q = 1, \dots, 2n).$$

At first, since a curvature is skew-symmetric, its complex coefficients satisfy

$$(5.2) \quad \begin{cases} a_{pq} = -a_{qp}, & a_{p\bar{q}} = -a_{\bar{q}p}, \\ b_{pq} = -b_{qp}, & b_{p\bar{q}} = -b_{\bar{q}p}, & b_{\bar{p}q} = -b_{q\bar{p}}, \\ c_{p\bar{q}} = -c_{\bar{q}p}, & c_{\bar{p}q} = -c_{q\bar{p}}, \\ a_{p\bar{q}} + a_{\bar{p}q} = b_{\bar{p}q}, & c_{p\bar{q}} + c_{\bar{p}q} = b_{pq}, \\ a_{pq} - b_{p\bar{q}} - b_{\bar{p}q} + c_{\bar{p}q} = 0 \end{cases} \quad (p, q = 1, \dots, 2n).$$

Next, the curvature R is real, i.e., $\overline{R(X, Y)} = R(\overline{X}, \overline{Y})$ for $X, Y \in TM^{\mathbb{C}}$, where $\overline{\ast}$ is the operation of complex conjugation, so that its coefficients also satisfy the following conditions (5.3) :

$$\begin{aligned} \overline{a_{jk}} &= \begin{cases} \widetilde{c_{j+nk+n}} & (1 \leq j, k \leq n) \\ -\widetilde{c_{j+nk-n}} & (1 \leq j \leq n, n+1 \leq k \leq 2n) \\ -\widetilde{c_{j-nk+n}} & (n+1 \leq j \leq 2n, 1 \leq k \leq n) \\ \widetilde{c_{j-nk-n}} & (n+1 \leq j, k \leq 2n) \end{cases} \\ \overline{a_{j\bar{k}}} &= \begin{cases} -\widetilde{c_{j+nk+n}} & (1 \leq j, k \leq n) \\ \widetilde{c_{j+nk-n}} & (1 \leq j \leq n, n+1 \leq k \leq 2n) \\ \widetilde{c_{j-nk+n}} & (n+1 \leq j \leq 2n, 1 \leq k \leq n) \\ -\widetilde{c_{j-nk-n}} & (n+1 \leq j, k \leq 2n) \end{cases} \\ \overline{b_{jk}} &= \begin{cases} -\widetilde{b_{j+nk+n}} & (1 \leq j, k \leq n) \\ \widetilde{b_{j+nk-n}} & (1 \leq j \leq n, n+1 \leq k \leq 2n) \\ \widetilde{b_{j-nk+n}} & (n+1 \leq j \leq 2n, 1 \leq k \leq n) \\ -\widetilde{b_{j-nk-n}} & (n+1 \leq j, k \leq 2n) \end{cases} \\ \overline{b_{j\bar{k}}} &= \begin{cases} \widetilde{b_{j+nk+n}} & (1 \leq j, k \leq n) \\ -\widetilde{b_{j+nk-n}} & (1 \leq j \leq n, n+1 \leq k \leq 2n) \\ -\widetilde{b_{j-nk+n}} & (n+1 \leq j \leq 2n, 1 \leq k \leq n) \\ \widetilde{b_{j-nk-n}} & (n+1 \leq j, k \leq 2n). \end{cases} \end{aligned}$$

Moreover, we assume that R is of type (1,1). From Lemma 2.1, we see that R^{∇} is of type (1,1) if and only if R satisfies the condition

$$(*) \quad \Pi^J([R^D(JX, JY) - R^D(X, Y), A]) = 0$$

for each $X, Y \in T_x M$ and $A \in \Theta_J$. We take a real basis

$$(5.4) \quad \begin{cases} X^j = e^j h + e^{j+n} \tilde{h}, \\ Y^j = \sqrt{-1}(e^j h - e^{j+n} \tilde{h}), \\ Z^j = e^{j+n} h - e^j \tilde{h}, \\ W^j = \sqrt{-1}(e^{j+n} h + e^j \tilde{h}) \end{cases} \quad (j = 1, \dots, n)$$

on $TM^{\mathbb{C}}$, and put $J = ah \cdot h + b\tilde{h} \cdot h + c\tilde{h} \cdot \tilde{h}$. Since J is a real operator, i.e., $\bar{J} = J$, and $\langle J, J \rangle = \sqrt{2}$, we have $c = \bar{a}$, $\bar{b} = -b$ and $4ac - b^2 = 1$. For each $A \in \Theta_J$, $A = dh \cdot h + e\tilde{h} \cdot h + f\tilde{h} \cdot \tilde{h}$, we also have $f = \bar{d}$, $\bar{e} = -e$, $4df - e^2 = 1$, and $2af - be + 2cd = 0$ (i.e., $\langle J, A \rangle = 0$). We compute the condition (*) for the basis (5.4), we obtain the following conditions for coefficients of R (5.5) :

$$\begin{aligned} a_{jk} + \widetilde{b_{jk+n}} + \widetilde{b_{j+nk}} - c_{j\bar{k}} &= 0, \\ a_{jk} - \widetilde{b_{jk+n}} + \widetilde{b_{j+nk}} - c_{j\bar{k}} &= 0, \\ a_{jk+n} - \widetilde{b_{j\bar{k}}} + \widetilde{b_{j+nk+n}} - c_{j\bar{k+n}} &= 0, \\ a_{jk+n} + \widetilde{b_{j\bar{k}}} + \widetilde{b_{j+nk+n}} - c_{j\bar{k+n}} &= 0, \\ b_{jk} + \widetilde{b_{j+nk+n}} &= 0, \\ b_{jk} - \widetilde{b_{j+nk+n}} &= 0, \\ b_{jk+n} - \widetilde{b_{j+nk}} &= 0, \\ b_{jk+n} + \widetilde{b_{j+nk}} &= 0, \\ b_{j+nk+n} + \widetilde{b_{j\bar{k}}} &= 0, \\ b_{j+nk+n} - \widetilde{b_{j\bar{k}}} &= 0 \end{aligned} \quad (j, k = 1, \dots, n).$$

For example, we compute (*) for X^j and X^k , then we have

$$\begin{aligned} & \Pi^J([R^D(JX^j, JX^k) - R^D(X^j, Y^k), A]) \\ &= \{2e(a_{jk} + \widetilde{b_{jk+n}} + \widetilde{b_{j+nk}} - c_{j\bar{k}}) - 4d(b_{jk} + \widetilde{b_{j+nk+n}})\} h \cdot h \\ & \quad + \{4f(a_{jk} + \widetilde{b_{jk+n}} + \widetilde{b_{j+nk}} - c_{j\bar{k}}) + 4d(a_{j+nk+n} - b_{jk+n} - b_{j+nk} - c_{j\bar{k+n}})\} \tilde{h} \cdot h \\ & \quad + \{2e(a_{j+nk+n} - b_{jk+n} - b_{j+nk} - c_{j\bar{k+n}}) + 4f(b_{jk} + \widetilde{b_{j+nk+n}})\} \tilde{h} \cdot \tilde{h} \\ &= 0, \end{aligned}$$

for each A . So we get some equations in (5.5).

From (5.2), (5.3) and (5.5), we obtain

$$(5.6) \quad a_{pq} = c_{\bar{p}\bar{q}} \quad \text{and} \quad b_{pq} = b_{\bar{p}\bar{q}} = 0 \quad (p, q = 1, \dots, 2n).$$

Using the relation (5.1), we may rewrite the conditions (5.3) and (5.6) as the following (5.7) :

$$\begin{aligned} \overline{\alpha_{pq}} &= \begin{cases} \widetilde{\alpha_{p+nq+n}} & (1 \leq p, q \leq n) \\ -\widetilde{\alpha_{p+nq-n}} & (1 \leq p \leq n, n+1 \leq q \leq 2n) \\ -\widetilde{\alpha_{p-nq+n}} & (n+1 \leq p \leq 2n, 1 \leq q \leq n) \\ \widetilde{\alpha_{p-nq-n}} & (n+1 \leq p, q \leq 2n) \end{cases} \\ \overline{\alpha_{p\bar{q}}} &= \begin{cases} -\widetilde{\alpha_{p+nq+n}} & (1 \leq p, q \leq n) \\ \widetilde{\alpha_{p+nq-n}} & (1 \leq p \leq n, n+1 \leq q \leq 2n) \\ \widetilde{\alpha_{p-nq+n}} & (n+1 \leq p \leq 2n, 1 \leq q \leq n) \\ -\widetilde{\alpha_{p-nq-n}} & (n+1 \leq p, q \leq 2n) \end{cases} \\ \alpha_{pq} &= \alpha_{qp}, \quad \alpha_{p\bar{q}} = \alpha_{\bar{q}p}, \\ \alpha_{p\bar{q}} - \alpha_{q\bar{p}} &= \alpha_{\bar{q}p} - \alpha_{\bar{p}q} \quad (p, q = 1, \dots, 2n). \end{aligned}$$

In (5.7), we note that the first 2 conditions correspond to (5.3), and the last 3 conditions correspond to (5.6). From (5.7), we see that the curvatures of type (1,1) are constructed by the vectors

$$\begin{aligned} A_{jk} &= e_j h e_k \tilde{h} - e_j \tilde{h} e_k h + e_{j+n} h e_{k+n} \tilde{h} - e_{j+n} \tilde{h} e_{k+n} h, \\ B_{jk} &= \sqrt{-1}(e_j h e_k \tilde{h} - e_j \tilde{h} e_k h - e_{j+n} h e_{k+n} \tilde{h} + e_{j+n} \tilde{h} e_{k+n} h), \\ C_{jk} &= e_j h e_{k+n} \tilde{h} - e_j \tilde{h} e_{k+n} h + e_{j+n} \tilde{h} e_k h - e_{j+n} h e_k \tilde{h}, \\ D_{jk} &= \sqrt{-1}(e_j h e_{k+n} \tilde{h} - e_j \tilde{h} e_{k+n} h - e_{j+n} \tilde{h} e_k h + e_{j+n} h e_k \tilde{h}), \\ E_{jk} &= e_j h e_k h + e_k h e_j h + e_{j+n} \tilde{h} e_{k+n} \tilde{h} + e_{k+n} \tilde{h} e_{j+n} \tilde{h}, \\ F_{jk} &= \sqrt{-1}(e_j h e_k h + e_k h e_j h - e_{j+n} \tilde{h} e_{k+n} \tilde{h} - e_{k+n} \tilde{h} e_{j+n} \tilde{h}), \\ G_{jk} &= e_j h e_{k+n} h + e_{k+n} h e_j h - e_{j+n} \tilde{h} e_k \tilde{h} - e_k \tilde{h} e_{j+n} \tilde{h}, \\ H_{jk} &= \sqrt{-1}(e_j h e_{k+n} h + e_{k+n} h e_j h + e_{j+n} \tilde{h} e_k \tilde{h} + e_k \tilde{h} e_{j+n} \tilde{h}), \\ I_{jk} &= e_{j+n} h e_{k+n} h + e_{k+n} h e_{j+n} h + e_j \tilde{h} e_k \tilde{h} + e_k \tilde{h} e_j \tilde{h}, \\ J_{jk} &= \sqrt{-1}(e_{j+n} h e_{k+n} h + e_{k+n} h e_{j+n} h - e_j \tilde{h} e_k \tilde{h} - e_k \tilde{h} e_j \tilde{h}), \\ K_{jk} &= e_k \tilde{h} e_j h + e_j \tilde{h} e_k h - e_{k+n} h e_{j+n} \tilde{h} - e_{j+n} h e_{k+n} \tilde{h}, \\ L_{jk} &= \sqrt{-1}(e_k \tilde{h} e_j h + e_j \tilde{h} e_k h + e_{k+n} h e_{j+n} \tilde{h} + e_{j+n} h e_{k+n} \tilde{h}), \\ M_{jk} &= e_{k+n} h e_j \tilde{h} + e_j \tilde{h} e_{k+n} h + e_k \tilde{h} e_{j+n} h + e_{j+n} h e_k \tilde{h}, \\ N_{jk} &= \sqrt{-1}(e_{k+n} h e_j \tilde{h} + e_j \tilde{h} e_{k+n} h - e_k \tilde{h} e_{j+n} h - e_{j+n} h e_k \tilde{h}) \quad (j, k = 1, \dots, n) \end{aligned}$$

over \mathbb{R} . We note that $\mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH \cong S^2 E \oplus \Lambda^2 E \oplus (S^2 E \oplus \Lambda^2 E) S^2 H$. We may see that the above vectors are all in $S^2 E \oplus \Lambda^2 E \oplus S^2 E S^2 H$ and span $S^2 E \oplus$

$\Lambda^2 E \oplus S^2 ES^2 H$. More precisely, $S^2 E$, $\Lambda^2 E$, and $S^2 ES^2 H$ are spanned by $\{A_{jk} + A_{kj}, B_{jk} + B_{kj}, C_{jk} - C_{kj}, D_{jk} + D_{kj}\}$, $\{A_{jk} - A_{kj}, B_{jk} - B_{kj}, C_{jk} + C_{kj}, D_{jk} - D_{kj}\}$, and $\{E_{jk}, F_{jk}, G_{jk}, H_{jk}, I_{jk}, J_{jk}, A_{jk} + A_{kj} + 2K_{jk}, B_{jk} + B_{kj} + 2L_{jk}, C_{jk} + C_{kj} + M_{jk} + M_{kj}, D_{jk} - D_{kj} + N_{jk} - N_{kj}\}$, respectively. Hence we obtain the following

Theorem 5.1. *Let M be a quaternionic manifold with a torsion-free connection D , and Θ the vertical bundle of the twistor fibration Z . Then the linear connection ∇ on Θ induced by D is a Chern connection if and only if the curvature R^D of D has no component in $\Lambda^2 ES^2 H$.*

Remark. The condition for the curvature R^D of D to have no component in $\Lambda^2 ES^2 H$ in Theorem 5.1 corresponds to the condition for a Weyl structure to be self-dual in the case of a 4-dimensional self-dual manifold (cf. Remark in Section 4).

Example. If (M, g) is a quaternionic Kähler manifold with Levi-Civita connection D , then D induces a Chern connection. Because the components of the curvature R^D lie in $\Lambda^2 E \oplus U$.

6. HYPERCOMPLEX MANIFOLDS

A $4n$ -dimensional manifold M with a $GL(n, \mathbb{H})$ -structure admitting a torsion-free connection is a *hypercomplex manifold*. Therefore the family of quaternionic manifolds contains that of hypercomplex manifolds. Applying the results of Sections 3 and 4 with the Lie algebra $\mathfrak{gl}(n, \mathbb{H})$, we obtain

Theorem 6.1 ([8]).

$$\mathfrak{gl}(n, \mathbb{H})^{(1)} = 0,$$

$$H^{1,2}(\mathfrak{gl}(n, \mathbb{H})) \cong U \oplus S^2 E.$$

For any two torsion-free G -connections $\nabla^{(1)}$ and $\nabla^{(2)}$, the difference $\nabla^{(1)} - \nabla^{(2)}$ belongs to $\mathfrak{g}^{(1)}$. From the first equation in Theorem 6.1, we see that a torsion-free $GL(n, \mathbb{H})$ -connection is unique if it exists. We call it the *Obata connection*. And we also see that the curvature of an Obata connection has the components in $U \oplus S^2 E$. Hence an Obata connection induces a Chern connection.

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