CONTRIBUTIONS TO

THE OPTIMAL INVENTORY PROBLEMS

Junichi NAKAGAMI

Department of Mathematics

Faculty of Science

Chiba University

ABSTRACT

This dissertation discusses the fundamental aspects of optimal inventory problems, taking notice of the uncertainty on the demand distribution F.

When F is known, the author derives an existence and uniqueness theorem and the related properties for the optimal inventory equation under considerably mild assumptions by using the technique of contraction mappings. Since the theorem developed guarantees a unique solution and the optimal inventory policy, the author determines some particular types of optimal policies for two examples of practical inventory problems and derives their properties by using the usual approach of successive approximations.

When F is unknown, i.e., most of informations about it are insufficient for a definite probability distribution, a Bayesian approach or a minimax procedure is sometimes used. The author sets up a Bayesian statistical inventory equation under a general class of prior distributions and derives some important properties of the Bayes solution. Though the Bayes solution and the inventory policies are themselves quite difficult to derive even in the case of linear costs, he approximates them by those which are asymptotically optimal if the sample size approaches infinity. As an example, a Bayesian nonparametric problem is discussed. The author also considers a minimax inventory problem as a two-person zero-sum game and shows the necessary and sufficient conditions for the existence of saddle points and a saddle value.

CONTENTS

CHAPTER 1

INTRODUCTION

In this dissertation we shall discuss the fundamental aspects of the structure of optimal inventory problems, which are the classical and typical problems of operations research. Dynamic inventory problems of the types mentioned here have received considerable attention during the past twenty years. Important contributions to the mathematical inventory theory may be found in the literatures listed at the end of this dissertation. Since the comprehensive discussions of the inventory problems are given by many authors, we limit our introductory remarks to a mathematical description of the structure of the model that we shall consider and a statement of previous results that we shall use.

An inventory problem is a sequential decision problem where decisions must be made at repeated intervals whether or not to raise the inventory level. The demands t in successive periods are independently distributed random variables with a common distribution F. Three kinds of costs are incurred during each period. Here, $c(z)$ represents the cost of ordering amount z of the good, $h(.)$ the holding cost for inventories on hand and $p(.)$ the penalty cost associated with the failure to meet demand. Holding and penalty costs are charged at the end of each period. The constant α represents the discounting factor. Costs experienced N periods after the current period are discounted by α^N.

Inventory problems can be broadly divided into two cases according to whether the demand distribution F is known or not known to us.

When F is known, two distinct approaches have been employed in analysis of this inventory model. One approach, which considers inventory problems as functional equations, is to derive an existence and uniqueness theorem for the solution of the functional equation and also to find the conditions sufficient to ensure that the optimal ordering policy exists in a simple form. A second approach, which considers inventory problems as multistage decision processes of dynamic programming, is to derive some quantitative properties of its solution and optimal inventory policy by using the technique of the successive approximation. A major drawback of this approach is to put rather strong conditions in order to guarantee the convergence of the solutions.

In Chapter 2 we shall derive an existence and uniqueness theorem for optimal inventory problems with bounded and unbounded cost functions by the first approach in terms of the functional equation. Until now previous discussions of this fundamental theorem were based on the successive approximations approach and they required relatively strong assumptions that would guarantee the uniform convergence of the continuous functions. See, e.g., Iglehart [9] and Boylan [3]. Recently, Lippman [16] and Van Nunen and Wessels [33] presented sufficient conditions for contraction mappings in semi-Markov decision processes with unbounded rewards. But their conditions could not be applied to the inventory problems directly, because both the state space and the action space in inventory problems were not countable. According to Nakagami [24], we, therefore, prove this theorem under considerably mild assumptions by using the technique of Lippman's contraction mappings.

In Chapter 3 we shall derive the particular types of optimal policies for the practical inventory problems. Since the theory developed in Chapter 2 ensures the existence of a unique continuous solution and the optimal inventory policy in our practical inventory problems, we shall determine the particular type of the optimal policy and discuss its quantitative properties by using the second approach of successive approximations.

First, let us consider the problem where the ordering cost function is linear with multiple set-up rather than one with a single set-up. This type of cost is neither convex nor concave, but has a practical meaning when the ordered quantity in each period is delivered by a transportation vehicle which has a certain limited capacity. In general, an optimal inventory policy is sensitive to the form of the ordering cost, so that until now some types of inventory policies have examined and studied by several authors. Scarf [31] proved that an (s,S) policy is optimal for a linear cost with a single set-up, and this case was investigated in detail by Iglehart [9], Veinott [35] et al. Porteus [28] proved that a generalized (s,S) policy is optimal for a concavely increasing cost. In Nakagami [20], the purpose for this problem is to derive the particular type of the optimal policy in the model when the ordering cost function has multiple set-up.

Next we consider the perishable inventory problem. One of the important aspects of it is that the perishable goods have a fixed lifetime and become useless in satisfying demands after a fixed length of periods. Several authors (Fries [7], Nahmias and Pierskalla [17],[19] and Nahmias [18]) investigated the model in which the goods perish exactly m periods after receipt on order. And they derived optimal ordering policies and some qualitative properties of policies by solving a dynamic program with a state variable of dimension m-1. Another remarkable aspect of the perishability is that one can preserve the perishable goods in a special warehouse, which keeps them in the almost same quality and extends their life time for a pretty long periods. For example, perishable foodstuffs like fresh fish and meat deteriorate in a week when refrigerated. But they will keep for half an year when frozen, which for the practical purpose is non-perishable. The objective for the second problem is to analyze this realistic model in order to derive an optimal preserving policy as well as an ordering one with their properties according to Nakagami [22].

When the demand distribution F is unknown to us, that is, most of the informations about it are insufficient for a definite probability distribution, a Bayesian approach or a minimax procedure is sometimes used. A Bayesian approach is very attractive because it allows the explicit incorporation of prior and new informations into the structure of the model. And its solution and inventory policy have quite realistic features of one's intuition and experience. But a major defect of Bayesian methods is the restriction of the class of prior distributions needed. A minimax procedure is very interesting from the mathematical point of view, and it gives the critical upper bound of the solution over a prescribed class of distribution even if the prior information is quite poor. However the minimax inventory policy has pessimistic features as compared with that of a Bayesian approach. A minimax procedure is naturally extended to a game theoretic procedure.

On the Bayesian inventory problem, Scarf [30] and Iglehart [10] have analyzed an inventory model with linear costs where they assume a demand distribution from the exponential family and derive the optimal inventory policy. They also show the convergence to the true optimal policy if the sample size approaches infinity. We shall consider the general treatments of the Bayesian inventory problems. In Chapter 4, according to Nakagami [25],[26], we shall set up a statistical inventory equation under a general class of prior distributions and discuss some important properties of its Bayes solution. Though the Bayes solution and the inventory policies are themselves quite difficult to derive even in the case of linear costs, we approximate them by those which are asymptotically optimal if the sample size approaches infinity.

As an example, a Bayesian nonparametric problem is discussed.

On the minimax inventory problem, one can refer the works done by Ben-Tal and Hochman [1], Jagannathan [12], [13], Kasugai and Kasegai [15], Nakagami [21], Odanaka [27] and Scarf [32]. A minimax policy is one that minimizes the maximum expected costs, where the maximum is taken over a prescribed class of distributions. For example, Scarf [32] assumed that only the mean and the variance of the distribution were known. In Chapter 5, according to Nakagami and Yasuda [23], we extend this minimax inventory problem to a two-person zero-sum game, in which one player (manager) decides his ordering level and other player (nature) chooses her demand distribution in the prescribed class of distributions. By a game theoretic approach, we first show the necessary and sufficient conditions for the existence of saddle points and a saddle value. As an example, we reconsider the minimax problems given by the above literatures. Since the classes of the demand distributions in these problems are easily checked to satisfy our conditions, the minimax policy is consistent with the player 1's strategy of the game. We, therefore, determine a set of all saddle points and a saddle value respectively in the explict form by solving the dual maximin problem, which enables us to derive the player 2's maximin strategy as well as the player 1's minimax one.

CHAPTER 2

AN EXISTENCE THEOREM
IN THE OPTIMAL INVENTORY PROBLEM

In this chapter we shall derive an existence and uniqueness theorem for optimal inventory problems with bounded and unbounded cost functions, when the demand distribution is completely known to us.

Until now discussions of this fundamental theorem were based on the successive approximations approach and they required relatively strong assumptions and tedious arguments that would guarantee the uniform convergence of the continuous functions. See, e.g., Iglehart [9] and Boylan [3].

Recently, Lippman [16] and Van Nunen and Wessels [33] presented sufficient conditions for contraction mappings in semi-Markov decision processes with unbounded rewards. But their conditions could not be applied to the inventory problems directly, because both a state space and an action space of inventory problems were not countable.

According to Nakagami [24], we, therefore, prove this theorem under a considerably mild assumption and a simple argument by using the technique of Lippman's contraction mappings.

A Contraction Theorem

We let $f(x,F)$ denote the optimal expected costs for the initial level x when the demand distribution is known to be F. Then $f(x,F)$ satisfies the following optimal inventory equation which is given by

$$(2\text{-}1) \qquad f(x,F) = \inf_{y \geqq x} [c(y-x) + L(y) + \alpha \int_0^\infty f(y-t,F)F(dt)],$$

where $L(y) = H(y) + P(y)$ and $H(y) = \int_0^\infty h((y-t)^+)F(dt)$,

$$P(y) = \int_0^\infty p((y-t)^-)F(dt).$$

An inventory problem is a sequential decision problem where decisions must be made at repeated intervals whether or not to raise the inventory level. The demands t in successive periods are independently distributed random variables with a common distribution F. Three kinds of costs are incurred during each period. Here, $c(z)$ represents the cost of ordering amount z of the good, $h(.)$ the holding cost for inventories on hand and $p(.)$ the penalty cost associated with the failure to meet demand. Holding and penalty costs are charged at the end of each period. The constant α represents the discounting factor. Costs experienced N periods after the current period are discounted by α^N.

We always put the following assumptions.

Assumption 2-1.

(a) $0 < \alpha < 1$. (b) F is a distribution on $R^+ = [0, \infty)$.

(c) $c,h,p : R^+ \to R^+$ are nondecreasing and $c(0) = h(0) = p(0) = 0$, and $c(x+y) \leqq c(x) + c(y)$ for any $x, y \geqq 0$.

(d) $0 < C(0) = \int_0^\infty c(t)F(dt) < \infty$, $0 < P(0) = \int_0^\infty p(t)F(dt) < \infty$.

(e) $P(x) \geqq c(-x)$ for any $x < 0$.

Let us define $\tilde{H} : R \to R^+$

$$(2\text{-}2) \qquad \tilde{H}(x) = H(x) + \alpha \int_0^\infty H(x-t)\tilde{F}(dt),$$

where $\tilde{F}(t) = \sum_{i=1}^{\infty} \alpha^{i-1} F^{i*}(t)$ and F^{i*} is the i-fold convolution of F. Then, from the renewal theory, it is well known that $\tilde{H}(x)$ of (2-2) is a unique solution of the following renewal equation.

$$\tilde{H}(x) = H(x) + \alpha \int_0^\infty \tilde{H}(x-t)F(dt).$$

In order to use a technique of a contraction mapping we define a complete metric space of functions on $R = (-\infty, \infty)$.

To begin with, let us define v_0, $v : R \to R^+$

$$(2\text{-}3) \qquad v_0(x) = (1-\beta)\tilde{H}(x) + c(x^-), \quad v(x) = \beta\tilde{H}(x) + K,$$

where β and K are positive numbers which satisfy

$$\beta < (1-\alpha), \qquad K \geqq \{P(0) + \alpha C(0)\}/(1-\alpha).$$

For each function $w : R \to R$, set

$$(2\text{-}4) \qquad \|w\| = \sup_{x \in R} |w(x)|/v(x) .$$

Let us define E to be the set of all nonnegative functions w for which $\|w\| < \infty$. The metric ρ is given by $\rho(w_1, w_2) = \|w_1 - w_2\|$. Also let us define a ball B in E

$$(2\text{-}5) \qquad B = \{ u = w + v_0 \ ; \ \|w\| \leqq 1, \ w \in E \}.$$

We let define the operator T on E by

$$(2\text{-}6) \qquad (Tu)(x) = \inf_{y \geqq x} [c(y-x) + L(y) + \alpha \int_0^\infty u(y-t)F(dt)] .$$

For every x, let $y^*(x;u)$ satisfy

$$(2\text{-}7)\qquad (Tu)(x) = c(y^*(x;u)-x) + L(y^*(x;u)) + \alpha\int_0^\infty u(y^*(x;u)-t)F(dt),$$

which depends on $u \in B$. Strictly speaking, no such $y^*(x;u)$ may exist. To avoid troublesome ε-arguments, however, we will assume all infimums are achieved.

According to Nakagami [24], we will examine the condition in which T is a contraction with respect to the metric ρ by the following lemmas.

Lemma 2-1. If $u \in B$, then $Tu \in B$.

Proof. For $x < 0$ we have from (2-6)

$$(Tu)(x) \leqq c(-x) + P(0) + \alpha\int_0^\infty u(-t)F(dt).$$

By (2-5) it holds that $u \leqq v_0 + v$, we obtain

$$(Tu)(x) \leqq c(-x) + P(0) + \alpha C(0) + \alpha K \leqq v_0(x) + v(x).$$

From (c) and (e) we also obtain

$$(Tu)(x) \geqq \inf_{y \geqq x} [c(y-x) + P(y)]$$

$$\geqq \min\left\{\begin{array}{l} \inf\limits_{0>y\geqq x} [c(y-x) + c(-y)] \\ \inf\limits_{y\geqq 0} [c(y-x) + P(y)] \end{array}\right\} \geqq c(-x) \geqq v_0(x).$$

For $x \geqq 0$, we have from (2-6)

$$(Tu)(x) \leqq H(x) + P(0) + \alpha\int_0^\infty u(x-t)F(dt).$$

By (2-3), (2-5) and (2-2) we obtain

$$(Tu)(x) \leqq H(x) + P(0) + \alpha K + \alpha(1-\beta)\int_0^\infty \tilde{H}(x-t)F(dt)$$

$$+ \alpha\beta\int_0^\infty \tilde{H}(x-t)F(dt) + \alpha\int_x^\infty c(t-x)F(dt)$$

$$\leqq \tilde{H}(x) + P(0) + \alpha C(0) + \alpha K \leqq v_0(x) + v(x).$$

We also obtain

$$(Tu)(x) \geqq \inf_{y \geqq x} [\, H(y) + \alpha\int_0^\infty u(y-t)F(dt) \,]$$

$$\geqq H(x) + \alpha(1-\beta) \int_0^\infty \tilde{H}(x-t)F(dt)$$

$$\geqq (1-\alpha\beta)\tilde{H}(x) \geqq v_0(x).$$

Hence, $0 \leqq Tu - v_0 \leqq v$, so $Tu \in B$. □

For any $x \in R$, let us define

$$\text{(2-8)} \qquad X^*(x) = \{z \,; z \geqq x,\ c(z-x)+v_0(z) \leqq v_0(x)+v(x)\},$$

$$x^*(x) = \sup\{z \,;\ z \in X^*(x)\}.$$

It holds that $x^* \geqq x^+$ since $x \in X^*(x)$ if $x \geqq 0$ and $0 \in X^*(x)$ if $x < 0$ and that x^* is contained in a finite interval when the unbounded costs case of $\lim_{x\to\infty} \{c(x)+h(x)\} = \infty$ and x^* may be infinite otherwise.

Lemma 2-2. For any given $u \in B$ and every $x \in R$,

$y^*(x;u) \in X^*(x)$ and $v_0(y^*)$ and $v(y^*)$ are finite.

Proof. If $x^*(x) = \infty$, then $\lim_{x\to\infty} \{c(x)+h(x)\} < \infty$ is satisfied. The result is obvious. If $x^*(x) < \infty$,then by Lemma 2-1 and (2-8), we have for any y with $y > x^*(x)$

$$(Tu)(x) \leqq v_0(x) + v(x) < c(y-x) + v_0(y)$$

$$\leqq c(y-x) + L(y) + \alpha\int_0^\infty u(y-t)F(dt).$$

Thus, we have the desired result. □

Lemma 2-3. $\rho(Tu_1, Tu_2) \leqq \gamma\, \rho(u_1, u_2)$ for any u_1, $u_2 \in B$ where $\gamma = \alpha(1-\beta)^{-1} < 1$.

Proof. For any u_1, u_2, $y_1 = y^*(x;u_1)$ and $y_2 = y^*(x;u_2)$ satisfy (2-5) respectively. We observe from Lemma 2-2

$$(Tu_1)(x) - (Tu_2)(x) \leqq \alpha\int_0^\infty [u_1(y_2-t) - u_2(y_2-t)]F(dt).$$

Then, we obtain from (2-5)

$$\begin{aligned}|Tu_1 - Tu_2|(x) &\leqq \max_{i=1,2} \alpha\int_0^\infty |u_1-u_2|(y_i-t)F(dt)\\ &\leqq \alpha\|u_1-u_2\| \max_{i=1,2}\int_0^\infty v(y_i-t)F(dt)\\ &\leqq \alpha\|u_1-u_2\|\max[v(y_1),v(y_2)].\end{aligned}$$

If $y^* = \max(y_1,y_2) < 0$,

$$|Tu_1-Tu_2|(x) \leqq \alpha K\|u_1-u_2\|.$$

If $y^* \geqq 0$, we have from (2-8)

$$(1-\beta)\tilde{H}(y^*) \leqq \begin{cases} c(-x) - c(y^*-x) + K & \text{for } x < 0,\\ \tilde{H}(x) + K & \text{for } x \geqq 0.\end{cases}$$

$$\tilde{H}(y^*) \leqq \{\tilde{H}(x) + K\}/(1-\beta).$$

Then,

$$v(y^*) \leqq \{\beta\tilde{H}(x) + K\}/(1-\beta).$$

Hence, we have

$$\|Tu_1 - Tu_2\| \leqq \gamma\|u_1 - u_2\|. \quad \square$$

Since we have constructed the contraction mapping by Lemma 2-3, we can state the existence and uniqueness theorem. Proof is found in Èlsgol'c [5].

Theorem 2-1. Under Assumption 2-1, there exists a unique fixed point $f \in B$ to (2-1) : $Tf = f$. This point can be found by the method of successive approximations,

$$f = \lim f^N \quad \text{where } f^N = Tf^{N-1}, \; N=1,2,\ldots,$$

with the point f^0 chosen arbitrarily in the set B. Note that, if we choose $f^0 = v_0$ ($v_0 \in B$), then the sequence f^N, $N=1,2,\ldots$, is nonnegative and converges nondecreasingly to f.

If we choose $f^0 = 0$ ($0 \in E, 0 \notin B$), then we have from the argument of Lemma 2-1,

$$f^N(x)=(T^N 0)(x) \geqq \begin{cases} H(x) + \sum_{i=1}^{N-1}\alpha^i \int_0^\infty H(x-t)F^{i*}(dt) & \text{for } x \geqq 0, \\ c(-x) & \text{for } x < 0. \end{cases}$$

Now we let $\underline{N}$ be the smallest number such that $\alpha^{\underline{N}}/(1-\alpha) \leqq \beta$, then we have

$$v_0(x) \leqq f^{\underline{N}}(x) \leqq v_0(x) + v(x),$$

that is, $f^{\underline{N}}$ is contained in B.

Corollary 2-1. The unique fixed point $f \in B$ to $Tf = f$ may also be found by putting $f^0 = 0$ and the sequence f^N, $N=1,2,\ldots$, is nonnegative and converges nondecreasingly to f.

2.2 Further Results for the Existence Theorem

Two special cases of Theorem 2-1 are considered, and the existence of the optimal inventory policy $y^*(x)$ is shown, which actually minimizes (2-1). First, we put the following assumption.

Assumption 2-2.

(f) $\lim_{x\to\infty} \{c(x)+h(x)\} = \infty$.

(g) $c,h,p : R^+ \to R^+$ are lower-semicontinuous(LSC).

Note that most of the cost functions practically considered in inventory problems are left-continuous, so (g) is satisfied by (c). Assumption 2-2 is of practical use.

Let us define

$$\text{(2-9)} \qquad g(x) = \inf_{y \geqq x} [c(y-x) + L(y)].$$

If Assumption 2-2 is satisfied, then the brackets of (2-9) are LSC in y for any fixed x, and the minimum of (2-9) is achieved by $y'(x)$.

Lemma 2-4. Under Assumption 2-2, g is LSC on R.

Proof. For each net $\{x_n\}$ converging to a point $x \in R$, let $y'(x_n)$ minimize (2-9) for x_n such that $y'(x_n) \geqq x_n$. Then any accumulation point y'' of $\{y'(x_n)\}$ satisfies $y'' \geqq x$. And

$$\liminf g(x_n) = \liminf [c(y'(x_n)-x_n) + L(y'(x_n))]$$
$$\geqq [c(y''-x) + L(y'')] \geqq g(x).$$

Then, g is LSC on R. □

Theorem 2-2. Under Assumptions 2-1 and 2-2, a unique fixed point $f \in B$ is LSC on R, and $y^*(x)$ exists.

Proof. Let the sequence $\{f^N\}$ be defined by $f^0 = v_0$ in Theorem 2-1. By Lemma 2-4, f^N, N=1,2,..., are LSC, and $\sup f^N$ is LSC in general. Then, $\sup f^N = \lim f^N = f$ is LSC. □

Second, we put the following assumption, which is a stronger condition than Assumption 2-2.

Assumption 2-3.

(f) $\lim_{x\to\infty} \{c(x)+h(x)\} = \infty$.

(g') $c : R^+ \to R^+$ is LSC.

(g") $L : R \to R^+$ is uniformly continuous.

Lemma 2-5. Under Assumption (g"), L and g are equicontinuous.

Proof. For any $\varepsilon > 0$, let $\delta > 0$ be such that $|x_2-x_1| < \delta$ implies $|L(x_2)-L(x_1)| < \varepsilon$. And let y_i' minimize (2-9) for x_i. Then

$$g(x_1) = c(y_1'-x_1) + L(y_1') \leqq c(x_1+y_2'-x_2-x_1) + L(x_1+y_2'-x_2),$$

$$g(x_2) = c(y_2'-x_2) + L(y_2') \leqq c(x_2+y_1'-x_1-x_2) + L(x_2+y_1'-x_1).$$

Therefore, we have $|g(x_1) - g(x_2)| < \varepsilon$. □

The above Lemma is refered to the theorem 3 in Boylan [3].

Theorem 2-3. Under Assumptions 2-1 and 2-3, a unique fixed point $f \in B$ is continuous on R, and $y^*(x)$ exists.

Proof. Let the sequence $\{f^N\}$ be defined by $f^0 = 0$ in Corollary 2-1. By Lemma 2-5 and induction on N, $|L(x_2)-L(x_1)| < \varepsilon$ implies that

(2-10) $$|f^N(x_2)-f^N(x_1)| \leqq (1+\alpha + \dots + \alpha^{N-1})\varepsilon ,$$

then L and f^N , N=1,2,..., are equicontinuous family of functions.

Next, from Lemma 2-1, we find that for N = 1,2,...

(2-11) $$f^N(x) = (T^N 0)(x) \leqq v_0(x) + v(x).$$

Then, for any positive number b, $f^N(x)$ remains bounded for all N whenever $|x| \leqq b$. Thus, together with (2-10) the limit function f(x) is continuous in any finite interval. □

CHAPTER 3

OPTIMAL POLICIES
FOR THE PRACTICAL INVENTORY PROBLEMS

In this chapter we shall derive the particular types of optimal policies for the practical inventory problems.

We use the same notations given in the chapter 2 and assume that Assumption 2-1 and 2-3 are satisfied, then from Theorem 2-3 the optimal inventory equation has a unique fixed point which is continuous and the optimal inventory policy exists. We, therefore, determine the particular type of the optimal policy and discuss its quantitative properties in our practical inventory problems by using the technique of successive approximations.

First, let us consider the problem where the ordering cost function is linear with multiple set-up rather than one with a single set-up. This type of cost is neither convex nor concave, but has a practical meaning when the ordered quantity in each period is delivered by a transportation vehicle which has a certain limited capacity. In general, an optimal inventory policy is sensitive to the form of the ordering cost, so that until now some types of inventory policies have examined and studied by several authors. Scarf [31] proved that an (s,S) policy is optimal for a linear cost with a single set-up, and this case was investigated in detail by Iglehart [9], Veinott [35] et al. Porteus [28] proved that a generalized (s,S) policy is optimal for a concavely increasing cost. In Nakagami [20], the purpose of this problem is to derive the particular type of the optimal policy in the model when the ordering cost function has multiple set-up.

Next we consider the perishable inventory problem. One of the important aspects of it is that the perishable goods have a fixed lifetime and become useless in satisfying demands after a fixed length of periods. Several authors (Fries [7], Nahmias and Pierskalla [17],[19] and Nahmias [18]) investigated the model in which the goods perish exactly m periods after receipt on order. And they derived optimal ordering policies and some qualitative properties of policies by solving a dynamic program with a state variable of dimension m-1.

Another remarkable aspect of the perishability is that one can preserve the perishable goods in a special warehouse, which keeps them in the almost same quality and extends their life time for a pretty long periods. For example, perishable foodstuffs like fresh fish and meat deteriorate in a week when refrigerated. But they will keep for half an year when frozen, which for the practical purpose is non-perishable. The objective of this problem is to analyze this realistic model in order to derive an optimal preserving policy as well as an ordering one with their properties according to Nakagami [22].

3.1 An Optimal Batch (s,S) Policy

Let $c(z)$ denote the ordering cost function with multiple set-ups as follows:

$$(3\text{-}1) \qquad c(z) = K\left\{\frac{z}{M}\right\} + cz \qquad \text{for } z \geqq 0,$$

where $c \geqq 0$, K, $M > 0$ and $\{z\}$ is the minimum integer not smaller than z. When we interpret M as the capacity of a transportation vehicle, K as the cost of its use and c as the unit cost of the treated good, then $c(z)$ is more reasonable, for if vehicles of the transportation are trucks the ordering cost is a function only of the number of trucks required to satisfy the order and not of the fraction of truck space used (if exess space cannot be used).

We use the same notations given in Chapter 2 except the ordering cost (3-1) and assume that Assumption 2-1 and 2-3 are satisfied, then from Theorem 2-3 the optimal inventory equation which is given by

$$(3\text{-}2) \qquad f(x) = \inf_{y \geqq x} \left[c(y-x) + L(y) + \alpha \int_0^{\infty} f(y-t)F(dt) \right]$$

has a unique fixed point $f(x)$ which is continuous and the optimal inventory policy $y^*(x)$ exists. Now, we shall determine the particular type of the optimal policy and discuss its property in our practical inventory problem by using the technique of successive approximations.

We let $f^N(x)$ be the optimal expected costs as a function of the level x of inventory before ordering if the inventory problem is engaged in for a total of N periods. We have

$$(3\text{-}3) \qquad f^N(x) = \inf_{y \geqq x} [\, c(y-x) + L(y) + \alpha \int_0^\infty f^{N-1}(y-t)F(dt)]$$

for N = 1,2,... and $f^0(x) = 0$, and

$$(3\text{-}4) \qquad f(x) = \lim_{N\to\infty} f^N(x).$$

Let us define

$$(3\text{-}5) \qquad G^N(x) = L(x) + cx + \alpha\int_0^\infty f^{N-1}(x-t)F(dt) \ .$$

Then

$$(3\text{-}6) \qquad f^N(x) = \inf_{y \geqq x} [\, G^N(y) + K \{\tfrac{y-x}{M}\}] \ - cx.$$

We let $y^N(x)$ denote the optimal inventory policy, i.e., the optimal level of inventory after ordering in the first of N periods when the level of inventory before ordering is x.

Now we shall give a definition of a batch (s,S) policy and some sufficient conditions under which this policy is optimal in the N finite horizon problem (3-6).

Definition 3-1. A batch (s,S) policy is an inventory policy y(x) defined by parameters s, S with s < S and M(> 0), such that

$$\begin{aligned} y(x) &= x & &\text{for } x \geqq s, \\ y(x) &= \min\,(\, S,\ x + M\{\tfrac{s-x}{M}\}\,) & &\text{for } x < s. \end{aligned}$$

A batch (s,S) policy has a following economic interpretation. In case 2, where $\{\frac{S-s}{M}\} \geqq 2$, M is smaller than S-s, i.e., the manager has smallsized trucks for the transportation use as compared with a satisfying level region (s,S]. Then he orders a minimum amount of the good with full-loaded trucks so as to raise the inventory level up to the region (s,S] if the initial level is less than s (batch policy). In case 1, where $\{\frac{S-s}{M}\} = 1$, the manager has large-sized trucks. Then he cannot order the good with full-loaded trucks so as to raise the inventory level into the region (s,S]. So that he raise the inventory level not to exceed S with trucks which are not always full-loaded if the initial level is less than s (batch policy + (s,S) policy). If M goes to infinity this policy is identical to the well-known (s,S) policy.

Lemma 3-1. Let $M > 0$, if the continuous function $G:R \to R$ satisfy the condition:

$$(3\text{-}7) \qquad \Delta_M G(x) = \max_{0 \leqq m \leqq M} [G(x) - G(x+m)] \quad \text{is non-increasing,}$$

then the following properties hold.

(i) If a function $H: R \to R$ is convex, then it satisfies (3-7).

(ii) If G(x) satisfies (3-7), then so is G(x+h) for all h.

(iii) If G satisfies (3-7), then -G is unimordal.

That is, there exists a number S (which may be $\pm\infty$) such that G(x) is non-increasing on $(-\infty, S)$ and non-decreasing on (S, ∞).

Proof. Properties (i) and (ii) is trivial, then we will give the proof of (iii). From (3-7) there exists a number S (which may be $\pm\infty$) which satisfies $\Delta_M G(S) = 0$. First, let us consider the interval $[S, \infty)$. It holds that $G(x) \leqq G(y)$ for any x and y, $S \leqq x < y < x+M$, that is, G is non-decreasing on $[S, \infty)$. Second, we will show that G is non-increasing on $(-\infty, S)$. Assume that there exists an interval (a,b) on which G is strictly increasing. Since $\Delta_M G(a) = \max_{0 \leqq m \leqq M} (G(a) - G(a+m)) > 0$, then

$$\Delta_M G(a) = \max_{0 \leqq m \leqq M-(b-a)} (G(a) - G(b+m))$$

$$< \max_{0 \leqq m \leqq M} (G(b) - G(b+m)) = \Delta_M G(b) .$$

This contradicts the condition (3-7). □

Theorem 3-1. If $G^N(x)$ is continuous and satisfies the condition(3-7), then $y^N(x)$ is of the batch (s,S) type.

Proof. By Lemma 3-1, there exist the smallest numbers s^N and S^N with $s^N < S^N$ (which may be $\pm\infty$), such that

(3-8) $\quad G^N(S) \leqq G^N(x) \qquad$ for all x,

(3-9) $\quad \Delta_M G^N(x) \leqq K \qquad$ for all $x \geqq s^N$.

Let us consider the optimal inventory equation (3-6) for the N finite horizon problem. From notational convenience, we abbreviate the superscript N.

Case 1, $\{\frac{S-s}{M}\} = 1$:

We have, for any x and y with $s \leqq x < y$,

$$G(y) + K\{\frac{y-x}{M}\} \geqq G(y) + K \geqq G(x) \qquad \text{by (3-9).}$$

Thus it follows that

$$(3\text{-}10) \quad y(x) = x \qquad \text{on} \quad [s, \infty).$$

We have, for any x and y with $S-M \leqq x < s$, $x < y$,

$$G(y) + K\{\frac{y-x}{M}\} \geqq G(y) + K \geqq G(S) + K \qquad \text{by (3-8).}$$

Hence we get

$$(3\text{-}11) \quad y(x) = S = \min (S, x+M\{\frac{s-x}{M}\}) \qquad \text{on } [S-M, s).$$

For any x and y with $S-2M \leqq x < S-M$, $x < y \leqq x+M$, we have

$$G(y) + K \geqq G(x+M) + K \qquad \text{(equality holds iff } y = x+M).$$

Thus it is easily shown by induction that for any x with $S-(d+1)M \leqq x < S-dM$, $d = 1,2,\ldots,$

$$(3\text{-}12) \quad \min_{x+dM \geqq y \geqq x} [G(y) + K\{\frac{y-x}{M}\}] \geqq G(x+dM) + dK .$$

Therefore we have, for any x with $S-(d+1)M \leqq x < S-dM$, $d = 1,2,\ldots,$

$$\min_{y \geqq x} [G(y) + K\{\frac{y-x}{M}\}]$$

$$= \min_{y \geqq x+dM} [G(y) + K\{\frac{y-x-dM}{M}\} + dK] \qquad \text{by (3-12)}$$

$$= \begin{cases} G(S) + (d+1)K & \text{if } S-M \leqq x+dM < s \quad \text{by (3-11)} \\ G(x+dM) + dK & \text{if } s \leqq x+dM < S \quad \text{by (3-10).} \end{cases}$$

Hence, if $S-M \leqq x+dM < s$ then $\{\frac{S-x}{M}\} = d+1$, so that $y(x) = S \leqq x + M\{\frac{S-x}{M}\}$, and if $s \leqq x+dM < S$ then $\{\frac{S-x}{M}\} = d$, so that $y(x) = M\{\frac{S-x}{M}\} < S$. Then

$$y(x) = \min\ (S,\ x + M\{\frac{S-x}{M}\}) \qquad \text{on} \quad (-\infty, s).$$

Case 2, $\{\frac{S-s}{M}\} \geqq 2$:

Similarly to Case 1, it is clearly shown that $y(x) = x$ on $[s, \infty)$, $y(x) = x + M\{\frac{S-x}{M}\} < S$ on $(-\infty, s)$, and $y(x) = \min\ (S, x + M\{\frac{S-x}{M}\})$ on $(-\infty, s)$. This completes the proof of the theorem. □

We notice that if L is convex and the ordering cost is given by (3-1), then Assumption 2-3 is automatically satisfied and $f^N(x)$ and $G^N(x)$ are continuous for all N. But, unfortunately, the condition (3-7) is not closed under sums and integrals, then it cannot be carried out the inductive arguments that $G^N(x)$ satisfies (3-7) for general types of distributions.

Hence we need a rather strong condition for the demand distribution F such that F has a density F' and satisfies the following definition.

Definition 3-2. A density F' is called M-indifferent, if it satisfies

$$\sum_{i=0}^{\infty} F'(t+iM) = \text{const.}\ (= 1/M) \quad \text{for } 0 \leqq t < M.$$

If we divide the demanded quantities by M, the M-indifferent densities give no information about which quantities left are likely to occur, that is, such densities are indifferent (ignorant) of the remaining quantities. For example, let, for $s \geqq 0$, $v \geqq 0$,

$$k(s,v,t) = v \quad \text{if } s \leqq t < s+M$$
$$= 0 \quad \text{otherwise,}$$

then an M-indifferent density $F'(t)$ is given by,

$$F'(t) = \int_0^{\infty} k(s,v,t)v(ds) \ ,$$

where $v(.)$ is a non-null measure defined on $[0, \infty)$ and $\int v(ds) = 1/M$.

Theorem 3-2. If L is convex and F' is M-indifferent, then a batch (s,S) policy is optimal in (3-3) and (3-2).

Proof. Here, we will show by induction that $G^N(.)$ is convex for all N. For $N = 1$, $G^1(x) = L(x) + cx$ is convex. Assume that $G^N(.)$ is convex. Then by Theorem 3-1 there exist two levels s^N, S^N with $s^N < S^N$ such that

$$(3\text{-}13) \quad f^N(x) = G^N(\min(S^N, x+dM)) + dK - cx$$
$$\text{on } [s^N-dM, s^N-(d-1)M\ ),\ d=1,2,\ldots,$$
$$= G^N(x) \quad \text{on } [s^N, \infty\ ).$$

Hence we have

$$(3\text{-}14) \quad f^N(x) - f^N(x-dM) = -dK \quad \text{if } x \leqq \min(S^N, s^N+M) = \tilde{s}^N,$$

and $f^N(x) - f^N(x-dM)$ is nondeceasing in x.

Now we examine $G^{N+1}(x)$ defined by (3-5).

$$G^{N+1}(x) = L(x) + cx + \alpha \int_0^\infty f^N(x-t)F'(t)dt.$$

By the notational convenience, let us abbreviate the supperscript N for simplicity. We have from continuity and piecewise convexity of f(x)

$$(3\text{-}15) \quad \frac{d}{dx}\int_0^\infty f(x-t)F'(t)dt = \int_0^\infty f'(x-t)F'(t)dt.$$

For $x \leqq \tilde{s}$ we have

$$(3\text{-}15) = \sum_{i=0}^\infty \int_{[0,M)} f'(x-t-iM)F'(t+iM)dt$$

$$= \sum_{i=0}^\infty \int_{[0,M)} f'(x-t)F'(t+iM)dt \qquad (\text{ by } (3\text{-}14) \text{ })$$

$$= (1/M)\int_{[0,M)} f'(x-t)dt \quad (\text{ by the M-indifference of } F' \text{ })$$

$$= (1/M)\int_{(s,\tilde{s})} G'(t)dt - c = \text{const. } (= -C) \text{ by } (3\text{-}13).$$

For $x > \tilde{s}$, let $\tilde{d} = \{\frac{x-\tilde{s}}{M}\}$. Then we have

$$(3\text{-}15) = \int_0^{x-\tilde{s}} f'(x-t)F'(t)dt + \int_{x-\tilde{s}}^\infty f'(x-t)F'(t)dt$$

$$- \int_0^{x-\tilde{s}} f'(x-t-\tilde{d}M)F'(t)dt + \int_0^{x-\tilde{s}} f'(x-t-\tilde{d}M)F'(t)dt$$

$$= \int_0^{x-\tilde{s}} [f'(x-t) - f'(x-t-\tilde{d}M)]\, F'(t)dt$$

$$+ \int_0^\infty f'(x-t-\tilde{d}M)F'(t)dt \quad (\text{ by } (3\text{-}14) \text{ }).$$

The second term is -C, it is therefore sufficient to show that the integrand of the first term is non-negative and non-deceasing in x.

For $x_1 < x_2$, let $\tilde{d}_1 = \{(x_1-\tilde{s})/M\}$, $\tilde{d}_2 = \{(x_2-\tilde{s})/M\}$ respectively, then $\tilde{d}_1 \leqq \tilde{d}_2$.

$$0 \leqq f'(x_1-t) - f'(x_1-t-\tilde{d}_1M) \quad (\text{ by } (3\text{-}14)\ )$$

$$= f'(x_1-t) - f'(x_1-t-\tilde{d}_2M) \quad (\ x_1-t-\tilde{d}_1 \leqq \tilde{s}\ )$$

$$\leqq f'(x_2-t) - f'(x_2-t-\tilde{d}_2M) \quad (\text{ by } (3\text{-}14)\ ).$$

Thus the proof of Theorem 3-2 is completed. □

Remark. Unfortunately, $G^N(x)$, $N=1,2,\ldots$, do not satisfy the condition (3-7) for general demand distributions, and hence any batch (s,S) policy may not be optimal. However, in many practical cases, the demand distribution is not determined precisely, and an M-indiffernt density gives a good approximation to the true demand distribution by exploiting a least-square method.

3.2 The Perishable Inventory Problem

Let a state variable x be the amount of perishable goods stored in the preservable warehouse. After observing x, two decisions are made as follows: The amount z-x is ordered at unit ordering cost c in order to raise the inventory level up to z. The amount y is stored in the preservable warehouse at unit cost h_1, the amount z-y is placed in the ordinary warehouse. The decision variable y, z represent the starting amount of preserving inventory and the starting amount of the total inventory, respectively ($0 \leqq x \leqq z$, $0 \leqq y \leqq z$). The inventory is first depleted from the ordinary warehouse, and depleted from the preserving one at unit emergency issuing cost k_2. The unsatisfied demand is lost for sales at unit penalty cost p. When the demand is over at the period, the amount remaining in the ordinary is disposed, and the amount remaining in the preserving is brought to the next period at unit holding cost h_2.

Let a random variable T with a distribution F represent a demand in the period. Then an amount H remaining in the preserving and an amount K issued from the preserving are given by

$$(3\text{-}16) \quad H = (z-(z-y) \vee T)^{+}, \quad K = (T \wedge z-(z-y))^{+}.$$

This can be seen most easily by considering the three possibilities $0 \leqq T \leqq z-t$, $z-y \leqq T \leqq z$ and $z \leqq T$ separatedly, where $x^{+} = \max(0,x)$, $x \vee y = \max(x,y)$ and $x \wedge y = \min(x,y)$.

The following cost structure in the period is summarized:

$c(z-x)^+$ = ordering cost; h_1y = preserving cost;

h_2H = holding cost; k_2K = issuing cost;

$p(T-z)^+$ = penalty cost.

Now, $H+K = y$, the preserving cost is included to the holding and issuing costs by putting $h = h_1+h_2$, $k = h_1+k_2$ respectively. Before giving our formulation, we examine random variables H and K without proof.

Lemma 3-2. The random variable H and K have distribution functions $H_{y,z}(.)$ and $K_{y,z}(.)$, $0 \leqq y \leqq z$, respectively given by

$$(3\text{-}17)\quad \begin{aligned} H_{y,z}(s) = P[H \leqq s] &= 1 - F(z-s) \quad (0 \leqq s < t) \\ &= 1 \qquad (y \leqq s), \\[1em] K_{y,z}(s) = P[K \leqq s] &= F(z-y+s) \quad (0 \leqq s < y) \\ &= 1 \qquad (y \leqq s). \end{aligned}$$

Moreover, if the density F' of F is continuous and any given function $g : R^+ \to R^+$ is continuously differentiable, then

$$(3\text{-}18)\quad H(y,z) = \int_0^\infty g(s)H_{y,z}(ds), \quad K(y,z) = \int_0^\infty g(s)K_{y,z}(ds)$$

have partial derivatives.

We let $f(x)$ (or $f^N(x)$) denote the optimal expected costs for the intial level x of the preserving inventory (if the problem is engaged in for a total of N periods). We have

$$(3\text{-}19) \quad f^N(x) = \min_{z \geq x,\, z \geq y \geq 0} [\, B^N(y,z) - cx\,],$$

$$(3\text{-}20) \quad B^N(y,z) = cz + \int_0^\infty p(t-z)^+ F(dt) + \int_0^\infty ks\, K_{y,z}(ds) + \int_0^\infty (hs + \alpha f^{N-1}(s)) H_{y,z}(ds),$$

for $N=1,2,\ldots$ and $f^0(x) = 0$.

We let $y^N(x)$ and $z^N(x)$ (or $y^*(x)$ and $z^*(x)$) denote the optimal preserving and ordering policies (in the period N) when the initial level of the preserving inventory before ordering is x.

It is noted that the non-perishable problem $\hat{f}(x)$ ($\hat{f}^N(x)$) is obtained by putting $z = y$ in (3-19) and (3-20).

$$(3\text{-}21) \quad \hat{f}^N(x) = \min_{z \geq x} [\, c(z-x) + \int_0^\infty \{ p(t-z)^+ + k(t \wedge z) + h(z-t)^+ + \alpha \hat{f}^{N-1}((z-t)^+) \} F(dt) \,],$$

for $N=1,2,\ldots$ and $\hat{f}^0(x) = 0$, and

$$(3\text{-}22) \quad \hat{f}(x) = \lim_{N \to \infty} \hat{f}^N(x)$$

is satisfied under Assumptions 2-1 and 2-3, i.e., $c > 0$ and $\int tF(dt) < \infty$ for the linear cost case. It follows that $\hat{f}(x)$ is continuous and the optimal policy $\hat{z}^*(x)$ exists.

Let

$$L(x,y,z) = c(z-x) + \int_0^\infty p(t-z)^+ F(dt) + \int_0^\infty ks\, K_{y,z}(ds) + \int_0^\infty hs\, H_{y,z}(ds)$$

for the perishable problem, and

$$\hat{L}(x,z) = c(z-x) + \int^{\infty}\{p(t-z)^{+}$$

$$+ k(t \wedge z) + h(z-t)^{+}\} F(dt)$$

for the non-perishable problem.

Then it follows that from Lemma 2-2

$$L(x, y_u^*(x), z_u^*(x)) \leqq \hat{L}(x, \hat{z}_u^*(x)) \quad .$$

Hence, we have a contraction theorem for the perishable problem (3-19) and (3-20). It holds that

$$(3\text{-}23) \quad f(x) = \lim_{N\to\infty} f^N(x)$$

satisfies under the assumption $c > 0$ and $\int tF(dt) < \infty$, and that $f(x)$ is continuous and the optimal preserving and ordering policies $y^*(x)$ and $z^*(x)$ exist.

Since the purpose of this section is to derive some properties for $y^N(x)$ and $z^N(x)$, $N=1,2,\ldots$, we put the following assumption.

Assumption 3-1.

(a) $\int F'(t)dt < \infty$.

(b) $F'(t)$ is continuous for all $t \geqq 0$. (It is necessary for Lemma 3-2.)

(b') $F'(t) > 0$ for all $t \geqq 0$. (It is not necessary but only for the uniqueness of the optimal policies.)

(c) $\alpha c > h+k$, $p > c+k$. (It would be optimal to order and preserve the goods at least.)

Theorem 3-3. Under Assumption 3-1, there exists a sequence $\tilde{z}^N$, n=1,2,..., such that

$$F^{-1}(\frac{p-c}{p}) = \tilde{z}^1 < \tilde{z}^2 < \ldots < \tilde{z} = F^{-1}(\frac{p-c-k}{p-\alpha c-k+h}).$$

If $\tilde{z}^n = \tilde{z}$ for some n, then $\tilde{z}^n = \tilde{z}^{n+1} = \ldots = \tilde{z}$.

(i) Results on $f^N(x)$.

It is convex on R^+, and strictly convex on $[\tilde{z}^N, \infty)$.

It is non-increasing and non-negative.

(ii) Results on $\frac{df^N(x)}{dx}$.

It is continuous and non-decreasing.

$\frac{df^N(x)}{dx} = c$ on $[0,\tilde{z}^N]$ and $\lim_{x\to\infty} \frac{df^N(x)}{dx} = 0$.

(iii) Results on $y^N(x)$ and $z^N(x)$.

$z^N(x) = \tilde{z}^N \vee x$, $y^N(x) = \tilde{y}^N(\tilde{z}^N \vee x)$,

where $\tilde{y}^N(x)$ is a unique solution of

$$k + (h-k+\alpha f^{N-1}(y))F(x-y) = 0 \quad \text{for all } x \geqq \tilde{y} = F^{-1}(\frac{k}{\alpha c-h+k}),$$

and $\tilde{y}^1(x) = 0$, $\tilde{y}^N(\tilde{z}^N) = \tilde{z}^N - \tilde{y}$, N=2,3,....

$0 \leqq \frac{d^N y(x)}{dx} \leqq 1$ and $y^N(x)$ is bounded for all $x \geqq 0$.

Proof. The proof is given in Nakagami [22].

Theorem 3-4.

(i) $f^{N+1}(x) \geqq f^N(x)$, $\frac{df^{N+1}(x)}{dx} \geqq \frac{df^N(x)}{dx}$

,N=1,2,..., for all $x \geqq 0$.

(ii) $\tilde{y}^{N+1}(x) \geqq \tilde{y}^N(x)$,N=1,2,..., for all $x \geqq \tilde{y}$.

Proof. The proof is given in Nakagami [22].

Remark. We interpret the optimal ordering policy $z^N(x)$ and the optimal preserving policy $y^N(x)$ as follows:

Case 1. If $x < \tilde{z}^N$, it is optimal to order the amount $\tilde{z}^N - x$ so as to raise the total inventory up to $\tilde{z}^N$. It is optimal to preserve the amount $\tilde{z}^N - \tilde{y}$ in the preserving and to place the amount $\tilde{y}$ in the ordinary when $N=2,3,\ldots$: Or to preserve none of the amount and to place all the amount $\tilde{z}^1$ in the ordinary when $N=1$. Note that the penalty cost p depends only on the total inventory $\tilde{z}^N$, but is independent of the amount $\tilde{y}$ in the ordinary.

Case 2. If $x \geqq \tilde{z}^N$, it is optimal to order none of the amount so as to keep the total inventory to x. It is optimal to preserve the amount $\tilde{y}^N(x)$ in the preserving and to place the amount $x - \tilde{y}^N(x)$ in the ordinary when $N=2,3,\ldots$: Or to preserve none of the amount and to place all the amount x in the ordinary when only $N=1$. Note that the amount $\tilde{y}^N(x)$ in the preserving is bounded. This indicates that the system is forced to go quickly to the steady state of Case 1 when the intial level is sufficiently large.

CHAPTER 4

A BAYESIAN STATISTICAL INVENTORY PROBLEM

In this chapter we now turn our attention to the inventory problems with uncertainty on the demand distribution.

When the demand distribution is unknown to us, that is, most of the informations about it are insufficient for a definite probability distribution, a Bayesian approach or a minimax procedure is sometimes used. A Bayesian approach is very attractive because it allows the explicit incorporation of prior and new informations into the structure of the model. But a major defect of Bayesian methods is the restriction of the class of prior distributions needed. A minimax procedure will be mentioned at the succeeding chapter.

On the Bayesian inventory problem, Scarf [30] and Iglehart [10] have already analyzed this model with linear costs, where they assume the demand distribution from the exponential family. According to Nakagami [25],[26], we shall set up a statistical inventory equation under a general class of prior distributions and discuss some important properties of its Bayes solution. The nonparametric application of the Bayes solution will be mentioned at the end of this chapter.

4.1 A Bayesian Approach

Let us introduce the terminology of a Bayesian statistics to describe our problem by the following notations given by Rieder [29] and Ferguson [6]. $(R^+, \mathcal{B})$ is the sample space where $R^+ = [0, \infty)$ and $\mathcal{B}$ is the σ-algebra of Borel subsets of R^+. P is a probability measure on $(R^+, \mathcal{B})$, F is a corresponding distribution and E denotes the expectation with respect to P. Ω is some collection of P on $(R^+, \mathcal{B})$, and $\mathcal{A}$ is some suitable σ-algebra of subsets of Ω, for example the Borel sets with respect to the topology of weak convergence. $(\Omega, \mathcal{A})$ is the parameter space. $\mathcal{P}$ is a probability measure on $(\Omega, \mathcal{A})$ and $\mathcal{E}$ denotes the expectation with respect to $\mathcal{P}$.

We assume that P_ω, defined by $P(\omega, .) : \Omega \to R^+$, is the probability measure such that

$$(4\text{-}1) \qquad P(\omega, B) = \int_B p(\omega, t)\, \lambda(dt) \qquad \text{for } B \in \mathcal{B},$$

where λ is a σ-finite measure on $\mathcal{B}$ and $p : \Omega \times R^+ \to R^+$ is a non-negative measurable function. And we let F_ω be the corresponding distribution which is called the demand distribution.

At the beginning of the n-th period, the history of the previous demands, $h_n = (t_1, t_2, \ldots, t_{n-1})$ on $H_n = \times_{k=1}^{n-1} R^+$, is to be used to make inferences about the true value of P.

We let $\mathcal{P}_0$ on $(\Omega, \mathcal{A})$ be a prior distribution. Then the sequence $\{\mathcal{P}_n\}$ of distributions $\mathcal{P}_n : H_n \to \Omega$ is defined recursively.

(4-2) $\mathcal{P}_1 = \mathcal{P}_0$,

$$\mathcal{P}_{n+1}(h_n \circ t_n, A) = \frac{\int_A P(\omega, t_n)\ \mathcal{P}_n(h_n, d\omega)}{\int_\Omega P(\omega, t_n)\ \mathcal{P}_n(h_n, d\omega)} \quad \text{for } A \in \mathcal{A},$$

if the denominator is positive and finite, and $\mathcal{P}_{n+1}(h_n \circ t_n, A) = \mathcal{P}_n(h_n, A)$ otherwise, where $h_n \circ t_n$ is to be interpreted as $(t_1, t_2, \ldots t_{n-1}, t_n)$. We shall call $\mathcal{P}_n$ a posterior distribution for the history h_n. $\mathcal{E}_n$ denotes the expectation with respect to $\mathcal{P}_n$.

The sequence $\{Q_n\}$ of distributions $Q_n : H_n \to R^+$ is defined by

(4-3) $Q_1(B) = \int_\Omega \mathcal{P}_1(d\omega) P(\omega, B)$ for $B \in \mathcal{B}$,

$$Q_n(h_n, B) = \int_\Omega \mathcal{P}_n(h_n, d\omega) P(\omega, B) \quad \text{for } B \in \mathcal{B} .$$

We call Q_n the marginal distribution for the history h_n. E_n denotes the expectation with respect to $Q_n(h_n, .)$.

The following proposition holds by Fubini's theorem.

Proposition 4-1. If $u : \Omega \times H_{n+1} \to R^+$ is a non-negative function, then

$$\int (\mathcal{P}_n P)(h_n, d(\omega, t_n)) u(\omega, h_n \circ t_n)$$

$$= \int Q_n(h_n, dt_n) \int \mathcal{P}_{n+1}(h_n \circ t_n, d\omega) u(\omega, h_n \circ t_n) .$$

4.2 The Statistical Inventory Problem

We now return to our discussion of the inventory problem. We let $f_n(x,h_n)$ denote the optimal expected costs for the initial level x and the history h_n when the demand distribution is not known. Then $f_n(x,h_n)$ satisfies the statistical inventory equation which is given by

$$(4\text{-}4)\qquad f_n(x,h_n) = \inf_{y \geq x} [c(y-x) + \int_0^\infty \{ l(y-t_n) + \alpha f_{n+1}(y-t_n, h_n \circ t_n)\} Q_n(h_n, dt_n)] .$$

In order to analyze (4-4) we first define $f_n^N(x,h_n)$ to be the optimal expected costs if the statistical inventory problem is engaged in for a total of N periods.

$$(4\text{-}5)\qquad f_n^N(x,h_n) = \inf_{y \geq x} [c(y-x) + \int_0^\infty \{l(y-t_n) + \alpha f_{n+1}^{N-1}(y-t_n, h_n \circ t_n)\} Q_n(h_n, dt_n)] ,$$

$$f_n^N(x,h_n) = 0 \qquad \text{for } N \leq 0.$$

We also define $f(x,F_\omega)$ and $f(x,Q_n)$ to satisfy (2-1) when the demand distributions are known to be F_ω and $Q_n(h_n,.)$ respectively.

$$(4\text{-}6)\qquad f(x,F_\omega) = \inf_{y \geq x} [c(y-x) + \int_0^\infty \{ l(y-t) + \alpha f(y-t, F_\omega)\} F(\omega, dt)].$$

$$(4\text{-}7) \qquad f(x,Q_n) = \inf_{y \geq x} [\, c(y-x) + \int_0^\infty \{\, l(y-t_n) + \alpha f(y-t_n, Q_n)\} Q_n(h_n, dt_n)\,].$$

Let $f^N(x,F_\omega)$ and $f^N(x,Q_n)$ denote the optimal expected costs if the inventory problems are engaged in for a total of N periods. Then

$$(4\text{-}8) \qquad f^N(x,F_\omega) = \inf_{y \geq x} [\, c(y-x) + \int_0^\infty \{\, l(y-t) + \alpha f^{N-1}(y-t, F_\omega)\} F(\omega, dt)\,],$$

$$f^N(x,F_\omega) = 0 \qquad \text{for } N \leqq 0.$$

$$(4\text{-}9) \qquad f^N(x,Q_n) = \inf_{y \geq x} [\, c(y-x) + \int_0^\infty \{\, l(y-t_n) + \alpha f^{N-1}(y-t_n, Q_n)\} Q_n(h_n, dt_n)\,],$$

$$f^N(x,Q_n) = 0 \qquad \text{for } N \leqq 0.$$

We can derive the following theorem which states the existence and uniqueness of the Bayes solution $f_n(x,h_n)$ of the statistical inventory equation (4-4). The proof follows directly from Theorem 2-1.

Theorem 4-1. If Assumption 2-1 is true for all $F \varepsilon \Omega$, then

$$\lim_{N\to\infty} f_n^N(x,h_n) = f_n(x,h_n), \qquad \lim_{N\to\infty} f^N(x,F_\omega) = f(x,F_\omega)$$

and $\lim_{N\to\infty} f^N(x,Q_n) = f(x,Q_n)$.

The following lemmas give the lower and upper bounds for the Bayes solution $f_n^N(x,h_n)$ of (4-5).

Lemma 4-1. $\mathcal{E}_n f^N(x,F_\omega) \leqq f_n^N(x,h_n)$ for all N and n,

$\mathcal{E}_n f(x,F_\omega) \leqq f_n(x,h_n)$ for all n.

Proof. If the 1st result is proved by induction on N for all n, the 2nd result holds from Theorem 4-1. The 1st is true for N=0 for all n. Assume it is true for N-1 and all n. The inside of the brackets of (4-5) is rewritten as follows,

$$c(y-x) + \int Q_n(h_n,dt_n)\{l(y-t_n) + \alpha f_{n+1}^{N-1}(y-t_n,h_n \circ t_n)\}$$

$$\geqq c(y-x) + \int Q_n(h_n,dt_n)\{l(y-t_n) + \alpha \mathcal{E}_{n+1} f^{N-1}(y-t_n,F_\omega)\}$$

(by the inductive assumption)

$$= c(y-x) + \int l(y-t_n) Q_n(h_n,dt_n)$$

$$+ \alpha \int (\mathcal{P}_n F)(h_n,d(\omega,t_n)) f^{N-1}(y-t_n,F_\omega)$$

(by Proposition 4-1). Hence, we obtain

$$f^N(x,h_n) = \inf_{y \geqq x} \int \mathcal{P}_n(h_n,d\omega)[c(y-x)$$

$$+ \int F(\omega,dt_n)\{l(y-t_n) + \alpha f^{N-1}(y-t_n,F_\omega)\}]$$

$$\geqq \int \mathcal{P}_n(h_n,d\omega) \inf_{y \geqq x} [c(y-x)$$

$$+ \int F(\omega,dt)\{l(y-t) + \alpha f^{N-1}(y-t,F_\omega)\}]$$

$$= \int \mathcal{P}_n(h_n,d\omega) f^N(x,F_\omega) . \quad \square$$

Note that the left hand sides of the equations in Lemma 4-1 are interpreted as the optimal expected costs when the true value of F_ω is revealed with $P_n(h_n,.)$ after the history $h_n = (t_1, t_2, \ldots, t_{n-1})$ is experienced.

Lemma 4-2. $E_n f^N(y-t_n, Q_n) \geqq E_n f^N_{n+1}(y-t_n, h_n o\, t_n)$.

Proof. It is proved by induction on N for all n. It is true for N=0 and all n. Assume it is true for N-1 and all n. We have by (4-9)

$$E_n f^N(y-t_n, Q_n) = E_n \inf_{y' \geqq y-t_n} [\, c(y'-y+t_n) + E_{n+1}\{l(y'-t_{n+1}) + \alpha f^{N-1}(y'-t_{n+1}, Q_n)\}] \; .$$

Using the optimality properties of $y^N_n{*}(y-t_n)$, abr. by y^*, for the above equation, we obtain

$$= E_n[c(y^*-y+t_n) + E_{n+1}\{l(y^*-t_{n+1}) + \alpha f^{N-1}(y^*-t_{n+1}, Q_n)\}]$$

$$\geqq E_n[c(y^*-y+t_n) + E_{n+1}\{l(y^*-t_{n+1}) + \alpha f^{N-1}_{n+2}(y^*-t_{n+1}, h_n o\, t_n o\, t_{n+1})\}]$$

(by the inductive assumption)

$$\geqq E_n \inf_{y' \geqq y-t_n} [\, c(y'-y+t_n) + E_{n+1}\{\, l(y'-t_{n+1}) + \alpha f^{N-1}_{n+1}(y'-t_{n+1},\; h_n o t_n o t_{n+1})\}\,]$$

$$= E_n f^N_{n+1}(y-t_n, h_n o t_n) \; . \quad \square$$

Lemma 4-3. $f_n^N(x,h_n) \leqq f^N(x,Q_n)$ for all N and n,

$f_n(x,h_n) \leqq f(x,Q_n)$ for all n.

Proof. It is proved by induction on N for all n. It is true for N=0 and all n. Assume it is true for N-1 and all n. We have by (4-9)

$$f^N(x,Q_n) = \inf_{y \geqq x}[c(y-x) + E_n\{l(y-t_n) + \alpha f^{N-1}(y-t_n,Q_n)\}]$$

$$\geqq \inf_{y \geqq x}[c(y-x) + E_n\{l(y-t_n) + \alpha f_{n+1}^{N-1}(y-t_n,h_n)\}]$$

(by Lemma 4-2)

$$= f_n^N(x,h_n) \ . \ \square$$

Note that the right hand sides of the equations in Lemma 4-3 are interpreted as the optimal expected costs when no additional informations are available after the history $h_n = (t_1,t_2,\ldots,t_{n-1})$ is experienced.

We have the following theorem which attains the lower and upper bounds for the Bayes solution of (4-4).

Theorem 4-2.

$$E_n f(x,F_\omega) \leqq f_n(x,h_n) \leqq f(x,Q_n) \quad \text{for all n.}$$

4.3 A linear Cost Case

To conclude this chapter we restrict our attention to the case of linear costs : $c(x)=cx$, $h(x)=hx$ and $p(x)=px$, and it is assumed that $p > c \geqq 0$, $c+h > 0$. So that, the Bayes solution $f_n(x,h_n)$ is uniformly continuous and bounded for x in any finite interval and the Bayesian inventory policy $y_n^o(x)$ exists which actually minimizes (3-4). Though the sequence $\{y_n^o(x)\}$ is itself quite difficult to be obtained analytically even in this case, it is known that the stationary and optimal inventory policy $y_n^*(x)$ for the equation (4-7) of $f(x,Q_n)$ has the simple form such that

$$(4\text{-}10) \quad y_n^*(x) = \max\,(x, t_n(q))\ ,$$

where $t_n(q)$ is the q-th quantile of Q_n and $q = \{p-c(1-\alpha)\}/(p+h)$ i.e., $t_n(q) \in Q_n^{-1}(q) = \{t\ ;\ Q_n(t)=q,\ t \in [0, \infty)\}$ if it is not empty, and $t_n(q)$ = the minimum t which satisfies $Q_n(t) > q$ if $Q_n^{-1}(q) = \emptyset$. Section 5-1 can be referred to about this fact. Moreover $f(x,Q_n)$ has been calculated by Scarf [30].

We therefore assume that $\int_0^\infty tF(dt) < \infty$ for all $F \in \Omega$ from the remark of Assumption 2-1 in case of linear costs, and that the marginal distribution Q_n converges weakly to the true demand distribution F_0 by the definition of the sample space (Ω, A). Then the following proposition holds from the Helly-Bray's theorem of the weak convergence of distribution functions.

Proposition 4-2.

$$\lim_{n\to\infty} E_n f(x,F_\omega) = \lim_{n\to\infty} f(x,h_n) = \lim_{n\to\infty} f(x,Q_n) = f(x,F_0).$$

It follows from the usual arguments of the inventory problem given by Scarf [30] that $f_n(x,h_n)$ is a convex function with respect to x, and also that its derivative with respect to x is no less than $-c$. Since the arg. minimum y of

$$\text{(4-11)} \quad cy + E_n\{\, l(y-t_n) + \alpha f_{n+1}(y-t_n, h_n \circ t_n)\,\}$$

is not greater than that of (4-11) when f_{n+1} is replaced by $-cx$, it yields the following results.

Theorem 4-3. In case of linear costs,

$$y_n^o(x) \leqq y_n^*(x) \qquad n=1,2,\ldots\ .$$

Moreover if the q-th quantile $t_0(q)$ of F_0 is unique, then

$$\lim_{n\to\infty} y_n^o(x) = \lim_{n\to\infty} y_n^*(x) = \max\,(x, t_0(q))\ .$$

Remark. Though the Bayes solution $f_n(x,h_n)$ and its inventory policy $y_n^o(x)$ for (4-4) are themselves quite difficult to derive, they can be approximated by $f(x,Q_n)$ and $y_n^*(x)$ for (4-7) respectively, which are asymptotically optimal from Proposition 4-2 and Theorem 4-3.

4.4 A Nonparametric Bayes Example

The following definition and theorem of the Dirichlet process are found in Ferguson [6], and their applications for the practical models are mentioned in Joe [14].

Definition. Let $\alpha(.)$ be a finite measure on $(R^+, \mathcal{B})$. We say P is a Dirichlet Process with parameter α and write P $\varepsilon\ \mathcal{D}(\alpha)$, if for every finite measurable partition $(B_1,\ldots,B_k)$ of R^+, the distribution of $(P(B_1),\ldots,P(B_k))$ is a Dirichlet distribution with parameter $(\alpha(B_1),\ldots,\alpha(B_k))$. In particular, $F(t) = P((-\infty,t])$ has a beta distribution with parameter $(\alpha(t),\alpha(R^+)-\alpha(t))$, where $\alpha(t) = \alpha((-\infty,t])$.

Theorem. If $F\ \varepsilon\ \mathcal{D}(\alpha)$, then the posterior distribution $\mathcal{P}_{n+1}$ for the history $h_{n+1} = (t_1,\ldots,t_n)$ is $\mathcal{D}(\alpha + \sum_{i=1}^{n}\delta(t_i))$, where $\delta(t_i)$ is the measure giving mass one to t_i.

We can apply Theorem to our problem.

The marginal distribution Q_{n+1} is given by

$$(4\text{-}12)\qquad Q_{n+1}(t) = \mathcal{E}_{n+1}F(t) = (1-p_n)Q_1(t) + p_n\hat{F}_n(t),$$

where $\hat{F}_n$ is the sample distribution and $p_n = n/(\alpha(R^+)+n))$.

If we choose that the initial guess of F i.e., $Q_1(t) = \alpha(t)/\alpha(R^+)$ is strictly increasing, the q-th quantile $t_n(q)$ of the marginal distribution Q_n is unique, and we can easily obtain $t_n(q)$ from the graph of Q_n.

It is known that the sample distribution $\hat{F}_n$ converges weakly to the true demand distribution F_0. And if $t_0(q)$ of F_0 is unique, the results of this chapter hold.

The asymptotic property of $t_n(q)$ is derived by the simple modification of a theorem in Wretman [36].

Theorem 4-4. If F_0 has a density $f_0(>0)$ at the neighborhood of $t_0(q)$, then $\sqrt{n}(t_n(q)-t_0(q))$ converges in law to Z as n increases infinitely, where Z has normal distribution with mean 0 and variance $q(1-q)/f_0^2(t_0(q))$.

CHAPTER 5

SADDLE POINTS IN INVENTORY PROBLEMS

In this chapter we shall mention the facts about the existence of saddle points for inventory problems with uncertainty on the demand distribution. When most of the informations about it are insufficient for a definite probability distribution, a minimax procedure is used as well as a Bayesian approach which has been explained in Chapter 4.

On the minimax inventory problem, one can refer the works done by Ben-Tal and Hochman [1], Jagannathan [12], [13], Kasugai and Kasegai [15], Nakagami [21], Odanaka [27] and Scarf [32]. A minimax policy is one that minimizes the maximum expected costs, where the maximum is taken over a prescribed class of distributions. For example, Scarf [32] assumed that only the mean and the variance of the distribution were known.

According to Nakagami and Yasuda [23], we consider this minimax inventory problem as a two-person zero-sum game, in which one player (manager) decides his ordering level and other player (nature) chooses her demand distribution in the prescribed class of distributions. By a game theoretic approach, we shall derive the necessary and sufficient conditions for the existence of saddle points and a saddle value.

5.1 Conditions for Saddle Points

Let us define the one-period expected cost function L as an inventory level y after ordering and a demand distribution F by the following:

$$(5\text{-}1) \qquad L(y,F) = cy + \int_{[0,\infty)} \{h(y-t)^+ + p(t-y)^+\}F(dt)$$

where $x^+ = \max(x,0)$. Here, c, h and p are the unit ordering cost, the unit holding cost and the unit penalty cost, respectively. And it is assumed that $p > c \geqq 0$, $c + h > 0$ and that the demand t is a non-negative random variable with the distribution F ($F(0-)=0$).

When F is known precisely, the value of y^o which minimizes (5-1) is easily solved. Let $F^{-1}(z) = \{y; F(y)=z,\ y \in [0,\infty)\}$, $0 \leqq z \leqq 1$, and $q = (p-c)/(p+h)$. Then

$$(5\text{-}2) \qquad \begin{cases} y^o \in F^{-1}(q) \quad \text{if } F^{-1}(q) \neq \emptyset, \quad \text{and} \\ y^o = \text{the minimum } y \text{ which satisfies } F(y) > q \\ \qquad \text{if } F^{-1}(q) = \emptyset. \end{cases}$$

In any specific problem, most of the informations of the demand distribution F are insufficient for a definite probability distribution. So it is natural to asssume that F is unknown but belongs to some class $\mathcal{F}$ of distributions, which is given by a prior information prescribed.

The minimax problem

$$(5\text{-}3) \qquad \inf_{y \geq 0}\ \sup_{F \in \mathcal{F}}\ L(y,F)$$

was considered for several class of distributions ([1],[12], [13],[15],[21],[27],[32]). In these literatures, a minimax value of L and a minimax policy of y in (5-3) were derived. It is naturally extended to consider a saddle point of this problem.

$$\text{(5-4)}\qquad \inf_{y \geqq 0} \sup_{F \varepsilon \mathcal{F}} L(y,F) = \sup_{F \varepsilon \mathcal{F}} \inf_{y \geqq 0} L(y,F) \; (= L(y^*,F^*) = L^*).$$

As we shall shown, all the classes $\mathcal{F}$ given by the above literatures are known to satisfy the conditions for the existence of saddle points. Hence, instead of deriving a minimax solution of (5-3), in which treatments F* does not come out, what we do in this chapter is to derive a saddle value L* and a set of saddle points (y*,F*) of (5-4) in the explicit form by solving the maximin problem. This is reduced by using (5-2) to the maximization problem

$$\text{(5-5)}\qquad \sup_{F \varepsilon \mathcal{F}} L(y^{o},F) \; .$$

The general treatments of a duality theorem are given in Isii [11].

For the class $\mathcal{F}$ of distributions, we assume that

$$\text{(a)}\qquad \sup_{F \varepsilon \mathcal{F}} \int_{[0,\infty)} tF(dt) < \infty \; .$$

This assumption (a) only makes L(y,F) finite for any fixed y to avoid a trivial case. The function L(y,F) is defined on $[0,\infty) \times \mathcal{F}$ with finite non-negative values, which is the same assumption given in Section 4.4 for the case of linear costs.

For any fixed $F \in \mathcal{F}$, it is seen that $L(y,F)$ is convex in y and $\lim_{|y| \to \infty} L(y,F) = \infty$. Then we may restrict the domain of y is compact and convex. Furthermore, we assume that

(b) $\mathcal{F}$ is convex,

i.e., if $F_1, F_2 \in \mathcal{F}$, then $bF_1 + (1-b)F_2 \in \mathcal{F}$ for any real number b $(0 \leqq b \leqq 1)$. The following lemmas are easily derived from the general minimax theorem in Ekaland and Terman [4].

Lemma 5-1. If the assumptions (a) and (b) are satisfied, then

$$\text{(5-6)} \qquad \min_y \sup_F L(y,F) = \sup_F \min_y L(y,F) \quad (= L^*),$$

and there exists a saddle value L^*. Moreover if we additionally assume that

(c) $\mathcal{F}$ is compact with respect to the Lévy metric,

then the function $L(y,F)$ possesses at least one saddle point (y^*,F^*) on $[0, \infty) \times \mathcal{F}$ and

$$\text{(5-7)} \qquad \min_y \max_F L(y,F) = \max_F \min_y L(y,F) = L(y^*,F^*) = L^*.$$

Lemma 5-2. If the assumptions (a), (b) and (c) are satisfied, then $(y^*,F^*) \in [0, \infty) \times \mathcal{F}$ is a saddle point of $L(y,F)$ if and only if

(i) $L'(y^*,F^*;y-y^*)$

$$= \begin{cases} (y-y^*)[c-p+(p+h)F^*(y^*)] \geqq 0 & \text{for any } y > y^*, \\ (y-y^*)[c-p+(p+h)F^*(y^*-)] \geqq 0 & \text{for any } y < y^*, \end{cases}$$

where L' is a Gâteaux-differential. That is,

$$\begin{cases} y^* = 0 \text{ and } F^*(0) \geqq (p-c)/(p+h) = q, \\ y^* > 0 \text{ and } F^*(y^*) \geqq q,\ F^*(y^*-) \leqq q. \end{cases}$$

(ii) $L(y^*,F^*) \geqq L(y^*,F)$ for any $F \in \mathcal{F}$.

From Lemma 5-2, we can see that F^* has more than q of probability on the interval $[0,y^*]$ and has less than $\tilde{q} = 1-q$ of probability on the interval (y^*,∞).

Now, we consider the next class $\mathcal{F}_\mu$ of distributions in a class $\mathcal{F}$ which satisfies assumptions (a), (b) and (c)

$$\mathcal{F}_\mu = \{ F \in \mathcal{F} \mid \int F(dt)=1,\ \int tF(dt)= \mu \}. \tag{5-8}$$

The equation (5-1) is reduced to as follows by the condition (i) of Lemma 5-2.

$$\begin{aligned} L^* &= (p+h) [y^* \{ F^*(y^*)-(p-c)/(p+h)\} \\ &\quad - \int_{[0,y^*]} tF^*(dt)] + p\mu \\ &= p\mu - (p+h) \int_{[0,y^*\}} tF^*(dt), \end{aligned}$$

where $\int_{[0,y^*\}} F^*(dt)$ represents the integral from 0 to y^* until just q of probability with respect to the distributon F^*. Since a saddle value L^* is given by (5-5), the following theorem holds.

Theorem 5-1.

(5-9) $$L^* = p\mu - (p+h) \min_{F \varepsilon F_\mu} \int_{[0,y^o\}} tF(dt),$$

where

$$\begin{cases} y^o = 0 & \text{if } F(0) \geqq q, \\ y^o \varepsilon F^{-1}(q) & \text{if } F(0) < q \text{ and } F^{-1}(q) \neq \emptyset, \\ y^o = \text{the minimum } y \text{ which satisfies } F(y) > q \\ \qquad \text{if } F(0) < q \text{ and } F^{-1}(q) = \emptyset. \end{cases}$$

Now we can divide the theorem into the two cases.

Corollary 5-1. If there is a distribution F with $F(0) \geqq q$ in the class F_μ, then

(5-10) $$L^* = p\mu, \quad y^* = 0.$$

Corollary 5-2. If all the distributions F in the class F_μ are satisfied with $F(0) < q$, then there exists a unique y_F ($y_F > 0$) such that $\int_{[0,y^o\}} tF(dt) = qy_F$. And

(5-11) $$L^* = p\mu - (p-c) \min_{F \varepsilon F_\mu} y_F .$$

Hence, this case reduces to the problem which determines the minimum value of y_F by other conditions of the class F_μ .

5.2 Examples for Saddle Points

The examples mentioned in this section can be easily checked to satisfy assumptions (a), (b) and (c). Some of the examples were treated as a minimax problem (5-3) by the literatures, in which cases L* and y* have been derived on the guarantee of the existence of saddle points, but F* does not come out. We, therefore, derive an L* and a set of (y*,F*) from Corollary 5-1, or Corollary 5-2 and other conditions of the class $\mathcal{F}_\mu$.

Example 5-1. The class $\mathcal{F}(\mu)$ of distribution of which only the mean μ is assumed to be known.

$$\mathcal{F}(\mu) = \{ \int F(dt)=1, \int tF(dt)=\mu \} \quad (\mu > 0) .$$

The following results are immediately seen from Corollary 5-1.

$$(5\text{-}12) \qquad \begin{cases} F^*(t) = q + \tilde{q}G(t) & (t \geqq 0), \\ y^* = 0, \quad L^* = p\mu , \end{cases}$$

where $\tilde{q} = (1-q)$ and $G(t)$ is an arbitary distribution with $G(0-) = 0$ such that $\int tG(dt) = \mu/\tilde{q}$.

Example 5-2. (Jagannathan [12],[13], Odanaka [27], Scarf [32]). The class $\mathcal{F}(\mu,\sigma)$ of distributions of which only the mean μ and the variance σ^2 are assumed to be known.

$$\mathcal{F}(\mu,\sigma) = \{ \int F(dt)=1, \int tF(dt)= \mu , \int t^2F(dt)= \mu^2+ \sigma^2 \} \quad (\mu, \sigma > 0).$$

It is noted that the distribution G which has a minimum variance in (5-12) is one that degenerates at a point $t = \mu/\tilde{q}$, and F^* in (5-12) satisfies $\mu^2/(\mu^2+\sigma^2) \leqq \tilde{q}$. The analysis of this example is divided into the following two cases according to Corollary 5-1 and 5-2.

Case 1 : $\mu^2/(\mu^2+\sigma^2) \leqq \tilde{q}$.

$$(5\text{-}13) \quad \begin{cases} F^*(t) = q + \tilde{q}G(t) & (t \geqq 0), \\ y^* = 0, \quad L^* = p\mu , \end{cases}$$

where G(t) is an arbitary distribution with $G(0-) = 0$ such that $\int tG(dt) = \mu/\tilde{q}$, $\int t^2 G(dt) = (\mu^2+\sigma^2)/\tilde{q}$.

Case 2 : $\mu^2/(\mu^2+\sigma^2) > \tilde{q}$.

From the definition of y_F in Corollary 5-2, the following Schwartz inequalities hold.

$$\left(\int_{[0,y^o]} tF(dt)\right)^2 \leqq \left(\int_{[0,y^o]} F(dt)\right)\left(\int_{[0,y^o]} t^2F(dt)\right),$$

$$\left(\int_{\{y^o,\infty)} tF(dt)\right)^2 \leqq \left(\int_{\{y^o,\infty)} F(dt)\right)\left(\int_{\{y^o,\infty)} t^2F(dt)\right).$$

After some simple calculations,

$$\mu - \sigma\sqrt{\tilde{q}/q} \leqq y_F \leqq \mu + \sigma\sqrt{\tilde{q}/q} .$$

Then,

$$\min_{F\varepsilon \mathcal{F}(\mu,\sigma)} y_F = \mu - \sigma\sqrt{\tilde{q}/q} \quad (> 0),$$

and

$$(5\text{-}14) \quad L^* = c\mu + \sigma\sqrt{(c+h)(p-c)} .$$

Since the equalities hold in the Schwartz inequalities, F* is restricted to the following two point distribution.

(5-15) $\quad$ F* has mass q at $\underline{y} = \mu - \sigma\sqrt{\tilde{q}/q}$

and $\quad$ mass $\tilde{q}$ at $\tilde{y} = \mu + \sigma\sqrt{q/\tilde{q}}$.

Then y* must be contained in the interval $[\underline{y},\tilde{y})$ because of $L(y,F^*) = L^*$ for any $y \in [\underline{y},\tilde{y})$, and each $y \in [\underline{y},\tilde{y})$ is a candidate for y* if $L(y,F^*) \geqq L(y,F)$ for some F in $\mathcal{F}(\mu,\sigma)$ by the condition (ii) in Lemma 5-2.

Now, let us consider the two-point distribution F which is shifted to the left (or right) from F*, such that F has mass r at $\mu - \sigma\sqrt{\tilde{r}/r}$ and mass $\tilde{r}$ at $\mu + \sigma\sqrt{r/\tilde{r}}$ where $r < q$ (or $r > q$). The simple calculations concerning $L(y,F^*) \geqq L(y,F)$ for $y = \mu + \sigma x \in [\underline{y},\tilde{y})$ yield that when r tends to q from the left (or right),

(5-16) $\quad x \leqq (\text{or } \geqq) \left.\frac{d}{dr}(-\sqrt{r\tilde{r}}\,)\right|_{r=q}$.

By Lemma 5-1, there exists at least one saddle point. Therefore, from (5-16),

(5-17) $$y^* = \mu + \sigma\,\frac{(p-c) - (c+h)}{2\sqrt{(p-c)(c+h)}} .$$

Example 5-3. (Ben-Tal and Hochman [1]) The class $\mathcal{F}(\mu,\delta)$ of distributions of which only the mean μ and the mean absolutedeviation δ are assumed to be known.

$$F(\mu, \delta) = \{ \int F(dt)=1, \int tF(dt)=\mu, \int |t-\mu| F(dt)=\delta \}$$

$$(1 > \delta/2\mu \text{ and } \mu, \delta > 0).$$

The analysis of this example is divided into the following two cases according to Corollary 5-1 and 5-2.

Case 1 : $1-\delta/2\mu \leqq \tilde{q}$.

In this case F^* is in (5-12). Then

$$(5\text{-}18) \quad \begin{cases} F^*(t) = q + \tilde{q}G(t) \quad (t \geqq 0), \\ y^* = 0, \quad L^* = p\mu, \end{cases}$$

where $G(t)$ is an arbitary distribution with $G(0-) = 0$ such that $\int tG(dt) = \mu/\tilde{q}$, $\int |t-\mu| G(dt) = \mu + (\delta - \mu)/\tilde{q}$.

Case 2 : $1-\delta/2\mu > \tilde{q}$.

From the definition of y_F in Corollary 5-2, the following inequality holds.

$$\int_{[0,y^o\}}(\mu - t)F(dt) = q(\mu - y_F)$$

$$\leqq \int_{[0,\mu]}(\mu - t)F(dt) = \delta/2,$$

$$y_F \geqq \mu - \delta/2q \ (> 0).$$

The equality holds if and only if

$$\int_{[0,y^o\}}(\mu - t)F^*(dt) = \int_{[0,\mu]}(\mu - t)F^*(dt) .$$

Therefore,

$$(5\text{-}19)\quad \begin{cases} F^*(t) = qG_1(t) + \tilde{q}G_2(t), \\ y^* = \mu, \quad L^* = c\mu + (p+h)\,\delta/2, \end{cases}$$

where $G_1(t)$ is an arbitary distribution on $[0,\mu]$ such that $\int_{[0,\mu]} tG_1(dt) = \mu - \delta/2q$ and $G_2(t)$ ia an arbitrary distribution on (μ,∞) such that $\int_{(\mu,\infty)} tG_2(dt) = \mu + \delta/2\tilde{q}$.

Example 5-4. The class $F(\mu,M)$ of distributions of which only the mean μ and the domain $[0,M]$ are assumed to be known. Kasugai and Kasegai [15] and Nakagami [21] assumed the domain only, but they treated more practical dynamic inventory problems and derived the explicit forms of the minimax policies. Let us consider

$$F(\mu,M) = \{\int_{[0,M]} F(dt)=1,\ \int tF(dt)=\mu\} \quad (M > \mu > 0).$$

The analysis of this example is divided into the following two cases according to Corollary 5-1 and 5-2.

Case 1 : $\mu/M \leqq \tilde{q}$.

In this case F^* is in (5-12). Then

$$(5\text{-}20)\quad \begin{cases} F^*(t) = q + \tilde{q}G(t) \quad (t \geqq 0), \\ y^* = 0, \quad L^* = p\mu, \end{cases}$$

where $G(t)$ is an arbitary distribution on $[0,M]$ such that $\int_{[0,M]} tG(dt) = \mu/\tilde{q}$.

Case 2 : $\mu/M > \tilde{q}$.

From the definition of y_F in Corollary 5-2, the following inequalities hold.

$$\int_{[0,y^o]} tF(dt) = qy_F, \quad \int_{\{y^o,M]} tF(dt) \leqq \tilde{q}M.$$

$$y_F \geqq (\mu - \tilde{q}M)/q \quad (> 0).$$

The equality holds if and only if F* has mass $\tilde{q}$ at a point M. Therefore,

$$(5\text{-}21) \qquad \begin{cases} F^*(t) = qG(t) + \tilde{q}I_{[M,\infty)}(t), \\ y^* = M, \quad L^* = cM + h(M-\mu), \end{cases}$$

where $I_{[\ )}(t)$ is the indicator function and $G(t)$ is an arbitary distribution on $[0,M]$ such that $\int_{[0,M]} tG(dt) = (\mu - \tilde{q}M)/q$.

5.3 The Multi-Period Model

In this section, the demands in successive periods are assumed to form a sequence of random variables whose distribution are contained in $\mathcal{F}_\mu$ and can change from period to period. The multi-periods model was treated as a minimax problem in Jagannathan [13], Kasugai and Kasegai [15], Nakagami [21] and Odanaka [27].

As usual, we define that $f^N(x, F_\mu)$ is the discounted saddle valued costs over N periods as a function of the level x of inventory before ordering and a prescribed class F_μ of distributions. Then the following theorem obviously holds by induction in applying Lemma 5-1.

Theorem 5-2. $f^N(x, F_\mu)$ satisfies the following equation

$$(5\text{-}22)\quad f^N(x, F_\mu) = \operatorname*{value}_{y \geqq x, F \varepsilon F_\mu} [\, c(y-x) + \int_{[0,\infty)} \{ h(y-t)^+ + p(t-y)^+ + \alpha f^{N-1}(y-t, F_\mu) \} F(dt)],$$

where $0 \leqq \alpha < 1$ and $f^0(x, F_\mu) = -cx$.

We have assumed $f^0(x, F_\mu) = -cx$ followd by Veinott [34], because a myopic policy for the multi-periods inventory problem is optimal when the demand distribution is known.

Now, we calculate $f^N(x, F_\mu)$ in this myopic case. Let us define

$$(5\text{-}23)\quad \tilde{L}(y,F) = (1-\alpha)cy + \int_{[0,\infty)} \{h(y-t)^+ + p(t-y)^+\}F(dt)$$

similarly as (5-1). Then, (5-22) is reformed for N = 1 as

$$f^1(x, F_\mu) = \operatorname*{value}_{y \geqq x, F \varepsilon F_\mu} [\, \tilde{L}(y,F)] - cx + \alpha c\mu \,.$$

If F_μ is one of four examples in Section 5-4, a saddle value $\tilde{L}^*$ and $\tilde{y}^*$ of a set of saddle points $(\tilde{y}^*, \tilde{F}^*)$ of (5-23) are unique and explicitly derived. Clearly,

$$f^1(x, F_\mu) = \tilde{L}^* + \alpha c\mu - cx \qquad \text{for } x \leqq \tilde{y}^*.$$

So it holds by induction that for all $N \geqq 1$.

(5-24) $f^N(x, F_\mu) = (1+\alpha+\ldots+\alpha^{N-1})(\tilde{L}^*+\alpha c\mu)$ for $x \leqq \tilde{y}^*$.

Hence, the following theorem is established.

Theorem 5-3. For each class F_μ in Section 5-2, if the level x of inventory before ordering at period N is less than $\tilde{y}^*$, a set of saddle points for the N-periods problem (5-22) is $(\tilde{y}^*, \tilde{F}^*)$. That is, the inventory policy $\tilde{y}^*$ is stationary and myopic, and the discounted saddle valued costs $f^N(x, F_\mu)$ over N periods is given by (5-24).

Remark. If the level x of inventory before ordering is larger than $\tilde{y}^*$, the inventory policy $\tilde{y}^*$ is not feasible in the equation (5-22). Namely, a set of saddle point and a saddle value are not the same as $(\tilde{y}^*, \tilde{F}^*)$ and (5-24), and the inductive argument can not be used to construct the similar one of Theorem 5-3.

In case of $N = 1$, Scarf [32] calculates $f^1(x, F_\mu)$ for all value of x when $F_\mu = F(\mu, \sigma)$, where $f^1(x, F_\mu)$ is not linear for $x > \tilde{y}^*$.

ACKNOWLEDGMENT

The author would like to express his appreciation to Professor Minoru Sakaguchi of Osaka University, for suggesting the topic of this dissertation and providing constant guidance and encouragement during his researches.

The author also wishes to express his appreciation to Assistant Professors Masami Kurano and Masami Yasuda of Chiba University, for their valuable advices and discussions with warm friendships.

REFERENCES

1. Ben-Tal, A. and Hochman, E., "Stochastic Programs with Incomplete Information," Operations Reseach, Vol. 24, (1976), pp. 336-347.
2. Bertsekas, D. P., "Stochastic Optimization Problems with Nondifferentiable Cost Functions," J. Optimization Theory and Applications," Vol. 12, (1973), pp.218-231.
3. Boylan, E. S., "Existence and Uniqueness Theorem for the Optimal Inventory Equation: The Backloggimg Case," J. Math. Analysis and Applications, Vol. 57, (1977), pp. 625-637.
4. Ekaland, I. and Terman, R., Convex Analysis and Variational Problems, North-Holland, (1976).
5. Élsgol'c, L. E., Qualitative Methods in Mathematical Analysis, Trans. by A. A. Brown and J. M. Danskin, American Mathematical Society, Providence, 1964.
6. Ferguson, T. S., "A Bayesian Analysis of Some Nonparametric Problems," Annals Statistics, Vol. 1, (1973), pp. 209-230.
7. Fries, B. E., "Optimal Ordering Policies for a Perishable Commodity with Fixed Life Time," Operations Reseach, Vol. 23, (1975), pp. 46-61.
8. Iglehart, D. L., "Dynamic Programming and Stationary Analysis of Inventory Problems," in Multistage Inventory Model and Techniques, Stanf. Univ. Press, (1963).

9. Iglehart, D. L., "Optimality of (s,S) Policies in the Infinite Horizon Dynamic Inventory Problem," Management Science, Vol. 9, (1963), pp.259-267.

10. Iglehart. D. L., "The Dynamic Inventory Problem with Unknown Demand Distribution," Management Science, Vol. 10, (1964), pp. 429-440.

11. Isii, K., "Inequalities of the Types of Chebyshev and Cramér-Rao and Mathematical Programming," Ann. Inst. Statist. Math., Vol. 16, (1964), pp. 277-293.

12. Jagannathan, R., "Minimax Procedure for a Class of Linear Programs under Uncertainity," Operations Research, Vol. 25, (1977), pp. 173-177.

13. Jagannathan, R., "A Minimax Ordering for the Infinite Stage Dynamic Inventory Problem," Management Science, Vol. 24, (1978), pp. 1138-1149.

14. Joe, L., "Bayesian Nonparametric Policies in Markov Decision Models," Ph. D. Dissertation, Engineering, UCLA, (1978).

15. Kasugai, H. and Kasegai, T., "Note on Minimax Regret Ordering Policy : Static and Dynamic Solutions and a Comparizon to Maximin Policy :," J. Operations Research Soc. of Japan, Vol. 3, (1960), PP. 155-169.

16. Lippman, S. L., "On Dynamic Programming with Unbounded Rewards," Management Science, Vol. 21, (1975), pp. 1225-1233.

17. Nahmias, S. and Pierskalla, W. P., "Optimal Ordering Policies for a Product That Perishes in Two Periods Subject to Stochastic Demand," Naval Research Logistics Quarterly, Vol. 20, (1973), pp. 207-229.

18. Nahmias, S., Optimal Ordering Policies for a Perishable Inventory - II," Operations Research, Vol. 23, (1975), pp. 735-749.

19. Nahmias, S. and Pierskalla, W. P., "A Two-Product Perishable / Nonperishable Inventory Problen," SIAM J. Appl. Math., Vol. 30, (1976), 483-500.

20. Nakagami, J., "Optimal Batch (s,S) Policies for the Multiple Set-Up Ordering Costs in Inventory Problem," J. Operations Research Soc. of Japan, Vol. 19, (1976), pp. 91-98.

21. Nakagami, J., "Minimax Policies for the Production of Seasonal Style-Goods," J. Operations Research Soc. of Japan, Vol. 19, (1976), pp. 117-124.

22. Nakagami, J., "Perishable Inventory Problem with Two Types of Warehouses," J. Operations Research Soc. of Japan, Vol. 22, (1979), pp. 29-40.

23. Nakagami, J. and M. Yasuda, "A Saddle Point of an Inventory Problem," J. Information & Optimization Sciences, Vol. 2, (1981), pp. 181-191.

24. Nakagami, J., "A Contraction Theorem in Inventory Problems," J. Information & Optimization Sciences, Vol. 3, (1982), pp. 125-132.

25. Nakagami, J., "Existence Theorems and Bayesian Approaches in Inventory Problems," in Proceedings of 7. Symposium über Operations Research, Athenäum Printers, Germany, (1982).

26. Nakagami, J., "A Bayesian Approach in Inventory Problems," in Submission.

27. Odanaka, T., "Analysis and Computational Studies of Optimal Inventory Process - III," USC Tech. Report. USCEE, (1971), pp. 111-123.

28. Porteus, E. L., "On the Optimality of Generalized (s,S) Policies," Management Science, Vol. 17, (1971), pp. 411-426.

29. Rieder, U., "Bayesian Dynamic Programming," Adv. Appl. Prob., Vol. 7, (1975), pp. 330-348.

30. Scarf, H., "Bayes Solutions of the Statistical Inventory Problem," Annals of Math. Statistics, Vol. 30, (1959), pp. 490-508.

31. Scarf, H., " The Optimality of (S,s) Policies in a Dynamic Inventory Problem," in Mathematical Methods in the Social Siences, Stanf. Univ. Press, (1960).

32. Scarf, H., "A minimax Solution of an Inventory Problem," in Studies in Mathematical Theory and Inventory, Stanf. Univ. Press, (1963).

33. Van Nunen, J. A. E. E. and J. Wessels, "A Note on Dynamic Programming with Unbounded Rewards," Management Science, Vol. 24, (1978), pp. 576-580.

34. Veinott Jr, A. F., "Optimal Policy for a Multi-Product Dynamic Nonstationary Inventory Problem," Management Science, Vol. 12, (1965), pp. 206-277.

35. Veinott Jr, A. F., " On the Optimality of (s,S) Inventory Policies: New Conditions and a New Proof," J. SIAM Appl. Math., Vol. 14, (1966), pp. 1067-1083.

36. Wretman, J., "A Simple Derivation of the Asymptotic Distribution of a Sample Quantile," Scand. J. Statistics, Vol. 5, (1978), pp. 123-124.