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A PROBABILISTIC CONSTRUCTION OF THE HEAT KERNEL FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON A STRONGLY PSEUDOCONVEX SIEGEL DOMAIN

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1. Introduction

In this paper, we consider the heat equation for the $\bar{\partial}$-Neumann problem. This is an initial boundary value problem whose boundary condition includes *an imaginary directional differentiation.* In [7], Mallivain constructed the solution of the heat equation on a domain by using a method of singular perturbations and pointed out that a method related to the Fourier transform can be applied to this problem. On the other hand, a strongly pseudoconvex Siegel domain is well known as one of the most fundamental complex manifolds with boundary. This domain D can be regarded as the product of a Heisenberg group H_n and $\boldsymbol{R}^+$: $D=H_n\times\boldsymbol{R}^+$. In [4], Gaveau constructed explicitly the heat kernel for Kohn's Laplacian on H_n, by combining a probabilistic method and the Fourier transform. In this paper, by referring their works, we construct the heat kernel for the $\bar{\partial}$-Neumann problem on the Siegel domain D explicitly in terms of the theory of generalized Wiener functionals by Watanabe [13]. For the heat kernel on this domain, Stanton gave an explicit formula in the $(0, q)$-form case ($q>0$), by using methods of the partial differential equations [10], [11]. We here consider the general (p, q)-form case. The main part of our discussion is the proof of well-definedness of the heat kernel. In [12], our main results (Theorems 2.1 and 2.2 below) were announced.

We briefly explain our methods. The equation we consider is the following:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}F(t,X)=-\Box F(t,X), \quad t>0,\ X\in D\,, \\ \lim\limits_{t\to 0}F(t,X)=f(X)\in\Lambda_0^{p,q}(D), \text{ uniformly on } \bar{D}\,, \\ PF(t,X)=Q\left(\dfrac{\partial}{\partial r}-i\dfrac{\partial}{\partial u}\right)F(t,X)=0 \text{ on } bD\,, \end{cases}$$

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where

$$\square = -\frac{1}{4}\sum_{j=1}^{2n} X_j^2 - \frac{1}{2}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial r^2}\right)+i(n-2C)\frac{\partial}{\partial u}-\sum_{j=1}^{2n}\frac{A_j}{\sqrt{2}}X_j-B\,.$$

For the notations see Section 2 below. Except for the term $i\partial/\partial u$ in the boundary condition this equation has the same form as the d-Neumann boundary condition case that Airault [1], Ikeda and Watanabe [5] (see also [9]) considered. Hence, as in the d-Neumann case, we may expect the heat kernel of (1.1) can be expressed as

$$h_t(X, X') = E[m(t)\,M(t)\,K(t)\,\delta_{X'}(X(t),\, U(t)-i\phi(t),\, R(t))] \tag{1.2}$$

where $(X(t), U(t), R(t))$ is the diffusion process generated by $1/4\sum_{j=1}^{2n} X_j^2+1/2\,(\partial^2/\partial u^2+\partial^2/\partial r^2)$, $\phi(t)$ is the local time of the diffusion $\{R(t), t\geqq 0\}$ at 0 and $\delta_{X'}$ is the Dirac δ function on D. $m(t)$ and $M(t)$ are operator valued functionals for $-i(n-2C)\,\partial/\partial u$ and $\sum_{j=1}^{2n} A_j/\sqrt{2}X_j+B$, respectively (see (2.12) and (3.4)). $K(t)=I-PI\{\min_{0\leqq s\leqq t} R(s)=0\}$ is the functional for the boundary condition. However the process $(X(t), U(t)-i\phi(t), R(t))$ does not lie on D. Hence we do not know how to give a mathematical sense to the right hand side of (1.2).

Now we note that the metric, the differential structure and the complex structure of our case are *constant in the direction that we must consider the imaginary directional differentiation.* By using these facts, we can use the independence of the above processes and Fourier transform effectively. Then we can formally rewrite the expression (1.2) as

$$\begin{aligned} &h_t((x,u,r),(x',u',r')) \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{i\lambda(u'-u)}\,E[e^{-i\lambda u(t)}\,m(t)] \\ &\qquad \times \theta_t^\lambda(x,x')\,E[e^{-\lambda\phi(t)}\,K(t)\,|\,R(t)=r']\,r_t(r,r') \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} &u(t) = U(t)-u-S_x(t)\,,\\ &\theta_t^\lambda(x,x') = E[e^{-i\lambda S_x(t)}\,M(t)\,\delta_{x'}(X(t))]\,,\\ &S_x(t) = \sum_{j=1}^{n} 2\int_0^t (X^{n+j}(s)\circ dX^j(s)-X^j(s)\circ dX^{n+j}(s)) \end{aligned}$$

and $r_t(r,r')$ is the transition probability function of $R(t)$. We prove that this is the heat kernel for (1.1). In $(0,q)$-form case, $M(t)=I$ and $\theta_t^\lambda(x,x')$ is rewritten more explicitly ((2.18) below). However, in (p,q)-form case, we can not rewite $\theta_t^\lambda(x,x')$ explicitly as in $(0,q)$-form case. Then it is difficult to show the exponential decay of $\theta_t^\lambda(x,x')$ in λ, which is necessary to show the well-definedness of the right hand side of (1.3). Now we note that $\theta_t^\lambda(x,x')$ is, as a function of (t,x,x'), the heat kernel for the operator $\square_\lambda^0$ defined by (5.26) below.

By using the semigroup property of this heat kernel, we can prove the exponential decay of $\theta_t^{\lambda}(x, x')$ which is sufficient to prove the convergence of the right hand side of (1.3). This decay of $\theta_t^{\lambda}(x, x')$ is an interesting property of the oscillatory integral over the Wiener space. Furthermore, by using the Malliavin calculus, we can prove the smoothness of (1.3). We note that we need not the partial Malliavin calculus used for the boundary value problem (e.g., [9]), because of the independence of the processes.

The organization of this paper is as follows. In Section 2, we first formulate the problem by following Stanton [10] and then state our main theorem. In Section 3, we give a probabilistic explanation of our formula (2.15) of the heat kernel; we give a few lemmas by which we expect the formula (2.15). In Section 4, we prove Theorem 2.2 below. This theorem plays a crucial role in this paper. In Section 5, we prove our main theorem, Theorem 2.1. In Section 6, we consider the short time asymptotic behavior of the heat kernel on the diagonal. Finally, in Section 7, we generalize the above results to certain domains with nondegenerate indefinite Levi forms.

2. Preliminaries and Main theorems

In this section, first of all, following Stanton [10], we review the formulation of the heat equation on a strongly pseudoconvex Siegel domain. The strongly pseudoconvex Siegel domain is defined by

$$D = \{(z, w) \in \boldsymbol{C}^n \times \boldsymbol{C}, \operatorname{Im} w > |z|^2\} .$$

We consider the Hermitian metric for which $\{Z_1, Z_2, \cdots, Z_{n+1}\}$ is an orthonormal basis of $T^{(1,0)}(D)$, where

$$Z_j = \frac{\partial}{\partial z^j} + 2i\bar{z}^j \frac{\partial}{\partial w}, \quad j = 1, 2, \cdots, n, \quad Z_{n+1} = i\sqrt{2}\frac{\partial}{\partial w}. \tag{2.1}$$

We note that the volume form for the metric is the restriction of 2^n times the standard Euclidean volume element on $\boldsymbol{C}^{n+1}$. The differential forms given by

$$\omega^j = dz^j, \quad \omega^{n+1} = \frac{1}{i\sqrt{2}}(dw - 2i \sum \bar{z}^j dz^j) \tag{2.2}$$

form the dual basis for $T^*_{(1,0)}(D)$.

We now introduce several spaces of differential forms and several operators. Let $\Lambda_0^{p,q}(D)$ be the space of C^∞ (p, q) forms with compact support in D. Let $\mathcal{S}^{p,q}(\bar{D})$ be the space of (p, q) forms on D whose coefficients relative to $\omega^I \wedge \bar{\omega}^J$ can be extended to rapidly decreasing functions on $\boldsymbol{C}^{n+1}$ and $L_2^{p,q}(D)$ that of square integrable (p, q) forms on D. Then $\bar{\partial}$ maps $\mathcal{S}^{p,q}(\bar{D})$ to $\mathcal{S}^{p,q+1}(\bar{D})$ and has a smallest closed extention to $L_2^{p,q}(D)$. For simplicity we also denote by $\bar{\partial}$ its closed extension. Let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$ on $L_2^{(p,q)}(D)$. We define the $\bar{\partial}$-Laplacian

$\square$ by

$$
\begin{aligned}
\square &= \bar\partial\bar\partial^* + \bar\partial^*\bar\partial\,, \\
\text{Dom}\,\square &= \{f\in L_2^{p,q}(D);\, f\in \text{Dom}\,\bar\partial\cap \text{Dom}\,\bar\partial^*\,, \\
&\qquad \bar\partial f\in \text{Dom}\,\bar\partial^*,\ \bar\partial^* f\in \text{Dom}\,\bar\partial\}\,.
\end{aligned} \tag{2.3}
$$

Then the operator $\square$ is a positive selfadjoint operator on $L_2^{p,q}(D)$ (Folland-Kohn [3] Proposition 1.3.8).

To express the above operators explicitly, we prepare several notations. For $\omega\in T^*D\otimes \boldsymbol{C}$, let ext($\omega$) be the exterior multiplication, i.e.,

$$
\text{ext}(\omega)\,\eta = \omega\wedge\eta \quad \text{for} \quad \eta\in\Lambda(T^*D\otimes\boldsymbol{C})\,,
$$

and int($\bar\omega$) be interior multiplication, i.e., the dual operator of ext(ω). This int(ω) is complex linear in ω. We note that in some literature, e.g., Folland-Kohn [3], int(ω) is conjugate linear in ω. The condition for $f\in\mathcal{S}^{p,q}(\bar D)$ to be in Dom $\bar\partial^*$ is

$$
\text{int}(\omega^{n+1})f\restriction_{bD} = 0\,,
$$

where bD is the boundary of the domain D. Thus the condition for $f\in\mathcal{S}^{p,q}(\bar D)$ to be in Dom $\square$ are

$$
\text{int}(\omega^{n+1})f\restriction_{bD} = 0\,,\quad \text{int}(\omega^{n+1})\,\bar\partial f\restriction_{bD} = 0\,. \tag{2.4}
$$

These mean the $\bar\partial$-Neumann boundary conditions. For a differential operator A, we define $\underline{A}$ to be the operator acting only on the coefficients of $\omega^I\wedge\bar\omega^J$, i.e.,

$$
\underline{A}(\sum f_{IJ}\,\omega^I\wedge\bar\omega^J) = \sum (Af_{IJ})\,\omega^I\wedge\bar\omega^J\,.
$$

Since $\bar\partial\omega^j=\bar\partial\bar\omega^j=\bar\partial\bar\omega^{n+1}=0$ for $j=1, 2, \cdots, n$, and

$$
\bar\partial\omega^{n+1} = \sqrt{2}\sum_{j=1}^{n}\omega^j\wedge\bar\omega^j\,,
$$

in terms of the above notations, $\bar\partial$ and $\bar\partial^*$ can be represented as follows:

$$
\begin{aligned}
\bar\partial &= \sum_{j=1}^{n+1}\text{ext}(\bar\omega^j)\,\underline{\bar Z}_j + \sqrt{2}\sum_{j=1}^{n}\text{ext}(\omega^j)\,\text{ext}(\bar\omega^j)\,\text{int}(\bar\omega^{n+1})\,, \\
\bar\partial^* &= -\sum_{j=1}^{n+1}\text{int}(\omega^j)\,\underline{Z}_j + \sqrt{2}\sum_{j=1}^{n}\text{ext}(\omega^{n+1})\,\text{int}(\omega^j)\,\text{int}(\bar\omega^j)\,,
\end{aligned} \tag{2.5}
$$

on $\mathcal{S}^{p,q}(\bar D)$.

We use as coordinates on D (z, u, r) where $u=\text{Re}\,w$ and $r=\text{Im}\,w-|z|^2$ for $w\in\boldsymbol{C}$. Then we can regard D as the product of the boundary bD and $\boldsymbol{R}^+$. We identify bD with the Heisenberg group H_n. The group law on H_n is

$$
(z', u')\,(z, u) = (z'+z,\, u'+u+2\sum_{j=1}^{n}(x'^{n+j}\,x^j - x'^j\,x^{n+j})) \quad \text{for } (z', u'), (z, u)\in H_n\,, \tag{2.6}
$$

where $z^j = x^j + ix^{n+j}$. The metric we gave is the product metric of an invariant metric on H_n and the standard metric on $\boldsymbol{R}^+$. In terms of these coordinates,

$$(2.7)\qquad Z_j = \frac{\partial}{\partial z^j} + i\bar{z}^j \frac{\partial}{\partial u}, \quad j = 1, 2, \cdots, n, \quad Z_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r} + i\frac{\partial}{\partial u}\right).$$

Then we have

$$Z_j = 1/2\,(X_j - iX_{n+j})$$

where

$$(2.8)\qquad X_j = \frac{\partial}{\partial x^j} + 2x^{n+j}\frac{\partial}{\partial u}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - 2x^j \frac{\partial}{\partial u}.$$

We also define the projections P and Q by

$$P = \operatorname{ext}(\varpi^{n+1})\operatorname{int}(\omega^{n+1}), \quad Q = \operatorname{int}(\omega^{n+1})\operatorname{ext}(\varpi^{n+1}).$$

Then the $\bar{\partial}$-Neumann boundary condition (2.6) is rewritten as follows:

$$(2.9)\qquad Pf\restriction_{bD} = Q\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial u}\right)f\restriction_{bD} = 0$$

for $f \in \mathcal{S}^{p,q}(\bar{D})$. Now we see that this condition is similar to the absolute boundary condition except the term $i\partial/\partial u$ (cf. Ikeda-Watanabe [5]).

The operator $\square$ is expressed on $\mathcal{S}^{p,q}(\bar{D})$ as follows:

$$(2.10)\qquad \square = -\frac{1}{4}\sum_{j=1}^{2n} X_j^2 - \frac{1}{2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2}\right) + i(n-2C)\frac{\partial}{\partial u} - \sum_{j=1}^{2n}\frac{A_j}{\sqrt{2}}X_j - B$$

where

$$\begin{aligned}
C &= qQ + (q-1)P, \\
A_j &= \operatorname{ext}(\omega^{n+1})\operatorname{int}(\varpi^j) - \operatorname{ext}(\omega^j)\operatorname{int}(\varpi^{n+1}),\ j = 1, 2, \cdots, n, \\
A_{n+j} &= i(\operatorname{ext}(\omega^{n+1})\operatorname{int}(\varpi^j) + \operatorname{ext}(\omega^j)\operatorname{int}(\varpi^{n+1})),\ j = 1, 2, \cdots, n, \\
B &= -2\sum_{j,k=1}^{n}\operatorname{ext}(\omega^j)\operatorname{ext}(\varpi^j)\operatorname{int}(\omega^k)\operatorname{int}(\varpi^k) \\
&\quad -2\sum_{j=1}^{n}(\operatorname{int}(\omega^j)\operatorname{ext}(\varpi^j) - \operatorname{ext}(\omega^j)\operatorname{int}(\varpi^j))\operatorname{ext}(\omega^{n+1})\operatorname{int}(\varpi^{n+1}).
\end{aligned}$$

REMARK. 2.1. If $p=0$, $A_j = A_{n+j} = B = 0$ and $\square$ acts diagonally on $(0, q)$ forms. If $p \neq 0$, $\square$ does not act on diagonally but A_j, A_{n+j} and B commutes with P and Q. Thus $\square$ preserves the orthogonal decomposition $\Lambda^{p,q}(T^* D) = \operatorname{Ran} P \oplus \operatorname{Ran} Q$.

A fundamental solution of the heat equation is a one parameter family of bounded operators H_t, $t>0$, on $L_2^{p,q}(D)$ such that for $f\in\Lambda_0^{p,q}(D)$,

(2.11)
(i) for $t\in[0, T]$, $||H_t f||\leqq C$ where C is independent of t but may depend on T and f;
(ii) $H_t f$ is differentiable in t;
(iii) $H_t f\in \mathrm{Dom}\,\square$;
(iv) $(\partial/\partial t+\square)\,H_t f=0$;
(v) $H_t f\to f$ in $L_2^{p,q}(D)$ as $t\to 0$;
(iv) $\square H_t f=H_t\square f$.

Then we obtain the following results (Stanton [10]):

Proposition 2.1. *There is a unique fundamental solution H_t. Furthermore, H_t is the semigroup generated by $-\square$.*

Stanton proved this proposition in $(0, q)$-form case. Her proof is applicable to general (p, q)-form case.

Before we state our main theorem, we prepare a few notations. Let (W_0^{2n}, P) be a $2n$-dimensional Wiener space, i.e.,

$$W_0^{2n}=\{(x^1(\cdot), x^2(\cdot), \cdots, x^{2n}(\cdot)) \colon [0, \infty)\to \boldsymbol{R}^{2n}\colon \text{continuous}, x(0)=0\}$$

and P is the Wiener measure. For any $\varepsilon>0$ and $m\in \boldsymbol{N}$, we define the End $(\Lambda^{p,q}(T^* D))$-valued process $M_m^\varepsilon(t)$ by the solution of the SDE

$$\begin{cases} dM_m^\varepsilon(t)=M_m^\varepsilon(t)\,d\Xi_m^\varepsilon(t)\,, \\ M_m^\varepsilon(0)=I \end{cases} \tag{2.12}$$

where

$$\Xi_m^\varepsilon(t)=\sum_{j=1}^{2n}\frac{\varepsilon}{\sqrt{m}}A_j\,x^j(t)+\frac{\varepsilon^2}{m}Bt\,.$$

When $m=1$, we omit the subscript m: $\Xi_1^\varepsilon(t)=:\Xi^\varepsilon(t)$, $M_1^\varepsilon(t)=:M^\varepsilon(t)$. When $m=\varepsilon=1$, we also omit the superscript ε: $\Xi_1^1(t)=:\Xi(t)$, $M_1^1(t)=:M(t)$. For any $\alpha\in\boldsymbol{R}$, $\lambda\in\boldsymbol{R}$ and $(x, u)\in H_n$,

$$\boldsymbol{f}_t^\alpha(x, u, \lambda)=\frac{1}{2\pi}\exp(i\lambda u-t\lambda\alpha)\,E\left[e^{-i\lambda S_0(t)}\,M(t)\,\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\right] \tag{2.13}$$

where

$$S_0(t)=\sum_{j=1}^{n}\int_0^t (x^{n+j}(s)\circ dx^j(s)-x^j(s)\circ dx^{n+j}(s))\,. \tag{2.14}$$

Here, $\circ dx^j(t)$ means the Stratonovich differential. The right hand sice of (2.13) is well defined as a generalized expectation of a generalized Wiener functional [13]. Then our main theorem is stated as follows:

Theorem 2.1. *The unique fundamental solution H_t of the heat equation for the $\bar{\partial}$-Neumann problem on (p, q)-forms on D has the following smooth kernel:*

$$
\begin{aligned}
& h_t((x, u, r), (x', u', r')) \\
& \quad = \boldsymbol{p}_t^{n-2q+2}((x, u)^{-1}(x', u'))\, e_t^-(r, r')\, P \\
& \quad\quad + \boldsymbol{p}_t^{p-2q}((x, u)^{-1}(x', u')\, e_t^+(r, r')\, Q \\
& \quad\quad + \boldsymbol{q}_t^{n-2q}((x, u)^{-1}(x', u'), r+r')\, Q
\end{aligned} \tag{2.15}
$$

where

$$
\begin{aligned}
\boldsymbol{p}_t^{\alpha}(x,u) &= \int_{-\infty}^{\infty} d\lambda\, \boldsymbol{f}_t^{\alpha}(x, u, \lambda) \exp\left(-\frac{t}{2}\lambda^2\right), \\
\boldsymbol{q}_t^{\lambda}((x, u), r) &= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\lambda \cdot \lambda \boldsymbol{f}_t^{\alpha}(x, u, \lambda)\, e^{r\lambda} \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu\, e^{-\mu^2}
\end{aligned} \tag{2.16}
$$

and

$$
e_t^{\pm}(r, r') = \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) \pm \exp\left(-\frac{(r+r')^2}{2t}\right) \right\}.
$$

REMARK 2.2. In the case of $p=0$, $M(t)=I$ and so

$$
\begin{aligned}
\boldsymbol{f}_t^{\alpha}(x, u, \lambda) &= \frac{1}{2\pi} \exp(i\lambda u - t\lambda\alpha)\, E\left[e^{-i\lambda S_0(t)}\, \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)\right] I \\
&= f_t^{0,\alpha}(x, u, \lambda)\, I
\end{aligned} \tag{2.17}
$$

where

$$
f_t^{0,\alpha}(x, u, \lambda) = \frac{1}{2\pi^{n+1}} \left(\frac{\lambda}{\sinh(t\lambda)}\right)^n \exp(i\lambda u - t\lambda\alpha - |x|^2 \lambda \coth(t)) \tag{2.18}
$$

([5]). Now we see that the h_t in (2.15) coincides with the heat kernel obtained by Stanton [10].

In our proof of Theorem 2.1, the most essential part is to show the convergence of the λ-integration in the right hand side of (2.16). For this purpose, we show Theorem 2.2 below. Before we state the theorem, we prepare a few notations. We use the multi-index notation: $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \boldsymbol{Z}_+^{2n}$, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_{2n}$, $a^{\alpha}=a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_{2n}^{\alpha_{2n}}$ for $a=(a_1, a_2, \cdots, a_{2n}) \in \boldsymbol{R}^{2n}$ and $\partial_x^{\alpha}=\left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x^2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x^{2n}}\right)^{\alpha_{2n}}$. For $A=\sum_{I,I' \in \mathfrak{I}(p), J,J' \in \mathfrak{I}(q)} A_{IJI'J'} \operatorname{ext}(\omega^I \wedge \bar{\omega}^J) \operatorname{int}(\bar{\omega}^{I} \wedge \omega^{J'})$ $\in \operatorname{End}(\Lambda^{p,q}(T^*D))$, we set

$$
||A|| := \Big(\sum_{I, I' \in \mathfrak{I}(p), J, J' \in \mathfrak{I}(q)} |A_{IJI'J'}|^2 \Big)^{1/2}
$$

where $\mathcal{I}(p)=\{I=(i_1, i_2, \cdots, i_p);\ 1\leqq i_1<i_2<\cdots<i_p\leqq 2(n+1)\}$, i.e., $\|A\|$ denotes the Hilbert-Schmidt norm. Then we obtain the following:

Theorem 2.2. (i) *For any* $\varepsilon_0>0$, $\alpha, \beta\in \boldsymbol{Z}_+^{2n}$ *and* $k\in \boldsymbol{Z}_+$,

$$\varlimsup_{\lambda\to\pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x\in \boldsymbol{R}^{2n}\\ 0<\varepsilon\leq\varepsilon_0}} \|x^\beta \partial_x^\alpha E\,[S_0(1)^k e^{-i\lambda S_0(1)} M^\varepsilon(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]\| \tag{2.19}$$
$$\leqq -n\,.$$

(ii) *For any* $\varepsilon_0>0$, $\alpha, \beta\in \boldsymbol{Z}_+^{2n}$ *and* $k\in \boldsymbol{Z}_+$,

$$\varlimsup_{\lambda\to\pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x\in \boldsymbol{R}^{2n}\\ 0<\varepsilon\leqq\varepsilon_0}} \|x^\beta \partial_x^\alpha E\,[S_0(1)^k e^{-i\lambda S_0(1)} (M^\varepsilon(1)-I)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]/\varepsilon\| \tag{2.20}$$
$$\leqq -n\,.$$

By using scaling property of Brownian motions, we can easily prove the following:

Corollary. *For any* $\alpha, \beta\in \boldsymbol{Z}_+^{2n}$ *and* $k\in \boldsymbol{Z}_+$,

$$\varlimsup_{\lambda\to\pm\infty} \sup_{t_0\leqq t\leqq t_1} \frac{1}{t|\lambda|} \log \sup_{x\in \boldsymbol{R}^{2n}} \|x^\beta \partial_x^\alpha E[S_0(t)^k e^{-i\lambda S_0(t)} M(t)\, \delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)]\| \tag{2.21}$$
$$\leqq -n\,.$$

3. A formal construction of the heat kernel

The purpose of this section is to show that h_t in (2.15) is a candidate for the integral kernel of H_t. We first consider the initial value problem (1.1). Let $(W_0^{2(n+1)}, P)$ be a $2(n+1)$-dimensional Wiener space, i.e.,

$$W_0^{2(n+1)} = \{(x^1(\cdot), x^2(\cdot), \cdots, x^{2n}(\cdot), u(\cdot), r(\cdot)) : [0, \infty) \to \boldsymbol{R}^{2(n+1)} : \text{continuous},\ (x(0), u(0), r(0)) = 0\}$$

and P is the Wiener measure. Then the diffusion process $Z(t)=(X^1(t), X^2(t), \cdots, X^{2n}(t), U'(t), R'(t))$ defined by

$$\begin{aligned} X^j(t) &= x^j+\frac{x^j(t)}{\sqrt{2}}\,, \quad j=1, 2, \cdots, 2n\,,\\ U'(t) &= u+u(t)+S_x(t)\,,\\ R'(t) &= r+r(t)\,,\\ S_x(t) &= \sum_{j=1}^n \int_0^t \sqrt{2}\,(X^{n+j}(s)\circ dx^j(s)-X^j(s)\circ dx^{n+j}(s)) \end{aligned} \tag{3.1}$$

is a solution of the following SDE on $H_n\times \boldsymbol{R}$:

$$(3.2)\quad \begin{cases} dZ(t) = \dfrac{1}{\sqrt{2}} \sum_{j=1}^{2n} X_j(Z(t)) \circ dx^j(t) + \dfrac{\partial}{\partial u}(Z(t)) \circ du(t) + \dfrac{\partial}{\partial r}(Z(t)) \circ d\, r(t)\,, \\ Z(0) = (x^1, x^2, \cdots, x^{2n}, u, r)\,. \end{cases}$$

$R(t) := |r + r(t)|$ has the following decomposition by Skolohod:

$$(3.3)\qquad R(t) = r + B(t) + \phi(t)$$

where

$$B(t) = \int_0^t \operatorname{sgn}(r + r(s))\, dx(s)$$

and

$$\phi(t) = \lim_{\eta \searrow 0} \frac{1}{2\eta} \int_0^t I_{(-\eta, \eta)}(r + r(s))\, ds$$

is the local time of $R(t)$ at 0. We define the $\operatorname{End}(\Lambda^{p,q}(T^*D))$-valued process $m(t)$ by the solution of the SDE

$$(3.4)\qquad \begin{cases} dm(t) = m(t)\{-i(n-2C)\}\, du(t)\,, \\ m(0) = I \end{cases}$$

and define the $\operatorname{End}(\Lambda^{p,q}(T^*D))$-valued process $K(t)$ by

$$(3.5)\qquad K(t) = I_{\{\sigma \leqq t\}}\, Q + I_{\{\sigma > t\}}$$

where $\sigma = \inf\{s : R(s) = 0\}$. Then we have the following lemma:

Lemma 3.1. *Let $G(t, X)$ be a smooth form on $[0, \infty) \times \bar{D}$ such that $G(t, \cdot) \in \mathcal{S}^{p,q}(\bar{D})$ for each $t \geqq 0$, uniformly with respect to $t \in [0, T]$ for any $T > 0$, i.e., any seminorms of $G(t, \cdot)$ are bounded in $t \in [0, T]$ for any $T > 0$. Then it holds that*

$$(3.6)\qquad \begin{aligned} & m(t)\, M(t)\, K(t)\, e^{-i\lambda U(t)} \int_{-\infty}^{\infty} e^{i\lambda\zeta}\, G(s, X(t), \zeta, R(t))\, d\zeta \\ & \quad - K(0)\, e^{-i\lambda u} \int_{-\infty}^{\infty} e^{i\lambda\zeta}\, G(0, x, \zeta, r)\, d\zeta \\ & \quad = \textit{martingale} + \int_0^t m(s)\, M(s)\, K(s)\, e^{-i\lambda U(s)} \\ & \qquad \times \int_{-\infty}^{\infty} e^{i\lambda\zeta} \left\{ \frac{\partial G}{\partial s} - \Box G \right\} (s, X(s), \zeta, R(s))\, d\zeta ds \\ & \quad + \int_0^t m(s)\, M(s)\, K(s)\, e^{-i\lambda U(s)} \\ & \qquad \times \int_{-\infty}^{\infty} e^{i\lambda\zeta} \left\{ \frac{\partial G}{\partial r} - i \frac{\partial G}{\partial u} \right\} (s, X(s), \zeta, R(s))\, d\zeta d\phi(s)\,, \end{aligned}$$

where $X(t) = (X^1(t), X^2(t), \cdots, X^{2n}(t))$ and $U(t) = U'(t) - i\phi(t)$.

Proof. By using Itô's formula and noting that $m(t)$, $M(t)$ and $K(t)$ are commutative, we have

$$\begin{aligned}
&m(t)\,M(t)\,K(t)\,e^{-i\lambda U(t)}\,e^{i\lambda\zeta}\,G(t,X(t),\zeta,R(t))\\
&\quad -m(0)\,M(0)\,K(0)\,e^{-i\lambda U(0)}\,e^{i\lambda\zeta}\,G(0,X(0),\zeta,R(0))\\
&\quad =\int_0^t m(s)\,M(s)\,K(s)\,e^{-i\lambda U(s)}\,e^{i\lambda\zeta}\\
&\quad \times\{-i(n-2c+\lambda)\,G(s,X(s),\zeta,R(s))\,du(s)\\
&\quad +\sum_{j=1}^{2n}\Big(A_jG+\frac{1}{\sqrt{2}}\frac{\partial G}{\partial x^j}\Big)(s,X(s),\zeta,R(s))\,dx^j(s)\\
&\quad -i\lambda G(s,X(s),\zeta,R(s))\,dS_x(s)+\frac{\partial G}{\partial r}(s,X(s),\zeta,R(s))\,dB(s)\}\\
&\quad +\int_0^t m(s)\,M(s)\,K(s)\,e^{-i\lambda U(s)}\,e^{i\lambda\zeta}\Big\{\frac{\partial G}{\partial t}+\frac{1}{4}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x^j}-2i\lambda X^{n+j}(s)\Big)^2G\\
&\quad +\frac{1}{4}\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x^{n+j}}+2i\lambda X^j(s)\Big)^2G-\frac{\lambda^2}{2}G+\frac{1}{2}\frac{\partial^2G}{\partial r^2}-\lambda(n-2C)\,G\\
&\quad +\sum_{j=1}^{n}\frac{A_j}{\sqrt{2}}\Big(\frac{\partial}{\partial x^j}-2i\lambda X^{n+j}(s)\Big)G+\sum_{j=1}^{n}\frac{A_{n+j}}{\sqrt{2}}\Big(\frac{\partial}{\partial x^{n+j}}+2i\lambda X^j(s)\Big)G\\
&\quad +BG\Big\}(s,X(s),\zeta,R(s))\,ds+\int_0^t m(s)\,M(s)\,K(s)\,e^{-i\lambda U(s)}\,e^{i\lambda\zeta}\\
&\quad \times\Big\{\frac{\partial G}{\partial r}-\lambda G\Big\}(s,X(s),\zeta,R(s))\,d\phi(s)\,.
\end{aligned}\tag{3.7}$$

Since G is assumed to be smooth and rapidly decreasing in X, by integraring in ζ variables, exchanging the order of integration and using the integration by parts, we obtain (3.6) from (3.7). □

If the solution $F(t, X)$ *of* (1.1) *is smooth in* (t, X) *and rapidly decreasing in* X *unifromly with respect to* $t\in[\delta, T]$ *for any* $0<\delta<T$, by Lemma 3.1, we have

$$\begin{aligned}
&e^{-i\lambda u}\int_{-\infty}^{\infty}d\zeta e^{i\lambda\zeta}\,F(T,x,\zeta,r)\\
&\quad =E\Big[m(T)\,M(T)\,K(T)\,e^{-i\lambda U(T)}\int_{-\infty}^{\infty}d\zeta e^{i\lambda\zeta}f(X(T),\zeta,R(T))\Big]
\end{aligned}$$

and so

$$\begin{aligned}
&F(t,x,u,r)\\
&\quad =\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda E[m(t)\,M(t)\,K(t)\,e^{-i\lambda U(t)}\int_{-\infty}^{\infty}d\zeta e^{i\lambda\zeta}f(X(t),\zeta,R(t))]\,.
\end{aligned}\tag{3.8}$$

Now we set

$$\begin{aligned}
&H_{t,\lambda}((x,u,r),(x',u',r'))\\
&\quad :=\frac{1}{2\pi}E\Big[m(t)\,M(t)\,K(t)\,e^{-i\lambda U(t)}\int_{-\infty}^{\infty}d\zeta e^{i\lambda\zeta}\,\delta_{(x',u',r')}(X(t),\zeta,R(t))\Big]
\end{aligned}\tag{3.9}$$

$$= \frac{1}{2\pi} e^{i\lambda(u'-u)} E[e^{-i\lambda u(t)} m(t)] E[e^{-\lambda S_x(t)} M(t) \delta_{x'}(X(t))] \\ \times E[e^{-\lambda\phi(t)} K(t) | R(t) = r'] r_t(r, r'),$$

where $r_t(r, r')$ is the probability density function of $R(t)$. We can rewrite this $H_{t,\lambda}((x, u, r), (x', u', r'))$ as follows:

Lemma 3.2. *$H_{t,\lambda}((x, u, r), (x', u', r'))$ is expressed as follows:*

$$(3.10) \quad \begin{aligned} &H_{t,\lambda}((x, u, r), (x', u', r')) \\ &= f_t^{n-2q+2}((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2}\lambda^2\right) e_t^-(r, r') P \\ &+ f_t^{n-2q}((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2}\lambda^2\right) e_t^+(r, r') Q \\ &- \frac{2}{\sqrt{\pi}} \lambda f_t^{n-2q}((x, u)^{-1}(x', u'), \lambda) e^{r\lambda} \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu \exp(-\mu^2) Q \end{aligned}$$

where $e_t^{\pm}(r, r')$ and $f_t^{\alpha}(x, u, \lambda)$ are defined in (2.13) *and* (2.16), *respectively.*

Proof. Since

$$(3.11) \quad \begin{cases} m(t) Q = \exp\left(-i(n-2q) u(t) + (n-2q)^2 \frac{t}{2}\right) Q, \\ m(t) P = \exp\left(-i(n-2q+2) u(t) + (n-2q+2)^2 \frac{t}{2}\right) P, \end{cases}$$

we have

$$(3.12) \quad \begin{cases} E[e^{-i\lambda u(t)} m(t)] Q = \exp\left(-\frac{t^2}{2}\lambda^2 - t\lambda(n-2q)\right) Q, \\ E[e^{-i\lambda u(t)} m(t)] P = \exp\left(-\frac{t^2}{2}\lambda^2 - t\lambda(n-2q+2)\right) P. \end{cases}$$

We also have

$$(3.13) \quad \begin{aligned} &E[e^{-i\lambda S_x(t)} M(t) \delta_{x'}(X(t))] \\ &= \exp\left(-i\lambda \sum_{j=1}^{n} 2(x^{n+j} x'^j - x^j x'^{n+j})\right) E\left[e^{-i\lambda S_0(t)} M(t) \delta_{x'-x}\left(\frac{x(t)}{\sqrt{2}}\right)\right]. \end{aligned}$$

By using Lemma 3.3 below, we obtain

$$(3.14) \quad \begin{aligned} &E[e^{-\lambda\phi(t)} K(t) | R(t) = r'] r_t(r, r') \\ &= \frac{1}{\sqrt{2\pi t}} \left\{\exp\left(-\frac{(r-r')^2}{2t}\right) - \exp\left(-\frac{(r+r')^2}{2t}\right)\right\} \\ &\quad + \int_0^{\infty} db\, e^{-\lambda b} \frac{2(r+r'+b)}{\sqrt{2\pi^3}} \exp\left(-\frac{(r+r'+b)^2}{2t}\right) Q \\ &= \frac{1}{\sqrt{2\pi t}} \left\{\exp\left(-\frac{(r-r')^2}{2t}\right) - \exp\left(-\frac{(r+r')^2}{2t}\right)\right\} P \end{aligned}$$

$$+\frac{1}{\sqrt{2\pi t}}\left\{\exp\left(-\frac{(r-r')^2}{2t}\right)+\exp\left(-\frac{(r+r')^2}{2t}\right)\right\} Q$$
$$-\int_0^\infty db\, \lambda e^{-\lambda b} \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{(r+r'+b)^2}{2t}\right) Q\,.$$

By (3.12), (3.13) and (3.14), we conclude (3.10). □

Lemma 3.3. (*cf. Itô-McKean* [6] *p.* 45) *Let* $r(\cdot)$ *be a* 1-*dimensional Brownian motion. For* $r \geqq 0$, *we set*

$$R(t, r) = |r+r(t)|,$$
$$\phi(t, r) = \lim_{\eta \searrow 0} \frac{1}{2\eta} \int_0^t I_{(-\eta,\eta)}(r+r(s))\, ds\,,$$
$$\sigma(r) = \inf\{t \geqq 0,\, R(t, r) = 0\}.$$

Then the joint distribution of $(R(t, r), \phi(t, r))$ *is*

$$\begin{aligned} &P(R(t,r)\in da, \phi(t,r)\in db, \sigma(r)\leqq t) \\ &\quad = \frac{2(r+a+b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(r+a+b)^2}{2t}\right) da db\,, \\ &P(R(t,r)\in da, \phi(t,r)\in db, \sigma(r)> t) \\ &\quad = \frac{1}{\sqrt{2\pi t}}\left\{\exp\left(-\frac{(r-a)^2}{2t}\right)-\exp\left(-\frac{(r+a)^2}{2t}\right)\right\} \delta_0(b)\, da db\,, \end{aligned} \tag{3.15}$$

$a, b \geqq 0$.

By (3.8) and Lemma 3.2 we can expect that (2.15) holds.

4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. First we prepare a few lemmas. The main idea of the proof of Theorem 2.2 is to use semigroup property of a kernel as follows:

Lemma 4.1. *For any* $m \in \boldsymbol{N}$ *and* $\varepsilon > 0$,

$$\begin{aligned} &E\left[S_0(1)^k e^{im\lambda S_0(t)} M^\varepsilon(1)\, \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)\right] \\ &= \sum_{k_1+\cdots+k_m=k} \frac{k!}{k_1!\cdots k_m!} \\ &\quad \times \int_{\boldsymbol{R}^{2n}} dx_1\, E\left[\left(\frac{S_0(1)}{m}\right)^{k_1} e^{i\lambda S_0(1)} M^\varepsilon_m(1)\, \delta_{x_1}\left(\frac{x(1)}{\sqrt{2m}}\right)\right] \\ &\quad \times \int_{\boldsymbol{R}^{2n}} dx_2\, E\left[\left(\frac{S_{x_1}(1)}{m}\right)^{k_2} e^{i\lambda S_{x_1}(1)} M^\varepsilon_m(1)\, \delta_{x_2}\left(x_1+\frac{x(1)}{\sqrt{2m}}\right)\right] \\ &\quad \times \cdots \times \int_{\boldsymbol{R}^{2n}} dx_{m-1}\, E\left[\left(\frac{S_{x_{m-2}}(1)}{m}\right)^{k_{m-1}} e^{i\lambda S_{x_{m-2}}(1)} M^\varepsilon_m(1)\right. \end{aligned} \tag{4.1}$$

$$\times\delta_{x_{m-1}}\left(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\right)\Big]$$

$$\times E\left[\left(\frac{S_{x_{m-1}}(1)}{m}\right)^{k_m} e^{i\lambda S_{x_{m-1}}(1)} M_m^\varepsilon(1)\,\delta_x\left(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\right)\right].$$

Proof. By using scaling property and Markovian property of the Brownian motion, for any $\zeta\in\boldsymbol{R}$, we have

$$\begin{aligned}
&\sum_{l=0}^{\infty}\frac{(i\zeta)^l}{l!}E\left[S_0(1)^l e^{im\lambda S_0(1)} M^\varepsilon(1)\,\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)\right]\\
&=E\left[e^{i(m\lambda+\zeta)S_0(1)} M^\varepsilon(1)\,\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)\right]\\
&=E\left[e^{i(\lambda+\zeta/m)S_0(m)} M_m^\varepsilon(m)\,\delta_x\left(\frac{x(m)}{\sqrt{2m}}\right)\right]\\
&=\int_{\boldsymbol{R}^{2n}}dx_1\,E\left[e^{i(\lambda+\zeta/m)S_0(1)} M_m^\varepsilon(1)\,\delta_{x_1}\left(\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times\int_{\boldsymbol{R}^{2n}}dx_2\,E\left[e^{i(\lambda+\zeta/m)S_{x_1}(1)} M_m^\varepsilon(1)\,\delta_{x_2}\left(x_1+\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times\cdots\times\int_{\boldsymbol{R}^{2n}}dx_{m-1}\,E\left[e^{i(\lambda+\zeta/m)S_{x_{m-2}}(1)} M_m^\varepsilon(1)\,\delta_{x_{m-1}}\left(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times E\left[e^{i(\lambda+\zeta/m)S_{x_{m-1}}(1)} M_m^\varepsilon(1)\,\delta_x\left(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\right)\right]
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
&=\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{1}{l_1!\cdots l_m!}(i\zeta)^{l_1+\cdots+l_m}\\
&\times\int_{\boldsymbol{R}^{2n}}dx_1\,E\left[\left(\frac{S_0(1)}{m}\right)^{l_1} e^{i\lambda S_0(1)} M_m^\varepsilon(1)\,\delta_{x_1}\left(\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times\int_{\boldsymbol{R}^{2n}}dx_2\,E\left[\left(\frac{S_{x_1}(1)}{m}\right)^{l_2} e^{i\lambda S_0(1)} M_m^\varepsilon(1)\,\delta_{x_2}\left(x_1+\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times\cdots\times\int_{\boldsymbol{R}^{2n}}dx_{m-1}\,E\left[\left(\frac{S_{x_{m-2}}(1)}{m}\right)^{l_{m-1}} e^{i\lambda S_{x_{m-2}}(1)} M_m^\varepsilon(1)\right.\\
&\left.\times\delta_{x_{m-1}}\left(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\right)\right]\\
&\times E\left[\left(\frac{S_{x_{m-1}}(1)}{m}\right)^{l_m} e^{i\lambda S_{x_{m-1}}(1)} M_m^\varepsilon(1)\,\delta_x\left(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\right)\right].
\end{aligned}$$

Since ζ is arbitrary, we obtain (4.1). □

In the followings, we write $C(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_r)$ a constant which depends only on the parameter $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_r$ but may vary at each hand of the equations. We suppress the dependense on n. The next two lemmas give the necessary estimates in Malliavin calculus:

Lemma 4.2. *For any* $p>1$ *and* $s\in\boldsymbol{Z}_+$,

$$\|M_m^\varepsilon(1)\|_{p,s} \leqq C(p,s) \exp\left\{\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right\} \tag{4.3}$$

and

$$\|M_m^\varepsilon(1)-I\|_{p,s} \leqq C(p,s)\frac{\varepsilon}{\sqrt{m}} \exp\left\{\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right\} \tag{4.4}$$

where $\|\cdot\|_{p,s}$ *is the Sobolev norm on the Wiener space (see e.g.* [13]).

Proof. For any continuous adapted Hilbert space valued process $N(\cdot)$ and $0 \leqq t \leqq 1$, by using the inequality of Burkholder (Ikeda-Watanabe [5] Theorem III-3.1), we have

$$\begin{aligned} & E\left[\left\|\int_0^t N(s)\, d\Xi_m^\varepsilon(s)\right\|^p\right] \\ & \leqq \sum_{j=1}^{2n} E\left[\left\|\int_0^t N(s)\, A_j \frac{\varepsilon}{\sqrt{m}}\, dx^j(s)\right\|^p\right](2n+1)^{p-1} \\ & \quad + E\left[\left\|\int_0^t N(s)\, B\frac{\varepsilon^2}{m}\, ds\right\|^p\right](2n+1)^{p-1} \\ & \leqq C(p)\left\{\sum_{j=1}^{2n} E\left[\left(\int_0^t \|N(s)\|^2 \|A_j\|^2 \frac{\varepsilon^2}{m}\, ds\right)^{p/2}\right]\right. \\ & \quad \left. + E\left[\int_0^t \|N(s)\|^p \|B\|^p \frac{\varepsilon^{2p}}{m^p}\, ds\right] t^{p-1}\right\} \\ & \leqq C(p)\left(\frac{\varepsilon}{\sqrt{m}}\right)^p \int_0^t E[\|N(s)\|^p]\, ds\,. \end{aligned} \tag{4.5}$$

On the other hand, let H be the $2n$-dimensional Cameron-Martin space ([13]). It is easily seen that

$$\begin{aligned} & D\Xi_m^\varepsilon(s)\,[h] = \sum_{j=1}^{2n} \frac{\varepsilon}{\sqrt{m}} A_j h^j(s) \quad \text{for} \quad h \in H\,, \\ & D^r \Xi_m^\varepsilon(s) = 0 \quad \text{for} \quad r \geqq 2\,. \end{aligned}$$

Then we see that

$$E\left[\left\|\int_0^t N(s)\, d(D\Xi_m^\varepsilon)(s)\right\|^p\right] \leqq C(p)\left(\frac{\varepsilon}{\sqrt{m}}\right)^p \int_0^t E[\|N(s)\|^p]\, ds\,. \tag{4.6}$$

Since, by (3.2),

$$D^r M_m^\varepsilon(t) = \delta_{r0} I + \int_0^t D^r M_m^\varepsilon(s)\, d\Xi_m^\varepsilon(s) + r\int_0^t D^{r-1} M_m^\varepsilon(s)\, d(D\Xi_m)(s)\,,$$

we have

$$\sum_{r=0}^s E[\|D^r M_m^\varepsilon(t)\|^p] \leqq C(p,s)\left(1+\sum_{r=0}^s \left(\frac{\varepsilon}{\sqrt{m}}\right)^p \int_0^t E[\|D^r M_m^\varepsilon(s)\|^p]\, ds\right). \tag{4.7}$$

By using the equivalence of the Sobolev norms ([5] Theorem V-8.4), (4.7) is rewritten as follows:

$$(4.8) \qquad \|M_m^{\varepsilon}(t)\|_{p,s}^p \leqq C(p, s)\left(1+\left(\frac{\varepsilon}{\sqrt{m}}\right)^p \int_0^t \|M_m(s)\|_{p,s}^p\, ds\right).$$

By using the lemma of Gronwall, we obtain (4.3) easily. Similarly we obtain (4.4). □

Lemma 4.3. *For any $k\in \boldsymbol{N}$, $\alpha\in \boldsymbol{Z}_+^{2n}$ and $q>1$, there exists $s\in \boldsymbol{N}$ such that*

$$(4.9) \qquad \sup_{x\in \boldsymbol{R}^{2n}} (1+|x|)^k \|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leqq \infty .$$

Proof. For any $\kappa>0$, there exists $s\in \boldsymbol{N}$ such that

$$\|(1+|\cdot|-\Delta)^{-s}\phi(\cdot)\|_\infty \leqq C(s, \kappa)\|(1+|\cdot|-\Delta)^{-\kappa}\phi(\cdot)\|_2 \quad \text{for} \quad \phi\in \mathcal{S}(\boldsymbol{R}^{2n})$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are the L_∞-norm and L_2-norm on $\boldsymbol{R}^{2n}$, respectively. By combining this with Theorem 2.1 of [13], we have

$$\|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leqq C(q, s, k)\|(1+|\cdot|-\Delta)^{-\kappa}\partial^\alpha \delta_x(\cdot)\|_2 .$$

Furthermore the L_2-norm is estimated as follows:

$$\begin{aligned}
&(1+|x|^2)^k \|(1+|\cdot|-\Delta)^{-\kappa}\partial^\alpha \delta_x(\cdot)\|_2 \\
&= (1+|x|^2)^k \sup_{\substack{g\in \mathcal{S}(\boldsymbol{R}^{2n}) \\ \|g_2\|\leqq 1}} \int_{\boldsymbol{R}^{2n}} (1+|\xi|^2-\Delta_\xi)^{-\kappa}\partial_\xi^\alpha \delta_x(\xi) g(\xi)\, d\xi \\
&= \sup_g |(1+|x|^2)^k \partial_x^\alpha (1+|x|^2-\Delta_x)^{-\kappa} g(x)| \\
&= \sup_g |(1+|x|^2)^k \int_{\boldsymbol{R}^{2n}} \frac{e^{i\xi x}}{(2\pi)^n} \xi^\alpha (1+|\xi|^2-\Delta_\xi)^{-\kappa} \hat{g}(\xi)\, d\xi| \\
&\leqq C \sup_g \left(\int_{\boldsymbol{R}^{2n}} |(1+|\xi|^2)^n (1-\Delta_\xi^k) \xi^\alpha (1+|\xi|^2-\Delta_\xi)^{-\kappa} \hat{g}(\xi)|^2\, d\xi\right)^{1/2} \\
&\leqq C \sup_g \left(\int_{\boldsymbol{R}^{2n}} |(1+|\xi|^2-\Delta_\xi)^{\kappa(k,\alpha,n)-\kappa} \hat{g}(\xi)|^2\, d\xi\right)^{1/2}
\end{aligned}$$

for some $\kappa(k, \alpha, n)\geqq 0$ determined by k, α and n. Thus, if κ is large enough, we have

$$(4.10) \qquad (1+|x|^2)^k \|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leqq C(q, s, k, \alpha) .$$

Hence we conclude (4.9). □

Now we prove Theorem 2.2.

Proof of Theorem 2.2. We prove the theorem as $\lambda\to-\infty$ only. The case as $\lambda\to+\infty$ can be proved similarly.

We consider (i). We take $m\in\boldsymbol{N}$ arbitrary. We exchange the order of differetiations and integrations in (4.1) formally:

$$
\begin{aligned}
&x^{\beta}\,\partial^{\alpha}\,E\left[S_0(1)^k\,e^{im\lambda S_0(1)}\,M^{\varepsilon}(1)\,\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)\right] \\
&\quad=\int_{(\boldsymbol{R}^{2n})^{m-1}} dx_1\cdots dx_{m-1}\,\Phi^{\varepsilon}_{k,\alpha,\beta}(m,\lambda,x_1,\cdots,x_{m-1},x)
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
&\Phi^{\varepsilon}_{k,\alpha,\beta}(m,\lambda,x_1,\cdots,x_{m-1},x) \\
&\quad=\sum_{k_1+\cdots+k_m=k}\frac{k!}{k_1!\cdots k_m!}\,E\left[\left(\frac{S_0(1)}{m}\right)^{k_1}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\,\delta_{x_1}\left(\frac{x(1)}{\sqrt{2m}}\right)\right] \\
&\quad\times E\left[\left(\frac{S_{x_1}(1)}{m}\right)^{k_2}e^{i\lambda S_{x_1}(1)}\,M^{\varepsilon}_m(1)\,\delta_{x_2}\left(x_1+\frac{x(1)}{\sqrt{2m}}\right)\right] \\
&\quad\times\cdots\times E\left[\left(\frac{S_{x_{m-2}}(1)}{m}\right)^{k_{m-1}}e^{i\lambda S_{x_{m-2}}(1)}\,M^{\varepsilon}_m(1)\,\delta_{x_{m-1}}\left(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\right)\right] \\
&\quad\times x^{\beta}\,\partial^{\alpha}\,E\left[\left(\frac{S_{x_{m-1}}(1)}{m}\right)^{k_m}e^{i\lambda S_{x_{m-1}}(1)}\,M^{\varepsilon}_m(1)\,\delta_{x}\left(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\right)\right].
\end{aligned}
\tag{4.12}
$$

We will estimate the right hand side of (4.11). Let $\|\cdot\|$ be the Hilbert-Schmidt norm defined in Section 2. By using a property of δ-functions and making the change of variables $\sqrt{2m}\;x_j\to x_j$,

$$
\begin{aligned}
&\int_{(\boldsymbol{R}^{2n})^{m-1}} dx_1\cdots dx_{m-1}\,\|\Phi^{\varepsilon}_{k,\alpha,\beta}(m,\lambda,x_1,\cdots,x_{m-1},x)\| \\
&\leqq\sum_{k_1+\cdots+k_m=k}\frac{k!}{k_1!\cdots k_m!} \\
&\quad\times\int_{(\boldsymbol{R}^{2n})^{m-1}} dx_1\cdots dx_{m-1}\left\|E\left[\left(\frac{S_0(1)}{m}\right)^{k_1}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\delta_{\sqrt{2m}x_1}(x(1))\right]\right\|(2m)^n \\
&\quad\times\left\|E\left[\left(\frac{S_{x_1}(1)}{m}\right)^{k_1}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\,\delta_{\sqrt{2m}x_2}(\sqrt{2m}\,x_1+x(1))\right]\right\|(2m)^n \\
&\quad\times\cdots\times\left\|E\left[\left(\frac{S_{x_{m-2}}(1)}{m}\right)^{k_{m-1}}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\right.\right. \\
&\quad\left.\left.\times\,\delta_{\sqrt{2m}x_{m-1}}(\sqrt{2m}\,x_{m-2}+x(1))\right]\right\|(2m)^n \\
&\quad\times\left\|x^{\beta}\,\partial^{\alpha}\,E\left[\left(\frac{S_{x_{m-1}}(1)}{m}\right)^{k_m}e^{i\lambda S_{x_{m-1}}(1)}\,M^{\varepsilon}_m(1)\,\delta_x\left(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\right)\right]\right\| \\
&=\sum_{k_1+\cdots+k_m=k}\frac{k!m^k}{k_1!\cdots k_m!} \\
&\quad\times\int_{(\boldsymbol{R}^{2n})^{m-1}} dx_1\cdots dx_{m-1}\left\|E\left[\left(\frac{S_0(1)}{m^2}\right)^{k_1}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\,\delta_{x_1}(x(1))\right]\right\| \\
&\quad\times\left\|E\left[\left(\frac{S_{x_1/\sqrt{2m}}(1)}{m^2}\right)^{k_2}e^{i\lambda S_0(1)}\,M^{\varepsilon}_m(1)\,\delta_{x_2}(x_1+x(1))\right]\right\|
\end{aligned}
\tag{4.13}
$$

$$\times\cdots\times\left\|E\left[\left(\frac{S_{x_{m-2}/\sqrt{2m}}(1)}{m^2}\right)^{k_{m-1}} e^{i\lambda S_0(1)}\, M_m^{\varepsilon}(1)\, \delta_{x_{m-1}}(x_{m-2}+x(1))\right]\right\|$$
$$\times\left\|x^{\beta}\,\partial^{\alpha}\, E\left[\left(\frac{S_{x_{m-1}/\sqrt{2m}}(1)}{m^2}\right)^{k_m} e^{i\lambda S_{x_{m-1}/\sqrt{2m}}(1)}\, M_m^{\varepsilon}(1)\, \delta_x\left(\frac{x_{m-1}+x(1)}{\sqrt{2m}}\right)\right]\right\|.$$

Since

$$(1+|x_1|)\cdot(1+|x_2-x_1|)\cdots(1+|x_j-x_{j-1}|)\geqq 1+|x_j|$$

for any $j\geqq 1$, if we set $R_n=\int_{R^{2n}}(1+|x|)^{-2n-1}\,dx$,

the right hand side of (4.13)

$$\leqq\sum_{k_1+\cdots+k_m=k}\frac{k!m^k}{k_1!\cdots k_m!}\int_{(R^{2n})^{m-1}}dx_1\cdots dx_{m-1}$$
$$\times\left\|(1+|x_1|)^{2n+2k+|\alpha|+|\beta|+1}\, E\left[\left(\frac{S_0(1)}{m^2}\right)^{k_1} e^{i\lambda S_0(1)}\, M_m^{\varepsilon}(1)\,\delta_{x_1}(x(1))\right.\right.$$
$$\times\left\|(1+|x_2-x_1|)^{2n+2k+|\alpha|+|\beta|+1}\, E\left[\left(\frac{S_{x_1/\sqrt{2m}}(1)}{m^2(1+|x_1|)^2}\right)^{k_1} e^{i\lambda S_0(1)}\right.\right.$$
$$\left.\left.\times M_m^{\varepsilon}(1)\,\delta_{x_2}(x_1+x(1))\right]\right\|$$
$$\times\cdots\times\left\|(1+|x_{m-1}-x_{m-2}|)^{2n+2k+|\alpha|+|\beta|+1}\, E\left[\left(\frac{S_{x_{m-2}/\sqrt{2m}}(1)}{m^2(1+|x_{m-2}|)^2}\right)^{k_{m-1}}\right.\right. \tag{4.14}$$
$$\left.\left.\times e^{i\lambda S_0(1)}\, M_m^{\varepsilon}(1)\,\delta_{x_{m-1}}(x_{m-2}+x(1))\right]\right\|$$
$$\times\left\|\frac{x^{\beta}}{(1+|x_{m-1}|)^{|\alpha|+|\beta|}}\,\partial^{\alpha}\, E\left[\left(\frac{S_{x_{m-1}/\sqrt{2m}}(1)}{m^2(1+|x_{m-1}|)^2}\right)^{k_m} e^{i\lambda S_{x_{m-1}/\sqrt{2m}}(1)}\right.\right.$$
$$\left.\left.\times M_m^{\varepsilon}(1)\,\delta_x\left(\frac{x_{m-1}+x(1)}{\sqrt{2m}}\right)\right]\right\|$$
$$\times\frac{1}{(1+|x_1|)^{2n+1}}\cdot\frac{1}{(1+|x_2-x_1|)^{2n+1}}\cdots\frac{1}{(1+|x_{m-1}-x_{m-2}|)^{2n+1}}$$
$$\leqq\sum_{k_1+\cdots+k_m=k}\frac{k!m^k}{k_1!\cdots k_m!}$$
$$\times\left\{\prod_{j=1}^{m-1}R_n\sup_{x,x'}\left\|(1+|x'-x|)^{2n+2k+|\alpha|+|\beta|+1}\, E\left[\left(\frac{S_{x/\sqrt{2m}}(1)}{m^2(1+|x|)^2}\right)^{k_j}\right.\right.\right.$$
$$\left.\left.\left.\times e^{i\lambda S_0(1)}\, M_m^{\varepsilon}(1)\,\delta_{x'}(x+x(1))\right]\right\|\right\}$$
$$\times\sup_{x,x'}\left\|\frac{x'^{\beta}}{(1+|x|)^{|\alpha|+|\beta|}}\,\partial_{x'}^{\alpha}\, E\left[\left(\frac{S_{x/\sqrt{2m}}(1)}{m^2(1+|x|)^2}\right)^{k_m} e^{i\lambda S_{x/\sqrt{2m}}(1)}\right.\right.$$
$$\left.\left.\times M_m^{\varepsilon}(1)\,\delta_{x'}\left(\frac{x+x(1)}{\sqrt{2m}}\right)\right]\right\|.$$

The middle factor under the summation in the right hand side of (4.14) is estimated as follows:

$$\|(1+|x'-x|)^{2n+2k+|\alpha|+|\beta|+1} E\Big[\Big(\frac{S_{x/\sqrt{2m}}(1)}{m^2(1+|x|)^2}\Big)^l e^{i\lambda S_0(1)}$$
$$\times M^{\varepsilon}_m(1)\,\delta_{x'}(x+x(1))\Big]\|$$
$$\leqq \sum_{h=0}^{l} \frac{l!\cdot(1+|x'-x|)|^{2n+2k+|\alpha|+|\beta|+1}}{h!(l-h)!}\Big(\frac{\sum_{j=1}^n|(x^{n+j}x'^j-x^jx'^{n+j})|}{\sqrt{m}\,m^2(1+|x|)^2}\Big)^h$$
$$(4.15)\qquad \times\Big\|E\Big[\Big(\frac{S_0(1)}{m^2(1+|x|)^2}\Big)^{l-h} e^{i\lambda S_0(1)} M^{\varepsilon}_m(1)\,\delta_{x'}(x+x(1))\Big]\Big\|$$
$$\leqq\Big(\frac{4}{m}\Big)^l (1+|x'-x|)^{2n+3k+|\alpha|+|\beta|+1}$$
$$\times\sum_{h=0}^{l}\Big\|E\Big[\Big(\frac{S_0(1)}{m}\Big)^h e^{i\lambda S_0(1)} M^{\varepsilon}_m(1)\,\delta_{x'}(x+x(1))\Big]\Big\|.$$

On the other hand, since

$$\partial^{\alpha}_{x'} E[\Phi\delta_{ax'}(x+x(1))] = (-a)^{|\alpha|} E[\Phi\partial^{\alpha}\,\delta_{ax'}(x+x(1))]$$

for any $a>0$ and $\alpha\in \boldsymbol{Z}^{2n}_+$,

$$\partial^{\alpha}_{x'} E\Big[\Big(\frac{S_{x/\sqrt{2m}}(1)}{m^2(1+|x|)^2}\Big)^l e^{i\lambda S_{x/\sqrt{2m}}(1)} M^{\varepsilon}_m(1)\,\delta_{x'}\Big(\frac{x+x(1)}{\sqrt{2m}}\Big)\Big]$$
$$=\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}\,\partial^{\alpha_1}_{x'}\Big(\frac{\sqrt{2}\,\sum_{j=1}^n(x^{n+j}x'^j-x^jx'^{n+j})}{m^2(1+|x|)^2}\Big)^h$$
$$\times\partial^{\alpha_2}_{x'}\exp\Big(i\lambda\sqrt{2}\sum_{j=1}^{n}(x^{n+j}x'^j-x^jx'^{n+j})\Big)(-1)^{|\alpha_3|}(2m)^{n+|\alpha_3|/2}$$
$$\times E\Big[\Big(\frac{S_0(1)}{m^2(1+|x|)^2}\Big)^{l-h} e^{i\lambda S_0(1)} M^{\varepsilon}_m(1)\,\partial^{\alpha_3}\,\delta_{\sqrt{2m}x'}(x+x(1))\Big].$$

Thus the last factor under the summation of the right hand side of (4.4) is estimated as follows:

$$\Big\|\frac{x'^{\beta}}{(1+|x|)^{|\alpha|+|\beta|}}\,\partial^{\alpha}_{x'} E\Big[\Big(\frac{S_{x/\sqrt{2m}}(1)}{m^2(1+|x|)^2}\Big)^l e^{i\lambda S_{x/\sqrt{2m}}(1)} M^{\varepsilon}_m(1)\,\delta_{x'}\Big(\frac{x+x(1)}{\sqrt{2m}}\Big)\Big]\Big\|$$
$$\leqq\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\sum_{h=0}^{l}\frac{l!}{h!(l-h)!}\Big(\frac{|x'|}{1+|x|}\Big)^{|\beta|}\Big(\frac{2\sqrt{2}\,n|x|\,|x'|}{m^2(1+|x|)^2}\Big)^h\Big(\frac{\sqrt{2}\lambda|x|}{1+|x|}\Big)^{|\alpha_2|}$$
$$(4.16)\qquad\times(2m)^{n+|\alpha_3|/2}\Big\|E\Big[\Big(\frac{S_0(1)}{m^2(1+|x|)^2}\Big)^{l-h} e^{i\lambda S_0(1)} M^{\varepsilon}_m(1)\,\partial^{\alpha_3}\,\delta_{\sqrt{2m}x'}(x+x(1))\Big]\Big\|$$
$$\leqq 2^{|\alpha|+2k+n}\,m^{n+|\alpha|/2}\,n^k\,\lambda^{|\alpha|}(1+|\sqrt{2m}\,x'-x|)^{|\beta|+k}$$
$$\times\sum_{\substack{h\leqq l\\ \alpha'\leqq\alpha}}\Big\|E\Big[\Big(\frac{S_0(1)}{m}\Big)^h e^{i\lambda S_0(1)} M^{\varepsilon}_m(1)\,\partial^{\alpha'}\,\delta_{\sqrt{2m}x'}(x+x(1))\Big]\Big\|.$$

By combining (4.14), (4.15) and (4.16), we obtain

$$\sup_x\int_{(R^{2n})^{m-1}} dx_1\cdots dx_{m-1}\,\|\Phi^{\varepsilon}_{k,\alpha,\beta}(m,\lambda,x_1,\cdots,x_{m-1},x)\|$$

$$
\begin{aligned}
(4.17)\qquad &\leqq \sum_{k_1+\cdots+k_m=k} \frac{k!m^k}{k_1!\cdots k_m!}\left(\frac{4}{m}\right)^k 2^{|\alpha|+2k+n} m^{n+|\alpha|/2} n^k \lambda^{|\alpha|} \\
&\times\left\{\prod_{j=1}^{m-1} R_n \sum_{h=0}^{k_j} \sup_x \left\| (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E\left[\left(\frac{S_0(1)}{m}\right)^h \right.\right.\right. \\
&\left.\left.\left.\times e^{i\lambda S_0(1)} M_m^{\varepsilon}(1)\, \delta_x(x(1))\right]\right\|\right\} \\
&\times\left\{\sum_{\substack{0\leqq h\leqq k_m \\ \alpha'\leqq\alpha}} \sup_x \left\|(1+|x|)^{|\beta|+k} E\left[\left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m^{\varepsilon}(1)\right.\right.\right. \\
&\left.\left.\left.\times \partial^{\alpha'}\delta_x(x(1))\right]\right\|\right\} \\
&\leqq 2^{|\alpha|+4k+n} m^{n+k|\alpha|/2} \lambda^{|\alpha|} n^k \\
&\times\left\{R_n \sum_{h=0}^{k} \sup_x \left\| (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E\left[\left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)}\right.\right.\right. \\
&\left.\left.\left.\times M_m^{\varepsilon}(1)\,\delta_x(x(1))\right]\right\|\right\}^{m-1} \\
&\times\left\{\sum_{\substack{0\leqq b\leqq k \\ \alpha'\leqq\alpha}} \sup_x \left\|(1+|x|)^{|\beta|+k} E\left[\left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)}\right.\right.\right. \\
&\left.\left.\left.\times M_m^{\varepsilon}(1)\,\partial^{\alpha'}\delta_x(x(1))\right]\right\|\right\} \\
&\leqq 2^{|\alpha|+4k+n} m^{n+k+|\alpha|/2} \lambda^{|\alpha|} n^k \\
&\times\{C \sup_x |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda S_0(1)}\delta_x(x(1))]| \\
&\quad + C(p,s)\, \|e^{i\lambda S_0(1)}(M_m^{\varepsilon}(1)-I)\|_{p,s} \\
&\qquad \times \sup_x (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} \|\delta_x(x(1))\|_{q,-s} \\
&\quad + C(p,s) \sum_{h=1}^{k} \left\|\left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m^{\varepsilon}(1)\right\|_{p,s} \\
&\quad \times \sup_x (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} \|\delta_x(x(1))\|_{q,-s}\Big\}^{m-1} \\
&\qquad \times\left\{C(p,s)\sum_{h=0}^{k}\left\|\left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m(1)\right\|_{p,s}\right. \\
&\qquad\quad \left.\times \sum_{\alpha'\leqq\alpha} \sup_x (1+|x|)^{|\beta|+k}\, \partial^{\alpha'} \|\delta_x(x(1))\|_{q,-s}\right\}
\end{aligned}
$$

where $p, q>1$, $1/p+1/q=1$, $s\in \boldsymbol{N}$. Furthermore,

$$
\begin{aligned}
&\|e^{i\lambda S_0(1)}(M_m^{\varepsilon}(1)-I)\|_{p,s} \\
&\quad \leqq \|e^{i\lambda S_0(1)}\|_{2p,s}\, \|M_m^{\varepsilon}(1)-I\|_{2p,s} \\
&\quad \leqq C(p)\,\lambda^{N(s)}\, \|M_m^{\varepsilon}(1)-I\|_{2p,s}
\end{aligned}
$$

where $N(s)$ is an integer which is determined by the number s. Similarly,

$$
\|S_0(1)^h e^{i\lambda S_0(1)} M_m^{\varepsilon}(1)\|_{p,s} \leqq C(p,h)\,\lambda^{N(s)}\, \|M_m^{\varepsilon}(1)\|_{p,s}\,.
$$

Thus by Lemma 4.2, we have

$$(4.18)\quad \|e^{i\lambda S_0(1)}(M_m^\varepsilon(1)-I)\|_{p,s}\leqq\frac{\varepsilon}{\sqrt{m}}\lambda^{N(s)}C(p,s)\exp\left\{\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right\}$$

and

$$(4.19)\quad \|S_0(1)^h e^{i\lambda S_0(1)}M_m^\varepsilon(1)\|_{p,s}\leqq\lambda^{N(s)}C(p,s,h)\exp\left\{\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right\}.$$

By applying lemma 4.3, (4.18) and (4.19) to (4.17), we have

$$(4.20)\quad \begin{aligned}&\sup_{x\in R^{2n}}\int_{(R^{2n})^{m-1}}dx_1\cdots dx_{m-1}\|\Phi_{k,\alpha,\beta}^\varepsilon(m,\lambda,x_1,\cdots,x_{m-1},x)\|\\ &\leqq 2^{|\alpha|+4k+n}m^{n-k+|\alpha|/2}\lambda^{|\alpha|}n^k\\ &\quad\times\{C\sup_{x\in R^{2n}}|(1+|x|)^{2n+3k+|\alpha|+|\beta|+1}E[e^{i\lambda S_0(1)}\delta_x(x(1))]|\\ &\quad+\frac{\varepsilon}{\sqrt{m}}\lambda^{N(s)}C(p,s,\alpha,\beta,k)\exp\left(\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right)\\ &\quad+\frac{1}{m}\lambda^{N(s)}C(p,s,\alpha,\beta,k)\exp\left(\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right)\}^{m-1}\\ &\quad\times\{\lambda^{N(s)}C(p,s,\alpha,\beta,k)\exp\left(\left(\frac{\varepsilon}{\sqrt{m}}\right)^p C(p,s)\right)\}.\end{aligned}$$

Since the right hand side of (4.20) is finite, the exchanging of the order of differentiations and integrations in (4.11) is justified and we obtain

$$(4.21)\quad \begin{aligned}&\sup_{x\in R^{2n},0<\varepsilon\leqq\varepsilon_0}\|x^\beta\partial^\alpha E[S_0(1)^k e^{im\lambda S_0(1)}M^\varepsilon(1)\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\|\\ &\leqq 2^{|\alpha|+4k+n}m^{n+k+|\alpha|/2}\lambda^{|\alpha|}n^k\\ &\quad\times\{C\sup_{x\in R^{2n}}|(1+|x|)^{2n+3k+|\alpha|+|\beta|+1}E[e^{i\lambda S_0(1)}\delta_x(x(1))]|\\ &\quad+\frac{1}{\sqrt{m}}\lambda^{N(s)}C(p,s,\alpha,\beta,k,\varepsilon_0)\}^{m-1}\\ &\quad\times\{\lambda^{N(s)}C(p,s,\alpha,\beta,k,\varepsilon_0)\}.\end{aligned}$$

If we replace (λ, m) with $(\lambda\zeta/[\lambda], [\lambda])$ where $[\lambda]$ is the integral part of λ,

$$(4.22)\quad \begin{aligned}&\sup_{x\in R^{2n},0<\varepsilon\leqq\varepsilon_0}\|x^\beta\partial^\alpha E[S_0(1)^k e^{i\lambda S_0(1)}M^\varepsilon(1)\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\|\\ &\leqq 2^{|\alpha|+4k+n}[\lambda]^{n+k+|\alpha|/2}\left(\frac{\lambda}{[\lambda]}\zeta\right)^{|\alpha|}n^k\\ &\quad\times\{C\sup_{x\in R^{2n}}|(1+|x|)^{2n+3k+|\alpha|+|\beta|+1}E[e^{i\lambda S_0(1)}\delta_x(x(1))]|\\ &\quad+\frac{1}{\sqrt{[\lambda]}}\left(\frac{\lambda}{[\lambda]}\zeta\right)^{N(s)}C(p,s,\alpha,\beta,k,\varepsilon_0)\}^{[\lambda]-1}\end{aligned}$$

$$\times \{\Big(\frac{\lambda}{[\lambda]}\zeta\Big)^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0)\} .$$

Hence we have

$$\begin{aligned}
&\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \sup_{x\in \boldsymbol{R}^{2n}} ||x^\beta \partial^\alpha E[S_0(1)^k e^{i\lambda\zeta S_0(1)} M^\varepsilon(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]|| \\
&= \overline{\lim_{\lambda\to\infty}} \log \{C \sup_{x\in \boldsymbol{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda\zeta S_0(1)/[\lambda]}\, \delta_x(x(1))]| \\
&\quad + \frac{1}{\sqrt{[\lambda]}}\Big(\frac{\lambda}{[\lambda]}\zeta\Big)^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0)\} \\
&= \log \overline{\lim_{\lambda\to\infty}} \{C \sup_{x\in \boldsymbol{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda\zeta S_0(1)/[\lambda]}\, \delta_x(x(1))]|\} .
\end{aligned} \tag{4.23}$$

Since $\lambda \coth \lambda \geqq 1$ for $\forall \lambda \in \boldsymbol{R}$,

$$\begin{aligned}
&\sup_{x\in \boldsymbol{R}^{2n}} |(1+|x|)^h E[e^{i\lambda S_0(1)}\, \delta_x(x(1))]| \\
&= \sup_{x\in \boldsymbol{R}^{2n}} |(1+|x|)^h \Big(\frac{1}{2\pi}\Big)^n \Big(\frac{\lambda/2}{\sinh(\lambda/2)}\Big)^n \exp\Big(-\frac{|x|^2\lambda}{4}\coth\frac{\lambda}{2}\Big)| \\
&\leqq C(h)\Big(\frac{\lambda}{\sinh(\lambda)}\Big)^n ,
\end{aligned} \tag{4.24}$$

and so

$$\begin{aligned}
&\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \sup_{x\in \boldsymbol{R}^{2n}, 0<\varepsilon\leqq\varepsilon_0} ||x^\beta \partial^\alpha E[S_0(1)^k e^{i\lambda S_0(1)} M^\varepsilon(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]|| \\
&\leqq \log C(k, \alpha, \beta) \overline{\lim_{\lambda\to\infty}} \Big(\frac{\lambda\zeta/[\lambda]}{\sinh \lambda\zeta/[\lambda]}\Big)^n \\
&= \log C(k, \alpha, \beta)\Big(\frac{\zeta}{\sinh \zeta}\Big)^n .
\end{aligned}$$

Consequently we obtain

$$\begin{aligned}
&\overline{\lim_{\lambda\to\infty}} \frac{1}{\lambda} \log \sup_{x\in \boldsymbol{R}^{2n}, 0<\varepsilon\leqq\varepsilon_0} ||x^\beta \partial^\alpha E[S_0(1)^k e^{i\lambda S_0(1)} M^\varepsilon(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]|| \\
&\leqq \overline{\lim_{\zeta\to\infty}}\, \overline{\lim_{\lambda\to\infty}} \frac{1}{\zeta\lambda} \log \sup_{x\in \boldsymbol{R}^{2n}, 0<\varepsilon\leqq\varepsilon_0} ||x^\beta \partial^\alpha E[S_0(1)^k e^{i\zeta\lambda S_0(1)} M^\varepsilon(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]|| \\
&\leqq \overline{\lim_{\zeta\to\infty}} \frac{1}{\zeta} \log C(k, \alpha, \beta)\Big(\frac{\zeta}{\sinh \zeta}\Big)^n \\
&= \overline{\lim_{\zeta\to\infty}} \frac{n}{\zeta} \log C(k, \alpha, \beta)\Big(\frac{\zeta}{\sinh \zeta}\Big) \\
&= -n
\end{aligned} \tag{4.25}$$

which completes the proof of (i).

For the proof of (ii), we use the following (4.26), instead of (4.1): for any $m\in\boldsymbol{N}$ and $\varepsilon>0$,

$$
\begin{aligned}
&E[S_0(1)^k e^{im\lambda S_0(1)}(M^\varepsilon(1)-I)\,\delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)] \\
&=\sum_{k_1+\cdots+k_m=k}\sum_{l=1}^{m}\frac{k!}{k_1!\cdots k_m!} \\
&\quad\times\int_{\boldsymbol{R}^{2n}}dx_1\,E[\Big(\frac{S_0(1)}{m}\Big)^{k_1}e^{i\lambda S_0(1)}M^\varepsilon_m(1)\,\delta_{x_1}\Big(\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\quad\times\cdots\times\int_{\boldsymbol{R}^{2n}}dx_{l-1}\,E[\Big(\frac{S_{x_{l-2}}(1)}{m}\Big)^{k_{l-1}}e^{i\lambda S_{x_{l-2}}(1)}M^\varepsilon_m(1) \\
&\quad\times\delta_{x_{l-1}}\Big(x_{l-2}+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\quad\times\int_{\boldsymbol{R}^{2n}}dx_l\,E[\Big(\frac{S_{x_{l-1}}(1)}{m}\Big)^{k_l}e^{i\lambda S_{x_{l-1}}(1)}(M^\varepsilon_m(1)-I) \\
&\quad\times\delta_{x_l}\Big(x_{l-1}+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\quad\times\int_{\boldsymbol{R}^{2n}}dx_{l+1}\,E[\Big(\frac{S_{x_l}(1)}{m}\Big)^{k_{l+1}}e^{i\lambda S_{x_l}(1)}\,\delta_{x_{l+1}}\Big(x_l+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\quad\times\cdots\times\int_{\boldsymbol{R}^{2n}}dx_{m-1}\,E[\Big(\frac{S_{x_{m-2}}(1)}{m}\Big)^{k_{m-1}}e^{i\lambda S_{x_{m-2}}(1)} \\
&\quad\times\delta_{x_{m-1}}\Big(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\quad\times E[\Big(\frac{S_{x_{m-1}}(1)}{m}\Big)^{k_m}e^{\lambda S_{x_{m-1}}(1)}\,\delta_x\Big(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\Big)]\,.
\end{aligned}
\tag{4.26}
$$

This identity is easily proven from (4.1). □

5. Proof of Theorem 2.1 and properties of the heat kernels

In this section, we prove Theorem 2.1. Furthermore we prove some properties of the kernel. We name the right hand side of (2.12) as follows:

$$
\begin{aligned}
&h_t((x,u,r),(x',u',r')) \\
&\quad=\boldsymbol{p}_t^{n-2q+2}((x,u)^{-1}(x',u'))\,e_t^-(r,r')\,P+\boldsymbol{p}_t^{n-2q}((x,u)^{-1}(x',u'))\,e_t^+(r,r')\,Q \\
&\quad\quad+\boldsymbol{q}_t^{n-2q}((x,u)^{-1}(x',u'),r+r')\,Q \\
&\quad=:h_t^{(1)}((x,u,r),(x',u',r'))+h_t^{(2)}((x,u,r),(x',u',r')) \\
&\quad\quad+h_t^{(3)}(x,u,r),(x',u',r'))\,.
\end{aligned}
\tag{5.1}
$$

First we prove that $h_t^{(i)}$ has the following good regularity properties.

Proposition 5.1. *If $i\neq 3$ or $q\neq 0$ (resp. $i=3$ and $q=0$), $h_t^{(i)}(X,X')$ is the restriction to $\bar{D}$ of a rapidly decreasing function of X for each fixed $(t,X')\in(0,\infty)$*

$\times\bar{D}$ (*resp.* $(0,\infty)\times D$) *and the restriction to* $\bar{D}$ *of a rapidly decreasing function of* X' *for each fixed* $(t, X)\in(0,\infty)\times\bar{D}$ (*resp.* $(0,\infty)\times D$) *uniformly with respect to* $t\in[t_0, t_1]$ *for any* $0<t_0\leqq t_1$. *Morevoer* $h_t^{(i)}(X, X')$ *is a smooth function of* $(t, X, X')\in(0,\infty)\times D\times D$.

Proof. For any $\beta, \gamma\in \boldsymbol{Z}_+^{2n}$ and $k_1, k_2, k_3, k_4\in\boldsymbol{Z}_+$, we exchange the order of differentiations and integrate by parts in $\boldsymbol{q}_t$ formally:

$$x^\gamma u^{k_1} r^{k_2}\partial_x^\beta \frac{\partial^{k_3}}{\partial u^{k_3}}\frac{\partial^{k_4}}{\partial r^{k_4}}\boldsymbol{q}_t^\alpha(x,u,r)$$

$$(5.2)\qquad = \text{a linear combination of }\Big\{\int_{-\infty}^{\infty} d\lambda\cdot\lambda^{l_1}\exp(i\lambda u-t\lambda\alpha) \times x^\gamma\partial_x^\beta E[S_0(t)^{l_2}e^{-i\lambda S_0(t)}M(t)\,\delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)]\,r^{l_3}e^{r\lambda}\int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu e^{-\mu^2},$$

$$\int_{-\infty}^{\infty} d\lambda\cdot\lambda^{l_1}\exp(i\lambda u-t\lambda\alpha)\,x^\gamma\partial_x^\beta E[S_0(t)^{l_2}e^{-i\lambda S_0(t)}M(t)\,\delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)] \times r^{l_3}e^{r\lambda}\exp\Big(-\frac{t}{2}\Big(\frac{r}{t}+\lambda\Big)^2\Big)\Big\}_{0\leqq l_0,l_2,l_3\leqq k}$$

$$=:\underline{\boldsymbol{q}}_t(x,u,r),$$

where $k=1+k_1+k_2+k_3+k_4$. By using (2.21), the right hand side of (4.2) is estimated as follows:

$$\sup_{\substack{(x,u)\in\boldsymbol{R}^{2n+1}\\ r\geq r_0\\ t\in[t_0,t_1]}}\|\underline{\boldsymbol{q}}_t(x,u,r)\|$$

$$\leqq C(k,t_0,t_1)\sum_{0\leqq l_1,l_2,l_3\leqq k}\int_{-\infty}^{\infty} d\lambda\cdot|\lambda^{l_1}|\sup_{t\in[t_0,t_1]}\{\exp(-t\lambda\alpha)$$

$$(5.3)\qquad \times\sup_{x\in\boldsymbol{R}^{2n}}\|x^\gamma\partial_x^\beta E[S_0(t)^{l_2}e^{-i\lambda S_0(t)}M(t)\,\delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)]\|$$

$$\times\sup_{r\geqq r_0}|r|^{l_3}e^{r\lambda}\Big(\int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu e^{-\mu^2}+\exp\Big(-\frac{t}{2}\Big(\frac{r}{t}+\lambda\Big)^2\Big)\Big)\}$$

$$\leqq C(k,\gamma,\beta,t_0,t_1,\varepsilon,r_0)\sum_{0\leqq l_1,l_2,l_3\leqq 2k}\Big[\int_0^{\infty} d\lambda\cdot\lambda^{l_1}\sup_{t\in[t_0,t_1]}\{e^{-t\lambda\alpha} \times e^{\lambda(-n+\varepsilon)t}\exp\Big(-\frac{t}{4}\lambda^2\Big)\}$$

$$+\int_{-\infty}^{0} d\lambda\cdot(-\lambda)^{l_1}\sup_{t\in[t_0,t_1]}\{e^{-t\lambda\alpha}e^{r_0\lambda}e^{-\lambda(-n+\varepsilon)t}\}\Big]$$

for any $r_0\geqq 0$ and $\varepsilon>0$. The right hand side of (5.3) is finite if and only if $\alpha<n$ or $\alpha\leqq n$ and $r_0>0$. In this case the formal computations in (5.2) are justified and $\boldsymbol{q}_t^\alpha(x,u,r)$ is the restriction to $\boldsymbol{R}^{2n+1}\times[r_0,\infty)$ of a rapidly decreasing function uniformly with respect to $t\in[t_0,t_1]$ for any $0<t_0\leqq t_1$ and a smooth function of $(t,x,u,r)\in(0,\infty)\times\boldsymbol{R}^{2n+1}\times[r_0,\infty)$. By similar methods, we can prove that

$\boldsymbol{p}_t^\alpha(x, u)$ is a rapidly decreasing function of $(x, u)\in\boldsymbol{R}^{2n+1}$ and a smooth function of $(t, x, u)\in(0, \infty)\times\boldsymbol{R}^{2n+1}$. $\square$

Now, for any $f\in L_2^{p,q}(D)$, we define

$$(5.4)\quad (H_t^{(i)}f)(x, u, r):=\int_{\bar{D}} h_t^{(i)}((x, u, r), (x', u', r'))f(x', u', r')\, dx'\, du'\, dr' \quad (x, u, r)\in\bar{D},\ i=1, 2, 3\,.$$

For this $H_t^{(i)}$, we have the following (cf. [10] [11]):

Theorem 5.1. *$H_t^{(i)}$, $i=1, 2, 3$, are bounded operators on $L_2^{p,q}(D)$. Furthermore, for fixed $f\in\Lambda_0^{p,q}(D)$, the following holds:*

(i) *for $T>0$, there is a constant C depending only on f and T such that for $t\in(0, T]$, $||H_t^{(i)}f||\leqq C$, $i=1, 2, 3$;*
(ii) *$H_t^{(1)}f\to Pf$, $H_t^{(2)}f\to Qf$, $H_t^{(3)}f\to 0$ in $L_2^{p,q}(D)$ and uniformly as $t\to 0$;*
(iii) *$H_t^{(i)}f$, $i=1, 2, 3$, are the restrictions of rapidly decreasing forms;*
(iv) *$H_t^{(i)}f$, $i=1, 2, 3$, are differentiable in t;*
(v) *$H_tf:=H_t^{(1)}f+H_t^{(2)}f+H_t^{(3)}f\in\mathrm{Dom}\,\square$;*
(vi) *$(\partial/\partial t+\square)H_tf=0$;*
(vii) *$\square H_tf=H_t\square f$.*

Proof. For any $r_0\geqq 0$, we set $D_{r_0}:=\{(x, u, r)\in D; r\geqq r_0\}$. Then, for any $r_0\geqq 0$ and $f\in L_2^{p,q}(D)$, we have

$$(5.5)\quad \begin{aligned} &\int_{D_{r_0}} dX||(H_t^{(i)}f)(X)||^2 \\ &\leqq\int_{D_{r_0}} dX\int_D dY||h_t^{(i)}(X, Y)||\int_D dY||h_t^{(i)}(X, Y)||\,||f(Y)||^2 \\ &\leqq\Big(\sup_{x\in D_{r_0}}\int_D dY||h_t^{(i)}(X, Y)||\Big)\int_D dY||f(Y)||^2\int_{D_{r_0}} dX||h_t^{(i)}(X, Y)|| \\ &\leqq\Big(\sup_{X\in D_{r_0}}\int_D dY||h_t^{(i)}(X, Y(||\Big)\Big(\sup_{Y\in D}\int_{D_{r_0}} dX||h_t^{(i)}(X, Y||\Big)\int_D dY||f(Y)||^2 \\ &\leqq\Big(\int_{D_{r_0}} dY||h_t^{(i)}(0, Y)||\Big)^2\int_D dY||f(Y)||^2\,. \end{aligned}$$

Hence $H_t^{(i)}$ is a bounded linear operator from $L_2^{p,q}(D)$ to $L_2^{p,q}(D_{r_0})$ if $r_0>0$ or $(i, q)\neq(3, 0)$. Especially, if $(i, q)\neq(3, 0)$, $H_t^{(i)}$ is a bounded linear operator on $L_2^{p,q}(D)$. In the followings, let $f\in\Lambda_0^{p,q}(D)$. We will prove (i)—(vii) in order.

(i): From (5.5), it is enough to show that

$$(5.6)\quad \sup_{\varepsilon\in(0,T]}\int_D ||h_{\varepsilon 2}^{(i)}(0, Y)||\, dY<\infty\,.$$

To do this for $i=1$ and 2, it is enough to show that

$$(5.7)\quad \sup_{\varepsilon\in(0,T]}\int_{\boldsymbol{R}^{n+1}} ||\boldsymbol{p}_{\varepsilon 2}^\alpha(x, u)||\, dxdu<\infty$$

for any $\alpha\in \boldsymbol{Z}$. We use scaling property of the Brownian motion and make the change of the variables ($x\to\varepsilon x$, $u\to\varepsilon u$, $\varepsilon\lambda\to\lambda$):

$$
\begin{aligned}
&\int_{\boldsymbol{R}^{2n+1}} \|\boldsymbol{p}^{\alpha}_{\varepsilon^2}(x,u)\|\, dxdu \\
&\quad = \int_{\boldsymbol{R}^{2n+1}} dxdu\, \Big\| \int_{-\infty}^{\infty} d\lambda\, \frac{1}{2\pi} \exp\Big(i\lambda u - \varepsilon\lambda\alpha - \frac{\lambda^2}{2}\Big) \\
&\qquad \times E\Big[e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big]\Big\| \\
&\quad \leqq C \sup_{(x,u)\in \boldsymbol{R}^{2n+1}} (1+|x|^2)^{2n}(1+u)^2 \Big\| \int_{-\infty}^{\infty} d\lambda \exp\Big(i\lambda u - \varepsilon\lambda\alpha - \frac{\lambda^2}{2}\Big) \\
&\qquad \times E\Big[e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big]\Big\| \qquad (5.8)\\
&\quad = C \sup_{(x,u)\in \boldsymbol{R}^{2n+1}} (1+|x|^2)^{2n} \Big\| \int_{-\infty}^{\infty} d\lambda e^{i\lambda u}\Big(1-\frac{\partial^2}{\partial\lambda^2}\Big) \\
&\qquad \times \Big\{\exp\Big(-\varepsilon\lambda\alpha - \frac{\lambda^2}{2}\Big) E\Big[e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big]\Big\}\Big\| \\
&\quad \leqq C(\alpha)(1+\varepsilon^2) \int_{-\infty}^{\infty} d\lambda (1+|\lambda|^2) \exp\Big(-\varepsilon\lambda\alpha - \frac{\lambda^2}{2}\Big) \\
&\qquad \times \sum_{k=0}^{2} \sup_{x\in \boldsymbol{R}^{2n}} (1+|x|^2)^{2n} \Big\| E\Big[S_0(1)^k e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big]\Big\| \\
&\quad \leq C(\alpha, T)\,.
\end{aligned}
$$

The last inequality follows from Theorem 2.2 (i). From (5.8), we obtain (5.7). For $i=3$, we use the following inequality instead of (5.5): if R is the distance from supp f to bD,

$$
\|H_t^{(3)} f\|_2 \leqq C\Big(\int_{\boldsymbol{R}^{2n+1}} dxdu \int_{\boldsymbol{R}}^{\infty} dr\, \|\boldsymbol{q}_t^{n-2q}(x,u,r)\|\Big) \|Qf\|_2\,. \qquad (5.9)
$$

As in (5.8), we use scaling property and make the change of variables ($x\to\varepsilon x$, $u\to\varepsilon u$, $r\to\varepsilon r$, $\varepsilon\lambda\to\lambda$):

$$
\begin{aligned}
&\int_{\boldsymbol{R}^{2n+1}} dxdu \int_{\boldsymbol{R}}^{\infty} dr\, \|\boldsymbol{q}^{\alpha}_{\varepsilon^2}(x,u,r)\| \\
&\quad = \frac{1}{\pi^{n+3/2}} \int_{\boldsymbol{R}^{2n+1}} dxdu \int_{R/\varepsilon}^{\infty} dr\, \Big\| \int_{-\infty}^{\infty} d\lambda\cdot\lambda \exp(i\lambda u - \varepsilon\alpha) \\
&\qquad \times E\Big[e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big] e^{r\lambda} \int_{(r+\lambda)/\sqrt{2}} d\mu e^{-\mu^2}\Big\| \\
&\quad \leqq C \int_{R/\varepsilon}^{\infty} dr \sup_{(x,u)\in \boldsymbol{R}^{2n+1}} (1+|x|^2)^{2n}(1+u^2) \Big\| \int_{-\infty}^{\infty} d\lambda\cdot\lambda \exp(i\lambda u - \varepsilon\lambda\alpha) \\
&\qquad \times E\Big[e^{-i\varepsilon\lambda S_0(1)} M^{\varepsilon}(1)\, \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)\Big] e^{r\lambda} \int_{(r+\lambda)/\sqrt{2}} d\mu e^{-\mu^2}\Big\| \\
&\quad = C \int_{R/\varepsilon}^{\infty} dr \sup_{(x,u)\in \boldsymbol{R}^{2n+1}} (1+|x|^2)^{2n} \Big\| \int_{-\infty}^{\infty} d\lambda e^{i\lambda u} \Big(1-\frac{\partial^2}{\partial\lambda^2}\Big) \{\lambda e^{-\varepsilon\lambda\alpha} \qquad (5.10)
\end{aligned}
$$

$$\times E[e^{-i\varepsilon\lambda S_0(1)} M^\varepsilon(1)\,\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\, e^{r\lambda}\int_{(r+\lambda)\sqrt{2}} d\mu e^{-\mu^2}\}||$$

$$\leqq C(\alpha)(1+\varepsilon^2)\int_{R/\varepsilon}^\infty dr(1+r^2)\int_{-\infty}^\infty d\lambda(1+\lambda^2)\, e^{-\varepsilon\alpha}$$

$$\times\sum_{k=0}^{2}\sup_{x\in\boldsymbol{R}^{2n}}(1+|x|^2)^{2n}\,||E[S_0(1)^k e^{-i\varepsilon\lambda S_0(1)} M^\varepsilon(1)\,\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]||$$

$$\times e^{r\lambda}\left(\int_{(r+\lambda)/\sqrt{2}}^\infty d\mu e^{-\mu^2}+\exp\left(-\frac{1}{2}(r+\lambda)^2\right)\right)$$

$$\leqq C(\alpha,T,\eta)\int_{R/\varepsilon}^\infty dr(1+r^2)\int_{-\infty}^\infty d\lambda(1+\lambda^2)\exp(-\varepsilon\lambda\alpha-\varepsilon|\lambda|(n-\eta)$$

$$+r\lambda)\left(\int_{(r+\lambda)/\sqrt{2}}^\infty d\mu e^{-\mu^2}+\exp\left(-\frac{1}{2}(r+\lambda)^2\right)\right)$$

for any $\eta>0$. The last inequality follows from Theorem 2.2 (i). Futhermore,

$$\int_{R/\varepsilon}^\infty dr(1+r^2)\int_{-\infty}^\infty d\lambda(1+\lambda^2)\exp(-\varepsilon\lambda\alpha-\varepsilon|\lambda|(n-\eta)+r\lambda)$$

$$\times\exp(-\frac{1}{2}(r+\lambda)^2)$$

$$\leqq C(\alpha,T,\eta)\exp\left(-\frac{R^2}{4\varepsilon^2}\right),$$

and

$$\int_{R/\varepsilon}^\infty dr(1+r^2)\int_0^\infty d\lambda(1+\lambda^2)\exp(-\varepsilon\lambda\alpha-\varepsilon|\lambda|(n-\eta)+r\lambda)$$

$$\times\int_{(r+\lambda)/\sqrt{2}}^\infty d\mu e^{-\mu^2}$$

$$\leqq C(\alpha,T,\eta)\exp\left(-\frac{R^2}{4\varepsilon^2}\right).$$

For $0<\eta<R/T^2$ and $\alpha\leqq n$, we have

$$\int_{R/\varepsilon}^\infty dr(1+r^2)\int_{-\infty}^0 d\lambda(1+\lambda^2)\exp(-\varepsilon\lambda\alpha-\varepsilon|\lambda|(n-\eta)+r\lambda)\int_{(r+\lambda)/\sqrt{2}}^\infty d\mu e^{-\mu^2}$$

$$=\int_{R/\varepsilon}^\infty dr(1+r^2)\exp\left(-\frac{r^2}{2}\right)\int_0^\infty d\mu\exp(-\mu^2-\sqrt{2}\,r\mu)$$

$$\times\int_0^\infty d\lambda(1+\lambda^2)\exp\left(-\frac{\lambda^2}{2}+\lambda(\sqrt{2}\,\mu+\varepsilon(\alpha-n+\eta))\right)$$

$$\leqq C(\alpha,T,\eta)\int_{R/\varepsilon}^\infty dr(1+r^2)\exp\left(-\frac{r^2}{2}\right)\int_0^\infty d\mu\exp(\sqrt{2}\,\mu(\varepsilon(\alpha-n+\eta)-r))$$

$$+\frac{\varepsilon^2}{2}(\alpha-n+\eta)^2)$$

$$\leqq C(\alpha,T,\eta)\int_{R/\varepsilon}^\infty dr(1+r^2)\exp\left(-\frac{r^2}{2}\right)\int_0^\infty d\mu\exp\left(-\sqrt{2}\,\varepsilon\mu\left(\frac{R}{T^2}-\eta\right)\right)$$

$$\leqq C(\alpha, T, \eta, R)\frac{1}{\varepsilon}\int_{R/\varepsilon}^{\infty} dr(1+r^2)\exp\left(-\frac{r^2}{2}\right)$$
$$\leqq C(\alpha, T, \eta, R)\exp\left(-\frac{R^2}{4\varepsilon^2}\right).$$

Hence if $\alpha \leqq n$, we have

$$\int_{R^{2n+1}} dxdu \int_R^{\infty} dr\, \|q_{\varepsilon^2}^{\alpha}(x, u, r)\| \leqq C(\alpha, T, R)\exp\left(-\frac{R^2}{4\varepsilon^2}\right) \tag{5.11}$$

for any $\varepsilon \in [0, T]$. This lead to (i) for $i=3$.

(ii) To prove (ii) for $i=1$ and 2, it is enough to show that

$$\int_D p_t^{\alpha}((x, u)^{-1}(x', u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r-r')^2}{2t}\right)f(x', u', r')\,dx'du'dr' \to f(x, u, r) \tag{5.12}$$

and

$$\int_D p_t^{\alpha}((x, u)^{-1}(x', u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r+r')^2}{2t}\right)f(x', u', r')\,dx'\,du'\,dr' \to 0 \tag{5.13}$$

in $L_2^{p,q}(D)$ and uniformly as $t \to 0$. We set

$$p_t^{0,\alpha}(x, u) := \int_{-\infty}^{\infty} d\lambda\, f_t^{0,\alpha}(x, u, \lambda)\exp\left(-\frac{t}{2}\lambda^2\right),$$
$$p_t^{-,\alpha}(x, u) := p_t^{\alpha}(x, u) - p_t^{0,\alpha}(x, u)\, I.$$

As in (5.5), we have

$$\begin{aligned}&\int_D \Big\| \int_D p_t^{-,\alpha}((x, u)^{-1}(x'\, u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r\pm r')^2}{2t}\right)\\ &\qquad \times f(x'\, u'\, r')\,dx'\,du'\,dr'\Big\|^2 dx\,du\,dr\\ &\leqq C\left(\int_{R^{2n+1}} \|p_t^{-,\alpha}(x, u)\|\,dx\,du\right)^2 \|f\|_2^2.\end{aligned} \tag{5.14}$$

On the other hand, it is easy to see that

$$\begin{aligned}&\sup_{(x,u,r)\in D}\Big\|\int_D p_t^{-,\alpha}((x, u)^{-1}(x'\, u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r\pm r')^2}{2t}\right)\\ &\qquad \times f(x', u', r')\,dx'\,du'\,dr'\Big\|\\ &\leqq \int_{R^{2n+1}} \|p_t^{-,\alpha}(x, u)\|\,dxdu\, \|f\|_\infty.\end{aligned} \tag{5.15}$$

If we set $t=\varepsilon^2$, as in (5.8), we have

$$\int_{R^{2n+1}} \|p_{\varepsilon^2}^{-,\alpha}(x, u)\|\,dxdu$$

$$\leqq C(T,\alpha)\sum_{k=0}^{2}\sup_{\lambda\in\boldsymbol{R},x\in\boldsymbol{R}^{2n}}(1+|x|^2)^{2n}\|E[S_0(1)^k e^{-i\lambda S_0(1)} \times(M^\varepsilon(1)-I)\delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\| \leqq C(T,\alpha)\,\varepsilon \tag{5.16}$$

for $0<\varepsilon\leqq T$. Here we used Theorem 2.2 (ii). Hence

$$\int_D p_t^{-,\alpha}((x,u)^{-1}(x'\,u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r\pm r')^2}{2t}\right)f(x',u',r')\,dx'\,du'\,dr' \to 0 \tag{5.17}$$

in $L_2^{p,q}(D)$ and uniformly as $t\to 0$. The following facts are proved in Stanton [10]:

$$\int_D p_t^{0,\alpha}((x,u)^{-1}(x',u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r-r')^2}{2t}\right)f(x',u',r')\,dx'\,du'\,dr' \to f(x,u,r), \tag{5.18}$$

and

$$\int_D p_t^{0,\alpha}((x,u)^{-1}(x',u'))\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(r+r')^2}{2t}\right)f(x',u',r')\,dx'\,du'\,dr' \to 0 \tag{5.19}$$

in $L_2^{p,q}(D)$ and uniformly as $t\to 0$. From (5.16–18), we conclude (5.12) and (5.13).

Next we will consider $H_t^{(3)}f$. If R is the distance from $\operatorname{supp} f$ to bD,

$$\sup_{(x,u,r)\in D}\|(H_t^{(3)}f)(x,u,r)\| \leqq\left(\int_{\boldsymbol{R}^{2n+2}}dxdu\int_R^\infty dr\,\|q_t^{(n-2q)}(x,u,r)\|\right)\sup_{(x,u,r)\in D}\|Qf(x,u,r)\|. \tag{5.20}$$

Thus, by (5.9) and (5.20), to prove (ii) for $i=3$, it is enough to show that

$$\int_{\boldsymbol{R}^{2n+1}}dxdu\int_R^\infty dr\,\|q_t^\alpha(x,u,r)\|\to 0 \quad\text{as}\quad t\to 0 \tag{5.21}$$

for $\alpha\leqq n$. We set $t=\varepsilon^2$ and calculate like (5.10). Then for $\alpha\leqq n$ and $0<\varepsilon\leqq T$, we have

$$\int_{\boldsymbol{R}^{2n+1}}dxdu\int_R^\infty dr\,\|q_{\varepsilon^2}^\alpha(x,u,r)\|\leqq C(\alpha,T,R)\exp\left(-\frac{R^2}{4\varepsilon^2}\right), \tag{5.22}$$

which proves (5.21).

(iii) (iv): Since f has compact support, we can easily prove (iii) and (iv) by using Proposition 5.1.

(v): From (5.1), we easily see that

$$h_t^{(i)}((x, u, 0), (x', u', r')) = 0 . \tag{5.23}$$

On the other hand,

$$\begin{aligned}&\left(\frac{\partial}{\partial r}+i\frac{\partial}{\partial u}\right)\Bigg[\frac{1}{\sqrt{2\pi t}}\left\{\exp\left(-\frac{(r-r')^2}{2t}\right)+\exp\left(-\frac{(r+r')^2}{2t}\right)\right\}\\ &\quad\times\int_{-\infty}^{\infty} d\lambda\cdot \boldsymbol{f}_\alpha^t(x, u, r)\exp\left(-\frac{t}{2}\lambda^2\right)\\ &\quad-\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\lambda\cdot\lambda \boldsymbol{f}_t^\alpha(x, u, \lambda)\, e^{(r+r')\lambda}\int_{\sqrt{t/2}((r+r')/t+\lambda)}^{\infty} d\mu e^{-\mu^2}\Bigg]\Bigg|_{r=0}\\ &=0\end{aligned} \tag{5.24}$$

Hence we have

$$\left(\frac{\partial}{\partial r}-i\frac{\partial}{\partial u}\right)(h_t^{(2)}+h_t^{(3)})((x, u, 0), (x', u', r')) = 0 . \tag{5.25}$$

(vi): By Feynman-Kac formula, we see that

$$\frac{\partial}{\partial t}\boldsymbol{f}_t^\alpha((x, u)^{-1}(x', u'), \lambda) = -\Box_\lambda^\alpha \boldsymbol{f}_\lambda^\alpha((x, u)^{-1}(x', u'), \lambda) \tag{5.26}$$

where

$$\begin{aligned}\Box_\lambda^\alpha = &-\frac{1}{4}\sum_{j=1}^n\left\{\left(\frac{\partial}{\partial x^j}-2i\lambda\, x^{n+j}\right)^2+\left(\frac{\partial}{\partial x^{n+j}}+2i\lambda x^j\right)^2\right\}\\ &-\sum_{j=1}^n\frac{A_j}{\sqrt{2}}\left(\frac{\partial}{\partial x^j}-2i\lambda\, x^{n+j}\right)-\sum_{j=1}^n\frac{A_{n+j}}{\sqrt{2}}\left(\frac{\partial}{\partial x^{n+j}}+2i\lambda x^j\right)\\ &+B+\lambda\alpha .\end{aligned}$$

By using the fact that $\boldsymbol{f}_t^\alpha((x, u), \lambda)$ has good regularity, we have

$$\begin{aligned}&\frac{\partial}{\partial t}h_t^{(i)}((x, u, r), (x', u', r'))\\ &\quad= -\Box h_t^{(i)}(x, u, r), (x', u', r')),\quad t>0,\quad (x, u, r)\in\bar{D} ,\end{aligned} \tag{5.27}$$

for each $(x', u', r')\in\bar{D}$ and $i=1, 2, 3$ $((x', u', r')\in D$ for $q=0$ and $i=3)$.

(vii): By using Markovian property of the Brownian motion, we can prove that

$$\begin{aligned}&\Box_\lambda^\alpha\int_{H_n}\boldsymbol{f}_t^\alpha((x, u)^{-1}(x', u'), \lambda)\, g(x', u')\, dx'\, du'\\ &\quad=\int_{H_n}\boldsymbol{f}_t^\alpha((x, u)^{-1}(x', u'), \lambda)\,\Box_\lambda^\alpha g(x', u')\, dx'du'\end{aligned} \tag{5.28}$$

for any rapidly decreasing section on H_n to $\Lambda^{p,q}(T^*D)\restriction_{H_n}$ where $\Box_\lambda^\alpha$ is that of (5.26). Then we can esaily prove (vii).

Finally we prove that $H_t^{(3)}$ is a bounded operator on $L_2^{p,0}(D)$. By the above (i-vii) and the proof of Proposition 2.1, we see that

$$H_t f = e^{-t\Box} f \quad \text{for} \quad f \in \Lambda_0^{p,q}(\bar{D}) . \tag{5.29}$$

For any $r_0>0$, since both H_t and $e^{-t\Box}$ are bounded operators from $L_2^{p,0}(D)$ to $L_2^{p,0}(D_{r_0})$, we have

$$(H_t f)(X) = (e^{-t\Box} f)(X) \quad a.e. \qquad X \in D_{r_0}$$

for any $f \in L_2^{p,q}(D)$. Hence we have

$$H_t f = e^{-t\Box} f \quad \text{for any} \quad f \in L_2^{p,q}(D) . \tag{5.30}$$

Since $e^{-t\Box}$ is a bounded operator on $L_2^{p,0}(D)$, H_t is also a bounded operator on $L_2^{p,0}(D)$. $\Box$

By Proposition 5.1 and Theorem 5.1, we see that the integral operator involved by the kernel $h_t(X, Y)$ satisfies the conditions of a fundamental solution in Section 2. Hence we conclude Theorem 2.1.

6. Short time asymptotic behavior

In this section, we examine the behaviour of h_t on the diagonal of $\bar{D}\times\bar{D}$ as $t\to 0$ (cf. Stanton [11] §6, Beals-Stanton [2]). By (5.1), we have

$$\begin{aligned} &h_t((x, u, r), (x, u, r)) \\ &= \boldsymbol{p}_t^{n-2q+2}(0) \frac{1}{\sqrt{2\pi t}} P - \boldsymbol{p}_t^{n-2q+2}(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right) P \\ &\quad + \boldsymbol{p}_t^{n-2q}(0) \frac{1}{\sqrt{2\pi t}} Q + \boldsymbol{p}_t^{n-2q}(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right) Q \\ &\quad + \boldsymbol{q}_t^{n-2q}(0, 2r) Q \\ &=: \boldsymbol{k}_t^{n-2q+2}(0) P - \boldsymbol{k}_t^{n-2q+2}(r) P \\ &\quad + \boldsymbol{k}_t^{n-2q}(0) Q + \boldsymbol{k}_t^{n-2q}(r) Q + \boldsymbol{q}_t^{n-2q}(0, 2r) Q \end{aligned} \tag{6.1}$$

where

$$\boldsymbol{k}_t^{\alpha}(r) = \boldsymbol{p}_t^{\alpha}(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right).$$

We examine the behaviour of $\boldsymbol{k}_t^{\alpha}(r)$ for $\alpha \in \boldsymbol{Z}$ and $\boldsymbol{q}_t^{\alpha}(0, 2r)$ for $\alpha \leqq n$ as $t\to 0$.

Proposition 6.1. *For any $\alpha \in \boldsymbol{Z}$,*

$$\lim_{t\to 0} t^{n+1} \boldsymbol{k}_t^{\alpha}(0) = \frac{1}{2\pi^{n+1}} I . \tag{6.2}$$

Proof. $\boldsymbol{k}_t^{\alpha}(0)$ is decomposed as follows:

$$k_t^\alpha(0) = p_t^\alpha(0)\frac{1}{\sqrt{2\pi t}} \tag{6.3}$$
$$= p_t^{0,\alpha}(0)\frac{1}{\sqrt{2\pi t}} I + p_t^{-,\alpha}(0)\frac{1}{\sqrt{2\pi t}}.$$

As in Stanton [11] Proposition 6.1,

$$\lim_{t\to 0} t^{n+1} p_t^{0,\alpha}(0)\frac{1}{2\pi t} = \frac{1}{2\pi^{n+1}}. \tag{6.4}$$

For the second term of the right hand side of (6.3), we take $t=\varepsilon^2$ and use scaling property of the Brownian motion and Theorem 2.2 (ii): for any $\varepsilon\in(0, T]$,

$$\begin{aligned} &||\varepsilon^{2(n+1)} p_{\varepsilon^2}^{-,\alpha}(0)\frac{1}{\sqrt{2\pi\varepsilon^2}}|| \\ &= ||\varepsilon^2\int_{-\infty}^{\infty} d\lambda\frac{1}{2\pi}\exp\left(-\varepsilon^2\lambda\alpha-\frac{\varepsilon^2}{2}\lambda^2\right) \\ &\quad\times E[e^{-i\varepsilon^2\lambda S_0(1)}(M^\varepsilon(1)-I)\,\delta_0\left(\frac{x(1)}{\sqrt{2}}\right)]|| \\ &\leqq C(T)\,\varepsilon^2. \end{aligned} \tag{6.5}$$

Hence we have

$$\lim_{t\to 0} t^{n+1} p_t^{-,\alpha}(0)\frac{1}{\sqrt{2\pi t}} = 0. \tag{6.6}$$

By (6.4) and (6.6), we conclude (6.2). □

Next we consider the boundary correction term:

Proposition 6.2. *For any $\alpha\in\mathbf{Z}$, $r>0$ and $k\in\mathbf{Z}$,*

$$\lim_{t\to 0} t^k k_t^\alpha(r) = 0. \tag{6.7}$$

For any $\alpha\leqq n$, $r>0$ and $k\in\mathbf{Z}$,

$$\lim_{t\to 0} t^k q_t^\alpha(0, 2r) = 0. \tag{6.8}$$

Proof. We easily see that

$$k_t^\alpha(r) = k_t^\alpha(0)\exp\left(-\frac{2r^2}{t}\right) \tag{6.9}$$

and $\lim_{t\to 0} t^k \exp(-2r^2/t)=0$. Thus we conclude (6.7) from (6.2). Since

$$e^{2r\lambda}\int_{\sqrt{t/2}(2r/t+\lambda)} d\mu e^{-\mu^2} \leqq \begin{cases} \exp\left(-\dfrac{2r^2}{t}\right)\displaystyle\int_{\sqrt{t/2}\lambda}^{\infty} d\mu e^{-\mu}, \\ \exp\left(-\dfrac{r^2}{t}+\sqrt{2}\,r\lambda\right)\displaystyle\int_{\sqrt{t/2}(r/\sqrt{2t}+\lambda)}^{\infty} d\mu e^{-\mu^2}, \end{cases}$$

we have

$$(6.10)\quad \begin{aligned}&\|q_t^\alpha(0, 2r)\|\\ &\leqq C\{\exp(-2r^2/t)\int_0^\infty d\lambda\cdot\lambda\,\|f_t^\alpha(0,\lambda)\|\int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2}\\ &\quad+\exp\left(-\frac{r^2}{t}\right)\int_{-\infty}^0 d\lambda\cdot(-\lambda)\,\|f_t^\alpha(0,\lambda)\|\exp(\sqrt{2r}\,\lambda)\\ &\quad\times\int_{\sqrt{t/2}(r/\sqrt{2t}+\lambda)}^\infty d\mu e^{-\mu^2}\}\\ &\leqq C'\exp\left(-\frac{r^2}{t}\right)\{\int_0^\infty d\lambda\cdot\lambda\,\|f_t^\alpha(0,\lambda)\|\exp\left(-\frac{t\lambda^2}{4}\right)\\ &\quad+\int_{-\infty}^0 d\lambda\cdot(-\lambda)\,\|f_t^\alpha(0,\lambda)\|\,e^{\sqrt{2}r\lambda}\}\,.\end{aligned}$$

As in (5.15), we take $t=\varepsilon^2(\leqq T^2)$ and use scaling property of the Brownian motion and Theorem 2.2 (i): for any $\varepsilon\in(0, T]$,

$$(6.11)\quad \begin{aligned}&\int_0^\infty d\lambda\cdot\lambda\,\|f_{\varepsilon^2}^\alpha(0,\lambda)\|\exp\left(-\frac{\varepsilon^2\lambda^2}{4}\right)\\ &=C\varepsilon^{-2n}\int_0^\infty d\lambda\cdot\lambda e^{-\varepsilon^2\lambda\alpha}\,\|E[e^{-i\varepsilon^2\lambda S_0(1)}M^\varepsilon(1)\,\delta_0\left(\frac{x(1)}{\sqrt{2}}\right)]\|\exp\left(-\frac{\varepsilon^2\lambda^2}{4}\right)\\ &\leqq C(T,\alpha)\,\varepsilon^{-2(n+1)}\,.\end{aligned}$$

For $0<\eta<\sqrt{2}r/T^2$ and $\alpha\leqq n$,

$$(6.12)\quad \begin{aligned}&\int_{-\infty}^0 d\lambda\cdot(-\lambda)\,\|f_{\varepsilon^2}^\alpha(0,\lambda)\|\,e^{\sqrt{2}r\lambda}\\ &=C\varepsilon^{-2n}\int_{-\infty}^0 d\lambda\cdot(-\lambda)\,e^{-\varepsilon^2\lambda\alpha}\,\|E[e^{-i\varepsilon^2\lambda S_0(2)}M^\varepsilon(1)\,\delta_0\left(\frac{x(1)}{\sqrt{2}}\right)]\|\,e^{\sqrt{2}r\lambda}\\ &\leqq C(\alpha, T, \eta)\,\varepsilon^{-2(n+2)}\,.\end{aligned}$$

Hence, if $\alpha\leqq n$, for any $k\in\boldsymbol{Z}$,

$$(6.13)\quad \begin{aligned}&\|t^k q_t^\alpha(0, 2r)\|\\ &\leqq C(T,\alpha)\exp\left(-\frac{r^2}{t}\right)t^{-(n+2)+k}\to 0\quad\text{as}\quad t\to 0\,.\end{aligned}$$

□

Next result suggest the short time asymptotic behaviour of the trace of the heat semigroup by Beals and Stanton [2]:

Proposition 6.3. (i) *For any $\alpha\in\boldsymbol{Z}$,*

$$(6.14)\quad \lim_{t\to 0} t^{n+1/2}\int_0^\infty dr\,k_t^\alpha(r)=\frac{1}{4\sqrt{2}\pi^{n+1/2}}I\,.$$

(ii) *For any $\alpha<n$,*

$$\lim_{t\to 0} t^{n+1}\int_0^\infty dr\, \boldsymbol{q}_t^\alpha(0, 2r) \tag{6.15}$$
$$= \frac{1}{2\pi^{n+1}}\int_0^\infty d\tau \left(\frac{\tau}{\sinh \tau}\right)^n e^{\alpha\tau} I .$$

Proof. (i) follows from (6.2) and (6.9). We consider (ii), $\boldsymbol{q}_t^\alpha(0, 2r)$ is decomposed as follows:

$$\boldsymbol{q}_t^\alpha(0, 2r) = q_t^{0,\alpha}(0, 2r)\, I + \boldsymbol{q}_t^{-,\alpha}(0, 2r) . \tag{6.16}$$

As is shown in Stanton [11] Theorem 2.2 (i),

$$\lim_{t\to 0} t^{n+1}\int_0^\infty dr\, q_t^{0,\alpha}(0, 2r) = \frac{1}{2\pi^{n+1}}\int_0^\infty d\tau \left(\frac{\tau}{\sinh \tau}\right)^n e^{\alpha\tau} . \tag{6.17}$$

On the other hand,

$$\begin{aligned}
&\left\|\int_0^\infty dr\, \boldsymbol{q}_t^{-,\alpha}(0, 2r)\right\| \\
&\leqq C\int_{-\infty}^\infty d\lambda |\lambda|\, \|\boldsymbol{f}_t^{-,\alpha}(0, \lambda)\| \\
&\quad \times \int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2}\int_0^{\sqrt{t/2}\mu - t\lambda/2} dr e^{2r\lambda} \\
&= C\int_{-\infty}^\infty d\lambda\, \|\boldsymbol{f}_t^{-,\alpha}(0, \lambda)\| \\
&\quad \times \left|\frac{\sqrt{\pi}}{2}\exp\left(-\frac{t}{2}\lambda^2\right) - \int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2}\right| \\
&\leqq C'\left\{\int_0^\infty d\lambda \cdot \exp\left(-\frac{t}{4}\lambda^2\right) \|\boldsymbol{f}_t^{-,\alpha}(0, \lambda)\|\right. \\
&\quad \left. + \int_{-\infty}^0 d\lambda\, \|\boldsymbol{f}_t^{-,\alpha}(0, \lambda)\|\right\} .
\end{aligned} \tag{6.18}$$

Now we calculate as in (6.11) and (6.12) by using Theorem 2.2 (ii): for $\varepsilon\in [0, T]$,

$$\int_0^\infty d\lambda \cdot \exp\left(-\frac{\varepsilon_2}{4}\lambda^2\right) \|\boldsymbol{f}_{\varepsilon^2}^{-,\alpha}(0, \lambda)\| \leqq C(T)\, \varepsilon^{-2n} \tag{6.19}$$

and

$$\int_{-\infty}^0 d\lambda\, \|\boldsymbol{f}_{\varepsilon^2}^{-,\alpha}(0, \lambda)\| \leqq C(T)\, \varepsilon^{-2n-1} . \tag{6.20}$$

Hence we have

$$\left\| t^{n+1}\int_0^\infty dr\, \boldsymbol{q}_t^{-,\alpha}(0, 2r)\right\| \leqq C(T)\sqrt{t} \to 0 \quad \text{as} \quad t\to 0 . \tag{6.21}$$

From (6.17) and (6.21), we conclude (6.15). □

7. The analogue of the Siegel domain with a nondegenerate indefinite Levi form

For $0\leqq\kappa\leqq n$, let $D_\kappa=\{(z,w)\in C^n\times C\colon \operatorname{Im} w>\sum_{j=1}^n \varepsilon_j|z^j|^2\}$ where $\varepsilon_j=1$ for $1\leqq j\leqq\kappa$ and $\varepsilon_j=-1$ for $\kappa<j\leqq n$. This is an analogue of the Siegel domain D with a nondegenerate indefinite Levi from having κ positive and $n-\kappa$ negative eigenvalues. In [10], Stanton states that her results also apply to this domain in the $(0,q)$-form case. In the general (p,q)-form case, we need a few remarks to apply our methods (cf. Remark 7.1 (i) below). Our main theorem is Theorem 7.2 bellow.

We consider the Hermitian metric for which $\{Z_1,Z_2,\cdots,Z_{n+1}\}$ is an orthonormal basis of $T^{(1,0)}(D_\kappa)$, where

$$Z_j=\frac{\partial}{\partial z^j}+2i\varepsilon_j\bar{z}^j\frac{\partial}{\partial w},\quad j=1,2,\cdots,n,\quad Z_{n+1}=i\sqrt{2}\frac{\partial}{\partial w}. \tag{7.1}$$

The dual basis is given by

$$\omega^j=dz^j,\quad \omega^{n+1}=\frac{1}{i\sqrt{2}}(dw-2i\sum\varepsilon_j\bar{z}^jdz^j). \tag{7.2}$$

Then $\bar\partial$ and $\bar\partial^*$ can be represented as follows:

$$\begin{aligned}\bar\partial&=\sum_{j=1}^{n+1}\operatorname{ext}(\bar\omega^j)\bar{Z}_j+\sqrt{2}\sum_{j=1}^{n}\varepsilon_j\operatorname{ext}(\omega^j)\operatorname{ext}(\bar\omega^j)\operatorname{int}(\bar\omega^{n+1}),\\ \bar\partial^*&=-\sum_{j=1}^{n+1}\operatorname{int}(\omega^j)Z_j+\sqrt{2}\sum_{j=1}^{n}\varepsilon_j\operatorname{ext}(\omega^{n+1})\operatorname{int}(\omega^j)\operatorname{int}(\bar\omega^j),\end{aligned} \tag{7.3}$$

on $\mathcal{S}^{p,q}(\bar D_\kappa)$.

We use as coordinates on D_κ (z,u,r) where $u=\operatorname{Re} w$ and $r=\operatorname{Im} w-\sum_{j=1}^n\varepsilon_j|z^j|^2$. Then we can regard D_κ as product of the boundary bD_κ and $\boldsymbol{R}^+$. We identify bD_κ with the Heisenberg group H_n. The group law on H_n is

$$\begin{gathered}(z',u')(z,u)=(z'+z,u'+u+2\sum_{j=1}^n\varepsilon_j(x'^{n+j}x^j-x'^jx^{n+j}))\\ \text{for } (z',u'),(z,u)\in H_n,\end{gathered} \tag{7.4}$$

where $z^j=x^j+ix^{n+j}$. The metric we gave is the product metric of an invariant mectric on H_n and the standard metric on $\boldsymbol{R}^+$. In terms of these coordinates,

$$Z_j=\frac{\partial}{\partial z^j}+i\varepsilon_j\bar{z}^j\frac{\partial}{\partial u},\quad j=1,2,\cdots,n,\quad Z_{n+1}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r}+i\frac{\partial}{\partial u}\right). \tag{7.5}$$

Then we have

$$Z_j=1/2(X_j-iX_{n+j})$$

where

(7.6) $X_j = \frac{\partial}{\partial x^j} + 2\varepsilon_j x^{n+j} \frac{\partial}{\partial u}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - 2\varepsilon_j x^j \frac{\partial}{\partial u}, \quad j = 1, 2, \cdots, n.$

If we define the projections P and Q by

$$P = \operatorname{ext}(\bar{\omega}^{n+1}) \operatorname{int}(\omega^{n+1}) \quad \text{and} \quad Q = \operatorname{int}(\omega^{n+1}) \operatorname{ext}(\bar{\omega}^{n+1}),$$

the $\bar{\partial}$-Neumann boundary condition can be rewritten as

$$Pf \restriction_{bD_\kappa} = Q\Big(\frac{\partial}{\partial r} - i\frac{\partial}{\partial u}\Big) f \restriction_{bD_\kappa} = 0 \quad \text{for} \quad f \in \mathcal{S}^{p,q}(\bar{D}_\kappa). \tag{7.7}$$

The $\bar{\partial}$-Laplacian $\Box$ is expressed on $\mathcal{S}^{p,q}(\bar{D}_\kappa)$ as follows:

$$\Box = -\frac{1}{4}\sum_{j=1}^{2n} X_j^2 - \frac{1}{2}\Big(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2}\Big) + iU\frac{\partial}{\partial u} - \sum_{j=1}^{2n} \frac{A_j}{\sqrt{2}} X_j - B \tag{7.8}$$

where

$$U = \sum_{j=1}^{n} \varepsilon_j [\operatorname{int}(\omega^j), \operatorname{ext}(\bar{\omega}^j)],$$

$$A_j = \varepsilon_j(\operatorname{ext}(\omega^{n+1}) \operatorname{int}(\bar{\omega}^j) - \operatorname{ext}(\omega^j) \operatorname{int}(\bar{\omega}^{n+1})), \quad j = 1, 2, \cdots, n,$$

$$A_{n+j} = i\varepsilon_j(\operatorname{ext}(\omega^{n+1}) \operatorname{int}(\bar{\omega}^j) - \operatorname{ext}(\omega^j) \operatorname{int}(\bar{\omega}^{n+1})), \quad j = 1, 2, \cdots, n,$$

$$B = -2\sum_{j,k=1}^{n} \varepsilon_j \varepsilon_k \operatorname{ext}(\omega^j) \operatorname{ext}(\bar{\omega}^j) \operatorname{int}(\omega^k) \operatorname{int}(\bar{\omega}^k)$$

$$-2\sum_{j=1}^{n} (\operatorname{int}(\omega^j) \operatorname{ext}(\bar{\omega}^j) - \operatorname{ext}(\omega^j) \operatorname{int}(\bar{\omega}^j)) \operatorname{ext}(\omega^{n+1}) \operatorname{int}(\bar{\omega}^{n+1}).$$

REMARK 7.1. (i) The coefficient U of $\partial/\partial u$ in (7.8) does not commute with A_j and B. This is the only difficult point of the problem in the domain D_κ (cf. Lemma 7.2 and Theorem 7.1).

(ii) As in Remark 2.2, $\Box$ acts diagonally on $(0, q)$-forms and generally preserves the orthogonal decomposition $\Lambda^{p,q}(T^*D) = \operatorname{Ran} P \oplus \operatorname{Ran} Q$.

Now we consider the heat equation (1.1) for D_κ. Let $(W_0^{2(n+1)}, P)$ be a $2(n+1)$-dimensional Wiener space as in Section 3. We consider the following diffusion process $(X(t), U(t), R(t))$:

$$\begin{cases} X^j(t) = x^j + \dfrac{x^j(t)}{\sqrt{2}}, & j = 1, 2, \cdots, 2n, \\ U(t) = u + u(t) + S_x^\kappa(t) - i\phi(t), \\ R(t) = r + B(t) + \phi(t) \end{cases} \tag{7.9}$$

where

$$S_x^\kappa(t) = \sum_{j=1}^{n} \varepsilon_j \int_0^t \sqrt{2}(X^{n+j}(s)\circ dx^j(s) - X^j(s)\circ dx^{n+j}(s)),$$

$$B(t) = \int_0^t \operatorname{sgn}(r+r(s))dr(s)$$

and

$$\phi(t) = \lim_{\eta\searrow 0} \frac{1}{2\eta}\int_0^t I_{(-\eta,\eta)}(r+r(s))ds.$$

We define the End $(\Lambda^{p,q}(T^*D_\kappa))$-valued process $M(t, x(\cdot), u(\cdot))$ by the solution of the SDE

$$\text{(7.10)} \qquad \begin{cases} dM(t) = M(t)d\Xi(t), \\ M(0) = I \end{cases}$$

where

$$\Xi(t) = -iUu(t) + \sum_{j=1}^{2n} A_j x^j(t) + Bt.$$

Let $K(t)$ be the End $(\Lambda^{p,q}(T^*D_\kappa))$-valued process as in (3.5).

Then the heat kernel h_t is expected to be represented as follows

$$\text{(7.12)} \qquad h_t(X, X') = \int_{-\infty}^{\infty} d\lambda H_{t,\lambda}(X, X')$$

where

$$\text{(7.13)} \qquad \begin{aligned} &H_{t,\lambda}((x, u, r), (x', u', r')) \\ &= \frac{1}{2\pi} e^{i\lambda(u'-u)} E^x[e^{-i\lambda S_x^\kappa(t)} E^u[e^{-i\lambda u(t)} M(t, x(\cdot), u(\cdot))]\delta_{x'}(X(t))] \\ &\quad \times E[e^{-\lambda\phi(t)} K(t) \mid R(t) = r'] r_t(r, r') \end{aligned}$$

where E^u is the expectation with respect to the 1-dimensional Brownian motion $u(\cdot)$ and E^x is the generalized expectation of the generalized Wiener functional with respect to the $2n$-dimensional Brownian motion $x(\cdot)$.

For any $\varepsilon > 0$, $m \in \boldsymbol{N}$ and $\lambda \in \boldsymbol{R}$, we define the End $(\Lambda^{p,q}(T^*D_\kappa))$-valued processes $m_m^\lambda(t)$ and $M_m^{\lambda;\varepsilon}(t)$ by the solutions of

$$\text{(7.14)} \qquad \begin{cases} dm_m^\lambda(t) = m_m^\lambda(t)\left(-\dfrac{\lambda}{m} U\right) dt, \\ m_m^\lambda(0) = I \end{cases}$$

and

$$\text{(7.15)} \qquad \begin{cases} dM_m^{\lambda;\varepsilon}(t) = M_m^{\lambda;\varepsilon}(t) d\Xi_m^{\lambda;\varepsilon}(t), \\ M_m^{\lambda;\varepsilon}(0) = I \end{cases}$$

respectively, where

$$\Xi_m^{\lambda,\varepsilon}(t)=\sum_{j=1}^{2n}\frac{\varepsilon}{\sqrt{m}}A_jx^j(t)+\frac{\varepsilon^2}{m}Bt-\lambda\frac{\varepsilon^2}{m}Ut.$$

When $m=1$, we omit the subscript m: $\Xi_1^{\lambda,\varepsilon}(t)=:\Xi^{\lambda,\varepsilon}(t)$, $M_1^{\lambda,\varepsilon}(t)=:M^{\lambda,\varepsilon}(t)$, $m_1^{\lambda}(t)=:m^{\lambda}(t)$. When $m=\varepsilon=1$, we also omit the superscript ε: $\Xi_1^{\lambda,1}(t)=:\Xi^{\lambda}(t)$, $M_1^{\lambda,1}(t)=:M^{\lambda}(t)$.

Then $H_{t,\lambda}(X,X')$ is rewritten as follows:

Lemma 7.2. *$H_{t,\lambda}(X,X')$ is expressed as follows:*

$$\begin{aligned}(7.16)\qquad &H_{t,\lambda}((x,u,r),(x',u',r'))\\ &=\boldsymbol{f}_t((x,u)^{-1}(x',u'),\lambda)\exp\left(-\frac{t}{2}\lambda^2\right)e_t^-(r,r')P\\ &+\boldsymbol{f}_t((x,u)^{-1}(x',u'),\lambda)\exp\left(-\frac{t}{2}\lambda^2\right)e_t^+(r,r')Q\\ &-\frac{2}{\sqrt{\pi}}\lambda\boldsymbol{f}_t((x,u)^{-1}(x',u'),\lambda)e^{r\lambda}\int_{\sqrt{t/2}(r/t+\lambda)}^{\infty}d\mu\, e^{-\mu^2}Q\end{aligned}$$

where

$$\boldsymbol{f}_t(x,u,\lambda)=\frac{1}{2\pi}e^{i\lambda u}E[e^{-i\lambda S_0^{\kappa}(t)}M^{\lambda}(t)\delta_x\left(\frac{x(t)}{\sqrt{2}}\right)].$$

Proof. Generally for a square integrable continuous $\sigma(x(s),u(s)\colon s\leqq t)$-adapted process $\Phi(\cdot)$, it holds that

$$E^u\left[\int_0^t\Phi(s)dx^j(s)\right]=\int_0^t E^u[\Phi(s)]dx^j(s),\ 1\leqq j\leqq 2n.$$

Then, by the uniqueness of the solution of (7.15), we obtain

$$(7.17)\qquad M^{\lambda}(t)=E^u[e^{-i\lambda u(t)}M(t,x(\cdot),u(\cdot))]\exp\left(\frac{t}{2}\lambda^2\right).$$

Hence we have

$$\begin{aligned}(7.18)\qquad &E^x[e^{-\lambda S_x^{\kappa}(t)}E^u[e^{-i\lambda u(t)}M(t,x(\cdot),u(\cdot))]\,\delta_{x'}(x(t))]\\ &=\exp\left(-i\lambda\sum_{j=1}^{n}2\varepsilon_j(x^jx'^j-x^jx'^{n+j})-\frac{t}{2}\lambda^2\right)\\ &\times E[e^{-i\lambda S_0^{\kappa}(t)}M^{\lambda}(t)\delta_{x'-x}\left(\frac{x(t)}{\sqrt{2}}\right)].\end{aligned}$$

By combining (7.13),(7.18) and (3.14), we conclude (7.16). □

We first note the following theorem (cf. Theorem 2.2).

Theorem 7.1. *We take $\alpha, \beta \in \boldsymbol{Z}_+^{2n}$, $k \in \boldsymbol{Z}_+$ and $\varepsilon_0 > 0$ arbitrary.*

(i) *If $q \neq n-\kappa$,*

$$\overline{\lim_{\lambda\to-\infty}} \frac{1}{|\lambda|} \log \sup_{\substack{x\in\boldsymbol{R}^{2n} \\ 0<\varepsilon\leqq\varepsilon_0}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(1)} M^{\lambda/\varepsilon^2,\varepsilon}(1) \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)] Q\| \tag{7.19}$$
$$\leqq -2\,.$$

Generally

$$\overline{\lim_{\lambda\to\pm\infty}} \frac{1}{|\lambda|} \log \sup_{\substack{x\in\boldsymbol{R}^{2n} \\ 0<\varepsilon\leqq\varepsilon_0}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(1)} M^{\lambda/\varepsilon^2,\varepsilon}(1) \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)] \| \tag{7.20}$$
$$\leqq 0\,.$$

(ii) *If $q \neq n-\kappa$,*

$$\overline{\lim_{\lambda\to-\infty}} \frac{1}{|\lambda|} \log \sup_{\substack{x\in\boldsymbol{R}^{2n} \\ 0<\varepsilon\leqq\varepsilon_0}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(1)} (M^{\lambda/\varepsilon^2,\varepsilon}(1) - m^\lambda(1)) \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)] Q/\varepsilon\| \tag{7.21}$$
$$\leqq -2\,.$$

Generally,

$$\overline{\lim_{\lambda\to\pm\infty}} \frac{1}{|\lambda|} \log \sup_{\substack{x\in\boldsymbol{R}^{2n} \\ 0<\varepsilon\leqq\varepsilon_0}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(1)} M^{\lambda/\varepsilon^2,\varepsilon}(1) - m^\lambda(1)) \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)] /\varepsilon\| \tag{7.22}$$
$$\leqq 0\,.$$

Corollary. *We take $\alpha, \beta \in \boldsymbol{Z}_+^{2n}$ and ε_0 arbitrary. If $q \neq n-k$,*

$$\overline{\lim_{\lambda\to-\infty}} \sup_{t_0\leqq t\leqq t_1} \frac{1}{t|\lambda|} \log \sup_{x\in\boldsymbol{R}^{2n}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(t)} M^\lambda(t) \delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)] Q\| \tag{7.23}$$
$$\leqq -2\,.$$

Generally,

$$\overline{\lim_{\lambda\to\pm\infty}} \sup_{t_0\leqq t\leqq t_1} \frac{1}{t|\lambda|} \log \sup_{x\in\boldsymbol{R}^{2n}} \|x^\beta \partial_x^\alpha \frac{\partial^k}{\partial\lambda^k} E[e^{-i\lambda S_0^\kappa(t)} M^\lambda(t) \delta_x\Big(\frac{x(t)}{\sqrt{2}}\Big)] \| \tag{7.24}$$
$$\leqq 0\,.$$

The proof of Theorem 7.1 proceeds just like the proof of Theorem 2.2, using Lemmas 7.3 and 7.4 below: we omit the details.

Lemma 7.3. *For any $m \in \boldsymbol{N}$ and $\varepsilon > 0$,*

$$E[e^{i\lambda S_0^\kappa(1)} M^{\lambda/\varepsilon^2,\varepsilon}(1) \delta_x\Big(\frac{x(1)}{\sqrt{2}}\Big)]$$

$$
\begin{aligned}
&= \int_{\mathbf{R}^{2n}} dx_1\, E[e^{i\lambda S_0^\kappa(1)/m} M_m^{\lambda/\varepsilon^2,\varepsilon}(1)\, \delta_{x_1}\Big(\frac{x(1)}{\sqrt{2m}}\Big)] \\
(7.25) \qquad &\times \int_{\mathbf{R}^{2n}} dx_2\, E[e^{i\lambda S_{x_1}^\kappa(1)/m} M_m^{\lambda/\varepsilon^2,\varepsilon}(1)\, \delta_{x_2}\Big(x_1+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\times\cdots\times \int_{\mathbf{R}^{2n}} dx_{m-1}\, E[e^{i\lambda S_{x_{m-2}}(1)/m} M_m^{\lambda/\varepsilon^2,\varepsilon}(1) \\
&\times \delta_{x_{m-1}}\Big(x_{m-2}+\frac{x(1)}{\sqrt{2m}}\Big)] \\
&\times E[e^{i\lambda S_{x_{m-1}}(1)/m} M_m^{\lambda/\varepsilon^2,\varepsilon}(1)\, \delta_x\Big(x_{m-1}+\frac{x(1)}{\sqrt{2m}}\Big)]\,.
\end{aligned}
$$

Lemma 7.4. (i) *For any $p>1$, $s\in \mathbf{Z}_+$, $h\in \mathbf{Z}_+$ and $\lambda>0$,*

$$(7.26) \quad \Big\|\frac{\partial^h}{\partial\lambda^h} M_m^{\lambda/\varepsilon^2,\varepsilon}(1)\Big\|_{p,s} \leqq C(p,s,h)\, m^{-h} \exp\Big\{C(p,s)\Big(\Big(\frac{\varepsilon^2}{m}\Big)^{p/2}+\Big(\frac{|\lambda|}{m}\Big)\Big)\Big\}$$

and

$$
\begin{aligned}
(7.27) \quad &\Big\|\frac{\partial^h}{\partial\lambda^h}(M_m^{\lambda/\varepsilon^2,\varepsilon}(1)-m_m^\lambda(1))\Big\|_{p,s} \\
&\leqq C(p,s,h)\, m^{-h-1/2} \exp\Big\{C(p,s,h)\Big(\Big(\frac{\varepsilon^2}{m}\Big)^{p/2}+\Big(\frac{|\lambda|}{m}\Big)^p+\frac{|\lambda|}{m}\Big)\Big\}\,.
\end{aligned}
$$

(ii) *If $q\neq n-\kappa$,*

$$(7.28) \qquad \Big\|\frac{\partial^h}{\partial\lambda^h} m_m^\lambda(1)\, Q\Big\| \leqq Cm^{-h} \exp\Big\{\frac{|\lambda|(n-2)}{m}\Big\}$$

for any $\lambda\leqq 0$. Generally,

$$(7.29) \qquad \Big\|\frac{\partial^h}{\partial\lambda^h} m_m^\lambda(1)\Big\| \leqq Cm^{-h} \exp\Big\{\frac{|\lambda| n}{m}\Big\}\,.$$

By using Theorem 7.1, we can prove the following theorem:

Theorem 7.2. *The unique fundamental solution H_t of the heat equation for the $\bar\partial$-Neumann problem on (p, q)-forms on D has the following smooth kernel:*

$$
\begin{aligned}
&h_t((x,u,r),(x',u',r')) \\
(7.30) \qquad &= \boldsymbol{p}_t((x,u)^{-1}(x',u'))\, e_t^-(r,r')\, P + \boldsymbol{p}_t((x,u)^{-1}(x',u'))\, e_t^+(r,r')\, Q \\
&\quad + \boldsymbol{q}_t((x,u)^{-1}(x',u'), r+r')\, Q
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{p}_t(x,u) &= \int_{-\infty}^{\infty} d\lambda\, \boldsymbol{f}_t(x,u,\lambda)\exp\Big(-\frac{t}{2}\lambda^2\Big)\,, \\
\boldsymbol{q}_t((x,u),r) &= -\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\lambda\cdot\lambda\, \boldsymbol{f}_t(x,u,\lambda)\, e^{r\lambda}\int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu\, e^{-\mu^2}\,,
\end{aligned}
$$

and $\boldsymbol{f}_t(x, u, \lambda)$ *is defined in* (7.16).

As in Section 5, we can state properties of the heat kernel more precisely. We set

$$
\begin{aligned}
&h_t^{(1)}((x,u,r),(x',u',r')) = \boldsymbol{p}_t((x,u)^{-1}(x',u'))\, e_t^-(r,r')\, P\,,\\
(7.31)\qquad &h_t^{(2)}((x,u,r),(x',u',r')) = \boldsymbol{p}_t((x,u)^{-1}(x',u'))\, e_t^+(r,r')\, Q\,,\\
&h_t^{(3)}((x,u,r),(x',u',r')) = \boldsymbol{q}_t((x,u)^{-1}(x',u'), r+r')\, Q\,.
\end{aligned}
$$

Then we have the following:

Proposition 7.1. *If* $i\neq 3$ *or* $q\neq n-\kappa$ *(resp.* $i=3$ *and* $q=n-\kappa$*),* $h_t^{(i)}(X, X')$ *is the restriction to* $\bar{D}_\kappa$ *of a rapidly decreasing function of* X *for each fixed* $(t, X')\in(0,\infty)\times\bar{D}_\kappa$ *(resp.* $(0,\infty)\times D_\kappa$*) and the restriction to* $\bar{D}_\kappa$ *of a rapidly decreasing function of* X' *for each fixed* $(t, X)\in(0,\infty)\times\bar{D}_\kappa$ *(resp.* $(0,\infty)\times D_\kappa$*) uniformly with respect to* $t\in[t_0, t_1]$ *for any* $0<t_0<t_1$*. Moreover* $h_t^{(i)}(X, X')$ *is a smooth function of* $(t, X, X')\in(0,\infty)\times D_\kappa\times D_\kappa$.

For any $f\in L_2^{p,q}(D_\kappa)$, we define

$$
(H_t^{(i)}f)(X) := \int_{D_\kappa} h_t^{(i)}(X, Y) f(Y)\, dY, \quad X\in D_\kappa, \quad i=1,2,3\,.
$$

For this $H_t^{(i)}$, we have the following theorem:

Theorem 7.3. $H_t^{(i)}$, $i=1,2,3$, *are bounded operators on* $L_2^{p,q}(D_\kappa)$. *Furthermore, for fixed* $f\in\Lambda_0^{p,q}(D_\kappa)$*, the followings hold:*

(i) *for* $T>0$*, there is a constant* C *depending only on* f *and* T *such that for* $t\in(0, T]$, $\|H_t^{(i)}f\|\leqq C$, $i=1,2,3$;

(ii) $H_t^{(1)}f\to Pf$, $H_t^{(2)}f\to Qf$, $H_t^{(3)}f\to 0$ *in* $L_2^{p,q}(D_\kappa)$ *and uniformly as* $t\to 0$;

(iii) $H_t^{(i)}f$, $i=1,2,3$, *are the restrictions of rapidly decreasing forms*;

(iv) $H_t^{(i)}f$, $i=1,2,3$, *are differentiable in* t;

(v) $H_t f := H_t^{(1)}f + H_t^{(2)}f + H_t^{(3)}f \in \mathrm{Dom}\,\Box$;

(vi) $(\partial/\partial t + \Box) H_t f = 0$;

(vii) $\Box H_t f = H_t \Box f$.

Finally we investigate the behavior of $h_t(X, X')$ on the diagonal of $\bar{D}_\kappa\times\bar{D}_\kappa$ as $t\to 0$. As in (6.1), we have

$$
(7.32)\qquad h_t((x,u,r),(x,u,r)) = \boldsymbol{k}_t(0) - \boldsymbol{k}_t(r)\, P + \boldsymbol{k}_t(r)\, Q + \boldsymbol{q}_t(0, 2r)\, Q
$$

where

$$
\boldsymbol{k}_t(r) = \boldsymbol{p}_t(0)\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{2r^2}{t}\right).
$$

Let L be the Levi form on bD_κ and L^q be its extension to $\Lambda^q(T^{(1,0)}D_\kappa\cap \boldsymbol{C}T_z(bD_\kappa))$ (cf. Beals-Stanton [2] p. 407):

$$L_X^q(Z_{j_1}\wedge\cdots\wedge Z_{j_q}, Z_{k_1}\wedge\cdots\wedge Z_{k_q}) = (\sum_{l=1}^{q} 2\varepsilon_{j_l})\,\delta_{j_1k_1}\cdots\delta_{j_qk_q}$$

for $1\leqq j_1<\cdots<j_q\leqq n$ and $1\leqq k_1<\cdots<k_p\leqq n$. Then we have the followings:

Proposition 7.3. (i)

$$\lim_{t\to 0} t^{n+1}\, \boldsymbol{k}_t(0) = \frac{1}{2\pi^{n+1}} I\,. \tag{7.33}$$

(ii) *For any $r>0$ and $k\in \boldsymbol{Z}$,*

$$\lim_{t\to 0} t^k\, \boldsymbol{k}_t(r) = 0 \tag{7.34}$$

and

$$\lim_{t\to 0} t^k\, \boldsymbol{q}_t(0, 2r)\, Q = 0\,. \tag{7.35}$$

(iii)

$$\lim_{t\to 0} t^{n+1/2} \int_0^\infty dr\, \boldsymbol{k}_t(r) = \frac{1}{4\sqrt{2}\,\pi^{n+1/2}} I\,. \tag{7.36}$$

(iv) *When $q\neq n-\kappa$, we have*

$$\begin{aligned} &\lim_{t\to 0} t^{n+1}\int_0^\infty dr\cdot \mathrm{tr}\,[\boldsymbol{q}_t(0, 2r)\, Q] \\ &= \binom{n+1}{p}\frac{1}{2\pi^{n+1}}\int_0^\infty d\tau\ \mathrm{tr}\,[\exp(-\tau L^q)] \\ &\quad\times \exp(-\tau\ \mathrm{tr}^- L)\ \det\{\tau|L|(1-e^{-\tau|L|})^{-1}\}\,. \end{aligned} \tag{7.37}$$

where $\mathrm{tr}^- L$ *is the sum of the negative parts of the eigenvalues of L and* $|L|:=(L^*L)^{1/2}=(L^2)^{1/2}$.

Proof. The proofs of (i), (ii) and (iii) are similar to the corresponding results in Section 6. As in Proposition 6.3 (ii), we can prove the following: when $q\neq n-\kappa$,

$$\begin{aligned} &\lim_{t\to 0} t^{n+1}\int_0^\infty dr\, \boldsymbol{q}_t(0, 2r)\, Q \\ &= \sum_{0\vee(q+\kappa+n)\leqq l\leqq \kappa\wedge q} \frac{1}{2\pi^{n+1}}\int_0^\infty d\tau \left(\frac{\tau}{\sinh\tau}\right)^n e^{-(n-2\kappa-2q+4l)\tau}\, Q_l \end{aligned} \tag{7.38}$$

where Q_l is the projection onto the space

$$\mathrm{span}\,\{\overline{\omega}^{j_1}\wedge\cdots\wedge\overline{\omega}^{j_l}\wedge\overline{\omega}^{k_{l+1}}\wedge\cdots\wedge\overline{\omega}^{k_q} \mid j_1,\cdots,j_l\leqq\kappa<k_{l+1},\cdots,k_q\leqq n\}\,.$$

By taking the trace of the both hand sides in the equality (7.38), we have (7.37).
□

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