Decay of Solutions to the Cauchy Problem for the Linearized Boltzmann Equation with Some External-Force Potential

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§1 Introduction

The time evolution of rarefied gas in the external-force field derived from a potential $\phi = \phi(x)$ is described by the nonlinear Boltzmann equation with the external-force term

$$f_t + \Lambda(\phi)f = Q(f,f), \tag{1.1}$$

where $\Lambda(\phi) \equiv \xi\cdot\nabla_x - \nabla_x\phi\cdot\nabla_\xi$. $f = f(t,x,\xi)$ is an unknown function, which represents the density of gas particles at time $t \geqq 0$, at a point $x \in \mathbb{R}^3$ and with a velocity $\xi \in \mathbb{R}^3$. $\phi = \phi(x)$ is a sufficiently smooth real-valued function. $Q(\cdot,\cdot)$ is a bilinear symmetric operator defined as follows:

$$Q(g,h) \equiv (1/2)\int_{\xi' \in \mathbb{R}^3, s \in S^2} B(\theta, |\xi-\xi'|)\times$$

$$\times\{g(\eta)h(\eta')+g(\eta')h(\eta)-g(\xi)h(\xi')-g(\xi')h(\xi)\}d\xi' ds,$$

where $g(\eta) = g(t,x,\eta)$, etc., $\eta = \xi-((\xi-\xi')\cdot s)s$, $\eta' = \xi'+((\xi-\xi')\cdot s)s$, $\cos\theta = (\xi-\xi')\cdot s/|\xi-\xi'|$, $s \in S^2$. $B(\theta,V)$ is a non-negative given function of $(\theta,V) \in (-\pi,\pi)\times[0,+\infty)$. In the present paper we will impose the following assumption on $B(\theta,V)$ (cf. [2,(55)]):

<u>Assumption 1.1.</u> There exist constants $c_1>0$ and $0<\varepsilon_1<1$ such that

$$B(\theta,V)/|\sin\theta\cos\theta| \leqq c_1(V+V^{\varepsilon_1-1}).$$

Substituting $f = \Omega+\Omega^{1/2}u$, $\Omega \equiv \exp(-\phi(x)-|\xi|^2/2)$, in (1.1), and dropping the nonlinear term, we obtain the following linearized Boltzmann equation with the external-force term:

$$u_t = B(\phi)u, \tag{1.2}$$

where $B(\phi) \equiv A(\phi) + L_1(\phi)$, $A(\phi) \equiv -\Lambda(\phi) + \{\exp(-\phi)\}(-\nu)$, and $L_1(\phi) \equiv \{\exp(-\phi)\}K$. $\nu = \nu(\xi)$ is a multiplication operator, and K is a self-adjoint compact operator on $L^2(\mathbb{R}_\xi{}^3)$ (see [2]).

Our primary concern is to study decay of solutions to the Cauchy problem for (1.2) (we write $CP\phi$ for this Cauchy problem in what follows). It is important to obtain estimates for decay of the solutions, not only because they are interesting in themselves, but also because if it is proved that the solutions to $CP\phi$ decay sufficiently quickly, then we can very likely apply the decay estimates to demonstrate the existence of global solutions to the Cauchy problem for (1.1) with initial data sufficiently close to Ω (see Note 2.3).

It is conjectured that the order of decay of solutions to the Cauchy problem

$$\begin{cases} u_t = A(\phi)u, \\ u(0) = u_0, \end{cases} \tag{1.3}$$

has a close relation to that of the solutions to $CP\phi$. We also note that the solutions to $CP\phi$ are unlikely to decay more quickly than those to (1.3), because ν and K have properties described in Lemma 3.3,(i-iii) of the present paper. In addition, we can solve (1.3) more easily than $CP\phi$. For these reasons, we will first try to investigate the order of decay of the solutions to (1.3). It is clear that the semigroup $e^{tA(\phi)}$ has the form

$$(e^{tA(\phi)}f(\cdot,\cdot))(X,\Xi) = f(x(-t,X,\Xi),\xi(-t,X,\Xi))\exp\{-U(t,X,\Xi)\}, \qquad (1.4)$$

where

$$U(t,X,\Xi) \equiv \int_0^t e^{-\phi(x(-s,X,\Xi))}\nu(\xi(-s,X,\Xi))ds. \qquad (1.5)$$

$x = x(t,X,\Xi)$ and $\xi = \xi(t,X,\Xi)$ denote the solution to the Cauchy problem

$$\begin{cases} dx/dt = \xi, \quad d\xi/dt = -\nabla\phi(x), \\ (x,\xi)(0) = (X,\Xi). \end{cases} \qquad (1.6)$$

In view of (1.4-5), we conclude that the order of decay of $e^{tA(\phi)}$ varies considerably depending on whether the variable point $x = x(t,X,\Xi)$ runs in a domain of higher potential energy or of lower potential energy.

Based upon the considerations above, we conjecture that the order of decay of $e^{tB(\phi)}$ must also vary according to the behavior of the variable point $x = x(t,X,\Xi)$. Hence, we need to observe that variable point globally in time. In general, however, it is difficult to make such an observation, except in the case of some concrete potentials. For these reasons, in the present paper, we will investigate CPϕ with $\phi(x) = |x|^2/2$. The principal result is Theorem 2.2.

In what follows, we will write A and B for $A(\phi)$ and $B(\phi)$ with

$\phi(x) = |x|^2/2$, respectively. We note that B contains $\exp(-|x|^2/2)$, which is the multiplier of $(-\nu+K)$ and converges to 0 as $|x| \to +\infty$. Recalling the properties of ν and K stated in Lemma 3.3,(i-iii), we see that this fact presents a difficulty in trying to obtain estimates for decay of e^{tB}. We will be able to overcome this difficulty by a method similar to that in [1].

This paper consists of 7 sections. In §2 we introduce the notation and state the principal result. The purpose of §3 is to prove some lemmas which will be employed frequently later. In §4 we demonstrate some estimates for the semigroup generated by A and the resolvent operator of A. The aim of §5 is to obtain all the eigenvalues of B on the imaginary axis and their corresponding eigenfuctions. We prove estimates for some operators in §6. By making use of those estimates, we will prove Theorem 2.2 in §7.

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§2 The notation and the principal theorem

(1) Symbols. We write Λ and L_1 for $\Lambda(\phi)$ and $L_1(\phi)$ with $\phi(x) = |x|^2/2$, respectively. By $\mathbb{B}(X,Y)$ ($\mathbb{C}(X,Y)$ respectively) we denote the set of all bounded (compact respectively) linear operators from a Banach space X to a Banach space Y. The norm of operators of $\mathbb{B}(X,Y)$ is designated by $\| \ \|_{B(X,Y)}$. We write $\mathbb{B}(X)$ and $\mathbb{C}(X)$ for $\mathbb{B}(X,X)$ and $\mathbb{C}(X,X)$ respectively. By $\mathfrak{D}(F)$, $\sigma(F)$ and $\sigma_p(F)$, we denote the domain, the spectrum and the point spectrum of an operator F, respectively. For operators F_j, $j = 1,2$, we designate the product F_2F_1 by $\Pi_{j=1}^{2}F_j$. By $Q(\mathbb{R}^3)$, we denote the set of all one-rank operators of the form $(Mu(\cdot))(\xi) = (u(\cdot), f(\cdot))g(\xi)$, where $f(\xi)$ and $g(\xi)$, $\xi \in \mathbb{R}^3$, are infinitely partially differentiable complex-valued functions with compact support. The brackets $(\cdot,\cdot)$ denote the inner product in $L^2(\mathbb{R}_\xi{}^3)$.

(2) The harmonic oscillator. $\phi(x) = |x|^2/2$ is a potential of a harmonic oscillator. The motion of this harmonic oscillator is described by the Cauchy problem (1.6) with $\phi(x) = |x|^2/2$. This Cauchy problem is solved as follows:

$$\begin{cases} x = x(t,X,\Xi) \equiv (\cos t)X + (\sin t)\Xi, \\ \xi = \xi(t,X,\Xi) \equiv -(\sin t)X + (\cos t)\Xi. \end{cases} \tag{2.1}$$

The trajectory of $x = x(t,X,\Xi)$, $0 \leqq t \leqq 2\pi$, is an ellipse whose center is the origin 0. By $p_0(X,\Xi)$ ($q_0(X,\Xi)$ respectively) we denote the half of the length of the major axis (the minor axis respectively) of this ellipse. Set

$$p(X, \Xi) \equiv p_0{}^2(X, \Xi)/2, \quad q(X, \Xi) \equiv q_0{}^2(X, \Xi)/2,$$

$$e_{\alpha,\beta,\gamma} = e_{\alpha,\beta,\gamma}(x, \xi) \equiv e^{\alpha E+\beta q(x,\xi)}(1+E)^{\gamma/2},$$

where $E = E(x, \xi) \equiv (|x|^2+|\xi|^2)/2$, $\alpha, \beta, \gamma \in \mathbb{R}$.

Let $1 \leqq w \leqq +\infty$. By $\chi_w = \chi_w(x, \xi)$ and $\rho_w = \rho_w(x, \xi)$, we denote the characteristic functions of the domains $q(w) \equiv \{(x, \xi);\ q(x, \xi) < w\}$ and $\{(x, \xi);\ E(x, \xi) < w\}$, respectively. Set $\underline{\chi}_w \equiv 1-\chi_w$ and $\underline{\rho}_w \equiv 1-\rho_w$. Let H be a set of functions defined on $\mathbb{R}_x{}^3 \times \mathbb{R}_\xi{}^3$. We write $\chi_w H$ for the subset $\{f \in H;\ \underline{\chi}_w f = 0\}$.

<u>(3) Function spaces.</u> Let $\alpha, \beta, \gamma \in \mathbb{R}$. By $E_\alpha(\mathbb{R}_\xi{}^3)$ we denote a Hilbert space of complex-valued functions on $\mathbb{R}_\xi{}^3$ with the inner product

$$(u, v)_{E_\alpha(\mathbb{R}_\xi{}^3)} \equiv \int_{\mathbb{R}_\xi{}^3} u(\xi)\overline{v(\xi)}\{\exp(\alpha|\xi|^2)\}d\xi.$$

By $E_{\alpha,\beta,\gamma}$ we designate a Hilbert space of complex-valued functions on $\mathbb{R}_x{}^3 \times \mathbb{R}_\xi{}^3$ with the inner product

$$(u, v)_{\alpha,\beta,\gamma} \equiv \int_{\mathbb{R}_x{}^3 \times \mathbb{R}_\xi{}^3} u(x, \xi)\overline{v(x, \xi)}e^2{}_{\alpha,\beta,\gamma}(x, \xi)dxd\xi.$$

Set $\|u\|_{\alpha,\beta,\gamma} = ((u,u)_{\alpha,\beta,\gamma})^{1/2}$. For simplicity, we write E_α, $E_{\alpha,\beta}$, $\|u\|_\alpha$ and $\|u\|_{\alpha,\beta}$ for $E_{\alpha,0,0}$, $E_{\alpha,\beta,0}$, $\|u\|_{\alpha,0,0}$ and $\|u\|_{\alpha,\beta,0}$, respectively.

We define sets of subscripts as follows:

$$S_1 \equiv \{(\alpha, \beta);\ -1/2<\alpha<1/2,\ 0<\beta<1\},$$

$$S_2 \equiv \{(\alpha, \beta);\ -1/2<\alpha<1/2,\ 0<\alpha+\beta/2<1/2\},$$

$$S_3 \equiv \{(\alpha, \beta);\ -1/2<\alpha<1/2,\ -1<\beta<1\},$$

$$S_4 \equiv \{(\alpha, \beta);\ -1/2<\alpha<1/2,\ -1/2<\alpha+\beta/2<1/2\},$$

$$S_5 \equiv S_1 \cap S_2, \quad S_6 \equiv S_3 \cap S_4,$$

$$S_7 \equiv \{(\alpha, \beta);\ (\alpha, \beta+1/2)\in S_5\}, \quad S_8 \equiv \{(\alpha, \beta)\in S_7;\ \alpha, \beta \geqq 0\},$$

$$S_9 \equiv \{(\alpha, \alpha', \beta');\ \alpha' \geqq \alpha,\ (\alpha, 0), (\alpha', \beta')\in S_8\}.$$

We set

$$(\Phi_j)_{j=1,2,3} \equiv (x\times\xi)\Omega^{1/2}, \quad \Phi_4 \equiv \Omega^{1/2}, \quad \Phi_5 \equiv E\Omega^{1/2},$$

$$\Phi_6 \equiv (|x|^2-|\xi|^2+2ix\cdot\xi)\Omega^{1/2}, \quad \Phi_7 \equiv (|\xi|^2-|x|^2+2ix\cdot\xi)\Omega^{1/2},$$

$$\Phi_j \equiv (x_{j-7}+i\xi_{j-7})\Omega^{1/2}, \quad j = 8, 9, 10,$$

$$\Phi_j \equiv (x_{j-10}-i\xi_{j-10})\Omega^{1/2}, \quad j = 11, 12, 13,$$

where $\Omega \equiv \exp(-E)$, $E \equiv E(x, \xi)$. x_j and ξ_j denote the j-th components of the vectors $x \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$ respectively, $j = 1, 2, 3$. Any eigenspaces corresponding to eigenvalues of B on the imaginary axis are spanned by Φ_j, $j = 1, \cdots, 13$ (see §5). By $-L_2$ we denote the projector in $L^2(\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ upon the linear space spanned by Φ_j, $j = 1, \cdots, 13$. Let H be a set of functions defined on $\mathbb{R}_x{}^3 \times \mathbb{R}_\xi{}^3$. We denote the subset $\{f \in H;\ L_2 f = 0\}$ by $H_\perp$.

(4) An assumption. In addition to Assumption 1.1, we will impose the following assumption on $B(\theta, V)$:

Assumption 2.1. There exists a positive constant c_2 such that

$$c_2 V \leqq B(\theta, V)/|\sin\theta\cos\theta|.$$

We need this assumption only when proving the first inequality of Lemma 3.3, (i).

(5) Conventions. We set the following conventions:

(i) The letter c denotes a positive constant. By $c(\theta)$ we designate a positive constant depending on the parameter $0 < \theta < 1$ such that $c(\theta) \uparrow +\infty$ as $\theta \uparrow 1$. We will use c and $c(\theta)$ as generic constants, and so they are not the same at each occurence.

(ii) Let L be a formally defined operator. Then L may happen to have various realizations in a variety of spaces. For simplicity, we will use the same symbol L for all such realizations. No confusion will arise.

(6) The principal theorem. The following theorem is the main result of the present paper, which will be proved in §7:

Theorem 2.2. If $(\alpha, \beta) \in S_3$ and $\gamma \geqq 0$, then the operator B, $\mathcal{D}(B) \equiv \{u \in E_{\alpha,\beta,\gamma};\ Au \in E_{\alpha,\beta,\gamma}\}$, generates a C_0-semigroup in $E_{\alpha,\beta,\gamma}$ such that for any $(\delta, \delta', \varepsilon') \in S_9$ and $\gamma \geqq 0$,

$$\| e^{tB} \|_{\mathbb{B}(E_{\delta', \varepsilon', \gamma, \perp}, E_{\delta, 0, \gamma, \perp})} \leqq c(\theta)(1+t)^{-a\theta}, \qquad t \geqq 0, \tag{2.2}$$

where $a \equiv 2(\delta' - \delta) + \varepsilon'$. $c(\theta)$ is independent of t.

We conclude the present section with the following:

Note 2.3. (i) By making use of Theorem 2.2, we can prove that if the initial data are sufficiently close to Ω, then there exists a unique global solution to the Cauchy problem for (1.1) with $\phi(x) = |x|^2/2$. We will discuss this subject in another paper which is in preparation.

(ii) There have been several studies on the existence of global solutions to the Cauchy problem for the nonlinear Boltzmann equation without an external force. These studies are made with the aid of decay estimates for solutions to the Cauchy problem for the linearized Boltzmann equation without an external force. See, e.g., [1] and [3].

§3 Some lemmas

Lemma 3.1. (i) For any $X, \Xi \in \mathbb{R}^3$ there exists some $k \in [0, 2\pi)$ such that for any $t \in \mathbb{R}$

$$|x(t, X, \Xi)|^2/2 = p(X, \Xi)\cos^2(t+k) + q(X, \Xi)\sin^2(t+k),$$

$$|\xi(t, X, \Xi)|^2/2 = p(X, \Xi)\sin^2(t+k) + q(X, \Xi)\cos^2(t+k).$$

(ii) $|X|^2/2, |\Xi|^2/2 \geqq q(X, \Xi)$, $E(X, \Xi) \geqq 2q(X, \Xi)$.

(iii) $E(X, \Xi) = E(x(t, X, \Xi), \xi(t, X, \Xi))$, $q(X, \Xi) = q(x(t, X, \Xi), \xi(t, X, \Xi))$,

(iv) $|D(x(t, X, \Xi), \xi(t, X, \Xi))/D(X, \Xi)| = 1$.

(v) If $x_2, \eta_k \in \mathbb{R}^3$ and $\tau_k \in \mathbb{R}$, $k = 1, 2$, then

$$|D(x_0, \xi_0, \tau_1, \tau_2)/D(\eta_1, \eta_2, \tau_1, \tau_2)| = |\sin^3 \tau_2|,$$

where $x_{k-1} \equiv x(-\tau_k, x_k, \eta_k)$, $\xi_{k-1} \equiv \xi(-\tau_k, x_k, \eta_k)$, $k = 1, 2$. (3.1)

Proof. We obtain (i), (iv) and (v) from (2.1). (i) implies (ii). (iii) is clear.

We will make use of Lemma 3.1, (i) ((v) respectively) in order to prove Lemma 3.5 (Lemma 6.3 respectively).

Lemma 3.2. (i) If $\alpha \leqq \alpha'$, $\beta \leqq \beta'$, $\gamma \leqq \gamma'$ and $\delta \geqq 0$, then

$$\|f\|_{\alpha, \beta+2\delta, \gamma} \leqq \|f\|_{\alpha'+\delta, \beta', \gamma'}.$$

(ii) If $\alpha \leqq \alpha'$, $\beta \leqq \beta'$, $\gamma \in \mathbb{R}$ and $1 \leqq w < +\infty$, then

$$\| \chi_w \|_{\mathbb{B}(E_{\alpha', \beta', \gamma}, E_{\alpha, \beta, \gamma})} \leqq \exp\{-2(\alpha' - \alpha)w - (\beta' - \beta)w\}.$$

(iii) If $\alpha \leqq \alpha'$, $\beta, \gamma \in \mathbb{R}$ and $1 \leqq w < +\infty$, then

$$\| \rho_w \|_{\mathbb{B}(E_{\alpha', \beta, \gamma}, E_{\alpha, \beta, \gamma})} \leqq \exp\{-(\alpha' - \alpha)w\}.$$

Proof. We obtain (i-ii) by making use of the last inequality of Lemma 3.1, (ii). (iii) is clear.

Lemma 3.3. (i) There exist positive constants ν_j, $j = 1, 2$, such that

$$\nu_1(1+|\xi|) \leqq \nu(\xi), \quad \nu(\xi) \leqq \nu_2(1+|\xi|).$$

(ii) K is a self-adjoint compact operator on $L^2(\mathbb{R}_\xi^3)$.

(iii) $(-\nu+K)$ is a non-positive operator on $L^2(\mathbb{R}_\xi^3)$ such that $(-\nu+K)f = 0$ iff f is a linear combination of $\phi_j \equiv \xi_j \omega^{1/2}$, $j = 1, 2, 3$, $\phi_4 \equiv \omega^{1/2}$ and $\phi_5 \equiv |\xi|^2 \omega^{1/2}$, where $\omega \equiv \exp(-|\xi|^2/2)$.

(iv) $K \in \mathbb{C}(E_\alpha(\mathbb{R}_\xi^3))$, if $-1/2 < \alpha < 1/2$.

(v) If $-1/2 < \alpha < 1/2$, $-1 < \beta \leqq 0 \leqq \beta' < 1$, $\beta' - \beta < 1$ and $\gamma \geqq 0$, then $L_1 \in \mathbb{B}(E_{\alpha, \beta, \gamma}, E_{\alpha, \beta', \gamma+1})$.

Proof. See [2] for (ii), (iii) and the second inequality of (i). By Assumption 2.1, we can obtain the first inequality of (i), in the same way as that in proving the second inequality of (i).

Let us prove estimates for $K_\alpha(\xi, \eta)$, where $K_\alpha(\xi, \eta)$ is the integral kernel of the operator $K_\alpha \equiv \{\exp(\alpha|\xi|^2/2)\}K\exp(-\alpha|\xi|^2/2)$. Performing calculations similar to those in [2], we deduce, from Assumption 1.1, that

$$|K_\alpha(\xi,\eta)| \leqq c(|\xi-\eta|+|\xi-\eta|^{\varepsilon_1-1})\exp\{-C_\alpha(|\xi|^2+|\eta|^2)\} +$$

$$+c|\xi-\eta|^{-1}\exp\{-C_\alpha(|\xi-\eta|^2+(|\xi|^2-|\eta|^2)^2|\xi-\eta|^{-2})\}, \qquad (3.2)$$

where C_α is a positive constant such that $C_\alpha \downarrow 0$ as $\alpha \uparrow 1/2$ or $\alpha \downarrow -1/2$. It follows from (3.2) that

$$m_r K_\alpha(\xi,\eta) \in L^2(\mathbb{R}_\xi{}^3 \times \mathbb{R}_\eta{}^3), \qquad (3.3)$$

where $m_r = m_r(\xi)$ is the characteristic function of the domain $|\xi| \leqq r$. In the same way as that in [2], we conclude from (3.2) that for any $n \geqq 0$

$$\int |K_\alpha(\xi^1,\xi^2)|(1+|\xi^j|^2)^{-n/2}d\xi^j \leqq c(1+|\xi^k|^2)^{-(1+n)/2}, \qquad (3.4)$$

where $(j,k) = (1,2),(2,1)$. Applying (3.4) and Schwarz's inequality to the operator $\underline{m}_r K_\alpha$, $\underline{m}_r \equiv 1-m_r$, we see that $\|\underline{m}_r K_\alpha\|_{\mathbb{B}(L^2(\mathbb{R}_\xi{}^3))} \leqq c(1+r^2)^{-1/2}$. It follows from this inequality and (3.3) that $K_\alpha \in \mathbb{C}(L^2(\mathbb{R}_\xi{}^3))$, which implies (iv). With the aid of Lemma 3.1,(ii) and (3.4), we can get (v).

We will make use of the first inequality of Lemma 3.3,(i) only when proving Lemma 3.5.

<u>Lemma 3.4.</u> (i) If $\alpha \in \mathbb{R}$, $-1 \leqq \beta \leqq 0$ and $M \in \mathbb{B}(E_\alpha(\mathbb{R}_\xi{}^3))$, then

$$\|\{\exp(-|x|^2/2)\}M\|_{\mathbb{B}(E_{\alpha,\beta},E_{\alpha,\beta+1})} \leqq c\|M\|_{\mathbb{B}(E_\alpha(\mathbb{R}_\xi^3))},$$

where c is independent of M.

(ii) If $\alpha,\beta,\gamma\in\mathbb{R}$ and $M\in Q(\mathbb{R}_\xi^3)$, then $\{\exp(-|x|^2/2)\}M\in\mathbb{B}(E_{\alpha,\beta}, E_{\alpha+1,\gamma})$.

(iii) If $-1/2<\alpha<1/2$ and $-1\leqq\beta\leqq 0$, then $L_1\in\mathbb{B}(E_{\alpha,\beta},E_{\alpha,\beta+1})$.

(iv) Let $-1/2<\alpha<1/2$ and $\varepsilon>0$. Then there exists a decomposition $L_1 = \ell_1+\ell_2$ such that

$$\|\ell_1\|_{\mathbb{B}(E_{\alpha,\beta},E_{\alpha,\beta+1})} \leqq \varepsilon \quad \text{for any } -1\leqq\beta\leqq 0,$$

$$\ell_2 = \{\exp(-|x|^2/2)\}M,$$

where M is a finite sum of operators $\in Q(\mathbb{R}_\xi^3)$.

(v) If $(\alpha,\beta),(\alpha',\beta')\in S_4$ and $\gamma,\gamma'\in\mathbb{R}$, then $L_2\in\mathbb{C}(E_{\alpha,\beta,\gamma}, E_{\alpha',\beta',\gamma'})$.

<u>Proof.</u> With the aid of Lemma 3.1,(ii), we obtain (i), (ii) and (v). Lemma 3.3,(iv) and (i) of the present lemma imply (iii-iv).

<u>Lemma 3.5.</u> There exists a constant $c_{3.5}>0$ such that for any $t\geqq 0$ and any $(X,\Xi)\in\mathbb{R}_X^3\times\mathbb{R}_\Xi^3$

$$U(t,X,\Xi) \geqq c_{3.5}\{\exp(-q(X,\Xi))\}t - 2\pi c_{3.5},$$

where $U(t,X,\Xi)$ is that in (1.5) with $\phi(x) = |x|^2/2$.

<u>Proof.</u> Applying Lemma 3.1,(i) and the first inequality of Lemma 3.3,(i) to $U(2\pi,X,\Xi)$, we deduce that

$$U(2\pi, X, \Xi) \geqq ce^{-q}\int_0^{2\pi} e^{-(p-q)\cos^2 t}(1+(p-q)^{1/2}|\sin t|)dt \geqq c\exp(-q),$$

where $p = p(X, \Xi)$, $q = q(X, \Xi)$. c is independent of (X, Ξ). Noting that the integrand of $U(t, X, \Xi)$ is a periodic function whose period is equal to 2π, we see from the above inequality that there exists a constant $c_{3.5}>0$ such that

$$U(t, X, \Xi) \geqq 2\pi c_{3.5}\{\exp(-q(X, \Xi))\}[t/2\pi]. \tag{3.5}$$

This inequality implies the present lemma.

We will make use of this lemma in the next section.

§4 The operator A

Let $\alpha, \beta, \gamma \in \mathbb{R}$. With the aid of Lemma 3.1,(iii-iv), we see that A, $\mathcal{D}(A) \equiv \{u \in E_{\alpha,\beta,\gamma};\ Au \in E_{\alpha,\beta,\gamma}\}$, generates a C_0-semigroup in $E_{\alpha,\beta,\gamma}$, which has the form (1.4) with $\phi(x) = |x|^2/2$. Let us study this semigroup.

Lemma 4.1. (i) If $\alpha' \geqq \alpha$, $\beta' \geqq \beta$ and $\gamma \in \mathbb{R}$, then

$$\| e^{tA} \|_{\mathbb{B}(E_{\alpha',\beta',\gamma}, E_{\alpha,\beta,\gamma})} \leqq c(1+t)^{-a}, \qquad t \geqq 0,$$

where $a \equiv 2(\alpha'-\alpha)+(\beta'-\beta)$. c is independent of t.

(ii) If $\alpha, \beta, \gamma \in \mathbb{R}$ and $1 \leqq w \leqq +\infty$, then

$$\| e^{tA} \|_{\mathbb{B}(\chi_w E_{\alpha,\beta,\gamma})} \leqq c\exp\{-c_{3.5}(\exp(-w))t\}, \quad t \geqq 0$$

where c is independent of t and w. See Lemma 3.5 for $c_{3.5}$.

(iii) Let $\gamma \geqq -c_{3.5}\{\exp(-w)\}/2$, $1 \leqq w \leqq +\infty$, $\alpha, \beta \in \mathbb{R}$ and $f \in \chi_w E_{\alpha,\beta}$. For simplicity, we write A_j, $j = 1,2$, for A and $A^* \equiv \Lambda + \{\exp(-|x|^2/2)\}(-\nu)$, respectively. Then,

$$\int_0^{+\infty} e^{-2\gamma t} \| e^{tA_j} f \|^2_{\alpha,\beta-1/2} dt \leqq c \| f \|^2_{\alpha,\beta}, \quad j = 1,2,$$

where c is independent of γ, w and f.

Proof. We write (4.1) for (1.4) with $\phi(x) = |x|^2/2$. Replacing f by $e_{-m,-n,0} f$, $m,n > 0$, in (4.1), and applying Lemma 3.1,(ii-iii) and Lemma 3.5 to the right hand side, we deduce that

$$|(e^{tA}(e_{-m,-n,0}(\cdot,\cdot)f(\cdot,\cdot)))(X,\Xi)| \leq$$

$$c|f(x(-t,X,\Xi),\xi(-t,X,\Xi))|\exp\{-c_{3.5}(\exp(-q(X,\Xi)))t-bq(X,\Xi)\}, \qquad (4.2)$$

where $b \equiv 2m+n$. We easily see that $\max_{q\geqq 0} \exp\{-c_{3.5}(\exp(-q))t-bq\} \leq c(1+t)^{-b}$ for any $t\geqq 0$. Applying this inequality and Lemma 3.1,(iv) to (4.2), and noting that $e^{tA}e_{\alpha,\beta,\gamma} = e_{\alpha,\beta,\gamma}e^{tA}$, we obtain (i).

Let $m,n = 0$ and $f\in \chi_w L^2(\mathbb{R}_x{}^3\times\mathbb{R}_\xi{}^3)$ in (4.2). Recalling the definition of $\chi_w L^2(\mathbb{R}_x{}^3\times\mathbb{R}_\xi{}^3)$, we see that we can replace $q(X,\Xi)$ by w in this inequality. From the inequality thus obtained and Lemma 3.1,(iv), we have (ii) with $\alpha,\beta,\gamma = 0$. Hence, we get (ii) for any α, β and γ.

Let $m,n = 0$ and $f\in \chi_w E_{0,1/2}$ in (4.2), and raise both sides to the second power. Moreover, multiply both sides by $\exp(-2\gamma t)$, and integrate them over $\mathbb{R}_X{}^3\times\mathbb{R}_\Xi{}^3\times[0,+\infty)_t$. Because $\underline{\chi}_w f = 0$, we can apply the following fact to the inequality thus obtained:

If $(X,\Xi) \in q(w)$ and $\gamma \geq -c_{3.5}\{\exp(-w)\}/2$,

then $-2\gamma t-2c_{3.5}\{\exp(-q(X,\Xi))\}t \leqq -c_{3.5}\{\exp(-q(X,\Xi))\}t$.

Hence,

$$J \leqq c\int_0^{+\infty}\int_{X,\Xi}|f(x(-t,X,\Xi),\xi(-t,X,\Xi))|^2\times$$

$$\times(\exp(-c_{3.5}\{\exp(-q(X,\Xi))\}t))dXd\Xi dt,$$

where J denotes the left hand side of the inequality in (iii) with $(\alpha,\beta) =$

$(0,1/2)$, $j = 1$. Change the variables in the right hand side of this inequality as follows: $(X, \Xi) \to (x(-t,X,\Xi), \xi(-t,X,\Xi))$. By means of Lemma 3.1,(iii-iv), we have (iii) with $(\alpha,\beta) = (0,1/2)$, $j = 1$. Hence, recalling that $e^{tA}e_{\alpha,\beta,\gamma} = e_{\alpha,\beta,\gamma}e^{tA}$, we obtain (iii), $j = 1$, for any α and β. (iii) with $j = 2$ can be proved in the same way.

Lemma 4.2. Let $-1/2<\alpha<1/2$, $\gamma\geqq 0$, $0<a<1$ and $C_0\geqq 0$. If $f = f(t)$ is a continuous function from $[0,+\infty)_t$ to $E_{\alpha,0,\gamma}$, and satisfies

$$\| f(t) \|_{\alpha,0,\gamma} \leqq C_0(1+t)^{-a}, \quad t\geqq 0,$$

then
$$\| Z(f)(t) \|_{\alpha,0,\gamma+1} \leqq c(\theta)C_0(1+t)^{-a\theta}, \quad t\geqq 0,$$

where
$$Z(f)(t) \equiv \int_0^t e^{(t-s)A_{L_1}} f(s)ds.$$

Proof. Combining Lemma 4.1,(i) and Lemma 3.3,(v) with $\beta = 0$, we have

$$\| e^{(t-s)A_{L_1}} f(s) \|_{\alpha,0,\gamma+1} \leqq c(\beta')\{1+(t-s)\}^{-\beta'} \| f(s) \|_{\alpha,0,\gamma}.$$

Integrate both sides with respect to $s\in[0,t]$, and apply the inequality

$$\int_0^t \{1+(t-s)\}^{-m}(1+s)^{-n}ds \leqq c(1+t)^{1-m-n}, \quad 0 < m,n < 1.$$

By replacing $1-a-\beta'$ by $-a\theta$, $0<\theta<1$, in the inequality thus obtained, we get the present lemma (see §2,(5),(i)).

Let us pick out the integral kernel of $(\mu-A)^{-1}$. We will express $(\mu-A)^{-1}$ in terms of the Laplace transform of e^{tA}. Let $f\in\chi_w E_{\alpha,\beta,\gamma}$, $1\leqq w\leqq+\infty$ and $\alpha,\beta,\gamma\in\mathbb{R}$ in (4.1). Multiply both sides of this equality by $\exp(-\mu t)$, $\mathrm{Re}\,\mu > -c_{3.5}\exp(-w)$, and integrate them over $[0,+\infty)_t$. Moreover, decompose $[0,+\infty)_t$ as follows: $[0,+\infty)_t = \cup_{m=0}^{\infty}[2\pi m, 2\pi(m+1))_t$, and replace the variable of integration $t\in[2\pi m, 2\pi(m+1))$ by $2\pi m+s$ $(0\leqq s\leqq 2\pi)$ for each $m\in\mathbb{N}\cup\{0\}$. Substituting the equality $U(2\pi m+s,X,\Xi) = mU(2\pi,X,\Xi) + U(s,X,\Xi)$ in the integrals thus obtained, we formally get

$$((\mu-A)^{-1}f(\cdot,\cdot))(X,\Xi) =$$

$$\int_0^{2\pi} R(\mu,s,X,\Xi)f(x(-s,X,\Xi),\xi(-s,X,\Xi))ds, \qquad (4.3)$$

where

$$R(\mu,s,X,\Xi) \equiv \{\exp(-\mu s-U(s,X,\Xi))\}\Sigma_{m=0}^{\infty}\exp\{-m(2\pi\mu+U(2\pi,X,\Xi))\}. \qquad (4.4)$$

Let us consider operators of the following form:

$$(\mathbb{R}(\mu,M)u(\cdot,\cdot))(X,\Xi)$$

$$\equiv \int_{t\in M} R(\mu,t,X,\Xi)u(x(-t,X,\Xi),\xi(-t,X,\Xi))dt,\quad M\subseteq[0,2\pi). \qquad (4.5)$$

<u>Lemma 4.3.</u> Let $\alpha,\beta,\gamma\in\mathbb{R}$.

(i) $$\sup_{\mu\in D(w),1\leqq w\leqq+\infty}\|\mathbb{R}(\mu,M)\|_{\mathbb{B}(\chi_w E_{\alpha,\beta,\gamma},\,\chi_w E_{\alpha,\beta-1,\gamma})} \leqq cm(M),$$

for any $M \subseteq [0, 2\pi)$, where c is independent of M. m(M) denotes the Lebesgue measure of M. $D(w) \equiv \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu \geqq -c_{3.5}\{\exp(-w)\}/2\}$, $1 \leqq w \leqq +\infty$.

$$\text{(ii)} \quad \| \mathbb{R}(\mu, M) - \mathbb{R}(\lambda, M) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta,\gamma},\, \chi_w E_{\alpha,\beta-2,\gamma})} \leqq c|\mu - \lambda| m(M),$$

for any $\mu, \lambda \in D(w)$, $1 \leqq w \leqq +\infty$ and $M \subseteq [0, 2\pi)$, where c is independent of μ, λ, w and M.

Proof. First we will prove (ii). It follows from (3.5) with $t = 2\pi$ that if $(X, \Xi) \in q(w)$, $\mu \in D(w)$, and $1 \leqq w \leqq +\infty$, then

$$-2\pi \mathrm{Re}\,\mu - U(2\pi, X, \Xi) \leqq -\pi c_{3.5} \exp\{-q(X, \Xi)\}.$$

Applying this inequality to (4.4), we see that if $(X, \Xi) \in q(w)$, $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$, then

$$R(\mu, s, X, \Xi) = \{\exp(-\mu s - U(s, X, \Xi))\}/\{1 - \exp(-2\pi\mu - U(2\pi, X, \Xi))\}. \quad (4.6)$$

Hence, for $n = 0, 1$,

$$\sup_{\mu \in D(w),\, 0 \leqq t \leqq 2\pi} |(\partial/\partial\mu)^n R(\mu, t, X, \Xi)|$$

$$\leqq c\{1 - \exp(-\pi c_{3.5} \exp\{-q(X, \Xi)\})\}^{-(n+1)} \leqq c\exp\{(n+1) q(X, \Xi)\}. \quad (4.7)$$

It follows from this inequality with $n = 1$ that if $\mu, \lambda \in D(w)$, $(X, \Xi) \in q(w)$ and $1 \leqq w \leqq +\infty$, then

$$\sup_{0 \leqq t \leqq 2\pi} |R(\mu, t, X, \Xi) - R(\lambda, t, X, \Xi)| \leqq c|\mu - \lambda| \exp\{2q(X, \Xi)\}.$$

Applying this inequality and Schwarz's inequality to the integral

$$I \equiv \int_{t\in M} |R(\mu,t,X,\Xi)-R(\lambda,t,X,\Xi)||u(x(-t,X,\Xi),\xi(-t,X,\Xi))|dt,$$

where $u\in\chi_w E_{0,2}$, we conclude that

$$I^2 \leqq c|\mu-\lambda|^2 m(M)\int_{t\in M}\{\exp(4q(X,\Xi))\}|u(x(-t,X,\Xi),\xi(-t,X,\Xi))|^2 dt.$$

Integrate both sides over $\mathbb{R}_X{}^3\times\mathbb{R}_\Xi{}^3$, apply Lemma 3.1,(iii-iv), and note that $\mathbb{R}(\mu,M)$ commutes with χ_w and $e_{\alpha,\beta,\gamma}$. Then, we get (ii). We can obtain (i) by calcultions similar to, but easier than, those above.

Noting that $\mathbb{R}(\mu,\cdot)$ is an additive set function, and making use of (i) of the lemma above, we can approximate $(\mu-A)^{-1}$ with operators of the form (4.5).

From Lemma 4.1,(iii), we can obtain the following lemma in the same way as that in [5, Lemma 3.4]):

<u>Lemma 4.4.</u> Let $\alpha,\beta\in\mathbb{R}$, $\mu\in D(w)$, $f\in\chi_w E_{\alpha,\beta}$ and $1\leqq w\leqq+\infty$. Write $\mu=\gamma+i\delta$ $(\gamma,\delta\in\mathbb{R})$. Then,

$$\int_{-\infty}^{+\infty}\|(\mu-A_j)^{-1}f\|^2_{\alpha,\beta-1/2}d\delta \leqq c\|f\|^2_{\alpha,\beta}, \quad j=1,2,$$

where c is independent of γ, f and w. A_j, $j=1,2$, are the same as those in Lemma 4.1,(iii).

§5 The point spectrum of B

Let us obtain all the eigenvalues of B contained in $D(\infty)$ and their corresponding eigenspaces (see Lemma 4.3 for $D(\infty)$). Let $(\alpha, \beta) \in S_6$ be fixed in the present section. We can define the domain of A as follows: $\mathscr{D}(A) \equiv \{u \in E_{\alpha,\beta};\ Au \in E_{\alpha,\beta}\}$. In virtue of Lemma 3.4, (iii), we can define the domain of L_1 by $\mathscr{D}(L_1) \equiv E_{\alpha,\beta}$. Hence, we can define the domain of B as follows: $\mathscr{D}(B) \equiv \mathscr{D}(A)$. We will regard B as an operator on $E_{\alpha,\beta}$ in this section.

Lemma 5.1. If $\mu \in D(\infty)$ and $v \in E_{\alpha,\beta}$ satisfy $\mu v = Bv$, then $v \in L^2 \equiv L^2(\mathbb{R}_x{}^3 \times \mathbb{R}_\xi{}^3)$.

Proof. Let $\varepsilon > 0$ be sufficiently small, and decompose L_1 in the same way as that in Lemma 3.4, (iv) with $\alpha = 0$. Combining Lemma 4.3, (i) and Lemma 3.4, (ii), (iv), we see that $P_2 \equiv (\mu - A)^{-1} \ell_2 \in \mathbb{B}(E_{\alpha,\beta}, L^2)$ and $(1-P_1)^{-1} \in \mathbb{B}(L^2)$, where $P_1 \equiv (\mu - A)^{-1} \ell_1$. Hence, $P_3 \equiv (1-P_1)^{-1} P_2 \in \mathbb{B}(E_{\alpha,\beta}, L^2)$. Rewriting $\mu v = Bv$ as follows: $v = P_3 v$, we obtain the present lemma.

We write $e(\lambda)$ for the eigenspace corresponding to an eigenvalue λ of B. Combining Lemma 5.1 and [4, Theorem 3.2], we obtain the following:

Proposition 5.2. (i) $\sigma_P(B) \cap D(\infty) = \{0,\ i,\ -i,\ 2i,\ -2i\}$.

(ii) $e(0)$ is spanned by $\{\Phi_k\}_{k=1,\ldots,5}$. $e((-1)^{j+1}2i)$, $j = 1, 2$, are spanned by Φ_k, $k = 6, 7$, respectively. $e((-1)^{j+1}i)$, $j = 1, 2$, are spanned by $\{\Phi_k\}_{k=8,9,10}$, $\{\Phi_k\}_{k=11,12,13}$, respectively.

§6 Estimates for operators

The purpose of the present section is to obtain estimates for the operator $\mathbb{L}(w, \mu)$, which appears in

$$(\mu-\mathbb{B}_w)^{-1} = (\mu-A)^{-1} + (\mu-A)^{-1}(1-\mathbb{L}(w, \mu))^{-1}(L_{1,w}+L_{2,w})(\mu-A)^{-1}, \tag{6.1}$$

where $\mathbb{B}_w \equiv A+L_{1,w}+L_{2,w}$, $L_{j,w} \equiv \chi_w L_j \chi_w$, $\mathbb{L}(w, \mu) \equiv \mathbb{L}_1(w, \mu)+\mathbb{L}_2(w, \mu)$, $\mathbb{L}_j(w, \mu) \equiv L_{j,w}(\mu-A)^{-1}$, $j = 1,2$, $\mu \in D(w)$, $1 \leqq w \leqq +\infty$. The main result is the following proposition, which will be employed in showing Lemma 7.1:

Proposition 6.1. Let $(\alpha, \beta) \in S_5$.

(i) $\sup_{\mu \in D(w), 1\leqq w\leqq +\infty} \| \mathbb{L}_j(w, \mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} < +\infty$, $j = 1,2$.

(ii) Let $\mu_0 \in D(\infty)$. Then, $\| \mathbb{L}_j(w, \mu)-\mathbb{L}_j(+\infty, \mu_0) \|_{\mathbb{B}(E_{\alpha,\beta})} \to 0$ as $\mu \to \mu_0$, $w \to +\infty$, $\mu \in D(w)$, $j = 1,2$.

(iii) $\mathbb{L}^2(w, \mu) \in \mathbb{C}(\chi_w E_{\alpha,\beta})$, if $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$.

(iv) $\| \mathbb{L}^2(w, \mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \to 0$ as $|\mu| \to +\infty$, $w \to +\infty$, $\mu \in D(w)$.

We will prove this proposition, in the end of the present section, by approximating $\mathbb{L}_1(w, \mu)$ and $\mathbb{L}_1{}^2(w, \mu)$ with finite sums of operators of the following forms (6.2-3), respectively:

$$M(w, \mu, \delta, \varepsilon) \equiv \rho_\delta \chi_w \{\exp(-|x|^2/2)\} m \chi_w \mathbb{R}(\mu, s(\varepsilon)), \tag{6.2}$$

$$N(w, \mu, \delta, \varepsilon) \equiv \Pi_{k=1}^{2} N_k(w, \mu, \delta, \varepsilon), \tag{6.3}$$

where $\mu \in D(w)$, $1 \leqq w \leqq +\infty$, $1 \leqq \delta \leqq +\infty$, $0 \leqq \varepsilon \leqq \pi/4$, $m \in Q(\mathbb{R}_\xi{}^3)$ and $s(\varepsilon) \equiv [\varepsilon, \pi-\varepsilon] \cup [\pi+\varepsilon, 2\pi-\varepsilon]$. $N_k(w, \mu, \delta, \varepsilon)$, $k = 1,2$, are defined as follows:

$$N_k(w, \mu, \delta, \varepsilon) \equiv \rho_\delta \chi_w \{\exp(-|x|^2/2)\} r_k \chi_w \mathbb{R}(\mu, s(\varepsilon)), \quad k = 1,2,$$

where $r_k \in Q(\mathbb{R}_\xi{}^3)$, $k = 1,2$. For simplicity, we write $M(w, \mu)$ and $N(w, \mu)$ for $M(w, \mu, +\infty, 0)$ and $N(w, \mu, +\infty, 0)$, respectively.

First we will prove some estimates for operators of the forms (6.2-3).

Lemma 6.2. Let $\alpha, \beta \in \mathbb{R}$.

(i) $$\sup_{\mu \in D(w), 1 \leqq w, \delta \leqq +\infty, 0 \leqq \varepsilon \leqq \pi/4} \| M(w, \mu, \delta, \varepsilon) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} < +\infty.$$

(ii) $$\sup_{\mu \in D(w), 1 \leqq w \leqq +\infty} \| M(w, \mu, \delta, \varepsilon) - M(w, \mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \leqq c(\varepsilon + e^{-\delta}),$$

for any $0 \leqq \delta \leqq +\infty$ and $0 \leqq \varepsilon \leqq \pi/4$, where c is independent of δ and ε.

(iii) $$\| M(w, \mu) - M(w', \mu') \|_{\mathbb{B}(\chi_{w'} E_{\alpha,\beta})} \leqq c(\exp(-w) + |\mu - \mu'|),$$

for any w, w', μ and μ' such that $\mu \in D(w)$, $\mu' \in D(w')$, $1 \leqq w \leqq w' \leqq +\infty$, where c is independent of w, w', μ and μ'.

Proof. Lemma 3.4,(ii) and Lemma 4.3,(i) imply (i). By means of Lemma 3.4, (ii), Lemma 3.2,(iii) and Lemma 4.3,(i), we obtain (ii). Combining Lemma 3.4, (ii), Lemma 3.2,(ii) and Lemma 4.3, we get (iii).

Lemma 6.3. Let $\alpha, \beta \in \mathbb{R}$.

(i) $N(w, \mu) \in \mathbb{C}(\chi_w E_{\alpha,\beta})$, if $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$.

(ii) $\| N(w, \mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \to 0$ as $|\mu| \to +\infty$, $w \to +\infty$, $\mu \in D(w)$.

Proof. First we will pick out the integral kernel of $N(w, \mu, \delta, \varepsilon)$. By $r_k(\cdot,\cdot)$ we denote the integral kernel of $r_k \in Q(\mathbb{R}_\xi{}^3)$, $k = 1,2$. We write (x_2, ξ_2) for the variables of $N(w, \mu, \delta, \varepsilon)u$, i.e., $N(w, \mu, \delta, \varepsilon)u = (N(w, \mu, \delta, \varepsilon)u(\cdot,\cdot))(x_2, \xi_2)$. In view of (4.5) and (6.3), we see that $N(w, \mu, \delta, \varepsilon)u$ has the form

$$(N(w, \mu, \delta, \varepsilon)u(\cdot,\cdot))(x_2, \xi_2)$$

$$= \int_{\tau_k \in s(\varepsilon),\, \eta_k \in \mathbb{R}^3,\, k=1,2} n(w, \mu, \delta)u(x_0, \xi_0)\, d\eta\, d\tau, \tag{6.4}$$

where $d\eta \equiv d\eta_1 d\eta_2$, $d\tau \equiv d\tau_1 d\tau_2$. $n(w, \mu, \delta)$ is defined as follows:

$$n(w, \mu, \delta) = n(w, \mu, \delta; x_2, \xi_2; \eta_1, \eta_2, \tau_1, \tau_2)$$

$$\equiv \Pi_{k=1}^{2}\{\rho_\delta(x_k, \xi_k)\chi_w(x_k, \xi_k)\{\exp(-|x_k|^2/2)\}r_k(\xi_k, \eta_k)\times$$

$$\times \chi_w(x_k, \eta_k)R(\mu, \tau_k, x_k, \eta_k)\}.$$

x_k and ξ_k, $k = 0,1$, are the same as those in (3.1). Change the integration variables in (6.4) as follows: $(\eta_1, \eta_2, \tau_1, \tau_2) \to (x_0, \xi_0, \tau_1, \tau_2)$. We regard η_j, $j = 1,2$, in (6.4) as functions of x_k, ξ_k, $k = 0,2$, and τ_ℓ, $\ell = 1,2$. By Lemma 3.1,(v), we can rewrite (6.4) as follows:

$$(N(w, \mu, \delta, \varepsilon)u(\cdot,\cdot))(x_2, \xi_2) = \int_{x_0, \xi_0 \in \mathbb{R}^3} \mathbb{N}(w, \mu, \delta, \varepsilon)u(x_0, \xi_0)\, dx_0\, d\xi_0,$$

where

$$\mathbb{N}(w, \mu, \delta, \varepsilon) = \mathbb{N}(w, \mu, \delta, \varepsilon; x_2, \xi_2, x_0, \xi_0)$$

$$\equiv \int_{\tau_k \in s(\varepsilon), k=1,2} n(w, \mu, \delta)|\sin^{-3}\tau_2| d\tau. \tag{6.5}$$

Next we will obtain estimates for this kernel. Let $1 \leqq \delta < +\infty$ and $0 < \varepsilon \leqq \pi/4$. Note that if $\tau_2 \in s(\varepsilon)$, then $|\sin^{-3}\tau_2| \leqq c\varepsilon^{-3}$, and apply Lemma 3.1,(ii-iii) and (4.7) with $n = 0$ to (6.5). Then, we see that

$$\sup_{x_2, \xi_2, x_0, \xi_0, \mu \in D(w), 1 \leqq w \leqq +\infty} |\mathbb{N}(w, \mu, \delta, \varepsilon; x_2, \xi_2, x_0, \xi_0)| \leqq c, \tag{6.6}$$

and that there exists a compact set $S \subset \mathbb{R}^3_{x_2} \times \mathbb{R}^3_{\xi_2} \times \mathbb{R}^3_{x_0} \times \mathbb{R}^3_{\xi_0}$ such that for any $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$

$$\text{supp}\, \mathbb{N}(w, \mu, \delta, \varepsilon; \cdot, \cdot, \cdot, \cdot) \subseteq S. \tag{6.7}$$

c and S depend on ε and δ in (6.6-7).

Let us prove (i). It follows from (6.6-7) that for any $\mu \in D(w)$, $1 \leqq w \leqq +\infty$, $1 \leqq \delta < +\infty$ and $0 < \varepsilon \leqq \pi/4$

$$N(w, \mu, \delta, \varepsilon) \in \mathbb{C}(\chi_w E_{\alpha,\beta}). \tag{6.8}$$

Note that any operator of the form (6.3) is equal to the product of 2 operators of the form (6.2), and apply Lemma 6.2,(i-ii). Then, we can approximate $N(w, \mu)$ with $N(w, \mu, \delta, \varepsilon)$, $1 \leqq \delta < +\infty$, $0 < \varepsilon \leqq \pi/4$, uniformly for $\mu \in D(w)$ and

$1 \leqq w \leqq +\infty$. Hence, (6.8) implies (i).

Let us prove (ii). Let $\mu \in D(w)$. Let $1 \leqq w < +\infty$, $1 \leqq \delta < +\infty$ and $0 < \varepsilon \leqq \pi/4$ be fixed in what follows. Substituting (4.6) in (6.5), and applying Riemann-Lebesgue theorem, we deduce that for any x_k and ξ_k, k = 0, 2,

$$\mathbb{N}(w, \mu, \delta, \varepsilon; x_2, \xi_2, x_0, \xi_0) \to 0 \quad \text{as } |\mu| \to +\infty, \quad \mu \in D(w).$$

With the aid of this and (6.6-7), we conclude that

$$\| N(w, \mu, \delta, \varepsilon) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \to 0 \quad \text{as } |\mu| \to +\infty, \quad \mu \in D(w).$$

In virtue of this fact and Lemma 6.2, we obtain (ii).

Making use of Lemma 3.4, (v), we can demonstrate the following lemma in the same way as that in proving Lemma 6.3:

<u>Lemma 6.4.</u> Let $(\alpha, \beta) \in S_2$.

(i) $\mathbb{L}_2(w, \mu) \in \mathbb{C}(\chi_w E_{\alpha,\beta})$, if $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$.

(ii) $\| \mathbb{L}_2(w, \mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \to 0$ as $|\mu| \to +\infty$, $w \to +\infty$, $\mu \in D(w)$.

<u>Proof of Proposition 6.1.</u> Making use of Lemma 3.4, (iv) and Lemma 4.3, (i), we can approximate $\mathbb{L}_1(w, \mu)$ ($\mathbb{L}_1{}^2(w, \mu)$ respectively) in $\mathbb{B}(\chi_w E_{\alpha,\beta})$, $(\alpha, \beta) \in S_1$, with finite sums of operators of the form (6.2) ((6.3) respectively) uniformly for $\mu \in D(w)$ and $1 \leqq w \leqq +\infty$. Hence, Lemma 6.2, (i), (iii) imply (i-ii) with j = 1, respectively, and we obtain (iii-iv) from Lemmas 6.3-4. We can get (i-ii) with j = 2 in the same way as that in showing (i-ii) with j = 1.

§7 Proof of Theorem 2.2

The purpose of this section is to prove the main theorem. First we will obtain estimates for the operator $(1-\mathbb{L}(w, \mu))^{-1}$, which appears in (6.1).

Lemma 7.1. If $(\alpha, \beta) \in S_5$, then there exists a $W \geqq 1$ such that

$$\sup_{\mu \in D(w), W \leqq w < +\infty} \| (1-\mathbb{L}(w, \mu))^{-1} \|_{\mathbb{B}(\chi_w E_{\alpha, \beta})} < +\infty.$$

Proof. Let us regard $\mathbb{L}(w, \mu)$ as an operator on $\chi_w E_{\alpha, \beta}$ (see Proposition 6.1,(i)). We will prove, by a reduction to absurdity, that if $1 \leqq w < +\infty$ is sufficiently large, then $1 \notin \sigma(\mathbb{L}(w, \mu))$ for any $\mu \in D(w)$. We suppose that the assertion is false. It follows from Proposition 6.1,(iii) that for any $\mu \in D(w)$ and $w \geqq 1$, the continuous spectrum and the residual spectrum of $\mathbb{L}(w, \mu)$ are empty, i.e., $\sigma_p(\mathbb{L}(w, \mu)) = \sigma(\mathbb{L}(w, \mu))$. Hence, there exist sequences μ_n, w_n and u_n such that $\mu_n \in D(w_n)$, $w_n \to +\infty$, $\| u_n \|_{\alpha, \beta} = 1$ and $u_n = \mathbb{L}(w_n, \mu_n)u_n$. Iterating the last equality, we see that $u_n = \mathbb{L}^2(w_n, \mu_n)u_n$. Applying Proposition 6.1 to this equality, we deduce that there exist subsequences of μ_n and u_n, denoted again by μ_n and u_n, such that there exist $\mu \in D(\infty)$ and $u \in E_{\alpha, \beta}$ satisfying that $\mu_n \to \mu$, $u_n \to u$ and $u = \mathbb{L}(\infty, \mu)u$. Rewriting the last equality as follows: $\mu v = \mathbb{B}_\infty v$, $v \not\equiv 0$, where $v \equiv (\mu - A)^{-1}u \in E_{\alpha, \beta-1}$, $(\alpha, \beta-1) \in S_6$ (see Lemma 4.3,(i)), we conclude that this equality and Proposition 5.2 incur a contradiction. We have thus proved the assertion above. We can deduce, in the same way as that above, that there exists a W such that $(1-\mathbb{L}(w, \mu))^{-1}$ is bounded uniformly for $\mu \in D(w)$ and $w \geqq W$.

With the aid of Lemma 3.4,(iii),(v) and Lemma 4.1,(i), we see that $\mathbb{B}_w$ generates a C_0-semigroup in $\chi_w E_{\alpha,\beta,\gamma}$, $1 \leqq w \leqq +\infty$, $(\alpha,\beta) \in S_6$, $\gamma \geqq 0$. Let us obtain estimates for decay of this semigroup.

Lemma 7.2. Let $(\alpha,\beta) \in S_7$. There exists a $W \geqq 1$ such that if $W \leqq w < +\infty$, then

$$\| e^{t\mathbb{B}_w} \|_{\mathbb{B}(\chi_w E_{\alpha,\beta})} \leqq c\exp\{-c_{3.5}(\exp(-w))t/2\}, \quad t \geqq 0,$$

where c is independent of t and w (see Lemma 3.5 for $c_{3.5}$).

Proof. Applying Lemma 3.4,(iii),(v) and Lemma 7.1 to $P(w,\mu) \equiv (1-\mathbb{L}(w,\mu))^{-1}(L_{1,w}+L_{2,w})$, we see that there exists some $1 \leqq W < +\infty$ such that

$$\sup_{\mu \in D(w), W \leqq w < +\infty} \| P(w,\mu) \|_{\mathbb{B}(\chi_w E_{\alpha,\beta-1/2},\, \chi_w E_{\alpha,\beta+1/2})} < +\infty.$$

Applying this inequality, Lemma 4.4 and Lemma 4.1,(ii) to the inverse Laplace transform of (6.1) (the integration path is $\mathrm{Re}\,\mu = -c_{3.5}\{\exp(-w)\}/2$), we obtain the present lemma in the same way as that in [5, p. 182].

Lemma 7.3. If $(\alpha,\beta), (\alpha',\beta') \in S_8$, $\alpha \leqq \alpha'$ and $\beta \leqq \beta'$, then

$$\| e^{t\mathbb{B}} \|_{\mathbb{B}(E_{\alpha',\beta'},\, E_{\alpha,\beta})} \leqq c(\theta)(1+t)^{-a\theta}, \quad t \geqq 0,$$

where $\mathbb{B} \equiv \mathbb{B}_\infty$, $a \equiv 2(\alpha'-\alpha)+(\beta'-\beta)$. $c(\theta)$ is independent of t.

Proof. We will follow the lines in [1] and [3]. By Lemma 3.4,(iii),(v) and Lemma 3.2,(ii), we see that $\mathbb{B}_w \to \mathbb{B}$ in $\mathbb{B}(E_{\alpha,\beta})$ as $w \to +\infty$. Hence, Lemma 7.2 implies the present lemma with $a = 0$.

Let $\tau>0$ be a sufficiently large constant. Set

$$w \equiv \log\{\tau/\log(1+\tau)^C\}, \quad C \equiv 2a/c_{3.5}. \tag{7.1}$$

Multiply the following Cauchy problem by χ_w:

$$\begin{cases} u_t = \mathbb{B}u, & t>0, \\ u(0) = u_0, \end{cases} \tag{7.2}$$

and solve the equation thus obtained with respect to $g(t) \equiv \chi_w u(t)$. Noting that $\chi_w A = A\chi_w$, we conclude that

$$g(t) = e^{t\mathbb{B}_w}\chi_w u_0 + \int_0^t e^{(t-s)\mathbb{B}_w}\chi_w(L_1+L_2)h(s)ds, \tag{7.3}$$

where $h(s) \equiv \underline{\chi_w} u(s)$. It follows from the present lemma with $a = 0$ and Lemma 3.2,(ii) that

$$\|h(t)\|_{\alpha,\beta-k} \leqq c\{\exp(-(a+k)w)\}\|u_0\|_{\alpha',\beta'}, \quad k = 0,1, \tag{7.4}$$

$$\|h(t)\|_{-\theta'/2} \leqq c\{\exp(-(a+\theta')w)\}\|u_0\|_{\alpha',\beta'}, \quad 0<\theta'<1. \tag{7.5}$$

Applying Lemma 3.4,(iii),(v), (7.5), (7.4) with $k = 1$ and Lemma 7.2 to (7.3), we deduce that

$$\| g(t) \|_{\alpha,\beta} \leqq c \| u_0 \|_{\alpha,\beta} \exp\{-c_{3.5}(\exp(-w))t/2\} +$$

$$+ tc(\theta')\{\exp(-(\theta'+a)w)\} \| u_0 \|_{\alpha',\beta'}.$$

Substitute (7.1) in this inequality, let $t \to \tau$ and replace $\tau^{1-\theta'-a}(\log(1+\tau)^{C})^{\theta'+a}$ by $c(\theta)(1+\tau)^{-a\theta}$ in the second term of the right hand side. Moreover, substitute (7.1) in (7.4) with $k = 0$, let $t \to \tau$ and replace $\tau^{-a}(\log(1+\tau)^{C})^{a}$ by $c(\theta)(1+\tau)^{-a\theta}$ in the right hand side. From the two inequalities thus obtained, we get the present lemma.

Proof of Theorem 2.2. With the aid of Lemma 3.3,(v) and Lemma 4.1,(i), we see that B generates a C_0-semigroup in $E_{\alpha,\beta,\gamma}$.

By making use of Lemma 3.3,(iii), we easily deduce that if $u_0 \in E_{\delta',\varepsilon',1}$, then $L_2 e^{tB} u_0 = 0$ for any $t \geqq 0$. Hence, we conclude that $u = u(t) \equiv e^{tB} u_0$ satisfies (7.2). Therefore Lemma 7.3 implies (2.2) with $\gamma = 0$. Rewriting the Cauchy problem $u_t = Bu$, $u(0) = u_0$, as folows: $u(t) = e^{tA} u_0 + Z(u)(t)$, and applying Lemma 4.2, Lemma 4.1,(i) and (2.2) with $\gamma = 0$ to this equality, we obtain (2.2).

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