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THESIS

Evolution of the Non-Linear Density Fluctuations

in

the Expanding Universe

—Two-Point Spatial Correlation Function and its Self-Similarity—

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ABSTRACT

We have investigated the evolution of the density fluctuations in the expanding universe, especially in the non-linear regime. First, we have investigated the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime. We have derived the solutions and obtained the relation between the power-law index of the two-point spatial correlation function and the other statistical quantities, such as skewness of the velocity field, three-point correlation function, and the velocity parameter. We have also derived the index of the two-point spatial correlation function as a function of the index of the initial power spectrum and the velocity parameter $h \equiv -\langle v \rangle / \dot{a}x$ (mean relative peculiar velocity divided by the Hubble expansion velocity) when the self-similar evolution by connecting the non-linear solution to the well known linear solution is satisfied.

Second, we have investigated the stability of the scale-invariant solutions of the BBGKY equations in the non-linear regime because whether the solutions are suitable in the real world or not is not clear. We have found that there is no unstable mode when the skewness of the velocity field is equal to zero. The solutions have proved to be marginally stable. This means that there is no special value of the index of the two-point spatial correlation function from the viewpoint of stability of the solutions.

Finally, we have investigated self-similarity of the two-point spatial correlation function (power spectrum) in an one-dimensional system because if the self-similarity is satisfied, the two-point spatial correlation function in the non-linear regime can be expressed as a function of the index of the initial power spectrum. We have verified the self-similar evolution when the initial power spectrum is scale free. We have also investigated the evolution of the power spectrum when the initial power spectrum obeys the power-law with a cutoff. In this case, the self-similarity of the two-point spatial correlation function (power spectrum) is broken.

Contents

1. INTRODUCTION

1.1. General Introduction

In our universe, there are many kinds of structures, such as galaxies, clusters of galaxies and super-clusters of galaxies. It is not known clearly even today how galaxies and the large-scale structures were formed after the birth of the universe. This is one of the most important problems in cosmology. In the standard scenario of the large-scale structure formation, it is proposed that small density fluctuations at early times grow with time owing to the gravitational instability.

When the amplitude of density fluctuations is much smaller than unity at early times, their temporal evolution can be analyzed by making use of the linear theory. In this regime, we can understand analytically how the small fluctuations grow(Peebles 1980, 1993). When the amplitude of these small fluctuations becomes as large as unity, that is, when the fluctuations reach the quasi-linear stage, we cannot make use of linear theory. In this regime, higher order perturbative methods (Jain & Bertschinger 1996) and the Zel'dovich approximation (Zel'dovich 1970) are often used for the analysis.

Moreover the density fluctuations continue to grow with time, and then their amplitude becomes much larger than unity. At last, caustics of the density fields appear at every spatial field. The nonlinear phenomena of the self-gravity are not only very interesting and important for the large-scale structure formation, but also of academic interest in nonlinear dynamics. In this strongly nonlinear regime, however, analytical approach is difficult. It is necessary to understand clearly the nonlinear behavior of the density fluctuations in spite of its difficulty. For example, we are interested in the two-point spatial correlation functions in the strongly nonlinear regime, that is one of the statistical quantities for the estimation of the pattern of the distribution of dark mater. It is found from the N-body simulations that the two-point spatial correlation functions obey the power law. This result is reasonable because the self-gravity is scale-free. Then the power-law index of the two-point spatial correlation function is a good indicator, representing the nonlinear dynamics of self-gravity in this regime. In addition, the power-law index is related to the clustering pattern of collisionless matter. Hence it is very important to study what physical processes determine the power-index of the two-point spatial correlation in the strongly nonlinear regime. Moreover it is interesting to analyze whether the power-law index in this regime depends on the initial conditions or not.

These problems have usually been analyzed by N-body simulations (Frenk, White, & Davis 1983; Davis et.al. 1985; Suto 1993 and references therein). The method is straightforward for tracking the evolution of the density fluctuations in the nonlinear regime. In the simulations, high spatial resolution is necessary to estimate the correlation functions and the mean relative peculiar velocity on very small scales in the strongly nonlinear regime with good accuracy (Jain 1995). At the present time, however, the computational ability to achieve high enough resolution is lacking. Hence the dynamics in the strongly nonlinear regime has not been completely verified by N-body simulations.

There are other methods for the analysis of the nonlinear density fluctuations. One is analysis by BBGKY equations, pioneered in the work of Davis & Peebles (1977; hereafter DP). They showed the existence of self-similar solutions for correlation functions under some assumptions, where self-similarity means that the two-point spatial correlation function has a same scaling relation in all the regimes, that is, both in the linear regime and in the non-linear regime. It can then be shown that the power-law index γ of the two-point spatial correlation function ξ in the strongly nonlinear regime is related to the power-law index of the initial power spectrum n as follows:

$$\xi(r) \propto r^{-\gamma}(\xi \gg 1\ :\ \gamma = \frac{3(3+n)}{5+n}) \tag{1}$$

One of the assumptions that DP adopted is called the stability condition, which states that the mean relative physical velocity vanishes in the strongly nonlinear regime. This condition has been tested by N-body simulations (Efstathiou et al. 1988; Jain 1996) but has not been completely verified in the strongly nonlinear regime ($\xi \gg 10^3$), again because current computational ability does not allow high enough resolution.

As for physical processes determining the power-law index in the strongly nonlinear regime, there are other analyses besides that proposed by DP. One is given by Saslaw(1980), who concluded that the power-law index γ approaches to 2 by using the cosmic energy equation under some assumptions although numerical simulations do not support this result(Frenk, White & Davis 1983; Davis et al. 1985; Fry & Melott 1985).

There is another idea as follows: when the initial power spectrum has a sharp cutoff or is scale-free with negative and small initial power-law index, then caustics of the density fields appear everywhere. In these cases, the power-law index is independent of the detailed initial conditions after the first appearance of caustics on small scales, around the typical size of the thickness of caustics(in two-dimensional systems, they correspond to the filamentaly structures of highly clustered matter). The power-law index is determined by the type of the singularity, which is classified in accord with catastrophe theory. This idea has been verified in one-dimensional systems (Kotok & Shandarin 1988; Gouda & Nakamura 1988, 1989), spherically symmetric systems (Gouda 1989), two-dimensional systems (Gouda 1996), and also three-dimensional systems (Gouda 1998). In these cases, it is suggested that $\gamma \approx 0$ on small scales.

As can be seen from the above arguments, there are uncertainties about the physical processes that determine the value of the power-law index of the two-point spatial correlation function in the non-linear regime.

Whether the evolutions of the two-point spatial correlation function satisfy self-similarity or not is especially important. If the self-similarity is satisfied, the power-law index of the two-point spatial correlation function is expressed as a function of the index of the initial power spectrum n and the mean relative peculiar velocity. Furthermore we can describe completely the time evolution of the two-point spatial correlation function without numerical simulations because we can estimate the growth rate of the two-point spatial correlation function on all scales. Many authors have investigated the self-similarity of the two-point spatial correlation function (or the power spectrum). As mentioned above, DP investigated self-similarity of the two-point correlation function by using the BBGKY equations. It is well known that we can represent BBGKY equations by using a scaled variable $s \equiv x/t^{\alpha}$ (where α is a constant) because the gravity is scale-free. They showed the existence of the self-similarity by integrating this scaled BBGKY equations numerically under the assumptions shown below. The BBGKY equations have hierarchical structure, that is, the time evolution of the N-th order correlation function includes the $(N+1)$-th order correlation function. So, the cutoff of the hierarchy is needed in order to close these equations. DP assumed that the three-point correlation function can be expressed by the product of the two-point spatial correlation functions and that the skewness of the velocity field is equal to zero. Thus the existence of the self-similarity is still uncertain because we do not know whether these assumptions are correct or not. On the other hand, by using N-body simulations, time evolutions of the power spectrum can be calculated directly in principle. Recent works (e.g., Colombi,Bouchet & Hernquist (1996); Couchman & Peebles (1998); Jain, Mo, & White (1995); and others) have shown that the self-similar evolution of the power spectrum is satisfied when the initial power spectrum is scale-free. In the simulations, wide range of the scales, that is, high resolution is needed for investigating the self-similarity of the two-point correlation function. At present time, however, the computational ability does not allow high enough resolution. Therefore there are still uncertainties also about the self-similarity of the two-point spatial correlation function. Furthermore the relation between the index of the power spectrum and the mean relative peculiar velocity is still uncertain quantitatively, and only the scale-free cases are investigated. As can be seen from the above arguments, there are uncertainties also about the self-similarity of the evolution of the two-point correlation function of the density fluctuations in the expanding universe.

1.2. Organization of this Thesis

In this thesis, we investigate the non-linear density fluctuations in the expanding universe, especially the power-law index of the two-point spatial correlation function from the various points of view. First, we examine the conditions that determine the power-law index by analysing the scale-invariant solutions of the cosmological BBGKY equations. The analysis of nonlinear clustering in the strongly nonlinear regime with the BBGKY equations has an advantage over that of the N-body simulations. This is because the BBGKY equations deal directly with the statistical quantities such as correlation functions and because this analysis is free from the artificial collisionality due to finite numbers of particles that might appear on small scales in the N-body simulations. However, it is technically difficult to solve the BBGKY equations; in our analysis the BBGKY equations are translated to the moment equations by integrating them over velocity because we are interested in the statistical quantities such as the two-point correlation function ξ, mean relative peculiar velocity $\langle v \rangle$ and so on. These moment equations for time evolution of the N-point spatial correlation function include terms including $N+1$-point spatial correlation functions. Furthermore, N-th moment equation involves terms include the $N+1$-th moment. In general, these equations form an infinite hierarchy and cannot be closed at the lower order spatial correlation and moment. Hence, one must close the equations by making some assumptions. For example, DP assumed that the three-point spatial correlation function ζ can be represented by the products of the two-point spatial correlation functions ξ as follows:

$$\begin{aligned}\zeta_{123} &= Q(\xi_{12}\xi_{23} + \xi_{23}\xi_{31} + \xi_{31}\xi_{12}),\\ \xi_{ik} &\equiv \xi(x_i, x_k), \quad \zeta_{ijk} \equiv \zeta(x_i, x_j, x_k),\end{aligned}$$

where Q is a constant. Some observations suggest that this relation with $Q \sim 1$ holds at $\xi \sim 1$. DP furthermore assumed that the skewness of the velocity fields vanishes. Adding one more assumption, DP closed the BBGKY equations: as mentioned above, DP used the stability condition in deriving the power-law index γ given by eq.(1). In this thesis, we reexamine the scale-invariant solutions of the BBGKY equations and estimate the value of the power-law index and its dependence of the initial power-law index n when the above assumptions and stability condition are changed. Furthermore we analyze whether the stability condition is satisfied and how the mean relative peculiar velocity would behave in a real system. Indeed, Jain(1995) claimed that the stability condition has not yet been verified by N-body simulations.

Furthermore we investigate whether there is a possibility that the power-law index in the strongly nonlinear regime does not depend on the initial power index n even if the self-similarity of the solutions is satisfied. Recently Padmanabhan(1996) suggested the possibility that the power-law index is independent of n on the basis of the pair conservation equations.

Although we find the various scale-invariant solutions in §2, whether these are stable or not is another interesting problem. As a matter of fact, the values of the power-law index leading to unstable solutions do not appear in the real world. Ruamswan & Fry (1992) investigated the stability of the DP solutions with the help of the linear perturbation. They showed in this way that the perturbations of the solutions prove to be only marginally stable. Yet they did not investigate other solutions that we obtain in §2 (Yano & Gouda (1997): hereafter YG). Furthermore, Ruamswan & Fry misunderstood the way to perturb the skewness, and consider inapplicable perturbations which diverges on small or large scales. So, in this thesis, we investigate the stability of the general solutions that we obtain in §2 by applying an accurate method of perturbation. We derive the perturbation equation by perturbed BBGKY equation from the background scale-invariant solutions in the strongly nonlinear regime. And we show the solution to the linear perturbation equations when we assume a well-defined appropriate form of perturbation.

As shown in §3, the index of the two-point spatial correlation function cannot be determined from the viewpoint of stability of the solutions. This means that the index of the two-point spatial correlation function in the non-linear regime is related to the evolution of the density fluctuations in the linear and the quasi-linear regime. Furthermore, if the self-similarity of the two-point spatial correlation function is satisfied, the two-point spatial correlation function in the non-linear regime is related to the index of the initial power-law index n and the velocity parameter h as we comment in §2. Therefore whether self-similarity is satisfied or not is a very important problem. Therefore, we investigate the self-similarity of the two-point spatial correlation function in the expanding universe finally.

We would like to investigate not only the scale-free cases but also the cases that the initial power spectrum obeys the power-law with a cutoff. Hereafter we call this case the cutoff-case. In the cutoff-case, the power spectrum obeys the power law even on scales smaller than the cutoff scale after the first appearance of caustics. For example, in an one-dimensional system, the value of the power-law index is -1 which can be derived according to the catastrophe theory (Gouda & Nakamura (1988,1989)). Kotok & Shandarin (1988) also studied the nonlinear spectra in the cutoff case and showed that the value of the power index is -1. Of course, the power spectrum does not evolve in the self-similar form before the first appearance of caustics. However it has not been certain whether the time evolution is self-similar or not after the first appearance of caustics. Therefore we investigate the self-similarity not only in the case of the scale-free case but also in the cutoff-case. Then we have to investigate in detail the evolutions of the power spectrum on scales smaller than the initial cutoff scale and also those on larger scales. Thus we need a wide dynamic range in the wave number space. However, we cannot get such a wide dynamical range in the numerical simulation of three-dimensional systems due to the limit of the resolution. Then, as a first step, we consider one-dimensional sheet systems in order to investigate the self-similarity of the two-point spatial correlation function. and the physical process which determine the power-law index of the power spectrum or the two-point spatial correlation function. In the one-dimensional system we can get a wide range of resolution for calculating time evolutions of the power spectrum. Furthermore we can have the numerical method for the evolution of the power spectrum with a good accuracy in the one-dimensional sheet systems as shown later.

In §2 we will investigate the evolution of the density fluctuations in the expanding universe, especially in the non-linear regime. First, we derive the scale-invariant solutions by using the cosmological BBGKY equations in the strongly nonlinear regime. Whether all the solutions derived in §2 are suitable in the real situation or not is not clear, because we do not know whether these solutions are stable or not. Then we will investigate the stability of the scale-invariant solutions of the BBGKY equations in the non-linear regime in §3. Last, we will investigate the self-similarity of the two-point spatial correlation function (power spectrum) in §4. In this section, we will consider an one-dimensional system because we need the wide dynamical range in the wave number space. In §5, we will summarize the points of this thesis. In Appendix A, we will show the derivation of the BBGKY equations in detail.

2. SCALE-INVARIANT SOLUTION

In this section, we investigate the statistical quantities of the density fluctuations in the expanding universe such as the two-point spatial correlation function, the mean relative peculiar velocity, relative peculiar velocity dispersion, and so on. We derive the scale-invariant solutions in the strongly nonlinear regime by using the BBGKY equations. Furthermore we investigate the relation between the index of the two-point spatial correlation function and some other statistical quantities, such as mean relative peculiar velocity, skewness of the velocity field, and so on.

2.1. Basic Equations

DP and Ruamsuwan & Fry(1992) derived the cosmological BBGKY equations. In this subsection, we briefly review the derivation of the cosmological BBGKY equations according to DP and Ruamsuwan & Fry(1992) except for the notation.

2.1.1. Cosmological BBGKY Equations

Here we derive the BBGKY equations from the ensemble mean of the Vlasov equation in the expanding homogeneous and isotropic background universe. In this thesis, we consider only Einstein-de Sitter universe because we are interested in the scale-invariant solutions of the correlation functions and the self-similarity of the solutions and so it is necessary that the background universe is scale-free. The N-body correlation function is the statistical quantity which is given by the ensemble mean of the N-products of the one-body distribution functions. Then the BBGKY equations can be derived by the ensemble mean of the Vlasov equation (we show the detailed derivation in Appendix A).

The first and the second BBGKY equations are written as follows:

$$\frac{\partial b(1)}{\partial t} + \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int \frac{x_{21}^\alpha}{x_{21}^3} c(1,2) d^3x_2 d^3p_2 = 0 \qquad \text{(first BBGKY)} \tag{2}$$

$$\begin{aligned}\frac{\partial c(1,2)}{\partial t} &+ \frac{p_1^\alpha}{ma^2}\frac{\partial c(1,2)}{\partial x_1^\alpha} + \frac{Gm^2}{a}\frac{\partial b(1)}{\partial p_1^\alpha}\int \frac{x_{31}^\alpha}{x_{31}^3} c(2,3) d^3x_3 d^3p_3 \\ &+ \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int \frac{x_{31}^\alpha}{x_{31}^3} d(1,2,3) d^3x_3 d^3p_3 + (1 \leftrightarrow 2) = 0 \quad \text{(second BBGKY)}\end{aligned} \tag{3}$$

where m is a mass of a particle, G is a gravitational constant, a is a scale factor, and

$$\begin{aligned} b(1) &\equiv \langle f(\boldsymbol{x}_1, \boldsymbol{p}_1)\rangle \equiv \langle f(1)\rangle, \\ c(1,2) &\equiv \langle f(1)f(2)\rangle - b(1)b(2), \\ d(1,2,3) &\equiv \langle f(1)f(2)f(3)\rangle - b(1)c(2,3) - b(2)c(3,1) - b(3)c(1,2) - b(1)b(2)b(3), \\ \frac{x_{31}^\alpha}{x_{31}^3} &\equiv \frac{x_3^\alpha - x_1^\alpha}{|\boldsymbol{x}_3 - \boldsymbol{x}_1|^3}. \end{aligned} \tag{4}$$

2.1.2. Velocity Moment

We are interested in the power-law index of the two-point spatial correlation function in the strongly non-linear regime. Hence the equation which we use in our analysis is the second BBGKY equation. Moreover we use the velocity moment equations which are given by multiplying the second BBGKY equation by a power of moment and integrate them over all moment arguments, because we are interested in the statistical quantities such as the two-point spatial correlation fucntion, the mean relative peculiar velocity and so on.

The time evolution of the N-th moment depends on the $N+1$-th moment. So we should take assumptions in order to close these equations. DP used the assumption that the skewness of the velocity is equal to 0. In this thesis, we do not assume that the skewness of the velocity fields is equal to 0 in order to study the relation between the skewness and the two-point spatial correlation function.

Here we define the two-point spatial correlation function ξ, the mean relative peculiar velocity $\langle v^\alpha \rangle$, the relative peculiar velocity dispersion Π, Σ and the mean third moment $\langle v^\alpha v^\beta v^\gamma \rangle$ as follows:

$$
\begin{aligned}
\bar{n}^2 a^6 \xi &\equiv \int c(2,1) d^3p_1 d^3p_2, & (5)\\
\bar{n}^2 a^6 (1+\xi) ma \langle v^\alpha \rangle &\equiv \int c(2,1) p_{21}^\alpha d^3p_1 d^3p_2, & (6)\\
\bar{n}^2 a^6 (1+\xi)(ma)^2 [\Pi P_{\parallel}^{\alpha\beta} + \Sigma P_{\perp}^{\alpha\beta}] &\equiv \int c(2,1) p_{21}^\alpha p_{21}^\beta d^3p_1 d^3p_2, & (7)\\
\langle v^\alpha v^\beta v^\gamma \rangle &\equiv \frac{\dfrac{1}{(ma)^3} \displaystyle\int \rho_2 p_{21}^\alpha p_{21}^\beta p_{21}^\gamma d^3p_1 d^3p_2}{\displaystyle\int \rho_2 d^3p_1 d^3p_2} \\
&= \frac{\displaystyle\int c(2,1) p_{21}^\alpha p_{21}^\beta p_{21}^\gamma d^3p_1 d^3p_2}{(ma)^3 \bar{n}^2 a^6 (1+\xi)}, & (8)
\end{aligned}
$$

where $\bar{n}$ is the mean density of the universe, and

$$
\begin{aligned}
P_{\parallel}^{\alpha\beta} &= \frac{x^\alpha x^\beta}{x^2}, \qquad P_{\perp}^{\alpha\beta} = \delta^{\alpha\beta} - \frac{x^\alpha x^\beta}{x^2}, \\
x^\alpha &\equiv x_{21}^\alpha \equiv x_2^\alpha - x_1^\alpha, \qquad p_{21}^\alpha \equiv p_2^\alpha - p_1^\alpha. & (9)
\end{aligned}
$$

In eq.(7), Π and Σ are the parallel and transverse correlated parts of the relative peculiar velocity dispersion of correlated particles, respectively.

We define the skewness as follows:

$$
s^{\alpha\beta\gamma} \equiv \langle (v - \langle v \rangle)^\alpha (v - \langle v \rangle)^\beta (v - \langle v \rangle)^\gamma \rangle. \tag{10}
$$

From the symmetry of the universe(homogeneity and isotropy), we can write $\langle v^\alpha \rangle = \langle v \rangle x^\alpha / x$. Hence,

$$
\langle v^\alpha v^\beta v^\gamma \rangle = -2 \langle v \rangle^3 \frac{x^\alpha x^\beta x^\gamma}{x^3} + \langle v \rangle \left\{ \frac{x^\alpha}{x} \langle v^\beta v^\gamma \rangle + \frac{x^\beta}{x} \langle v^\gamma v^\alpha \rangle + \frac{x^\gamma}{x} \langle v^\alpha v^\beta \rangle \right\} + s^{\alpha\beta\gamma}. \tag{11}
$$

Furthermore, the skewness can be written by the symmetry of the universe as follows:

$$s^{\alpha\beta\gamma} = s_{\|}\, P^{\alpha\beta\gamma}_{ppp} + s_{\perp}\, P^{\alpha\beta\gamma}_{ptt}, \tag{12}$$

where the subscripts p and t denote the parallel and transverse components of the two particles, respectively. We have

$$P^{\alpha\beta\gamma}_{ppp} = \frac{x^\alpha x^\beta x^\gamma}{x^3}, \qquad P^{\alpha\beta\gamma}_{ptt} = \frac{x^\alpha}{x}\delta^{\beta\gamma} + \frac{x^\beta}{x}\delta^{\gamma\alpha} + \frac{x^\gamma}{x}\delta^{\alpha\beta} - 3\frac{x^\alpha x^\beta x^\gamma}{x^3}. \tag{13}$$

P_{ppt} and P_{ttt} vanish because of the symmetry of the universe.

From eqs.(8) and (11), we can obtain the following equation:

$$\begin{aligned}\int c(2,1) p^\alpha_{21} p^\beta_{21} p^\gamma_{21} d^3p_1 d^3p_2 = \bar{n}^2 a^6 (1+\xi)(ma)^3 \{ [3\langle v\rangle(\Pi - \Sigma) - 2\langle v\rangle^3] \frac{x^\alpha x^\beta x^\gamma}{x^3} \\ + \langle v\rangle(\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)})(\frac{x^\alpha}{x}\delta^{\beta\gamma} + \frac{x^\beta}{x}\delta^{\gamma\alpha} + \frac{x^\gamma}{x}\delta^{\alpha\beta}) + s^{\alpha\beta\gamma}.\}\end{aligned} \tag{14}$$

Then the moment equations are as follows.

$$\bar{n}^2 a^6 \frac{\partial \xi}{\partial t} + \frac{\bar{n}^2 a^6 (ma)}{ma^2} \frac{\partial}{\partial x^\alpha}[(1+\xi)\langle v^\alpha\rangle] = 0, \qquad \text{(0th moment)} \tag{15}$$

$$\begin{aligned}\bar{n}^2 a^6 \frac{\partial}{\partial t}[(1+\xi) ma \langle v^\beta\rangle] &+ \frac{\bar{n}^2 a^6 (ma)^2}{ma^2} \frac{\partial}{\partial x^\alpha}\left[(1+\xi)[\Pi P^{\alpha\beta}_{\|} + \Sigma P^{\alpha\beta}_{\perp}]\right] \\ &+ \frac{Gm^2}{a}\bar{n}a^3 \int c(2,3) \frac{x^\beta_{31}}{x^3_{31}} d^3x_3 d^3p_3 d^3p_2 \\ &- \frac{Gm^2}{a}\bar{n}a^3 \int c(3,1) \frac{x^\beta_{32}}{x^3_{32}} d^3x_3 d^3p_3 d^3p_1 \\ &+ \frac{Gm^2}{a}\bar{n}^3 a^9 \int \zeta(\frac{x^\beta_{31}}{x^3_{31}} - \frac{x^\beta_{32}}{x^3_{32}}) d^3x_3 = 0, \qquad \text{(1st moment)}\end{aligned} \tag{16}$$

$$\begin{aligned}&\bar{n}^2 a^6 \frac{\partial}{\partial t}\left[(ma)^2(1+\xi)[\Pi P^{\alpha\beta}_{\|} + \Sigma P^{\alpha\beta}_{\perp}]\right] \\ &+ \frac{\bar{n}^2 a^6 (ma)^3}{ma^2} \frac{\partial}{\partial x^\alpha}(1+\xi)[3\langle v\rangle(\Pi - \Sigma) - 2\langle v\rangle^3] \frac{x^\alpha x^\beta x^\gamma}{x^3}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{\partial}{\partial x^\alpha}(1+\xi)\langle v\rangle(\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)})(\frac{x^\alpha}{x}\delta^{\beta\gamma}+\frac{x^\beta}{x}\delta^{\gamma\alpha}+\frac{x^\gamma}{x}\delta^{\alpha\beta})\\
&+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{\partial}{\partial x^\alpha}s^{\alpha\beta\gamma}(1+\xi)\\
&+\frac{Gm^2}{a}\bar{n}a^3\int c(2,3)[p_{21}^\gamma\frac{x_{31}^\beta}{x_{31}^3}+p_{21}^\beta\frac{x_{31}^\gamma}{x_{31}^3}]d^3x_3d^3p_3d^3p_2\\
&-\frac{Gm^2}{a}\bar{n}a^3\int c(3,1)[p_{21}^\gamma\frac{x_{32}^\beta}{x_{32}^3}+p_{21}^\beta\frac{x_{32}^\gamma}{x_{32}^3}]d^3x_3d^3p_3d^3p_1\\
&+\frac{Gm^2}{a}\int d[p_{21}^\gamma(\frac{x_{31}^\beta}{x_{31}^3}-\frac{x_{32}^\beta}{x_{32}^3})+p_{21}^\beta(\frac{x_{31}^\gamma}{x_{31}^3}-\frac{x_{32}^\gamma}{x_{32}^3})]d^3x_3d^3p_3d^3p_1d^3p_2=0,
\end{aligned}
\qquad \text{(2nd moment)} \quad (17)
$$

where ζ is the three-point correlation function defined by

$$
\bar{n}^3a^9\zeta\equiv\int dd^3p_1d^3p_2d^3p_3. \quad (18)
$$

2.1.3. Contraction of the Equation

The zeroth and first moment equations are the time evolution equations of the ξ and $\langle v\rangle$, respectively. The second moment equation is the time evolution equation of the Π,Σ. For convenience, we transform these equations by taking the divergence of the first moment equation and by applying the following two operators to the second moment equation,

$$
\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma}, \qquad \Delta^{\beta\gamma}=\frac{1}{2}(\delta^{\beta\gamma}-\frac{x^\beta x^\gamma}{x^2}). \quad (19)
$$

Hence we get two equations from the second moment equation. We call them contraction 1 and contraction 2 equations hereafter, shown below as eqs.(23) and (24) , respectively. Here we assume the following relation,

$$
\zeta_{123}=Q(\xi_{12}\xi_{23}+\xi_{23}\xi_{31}+\xi_{31}\xi_{12}). \quad (20)
$$

Finally we obtain the following four equations:

$$
\bar{n}^2a^6\frac{\partial\xi}{\partial t}+\frac{\bar{n}^2a^6(ma)}{ma^2}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)\langle v\rangle]=0, \qquad \text{(0th moment)} \quad (21)
$$

$$\begin{aligned}
\bar{n}^2a^6\frac{\partial}{\partial t}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)(ma)\langle v\rangle] &+ \frac{\bar{n}^2a^6(ma)^2}{ma^2}\frac{1}{x^2}\frac{\partial}{\partial x}\{\frac{\partial}{\partial x}(x^2(1+\xi)\Pi)-2x(1+\xi)\Sigma\}\\
&+ \frac{Gm^2}{a}\bar{n}^3a^98\pi\xi\\
&+ 2\frac{Gm^2}{a}\bar{n}^3a^9QM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2]=0,
\end{aligned}$$

(1st moment) (22)

$$\begin{aligned}
&\bar{n}^2a^6\frac{\partial}{\partial t}\frac{(ma)^2}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[x^2(1+\xi)\Pi]-2x(1+\xi)\Sigma\right]\\
&+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}\left[x^2(1+\xi)\{3\langle v\rangle(\Pi-\Sigma)-2\langle v\rangle^3\}\right]\\
&+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)\langle v\rangle\{\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right]\\
&+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}[x^2(1+\xi)s_{||}]\\
&+\frac{16\pi Gm^2}{a}\bar{n}a^3\bar{n}^2a^6\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)(ma)\langle v\rangle]\\
&+4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta\partial x^\gamma}\int\frac{x_{31}^\beta}{x_{31}^3}x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12}+\frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}d^3x_3\\
&+4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta\partial x^\gamma}\int\frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z}-\frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3=0,
\end{aligned}$$

(2nd moment : contraction 1) (23)

$$\begin{aligned}
\bar{n}^2a^6\frac{\partial}{\partial t}[(ma)^2(1+\xi)\Sigma] &+ \frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4(1+\xi)\langle v\rangle\{\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right]\\
&+ \frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{1}{x^2}\frac{1}{x^4}\frac{\partial}{\partial x}[x^4(1+\xi)s_\perp]\\
&+ \frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(2,3)[p_{21}^\gamma\frac{x_{31}^\beta}{x_{31}^3}+p_{21}^\beta\frac{x_{31}^\gamma}{x_{31}^3}]d^3x_3d^3p_3d^3p_2\\
&- \frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(3,1)[p_{21}^\gamma\frac{x_{32}^\beta}{x_{32}^3}+p_{21}^\beta\frac{x_{32}^\gamma}{x_{32}^3}]d^3x_3d^3p_3d^3p_1\\
&+ 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int\frac{x_{31}^\beta}{x_{31}^3}x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12}+\frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}d^3x_3\\
&+ 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int\frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z}-\frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3=0,
\end{aligned}$$

(2nd moment : contraction 2) (24)

where M is defined as

$$x^{\beta}\xi(x)^2 M \equiv \int \frac{x_{31}^{\beta}}{x_{31}^{3}}\{\xi_{12}+\xi_{31}\}\xi_{23}d^3x_3, \qquad x \equiv x_{21}. \tag{25}$$

This definition is well defined for the vector component of $\boldsymbol{x}$ and $\boldsymbol{x}_{31}$ because of the symmetry of the universe.

And, where

$$\frac{Q^*}{Q}\bar{n}^3 a^9 \zeta ma\langle v_{21}\rangle \equiv \int dp_{21}^{\alpha} d^9 p \tag{26}$$

In general this relation (eq.[26]) is not satisfied. But DP showed the existence of d that satisfies this relation. Hansel et.al. (1986) have investigated whether this relation is suitable or not. And they showed its consistency domain. This relation is suitable in the strongly nonlinear regime. Furthermore, in the strongly nonlinear regime, which we are interested in, we find from the dimensional analysis that this relation is correct in general.

We are interested in the strongly nonlinear regime in which the solutions of the above equations are expected to obey the power law because self-gravity is scale-free (we investigate the solutions that are not the power law in §5.). We assume that ξ is represented by the power-law form

$$\xi = \xi_0 a^{\beta} x^{-\gamma}, \tag{27}$$

where ξ_0, β, and γ are constants.

Then we obtain from the dimensional analysis in eq.(21) in the strongly nonlinear regime that

$$\langle v\rangle = -h\dot{a}x, \tag{28}$$

where h is a constant.

Finally we obtain the following four equations.

$$\frac{\partial \xi}{\partial t} + \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)\langle v\rangle] = 0, \qquad \text{(0th moment)} \tag{29}$$

$$\begin{aligned}\frac{1}{a}\frac{\partial}{\partial t}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)a\langle v\rangle] \quad &+ \quad \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}\{\frac{\partial}{\partial x}(x^2(1+\xi)\Pi) - 2x(1+\xi)\Sigma\} \\ &+ \quad 8\pi Gm\bar{n}a\xi \\ &+ \quad 2Gm\bar{n}aQM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2] = 0, \\ &\qquad\qquad\qquad \text{(1st moment)}\end{aligned} \tag{30}$$

$$\begin{aligned}&\frac{1}{a^2}\frac{\partial}{\partial t}a^2\frac{1}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[x^2(1+\xi)\Pi] - 2x(1+\xi)\Sigma\right] \\ &+ \frac{1}{a}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}\left[x^2(1+\xi)\{3\langle v\rangle(\Pi-\Sigma) - 2\langle v\rangle^3\}\right] \\ &+ \frac{1}{a}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)\langle v\rangle\{\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\ &+ \frac{1}{a}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}[x^2(1+\xi)(s_{\parallel} - 3s_{\perp})] \\ &+ \frac{1}{a}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)s_{\perp}\right] \\ &+ 16\pi Gm\bar{n}a\frac{1}{x^2}\frac{\partial}{\partial x}x^2(1+\xi)\langle v\rangle \\ &- 4Gm\bar{n}aQ^*(\dot{a}h)\frac{1}{x^2}\frac{\partial^2}{\partial x^2}[x^4M\xi^2] = 0, \\ &\qquad \text{(2nd moment : contraction 1)}\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{1}{a^2}\frac{\partial}{\partial t}[a^2(1+\xi)\Sigma] \quad &+ \quad \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4(1+\xi)\langle v\rangle\{\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\ &+ \quad \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}[x^4(1+\xi)s_{\perp}] \\ &+ \quad J = 0, \\ &\qquad\qquad \text{(2nd moment : contraction 2)}\end{aligned} \tag{32}$$

where,

$$\begin{aligned}J \equiv \quad &\frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(2,3)[p_{21}^{\gamma}\frac{x_{31}^{\beta}}{x_{31}^3} + p_{21}^{\beta}\frac{x_{31}^{\gamma}}{x_{31}^3}]d^3x_3d^3p_3d^3p_2 \\ &- \frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(3,1)[p_{21}^{\gamma}\frac{x_{32}^{\beta}}{x_{32}^3} + p_{21}^{\beta}\frac{x_{32}^{\gamma}}{x_{32}^3}]d^3x_3d^3p_3d^3p_1\end{aligned}$$

This term is negligible in the strongly nonlinear regime. In deriving the above equations, we have made use of the symmetry of the background universe and assumed that the three-point correlation function can be written as the products of the two-point spatial correlation functions. In addition, ξ is assumed to be given by eq.(27) which is expected to be correct in the strongly nonlinear region (We also consider the the case that the two-point spatial correlation function does not have the power law in §5.).

2.2. Scale-Invariant Solutions in the Strongly Non-Linear Regime

In this subsection, we discuss how the BBGKY equations are approximated in the strongly nonlinear limit, and we discuss how the power-law index of the two-point spatial correlation function is related to the skewness, three-point correlation function, and the mean relative peculiar velocity. We also investigate whether the stability condition is correct.

2.2.1. BBGKY Equations in the Nonlinear Limit

In the strongly nonlinear regime, the two-point spatial correlation function is much greater than unity, $\xi \gg 1$ at $x \ll 1$. In this limit, the BBGKY equations are as follows.

$$\frac{\partial \xi}{\partial t} + \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2 \xi \langle v \rangle] = 0, \qquad \text{(0th moment)} \tag{33}$$

$$\frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}\{\frac{\partial}{\partial x}(x^2 \xi \Pi) - 2x\xi\Sigma\} + 2Gm\bar{n}aQM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2] = 0, \qquad \text{(1st moment)} \tag{34}$$

$$\begin{aligned}
&\frac{1}{a^2}\frac{\partial}{\partial t}a^2\frac{1}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[x^2\xi\Pi] - 2x\xi\Sigma\right] \\
&+\frac{1}{a}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}\left[x^2\xi\{3\langle v\rangle(\Pi - \Sigma) + (s_{\parallel} - 3s_{\perp})\}\right] \\
&+\frac{1}{a}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}\xi\{\langle v\rangle\Sigma + s_{\perp}\}\right]
\end{aligned}$$

$$-4Gm\bar{n}aQ^*(\dot{a}h)\frac{1}{x^2}\frac{\partial^2}{\partial x^2}[x^4M\xi^2] = 0,$$

$$\text{(2nd moment : contraction 1)} \qquad (35)$$

$$\frac{1}{a^2}\frac{\partial}{\partial t}[a^2\xi\Sigma] \;+\; \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4\xi\{\langle v\rangle\Sigma + s_\perp\}\right] = 0.$$

$$\text{(2nd moment : contraction 2)} \qquad (36)$$

These equations are equivalent to eqs.(39) $\sim$ (42) in Ruamsuwan & Fry(1992) with some differences in notation. The zeroth moment equation (33) is derived without any assumptions. Here it must be noted, however, that there is an assumption about the three-point correlation function involved in deriving the first moment equation (34). In addition, the second moment equations ((35) and (36)), involve terms of the skewness, so one usually requires the higher moment equations in order to solve eqs.(35) and (36), although we do not need them in our analysis.

2.2.2. Scale-Invariant Solutions in the Strongly Nonlinear Limit

We investigate the power law solutions for $\xi, \langle v\rangle, \Pi$, and Σ in this paragraph. Then we assume that the two-point spatial correlation function ξ is given by $\xi = \xi_0 a^\beta x^{-\gamma}$.

We can see from eq.(33) that the mean relative peculiar velocity $\langle v\rangle$ is given by the dimensional analysis as follows:

$$\langle v\rangle = -h\dot{a}x, \qquad (37)$$

$$\beta = (3-\gamma)h. \qquad (38)$$

The stability condition means that $h = 1$ because $\langle\dot{r}\rangle = \dot{a}x + \langle a\dot{x}\rangle = \dot{a}x + \langle v\rangle = 0$. Hereafter we call this parameter h, *relative velocity parameter*. We obtain the power law solutions for the other quantities from eqs.(34) and (35) as follows:

$$\begin{aligned}
\Pi &= \Pi_0 a^{\beta-1} x^{2-\gamma}, \\
\Sigma &= \Sigma_0 a n^{\beta-1} x^{2-\gamma} \\
s_{\|} &= s_{\|0} \dot{a} a^{\beta-1} x^{3-\gamma}, \\
s_{\perp} &= s_{\perp 0} \dot{a} a^{\beta-1} x^{3-\gamma}.
\end{aligned}$$

Here $\Pi_0, \Sigma_0, s_{\|0}$ and $s_{\perp 0}$ are constants.

With the above results, we obtain the next relation from eq.(36).

$$(1 + 2\beta) - (7 - 2\gamma)(h - \Delta) = 0, \tag{39}$$

where Δ is defined by $\Delta \equiv s_{\perp 0/\Sigma_0}$.

We find from eqs.(38) and (39) that

$$h = 1 + (7 - 2\gamma)\Delta. \tag{40}$$

Here it must be noted that eq.(40) is determined based on the assumption that the three-point correlation function can be written as the products of the two-point spatial correlation functions(see eq.[20]). If this assumption is correct and the skewness also vanishes, which means that $\Delta = 0$, we can see from eq.(40) that the relative velocity parameter h is equal to unity. This fact means that the stability condition $h = 1$ is *not* an assumption, but *should be* satisfied in the strongly nonlinear regime when the three-point correlation function can be represented by the products of the two-point spatial correlation functions and the skewness vanishes.

Indeed, Bouchet & Pellat(1984) have noted that the skewness must be zero under the same assumption regarding the three-point correlation function and the stability condition as DP. However they investigated the skewness only under the stability condition when the assumption of the three-point correlation function was satisfied. They did not investigate the case in which the stability condition or the assumption of the three-point correlation function was not satisfied. On the other hand, we derived analytically the explicit relation of the parameter h, skewness Δ, three-point correlation function, and the two-point spatial correlation function by using the analysis of the scale-invariant solution.

It may be physically natural that the skewness does not vanish in the nonlinear regime while the skewness equals zero in the linear regime. Then h should not be equal to 1. This means that the stability condition is satisfied for the unique case in which the skewness vanishes. Furthermore h varies when the assumption about the three-point correlation function ζ changes. For example, if we assume that $\zeta \propto \xi^{2(1+\delta)}$, where δ is a constant, then we can find from eqs.(33) $\sim$ (35) that

$$\begin{aligned}
\xi &= \xi_0 a^{\beta} x^{-\gamma}, \\
\langle v \rangle &= -h\dot{a}x, \\
\Pi &= \Pi_0 a^{\beta(1+2\delta)-1} x^{2-\gamma(1+2\delta)}, \\
\Sigma &= \Sigma_0 a^{\beta(1+2\delta)-1} x^{2-\gamma(1+2\delta)}, \\
s_{\parallel} &= s_{\parallel_0} \dot{a} a^{\beta(1+2\delta)-1} x^{3-\gamma(1+2\delta)}, \\
s_{\perp} &= s_{\perp_0} \dot{a} a^{\beta(1+2\delta)-1} x^{3-\gamma(1+2\delta)}.
\end{aligned}$$

From eq.(36), it is found that

$$2\beta(1+\delta) + 1 - \{7 - 2\gamma(1+\delta)\}(h - \Delta) = 0, \tag{41}$$

and so

$$h = \frac{1 + \{7 - 2\gamma(1+\delta)\}\Delta}{1 - 6\delta}. \tag{42}$$

Thus we can see that the value of the relative velocity parameter h also depends on the assumption about the three-point correlation function. It must be noted here that a small change in δ results in a large change in h because of the factor of $1 - 6\delta$ in eq.(42).

2.2.3. Self-Similar Solutions

DP demonstrated the existence of the self-similar solutions under the stability condition and the assumptions about the three-point correlation function and the skewness by integrating the BBGKY equations numerically. In the self-similar solutions, the power-law index of the two-point spatial correlation function in the strongly nonlinear regime is related to the power-law index n of the initial power spectrum, where n is defined by

$$P(k) \propto k^n. \tag{43}$$

In the linear regime, the two-point spatial correlation function is given by

$$\xi \propto a^2 x^{-(3+n)} \propto (\frac{x}{a^{\alpha}})^{-(3+n)} \tag{44}$$

where

$$\alpha \equiv \frac{2}{3+n}, \tag{45}$$

(Peebles 1980,1993). On the other hand, in the strongly nonlinear regime, the two-point spatial correlation function obeys the following evolution equation (0th moment, eq.[33]),

$$a\frac{\partial \xi}{\partial a} - h\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi] = 0, \tag{46}$$

where we have used the relation $\langle v \rangle = -ha\dot{x}$.

This equation can be rewritten by transforming the variables x and a to the scaling variable, $s \equiv x/a^{\alpha'}$,

$$\frac{d\xi}{ds} = -\frac{3h}{\alpha' + h}\frac{\xi}{s} \tag{47}$$

If the self-similarity is satisfied for the scaling variable s, α' should be equal to $\alpha = 2/(3+n)$.

Then it is found from eq.(47) that

$$\xi \propto s^{-\frac{3h}{\alpha+h}} = s^{-\gamma}, \tag{48}$$

and the power-law index γ is given by

$$\gamma = \frac{3h}{\alpha + h} = \frac{3h(3+n)}{2 + h(n+3)}. \tag{49}$$

If the relative velocity parameter $h = 1$ is satisfied, $\gamma = 3(3+n)/(5+n)$, which is the result shown by DP. And if we can find the value of the parameter h, we obtain the value of the power-law index γ.

2.3. Results and Discussion

In this section, we have investigated the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime. The mean relative peculiar velocity depends on the skewness and the three-point correlation function. The stability condition used by DP is satisfied for the unique case that the skewness vanishes and the three-point correlation function can be represented by the products of the two-point correlation functions. In general the power-law index of the two-point spatial correlation function in the strongly nonlinear regime depends on h, the skewness, and the three-point correlation function(see eqs.[40] and [42]).

In the hierarchical clustering, there are two extreme situations in the manner of the clustering. In one situation, the collapsed object can not be broken and clustered together to form the larger cluster. In this situation, the mean separation of the particles does not change as time increases and $h = 1$. This corresponds to the stability condition. At the other extreme case, the smaller objects have clustered and merged together and the completely virialized object is newly formed. In this case, the mean separation of the particles is expanding with the Hubble velocity and then $h = 0$ (comoving clustering). So, the parameter h becomes the indicator of the merger rate of the clusters. In the case of the collisional system, in which the two-body encounter occur, such as the core collapse of a globular star cluster, the mean interparticle spacing shrinks and the parameter h can become a little larger than 1. But when we consider the special case of collisionless systems, in which the time scale of two-body encounters is much longer than the Hubble time (e.g. in a dark matters-dominated system), the parameter h takes a value between 0 and 1. So, the parameter h has a value of order 1.

This condition constrains the skewness Δ and the three-point correlation function ζ. Eq.(42) suggests that $\delta \ll \frac{1}{6}(\delta \approx 0)$ because h has a value of order 1. This means that the relation $\zeta \propto \xi^2$ is almost correct even in the strongly nonlinear regime. This result is consistent with numerical results by Jain(1997). In general, the skewness does not vanish in the nonlinear regime and so it might be impossible for the stability condition($h = 1$) to be satisfied.

The self-similar solution is shown by DP under some assumptions and the stability condition. If there exist self-similar solutions under other conditions, then the power-law index γ is given by eq(49).

In general we may expect that the power-law index γ does not depend on n because the systems forget the initial conditions in the nonlinear regions as a result of the nonlinearity of gravity. As can be seen from eq.(49), if $h = c/(n+3)$, then γ is independent of n and is given by $3c/(2+c)$ (Padmanabhan(1996)). However h should have the value of order 1 for any probable value of n ($n > -3$). Then c should be zero, that is, $h = 0$. In this case, as mentioned before, the collapsed objects are hierarchically absorbed into larger objects and the substructures in the cluster cannot survive. Then the initial memories are erased. Thus it is physically reasonable that γ does not depend on the initial condition n in this case.

In general, the systems have substructure in some clusters and no subcluster in others, and so γ depends on n. However the relation between γ and n may not be given by the result shown by DP, because the skewness does not generally vanish in the nonlinear regions, and so h should not be equal to 1. Here we comment that $\gamma = 0$ is expected from the catastrophe theory. We can also see that if the solutions of the correlation function possess self-similarity, the power-law index $\gamma = 2$, predicted by Saslaw(1980), is not realized in the strongly nonlinear regime, irrespectively of the initial power-law index n, because in this case c should be eqaul to 4, and then $h > 1$ for $n < 1$.

Ruamsuwan & Fry tested the stability of the scale-invariant solutions derived by DP and found that they prove to be marginally stable. We intend to analyze the stability of the scale-invariant solutions shown in this section under assumptions other than those taken in DP in §3. Furthermore, it is still uncertain that whether self-similarity is satisfied or not. We will investigate whether there exist the self-similarity. We also investigate if the two-point spatial correlation function with the index $\gamma = 0$ evolves self-similarly in §4 because after the first caustic appeared, the two-point spatial correlation function in the non-linear regime have the index with $\gamma = 0$.

3. STABILITY OF THE SCALE-INVARIANT SOLUTION

We have investigated the scale-invariant solutions of the cosmological BBGKY equations in §2. Then, we obtain the various solutions in addition to the DP's solution. We have obtained the index of the two-point spatial correlation function as a function of the index of the initial power spectrum n and the velocity parameter h when we assume the self-similarity, and constrained the possible value of the index from the physical point of view in §2. However, whether these solutions are suitable in the real world or not is unclear. If there is a solution that is unstable, this solution can not be realized in the real situation. We investigate the power-law index of the two-point spatial correlation function from the viewpoint of the stability of the scale-invariant solution in this section. That is, we investigate the stability of the scale-invariant solutions of the BBGKY equations in the non-linear regime.

3.1. Basic Equations

In this subsection, we derive the equations of the linear perturbations in order to investigate the stability of them. We have considered two-point spatial correlation function in the non-linear regime. Then we use the four BBGKY equations in the non-linear limit shown bellow (see also eqs.(33)-(36) in §2):

$$\frac{\partial \xi}{\partial t} + \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2 \xi \langle v \rangle] = 0, \qquad \text{(0th moment)} \tag{50}$$

$$\begin{aligned} \frac{1}{ax^2}\frac{\partial}{\partial x}(x^2 \xi \Pi) \quad &- \quad \frac{2\xi\Sigma}{ax} \\ &+ \quad 2Gm\bar{n}aQ\frac{x^\beta}{x}\int \frac{x_{31}^\beta}{x_{31}^3}\{\xi(x)^{1+\delta} + \xi(z)^{1+\delta}\}\xi(z-x)^{1+\delta} d^3x_3 = 0, \\ &\qquad \text{(1st moment)} \end{aligned} \tag{51}$$

$$\frac{1}{a^2}\frac{\partial}{\partial t}[a^2\xi\Pi] + \frac{1}{ax^2}\frac{\partial}{\partial x}\left[x^2\xi\{3\langle v\rangle\Pi + s_{\|}\}\right] - \frac{4\xi}{ax}\{\langle v\rangle\Sigma + s_{\perp}\}$$
$$+ \ 4Gm\bar{n}aQ^*\frac{x^\beta x^\gamma}{x^2}\int\frac{x_{31}^\beta}{x_{31}^3}\{\xi(x)^{1+\delta} + \xi(z)^{1+\delta}\}\xi(z-x)^{1+\delta}\langle v^\gamma\rangle d^3x_3 = 0,$$
$$(\text{2nd moment : contraction 1}) \tag{52}$$

$$\frac{1}{a^2}\frac{\partial}{\partial t}[a^2\xi\Sigma] \ + \ \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4\xi\{\langle v\rangle\Sigma + s_{\perp}\}\right] = 0,$$
$$(\text{2nd moment : contraction 2}) \tag{53}$$

where we assumed that the three-point correlation function has the following form that is the same assumption with the one in §2.

$$\zeta_{123} = Q(\xi_{12}^{1+\delta}\xi_{23}^{1+\delta} + \xi_{23}^{1+\delta}\xi_{31}^{1+\delta} + \xi_{31}^{1+\delta}\xi_{12}^{1+\delta}). \tag{54}$$

With this aim in view, the two-point spatial correlation function in the scale-invariant solution is perturbed as follows;

$$\xi' = \xi(1 + \Delta_\xi), \tag{55}$$

where ξ is the scale-invariant solution, ξ' the perturbed one and $\Delta_\xi \ll 1$. We also perturb the other variables such as the mean relative peculiar velocity $\langle v\rangle$, the relative peculiar velocity dispersions Π, Σ and the skewness $s_{\|}$, $s_{\perp}$ in the same way.

The equations of the linear perturbations are the following;

$$\frac{\partial\xi\Delta_\xi}{\partial t} + \frac{1}{ax^2}\frac{\partial}{\partial x}[x^2\xi\langle v\rangle\{\Delta_\xi + \Delta_{\langle v\rangle}\}] = 0, \qquad (\text{0th moment}) \tag{56}$$

$$\frac{1}{ax^2}\frac{\partial}{\partial x}[x^2\xi\Pi\{\Delta_\xi + \Delta_\Pi\}] \ - \ \frac{2x\xi\Sigma}{ax}\{\Delta_\xi + \Delta_\Sigma\}$$
$$+ \ 2Gm\bar{n}aQx\xi^{2(1+\delta)}(1+\delta)M'_{\gamma(1+\delta),q}\Delta_\xi = 0,$$
$$(\text{1st moment}) \tag{57}$$

$$
\begin{aligned}
&\frac{1}{a^2}\frac{\partial}{\partial t}[a^2\xi\Pi]\{\Delta_\xi + \Delta_\Pi\} \\
&\quad + \frac{1}{ax^2}\frac{\partial}{\partial x}\left[x^2\xi\{3\langle v\rangle\Pi\{\Delta_\xi + \Delta_{\langle v\rangle} + \Delta_\Pi\} + s_{\|}\{\Delta_\xi + \Delta_{s_{\|}}\}\}\right] \\
&\quad - \frac{4\xi}{ax}\{\langle v\rangle\Sigma\{\Delta_\xi + \Delta_{\langle v\rangle} + \Delta_\Sigma\} + s_\perp\{\Delta_\xi + \Delta_{s_\perp}\}\} \\
&\quad + 4Gm\bar{n}aQ^* x\langle v\rangle\xi^{2(1+\delta)}[(1+\delta)M'_{\gamma(1+\delta),q}\Delta_\xi + M_{\gamma(1+\delta)}\Delta_{\langle v\rangle}] = 0, \\
&\qquad\qquad\qquad (\text{2nd moment : contraction 1})
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
&\frac{1}{a^2}\frac{\partial}{\partial t}[a^2\xi\Sigma\{\Delta_\xi + \Delta_\Sigma\}] \\
&\quad + \frac{1}{ax^4}\frac{\partial}{\partial x}\left[x^4\xi[\langle v\rangle\Sigma\{\Delta_\xi + \Delta_{\langle v\rangle} + \Delta_\Sigma\} + s_\perp\{\Delta_\xi + \Delta_{s_\perp}\}]\right] = 0, \\
&\qquad\qquad\qquad (\text{2nd moment : contraction 2})
\end{aligned} \tag{59}
$$

where $xQ\xi^2(M + M'\Delta_\xi)$ is the integral of the three-point spatial correlation function. Here it should be noted that in deriving the above equations (56)-(59) we neglect some terms; high order terms in the strongly non-linear limit, and higher order ones in the limit of small perturbations (larger than the first order perturbation). The ordering parameters in both limits are generally independent of each other. However we consider that higher order terms in the strongly non-linear limit are much smaller than the first order terms of the perturbation.

RF considered the following power law perturbation:

$$\Delta_\xi = \epsilon_\xi a^p x^q. \tag{60}$$

In this case, $M_{\gamma(1+\delta)}$ and $M'_{\gamma(1+\delta),q}$ are given by

$$
\begin{aligned}
M_{\gamma(1+\delta)} &= \int \frac{\mu}{y^2} s^{-\gamma(1+\delta)}(1 + y^{-\gamma(1+\delta)})d^3y, &(61)\\
M'_{\gamma(1+\delta),q} &= \int \frac{\mu}{y^2}[s^{q-\gamma(1+\delta)}(1 + y^{-\gamma(1+\delta)}) + s^{-\gamma(1+\delta)}(1 + y^{q-\gamma(1+\delta)})]d^3y, &(62)
\end{aligned}
$$

$$\mu \;=\; \frac{x^\alpha}{x}\frac{z^\alpha}{z}, \quad y = \frac{x_{31}}{x_{21}} = \frac{z}{x}, \quad s = \frac{x_{23}}{x_{21}} = (1 + y^2 - 2y\mu)^{1/2}. \tag{63}$$

The integrals should not diverge. Then, $2 - \gamma(1+\delta) > 0$ must be satisfied for $y \to 0$ and $\gamma(1+\delta) > 0$ for $y \to \infty$ in the M. Furthermore, $2 + q - \gamma(1+\delta) > 0$ must be satisfied for $y \to 0$ and $q - \gamma(1+\delta) < 0$ for $y \to \infty$ in the M'. As a result, the following relations must be satisfied:

$$0 < \gamma(1+\delta) < 2, \qquad \gamma(1+\delta) - 2 < q < \gamma(1+\delta). \tag{64}$$

The four perturbation equations (56)-(59) are a little different from the RF's equations. This is because in perturbing the three-body correlation term, RF divide artificially the $\langle v_{21}\rangle$ into $\langle v_{23}\rangle + \langle v_{31}\rangle$ and perturbed $\langle v_{23}\rangle$ and $\langle v_{31}\rangle$ independently, which is an incorrect treatment. Furthermore, although they obtained the q-dependence of M', they used the value of M' at only $q = 0$.

3.2. Solutions of the Perturbations

We consider the following form of perturbations. We have to consider perturbations that diverge neither at the strongly non-linear limit ($x \to 0$) nor at the linear limit ($x \to \infty$). The following perturbations of the two-point spatial correlation function are generated:

$$\Delta_\xi = \epsilon_\xi a^p x^q, \tag{65}$$

$$q = |q| i, \tag{66}$$

where $|q|$ is a real number (q is a pure imaginary number while RF adopted the real number). In this case, the perturbations do not diverge in the strongly non-linear limit, nor in the linear limit. If q is a real number, as RF adopted, the perturbations diverge on either limiting scales. When q is negative, the perturbation diverges in the non-linear limit

($x \to 0$). On the other hand, when q is positive, the perturbation diverges in the linear limit ($x \to \infty$). The perturbations of the other variables can also be written in the same form. From the dimensional analysis, all perturbations must have the same power (q and p) as those of the two-point spatial correlation function. When q is a pure imaginary number, $|s^q| = 1$ and $|y^q| = 1$. And the following relations are satisfied:

$$|M'_{\gamma(1+\delta),q}| \leq M'_{\gamma(1+\delta),q=0} = 2M_{\gamma(1+\delta)}. \tag{67}$$

When $0 < \gamma(1+\delta) < 2$, both $M_{\gamma(1+\delta)}$ and $M'_{\gamma(1+\delta),q}$ are finite for any value of the pure imaginary number q.

These perturbed BBGKY equations are not closed by themselves in general and higher moment equations are needed. But if the coefficients of the perturbation of the skewness ($\Delta_{s_\parallel}$ and $\Delta_{s_\perp}$) are equal to 0, the perturbation equations for the other variables have no relation with the perturbation of it. In this case we can solve these perturbation equations independently from the higher moment equations. As we can see from eqs.(58) and (59), when the skewness of the background velocity field is equal to 0, the coefficient of the perturbation of the skewness becomes 0. Here we consider only this particular case. Then we can neglect the higher moment equations. In this case, the above equations can be rewritten by using the power law perturbations given by eqs.(65) and (66);

$$(p - hq)\Delta_\xi - h(3 - \gamma + q)\Delta_{\langle v\rangle} = 0, \tag{68}$$

$$\begin{aligned}&[2\{\gamma(1+\delta) - 2 + \sigma\}(1+2\delta) + q + 2(1+\delta)Dkq]\Delta_\xi \\ &+ \{4 - 2\gamma(1+\delta) + q\}\Delta_\Pi - 2\sigma\Delta_\Sigma = 0,\end{aligned} \tag{69}$$

$$[1 - 15h + (6+4\gamma)h(1+\delta) + 4h\sigma + p - 3hq$$

$$
\begin{aligned}
&-8h\frac{Q^*}{Q}\{\gamma(1+\delta)-2+\sigma\}(1+\delta)-4h\frac{Q^*}{Q}(1+\delta)Dkq]\Delta_\xi \\
&+[-3h\{5-2\gamma(1+\delta)+q\}+4h\sigma-4h\frac{Q^*}{Q}\{\gamma(1+\delta)-2+\sigma\}]\Delta_{\langle v\rangle} \\
&+[1-15h+(6+4\gamma)h(1+\delta)+p-3hq]\Delta_\Pi+4h\sigma\Delta_\Sigma=0,
\end{aligned}
\tag{70}
$$

$$
(1-h+6h\delta+p-hq)\Delta_\xi-h\{7-2\gamma(1+\delta)+q\}\Delta_{\langle v\rangle}+(1-h+6h\delta+p-hq)\Delta_\Sigma=0, \tag{71}
$$

where

$$
D\equiv\gamma(1+\delta)-2+\sigma, \qquad \sigma=\frac{\Sigma}{\Pi}. \tag{72}
$$

It is difficult to treat exactly M' as a function of q. So, we approximate M' by a linear function of q. When $q=0$, M'/M is equal to 2. Then we use the following approximation of M'/M in the above equations (57) and (58);

$$
\frac{M'}{M}\equiv 2+kq, \tag{73}
$$

where

$$
k=\frac{\int\frac{\mu}{y^2}[s^{-\gamma(1+\delta)}(1+y^{-\gamma(1+\delta)})\log s+s^{-\gamma(1+\delta)}y^{-\gamma(1+\delta)}\log y]d^3y}{\int\frac{\mu}{y^2}s^{-\gamma(1+\delta)}(1+y^{-\gamma(1+\delta)})d^3y}. \tag{74}
$$

The integral $\int(\mu/y^2)s^{-\gamma(1+\delta)}(1+y^{-\gamma(1+\delta)})d^3y$ is dominated around s and $y\sim 1$. Around s and $y\sim 1$, $\log s$ and $\log y$ have values of the order of 1. Therefore, $k\sim 1$.

Furthermore we use the following relation that can be derived from the first moment equation [eq.(51)];

$$
4-2\gamma(1+\delta)-2\sigma+2Gm\bar{n}a^2Qx^2\frac{\xi^{1+2\delta}}{\Pi}M_{\gamma(1+\delta)}=0. \tag{75}
$$

The four equations (68)-(71) can be rewritten by using the matrix notation.

$$N_{ij}u_j = 0, \tag{76}$$

$$u_j = (\Delta_\xi, \Delta_{\langle v\rangle}, \Delta_\Pi, \Delta_\Sigma). \tag{77}$$

If there exists a non trivial solution, the determinant of the matrix N_{ij} should be equal to 0. Now we consider the zero skewness case. The relation between the three-point spatial correlation function and the mean relative peculiar velocity eq.(42) becomes

$$\delta = \frac{h-1}{6h}. \tag{78}$$

By using this relation, we can eliminate δ in eqs.(69)-(71). Furthermore, from the first moment equation(51) and the second moment (contraction 1) equation (52), we obtain the following relation:

$$(-h+1+6h\delta)\Pi + \{5-2\gamma(1+\delta)\}\frac{s_\parallel}{\dot{a}x} - \frac{4s_\perp}{\dot{a}x} - 4Gm\bar{n}a^2x^2M_{\gamma(1+\delta)}h(Q^*-Q) = 0. \tag{79}$$

As we can see, when the skewness is equal to 0, Q^*-Q must be 0, that is, Q^*/Q is equal to 1.

Then the determinant of $\boldsymbol{N}$ is given by

$$\det \boldsymbol{N} = \frac{1}{3}(p-hq)^2 f(q,\gamma',h,\sigma,kD), \tag{80}$$

where

$$\begin{aligned} f(q,\gamma',h,\sigma,kD) &= \{9h+kD(7h-1)\}q^2 \\ &+ (4-2\gamma'+29h-10\gamma'h-2\sigma+8h\sigma)q \\ &+ kD(-3+21h-6\gamma'h)q+12-6\gamma'-6\sigma, \end{aligned} \tag{81}$$

and

$$\gamma' \equiv \gamma(1+\delta). \tag{82}$$

Here f is a quadratic equation of q. Now we investigate whether the equation $f = 0$ has real solutions or not. Since we do not know the value γ', h, σ, and kD in the strongly non-linear regime, we treat these values as parameters. Here we consider the allowed range for these parameter. As we can see from eq.(64), the parameter γ' must satisfy $0 < \gamma' < 2$. Since we have considered $\zeta \gg \xi$ in the strongly non-linear regime, $\xi^{2(1+\delta)}$ should be of an order higher than ξ, so that $2(1+\delta) > 1$ must be satisfied. In this case, $h > \frac{1}{4}$ must be satisfied from eq.(78). YG showed the probable range of the mean relative peculiar velocity and obtained that the mean relative physical peculiar velocity must lies between 0 and the Hubble expansion velocity. This means that the parameter h(relative velocity parameter) has a value between 0 and 1 (§2). So in this case, the parameter h should be in the range $\frac{1}{4} < h < 1$. We do not know the range of the parameters σ and kD. But the parameter σ should have a value around 1. Hence, we investigate σ in the range $\frac{1}{2} < \sigma < 2$. The value of the parameters D and k also must be around 1 as seen from the eqs. (72) and (74), respectively. So we investigate kD in the range $-1 < kD < 1$.

In the above probable value of the parameters, we can easily ascertain that $f = 0$ has real solutions. In other words, the solutions of $f = 0$ are not complex. Since we consider the case that q is an imaginary number, $f = 0$ can not be satisfied. Therefore, $p - hq$ is equal to 0 to allow the determinant of the matrix $\boldsymbol{N}$ to be null. In this case, p has the following value

$$p = hq = h|q|i. \tag{83}$$

This means that the perturbations do not grow. And the solutions are stable.

Furthermore we consider the strict condition which determines the stability of the two-point spatial correlation function. In the strongly non-linear regime, the mean relative peculiar velocity takes the value $\langle v \rangle = -h\dot{a}x$ depending on the process of clustering. In this case, the scale of $a^h x$ for the two particles whose mean comoving distance is x does not change as we can see from the following relation:

$$\begin{aligned} \frac{d}{dt}(a^h x) &= a^{h-1}(a\dot{x} + h\dot{a}x) \\ &= a^{h-1}(\langle v \rangle + h\dot{a}x) \\ &= 0. \end{aligned} \tag{84}$$

Then we should determine the stability of the two-point spatial correlation function at the fixed scale $a^h x$. The solutions of the perturbations are rewritten as

$$\begin{aligned} \Delta_\xi &= \epsilon_\xi (a^h x)^q \\ &= \epsilon_\xi e^{i|q| log(a^h x)}. \end{aligned} \tag{85}$$

These perturbations do not grow, nor do they decay. At the fixed scale of $a^h x$, the perturbation never even oscillates. This means that the perturbations prove to be marginally stable. RF used the real number q in investigating the behavior of the perturbations. As we can see from eq.(85), the perturbation in the $p - hq = 0$ mode works well even when q is a real number. That is, the perturbation proves to be marginally stable in this mode. However there also exist a 'strange' growing mode for the real number of q because $f = 0$ is satisfied in this case. RF did not comment sufficiently about this 'strange' growing mode.

3.3. Results and Discussion

In this section, we have investigated the stability of the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime with the skewness of the velocity field being equal to zero. The reason we have considered only the case is that perturbed BBGKY equations for ξ, $\langle v \rangle$, Π and Σ can be closed independently of the higher moment perturbations. When power law perturbations have been put in the solutions as investigated by RF, in other words, when q is a real number, the perturbations in the linear limit or the non-linear limit diverges. As a matter of fact, when q is negative, the perturbations diverge and do not work in the non-linear limit. When q is positive, the perturbations behave well in the non-linear regime. However in the linear regime, those perturbations may diverge if the power law form of the perturbations are retained, and so the form of the perturbations should be changed in order to avoid the divergence in the linear limit. In this case, it is insufficient to solve the non-linear approximated equations, because we do not have information about the evolutions on large scales. Thus, we have investigated only the local stability of the non-linear regime. In investigating the local stability of the two-point spatial correlation function in the strongly non-linear regime, we should put perturbations only on the scales in which we are going to investigate. That is, we should put the wave packet-like perturbation. In order to put such a wave packet-like perturbation, the number q must be imaginary. In this case, we have found that there is no unstable mode. It seems stable for any value of the power-law index of the two-point spatial correlation function. However we do not know whether a global instability exists because we consider only the local stability. It is certain that there is no local instability. So, in the strongly non-linear regime, the solutions have proved to be marginally stable, and it does not seem that the power-law index of the two-point spatial correlation function approaches to some stable point values. The power-law index of the two-point spatial correlation function that was derived by DP, $\gamma = 3(3+n)/(5+n)$, is not the special one also from the viewpoint of stability of the solution. As a result, the argument of the stability does not determine the power index of the two-point spatial correlation function.

The power-law index of the two-point spatial correlation function is determined only by the clustering process, that is, the parameter h, if self-similar solutions exist. Hence it is very important to estimate the parameter h and investigate whether the self-similar solutions exist or not in the general scale-invariant solutions obtained by YG. In the following section, we investigate the self-similarity of the two-point spatial correlation function or the power spectrum by using the one-dimensional system.

4. SELF-SIMILARITY OF THE TWO-POINT SPATIAL CORRELATION FUNCTION

We have investigated the scale-invariant solutions of the cosmological BBGKY equations in §2. Then we obtained the power-law index of the two-point correlation function as a function of the index of the initial power spectrum n and the velocity parameter h when we assume self-similarity of the two-point spatial correlation function. In §3, we investigated the stability of the scale-invariant solutions in the non-linear regime. The index of the two-point spatial correlation function cannot be determined from the viewpoint of stability of the solutions. This means that the index of the two-point spatial correlation function in the non-linear regime is related to the evolution of the density fluctuations in the linear and the quasi-linear regime. Furthermore, if the self-similarity of the two-point spatial correlation function is satisfied, the two-point spatial correlation function in the non-linear regime is related to the index of the initial power-law index n and the velocity parameter h as we commented above. Therefore whether self-similarity is satisfied or not is a very important problem. Then, we investigate self-similarity of the two-point spatial correlation function (power spectrum) in this section. Here, we note that Fourier transform of the two-point spatial correlation function is the power spectrum. So, we investigate the evolution of the power spectrum instead of the two-point correlation function in this section. In order to investigate self-similarity of the power spectrum, we need a high spatial resolution. Therefore in this section, we consider one-dimensional systems. We have shown the possibility that there exist a self-similar solution with the index of the two-point spatial correlation function being 0. On the other hand, when the initial power spectrum has a sharp cutoff (the cutoff case), there appear caustics of the density field everywhere. In this case the power-law index is independent of the initial condition, and the index of the two-point spatial correlation function becomes 0. Then, when the cutoff case, there exist the possibility of the self-similarity of the two-point spatial correlation function which have the power-law index with $\gamma = 0$. Therefore, we will investigate not only the scale-free case but also the cutoff case.

4.1. Numerical method

We investigate time evolutions of the power spectrum and its self-similarity by using a numerical method. In this section we consider the one-dimensional system in which many plane parallel sheets move only in a perpendicular direction to the surface of these sheets. When two sheets cross, they are allowed to pass through each other freely. In this sheet system, there is an exact solution until two sheets cross over as follows (Sunyaev & Zel'dovich(1972), Doroshkevich et.al.(1973)):

$$\begin{aligned} x &= q + B_1(t)S_1(q) + B_2(t)S_2(q), \\ v &= \dot{B}_1(t)S_1(q) + \dot{B}_2(t)S_2(q), \end{aligned} \tag{86}$$

where q and x are the Lagrangian and the Eulerian coordinates, respectively. Here, $S_1(q)$ and $S_2(q)$ are arbitrary functions of q. $B_1(t)$, and $B_2(t)$ are the growing mode and the decaying mode of linear perturbation solutions, respectively. Since we consider the Einstein-de Sitter universe, $B_1(t) = a$ and $B_2(t) = a^{-\frac{3}{2}}$, where a is the scale factor of the universe. We can compute the crossing times of all neighboring pairs of sheets. We use the shortest of these crossing times as a time step. Then, we can compute the new positions and velocities for all sheets at this crossing time. After two sheets cross, we exchange the velocities of the two sheets that have just crossed. Then we again obtain $S_1(q), S_2(q)$, and therefore exact solutions as follows:

$$\begin{aligned} S_1(q) &= \frac{3}{5}a^{-1}(x-q) + \frac{2}{5}\dot{a}^{-1}v, \\ S_2(q) &= \frac{2}{5}a^{\frac{3}{2}}(x-q) - \frac{2}{5}\dot{a}^{-1}a^{\frac{5}{2}}v. \end{aligned} \tag{87}$$

These new exact solutions can be used until two sheets cross over. In this way we obtain the exact loci of the sheets by coupling these solutions. This numerical method has high accuracy because we connect the exact solutions. Throughout this section, we use 2^{13} sheets for numerical calculations. A periodic boundary condition is used and unit of period is fixed by a length of 2π. Therefore, the smallest wave number is 1.

4.2. Scale-free spectrum case

In Figs.1(a) and 1(b), we show the time evolutions of the power spectrum with the scale-free initial power spectrum given by

$$P(k, t_{ini}) \propto k^n, \tag{88}$$

where n is the power-law index of the initial spectrum.

It is found that the following results are satisfied for $-1 < n \leq 4$. The condition of $n > -1$ means the hierarchical clustering in the one-dimensional system. We are interested in this case ($n > -1$) for "scale-free" case. One of the reasons that we consider this case is that the hierarchical clustering picture is expected in the real world. Another reason is that the results for $n < -1$ is similar to those in the case of the single wave case. See also eq.(99). Then the results for $n < -1$ can be inferred from the results shown in the regimes 4 and 5 in the "cutoff case"(see §4.3). The condition $n \leq 4$ is required because we consider the situation in which the non-linear mode coupling from higher k to smaller k can be neglected and so linear perturbation theory can hold on smaller k than k_{nl} (Peebles (1980), Shandarin & Melott (1990), Gouda (1995)). In the following, we show the cases of $n = 1$ and 2 as examples. Fig.1(a) and(b) are the $n = 1$ case and the $n = 2$ case, respectively. We can obtain the same results qualitatively in the cases of the other indexes. Here t_{ini} is the initial time, and the initial scale factors of these two cases are 0.1 and 0.01, respectively.

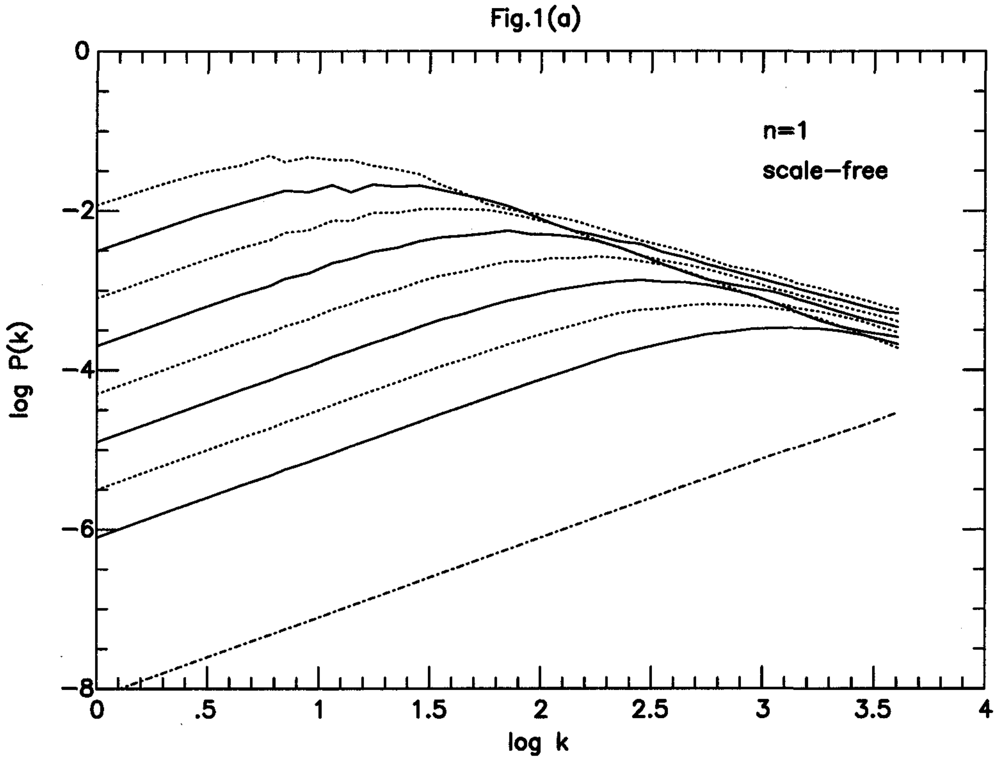

Fig.1.-(a) The time evolution of the power spectrum. The solid and dotted lines are used mutually in order to distinguish easily each line. The curves shown are (bottom to top) at $a = 0.1, 1, 2, 4, 8, 16, 32, 64, 128$. The initial power spectrum is scale-free with the power index $n = 1$(Dotted-dashed line).

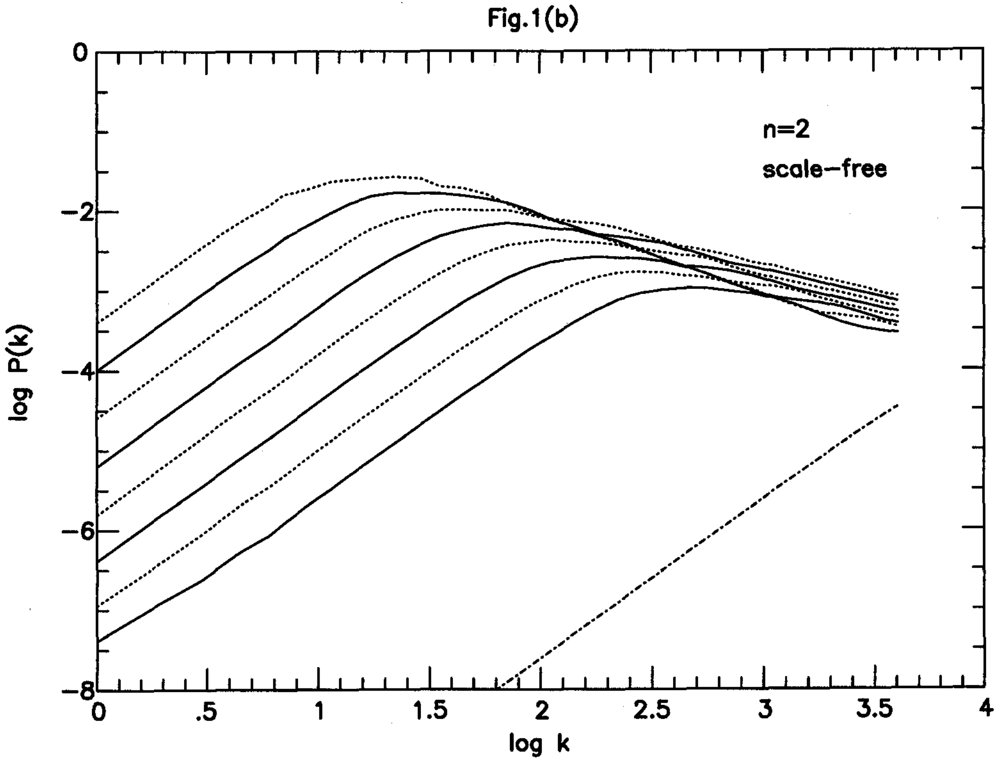

Fig.1.-(b) The same as (a) but for the initial power-law index $n = 2$ and the power spectra are shown at the scale factor $a = 0.01, 1, 2, 4, 8, 16, 32, 64, 128$.

We normalize the scale factor as follows: $a = 1$ when the first caustic has appeared. The phases of the initial Fourier spectrum are given at random. We have averaged 50 samples. Here, we scale the wave number and power spectrum as follows:

$$k_* \equiv \frac{k}{k_{nl}(t)}, \qquad P_*(k_*, t) \equiv \frac{P(k,t)}{P_{scale}(t)}, \tag{89}$$

where k_{nl} is defined by

$$\frac{1}{2\pi}\int_0^{k_{nl}(t)} P(k,t)dk = 1. \tag{90}$$

In the regime of $k < k_{nl}$, the power spectrum grows according to the solution of linear perturbation, that is, the power spectrum satisfies the relation $P(k,t) \propto a^2$. Therefore, k_{nl} is proportional to $a^{-2/(n+1)}$. $P_{scale}(t)$ is defined by

$$P_{scale}(t) \equiv P(k_{nl}(t), t). \tag{91}$$

Then $P_{scale}(t)$ is proportional to $a^2 k_{nl}^n \propto a^{2/(n+1)}$. This scaled power spectrum $P_*(k_*)$ is shown in Fig.2(a) for the case of $n = 1$. The same power spectrum for the case of $n = 2$ is shown in Fig.2(b).

We can see the coincidence of each power spectrum at each time for both cases with good accuracy. This means that the self-similarity is achieved in the scale-free case. Furthermore we can see the three different power-law indexes in three regimes: the linear regime ($k < k_{nl}$: regime 1), the single-caustic regime ($k_{nl} < k < k_{snl}$: regime 2), and the multi-caustics regime ($k > k_{snl}$: regime 3). Fig.3 shows a schematic general power spectrum at a time in the scale-free case.

The value of the power-law index in the linear regime, of course, remains unchanged. We explain the power-law index of the power spectrum in the other two regimes: the single-caustic regime and the multi-caustics regime in the following paragraphs.

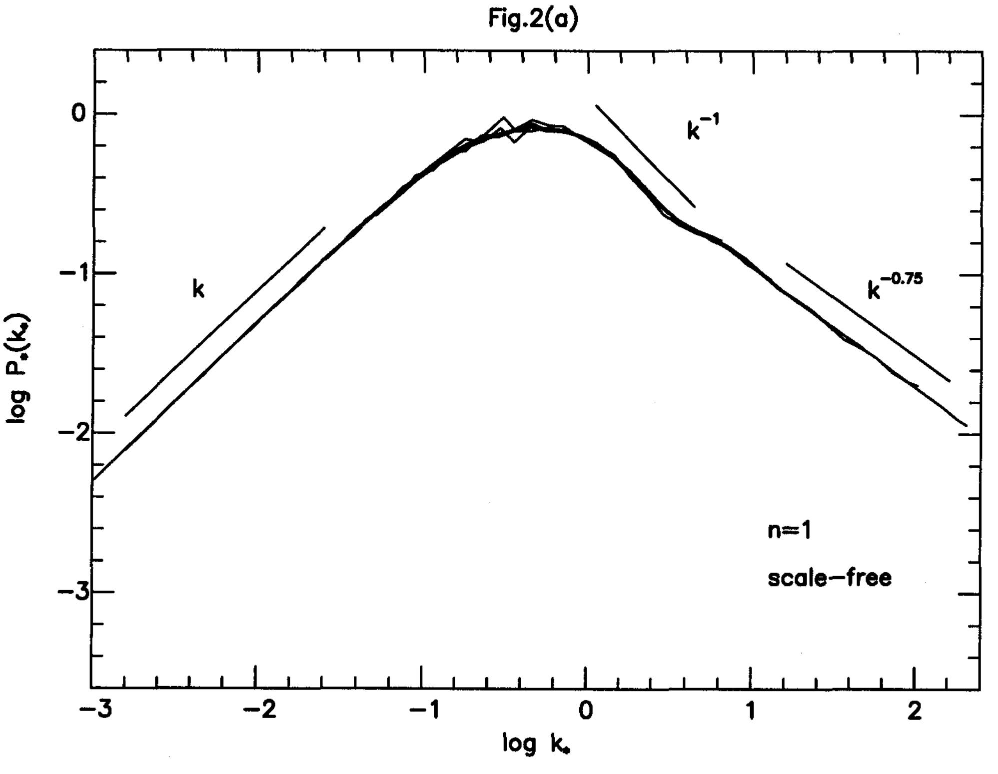

Fig.2.-(a) The scaled power spectrum $P_*(k_*)$. The initial power spectrum is scale-free with the power-law index $n = 1$. Three solid straight line shows the power law with the power-law index of 1,-1,-0.75. p

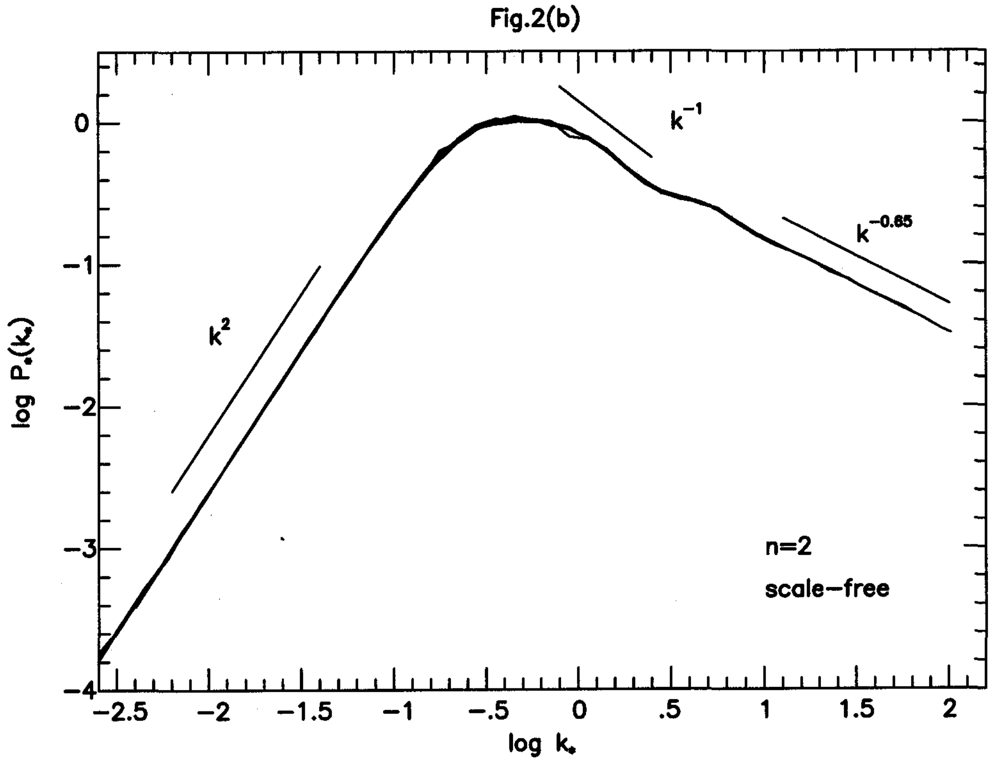

Fig.2.-(b) The same as (a) but for the initial power-law index $n = 2$. Three solid straight line shows the power law with the power-law index of 2,-1,-0.65.

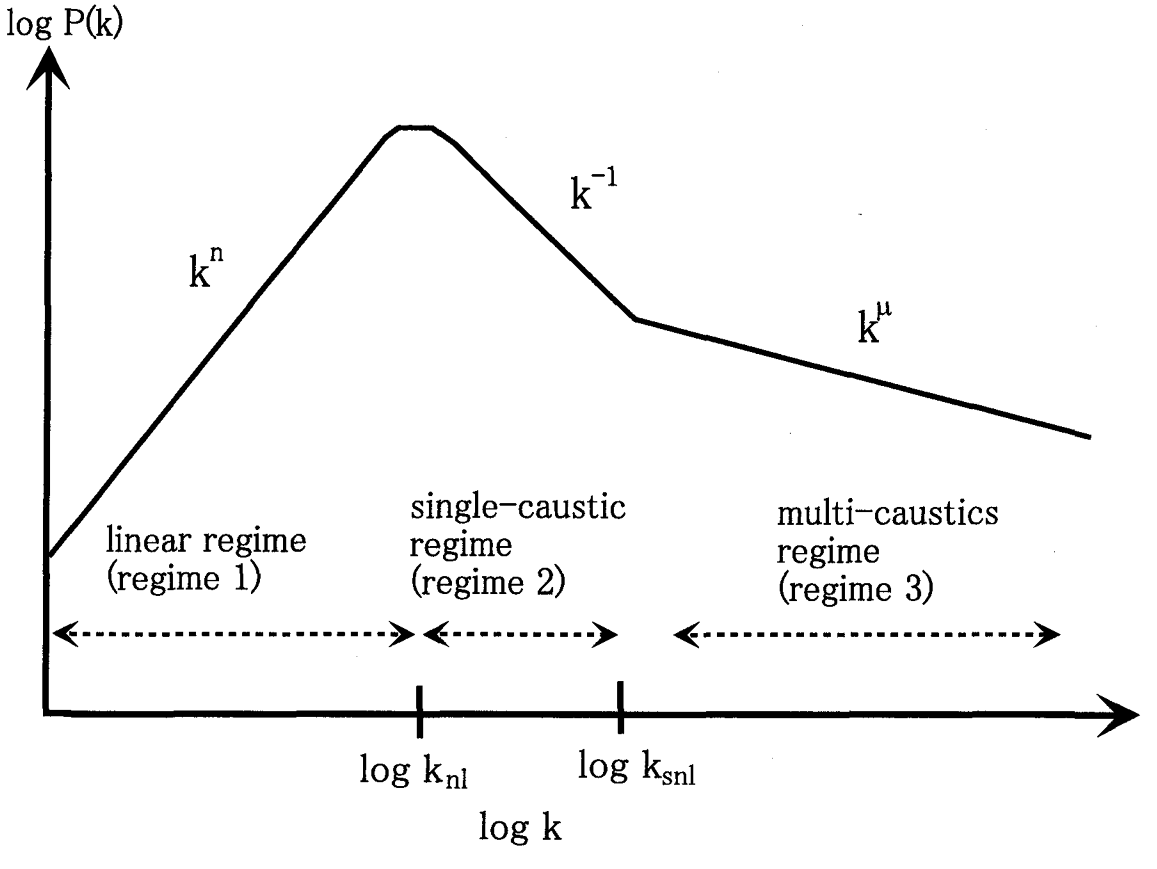

Fig. 3

Fig.3.- The schematic of the power spectrum in the scale-free case at a certain time.

4.2.1. Single-caustic regime (regime 2)

In the single-caustic regime ($k_{nl} < k < k_{snl}$: regime 2) we can see that the power-law index value becomes -1. The reason is as follows: The density perturbation with the scale that just entered the non-linear regime at a given time, that is, k_{nl}, makes "caustics". Strictly speaking, the caustics would appear if the initial spectrum was smoothed bellow the scales just entered the non-linear regime. Therefore, we must notice that the real caustics cannot be observed in the scale-free case because of smearing by the small scale fluctuations. However, we call these scale on $k_{nl} < k < k_{snl}$, "single-caustics regime" in this thesis. After the first appearance of "caustics", the value of the power-law index of the power spectrum on the scales smaller than that scale is -1 (Gouda & Nakamura (1989), Kotok & Shandarin (1988)): Here we briefly show why the value of the power-law index is -1 after the first appearance of caustics.

The Fourier spectrum δ_k of the density fluctuations is given by

$$\delta_k = \int \delta(x) e^{ikx} dx. \tag{92}$$

Then the density is given by

$$\rho(x) = \frac{\rho_0}{|\frac{dx}{dq}|}, \tag{93}$$

where ρ_0 is the mean density of the universe. At the Lagrangian singular point q_0, the following relation is satisfied.

$$(\frac{dx}{dq})_{q_0} = 0. \tag{94}$$

At the singular point q_0, the density $\rho(x)$ diverges. Therefore, the Eulerian coordinate x can be written

$$x = x_0 + \beta'(q - q_0)^2 + O((q - q_0)^3). \tag{95}$$

For simplicity, we can put $x_0 = q_0 = 0$ without losing generality. Therefore we can express

$x = \beta' q^2$ around the caustic. The density around the caustic is expressed by

$$\rho(x) = \rho_0 |\frac{dx}{dq}|^{-1} \propto (\beta' x)^{-\frac{1}{2}} \tag{96}$$

This density profile determines the proper index of the power spectrum. This type of singularity is called A2 type according to catastrophe theory (Gouda & Nakamura (1988,1989)). Eqs.(95), and (96) can be generally satisfied around the singular points in the multi-stream flow regimes where the Zel'dovich solution cannot hold. The A2 type of singularity is the only stable type in the one-dimensional system and eqs.(95) and (96) remain in the multi-stream flow regions after the first appearance of caustics. Here it must be noted that Lagrange coordinate q in the above argument is not the initial position of the sheet. See Roytvarf (1987) and Gouda & Nakamura (1989) for a detailed explanation. On the other hand, it is found that $\rho(x)$ is proportional to $x^{-2/3}$ around the singular points at the first appearance of caustics (Zel'dovich (1970), Arnold et.al.(1982), Gouda & Nakamura (1988)). This type of singularity is called type A3. The A3 Type is not structurally stable in the one-dimensional system and so the A3 type of singularities appears only for an instant. It disappears and quickly evolves into the A2 type, i.e. $\rho \propto x^{-1/2}$ singularity. Hence the contribution from the A3 type of singularity in estimating the power spectrum is negligible. The Fourier spectrum of the density is given by

$$\begin{aligned} \delta_k &= \int \delta(x) e^{ikx} dx \\ &= \int \frac{\rho}{\rho_0} e^{ikx} dx \\ &\propto \int (\beta' x)^{-\frac{1}{2}} e^{ikx} dx \\ &= \beta'^{-\frac{1}{2}} k^{\frac{1}{2}} k^{-1} \int t^{-\frac{1}{2}} e^{it} dt, \end{aligned} \tag{97}$$

where $t \equiv kx$. Then, we obtain

$$P(k) \propto (k\beta')^{-1}. \tag{98}$$

From eq.(98), we find the value of the power-law index is -1. Here we notice that the small value of β' contributes to the large amplitude of the power spectrum. Furthermore, the coefficient β' is very small when the caustics have just appeared. We can see this fact by the evolution of the "single-wave perturbation". Here, the single-wave perturbation means the density fluctuations whose initial condition is given by

$$x(q) = q + X sin(q), \tag{99}$$

where X is a constant value. Hereafter we call this case the single-wave case. The first caustics have just appeared at a center of the x axis ($x = 0$) in Fig.4(a). After the first appearance of this caustic, more caustics appear in the collapse regime by phase mixing (Fig.4(b)-(d)) (Doroshkevich et.al(1980), Melott (1983), Gouda & Nakamura (1989)).

A caustic (singularity of the density field) is located at the point at which the derivative of v with respect to x is infinity. The absolute value of the derivative of v with respect x around the singular points is proportional to β'^{-1}. Then as the phase mixing continues and so more caustics appear, it is found from Fig.4(a)-4(d) that β' increase because $|\frac{dv}{dx}|$ around the singular points decreases. Hereafter we call the structure in the phase space shown in Fig.4(a)-4(b) "the whirlpool". Just after the multi-caustics appear, the coefficient factor β' of this wavelength is very small and the effect of the amplitude of the power spectrum in these waves dominates. As a result, the index of -1 that is predicted by the catastrophe theory appears. Therefore we call this regime, single-caustic regime.

Here, we comment that *real* singularities do not occur in the real world. Of course, the arguments on caustics in this section cannot be applied to barionic gas because shocks appeared and they prohibited density divergence. Furthermore the maximum density is surprisingly low even in the neutrino-dominated model owing to the velocity dispersion of the neutrino (Zel'dovich & Shandarin (1982),Kotok & Shandarin (1987)). In this thesis, we consider the cold collisionless matter whose velocity dispersion is much small compared with the scales under consideration.

Fig.4

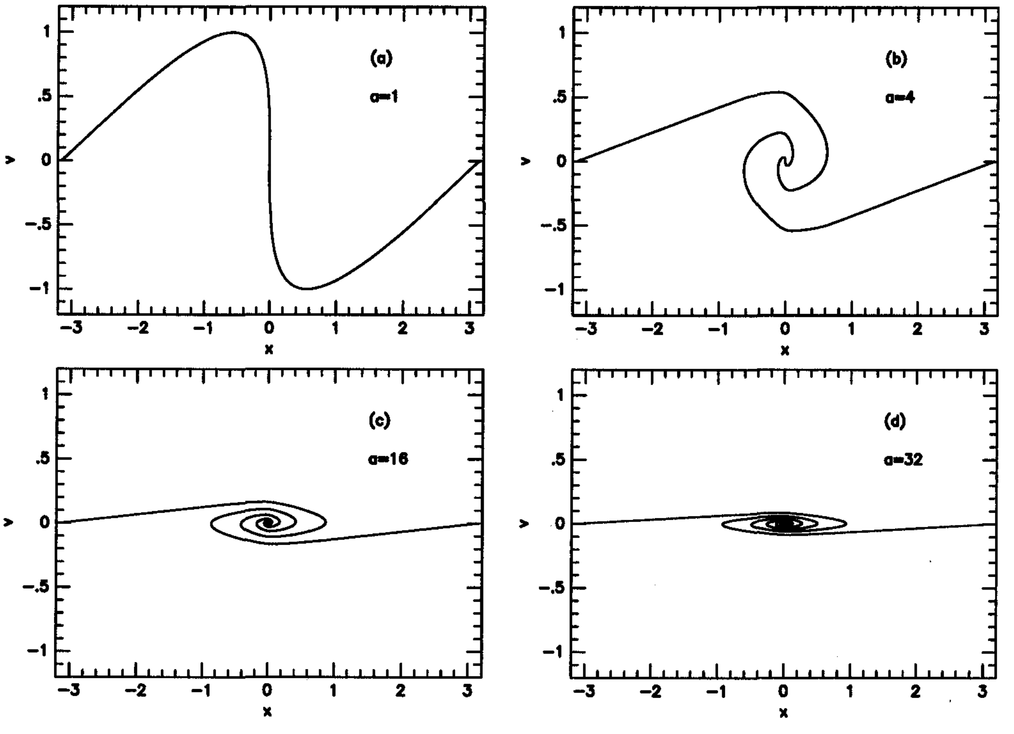

Fig.4.-(a) The distribution of plane parallel sheets in the phase space for the single-wave case at the first appearance of caustics. A horizontal axis and a vertical axis are the Eulerian coordinate x and the velocity, respectively. Scale factor a is normalized at this time ($a = 1$).

(b) The same as (a) but at $a = 4$.

(c) The same as (a) but at $a = 16$.

(d) The same as (a) but at $a = 32$.

4.2.2. Multi-caustics regime (regime 3)

In the multi-caustics regime ($k > k_{snl}$), we can see the power law spectrum. The power-law index of these regimes is different from -1 predicted by the catastrophe theory. In this regime, various small scale fluctuations ($k \gg k_{snl}$) have already collapsed and made singularities. Every singularity makes "the whirlpool" in phase space as shown in the single-wave case (Fig.4). Various size of "whirlpools" are made of the various scales of fluctuations. The distribution of the whirlpools determines the value of the power-law index. This distribution depends on the initial power-law index n. That is, the power-law index μ in this regime depends on n. Indeed, the value μ depends on the initial power spectrum as shown in Fig.2(a) and 2(b). Therefore we call this regime the multi-caustics regime. When self-similarity is satisfied, there is a relation between the power-law index of the two-point spatial correlation function, the initial power spectrum index, and the mean relative peculiar velocity. Here, we briefly show this relation in the w dimensional system. In the linear regime of the w dimensional system, the two-point correlation function is given by(Peebles 1980,1993, eq.(44)),

$$\xi \propto a^2 x^{-(w+n)} \propto (\frac{x}{a^\alpha})^{-(w+n)}, \tag{100}$$

where

$$\alpha \equiv \frac{2}{w+n}. \tag{101}$$

On the other hand, in the strongly nonlinear regime, the two-point spatial correlation function obeys the following evolution equation (0th moment equation of the 2nd BBGKY eq. (DP,YG):

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^{(w-1)}}\frac{\partial}{\partial x}[x^{(w-1)}(\xi+1)\langle v\rangle] = 0. \tag{102}$$

In the strongly non-linear limit, we can assume a power law solution for the two-point spatial correlation function (YG).

$$\xi \propto a^{\epsilon} x^{-\gamma}. \tag{103}$$

In this case, we obtain

$$\langle v \rangle = -h\dot{a}x, \qquad (\epsilon - h(w-\gamma) = 0) \tag{104}$$

and then,

$$\xi \propto a^{h(w-\gamma)} x^{-\gamma} = (\frac{x}{a^{\alpha'}})^{-\gamma}, \tag{105}$$

where h is the constant and

$$\alpha' \equiv \frac{h(w-\gamma)}{\gamma}. \tag{106}$$

As defined in §2, we call the parameter h the relative velocity parameter. If self-similarity is satisfied, α' must be equal to α. Then the power-law index γ is given by

$$\gamma = \frac{wh(n+w)}{2+h(n+w)}. \tag{107}$$

If the relative velocity parameter h is 1 (stability condition) as DP assumed, we obtain the $\gamma = (n+1)/(3+n)$ in the one-dimensional case ($w = 1$). In this case, the power-law index μ of the power spectrum is given by $\mu = \gamma - w = -2/(3+n)$. However the power-law index μ which is given by the numerical results of the power spectrum is different from $-2/(3+n)$. Therefore, the stability condition ($h = 1$) is not satisfied. We can estimate the velocity parameter h from the power-law index by using eq.(107).

$$h = \frac{2\gamma}{(n+w)(w-\gamma)} = -\frac{2(\mu+w)}{(n+w)\mu} = -\frac{2(\mu+1)}{(n+1)\mu}. \tag{108}$$

We show the results in the Table 1. YG discussed the fact that the velocity parameter h has a value between 0 and 1 (§2). Indeed the value of h stays in this range. We show the numerical result of h not only for the $n = 1$, and 2 cases but also for the $n = 0$, and 3 cases.

Table 1. n dependence of μ and h

n	μ	h
0	-0.88	0.29
1	-0.75	0.33
2	-0.65	0.36
3	-0.58	0.37

4.3. Cutoff-case

We consider the following initial spectrum, that is, the cutoff-case.

$$P(k)\begin{cases} \propto k & (k < k_{cut}) \\ = 0 & (k > k_{cut}) \end{cases} . \tag{109}$$

The time evolution of the power spectrum for the cutoff-case is shown in Fig.5.

It is shown only for the $n = 1$ case. However, we obtain essentially the same result for other cases ($n = 2, 3$, and 0). As can be seen from Fig.5, until the first appearance of caustics (lowest solid line of the power spectrum except for initial spectrum in Fig.5), the amplitude of the power spectrum grows even on scales smaller than the cutoff scale. The power-law index of these regime is predicted to be -1. This is because the wave number $k = k_{cut}$ makes the caustic and this caustic results in the predicted value of power-law index as shown in §4.2.1. Indeed, we find that the value of the index in these regimes is nearly equal to -1. The scaled power spectrum is also shown in Fig.6.

The definition of the scaling is the same as that in the scale-free case. As we see from Fig.6, the power spectra at the different times do not coincide. Therefore self-similarity is not satisfied.

The characteristic scale of separation of the caustics becomes smaller and smaller as

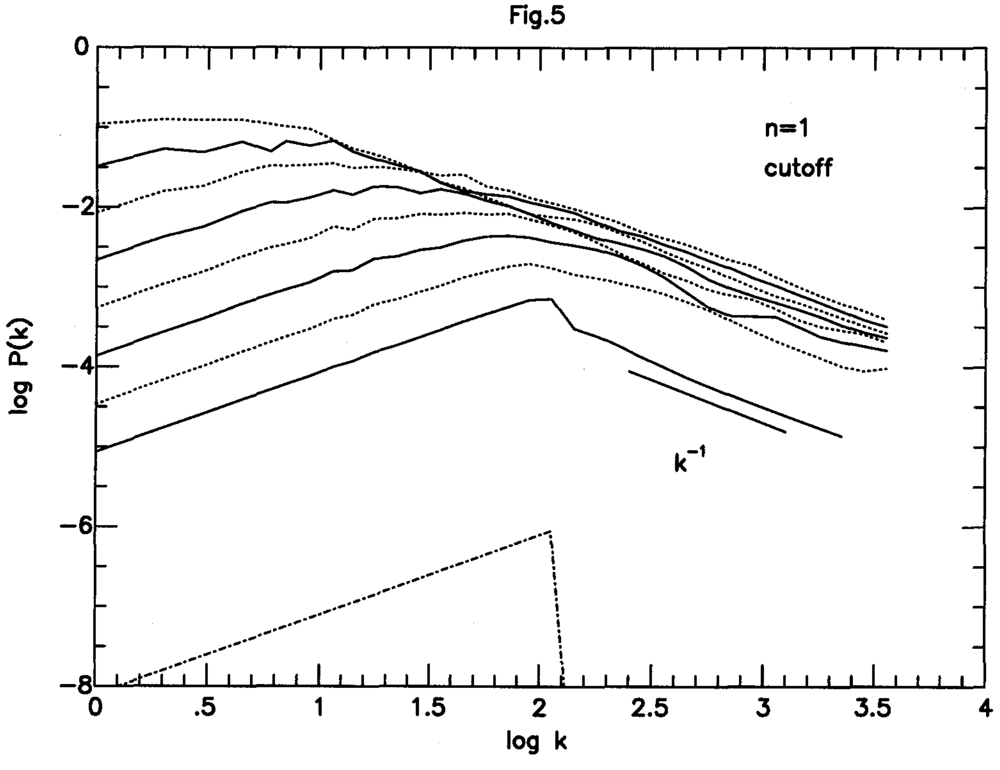

Fig.5.- The time evolution of the power spectrum. The solid and dotted lines are used mutually in order to distinguish easily each line. The curves shown are (bottom to top) at $a = 0.05, 1, 2, 4, 8, 16, 32, 64, 128$. The initial power spectrum obeys the power-law with the index $n = 1$ and it has a cutoff at $k_{cut} = 127$(Dotted-dashed line).

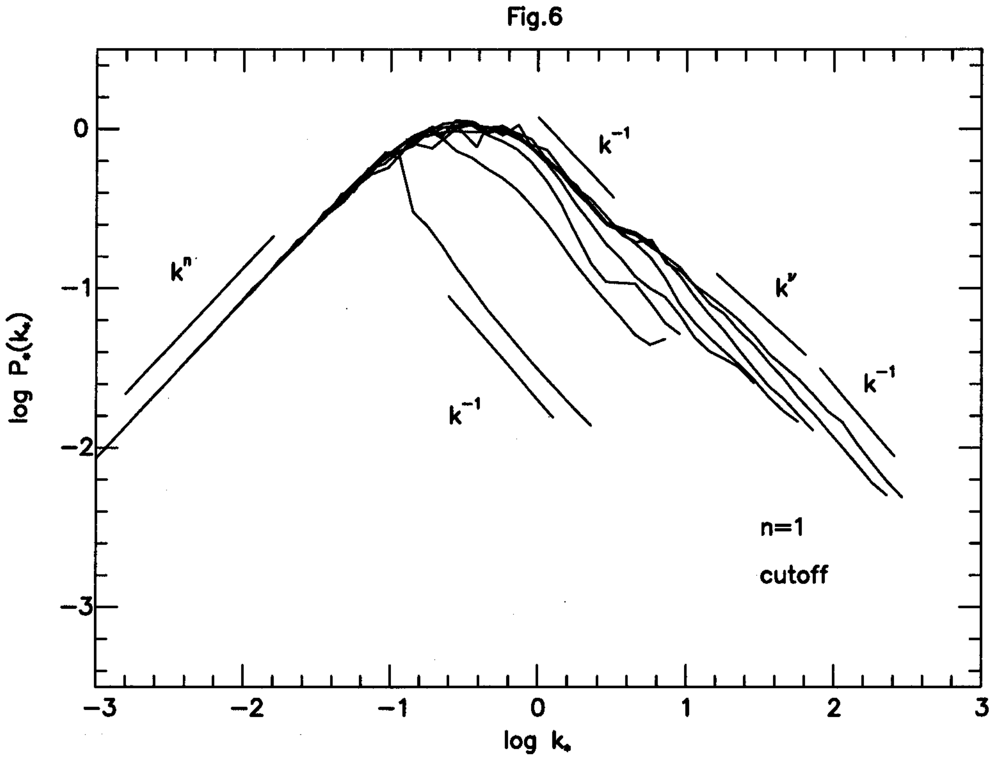

Fig.6.- The scaled power spectrum $P_*(k_*)$. The initial power spectrum obeys the power-law with the index $n = 1$ and it has a cutoff scale at $k_{cut} = 127$.

time increases (refer to the case of the single-wave: See Fig.4 (a)-4 (d)). Here, k_{cs} is defined as the wave number of the characteristic scale. On scales smaller than the characteristic scale $k > k_{cs}$(regime 5: see Fig.10), the power-law index of the power spectrum becomes -1. This is because on scales smaller than the characteristic scale, the density profile around one singular point determines the power spectrum. Therefore we call this regime the smallest single-caustic regime. On the other hand, in the regime $k_{cut} < k < k_{cs}$ (regime 4), the power-law index of the power spectrum is different from -1. The index has a certain value ν. This is because on the scales larger than the characteristic scale, smoothed density profile with the smoothing scale ($k_{cut} < k < k_{cs}$) determines the power spectrum. We call this regime the virialized regime. We show the schematic of the density distribution in Fig.7.

We notice that the wave number k_{cs} which represents the characteristic separation of caustics, becomes larger and larger as time increases. On the contrary, the non-linear scale wave number k_{nl} and k_{snl} become smaller and smaller as time increases. Thus, the power spectra of the different times cannot coincide on all scales. Therefore self-similar evolution in all the regimes including the scales smaller than the cutoff scale cannot be satisfied.

We compare the time evolutions of the power spectrum for the cutoff-case with those for the scale-free case in Fig.8.

Of course, the power spectrum for these two cases does not coincide with each other on all scales. On the other hand, on the scales larger than the cutoff scale, we can see the coincidence of power spectra for these two cases. This means that the power spectrum on the wave numbers larger than the cutoff scale k_{cut} does not affect the growth rate on wave numbers smaller than k_{cut}. We show the scaled power spectrum for the cutoff-case only on the wave numbers smaller than the cutoff scale k_{cut} in Fig.9. This means that even in the cutoff-case, the self-similar evolution can be satisfied on scales larger than the cutoff scale.

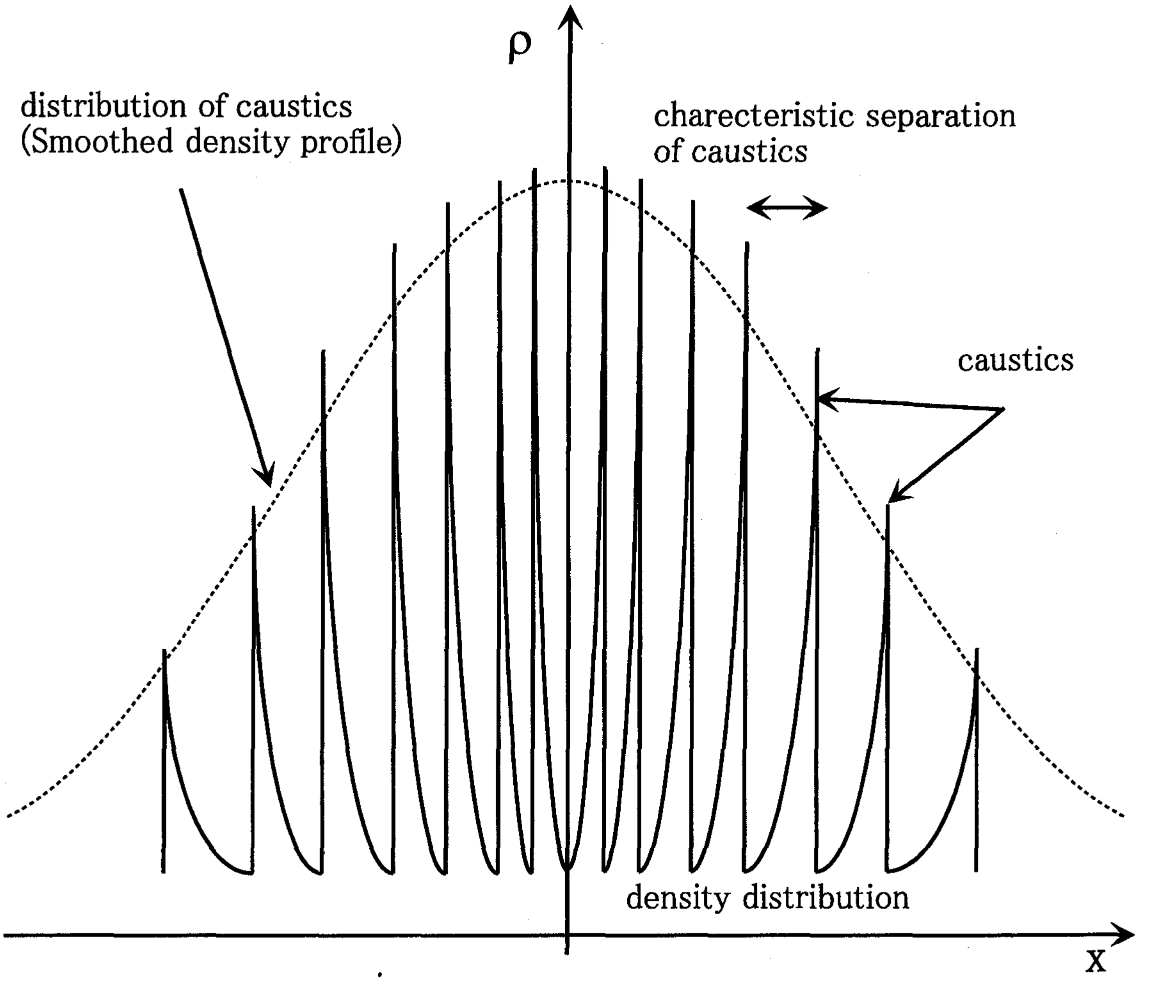

Fig.7

Fig.7.- The schematic of the density distribution in the single-wave case. This figure shows that the density profile around one caustic and the distribution of caustics, which determine the smoothed density profile on scales larger than the characteristic separation of caustics.

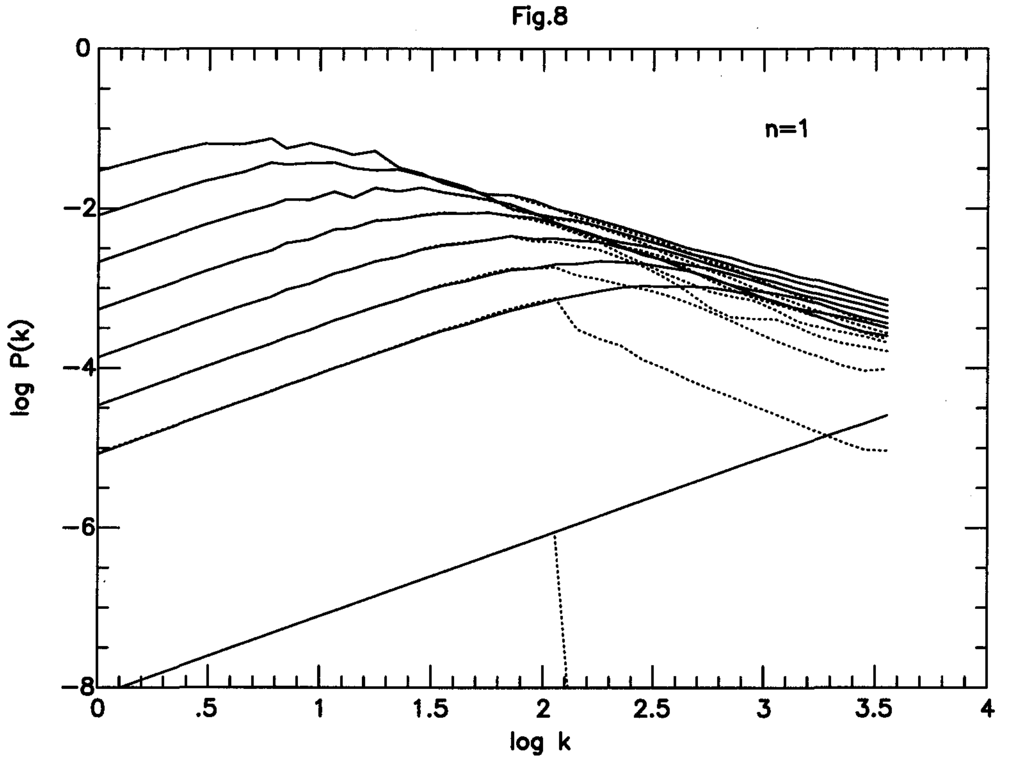

Fig.8.- Comparison of the evolution of the power spectrum between the scale-free case (solid line) and the cutoff-case (dashed line). Both the solid curves and dashed curves shown are (bottom to top) at $a = 0.05, 1, 2, 4, 8, 16, 32, 64$. The initial power spectrum obeys the power law with the power-law index $n = 1$ both for the scale-free case and for the cutoff-case. For the cutoff-case, the initial power spectrum has a cutoff at $k_{cut} = 127$.

Fig.9

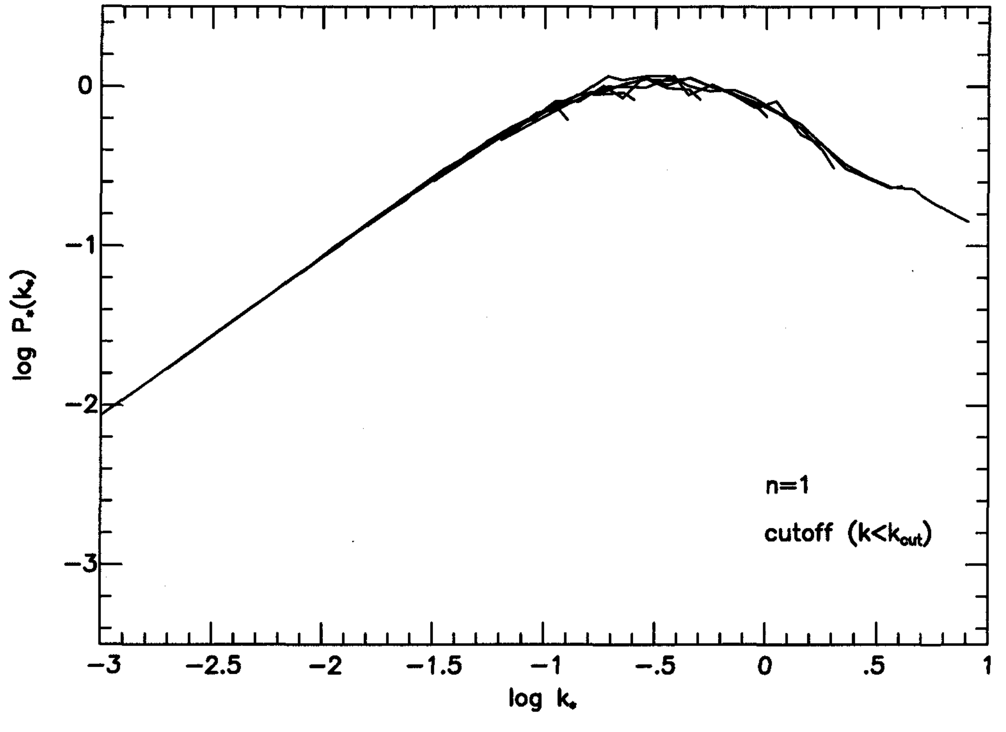

Fig.9.- The scaled power spectrum at a=1,2,4,8,16,32,64 on scales larger than the cutoff scale. The initial power spectrum obeys the power law with the power-law index n=1 and it has a cutoff scale at $k_{cut} = 127$.

4.4. Conclusions and Discussion

We have calculated the time evolution of the power spectrum for two cases of initial conditions. One is the scale-free case and the other is the cutoff case. In the case of the scale-free case, we can see self-similar evolution of the power spectrum. The scaled power spectrum $P_*(k_*)$ at each time coincides. We can roughly separate the power spectrum into three regimes. One is the linear regime ($k < k_{nl}$: regime 1). The value of the power-law index in this regime remains the initial power-law index, n. The second regime is the single-caustic regime ($k_{nl} < k < k_{snl}$: regime 2). In this regime, the power-law index becomes -1 and is independent of the initial conditions. This result is caused by the appearance of caustics at this scale, and these caustics determine the power-law index of the power spectrum in this regime. The third regime is the multi-caustics regime ($k > k_{snl}$: regime 3). The distribution of the "whirlpool" in phase space determine the value of the power-law index. Therefore in this scale the power-law index μ has the value that depends on the initial condition. We can estimate the velocity parameter h from the power-law index in the multi-caustics regime, which is around 0.5. YG discussed the probable value of h from the physical point of view and obtained that h takes a value between 0 and 1 (§2). Estimated values of h are consistent with this argument. Indeed, the index μ and the velocity parameter h depend on the initial power-law index n. The stability condition ($h = 1$) is not satisfied in this case.

In the cutoff-case, we find that there is no self-similarity on all scales. The scaled power spectrum do not coincide with each other in all regimes. However, the spectrum coincides on scales larger than the cutoff scale. After the appearance of the first caustics, the power-law index in the regime of the scale smaller than the cutoff scale becomes -1. This value is, as mentioned above, caused by the appearance of caustics as shown in Fig.3. More and more caustics appear one after another, and so the separation of caustics becomes smaller and smaller. Even after the appearance of many caustics, on scales smaller than the characteristic separation of the caustics ($k > k_{cs}$: regime 5), the power-law index of the power spectrum is obtained by the density profile around the singular point. Therefore the power-law index of these scales becomes -1, which can be derived according to the catastrophe theory. On the other hand, on scales larger than the characteristic separation of the caustics ($k_{cut} < k < k_{cs}$: regime 4), the distribution of the singular points determines the power-law index on these scales instead of the density profile around one singularity. This is because in this regime, the smoothed density profile with the smoothing scale ($k_{cut} < k < k_{cs}$) determine the power-law index. On these scales, the distribution of the singularity occurring in the evolution of the single-wave is important. Then, the power-law index on these scales is determined by this distribution of the singularity. Therefore, we can roughly separate two regimes on the scales smaller than the cutoff scale k_{cut}. One is the virialized regime ($k_{cut} < k < k_{cs}$: regime 4), and the power-law index has the value ν . Another is the smallest single-caustic regime ($k > k_{cs}$: regime 5) and the index has the value of -1. Both indexes are independent of the initial conditions.

In a real situation concerning the evolution of the power spectrum, the initial power spectrum has a cutoff on a certain scale. Therefore, in the case of a real situation, we can consider the evolution of the cutoff-case. We notice that even if the initial power spectrum does not have a cutoff, it decreases with the power-law index $n < -1$ on certain scales ($k > k_{dec}$). Therefore the time evolution of the power spectrum is the same with that in the cutoff-case after the appearance of caustics. Then, we can separate roughly five regimes. We show a schematic of the power spectrum at a time after the first appearance of caustics in Fig.10.

First regime is the linear regime ($k < k_{nl}$: regime 1). The index of the power spectrum in this regime is n. The second regime is the single-caustic regime ($k_{nl} < k_{snl}$: regime 2). The index in this regime is -1 and independent of the initial conditions. The third regime

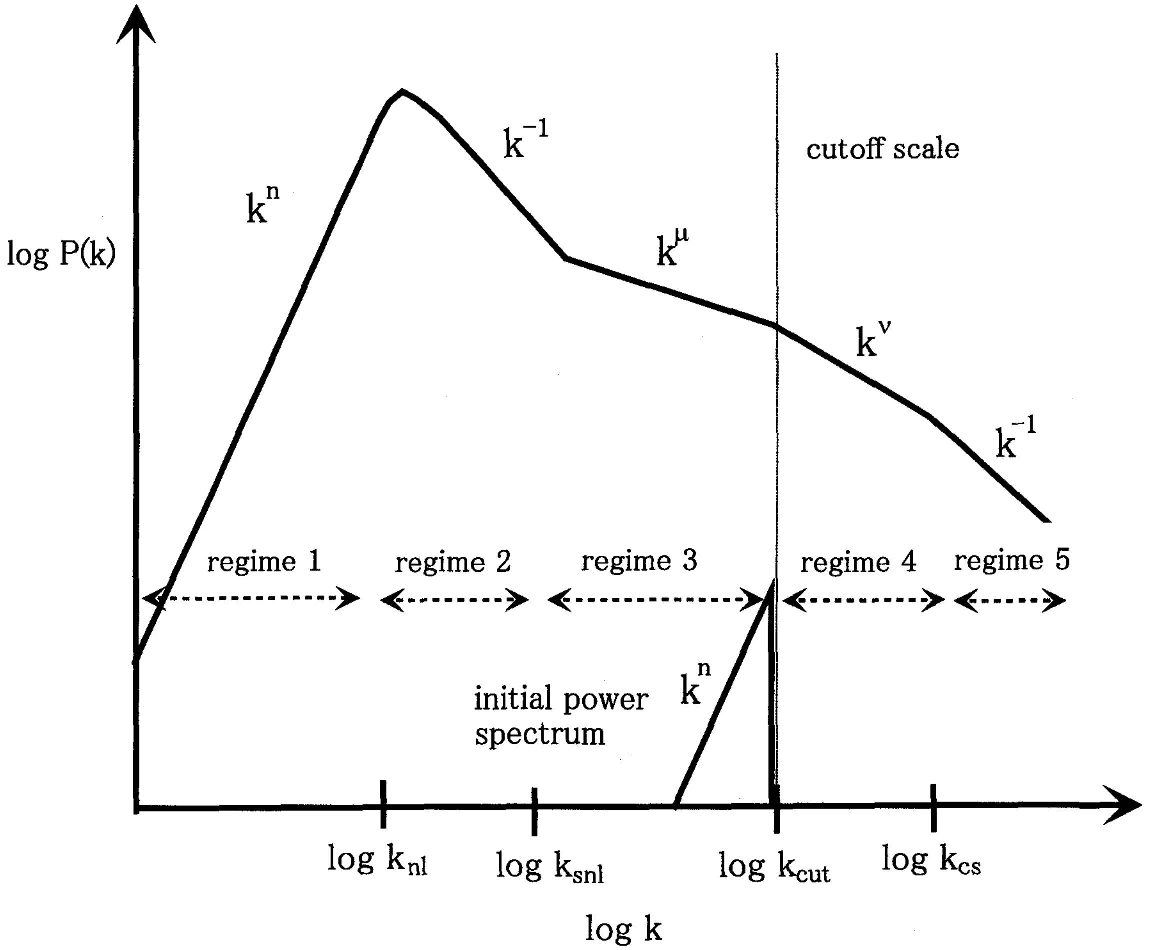

Fig. 10

Fig.10.- The schematic of the power spectrum with a general initial power spectrum at a certain time after the first appearance of caustics.

is the multi-caustics regime ($k_{snl} < k_{cut}$: regime 3). The power-law index is μ, which depends on the initial power-law index n. The fourth regime is virialized regime ($k_{cut} < k_{cs}$: regime 4). The value of the power-law index is ν, which is independent of the initial power-law index. The fifth regime is the smallest single-caustic regime ($k > k_{cs}$: regime 5). The value of the power-law index is -1. The self-similarity is satisfied only when we consider the evolution of the power spectrum on scales larger than the cutoff scale (i.e. the first, second, and third regimes).

5. CONCLUDING REMARKS

5.1. Summary

In this thesis, we have investigated the evolution of the density fluctuations in the expanding universe, especially in the non-linear regime. First, we have investigated the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime in §2. We have derived the solutions and then, we obtained the relation among the power-law index of the two-point spatial correlation function, the skewness of the velocity field, three-point correlation function, and the velocity parameter. We have also derived the index of the two-point spatial correlation function as a function of the index of the initial power spectrum n and the velocity parameter h when we assume that self-similar evolution of the two-point spatial correlation function is satisfied.

Next, it is not clear whether all the solutions derived in §2 are suitable in the real situation. We do not know whether these solutions are stable or not, and then we have investigated the stability of the scale-invariant solutions of the BBGKY equations in the non-linear regime in §3. We have found that there is no unstable mode in the case that the skewness of the velocity field is equal to zero. The solutions have proved to be marginally stable. This means that there is no special value of the index of the two-point spatial correlation function from the viewpoint of stability of the solutions.

Last, we have investigated whether the self-similarity of the two-point spatial correlation function (power spectrum) is satisfied in §4. In §4, we have considered the one-dimensional system because we can obtain a dynamic range as wide as possible. We have verified that the self-similar evolution is satisfied when the initial power spectrum is scale free. We have also investigated the evolution of the power spectrum when the initial power spectrum obeys the power-law with a cutoff. In this case, self-similarity of the power spectrum is not satisfied.

In the following, we summarize the above results again in detail.

First, in §2, we have investigated the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime. The mean relative peculiar velocity depends on the skewness and the three-point correlation function. The stability condition which DP used is satisfied for the unique case that the skewness vanishes and the three-point correlation function can be represented by the products of the two-point spatial correlation functions. In general the power-law index of the two-point spatial correlation function in the strongly nonlinear regime depends on h, the skewness and the three-point correlation function.

In general, the systems have substructures in some clusters and no subclusters in other clusters and so γ depends on n. If self-similar solutions of the two-point spatial correlation function exist, the power-law index γ is given as a function of the index of the initial power spectrum n and the velocity parameter h (eq.(49) in §2). However the relation between γ and n may not be given by the result shown by DP because the skewness does not vanish in general in the nonlinear regions and so h should not be equal to 1.

In §3, we have investigated the stability of the scale-invariant solutions of the cosmological BBGKY equations in the strongly nonlinear regime with the skewness of the velocity field being equal to zero. The reason we have considered only this case is that perturbed BBGKY equations for ξ, $\langle v \rangle$, Π and Σ can be closed independently of the higher moment perturbations. We have investigated the local stability of the non-linear regime. In investigating the local stability of the two-point spatial correlation function in the strongly non-linear regime, we should put perturbations only on these scales in which we are going to investigate. That is, we should put the wave packet-like perturbation. We have found that there is no unstable mode. It seems stable for any value of the power-law index of the two-point spatial correlation function. It is certain that there is no local instability. However we do not know whether a global instability exist or not because we consider only the local stability. In the strongly non-linear regime, the solutions have proved to be marginally stable, and it does not seem that the power-law index of the two-point spatial correlation function approaches to some stable point values. The power-law index of the two-point spatial correlation function that was derived by DP, $\gamma = 3(3+n)/(5+n)$, is not the special one also from the viewpoint of stability of the solution. As a result, the argument of the stability does not determine the power index of the two-point spatial correlation function. The power-law index of the two-point spatial correlation function is determined only by the clustering process, that is the parameter h, if self-similar solutions exist.

In §4, we have calculated the time evolution of the power spectrum for two cases of initial conditions in the one-dimensional system in order to investigate the self-similarity of the two-point spatial correlation function (power spectrum). One is the scale-free case and the other one is the cutoff-case. In the scale-free case, we can see the self-similar evolution of the power spectrum. The scaled power spectrum $P_*(k_*)$ at each time coincides with each other. We can separate roughly the power spectrum into three regimes. One is the linear regime ($k < k_{nl}$: regime 1). The value of the power-law index in this regime remains the initial power-law index, n. The second regime is the single-caustic regime ($k_{nl} < k < k_{snl}$: regime 2). In this regime, the power-law index becomes -1 and is independent of the initial conditions. This result is caused by the appearance of caustics at this scale, and these caustics determine the power-law index of the power spectrum in this regime. The third regime is the multi-caustics regime ($k > k_{snl}$: regime 3). The distribution of the "whirlpool" in phase space determines the value of the power-law index. Therefore in this scale the power-law index μ has the value which depends on the initial condition. We can estimate the velocity parameter h from the power-law index in the multi-caustics regime, which is around 0.5. As we discussed in §2, the velocity parameter h takes a value between 0 and 1. Estimated values of h in the one-dimensional systems are consistent with this argument. Indeed, the index μ and the velocity parameter h depend on the initial power-law index n. The stability condition $(h = 1)$ is not satisfied in this case.

In the cutoff-case, we find that there is no self-similarity on all scales. The scaled power spectrum does not coincide with each other in all regimes. However, the spectrum coincides on the scales larger than the cutoff scale. After the appearance of the first caustics, the power-law index in the regime of the scale smaller than the cutoff scale becomes -1. This value is, as mentioned above, caused by the appearance of caustics as shown in Fig.3. More and more caustics appear one after another and so the separation of caustics becomes smaller and smaller. Even after the appearance of many caustics, on scales smaller than the characteristic separation of the caustics ($k > k_{cs}$: regime 5), the power-law index of the power spectrum is obtained by the density profile around the singular point. Therefore the power-law index of these scales becomes -1 which can be derived according to the catastrophe theory. On the other hand, on the scales larger than the characteristic separation of the caustics ($k_{cut} < k < k_{cs}$: regime 4), the distribution of the singular points determines the power-law index on these scales instead of the density profile around one singularity. Because in this regime, the smoothed density profile with the smoothing scale ($k_{cut} < k < k_{cs}$) determines the power-law index. On these scales, the distribution of the singularity occurring in the evolution of the single-wave is important. The power-law index on these scales is determined by this distribution of the singularity. Therefore, we can roughly separate two regimes on the scales smaller than the cutoff scale k_{cut}. One is the virialized regime ($k_{cut} < k < k_{cs}$: regime 4), and the power-law index has the value ν . Another is the smallest single-caustic regime ($k > k_{cs}$: regime 5) and the index have the value of -1. Both indexes are independent of the initial conditions.

In the above argument, we can make it clear the difference between the power-law index that is produced by DP or derived by BBGKY equations in §2 and the index which is determined by the type of the singularity after the appearance of the caustics. The index which is proposed by DP or derived by BBGKY equation corresponds to the regime 3, that is, the self-similar evolution is satisfied. On the other hand, the index which is determined by the type of the singularity corresponds to the regime 5. This regime moves to the smaller scale because the more and more caustics appear and the characteristic scale of the separation of caustics becomes smaller and smaller. Thus, the self-similar evolution cannot be satisfied in the cutoff case.

In a real situation concerning the evolution of the power spectrum, the initial power spectrum has a cutoff at a certain scale. We notice that even if the initial power spectrum does not have a cutoff, there is a scales k ($k > k_{dec}$) in which the power-law index of the power spectrum with $n < -1$. Furthermore caustics appear in the scale k_{dec} at first. Therefore the time evolution of the power spectrum is the same with the one in the cutoff-case after the appearance of caustics. And then, we can separate roughly five regimes. We show a schematic of the power spectrum at a time after the first appearance of caustics in Fig.10. First regime is the linear regime ($k < k_{nl}$: regime 1). The index of the power spectrum in this regime is n. The second regime is the single-caustic regime ($k_{nl} < k_{snl}$: regime 2). The index in this regime is -1, and independent of the initial conditions. The third regime is the multi-caustics regime ($k_{snl} < k_{cut}$: regime 3). The power-law index is μ which depends on the initial power-law index n. The fourth regime is virialized regime ($k_{cut} < k_{cs}$: regime 4). The value of the power-law index is ν which is independent of the initial power-law index. The fifth regime is the smallest single-caustic regime ($k > k_{cs}$: regime 5). The value of the power-law index is -1. Only when we consider the evolution of the power spectrum on scales larger than the cutoff scale (i.e. the first, second, and third regime), the self-similarity is satisfied.

In §4, we investigated the evolution of the two-point spatial correlation function (power spectrum) in the one dimensional system. We believe, however, that the same physical process are realized even in the three-dimensional systems although the quantitative values of the index of the power spectrum in the three-dimensional systems are different from one in the one-dimensional systems.

5.2. Future Works

We have investigated the evolution of the density fluctuations in the expanding universe, especially in the non-linear regime in this thesis, and have derived the scale-invariant solutions. We have investigated the stability of the solutions of the BBGKY equations in the non-linear regime in §3, and have found that there is no unstable mode when the skewness of the velocity field is equal to zero. We would like to investigate the stability of the solutions in general case in the future. In §4, we have investigated the evolution of the power spectrum in an one-dimensional system. We have found that the power spectrum can be separated roughly into five regimes according to the shape of the power spectrum when the initial power spectrum obeys the power-law with a cutoff. We have shown the values of the power-law index of the power spectrum in the regimes 1, 2, and 5. Each index of the power spectrum is n, -1, and -1, respectively. In the near future, we would like to investigate how to determine the values of the power-law index in the regimes 3 and 4. Furthermore, we would like to verify the self-similarity of the two-point spatial correlation function (power spectrum) in three-dimensional systems.

APPENDIX

A. BBGKY EQUATIONS

In Appendix A, we show the detail of the derivation of the BBGKY equations that we used in this thesis.

Derivation of the BBGKY equation The BBGKY equations can be derived by the ensemble mean of the Vlasov equation. The Vlasov equation for the one-body distribution function $f(\boldsymbol{x}, \boldsymbol{p})$ is given by

$$\frac{\partial f}{\partial t} + \frac{p^\alpha}{ma^2}\frac{\partial f}{\partial x^\alpha} - m\frac{\partial \phi}{\partial x^\alpha}\frac{\partial f}{\partial p^\alpha} = 0, \tag{A1}$$

$$\phi(\boldsymbol{x}_1) = \frac{Gm}{a}\int \frac{f(\boldsymbol{x}_2, \boldsymbol{p}_2)}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|}d^3x_2 d^3p_2, \tag{A2}$$

where m is the mass of a particle, a is the scale factor and G is the gravitational constant. The ensemble mean of the Vlasov equation is

$$\frac{\partial \langle f\rangle}{\partial t} + \frac{p_1^\alpha}{ma^2}\frac{\partial \langle f\rangle}{\partial x_1^\alpha} - m\langle\frac{\partial \phi}{\partial x_1^\alpha}\frac{\partial f}{\partial p_1^\alpha}\rangle = 0. \tag{A3}$$

We define the statistical functions which are given by the ensemble mean as follows:

$$\begin{aligned} b(1) &= \langle f(\boldsymbol{x}_1, \boldsymbol{p}_1)\rangle, && \text{(A4)}\\ \rho_2(1,2) &= \langle f(\boldsymbol{x}_1, \boldsymbol{p}_1)f(\boldsymbol{x}_2, \boldsymbol{p}_2)\rangle, && \text{(A5)}\\ \rho_3(1,2,3) &= \langle f(\boldsymbol{x}_1, \boldsymbol{p}_1)f(\boldsymbol{x}_2, \boldsymbol{p}_2)f(\boldsymbol{x}_3, \boldsymbol{p}_3)\rangle, && \text{(A6)}\\ &\vdots \end{aligned}$$

where $b(1)$ is a function of only the momentum because of the homogeneity in the background universe. The ensemble mean of the Vlasov equation (A3) is rewritten by using above functions(eqs.[A4] $\sim$ [A5]) as follows:

$$\frac{\partial b}{\partial t}+\frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int\frac{x_2^\alpha-x_1^\alpha}{|x_2-x_1|^3}\rho_2(1,2)d^3x_2d^3p_2=0 \tag{A7}$$

This is the first BBGKY equation. As can be seen from eq.(9), the time evolution of the one-body distribution function depends on the two-body correlation function.

The following is the second BBGKY equation for the two-body correlation function:

$$\begin{aligned}\frac{\partial\rho_2(1,2)}{\partial t} &= \frac{\partial}{\partial t}\langle f(\boldsymbol{x}_1,\boldsymbol{p}_1)f(\boldsymbol{x}_2,\boldsymbol{p}_2)\rangle \\ &= -\frac{p_1^\alpha}{ma^2}\frac{\partial\rho_2(1,2)}{\partial x_1^\alpha}-\frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int\frac{x_{31}^\alpha}{x_{31}^3}\rho_3(1,2,3)d^3x_3d^3p_3+(1\leftrightarrow 2),\end{aligned} \tag{A8}$$

$$\frac{x_{31}^\alpha}{x_{31}^3}\equiv\frac{x_3^\alpha-x_1^\alpha}{|\boldsymbol{x}_3-\boldsymbol{x}_1|^3}. \tag{A9}$$

We can obtain the N-body correlation function by the same way as follows:

$$\begin{aligned}\frac{\partial\rho_3(1,2,3)}{\partial t} &= \frac{\partial}{\partial t}\langle f(\boldsymbol{x}_1,\boldsymbol{p}_1)f(\boldsymbol{x}_2,\boldsymbol{p}_2)f(\boldsymbol{x}_3,\boldsymbol{p}_3)\rangle \\ &= \langle\frac{\partial f(1)}{\partial t}f(2)f(3)\rangle+\langle f(1)\frac{\partial f(2)}{\partial t}f(3)\rangle+\langle f(1)f(2)\frac{\partial f(3)}{\partial t}\rangle \\ &= \cdots\end{aligned} \tag{A10}$$

$$\begin{aligned}\frac{\partial\rho_4(1,2,3,4)}{\partial t} &= \frac{\partial}{\partial t}\langle f(\boldsymbol{x}_1,\boldsymbol{p}_1)f(\boldsymbol{x}_2,\boldsymbol{p}_2)f(\boldsymbol{x}_3,\boldsymbol{p}_3)f(\boldsymbol{x}_4,\boldsymbol{p}_4)\rangle \\ &= \langle\frac{\partial f(1)}{\partial t}f(2)f(3)f(4)\rangle+\langle f(1)\frac{\partial f(2)}{\partial t}f(3)f(4)\rangle \\ &\quad+\langle f(1)f(2)\frac{\partial f(3)}{\partial t}f(4)\rangle+\langle f(1)f(2)f(3)\frac{\partial f(4)}{\partial t}\rangle \\ &= \cdots\end{aligned} \tag{A11}$$

In our analysis, however, we use the only second BBGKY equation.

The second BBGKY equation is the time evolution equation of the two-body correlation function. In this equation, the three-body correlation function is involved. In general, the time evolution of the N-body correlation function depends on the $N+1$-body correlation function.

We define the following irreducible correlation functions, c and d:

$$\rho_2(1,2) \equiv b(1)b(2) + c(1,2), \tag{A12}$$

$$\rho_3(1,2,3) \equiv b(1)b(2)b(3) + b(1)c(2,3) + b(2)c(3,1) + b(3)c(1,2) + d(1,2,3). \tag{A13}$$

Here b, c and d mean

$$b(i) = b(\boldsymbol{p}_i), \tag{A14}$$

$$c(i,j) = c(\boldsymbol{x}_i, \boldsymbol{p}_i, \boldsymbol{x}_j, \boldsymbol{p}_j), \tag{A15}$$

$$d(i,j,k) = d(\boldsymbol{x}_i, \boldsymbol{p}_i, \boldsymbol{x}_j, \boldsymbol{p}_j, \boldsymbol{x}_k, \boldsymbol{p}_k). \tag{A16}$$

The first and the second BBGKY equations are rewritten by using the above functions, respectively, as follows:

$$\frac{\partial b(1)}{\partial t} + \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int \frac{x_{21}^\alpha}{x_{21}^3} c(1,2) d^3x_2 d^3p_2 = 0 \qquad \text{(first BBGKY)} \tag{A17}$$

$$\begin{aligned}\frac{\partial c(1,2)}{\partial t} &+ \frac{p_1^\alpha}{ma^2}\frac{\partial c(1,2)}{\partial x_1^\alpha} + \frac{Gm^2}{a}\frac{\partial b(1)}{\partial p_1^\alpha}\int \frac{x_{31}^\alpha}{x_{31}^3} c(2,3) d^3x_3 d^3p_3 \\ &+ \frac{Gm^2}{a}\frac{\partial}{\partial p_1^\alpha}\int \frac{x_{31}^\alpha}{x_{31}^3} d(1,2,3) d^3x_3 d^3p_3 + (1 \leftrightarrow 2) = 0 \quad \text{(second BBGKY)}\end{aligned} \tag{A18}$$

Velocity Moment In this thesis, we use the second BBGKY equation. Then we show the derivation of the velocity moment equation of the second BBGKY equation that are given by multiplying the equation by a power of moment and integrate them over all moment arguments.

The zeroth moment equation is given by

$$\frac{\partial}{\partial t}\int c(1,2) d^3p_1 d^3p_2 + \frac{\partial}{\partial x^\alpha}\int \frac{1}{ma^2} p_{21}^\alpha c(1,2) d^3p_1 d^3p_2 = 0, \quad \text{(0th moment)} \tag{A19}$$

$$x^\alpha \equiv x_{21}^\alpha \equiv x_2^\alpha - x_1^\alpha, \qquad p_{21}^\alpha \equiv p_2^\alpha - p_1^\alpha. \tag{A20}$$

As we can see, the time evolution of the zeroth moment depends on the first moment in the BBGKY equation.

The first moment equation is given by

$$\begin{aligned}\frac{\partial}{\partial t}\int c(1,2)p_{21}^{\beta}d^3p_1d^3p_2 &+ \frac{\partial}{\partial x^{\alpha}}\int \frac{1}{ma^2}p_{21}^{\alpha}p_{21}^{\beta}c(1,2)d^3p_1d^3p_2 \\ &+ \int \frac{Gm^2}{a}\{b(1)c(2,3)+d\}\frac{x_{31}^{\beta}}{x_{31}^3}d^3x_3d^3p_3d^3p_1d^3p_2 \\ &- \int \frac{Gm^2}{a}\{b(2)c(3,1)+d\}\frac{x_{32}^{\beta}}{x_{32}^3}d^3x_3d^3p_3d^3p_1d^3p_2 = 0. \\ &\qquad\qquad\qquad\qquad\text{(1st moment)} \qquad\qquad \text{(A21)}\end{aligned}$$

This equation includes the second moment and then we need the second moment equation as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\int c(1,2)p_{21}^{\beta}p_{21}^{\gamma}d^3p_1d^3p_2 &+ \frac{\partial}{\partial x^{\alpha}}\int \frac{1}{ma^2}p_{21}^{\alpha}p_{21}^{\beta}p_{21}^{\gamma}c(1,2)d^3p_1d^3p_2 \\ &+ \int \frac{Gm^2}{a}\{b(1)c(2,3)+d\}\{p_{21}^{\gamma}\frac{x_{31}^{\beta}}{x_{31}^3} - p_{21}^{\beta}\frac{x_{31}^{\gamma}}{x_{31}^3}\}d^3x_3d^3p_3d^3p_1d^3p_2 \\ &- \int \frac{Gm^2}{a}\{b(2)c(3,1)+d\}\{p_{21}^{\gamma}\frac{x_{32}^{\beta}}{x_{32}^3} - p_{21}^{\beta}\frac{x_{32}^{\gamma}}{x_{32}^3}\}d^3x_3d^3p_3d^3p_1d^3p_2 = 0. \\ &\qquad\qquad\qquad\qquad\text{(2nd moment)} \qquad\qquad \text{(A22)}\end{aligned}$$

As we can see above, the time evolution of the N-th moment depends on the $N+1$-th moment. So we should take assumptions in order to close these equations. DP used the assumption that the skewness of the velocity is equal to 0. In this thesis, we do not assume about the skewness of the velocity fields in order to study the relation between the skewness and the two-point correlation function.

We defined in §2 the two-point correlation function ξ, the mean relative peculiar velocity $\langle v^{\alpha}\rangle$, the relative peculiar velocity dispersion Π, Σ and the mean third moment $\langle v^{\alpha}v^{\beta}v^{\gamma}\rangle$ as follows:

$$\bar{n}^2a^6\xi \equiv \int c(2,1)d^3p_1d^3p_2, \qquad \text{(A23)}$$

$$\begin{aligned}
\bar{n}^2a^6(1+\xi)ma\langle v^\alpha\rangle &\equiv \int c(2,1)p_{21}^\alpha d^3p_1d^3p_2, &\text{(A24)}\\
\bar{n}^2a^6(1+\xi)(ma)^2[\Pi P_\parallel^{\alpha\beta}+\Sigma P_\perp^{\alpha\beta}] &\equiv \int c(2,1)p_{21}^\alpha p_{21}^\beta d^3p_1d^3p_2, &\text{(A25)}\\
\langle v^\alpha v^\beta v^\gamma\rangle &\equiv \frac{\frac{1}{(ma)^3}\int \rho_2 p_{21}^\alpha p_{21}^\beta p_{21}^\gamma d^3p_1d^3p_2}{\int \rho_2 d^3p_1d^3p_2}\\
&= \frac{\int c(2,1)p_{21}^\alpha p_{21}^\beta p_{21}^\gamma d^3p_1d^3p_2}{(ma)^3\bar{n}^2a^6(1+\xi)}, &\text{(A26)}
\end{aligned}$$

where, $\bar{n}$ is the mean density of the universe, and

$$P_\parallel^{\alpha\beta}=\frac{x^\alpha x^\beta}{x^2},\qquad P_\perp^{\alpha\beta}=\delta^{\alpha\beta}-\frac{x^\alpha x^\beta}{x^2}. \tag{A27}$$

The ξ which is defined in eq.(A23) is equivalent to the two-point correlation function $\xi_2(x)$ which is defined by another familiar definition as follows:

$$\begin{aligned}
\xi_2(x) &\equiv \langle\delta(\boldsymbol{r})\delta(\boldsymbol{r}+\boldsymbol{x})\rangle_{\boldsymbol{r},\,|\boldsymbol{x}|=x}\\
&= \frac{1}{\langle\rho\rangle^2}\langle(\rho(\boldsymbol{r})-\langle\rho\rangle)(\rho(\boldsymbol{r}+\boldsymbol{x})-\langle\rho\rangle)\rangle_{\boldsymbol{r},\,|\boldsymbol{x}|=x}\\
&= \frac{1}{\bar{n}^2a^6}[\langle\rho(\boldsymbol{r})\rho(\boldsymbol{r}+\boldsymbol{x})\rangle-\langle\rho\rangle\langle\rho\rangle]\\
&= \frac{1}{\bar{n}^2a^6}\int(\rho_2(1,2)-b(1)b(2))d^3p_1d^3p_2\\
&= \frac{1}{\bar{n}^2a^6}\int c(1,2)d^3p_1d^3p_2\\
&= \xi(x_{21}). &\text{(A28)}
\end{aligned}$$

where $\langle\rho\rangle$ is the mean density of the universe.

In eq.(A24), $\langle v^\alpha\rangle$ is the mean relative peculiar velocity given by

$$\begin{aligned}
\langle v^\alpha\rangle &\equiv \frac{\int \rho_2(2,1)\frac{1}{ma}p_{21}^\alpha d^3p_1d^3p_2}{\int \rho_2(2,1)d^3p_1d^3p_2}\\
&= \frac{\int c(2,1)p_{21}^\alpha d^3p_1d^3p_2}{\bar{n}^2a^6(ma)(1+\xi)}. &\text{(A29)}
\end{aligned}$$

In eq.(A25), Π and Σ are the parallel and transverse correlated parts of the relative peculiar velocity dispersion of correlated particles, respectively.

As shown in §2, the skewness is written as follows:

$$s^{\alpha\beta\gamma} \equiv \langle(v - \langle v\rangle)^{\alpha}\rangle\langle(v - \langle v\rangle)^{\beta}\rangle\langle(v - \langle v\rangle)^{\gamma}\rangle. \tag{A30}$$

From the symmetry of the universe (homogeneity and isotropy), we can write $\langle v^{\alpha}\rangle = \langle v\rangle x^{\alpha}/x$. Hence,

$$\langle v^{\alpha}v^{\beta}v^{\gamma}\rangle = -2\langle v\rangle^3\frac{x^{\alpha}x^{\beta}x^{\gamma}}{x^3} + \langle v\rangle\left\{\frac{x^{\alpha}}{x}\langle v^{\beta}v^{\gamma}\rangle + \frac{x^{\beta}}{x}\langle v^{\gamma}v^{\alpha}\rangle + \frac{x^{\gamma}}{x}\langle v^{\alpha}v^{\beta}\rangle\right\} + s^{\alpha\beta\gamma}. \tag{A31}$$

Furthermore the skewness can be written by the symmetry of the universe as follows:

$$s^{\alpha\beta\gamma} = s_{\parallel}\ P^{\alpha\beta\gamma}_{ppp} + s_{\perp}\ P^{\alpha\beta\gamma}_{ptt}, \tag{A32}$$

where the subscripts p and t represent the parallel and transverse component of the two particles, respectively. We have

$$P^{\alpha\beta\gamma}_{ppp} = \frac{x^{\alpha}x^{\beta}x^{\gamma}}{x^3}, \qquad P^{\alpha\beta\gamma}_{ptt} = \frac{x^{\alpha}}{x}\delta^{\beta\gamma} + \frac{x^{\beta}}{x}\delta^{\gamma\alpha} + \frac{x^{\gamma}}{x}\delta^{\alpha\beta} - 3\frac{x^{\alpha}x^{\beta}x^{\gamma}}{x^3}. \tag{A33}$$

P_{ppt} and P_{ttt} vanish because of the symmetry of the universe.

From eqs.(A26) and (A31), we can obtain the following equations:

$$\begin{aligned}\int c(2,1)p^{\alpha}_{21}p^{\beta}_{21}p^{\gamma}_{21}d^3p_1d^3p_2 = \bar{n}^2a^6(1+\xi)(ma)^3\{[3\langle v\rangle(\Pi - \Sigma) - 2\langle v\rangle^3]\frac{x^{\alpha}x^{\beta}x^{\gamma}}{x^3} \\ +\langle v\rangle(\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)})(\frac{x^{\alpha}}{x}\delta^{\beta\gamma} + \frac{x^{\beta}}{x}\delta^{\gamma\alpha} + \frac{x^{\gamma}}{x}\delta^{\alpha\beta}) + s^{\alpha\beta\gamma}.\}\end{aligned} \tag{A34}$$

Then moment equations are following:

$$\bar{n}^2a^6\frac{\partial\xi}{\partial t} + \frac{\bar{n}^2a^6(ma)}{ma^2}\frac{\partial}{\partial x^{\alpha}}[(1+\xi)\langle v^{\alpha}\rangle] = 0, \qquad \text{(0th moment)} \tag{A35}$$

$$\bar{n}^2a^6\frac{\partial}{\partial t}[(1+\xi)ma\langle v^\beta\rangle] + \frac{\bar{n}^2a^6(ma)^2}{ma^2}\frac{\partial}{\partial x^\alpha}\left[(1+\xi)[\Pi P_{\parallel}^{\alpha\beta}+\Sigma P_{\perp}^{\alpha\beta}]\right]$$
$$+ \frac{Gm^2}{a}\bar{n}a^3\int c(2,3)\frac{x_{31}^\beta}{x_{31}^3}d^3x_3d^3p_3d^3p_2$$
$$- \frac{Gm^2}{a}\bar{n}a^3\int c(3,1)\frac{x_{32}^\beta}{x_{32}^3}d^3x_3d^3p_3d^3p_1$$
$$+ \frac{Gm^2}{a}\bar{n}^3a^9\int \zeta(\frac{x_{31}^\beta}{x_{31}^3}-\frac{x_{32}^\beta}{x_{32}^3})d^3x_3 = 0, \quad \text{(1st moment)} \qquad \text{(A36)}$$

$$\bar{n}^2a^6\frac{\partial}{\partial t}\left[(ma)^2(1+\xi)[\Pi P_{\parallel}^{\alpha\beta}+\Sigma P_{\perp}^{\alpha\beta}]\right]$$
$$+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{\partial}{\partial x^\alpha}(1+\xi)[3\langle v\rangle(\Pi-\Sigma)-2\langle v\rangle^3]\frac{x^\alpha x^\beta x^\gamma}{x^3}$$
$$+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{\partial}{\partial x^\alpha}(1+\xi)\langle v\rangle(\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)})(\frac{x^\alpha}{x}\delta^{\beta\gamma}+\frac{x^\beta}{x}\delta^{\gamma\alpha}+\frac{x^\gamma}{x}\delta^{\alpha\beta})$$
$$+\frac{\bar{n}^2a^6(ma)^3}{ma^2}\frac{\partial}{\partial x^\alpha}s^{\alpha\beta\gamma}$$
$$+\frac{Gm^2}{a}\bar{n}a^3\int c(2,3)[p_{21}^\gamma\frac{x_{31}^\beta}{x_{31}^3}+p_{21}^\beta\frac{x_{31}^\gamma}{x_{31}^3}]d^3x_3d^3p_3d^3p_2$$
$$-\frac{Gm^2}{a}\bar{n}a^3\int c(3,1)[p_{21}^\gamma\frac{x_{32}^\beta}{x_{32}^3}+p_{21}^\beta\frac{x_{32}^\gamma}{x_{32}^3}]d^3x_3d^3p_3d^3p_1$$
$$+\frac{Gm^2}{a}\bar{n}a^3\int d[p_{21}^\gamma(\frac{x_{31}^\beta}{x_{31}^3}-\frac{x_{32}^\beta}{x_{32}^3})+p_{21}^\beta(\frac{x_{31}^\gamma}{x_{31}^3}-\frac{x_{32}^\gamma}{x_{32}^3})]d^3x_3d^3p_3d^3p_1d^3p_2 = 0.$$
$$\text{(2nd moment)} \qquad \text{(A37)}$$

As shown in §2, ζ is the three-point correlation function defined by

$$\bar{n}^3a^9\zeta \equiv \int d d^3p_1d^3p_2d^3p_3. \qquad \text{(A38)}$$

Contraction of the Equation We transform these equations by taking divergence of the first moment equation and by operating the following two operators to the second moment equation:

$$\frac{\partial}{\partial x^\beta}\frac{\partial}{\partial x^\gamma}, \qquad \Delta^{\beta\gamma} = \frac{1}{2}(\delta^{\beta\gamma}-\frac{x^\beta x^\gamma}{x^2}). \qquad \text{(A39)}$$

Hence we obtain two equations from the second moment equation. We call them contraction 1and contraction 2 equations hereafter, shown later in eqs.(A48) and (A49) , respectively. Furthermore, we assume the following relation,

$$\zeta_{123} = Q(\xi_{12}\xi_{23} + \xi_{23}\xi_{31} + \xi_{31}\xi_{12}). \tag{A40}$$

Then the fifth term of the first moment equation (A36) is rewritten by

$$\begin{aligned}
\frac{Gm^2}{a}\bar{n}^3a^9\frac{\partial}{\partial x^\beta}\int \zeta(\frac{x_{31}^\beta}{x_{31}^3} - \frac{x_{32}^\beta}{x_{32}^3})d^3x_3 &= 2\frac{Gm^2}{a}\bar{n}^3a^9\frac{\partial}{\partial x^\beta}\int \zeta(\frac{x_{31}^\beta}{x_{31}^3})d^3x_3 \\
&= 2\frac{Gm^2}{a}\bar{n}^3a^9Q\frac{\partial}{\partial x^\beta}\int \frac{x_{31}^\beta}{x_{31}^3}d^3x_3(\xi_{12}\xi_{23} + \xi_{23}\xi_{31} + \xi_{31}\xi_{12}) \\
&= 2\frac{Gm^2}{a}\bar{n}^3a^9Q\frac{\partial}{\partial x^\beta}\int \frac{x_{31}^\beta}{x_{31}^3}d^3x_3(\xi_{12} + \xi_{31})\xi_{23} \\
&\equiv 2\frac{Gm^2}{a}\bar{n}^3a^9Q\frac{\partial}{\partial x^\beta}[x^\beta T] \\
&= 2\frac{Gm^2}{a}\bar{n}^3a^9Q[\frac{\partial T}{\partial x}x + T] \\
&= 2\frac{Gm^2}{a}\bar{n}^3a^9Q\frac{1}{x^2}\frac{\partial}{\partial x}[x^3T] \\
&= 2\frac{Gm^2}{a}\bar{n}^3a^9QM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2],
\end{aligned} \tag{A41}$$

where

$$x^\beta T \equiv x^\beta \xi(x)^2 M \equiv \int \frac{x_{31}^\beta}{x_{31}^3}\{\xi_{12} + \xi_{31}\}\xi_{23}d^3x_3, \qquad x \equiv x_{21}. \tag{A42}$$

This definition is well defined for the vector component of $\boldsymbol{x}$ and $\boldsymbol{x}_{31}$ because of the symmetry of the universe.

The seventh term in eq.(A37) is rewritten by

$$\begin{aligned}
&\frac{Gm^2}{a}\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int d[p_{21}^\gamma(\frac{x_{31}^\beta}{x_{31}^3} - \frac{x_{32}^\beta}{x_{32}^3}) + p_{21}^\beta(\frac{x_{31}^\gamma}{x_{31}^3} - \frac{x_{32}^\gamma}{x_{32}^3})]d^3x_3d^3p_3d^3p_1d^3p_2 \\
&= 4\frac{Gm^2}{a}\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int dp_{21}^\gamma\frac{x_{31}^\beta}{x_{31}^3}d^3x_3d^3p_3d^3p_1d^3p_2 \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}[\langle v_{21}^\gamma\rangle\xi_{12}\xi_{23} + (\langle v_{23}^\gamma\rangle + \langle v_{31}^\gamma\rangle)\xi_{23}\xi_{31} + \langle v_{21}^\gamma\rangle\xi_{31}\xi_{12}]d^3x_3 \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}[x^\gamma\frac{\langle v_{21}\rangle}{x}\xi_{12}\xi_{23} + \{(x^\gamma - z^\gamma)\frac{\langle v_{23}\rangle}{|x-z|} + z^\gamma\frac{\langle v_{31}\rangle}{z}\}\xi_{23}\xi_{31}]d^3x_3
\end{aligned}$$

$$= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}d^3x_3$$
$$+4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3 \qquad \text{(A43)}$$

where

$$\frac{Q^*}{Q}\bar{n}^3a^9\zeta ma\langle v_{21}\rangle \equiv \int dp_{21}^\alpha d^9p \qquad \text{(A44)}$$

In general this relation (eq.[A44]) is not satisfied. But DP showed the existence of d which satisfies this relation. Furthermore, in the strongly nonlinear regime, which we are interested in, we find from the dimensional analysis that this relation is correct in general.

We can see from eqs.(A37) and (A43) that the seventh term of the second moment equation is transformed by operator $\Delta^{\beta\gamma}$ as follows,

$$\frac{Gm^2}{a}\Delta^{\beta\gamma}\int d[p_{21}^\gamma(\frac{x_{31}^\beta}{x_{31}^3} - \frac{x_{32}^\beta}{x_{32}^3}) + p_{21}^\beta(\frac{x_{31}^\gamma}{x_{31}^3} - \frac{x_{32}^\gamma}{x_{32}^3})]d^3x_3d^3p_3d^3p_1d^3p_2$$
$$= 4\frac{Gm^2}{a}\Delta^{\beta\gamma}\int dp_{21}^\gamma\frac{x_{31}^\beta}{x_{31}^3}d^3x_3d^3p_3d^3p_1d^3p_2$$
$$= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}d^3x_3$$
$$+4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3. \qquad \text{(A45)}$$

Finally we obtain the following four equations:

$$\bar{n}^2a^6\frac{\partial\xi}{\partial t} + \frac{\bar{n}^2a^6(ma)}{ma^2}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)\langle v\rangle] = 0. \qquad \text{(0th moment)} \qquad \text{(A46)}$$

$$\bar{n}^2a^6\frac{\partial}{\partial t}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)(ma)\langle v\rangle] + \frac{\bar{n}^2a^6(ma)^2}{ma^2}\frac{1}{x^2}\frac{\partial}{\partial x}\{\frac{\partial}{\partial x}(x^2(1+\xi)\Pi) - 2x(1+\xi)\Sigma\}$$
$$+ \frac{Gm^2}{a}\bar{n}^3a^9 8\pi\xi$$
$$+ 2\frac{Gm^2}{a}\bar{n}^3a^9QM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2] = 0$$
$$\text{(1st moment)} \qquad \text{(A47)}$$

$$
\begin{aligned}
&\bar{n}^2 a^6 \frac{\partial}{\partial t}\frac{(ma)^2}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[x^2(1+\xi)\Pi] - 2x(1+\xi)\Sigma\right] \\
&+ \frac{\bar{n}^2 a^6 (ma)^3}{ma^2}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}\left[x^2(1+\xi)\{3\langle v\rangle(\Pi-\Sigma) - 2\langle v\rangle^3\}\right] \\
&+ \frac{\bar{n}^2 a^6 (ma)^3}{ma^2}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)\langle v\rangle\{\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\
&+ \frac{\bar{n}^2 a^6 (ma)^3}{ma^2}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}[x^2(1+\xi)s_{\|}] \\
&+ \frac{16\pi G m^2}{a}\bar{n}a^3\bar{n}^2 a^6 \frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)(ma)\langle v\rangle] \\
&+ 4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^* \frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3} x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23} d^3x_3 \\
&+ 4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^* \frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3} z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31} d^3x_3 = 0
\end{aligned}
$$

(2nd moment : contraction 1) (A48)

$$
\begin{aligned}
\bar{n}^2 a^6 \frac{\partial}{\partial t}[(ma)^2(1+\xi)\Sigma] \quad &+ \quad \frac{\bar{n}^2 a^6 (ma)^3}{ma^2}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4(1+\xi)\langle v\rangle\{\Sigma + \frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\
&+ \quad \frac{\bar{n}^2 a^6 (ma)^3}{ma^2}\frac{1}{x^2}\frac{1}{x^4}\frac{\partial}{\partial x}[x^4(1+\xi)s_\perp] \\
&+ \quad \frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(2,3)[p_{21}^\gamma \frac{x_{31}^\beta}{x_{31}^3} + p_{21}^\beta \frac{x_{31}^\gamma}{x_{31}^3}] d^3x_3 d^3p_3 d^3p_2 \\
&- \quad \frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(3,1)[p_{21}^\gamma \frac{x_{32}^\beta}{x_{32}^3} + p_{21}^\beta \frac{x_{32}^\gamma}{x_{32}^3}] d^3x_3 d^3p_3 d^3p_1 \\
&+ \quad 4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3} x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23} d^3x_3 \\
&+ \quad 4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3} z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31} d^3x_3 = 0.
\end{aligned}
$$

(2nd moment : contraction 2) (A49)

The sixth term of the second moment equation(contraction 1, eq.A48), is rewritten by

$$
\begin{aligned}
&4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^* \frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}[x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}] d^3x_3 \\
&= \quad 4\frac{Gm^2}{a}(ma)\bar{n}^3 a^9 Q^*(-\dot{a}h)\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3} x^\gamma\{\xi_{12} + \xi_{31}\}\xi_{23} d^3x_3
\end{aligned}
$$

$$
\begin{aligned}
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*(-\dot{a}h)\frac{\partial^2}{\partial x^\beta \partial x^\gamma}[x^\gamma x^\beta T] \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*(-\dot{a}h)\frac{1}{x^2}\frac{\partial^2}{\partial x^2}[x^4 T] \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*M(-\dot{a}h)\frac{1}{x^2}\frac{\partial^2}{\partial x^2}[x^4\xi^2]. \qquad \text{(A50)}
\end{aligned}
$$

The seventh term of the second moment equation(contraction 1, eq.A48) is rewritten by

$$
4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\frac{\partial^2}{\partial x^\beta \partial x^\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3 = 0. \qquad \text{(A51)}
$$

The sixth term of the second moment equation(contraction 2, eq.A49) is rewritten by

$$
\begin{aligned}
&4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}[x^\gamma\{\frac{\langle v_{21}\rangle}{x}\xi_{12} + \frac{\langle v_{23}\rangle}{|x-z|}\xi_{31}\}\xi_{23}]d^3x_3. \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*(-\dot{a}h)\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}x^\gamma\{\xi_{12}+\xi_{31}\}\xi_{23}d^3x_3 \\
&= 4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*(-\dot{a}h)\Delta^{\beta\gamma}[x^\gamma x^\beta T] \\
&= 0. \qquad \text{(A52)}
\end{aligned}
$$

The seventh term of the second moment equation (contraction 2, eq.A49) is rewritten by,

$$
4\frac{Gm^2}{a}(ma)\bar{n}^3a^9Q^*\Delta^{\beta\gamma}\int \frac{x_{31}^\beta}{x_{31}^3}z^\gamma\{\frac{\langle v_{31}\rangle}{z} - \frac{\langle v_{23}\rangle}{|x-z|}\}\xi_{23}\xi_{31}d^3x_3 = 0. \qquad \text{(A53)}
$$

Finally we obtain the following four equations:

$$
\frac{\partial \xi}{\partial t} + \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)\langle v\rangle] = 0, \qquad \text{(0th moment)} \qquad \text{(A54)}
$$

$$
\begin{aligned}
\frac{1}{a}\frac{\partial}{\partial t}\frac{1}{x^2}\frac{\partial}{\partial x}[x^2(1+\xi)a\langle v\rangle] &+ \frac{1}{a}\frac{1}{x^2}\frac{\partial}{\partial x}\{\frac{\partial}{\partial x}(x^2(1+\xi)\Pi) - 2x(1+\xi)\Sigma\} \\
&+ 8\pi Gm\bar{n}a\xi \\
&+ 2Gm\bar{n}aQM\frac{1}{x^2}\frac{\partial}{\partial x}[x^3\xi^2] = 0, \\
&\qquad \text{(1st moment)} \qquad \text{(A55)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{a^2}\frac{\partial}{\partial t}a^2\frac{1}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[x^2(1+\xi)\Pi]-2x(1+\xi)\Sigma\right] \\
&\quad+\frac{1}{a}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}\left[x^2(1+\xi)\{3\langle v\rangle(\Pi-\Sigma)-2\langle v\rangle^3\}\right] \\
&\quad+\frac{1}{a}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)\langle v\rangle\{\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\
&\quad+\frac{1}{a}\frac{1}{x^2}\frac{\partial^3}{\partial x^3}[x^2(1+\xi)(s_{\|}-3s_{\perp})] \\
&\quad+\frac{1}{a}\frac{3}{x^2}\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x}\left[x^4\frac{\partial}{\partial x}\frac{1}{x}(1+\xi)s_{\perp}\right] \\
&\quad+16\pi Gm\bar{n}a\frac{1}{x^2}\frac{\partial}{\partial x}x^2(1+\xi)\langle v\rangle \\
&\quad+4Gm\bar{n}aQ^*(\dot{a}h)\frac{1}{x^2}\frac{\partial^2}{\partial x^2}[x^4M\xi^2]=0, \\
&\qquad\qquad\text{(2nd moment : contraction 1)}
\end{aligned}
\tag{A56}
$$

$$
\begin{aligned}
\frac{1}{a^2}\frac{\partial}{\partial t}[a^2(1+\xi)\Sigma] &+ \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}\left[x^4(1+\xi)\langle v\rangle\{\Sigma+\frac{2\langle v_1^2\rangle}{3(1+\xi)}\}\right] \\
&+ \frac{1}{a}\frac{1}{x^4}\frac{\partial}{\partial x}[x^4(1+\xi)s_{\perp}] \\
&+ J=0 \\
&\qquad\text{(2nd moment : contraction 2)}
\end{aligned}
\tag{A57}
$$

where,

$$
\begin{aligned}
J\equiv\quad &\frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(2,3)[p_{21}^{\gamma}\frac{x_{31}^{\beta}}{x_{31}^3}+p_{21}^{\beta}\frac{x_{31}^{\gamma}}{x_{31}^3}]d^3x_3d^3p_3d^3p_2 \\
&-\frac{Gm^2}{a}\bar{n}a^3\Delta^{\beta\gamma}\int c(3,1)[p_{21}^{\gamma}\frac{x_{32}^{\beta}}{x_{32}^3}+p_{21}^{\beta}\frac{x_{32}^{\gamma}}{x_{32}^3}]d^3x_3d^3p_3d^3p_1
\end{aligned}
\tag{A58}
$$

This term is negligible in the strongly nonlinear regime.

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