ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS

(非線形 Schrödinger 方程式系の解の漸近挙動)

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CHAPTER 1

Prologue

Nonlinear Schrödinger equations describe a number of important physical phenomena. Some nonlinear Schrödinger systems are models for different physical phenomena (plasma physics, nonlinear optics and others) (see [**4**], [**5**] and [**41**]). Asymptotic behavior of the nonlinear Schrödinger systems has attracted a lot of attention.

This thesis is concerned with the asymptotic behavior in time of small solutions for the following system of nonlinear Schrödinger equations with quadratic nonlinearities in two space dimensions

$$i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = F_j\left(v_1, \cdots, v_l\right), \quad \text{in } \mathbb{R}\times\mathbb{R}^2, \tag{1.0.1}$$

for $1 \leq j \leq l$, where $\Delta = \sum_{j=1}^{2}\partial_j^2$, $\partial_j = \frac{\partial}{\partial x_j}$, v_j is a complex valued unknown function, $\overline{v_j}$ is the complex conjugate of v_j, m_j is a mass of a particle and $F_j\left(v_1, \cdots, v_l\right)$ is a quadratic term formed from the set

$$\mathcal{A} = \left[v_1, \cdots, v_l, \overline{v_1}, \cdots, \overline{v_l}\right] = \left[v_1, \cdots, v_l, v_{l+1}, \cdots, v_{2l}\right].$$

More precisely, we can write

$$F_j\left(v_1, \cdots, v_l\right) = \sum_{1\leq m\leq k\leq 2l}\lambda_{m,k}^{j} v_m v_k,$$

where $\lambda_{m,k}^{j} \in \mathbb{C}$. This system relates to the Raman amplification in a plasma (see [**4**] and [**5**]). We study the initial and final value problems of System (1.0.1).

This thesis is organized as follows: In Chapter 1, we introduce the overall frame of this thesis and fix some terminology that will be used throughout this thesis.

In Chapter 2, we consider the initial value problem of System (1.0.1)

$$\begin{cases} i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = F_j\left(v_1, \cdots, v_l\right), & \text{in } \mathbb{R}\times\mathbb{R}^2, \\ v_j\left(0, x\right) = \phi_j\left(x\right), & \text{for } x \in \mathbb{R}^2, \end{cases} \tag{1.0.2}$$

for $1 \leq j \leq l$. We prove global existence in time of solutions with small initial data. Especially, we present a proof of $\mathbf{L}^{\infty}$-time decay estimates of small solutions for System (1.0.2). We show that an $\mathbf{L}^2$ conservation law is important for obtaining the time decay estimates of solutions for System (1.0.2). At last using this time decay results, we show nonexistence of the asymptotically free solutions to a special case of System (1.0.2). $\mathbf{L}^{\infty}$-time decay estimates of small solutions for this system are our main results of this chapter. These estimates are obtained by showing a priori estimates of local in time of solutions.

In Chapter 3, we consider the asymptotic behavior in time of solutions to System (1.0.1)

$$i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = F_j\left(v_1, \cdots, v_l\right), \quad \text{in } \mathbb{R}\times\mathbb{R}^2.$$

We prove existence of wave operators for special support conditions on the final data. Moreover we describe the approximate solutions to a special form of the above system to the Cauchy problem. We show a necessary condition of existence of asymptotically free solutions of System (1.0.1) by using the time decay estimates obtained in Chapter 2.

In Chapter 4, we consider a special case of System (1.0.1)

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \lambda \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \mu v_1^2, \end{cases} \tag{1.0.3}$$

where $\lambda, \mu \in \mathbb{C}\backslash\{0\}$, $m_1, m_2 > 0$ are the masses of particles. We prove existence of modified wave operators or wave operators for System (1.0.3) under different mass conditions. We use approximate solutions of this system to study existence of modified wave operators and prove the existence of modified wave operators without any restriction on support condition.

In what follows, we use the same notations both for the vector function spaces and the scalar ones. For any p with $1 \le p \le \infty$, $\mathbf{L}^p$ denotes the Lebesgue space on $\mathbb{R}^2$ and $\|\ \ \|_{\mathbf{L}^p}$ denotes the $\mathbf{L}^p$ norm of $\mathbb{R}^2$. For any $m, s \in \mathbb{R}$, weighted Sobolev space $\mathbf{H}^{m,s}$ on $\mathbb{R}^2$ is defined by

$$\mathbf{H}^{m,s} = \left\{ f = (f_1, ..., f_l) \in \mathbf{L}^2; \|f\|_{\mathbf{H}^{m,s}} = \sum_{j=1}^{l} \|f_j\|_{\mathbf{H}^{m,s}} < \infty \right\},$$

where the Sobolev norm is defined as $\|f_j\|_{\mathbf{H}^{m,s}} = \left\|(1-\Delta)^{\frac{m}{2}}(1+|x|^2)^{\frac{s}{2}} f_j\right\|_{\mathbf{L}^2}$. Also we define the homogeneous Sobolev seminorm of $\mathbb{R}^2$ as $\|f_j\|_{\dot{\mathbf{H}}^{m,s}} = \left\|(-\Delta)^{\frac{m}{2}}|x|^s f_j\right\|_{\mathbf{L}^2}$. We write $\|f_j\|_{\mathbf{L}^2} = \|f_j\|$, $\mathbf{H}^m = \mathbf{H}^{m,0}$ and $\dot{\mathbf{H}}^m = \dot{\mathbf{H}}^{m,0}$ for simplicity. We write $\mathbf{C}\left(I, Y\right)$ for the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space Y. The Fourier transform is defined by

$$\mathcal{F}[\phi](\xi) = \hat{\phi}\left(\xi\right) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(x\cdot\xi)}\phi\left(x\right)dx$$

and the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}[\phi](x) = \check{\phi}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{i(x\cdot\xi)}\phi\left(\xi\right)d\xi.$$

We write Rev and Imv for the real part and the imaginary part of v respectively. We denote by the same letter C various positive constants. Unless otherwise specified, we assume the space dimension is two all through this thesis.

CHAPTER 2

Time decay estimates of solutions

2.1. Introduction

We consider the initial value problem of a system of nonlinear Schrödinger equations

$$L_{m_j} v_j = F_j\left(v_1, \cdots, v_l\right), \tag{2.1.1}$$

$$v_j\left(0, x\right) = \phi_j\left(x\right), \tag{2.1.2}$$

in $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ for $1 \leq j \leq l$, where

$$L_{m_j} = i\partial_t + \frac{1}{2m_j}\Delta,$$

v_j is a complex valued unknown function, $\overline{v_j}$ is the complex conjugate of v_j, m_j is a mass of a particle and $F_j\left(v_1, \cdots, v_l\right)$ is a quadratic term formed from the set

$$\mathcal{A} = \left[v_1, \cdots, v_l, \overline{v_1}, \cdots, \overline{v_l}\right] = \left[v_1, \cdots, v_l, v_{l+1}, \cdots, v_{2l}\right].$$

More precisely, we can write

$$F_j\left(v_1, \cdots, v_l\right) = \sum_{1 \leq m \leq k \leq 2l} \lambda_{m,k}^j v_m v_k,$$

where $\lambda_{m,k}^j \in \mathbb{C}$. The investigation of the Cauchy problem (2.1.1)-(2.1.2) is important, since a system of Schrödinger equations with quadratic nonlinearities describes some physical models such as a model of particles interacting each other (see [**10**]). For simplicity, we shall write F_j for $F_j\left(v_1, \cdots, v_l\right)$ if it does not yield confusion.

We assume that there exist positive constants c_j for $1 \leq j \leq l$ such that

$$\operatorname{Im} \sum_{j=1}^{l} c_j F_j \overline{v_j} = 0. \tag{2.1.3}$$

Condition (2.1.3) is a sufficient condition under which System (2.1.1) satisfies an $\mathbf{L}^2$ conservation law. In order to prove the $\mathbf{L}^2$ conservation law of solutions, it is sufficient to assume that there exist positive constants c_j for $1 \leq j \leq l$ such that

$$\operatorname{Im} \sum_{j=1}^{l} \int_{\mathbb{R}^2} c_j F_j \overline{v_j} dx = 0. \tag{2.1.4}$$

Therefore Assumption (2.1.3) is a stronger condition from the physical point of view. Under Condition (2.1.4) we have the $\mathbf{L}^2$ conservation law

$$\partial_t \sum_{j=1}^{l} c_j \left\|v_j\right\|^2 = 0. \tag{2.1.5}$$

Here we need Condition (2.1.3) to prove time decay of solutions by our method. We also assume the gauge condition such that

$$F_j(v_1,\cdots,v_l)=e^{im_j\theta}F_j\left(e^{-im_1\theta}v_1,\cdots,e^{-im_l\theta}v_l\right) \tag{2.1.6}$$

for any $\theta\in\mathbb{R}$ and F_j is a quadratic term with respect to variables. This condition allows us to use the operator $J_{\frac{1}{m_j},k}=x_k+i\frac{t}{m_j}\partial_k$. The operator $J_{\frac{1}{m_j},k}$ commutes with $L_{\frac{1}{m_j}}=i\partial_t+\frac{1}{2m_j}\Delta$ and is useful to get time decay of solutions (see Lemma 2.2.1 -Lemma 2.2.3) in the standard energy method.

It is interesting to compare System (2.1.1) with the system of nonlinear Klein-Gordon equations

$$\frac{1}{2c^2m_j}\partial_t^2u_j-\frac{1}{2m_j}\Delta u_j+\frac{m_jc^2}{2}u_j=-F_j(u_1,\cdots,u_l) \tag{2.1.7}$$

in $(t,x)\in\mathbb{R}\times\mathbb{R}^2$ for $1\le j\le l$, under Condition (2.1.6), where c is the speed of light. If we make a change of variables $u_j=e^{-itm_jc^2}v_j$ in (2.1.7), then by Condition (2.1.6) we find that v_j satisfies

$$\begin{aligned}\frac{1}{2c^2m_j}\partial_t^2v_j-i\partial_tv_j-\frac{1}{2m_j}\Delta v_j &= -e^{im_j\theta}F_j\left(e^{-im_1\theta}v_1,\cdots,e^{-im_l\theta}v_l\right)\\ &= -F_j(v_1,\cdots,v_l)\end{aligned} \tag{2.1.8}$$

with $\theta=tc^2$ for $1\le j\le l$. Therefore the non relativistic version of System (2.1.7) can be obtained by letting $c\to\infty$ in System (2.1.8) formally, which is the system of equations (2.1.1).

Strauss [**36**], Klainerman [**24**] and Shatah [**32**] studied global existence of small solutions to nonlinear evolution equations with power nonlinearity including nonlinear Schrödinger equations in the beginning of 80's. Since their works, there is a large body of literature discussing the problems of nonlinear Schrödinger equations and nonlinear Klein-Gordon equations in different space dimensions. We survey the following Schrödinger equation with the nonlinearity involving no gradient terms

$$i\partial_tu+\frac{1}{2}\Delta u=\mathcal{N}(u), \tag{2.1.9}$$

in $(t,x)\in\mathbb{R}\times\mathbb{R}^n$, where $\mathcal{N}(u)$ is a homogeneous *p-th* order nonlinear term. Let $S(n)=[n+2+(n^2+12n+4)^{\frac{1}{2}}]/2n$ be the Strauss exponent. In [**36**] Strauss showed if the nonlinear term $\mathcal{N}(u)$ satisfies $|\mathcal{N}'(u)|\le C|u|^{p-1}$ and $p>S(n)$, then global small solutions of the nonlinear Schrödinger equation (2.1.9) exist for suitable initial data. If we focus our attention on the quadratic nonlinearity, the problem on global existence of solutions becomes harder in the case of low space dimensions since $S(3)=2$ and $S(2)=1+\sqrt{2}\simeq 2.414$. Let space dimension $n=2$. If nonlinear term $\mathcal{N}(u)=\lambda|u|u$, Hayashi and Naumkin [**13**] obtained asymptotic formula of small solutions to the Cauchy problem for the nonlinear Schrödinger equation (2.1.9) in the case of $\lambda\in\mathbb{R}\backslash\{0\}$. Furthermore they proved sharp time decay estimates of the small solutions. Shimomura [**34**] considered the same problem in the case of $\lambda\in\mathbb{C}$ and $\mathrm{Im}\lambda<0$. Moreover he showed the nonlinearity of Equation (2.1.9) is dissipative in this case.

We are interested in the asymptotic behavior in time of small solutions for initial value problem of the system of nonlinear Schrödinger equations (2.1.1) in two space dimensions. Global existence in time of solutions for System (2.1.1) with small initial data can be obtained by the method of [**42**] and [**30**]. The chief purpose in this chapter is to present a proof of $\mathbf{L}^{\infty}$ -time decay estimates of small solutions for System (2.1.1) which is the same as those of solutions to linear problems. Main point in our proof is to derive the following ordinary differential equations under Condition (2.1.6) by using the factorization technique of Schrödinger evolution group.

$$i\partial_t u_j = \frac{1}{t} F_j\left(u_1, \cdots, u_l\right) + D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j},$$

where

$$D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}\left(-t\right) v_j = u_j.$$

Then we show $D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j}$ for $1 \leq j \leq l$ are remainder terms in our function space (see Lemma 2.3.3 in details), where

$$\left(D_m \phi\right)(x) = \frac{1}{im} \phi\left(\frac{x}{m}\right)$$

is the dilation operator, $\mathcal{F}, \mathcal{F}^{-1}$ denote the Fourier and its inverse transform operators, respectively, and $U_\delta\left(t\right)$ is the Schrödinger evolution group defined by $U_\delta\left(t\right) = \mathcal{F}^{-1} E^\delta \mathcal{F}$ with $E = e^{-\frac{i}{2}t|\xi|^2}$ and $\delta \neq 0$.

We now give some physical examples satisfying Conditions (2.1.3) and (2.1.6).

Example 2.1.1. *In* [**5**], *the system*

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = v_2 \overline{v_3}, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = 2 v_3 v_1, \\ i\partial_t v_3 + \frac{1}{2m_3}\Delta v_3 = \overline{v_1} v_2, \end{cases}$$

is considered in $(t,x) \in [0,\infty) \times \mathbb{R}^2$. *Obviously this system satisfies Condition (2.1.3). If the mass relation of* $m_2 = m_1 + m_3$ *is satisfied, then this system obeys Condition (2.1.6). This is a system for interacting fields. This model can be also found in the field of plasma physics.*

Example 2.1.2. *More general system*

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = i v_2 \partial_1 \overline{v_3} + v_2 \overline{v_3}, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = -i\partial_1\left(v_1 v_3\right) + 2 v_3 v_1, \\ i\partial_t v_3 + \frac{1}{2m_3}\Delta v_3 = i v_2 \partial_1 \overline{v_1} + \overline{v_1} v_2, \end{cases}$$

in $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, *under the mass relation of* $m_2 = m_1 + m_3$, *is also considered by our method. Physical meaning of this system is given in* [**4**] *and* [**5**]. *It is easy to see that*

$$\operatorname{Re} \int_{\mathbb{R}^2} \overline{v_1} v_2 \partial_1 \overline{v_3} - \overline{v_2} \partial_1\left(v_1 v_3\right) + \overline{v_3} v_2 \partial_1 \overline{v_1} dx = 0,$$

which ensures the $\mathbf{L}^2$ *conservation law. Namely we have Condition (2.1.3). Under the mass relation of* $m_2 = m_1 + m_3$, *the system obeys Condition (2.1.6). The desired ordinary differential equations will be*

$$\begin{cases} i\partial_t u_1 = \frac{1}{t}\left(1 - \xi_1\right) u_2 \overline{u_3}, \\ i\partial_t u_2 = \frac{1}{t}\left(2 + \xi_1\right) u_3 u_1, \\ i\partial_t u_3 = \frac{1}{t}\left(1 - \xi_1\right) \overline{u_1} u_2, \end{cases}$$

for $\xi_1 \in \mathbb{R}$.

Our method can be applied to a system of nonlinear Schrödinger equations with cubic nonlinearities in one space dimension

$$i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = F_j\left(v_1, \cdots, v_l\right), \quad 1 \leq j \leq l, \quad l \geq 3, \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

where F_j is a cubic nonlinear term satisfying Conditions (2.1.3) and (2.1.6). For example when $l = 3$,

$$F_1 = \overline{v_1}^2 v_3, F_2 = |v_1|^2 v_2, F_3 = v_1^3$$

satisfy these two conditions under the mass relation of $m_3 = 3m_1$. We note here that the Strauss exponent $S(1) = \frac{3+\sqrt{17}}{2} > 3$.

The main result in this chapter comes from the author's paper [**25**] and is stated as follows:

Theorem 2.1.3. *We assume that* $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{H}^{2,2}$ *and* F_j *satisfies Conditions (2.1.3) and (2.1.6) for each* $j \in \{1, \cdots, l\}$*. Then for some* $\varepsilon > 0$ *there exists a unique global solution* $v = (v_1, \cdots, v_l)$ *to System (2.1.1) such that*

$$v = (v_1, \cdots, v_l) \in \mathbf{C}\left(\mathbb{R}; \mathbf{H}^{2,2}\right)$$

and

$$\|v(t)\|_{\mathbf{L}^\infty} = \sum_{i=1}^{l} \|v_i(t)\|_{\mathbf{L}^\infty} \leq C\left(1 + |t|\right)^{-1}$$

for any $\phi = (\phi_1, \cdots, \phi_l)$ *satisfying*

$$\|\phi\|_{\mathbf{H}^{2,2}} = \sum_{i=1}^{l} \|\phi_i\|_{\mathbf{H}^{2,2}} \leq \varepsilon.$$

From Theorem 2.1.3, we have global existence of solutions to System (2.1.1) for small initial data. Moreover we find the solutions have the same time decay rate as that of free solutions for large time. The global existence result in time of small solutions for System (2.1.1) is not new. For convenience, we will prove it in more details in the appendix. $\mathbf{L}^\infty$-time decay of small solutions for System (2.1.1) is new and will be proved by showing a priori estimates of local in time of solutions. This kind of idea has been used for construction of $\mathbf{H}^{2,2}$ solutions to nonlinear Schrödinger equations by Hayashi and Naumkin [**13**]. We extended this idea to System (2.1.1).

Theorem 2.1.3 shows System (2.1.1) has a unique global strong solution $v = (v_1, \cdots, v_l) \in \mathbf{C}\left(\mathbb{R}; \mathbf{H}^{2,2}\right)$. We can construct a unique global solution $v = (v_1, \cdots, v_l) \in \mathbf{C}\left(\mathbb{R}; \mathbf{H}^{\beta} \cap \mathbf{H}^{0,\beta}\right)$ of the integral equations associated with System (2.1.1), where $1 < \beta$, by the similar method to that of Theorem 2.1.3 (See [**26**]).

2.2. Preliminary estimates

In this section, we give some estimates as preliminaries. Since $U_\delta(t)$ is the Schrödinger evolution group defined by $U_\delta(t) = \mathcal{F}^{-1}E^\delta\mathcal{F}$ with $\delta \neq 0$ and $E = e^{-\frac{i}{2}t|\xi|^2}$, we have

$$\overline{M^{\frac{1}{\delta}}}\left(i\delta t\partial_j\right) M^{\frac{1}{\delta}} = U_\delta(t)\, x_j U_\delta(-t) = x_j + i\delta t\partial_j,$$

where $M = e^{-\frac{i}{2t}|x|^2}$ for $t \neq 0$.

We first prove Sobolev type inequalities.

LEMMA 2.2.1. *Let $f \in \mathbf{H}^{0,2}, \delta \neq 0$. Then*

$$\|f\|_{\mathbf{L}^\infty} \leq C|t|^{-1} \|U_\delta(-t) f\|_{\mathbf{H}^{0,2}}^{\frac{1}{2}} \|f\|^{\frac{1}{2}}, \quad \textit{for} \ \ t \neq 0.$$

PROOF. By the Sobolev inequality $\|f\|_{\mathbf{L}^\infty} \leq C\|\Delta f\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}}$, we obtain

$$\|f\|_{\mathbf{L}^\infty} = \left\|M^{\frac{1}{\delta}} f\right\|_{\mathbf{L}^\infty} \leq C|t|^{-1} \left\|M^{-\frac{1}{\delta}} t^2 \delta^2 \Delta \left(M^{\frac{1}{\delta}} f\right)\right\|^{\frac{1}{2}} \|f\|^{\frac{1}{2}}$$

for $t \neq 0$.

We have the lemma by the identity $\overline{M^{\frac{1}{\delta}}} (i\delta t \partial_j) M^{\frac{1}{\delta}} = U_\delta(t) x_j U_\delta(-t)$. This completes the proof of the lemma. ∎

LEMMA 2.2.2. *Let $f, g \in \mathbf{H}^{0,2}, \alpha\delta \neq 0$. Then*

$$\begin{aligned} \|(J_{\delta,j} f)(J_{\alpha,k} g)\| \quad &\leq \quad C \|U_\delta(-t) f\|_{\mathbf{H}^{0,2}}^{\frac{1}{2}} \|U_\alpha(-t) g\|_{\mathbf{H}^{0,2}}^{\frac{1}{2}} \\ &\quad \times \|f\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \|g\|_{\mathbf{L}^\infty}^{\frac{1}{2}}, \end{aligned}$$

where $J_{\delta,j} = \overline{M^{\frac{1}{\delta}}} (i\delta t \partial_j) M^{\frac{1}{\delta}}$.

PROOF. We have by the Hölder inequality

$$\begin{aligned} & \|(J_{\delta,j} f)(J_{\alpha,k} g)\| \\ = \ & \left\|\left((i\delta t \partial_j) M^{\frac{1}{\delta}} f\right)\left((i\alpha t \partial_k) M^{\frac{1}{\alpha}} g\right)\right\| \\ = \ & |\alpha\delta|\, t^2 \left\|\partial_j \left(M^{\frac{1}{\delta}} f\right) \partial_k \left(M^{\frac{1}{\alpha}} g\right)\right\| \\ \leq \ & C\, |\alpha\delta|\, t^2 \left\|\partial_j \left(M^{\frac{1}{\delta}} f\right)\right\|_{\mathbf{L}^4} \left\|\partial_k \left(M^{\frac{1}{\alpha}} g\right)\right\|_{\mathbf{L}^4}. \end{aligned}$$

By the Sobolev inequality

$$\left\|(-\Delta)^{\frac{1}{2}} f\right\|_{\mathbf{L}^4} \leq C\|\Delta f\|^{\frac{1}{2}} \|f\|_{\mathbf{L}^\infty}^{\frac{1}{2}},$$

we find

$$\begin{aligned} & \|(J_{\delta,j} f)(J_{\alpha,k} g)\| \\ \leq \ & C\, |\alpha\delta|\, t^2 \left\|\Delta\left(M^{\frac{1}{\delta}} f\right)\right\|^{\frac{1}{2}} \left\|\Delta\left(M^{\frac{1}{\alpha}} g\right)\right\|^{\frac{1}{2}} \|f\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \|g\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \\ \leq \ & C \left\|\delta^2 t^2 \Delta\left(M^{\frac{1}{\delta}} f\right)\right\|^{\frac{1}{2}} \left\|\alpha^2 t^2 \Delta\left(M^{\frac{1}{\alpha}} g\right)\right\|^{\frac{1}{2}} \|f\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \|g\|_{\mathbf{L}^\infty}^{\frac{1}{2}} \end{aligned}$$

from which we have the lemma. ∎

LEMMA 2.2.3. *Let $f, g \in \mathbf{H}^2, \delta \neq 0$. Then*

$$\left\|\mathcal{F}\left(M^\delta - 1\right)\mathcal{F}^{-1} f\right\|_{\mathbf{L}^\infty} \leq C|t|^{-\frac{1}{2}} \|\Delta f\|,$$

$$\left\|\left(\mathcal{F} M^\alpha \mathcal{F}^{-1} f\right)\left(\mathcal{F} M^\delta \mathcal{F}^{-1} g\right) - fg\right\|_{\mathbf{L}^\infty} \leq C|t|^{-\frac{1}{2}} \|f\|_{\mathbf{H}^2} \|g\|_{\mathbf{H}^2},$$

for $t \neq 0$.

Proof. By the Sobolev inequality $\|f\|_{\mathbf{L}^\infty} \le C\|\Delta f\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}}$, we obtain

$$\begin{aligned} & \left\|\mathcal{F}\left(M^\delta-1\right)\mathcal{F}^{-1}f\right\|_{\mathbf{L}^\infty} \\ \le\ & C\left\|\Delta\mathcal{F}\left(M^\delta-1\right)\mathcal{F}^{-1}f\right\|^{\frac{1}{2}}\left\|\mathcal{F}\left(M^\delta-1\right)\mathcal{F}^{-1}f\right\|^{\frac{1}{2}} \\ \le\ & C|t|^{-\frac{1}{2}\gamma}\|\Delta f\|^{\frac{1}{2}}\|(-\Delta)^\gamma f\|^{\frac{1}{2}} \end{aligned}$$

for $t \neq 0$, where $0 \le \gamma \le 1$. This is the first result.
By the identity

$$\begin{aligned} & \left(\mathcal{F}M^\alpha\mathcal{F}^{-1}f\right)\left(\mathcal{F}M^\delta\mathcal{F}^{-1}g\right) - fg \\ =\ & \left(\mathcal{F}\left(M^\alpha-1\right)\mathcal{F}^{-1}f\right)\left(\mathcal{F}M^\delta\mathcal{F}^{-1}g\right) + f\left(\mathcal{F}\left(M^\delta-1\right)\mathcal{F}^{-1}g\right) \end{aligned}$$

and the first estimate, we obtain

$$\begin{aligned} & \left\|\mathcal{F}\left(M^\alpha-1\right)\mathcal{F}^{-1}f\right\|_{\mathbf{L}^\infty}\left\|\mathcal{F}M^\delta\mathcal{F}^{-1}g\right\|_{\mathbf{L}^\infty} \\ & +\left\|\mathcal{F}\left(M^\delta-1\right)\mathcal{F}^{-1}g\right\|_{\mathbf{L}^\infty}\|f\|_{\mathbf{L}^\infty} \\ \le\ & C|t|^{-\frac{1}{2}}\|\Delta f\|\,\|g\|_{\mathbf{H}^2} + C|t|^{-\frac{1}{2}}\|\Delta g\|\,\|f\|_{\mathbf{H}^2} \end{aligned}$$

for $t \neq 0$.
This is the second one. ∎

2.3. A priori estimates and proof of Theorem 2.1.3

In this section, we prove Theorem 2.1.3 by using a priori estimates of local solutions. For any $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{H}^{2,2}$, we let $v = (v_1, \cdots, v_l)$ be a solution of System (2.1.1) in the space

$$X_T = \mathbf{C}\left([0,T];\mathbf{H}^{2,2}\right)$$

with norm

$$\|v\|_{X_T} = \sum_{j=1}^{l}\|v_j\|_{X_T} = \sup_{t\in[0,T]}\sum_{j=1}^{l}(1+t)^{-\theta}\left\|U_{\frac{1}{m_j}}(-t)\,v_j\right\|_{\mathbf{H}^{0,2}},$$

where $T > 0$ and $0 < \theta < \frac{1}{2}$. Existence of local in time of solutions is obtained with a standard method (see Appendix).

Theorem 2.3.1. *Let $T > 1$, then there exists a small $\varepsilon > 0$ such that for any $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{H}^{2,2}$ with $\|\phi\|_{\mathbf{H}^{2,2}} \le \varepsilon$, System (2.1.1) has a unique pair of solutions $v = (v_1, \cdots, v_l) \in X_T$ such that $\|v\|_{X_T} \le 2\varepsilon$.*

In what follows we let v be a solution given by the above theorem. We define the dilation operator by

$$\left(D_\alpha\phi\right)(x) = \frac{1}{i\alpha}\phi\left(\frac{x}{\alpha}\right), \quad \text{for } \alpha \neq 0,$$

and

$$E = e^{-\frac{i}{2}t|\xi|^2},\ M = e^{-\frac{i}{2t}|x|^2}, \quad \text{for } t \neq 0.$$

The evolution operator $U_\alpha(t)$ and inverse evolution operator $U_\alpha(-t)$, for $t \neq 0$, are written as

$$\left(U_\alpha(t)\,\phi\right)(x) = M^{-\frac{1}{\alpha}}D_{\alpha t}\left(\mathcal{F}M^{-\frac{1}{\alpha}}\phi\right)(x)$$

and

$$(U_\alpha(-t)\phi)(x) = M^{\frac{1}{\alpha}}\left(\mathcal{F}^{-1}D_{\alpha t}^{-1}M^{\frac{1}{\alpha}}\phi\right)(x) = -M^{\frac{1}{\alpha}}\left(\mathcal{F}^{-1}E^{\alpha}D_{\frac{1}{\alpha t}}\phi\right)(x)$$

respectively. Using the notation $\mathcal{M}_\alpha = \mathcal{F}M^\alpha\mathcal{F}^{-1}$, we have

$$U_\alpha(t)\mathcal{F}^{-1} = M^{-\frac{1}{\alpha}}D_{\alpha t}\mathcal{M}_{-\frac{1}{\alpha}}$$

and

$$\mathcal{F}U_\alpha(-t) = -\mathcal{M}_{\frac{1}{\alpha}}E^{\alpha}D_{\frac{1}{\alpha t}}.$$

These formulas were used in [**18**] first.

We estimate difference between the free Schrödinger solution and its main term.

Lemma 2.3.2. *Let $f \in \mathbf{H}^{0,2}, \delta \neq 0$. Then*

$$\left\| f - M^{-\frac{1}{\delta}}D_{\delta t}\mathcal{F}U_\delta(-t)f \right\|_{\mathbf{L}^\infty} \leq C|t|^{-\frac{3}{2}}\left\|U_\delta(-t)f\right\|_{\mathbf{H}^{0,2}},$$

for $t \neq 0$.

Proof. By the identity

$$\begin{aligned} f &= U_\delta(t)U_\delta(-t)f = U_\delta(t)\mathcal{F}^{-1}\mathcal{F}U_\delta(-t)f \\ &= M^{-\frac{1}{\delta}}D_{\delta t}\mathcal{M}_{-\frac{1}{\delta}}\mathcal{F}U_\delta(-t)f \\ &= M^{-\frac{1}{\delta}}D_{\delta t}\mathcal{F}U_\delta(-t)f + R, \end{aligned}$$

where

$$R = M^{-\frac{1}{\delta}}D_{\delta t}\left(\mathcal{M}_{-\frac{1}{\delta}} - 1\right)\mathcal{F}U_\delta(-t)f.$$

By Lemma 2.2.3

$$\begin{aligned} \|R\|_{\mathbf{L}^\infty} &= \left\| D_{\delta t}\left(\mathcal{M}_{-\frac{1}{\delta}} - 1\right)\mathcal{F}U_\delta(-t)f \right\|_{\mathbf{L}^\infty} \\ &\leq C|t|^{-1}\left\| \mathcal{F}\left(M^{-\frac{1}{\delta}} - 1\right)U_\delta(-t)f \right\|_{\mathbf{L}^\infty} \\ &\leq C|t|^{-\frac{3}{2}}\left\| x^2 U_\delta(-t)f \right\| \end{aligned}$$

for $t \neq 0$.

This completes the proof of the lemma. ∎

Note that if we put $g = U_\delta(-t)f$, then Lemma 2.3.2 says the estimate of the difference between $U_\delta(t)g$ and its main term.

We now show that if we multiply both sides of System (2.1.1) by $\mathcal{F}U_{\frac{1}{m_j}}(-t)$, then we can divide the nonlinear terms into the main terms and the remainder terms under the gauge condition (2.1.6). By the inverse factorization formula $\mathcal{F}U_{\frac{1}{m_j}}(-t) = -\mathcal{M}_{m_j}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}$, we have

$$\mathcal{F}U_{\frac{1}{m_j}}(-t)F_j = -\mathcal{M}_{m_j}E^{\frac{1}{m_j}}D_{\frac{m_j}{t}}F_j.$$

Using the identity operator $I = -D_{\frac{t}{m_j}} D_{\frac{m_j}{t}}$ and definition of the dilation operator, we can show

$$
\begin{aligned}
& D_{\frac{m_j}{t}} F_j \left(v_1, \cdots, v_l\right) \\
= \; & D_{\frac{m_j}{t}} F_j \left(-D_{\frac{t}{m_1}} D_{\frac{m_1}{t}} v_1, \cdots, -D_{\frac{t}{m_l}} D_{\frac{m_l}{t}} v_l\right) \\
= \; & \frac{t}{i m_j} F_j \left(-i\frac{m_j}{t} D_{\frac{m_j}{t}} D_{\frac{t}{m_1}} D_{\frac{m_1}{t}} v_1, \cdots, -i\frac{m_j}{t} D_{\frac{m_j}{t}} D_{\frac{t}{m_l}} D_{\frac{m_l}{t}} v_l\right) \\
= \; & \frac{t}{i m_j} F_j \left(-\frac{m_j}{t} D_{\frac{m_j}{m_1}} D_{\frac{m_1}{t}} v_1, \cdots, -\frac{m_j}{t} D_{\frac{m_j}{m_l}} D_{\frac{m_l}{t}} v_l\right).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
& \mathcal{F} U_{\frac{1}{m_j}} (-t) F_j \left(v_1, \cdots, v_l\right) \\
= \; & i \mathcal{M}_{m_j} E^{\frac{1}{m_j}} \frac{t}{m_j} F_j \left(-\frac{m_j}{t} D_{\frac{m_j}{m_1}} D_{\frac{m_1}{t}} v_1, \cdots, -\frac{m_j}{t} D_{\frac{m_j}{m_l}} D_{\frac{m_l}{t}} v_l\right).
\end{aligned}
$$

We now use the identity $D_a E^{-b} = E^{-\frac{b}{a^2}} D_a$ for $a \neq 0$ which is obtained by a direct computation to get

$$
D_{\frac{m_j}{m_k}} D_{\frac{m_k}{t}} v_k = D_{\frac{m_j}{m_k}} E^{-\frac{1}{m_k}} E^{\frac{1}{m_k}} D_{\frac{m_k}{t}} v_k = E^{-\frac{m_k}{m_j^2}} D_{\frac{m_j}{m_k}} E^{\frac{1}{m_k}} D_{\frac{m_k}{t}} v_k.
$$

By Condition (2.1.6) with $\theta_j = -\frac{1}{2m_j^2} t \left|\xi\right|^2$ and

$$
E^{-\frac{m_k}{m_j^2}} = e^{\frac{i}{2} t \frac{m_k}{m_j^2} |\xi|^2} = e^{-i m_k \theta_j},
$$

it follows that

$$
\begin{aligned}
& \mathcal{F} U_{\frac{1}{m_j}} (-t) F_j \left(v_1, \cdots, v_l\right) \\
= \; & i \mathcal{M}_{m_j} E^{\frac{1}{m_j}} \frac{t}{m_j} F_j \left(-\frac{m_j}{t} E^{-\frac{m_1}{m_j^2}} D_{\frac{m_j}{m_1}} E^{\frac{1}{m_1}} D_{\frac{m_1}{t}} v_1, \cdots, -\frac{m_j}{t} E^{-\frac{m_l}{m_j^2}} D_{\frac{m_j}{m_l}} E^{\frac{1}{m_l}} D_{\frac{m_l}{t}} v_l\right) \\
= \; & i \mathcal{M}_{m_j} E^{\frac{1}{m_j}} \frac{t}{m_j} F_j \left(-\frac{m_j}{t} e^{-i m_1 \theta_j} D_{\frac{m_j}{m_1}} \widetilde{v}_1, \cdots, -\frac{m_j}{t} e^{-i m_l \theta_j} D_{\frac{m_j}{m_l}} \widetilde{v}_l\right) \\
= \; & i \mathcal{M}_{m_j} E^{\frac{1}{m_j}} \frac{t}{m_j} e^{-i m_j \theta_j} F_j \left(-\frac{m_j}{t} D_{\frac{m_j}{m_1}} \widetilde{v}_1, \cdots, -\frac{m_j}{t} D_{\frac{m_j}{m_l}} \widetilde{v}_l\right),
\end{aligned}
$$

where $\widetilde{v}_j = E^{\frac{1}{m_j}} D_{\frac{m_j}{t}} v_j$. Since F_j is a quadratic nonlinearity, we arrive at

$$
\mathcal{F} U_{\frac{1}{m_j}} (-t) F_j \left(v_1, \cdots, v_l\right) = i \mathcal{M}_{m_j} \frac{m_j}{t} F_j \left(D_{\frac{m_j}{m_1}} \widetilde{v}_1, \cdots, D_{\frac{m_j}{m_l}} \widetilde{v}_l\right).
$$

We again use the identity $\mathcal{F} U_{\frac{1}{m_j}} (-t) = -\mathcal{M}_{m_j} E^{\frac{1}{m_j}} D_{\frac{m_j}{t}}$ on the right hand of the above identity to get

$$
\begin{aligned}
& \mathcal{F} U_{\frac{1}{m_j}} (-t) F_j \left(v_1, \cdots, v_l\right) \\
= \; & i \mathcal{M}_{m_j} \frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{M}_{m_1}^{-1} \mathcal{F} U_{\frac{1}{m_1}} (-t) v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}_{m_l}^{-1} \mathcal{F} U_{\frac{1}{m_l}} (-t) v_l\right),
\end{aligned}
$$

where $\mathcal{M}_{m_j}^{-1} = \mathcal{F} M^{-m_j} \mathcal{F}^{-1}$. We now divide the right hand side into the main term and remainder ones. We define the remainder terms by

$$\begin{aligned} & R_{1,j} \\ = \ & i\left(\mathcal{M}_{m_j} - 1\right) \frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{M}_{m_1}^{-1} \mathcal{F} U_{\frac{1}{m_1}}(-t) v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}_{m_l}^{-1} \mathcal{F} U_{\frac{1}{m_l}}(-t) v_l \right) \end{aligned}$$

and

$$\begin{aligned} & R_{2,j} \\ = \ & i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{M}_{m_1}^{-1} \mathcal{F} U_{\frac{1}{m_1}}(-t) v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}_{m_l}^{-1} \mathcal{F} U_{\frac{1}{m_l}}(-t) v_l \right) \\ & -i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{F} U_{\frac{1}{m_1}}(-t) v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{F} U_{\frac{1}{m_l}}(-t) v_l \right). \end{aligned}$$

We let

$$D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t) v_j = u_j, \tag{2.3.1}$$

then

$$\begin{aligned} & \mathcal{F} U_{\frac{1}{m_j}}(-t) F_j (v_1, \cdots, v_l) \\ = \ & i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{F} U_{\frac{1}{m_1}}(-t) v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{F} U_{\frac{1}{m_l}}(-t) v_l \right) + \sum_{i=1}^{2} R_{i,j} \\ = \ & i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} D_{\frac{1}{m_1}}^{-1} u_1, \cdots, -D_{\frac{m_j}{m_l}} D_{\frac{1}{m_l}}^{-1} u_l \right) + \sum_{i=1}^{2} R_{i,j} \\ = \ & i\frac{m_j}{t} F_j \left(-iD_{m_j} u_1, \cdots, -iD_{m_j} u_l \right) + \sum_{i=1}^{2} R_{i,j}. \end{aligned}$$

We multiply both sides of the above by $D_{\frac{1}{m_j}}$ and use the quadratic property of F_j to get

$$\begin{aligned} & D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t) F_j (v_1, \cdots, v_l) \\ = \ & \frac{m_j^2}{t} F_j \left(-\frac{1}{m_j} u_1, \cdots, -\frac{1}{m_j} u_l \right) + D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j} \\ = \ & \frac{1}{t} F_j (u_1, \cdots, u_l) + D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j}. \end{aligned}$$

Therefore the nonlinear term is divided into two parts such that

$$i\partial_t u_j = \frac{1}{t} F_j (u_1, \cdots, u_l) + D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j}. \tag{2.3.2}$$

We multiply both sides of (2.3.2) by $c_j\overline{u}_j$, take the imaginary parts and use Condition (2.1.3) to obtain

$$\begin{aligned}
&\partial_t\left(\sum_{j=1}^{l} c_j\left|u_j\right|^2\right)\\
&= 2\,\mathrm{Im}\left(\sum_{j=1}^{l}\frac{1}{t}c_jF_j(u_1,\cdots,u_l)\overline{u_j}\right)\\
&\quad +2\,\mathrm{Im}\left(\sum_{j=1}^{l}c_j\left(D_{\frac{1}{m_j}}\sum_{i=1}^{2}R_{i,j}\right)\overline{u_j}\right)\\
&= 2\,\mathrm{Im}\left(\sum_{j=1}^{l}c_j\left(D_{\frac{1}{m_j}}\sum_{i=1}^{2}R_{i,j}\right)\overline{u_j}\right), \qquad (2.3.3)
\end{aligned}$$

where $c_j > 0$ for $1 \leq j \leq l$. We prove the second term of the right hand side of (2.3.2) for each $j \in \{1,\cdots,l\}$, where $l \geq 2$, is a remainder term.

LEMMA 2.3.3. *We have*

$$\sum_{j=1}^{l}\sum_{i=1}^{2}\|R_{i,j}\|_{\mathbf{L}^\infty} \leq C|t|^{-\frac{3}{2}}\left\|U_{\frac{1}{m}}(-t)\,v\right\|^2_{\mathbf{H}^{0,2}},$$

for $t \neq 0$, *where*

$$\left\|U_{\frac{1}{m}}(-t)\,v\right\|_{\mathbf{H}^{0,2}} = \sum_{j=1}^{l}\left\|U_{\frac{1}{m_j}}(-t)\,v_j\right\|_{\mathbf{H}^{0,2}}.$$

PROOF. By the definition of the remainder terms and the first estimate of Lemma 2.2.3, we have

$$\begin{aligned}
&\|R_{1,j}\|_{\mathbf{L}^\infty}\\
\leq\;& C|t|^{-\frac{3}{2}}\left\|\Delta F_j\left(-D_{\frac{m_j}{m_l}}\mathcal{F}M^{-m_1}U_{\frac{1}{m_1}}(-t)\,v_1,\cdots,-D_{\frac{m_j}{m_l}}\mathcal{F}M^{-m_l}U_{\frac{1}{m_l}}(-t)\,v_l\right)\right\|
\end{aligned}$$

for $t \neq 0$.
By the Sobolev inequality

$$\|f\|_{\mathbf{L}^\infty} \leq C\|\Delta f\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}}$$

and the fact that F_j is quadratic with respect to variables, we obtain

$$\begin{aligned}
\|R_{1,j}\|_{\mathbf{L}^\infty} &\leq C|t|^{-\frac{3}{2}}\sum_{j=1}^{l}\left\|U_{\frac{1}{m_j}}(-t)\,v_j\right\|^2_{\mathbf{H}^{0,2}}\\
&\leq C|t|^{-\frac{3}{2}}\left\|U_{\frac{1}{m}}(-t)\,v\right\|^2_{\mathbf{H}^{0,2}}
\end{aligned}$$

for $t \neq 0$.
We again use the first estimate of Lemma 2.2.3 to find

$$\begin{aligned} & \|R_{2,j}\|_{\mathbf{L}^\infty} \\ \leq \ & C|t|^{-\frac{3}{2}} \sum_{i,j=1}^{l} \left\| U_{\frac{1}{m_i}}(-t)\, v_i \right\|_{\mathbf{H}^{0,2}} \left\| U_{\frac{1}{m_j}}(-t)\, v_j \right\|_{\mathbf{H}^{0,2}} \\ \leq \ & C|t|^{-\frac{3}{2}} \left\| U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{H}^{0,2}}^2 \end{aligned}$$

for $t \neq 0$.
Therefore we have the result. ∎

We show the desired a priori estimates of local solutions. We first prove

Lemma 2.3.4. *We have*

$$G(t) \leq CG(1) + C\int_1^t \tau^{-\frac{3}{2}} H(\tau)^2 \, d\tau,$$

for any $t \in [1, T]$, *where*

$$H(t) = \left\| U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{H}^{0,2}},$$

$$G(t) = \left\| \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} = \sum_{j=1}^{l} \left\| \mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j \right\|_{\mathbf{L}^\infty}.$$

Proof. By (2.3.3) and the estimate of Lemma 2.3.3, we have

$$\begin{aligned} & \left\| \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} \\ \leq \ & C \left\| \mathcal{F} U_{\frac{1}{m}}(-1)\, v \right\|_{\mathbf{L}^\infty} + C\int_1^t \tau^{-\frac{3}{2}} \left\| U_{\frac{1}{m}}(-\tau)\, v \right\|_{\mathbf{H}^{0,2}}^2 d\tau \end{aligned}$$

for all $t \in [1, T]$, since

$$\|f\|_{\mathbf{L}^\infty} \leq C \|D_m f\|_{\mathbf{L}^\infty}.$$

This completes the proof of the lemma. ∎

Note that

$$\|U_m(-\tau) f\|_{\mathbf{H}^{0,2}}^2 = \|f\|^2 + 2\sum_{j=1}^{2} \|J_{m,j} f\|^2 + \sum_{j,k=1}^{2} \|J_{m,j} J_{m,k} f\|^2,$$

where

$$J_{m,j} = U_m(t)\, x_j U_m(-t) = e^{\frac{i|x|^2}{2mt}} (imt\partial_j)\, e^{\frac{-i|x|^2}{2mt}} = \overline{M^{\frac{1}{m}}} (imt\partial_j)\, M^{\frac{1}{m}}.$$

We have commutation relations with

$$J_{m,j} = U_m(t) x_j U_m(-t) = x_j + imt\partial_j \tag{2.3.4}$$

and $L_m = i\partial_t + \frac{m}{2}\Delta$ such that

$$[L_m, J_{m,j}] = 0.$$

We evaluate the derivative of $H(t)$ with respect to t.

Lemma 2.3.5. *We have*

$$\frac{d}{dt}H(t) \le C\left(t^{-1}G(t)H(t) + t^{-\frac{3}{2}}H(t)^2\right)$$

for any $t \in [1,T]$.

Proof. Multiplying both sides of (2.1.1) by $J_{\frac{1}{m_j},k}$, we can obtain

$$J_{\frac{1}{m_j},k}\left(i\partial_t + \frac{1}{2m_j}\Delta\right)v_j = J_{\frac{1}{m_j},k}\left(F_j\left(v_1,\cdots,v_l\right)\right).$$

Since

$$\left[L_{\frac{1}{m_j}}, J_{\frac{1}{m_j},k}\right] = 0,$$

then

$$\left(i\partial_t + \frac{1}{2m_j}\Delta\right)J_{\frac{1}{m_j},k}v_j = J_{\frac{1}{m_j},k}\left(F_j\left(v_1,\cdots,v_l\right)\right).$$

By the same way, we have

$$\left(i\partial_t + \frac{1}{2m_j}\Delta\right)J^{\alpha}_{\frac{1}{m_j}}v_j = J^{\alpha}_{\frac{1}{m_j}}\left(F_j\left(v_1,\cdots,v_l\right)\right), \tag{2.3.5}$$

where we have used the multi-index notation

$$J^{\alpha}_{\frac{1}{m_j}} = J^{\alpha_1}_{\frac{1}{m_j},1}J^{\alpha_2}_{\frac{1}{m_j},2},$$

where $|\alpha| = \alpha_1+\alpha_2, \alpha_i \in \mathbb{N}$, for $i = 1,2$. We multiply both sides of (2.3.5) by $\overline{J^{\alpha}_{\frac{1}{m_j}}v_j}$ with $|\alpha| = 2$, integrate in space and take the imaginary parts to obtain

$$\frac{d}{dt}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\| \le C\left\|J^{\alpha}_{\frac{1}{m_j}}F_j\left(v_1,\cdots,v_l\right)\right\|.$$

By the fact that F_j is a quadratic nonlinearity with respect to variables and the gauge condition (2.1.6) we find that

$$\begin{aligned}
&\sum_{|\alpha|=2}\frac{d}{dt}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\| \\
\le\ & C\left\|v\right\|_{\mathbf{L}^\infty}\sum_{j=1}^{l}\sum_{|\alpha|=2}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\| \\
+\ & C\left(\sum_{j=1}^{l}\sum_{|\alpha|=1}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\|_{\mathbf{L}^{p_1}}\right)\left(\sum_{j=1}^{l}\sum_{|\alpha|=1}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\|_{\mathbf{L}^{p_2}}\right),
\end{aligned}$$

with $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}, p_1, p_2 \in [2,\infty]$. We apply Lemma 2.2.2 to the last term of the right hand side to get

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=2}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\| \\
\le\ & C\left\|v\right\|_{\mathbf{L}^\infty}\sum_{|\alpha|=2}\sum_{j=1}^{l}\left\|J^{\alpha}_{\frac{1}{m_j}}v_j\right\| \le C\left\|v\right\|_{\mathbf{L}^\infty}\sum_{|\alpha|=2}\left\|J^{\alpha}_{\frac{1}{m}}v\right\|.
\end{aligned}$$

By Lemma 2.3.2, we have

$$\begin{aligned}\|v\|_{\mathbf{L}^\infty} &\le \left\| M^{-m} D_{\frac{t}{m}} \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} \\ &\quad + \left\| v - M^{-m} D_{\frac{t}{m}} \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} \\ &\le Ct^{-1} \left\| \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} + Ct^{-\frac{3}{2}} \left\| U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{H}^{0,2}}\end{aligned}$$

for all $t \in [1, T]$, which we apply the above inequality to get the lemma. ∎

We are now in a position to prove other a priori estimates of local solutions.

Lemma 2.3.6. *There exists a small $\varepsilon > 0$ such that*

$$(1+t)^{-\varepsilon^{\frac{2}{3}}} H(t) + G(t) < \varepsilon^{\frac{3}{4}}, \sum_{j=1}^{l} \left\|\phi_j\right\|_{\mathbf{H}^{2,2}} \le \varepsilon$$

for any $t \in [1, T]$.

Proof. From Lemma 2.3.4 and Lemma 2.3.5 we get

$$G(t) \le CG(1) + C \int_1^t \tau^{-\frac{3}{2}} H(\tau)^2 \, d\tau, \tag{2.3.6}$$

$$\frac{d}{dt} H(t) \le C \left(t^{-1} G(t) H(t) + t^{-\frac{3}{2}} H(t)^2 \right) \tag{2.3.7}$$

for all $t \in [1, T]$.
From the existence of local solutions stated in the above we obtain $G(1) + H(1) \le C\varepsilon$, hence

$$G(t) \le C \left(\varepsilon + \int_1^t \tau^{-\frac{3}{2}} H(\tau)^2 \, d\tau \right)$$

for all $t \in [1, T]$.
We let

$$(1+t)^{-\varepsilon^{\frac{2}{3}}} H(t) = \widetilde{H}(t).$$

Then

$$G(t) \le C \left(\varepsilon + \int_1^t \tau^{-\frac{3}{2} + 2\varepsilon^{\frac{2}{3}}} \widetilde{H}(\tau)^2 \, d\tau \right) \tag{2.3.8}$$

and

$$\frac{d}{dt} \widetilde{H}(t) + \varepsilon^{\frac{2}{3}} t^{-1} \widetilde{H}(t) \le Ct^{-\frac{3}{2} + \varepsilon^{\frac{2}{3}}} \widetilde{H}(t)^2 + Ct^{-1} G(t) \widetilde{H}(t)$$

from which it follows that

$$\begin{aligned} &\widetilde{H}(t) + \varepsilon^{\frac{2}{3}} \int_1^t \tau^{-1} \widetilde{H}(\tau) \, d\tau \\ &\le C \left(\varepsilon + \int_1^t \tau^{-\frac{3}{2} + \varepsilon^{\frac{2}{3}}} \widetilde{H}(\tau)^2 \, d\tau \right) + C \int_1^t \tau^{-1} G(\tau) \widetilde{H}(\tau) \, d\tau \end{aligned} \tag{2.3.9}$$

for all $t \in [1, T]$.
There exists an $\varepsilon > 0$ such that

$$\widetilde{H}(t) + G(t) < \varepsilon^{\frac{3}{4}} \tag{2.3.10}$$

for any $t \in [1, T]$. Indeed, if we assume that there exists a time $t_1 \in [1, T]$ such that $\widetilde{H}(t_1) + G(t_1) \le \varepsilon^{\frac{3}{4}}$, then by (2.3.8)

$$G(t_1) \le C\left(\varepsilon + \varepsilon^{\frac{3}{2}}\right),$$

and by (2.3.9)

$$\widetilde{H}(t_1) + \varepsilon^{\frac{2}{3}} \int_1^{t_1} \tau^{-1} \widetilde{H}(\tau)\, d\tau \le C\left(\varepsilon + \varepsilon^{\frac{3}{2}}\right) + C\varepsilon^{\frac{3}{4}} \int_1^{t_1} \tau^{-1} \widetilde{H}(\tau)\, d\tau.$$

Therefore if we choose $\varepsilon > 0$ small enough, we see that

$$\widetilde{H}(t_1) \le C\left(\varepsilon + \varepsilon^{\frac{3}{2}}\right).$$

Thus we have

$$\widetilde{H}(t_1) + G(t_1) \le C\left(\varepsilon + \varepsilon^{\frac{3}{2}}\right) < \varepsilon^{\frac{3}{4}}.$$

This contradicts the assumption that there exists a time t_1 such that $\widetilde{H}(t_1) + G(t_1) \le \varepsilon^{\frac{3}{4}}$. This completes the proof of the lemma. ∎

We prove Theorem 2.1.3 by using a priori estimates of local solutions obtained above.

Proof of Theorem 2.1.3. By the standard continuation argument we have a unique time global solution such that

$$\begin{aligned} H(t) &= \left\| U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{H}^{0,2}} \le \varepsilon^{\frac{3}{4}} (1+t)^{\varepsilon^{\frac{2}{3}}}, \\ G(t) &= \left\| \mathcal{F} U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{L}^\infty} \le \varepsilon^{\frac{3}{4}} \end{aligned}$$

for any $t \ge 1$. Therefore it is sufficient to prove the time decay estimates of solutions. By Lemma 2.3.2 and the definition of $M^{-m_j}, D_{\frac{t}{m_j}}$ we have

$$\begin{aligned} & \left\| v_j(t) - M^{-m_j} D_{\frac{t}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j(t) \right\|_{\mathbf{L}^\infty} \\ = & \left\| v_j(t) - e^{\frac{i m_j}{2t}|x|^2} \frac{m_j}{it} \left(\mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j \right)\left(t, \frac{m_j}{t} x\right) \right\|_{\mathbf{L}^\infty} \\ \le & \; C t^{-\frac{3}{2}} \left\| U_{\frac{1}{m_j}}(-t)\, v_j(t) \right\|_{\mathbf{H}^{0,2}} \end{aligned}$$

for $t > 0$. It follows that

$$\|v_j\|_{\mathbf{L}^\infty} \le C t^{-1} \left\| \mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j \right\|_{\mathbf{L}^\infty} + C t^{-\frac{3}{2}} \left\| U_{\frac{1}{m_j}}(-t)\, v_j \right\|_{\mathbf{H}^{0,2}}$$

for $t > 0$. Namely

$$\begin{aligned} \|v\|_{\mathbf{L}^\infty} &= \sum_{j=1}^{l} \|v_j\|_{\mathbf{L}^\infty(\mathbb{R}^2)} \le C t^{-1} G(t) + C t^{-\frac{3}{2}} H(t) \\ &\le C\left(\varepsilon^{\frac{3}{4}} t^{-1} + t^{-\frac{3}{2} + \varepsilon^{\frac{2}{3}}} \right) \le C \varepsilon^{\frac{3}{4}} t^{-1} \end{aligned}$$

for $t \ge 1$. If $t \in [0, 1)$, we have $\|v\|_{\mathbf{L}^\infty} \le C\varepsilon$ by $\|\phi\|_{\mathbf{H}^{2,2}} \le \varepsilon$. If $t < 0$, the theorem follows by the same method. This completes the proof of the theorem. ∎

2.4. A system without the $\mathbf{L}^2$ conservation

In this section we show that the $\mathbf{L}^2$ conservation law is important for obtaining the time decay estimates of solutions (see [**10**] and [**22**]). Let us consider the following system

$$\left\{ \begin{array}{l} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = 0, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = v_1^2, \end{array} \right. \tag{2.4.1}$$

with the initial conditions

$$v_1(0) = \phi_1, v_2(0) = \phi_2,$$

in $(t,x) \in \mathbb{R}\times\mathbb{R}^2$, where $\Delta = \sum_{j=1}^2 \partial_j^2$, $\partial_j = \frac{\partial}{\partial x_j}$, m_1, m_2 are the masses of particles.

Since the first equation of this system can be solved explicitly $v_1 = U_{\frac{1}{m_1}}(t)\,\phi_1$, we get the following Cauchy problem for v_2

$$\left\{ \begin{array}{l} i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \left(U_{\frac{1}{m_1}}(t)\,\phi_1\right)^2, \\ v_2(0) = \phi_2. \end{array} \right. \tag{2.4.2}$$

Equation (2.4.2) is a linear Schrödinger equation and the solution can be represented explicitly by (ϕ_1, ϕ_2). More precisely, we have

Proposition 2.4.1. *Suppose that $\phi_1 \in \mathbf{H}^{0,2}$, $\phi_2 \in \mathbf{L}^2$. Let $v_2 \in \mathbf{C}([0,\infty);\mathbf{L}^2)$ be a global solution of the Cauchy problem for Equation (2.4.2). Then the following estimate is true*

$$\|v_2(t)\| \geq m_1 \left\|\widehat{\phi_1}\right\|_{\mathbf{L}^4}^2 \log t - C\|\phi_1\|_{\mathbf{H}^{0,2}}^2$$

for $t > 1$.

This fact was pointed first in [**39**] and [**40**] in the case of Klein-Gordon equations and in Remark 3 of [**22**] in the case of Schrödinger equations. Therefore Proposition 2.4.1 is not new. However we give a short proof for the convenience of the readers.

Proof. Using the factorization properties of the inverse free Schrödinger evolution group we obtain

$$D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)\left(U_{\frac{1}{m_1}}(t)\,\phi_1\right)^2 = t^{-1}\left(D_{\frac{1}{m_1}}\widehat{\phi_1}\right)^2 + R_3,$$

where

$$R_3 = t^{-1}\mathcal{M}_{\frac{1}{m_2}}\left(D_{\frac{1}{m_1}}\mathcal{M}_{-m_1}\widehat{\phi_1}\right)^2 - t^{-1}\left(D_{\frac{1}{m_1}}\widehat{\phi_1}\right)^2.$$

As in the previous section, R_3 can be easily estimated

$$\|R_3\| \leq Ct^{-2}\|\phi_1\|_{\mathbf{H}^{0,2}}^2$$

for $t > 1$.

Multiplying both sides of Equation (2.4.2) by $D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)$ we get

$$i\partial_t\left(D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)\,v_2\right) = t^{-1}\left(D_{\frac{1}{m_1}}\widehat{\phi_1}\right)^2 + R_3.$$

Integration with respect to t yields

$$\begin{aligned} & D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)v_2(t) \\ = \ & D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-1)v_2(1) + im_1^2\widehat{\phi_1}^2(m_1x)\log t - i\int_1^t R_3 d\tau. \end{aligned}$$

Since $\|v_2(1)\| \leq C\left(\|\phi_2\| + \|\phi_1\|^2_{\mathbf{H}^{0,2}}\right)$ and

$$\int_1^t \|R_3\| \, d\tau \leq C \|\phi_1\|^2_{\mathbf{H}^{0,2}},$$

we find the desired estimate

$$\|v_2(t)\| \geq m_1 \left\|\widehat{\phi_1}\right\|^2_{\mathbf{L}^4} \log t - C \|\phi_1\|^2_{\mathbf{H}^{0,2}}$$

for $t > 1$. Proposition 2.4.1 is proved. ∎

2.5. Nonexistence of the usual scattering states

In this section we state nonexistence of the usual scattering states. This result (joint with Professor Nakao Hayashi and Professor Pavel I. Naumkin) comes from [**10**].

We consider the following system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = v_1^2, \end{cases} \tag{2.5.1}$$

in $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, where $\Delta = \sum_{j=1}^2 \partial_j^2$, $\partial_j = \partial/\partial x_j$, and m_1, m_2 are the masses of particles.

Definition 2.5.1. *A solution to System (2.5.1) is asymptotically free if there exists an $\mathbf{L}^2$ solution (w_{1+}, w_{2+}) to the system of corresponding free equations such that*

$$\sum_{j=1}^2 \|v_j(t) - w_{j+}(t)\| \to 0$$

as $t \to \infty$.

Since System (2.5.1) for small initial data with $2m_1 = m_2$ satisfies all conditions of Theorem 2.1.3, we have global existence and time decay estimates of small solutions to the Cauchy problem for this system. We show nonexistence of the asymptotically free solutions for System (2.5.1) by using time decay estimates of small solutions to the system.

Theorem 2.5.2. *Let $2m_1 = m_2$ and $(v_1, v_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)$ be a global solution of System (2.5.1) obtained in Theorem 2.1.3. Then there does not exist any nontrivial scattering state $(v_{1+}, v_{2+}) \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$ such that $v_{1+} \neq 0$ and*

$$\sum_{j=1}^2 \left\| v_j(t) - U_{\frac{1}{m_j}}(t) v_{j+} \right\| \to 0$$

as $t \to \infty$.

Proof. By the contrary we assume that there exists the nontrivial final state

$$(v_{1+}, v_{2+}) \in \mathbf{H}^2 \cap \mathbf{H}^{0,2}$$

such that $v_{1+} \neq 0$ and

$$\sum_{j=1}^2 \left\| U_{\frac{1}{m_j}}(-t) v_j - v_{j+} \right\| \to 0$$

as $t \to \infty$.
Let

$$(v_1, v_2) \in \mathbf{C}\left([0,\infty);\mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)$$

be a global solution obtained in Theorem 2.1.3 which satisfies the time decay estimate

$$\|v_1(t)\|_{\mathbf{L}^\infty} \le C(1+t)^{-1}$$

for $t \ge 0$.
Multiplying the second equation of System (2.5.1) by $D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)$ and integrating the result with respect to t we get

$$D_{\frac{1}{m_2}}\mathcal{F}\left(U_{\frac{1}{m_2}}(-t)v_2 - U_{\frac{1}{m_2}}(-s)v_2\right) = -i\int_s^t D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-\tau)\left(v_1^2\right)d\tau, \tag{2.5.2}$$

where $0 < s < t < \infty$.
We decompose the nonlinear term as follows

$$U_{\frac{1}{m_2}}(-t)v_1^2 = U_{\frac{1}{m_2}}(-t)\left(v_1^2 - \left(U_{\frac{1}{m_1}}(t)v_{1+}\right)^2\right) + U_{\frac{1}{m_2}}(-t)\left(U_{\frac{1}{m_1}}(t)v_{1+}\right)^2.$$

By the factorization formulas for the free Schrödinger evolution group

$$U_{\frac{1}{m_j}}(t) = M^{-m_j}D_{\frac{t}{m_j}}\mathcal{M}_{-m_j}\mathcal{F}$$

and

$$\mathcal{F}U_{\frac{1}{m_j}}(-t) = -i\mathcal{M}_{m_j}D_{m_j}E^{m_j}D_{\frac{1}{t}},$$

where $\mathcal{M}_\alpha = \mathcal{F}M^\alpha\mathcal{F}^{-1}$ and $D_{\frac{1}{m_j}}\mathcal{M}_{m_j}D_{m_j} = -\mathcal{M}_{\frac{1}{m_j}}$ for $1 \le j \le 2$, we find

$$\begin{aligned}
& D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)\left(U_{\frac{1}{m_1}}(t)v_{1+}\right)^2 \\
= \; & -iD_{\frac{1}{m_2}}\mathcal{M}_{m_2}D_{m_2}E^{m_2}D_{\frac{1}{t}}\left(M^{-m_1}D_{\frac{t}{m_1}}\mathcal{M}_{-m_1}\widehat{v_{1+}}\right)^2 \\
= \; & -t^{-1}D_{\frac{1}{m_2}}\mathcal{M}_{m_2}D_{m_2}\left(D_{\frac{1}{m_1}}\mathcal{M}_{-m_1}\widehat{v_{1+}}\right)^2 \\
= \; & t^{-1}\mathcal{M}_{\frac{1}{m_2}}\left(D_{\frac{1}{m_1}}\mathcal{M}_{-m_1}\widehat{v_{1+}}\right)^2 = t^{-1}\left(D_{\frac{1}{m_1}}\widehat{v_{1+}}\right)^2 + R_3
\end{aligned}$$

since we assume that $2m_1 = m_2$, where

$$R_3 = t^{-1}\mathcal{M}_{\frac{1}{m_2}}\left(D_{\frac{1}{m_1}}\mathcal{M}_{-m_1}\widehat{v_{1+}}\right)^2 - t^{-1}\left(D_{\frac{1}{m_1}}\widehat{v_{1+}}\right)^2.$$

By the similar method of the proof of Lemma 2.3.3 we find

$$\|R_3\| \le Ct^{-2}\|v_{1+}\|^2_{\mathbf{H}^{0,2}} \tag{2.5.3}$$

for $t > 0$.
We have by (2.5.2) and (2.5.3)

$$\begin{aligned}
\left\|U_{\frac{1}{m_2}}(-t)v_2 - U_{\frac{1}{m_2}}(-s)v_2\right\| &\ge \left\|\left(D_{\frac{1}{m_1}}\widehat{v_{1+}}\right)^2\right\|\int_s^t \tau^{-1}d\tau \\
& -C\int_s^t\left\|v_1^2 - \left(U_{\frac{1}{m_1}}(\tau)v_{1+}\right)^2\right\|d\tau \\
& -C\left(\|v_{1+}\|_{\mathbf{H}^{0,2}} + \|v_{2+}\|_{\mathbf{H}^{0,2}}\right)^2\int_s^t \tau^{-2}d\tau \qquad (2.5.4)
\end{aligned}$$

where $0 < s < t < \infty$.
By the Cauchy-Schwarz inequality we get

$$\begin{aligned} & \left\| v_1^2 - \left(U_{\frac{1}{m_1}}(t) v_{1+} \right)^2 \right\| \\ \le \quad & Ct^{-1} \left(\| v_{1+} \|_{\mathbf{L}^1} + t \| v_1 \|_{\mathbf{L}^\infty} \right) \left\| v_1 - U_{\frac{1}{m_1}}(t) v_{1+} \right\|_{\mathbf{L}^2} \le C\delta t^{-1} \end{aligned}$$

for $t > 0$.
Here we can choose $\delta > 0$ such that $\left\| D_{\frac{1}{m_1}} \widehat{v_{1+}} \right\|_{\mathbf{L}^4}^2 - C\delta > 0$.
Therefore by (2.5.4) we obtain

$$\left\| U_{\frac{1}{m_2}}(-t) v_2 - U_{\frac{1}{m_2}}(-s) v_2 \right\| \ge \left(\left\| D_{\frac{1}{m_1}} \widehat{v_{1+}} \right\|_{\mathbf{L}^4}^2 - C\delta \right) \int_s^t \tau^{-1} d\tau \to \infty,$$

as $t \to \infty$.
This contradicts with our assumption

$$\left\| v_2(t) - U_{\frac{1}{m_2}}(t) v_{2+} \right\| \to 0$$

as $t \to \infty$. Theorem 2.5.2 is proved. ∎

2.6. Appendix

This appendix provides a proof of global existence in time of solutions to System (2.1.1). If we restrict our attention to quadratic nonlinear Schrödinger equations, our proof is based on the standard Strichartz type estimates for the Schrödinger equations and the $\mathbf{L}^2$ conservation law. We introduce the following space-time norm

$$\| \phi \|_{\mathbf{L}_t^q(\mathbf{I};\mathbf{L}^r)} = \| \| \phi(t) \|_{\mathbf{L}^r} \|_{\mathbf{L}_t^q(\mathbf{I})},$$

where $\mathbf{I}$ is a bounded or unbounded time interval.

Define

$$\mathcal{G}[g](t) = \int_T^t U_{\frac{1}{m}}(t - \tau) g(\tau) d\tau$$

for any $T \in \bar{\mathbf{I}}$, where $U_{\frac{1}{m}}(t)$ is the Schrödinger evolution group defined by $U_{\frac{1}{m}}(t) = \mathcal{F}^{-1} e^{-\frac{i}{2m} t |\xi|^2} \mathcal{F}$. The Strichartz type estimates (see [**3**]) for the case of two spatial dimensions can be formulated as follows.

LEMMA 2.6.1. *Let $2 \le r < \infty$ and $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$. Then for any time interval $\mathbf{I}$ the following estimates are true*

$$\| \mathcal{G}[g] \|_{\mathbf{L}_t^q(\mathbf{I};\mathbf{L}^r)} \le C \| g \|_{\mathbf{L}_t^{q'}(\mathbf{I};\mathbf{L}^{r'})}$$

and

$$\left\| U_{\frac{1}{m}}(t) \phi \right\|_{\mathbf{L}_t^q(\mathbf{I};\mathbf{L}^r)} \le C \| \phi \|,$$

where $1 < r' \le 2$, $\frac{1}{q'} + \frac{1}{r'} = \frac{3}{2}$.

First we consider the local existence of $\mathbf{L}^2$ solutions to the integral equations

$$(2.6.1) \qquad \begin{cases} v_1(t) = U_{\frac{1}{m_1}}(t) \phi_1 - i \int_0^t U_{\frac{1}{m_1}}(t - t') F_1(v_1, \cdots, v_l)(t') dt', \\ \qquad \cdots\cdots \\ v_l(t) = U_{\frac{1}{m_l}}(t) \phi_l - i \int_0^t U_{\frac{1}{m_l}}(t - t') F_l(v_1, \cdots, v_l)(t') dt', \end{cases}$$

where $F_j \in \mathbf{C}^1\left(\mathbb{C}^l;\mathbb{C}\right)$, $U_{\frac{1}{m_j}}(t) = \exp\left(i\frac{t}{2m_j}\Delta\right)$ for $j = 1,\cdot\cdot\cdot,l$ are the free Schrödinger evolution group. From the point of the results of the Cauchy problem for a single nonlinear Schrödinger equation with power nonlinearities, we can treat the problem in the space $\mathbf{L}^2$. The method is similar to that by [**42**].

For any $\phi = (\phi_1,\cdot\cdot\cdot,\phi_l) \in \mathbf{L}^2$ we find a solution $v = (v_1,\cdot\cdot\cdot,v_l)$ of System (2.6.1) in the space

$$X_0(I) = (\mathbf{C}\cap\mathbf{L}_t^\infty)\left(I;\mathbf{L}^2\right)\cap\mathbf{L}_t^{q_0}(I;\mathbf{L}^{r_0})$$

on a small time interval $I = [-T,T]$, where $0 \le 2/q_0 = 1-2/r_0$. In order to do that we introduce the norm

$$\|v\|_{X_0(I)} = \sum_{j=1}^{l}\|v_j\|_{X_0(I)} = \sum_{j=1}^{l}\left(\|v_j\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|v_j\|_{\mathbf{L}_t^{q_0}(\mathbf{I};\mathbf{L}^{r_0})}\right),$$

where $0 \le 2/q_0 = 1-2/r_0$. We prove

Theorem 2.6.2. *For any $\rho > 0$ there exists $T(\rho) > 0$ such that for any $\phi = (\phi_1,\cdot\cdot\cdot,\phi_l) \in \mathbf{L}^2$ with $\|\phi\| \le \rho$, System (2.6.1) has a unique pair of solutions $v = (v_1,\cdot\cdot\cdot,v_l) \in X_0(I)$ with $I = [-T(\rho),T(\rho)]$.*

Proof. We denote the right hand sides of System (2.6.1) by $\Phi_j(v_1,\cdot\cdot\cdot,v_l)$ for $1 \le j \le l$. By Lemma 2.6.1, we estimates $\Phi_j(v_1,\cdot\cdot\cdot,v_l)$ for $1 \le j \le l$ as

$$\begin{aligned}
& \|\Phi_j(v_1,\cdot\cdot\cdot,v_l)\|_{X_0(I)} \\
\le\ & C\|\phi\| + C\|F_j(v_1,\cdot\cdot\cdot,v_l)\|_{\mathbf{L}_t^{q_0'}\left(\mathbf{I};\mathbf{L}^{r_0'}\right)} \\
\le\ & C\|\phi\| + CT^{1/2}\|v\|_{X_0(I)}^2,
\end{aligned}$$

where we have used the Hölder inequalities in time and space with $1/r_0' = 1/2 + 1/r_0, 1/q_0' = 1/2 + 1/q_0$.
Similarly,

$$\|\Phi_j(v_1,\cdot\cdot\cdot,v_l) - \Phi_j(v_1',\cdot\cdot\cdot,v_l')\|_{X_0(I)} \le CT^{1/2}\|v\|_{X_0(I)}\|v-v'\|_{X_0(I)}$$

holds for $1 \le j \le l$.

Therefore the conclusion follows from a contraction argument by taking $T > 0$ small in connection with the size of data ρ. ∎

Next we consider global existence of $\mathbf{L}^2$ solutions to the problem. Let $v = (v_1,\cdot\cdot\cdot,v_l) \in X_0(I)$ be the local solution of System (2.6.1) given by Theorem 2.6.2. Then we have

$$\|v_j(t)\|^2 = \left\|\phi_j\right\|^2 + 2\,\mathrm{Im}\int_0^t (F_j(t'),v_j(t'))\,dt'$$

for all $j \in \{1,\cdot\cdot\cdot,l\}$, where the last term of the right hand side is understood to be a duality between $\mathbf{L}_t^{q_0'}\left(I;\mathbf{L}^{r_0'}\right) \times \mathbf{L}_t^{q_0}(I;\mathbf{L}^{r_0})$ (see [**30**]).

Lemma 2.6.3. *Let $v = (v_1,\cdot\cdot\cdot,v_l) \in X_0(I)$ be the local solution of System (2.6.1) given in Theorem 2.6.2. If there exist constants $c_j > 0$ for $1 \le j \le l$ such that*

$$\mathrm{Im}\sum_{j=1}^{l}\int_{\mathbb{R}^2} c_jF_j\overline{v_j}dx \le 0, \tag{2.6.2}$$

then

$$\sum_{j=1}^{l} c_j \|v_j\|^2 \leq \sum_{j=1}^{l} c_j \|\phi_j\|^2$$

for all $t \in I$.

PROOF. The lemma follows from

$$\sum_{j=1}^{l} c_j \|v_j(t)\|^2 = \sum_{j=1}^{l} c_j \|\phi_j\|^2 + 2\,\mathrm{Im}\left[\sum_{j=1}^{l} \int_0^t c_j \left(F_j(t'), v_j(t')\right) dt'\right].$$

∎

THEOREM 2.6.4. *Let* $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{L}^2$. *If there exist constants* $c_j > 0$ *for* $1 \leq j \leq l$ *such that*

$$\mathrm{Im} \sum_{j=1}^{l} \int_{\mathbb{R}^2} c_j F_j \overline{v_j} dx \leq 0,$$

then System (2.6.1) has a unique pair of solutions $v = (v_1, \cdots, v_l) \in X_0(\mathbb{R})$, *where*

$$X_0(\mathbb{R}) = \left(\mathbf{C} \cap \mathbf{L}_t^{\infty}\right)\left(\mathbb{R}; \mathbf{L}^2\right) \cap \mathbf{L}_t^{q_0}\left(\mathbb{R}; \mathbf{L}^{r_0}\right).$$

Moreover,

$$\sum_{j=1}^{l} c_j \|v_j\|^2 \leq \sum_{j=1}^{l} c_j \|\phi_j\|^2$$

for all $t \in \mathbb{R}$.

PROOF. The theorem follows from Theorem 2.6.2 and Lemma 2.6.3 by the standard continuation argument of local solutions since the existence time depends only on $\|\phi\|$. ∎

By Theorem 2.6.2 and Theorem 2.6.4, we have

PROPOSITION 2.6.5. *If* $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{H}^{2,2}$ *and the nonlinear terms in System (2.1.1) satisfy Condition (2.6.2), then there exists a unique global solution* $v = (v_1, v_2, \cdots, v_l)$ *of System (2.1.1) such that*

$$v = (v_1, v_2, \cdots, v_l) \in \mathbf{C}\left([0, \infty); \mathbf{H}^{2,2}\right).$$

PROOF. If we take a time derivative of nonlinearities into account, in the same way as in the proof of Theorem 2.6.2, we have a unique pair of local solutions

$$\left(\mathbf{C} \cap \mathbf{L}_t^{\infty}\right)\left(I; \mathbf{H}^2\right)$$

and if we note that the existence time of solutions depends on $\|\phi\|_{\mathbf{H}^1}$. we can extend this existence time to infinity as in the proof of Theorem 2.6.4. Since $v \in \mathbf{C}\left([0, \infty); \mathbf{H}^2\right)$ and $\phi \in \mathbf{H}^{2,2}$, we get the desired result. Indeed we multiply $|x|^2$ both sides of equations and apply the energy method to the resulting systems to have a priori estimates of $\left\||x|^2 v\right\|$. ∎

CHAPTER 3

Wave operators and approximate solutions

3.1. Introduction

We consider the following system of nonlinear Schrödinger equations in two space dimensions

$$i\partial_t v_j + \frac{1}{2m_j}\Delta v_j = F_j\left(v_1, \cdots, v_l\right), \quad \text{in } \mathbb{R}\times\mathbb{R}^2, \tag{3.1.1}$$

for $1 \leq j \leq l$, where $\Delta = \sum_{j=1}^{2}\partial_j^2$, $\partial_j = \frac{\partial}{\partial x_j}$, v_j is a complex valued unknown function, $\overline{v_j}$ is the complex conjugate of v_j, m_j is a mass of a particle and $F_j\left(v_1, \cdots, v_l\right)$ is a quadratic term formed from the set

$$\mathcal{A} = \left[v_1, \cdots, v_l, \overline{v_1}, \cdots, \overline{v_l}\right] = \left[v_1, \cdots, v_l, v_{l+1}, \cdots, v_{2l}\right].$$

More precisely, we can write

$$F_j\left(v_1, \cdots, v_l\right) = \sum_{1\leq m\leq k\leq 2l} \lambda_{m,k}^j v_m v_k,$$

where $\lambda_{m,k}^j \in \mathbb{C}$.

Let $\psi_+ = (\psi_{1+}, \cdots, \psi_{l+})$, where $\psi_{j+} \in \mathbf{L}^2$ for each $j \in \{1, \cdots, l\}$, and $U_{\frac{1}{m}}(t)\psi_+ = \left(U_{\frac{1}{m_1}}(t)\,\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\,\psi_{l+}\right)$. Note that $U_{\frac{1}{m}}(t)\psi_+$ is a solution of the system of the corresponding free Schrödinger equations. If there exists a unique global solution $v = (v_1, \cdots, v_l) \in \mathbf{C}([1,\infty); \mathbf{L}^2)$ of System (3.1.1) such that

$$\left\|v(t) - U_{\frac{1}{m}}(t)\psi_+\right\| = \sum_{j=1}^{l}\left\|v_j(t) - U_{\frac{1}{m_j}}(t)\,\psi_{j+}\right\| \to 0,$$

as $t \to \infty$, then the map

$$\mathbf{W}_+ : \quad \psi_+ = \left(\psi_{1+}, \cdots, \psi_{l+}\right) \mapsto v = \left(v_1, \cdots, v_l\right)(1)$$

is well-defined on $\mathbf{L}^2$. We call the map $\mathbf{W}_+$ a wave operator. The given ψ_+ is called a final state.

We recall some of the results to the following Schrödinger equation with the nonlinearity involving no gradient terms

$$i\partial_t u + \frac{1}{2}\Delta u = \mathcal{N}(u), \tag{3.1.2}$$

in $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, where $\mathcal{N}(u)$ is a nonlinear term. Let the short range exponent $P(n) = 1 + \frac{2}{n}$. We assume nonlinear term $\mathcal{N}(u) = \lambda\,|u|^{p-1}\,u$ with $\lambda \in \mathbb{R}_+$. For $p > P(n)$, existence of asymptotically free solutions was proved by Lin and Strauss [**27**] for $p > \frac{8}{3}, n = 3$ and Y. Tsutsumi and K. Yajima [**43**] for $p > P(n)$. For $1 \leq p \leq P(n)$, Strauss [**37**] and Barab [**2**] showed nonexistence of asymptotically free solutions. Since $P(2) = 1 + \frac{2}{2} = 2$, quadratic nonlinearity in two space dimensions is critical. Let space dimension $n = 2$. If nonlinear term $\mathcal{N}(u) = \lambda |u| u$ with $\lambda \in \mathbb{R}\backslash\{0\}$, Ginibre and Ozawa [**7**] proved existence of modified wave operators in $\mathbf{L}^2$ sense. Hayashi, Naumkin [**13**] and Shimomura [**34**] studied asymptotic behavior of small solutions for the initial value problem of Equation (3.1.2) with $\mathcal{N}(u) = \lambda |u| u$ in different cases for λ. If nonlinear term $\mathcal{N}(u) = \lambda u^2$ or $\lambda \overline{u}^2$, where $\lambda \in \mathbb{C}\backslash\{0\}$, the existence of modified wave operators are shown in [**29**], [**17**] and [**35**]. If the nonlinear term $\mathcal{N}(u) = \lambda |u|^2$ with $\lambda \in \mathbb{C}\backslash\{0\}$, the initial value and scattering problems were considered in [**20**] and [**33**].

Our aim is to study final and initial problems of System (3.1.1). We prove existence of wave operators for special support conditions on the final data. Moreover we show asymptotic behavior of solutions to the initial value problem for a special system.

3.2. Existence of wave operators

In this section, we investigate existence of wave operators for System (3.1.1). First, we give a necessary condition of existence of asymptotically free solutions.

THEOREM 3.2.1. *Let $\phi = (\phi_1, \cdots, \phi_l) \in \mathbf{H}^{2,2}$ and v be global in time of solutions of System (3.1.1) satisfying a priori estimates*

$$\int_1^\infty \frac{1}{s^2} \left\| U_{\frac{1}{m}} (-s)\, v \right\|^2_{\mathbf{H}^{0,2}} ds < \infty, \left\| \mathcal{F} U_{\frac{1}{m}} (-t)\, v \right\|_{\mathbf{L}^\infty} \leq C.$$

We assume that the gauge condition (2.1.6) holds for each $j \in \{1, \cdots, l\}$. If there exists a $\widehat{\psi_+} = \left(\widehat{\psi_{1+}}, \cdots, \widehat{\psi_{l+}}\right) \in \mathbf{L}^2 \cap \mathbf{L}^\infty$ such that

$$\lim_{t \to \infty} \left\| v\,(t) - U_{\frac{1}{m}}\,(t)\, \psi_+ \right\| = 0.$$

Then

$$F_j \left(D_{\frac{1}{m_1}} \widehat{\psi_{1+}}, \cdots, D_{\frac{1}{m_l}} \widehat{\psi_{l+}} \right) = 0 \tag{3.2.1}$$

for every $j \in \{1, \cdots, l\}$, where $\widehat{\psi_+} = \mathcal{F} \psi_+$.

From Theorem 3.2.1, we find that if (3.2.1) does not hold, then the solutions are not asymptotically free. If the support condition

$$\cap_{j=1}^{l} \mathrm{supp} \widehat{\psi_{j+}}\,(m_j \xi) \text{ is empty,}$$

is satisfied, we have (3.2.1). The support condition means each particle does not work for any other particles. This effect does not occur in a single nonlinear Schrödinger equation.

The following theorem says existence of wave operators.

THEOREM 3.2.2. *Let $\widehat{\psi_+} = \left(\widehat{\psi_{1+}}, \cdots, \widehat{\psi_{l+}}\right) \in \mathbf{H}^{2,2}$ satisfy the so-called support condition (3.2.1). Assume that F_j satisfies the gauge condition (2.1.6) for each $j \in \{1, \cdots, l\}$. Then for some $\varepsilon > 0$ there exists a unique global solution $v = (v_1, \cdots, v_l)$ of System (3.1.1) such that*

$$v \in \mathbf{C}\left([1, \infty); \mathbf{L}^2\right)$$

and

$$\left\| v(t) - U_{\frac{1}{m}}(t)\,\psi_+ \right\| \le Ct^{-b}, \quad \frac{1}{2} < b < 1$$

for large t and any $\widehat{\psi_+}$ satisfying

$$\left\| \psi_+ \right\|_{\mathbf{H}^{2,2}} \le \varepsilon.$$

Theorem 3.2.2 follows from the fact that asymptotic behavior of each particle is determined by the leading term of $U_{\frac{1}{m_j}}(t)\,\psi_{j+}$ which is given by $e^{\frac{i|x|^2 m_j}{2t}} \frac{m_j}{it} \widehat{\psi_{j+}}\left(\frac{xm_j}{t}\right)$.

3.3. Proof of theorems

In this section we give the proof of existence of wave operators for System (3.1.1). First we consider the necessary condition of existence of asymptotically free solutions.

Proof of Theorem 3.2.1. We assume that (3.2.1) does not hold. Namely there exist $j \in \{1, \cdots, l\}$ and $\eta > 0$ such that

$$\left\| F_j\left(D_{\frac{1}{m_1}} \widehat{\psi_{1+}}, \cdots, D_{\frac{1}{m_l}} \widehat{\psi_{l+}} \right) \right\| = \eta.$$

By (2.3.2) we have

$$\begin{aligned} & i\partial_t \left(D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j \right) \\ = {} & \frac{1}{t}\left(F_j(u_1, \cdots, u_l) - F_j\left(D_{\frac{1}{m_1}} \widehat{\psi_{1+}}, \cdots, D_{\frac{1}{m_l}} \widehat{\psi_{l+}} \right) \right) \\ & + \frac{1}{t} F_j \left(D_{\frac{1}{m_1}} \widehat{\psi_{1+}}, \cdots, D_{\frac{1}{m_l}} \widehat{\psi_{l+}} \right) + D_{\frac{1}{m_j}} \sum_{i=1}^{2} R_{i,j}, \end{aligned} \tag{3.3.1}$$

where

$$u_j = D_{\frac{1}{m_j}} \mathcal{F} U_{\frac{1}{m_j}}(-t)\, v_j,$$

$$\begin{aligned} & R_{1,j} \\ = {} & i\left(\mathcal{M}_{m_j} - 1 \right) \frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{M}_{m_1}^{-1} \mathcal{F} U_{\frac{1}{m_1}}(-t)\, v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}_{m_l}^{-1} \mathcal{F} U_{\frac{1}{m_l}}(-t)\, v_l \right) \end{aligned}$$

and

$$\begin{aligned} & R_{2,j} \\ = {} & i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{M}_{m_1}^{-1} \mathcal{F} U_{\frac{1}{m_1}}(-t)\, v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}_{m_l}^{-1} \mathcal{F} U_{\frac{1}{m_l}}(-t)\, v_l \right) \\ & - i\frac{m_j}{t} F_j \left(-D_{\frac{m_j}{m_1}} \mathcal{F} U_{\frac{1}{m_1}}(-t)\, v_1, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{F} U_{\frac{1}{m_l}}(-t)\, v_l \right). \end{aligned}$$

By the same way to the proof of Lemma 2.3.3, we have

$$\sum_{j=1}^{l} \sum_{i=1}^{2} \left\| R_{i,j} \right\| \le Ct^{-2} \left\| U_{\frac{1}{m}}(-t)\, v \right\|_{\mathbf{H}^{0,2}}^2$$

for $t \neq 0$.
Integrating (3.3.1) in time and taking $\mathbf{L}^2$ norm, we obtain

$$\begin{aligned}
& \left\| D_{\frac{1}{m_j}} \mathcal{F}\left(U^{-1}_{\frac{1}{m_j}} v_j\right)(2t) - D_{\frac{1}{m_j}} \mathcal{F}\left(U^{-1}_{\frac{1}{m_j}} v_j\right)(t) \right\| \\
= & \left\| \left(U^{-1}_{\frac{1}{m_j}} v_j\right)(2t) - \left(U^{-1}_{\frac{1}{m_j}} v_j\right)(t) \right\| \\
\geq & \ \eta \int_t^{2t} \frac{1}{s} ds - C \int_t^{2t} \frac{1}{s} \left\| U_{\frac{1}{m}}(-s) v - \psi_+ \right\| \\
& \times \left(\left\| \mathcal{F} U_{\frac{1}{m}}(-s) v \right\|_{\mathbf{L}^\infty} + \left\| \widehat{\psi_+} \right\|_{\mathbf{L}^\infty} \right) ds \\
- & \ C \sum_{i=1}^{2} \int_t^{2t} \left\| R_{i,j}(s) \right\| ds \\
\geq & \ \eta \log 2 - C \int_t^{2t} \frac{1}{s} \left\| U_{\frac{1}{m}}(-s) v - \psi_+ \right\| ds \\
& - C \int_t^{2t} \frac{1}{s^2} \left\| U_{\frac{1}{m}}(-s) v \right\|^2_{\mathbf{H}^{0,2}} ds
\end{aligned}$$

for $t \geq 1$.
From the assumption, we have $\left\| U_{\frac{1}{m}}(-t) v - \psi_+ \right\| \to 0$ as $t \to \infty$, so there exists a time $T \geq 1$ such that

$$\left\| \left(U^{-1}_{\frac{1}{m_j}} v_j\right)(2t) - \left(U^{-1}_{\frac{1}{m_j}} v_j\right)(t) \right\| \geq \frac{\eta}{2} \log 2$$

for all $t \geq T$. This contradicts $\left\{ U_{\frac{1}{m_j}}(-t) v_j \right\}$ is a Cauchy sequence in $\mathbf{L}^2$. This completes the proof of the theorem. ∎

Proof of Theorem 3.2.2. Let us investigate

$$\begin{aligned}
L_{m_j} v_j &= F_j(v_1, \cdots, v_l) - F_j\left(U_{\frac{1}{m_1}}(t)\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\psi_{l+}\right) \\
&\quad + F_j\left(U_{\frac{1}{m_1}}(t)\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\psi_{l+}\right) \qquad (3.3.2)
\end{aligned}$$

for $j = 1, \cdots, l$.
We consider the last term of the right hand side of (3.3.2) to have

$$\begin{aligned}
& F_j\left(U_{\frac{1}{m_1}}(t)\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\psi_{l+}\right) \\
= & \ U_{\frac{1}{m_j}}(t) \mathcal{F}^{-1} \mathcal{F} U_{\frac{1}{m_j}}(-t) F_j\left(U_{\frac{1}{m_1}}(t)\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\psi_{l+}\right).
\end{aligned}$$

In the same way as in the derivation of (2.3.2) we obtain

$$\begin{aligned}
& \mathcal{F} U_{\frac{1}{m_j}}(-t) F_j\left(U_{\frac{1}{m_1}}(t)\psi_{1+}, \cdots, U_{\frac{1}{m_l}}(t)\psi_{l+}\right) \\
= & \ i \mathcal{M}_{m_j} \frac{m_j}{t} F_j\left(-D_{\frac{m_j}{m_1}} \mathcal{M}^{-1}_{m_1} \widehat{\psi_{1+}}, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}^{-1}_{m_l} \widehat{\psi_{l+}}\right),
\end{aligned}$$

where $\mathcal{M}^{-1}_{m_j} = \mathcal{F} M^{-m_j} \mathcal{F}^{-1}$. We define

$$R_{3,j} = i\left(\mathcal{M}_{m_j} - 1\right) \frac{m_j}{t} F_j\left(-D_{\frac{m_j}{m_1}} \mathcal{M}^{-1}_{m_1} \widehat{\psi_{1+}}, \cdots, -D_{\frac{m_j}{m_l}} \mathcal{M}^{-1}_{m_l} \widehat{\psi_{l+}}\right)$$

and

$$\begin{aligned} R_{4,j} &= i\frac{m_j}{t}F_j\left(-D_{\frac{m_j}{m_1}}\mathcal{M}_{m_1}^{-1}\widehat{\psi_{1+}},\cdots,-D_{\frac{m_j}{m_l}}\mathcal{M}_{m_l}^{-1}\widehat{\psi_{l+}}\right) \\ &\quad -i\frac{m_j}{t}F_j\left(-D_{\frac{m_j}{m_1}}\widehat{\psi_{1+}},\cdots,-D_{\frac{m_j}{m_l}}\widehat{\psi_{l+}}\right). \end{aligned}$$

Finally, we get

$$\begin{aligned} &D_{\frac{1}{m_j}}\mathcal{F}U_{\frac{1}{m_j}}(-t)\,F_j\left(U_{\frac{1}{m_1}}(t)\,\psi_{1+},\cdots,U_{\frac{1}{m_l}}(t)\,\psi_{l+}\right) \\ =\;& \frac{1}{t}F_j\left(D_{\frac{1}{m_1}}\widehat{\psi_{1+}},\cdots,D_{\frac{1}{m_l}}\widehat{\psi_{l+}}\right)+D_{\frac{1}{m_j}}\sum_{i=3}^{4}R_{i,j}. \end{aligned}$$

We substitute the formula into (3.3.2) to obtain

$$\begin{aligned} &L_{m_j}\left(v_j-U_{\frac{1}{m_j}}(t)\,\psi_{j+}\right) \\ =\;& F_j(v_1,\cdots,v_l)-F_j\left(U_{\frac{1}{m_1}}(t)\,\psi_{1+},\cdots,U_{\frac{1}{m_l}}(t)\,\psi_{l+}\right) \\ &+\frac{1}{t}U_{\frac{1}{m_j}}(t)\,\mathcal{F}^{-1}D_{\frac{1}{m_j}}^{-1}F_j\left(D_{\frac{1}{m_1}}\widehat{\psi_{1+}},\cdots,D_{\frac{1}{m_l}}\widehat{\psi_{l+}}\right) \\ &+U_{\frac{1}{m_j}}(t)\,\mathcal{F}^{-1}\sum_{i=3}^{4}R_{i,j} \end{aligned} \tag{3.3.3}$$

since $L_{m_j}\left(U_{\frac{1}{m_j}}(t)\,\psi_{j+}\right)=0$. We define the following function space

$$\mathbf{X}=\left\{f=(f_1,\cdots,f_l)\in\mathbf{C}\left([1,\infty);\mathbf{L}^2\right);\|f\|_{\mathbf{X}}<\infty\right\},$$

with the norm

$$\|f\|_{\mathbf{X}}=\sum_{j=1}^{2}\sup_{t\in[1,\infty)}t^b\left(\left\|f-U_{\frac{1}{m}}(\cdot)\,\psi_+\right\|_{\mathbf{L}^\infty([t,\infty);\mathbf{L}^2)}+\left\|f-U_{\frac{1}{m}}(\cdot)\,\psi_+\right\|_{\mathbf{L}^4([t,\infty);\mathbf{L}^4)}\right),$$

where $\frac{1}{2}< b<1$. We also denote the closed ball $\mathbf{X}_\rho=\{f\in\mathbf{X};\|f\|_{\mathbf{X}}\le\rho\}$. The linearized system of (3.3.3) is written by replacing $F_j(v_1,\cdots,v_l)$ by $F_j(w_1,\cdots,w_l)$ with $w\in\mathbf{X}_\rho$. We get from the linearized system of (3.3.3) and the Strichartz estimate

$$\begin{aligned} &\left\|v-U_{\frac{1}{m}}(\cdot)\,\psi_+\right\|_{\mathbf{L}^\infty([t,\infty);\mathbf{L}^2)}+\left\|v-U_{\frac{1}{m}}(\cdot)\,\psi_+\right\|_{\mathbf{L}^4([t,\infty);\mathbf{L}^4)} \\ \le\;& C\rho^2t^{\frac{1}{2}-2b}+C\varepsilon\rho t^{-b}+C\sum_{j=1}^{l}\sum_{i=3}^{4}\|R_{i,j}\|_{\mathbf{L}^1([t,\infty);\mathbf{L}^2)} \end{aligned}$$

for $t\ge1$, where $b\in(\frac{1}{2},1)$, since Condition (3.2.1) means the third term of the right hand side of System (3.3.3) vanishes.
In the same way as in the proof of Lemma 2.3.3 we find that

$$\sum_{j=1}^{l}\sum_{i=3}^{4}\|R_{i,j}\|\le Ct^{-2}\|\psi_+\|_{\mathbf{H}^{2,2}}^2\le C\varepsilon^2t^{-2}$$

for $t\ge1$.
Therefore there exist $\varepsilon>0$ and positive ρ such that $\|v\|_{\mathbf{X}}\le\rho$. In the same way we

have for the differences

$$\|v - \widetilde{v}\|_{\mathbf{X}} \le \frac{1}{2} \|w - \widetilde{w}\|_{\mathbf{X}},$$

where $\widetilde{w} \in \mathbf{X}_\rho$ and $\widetilde{v}$ is the corresponding solution of the linearized system of (3.3.3). This completes the proof of the theorem. ∎

REMARK 3.3.1. *Asymptotic behavior of solutions to the system $L_{m_j} v_j = F_j(v_1, \cdots, v_l)$ for $1 \le j \le l$ is determined by solutions of the ordinary differential equation*

$$i\partial_t \Phi_j = \frac{1}{t} F_j(\Phi_1, \cdots, \Phi_l), \quad 1 \le j \le l,$$

where $\Phi_j = D_{\frac{1}{m_j}} \mathcal{F} \psi_{j+}$. However asymptotic behavior of solutions to the system of ordinary differential equations is not well known. That is the main reason why we can not prove asymptotic behavior of solutions to System (3.1.1) without Condition (3.2.1).

3.4. Approximate solutions

We consider the following special form of System (3.1.1)

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \gamma \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = v_1^2, \end{cases} \tag{3.4.1}$$

in $(t, x) \in \mathbb{R} \times \mathbb{R}^2$, where $\gamma \in \mathbb{C}$, $|\gamma| = 1$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $\partial_j = \frac{\partial}{\partial x_j}$, m_1, m_2 are the masses of particles.

If we assume that

$$\gamma = 1, \tag{3.4.2}$$

then we find the $\mathbf{L}^2$ conservation law

$$\frac{d}{dt}\left(\|v_1\|^2 + \|v_2\|^2\right) = 0.$$

Under Condition (3.4.2) System (3.4.1) becomes

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = v_1^2, \end{cases} \tag{3.4.3}$$

The global existence of small solutions for System (3.4.3) is proved in Chapter 2 since System (3.4.3) is included to System (3.1.1) (see Theorem 2.1.3). Here, we focus our attention on the asymptotic behavior of solutions for System (3.4.3) under some special conditions. This work is done jointly with Professor Nakao Hayashi and Professor Pavel I. Naumkin (see [**10**]).

THEOREM 3.4.1. *Let $(v_1, v_2) \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,2}\right)$ be a global solution obtained in Theorem 2.1.3. Then there exist initial data $\Psi_1(\xi) \in \mathbf{L}^\infty$ and $\Psi_2(\xi) \in \mathbf{L}^\infty$ for the Cauchy problem*

$$\begin{cases} \partial_t \psi_1 = t^{-1} \overline{\psi_1} \psi_2, \quad t > 1, \\ \partial_t \psi_2 = -t^{-1} \psi_1^2, \quad t > 1, \\ \psi_1(1, \xi) = \Psi_1(\xi), \quad \psi_2(1, \xi) = \Psi_2(\xi), \end{cases} \tag{3.4.4}$$

such that the asymptotics for the solution (v_1, v_2) is true

$$v_j(t, x) = -it^{-1} e^{\frac{im_j}{2t}|x|^2} (\mathcal{M}_{-\frac{1}{m_j}} \psi_j)\left(t, \frac{x}{t}\right) + O\left(\varepsilon^2 t^{-\frac{3}{2}+\theta}\right)$$

for $j = 1, 2$ *as* $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}^2$, *where* $\theta \in \left(0, \frac{1}{2}\right)$.

We note that the asymptotic behavior of solutions to the Cauchy problem (3.4.4) is defined by the amplitude $|\Psi_j(\xi)|$ and the angular $\arg \Psi_j(\xi)$ which depend on the global solution (v_1, v_2) (see the proof of Lemma 3.4.2 below). However we can not find the exact representations of $\Psi_1(\xi)$ and $\Psi_2(\xi)$ for the Cauchy problem (3.4.4) and so it is difficult to find an exact asymptotic behavior of solutions to System (3.4.3) with initial data $v_j(1) = \phi_j(x)$ for $j = 1, 2$. Below in Section 3.5 we will show that the asymptotic behavior of solutions to the Cauchy problem (3.4.4) is divided into three cases (asymptotically free, asymptotically free with a phase modification, and a periodic in time oscillation of the amplitude of the solution). If we concentrate our attention to the final state problem for System (3.4.3), it is possible to find an exact asymptotic behavior of solutions to System (3.4.3) in the neighborhood of a special solution to the Cauchy problem (3.4.4). It is also interesting to consider System (3.4.1) without Condition (3.4.2). In this case, it is expected that System (3.4.1) has dissipative property under some angular conditions. We will discuss these problems in Chapter 4.

Define the norm

$$\|v\|_X = \sup_{t\geq 1} \sum_{j=1}^{2} \left((1+t) \|v_j(t)\|_{\mathbf{L}^\infty} + \|v_j(t)\| + (1+t)^{-\theta} \left\| J^{\alpha}_{\frac{1}{m_j}} v_j \right\| \right),$$

where $|\alpha| = 2$ and $\theta > 0$ is small.

By Theorem 2.1.3, there exists a small $\varepsilon > 0$ such that System (3.4.3) with initial data $v_j(1) = \phi_j(x)$ for $j = 1, 2$ has the form

$$\left\{ \begin{array}{c} \frac{d}{dt}\varphi_1 = t^{-1}\overline{\varphi_1}\varphi_2 + O\left(\varepsilon^2 t^{-1-\beta}\right), t > 1, \\ \frac{d}{dt}\varphi_2 = -t^{-1}\varphi_1^2 + O\left(\varepsilon^2 t^{-1-\beta}\right), t > 1, \\ \varphi_1(1) = \Phi_1, \varphi_2(1) = \Phi_2, \end{array} \right. \tag{3.4.5}$$

for all $\sum_{j=1}^{2} \left(\|\phi_j\|_{\mathbf{H}^2} + \|\phi_j\|_{\mathbf{H}^{0,2}}\right) < \varepsilon$, where $\varphi_j = -\mathcal{M}_{\frac{1}{m_j}} E^{m_j} D_{\frac{1}{t}} v_j$, $\beta = \frac{1}{2} - \theta$ and $\theta \in \left(0, \frac{1}{2}\right)$. We now compare the asymptotic behavior of solutions to the Cauchy problem (3.4.5) with the large time asymptotics of solutions to the unperturbed system of the ordinary differential equations depending on $\xi \in \mathbb{R}^2$ as a parameter

$$\left\{ \begin{array}{c} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, t > 1, \\ \frac{d}{dt}\psi_2 = -t^{-1}\psi_1^2, t > 1, \\ \psi_1(1) = \Psi_1, \psi_2(1) = \Psi_2. \end{array} \right. \tag{3.4.6}$$

The unperturbed system (3.4.6) will be studied in Section 3.5.

In this section we prove the following result.

Lemma 3.4.2. *Let the initial data for System (3.4.5) be such that*

$$\sum_{j=1}^{2} \left\|\varphi_j(1)\right\|_{\mathbf{L}^\infty} \leq \varepsilon,$$

where $\varepsilon > 0$ *is small enough. Then there exist initial data* $\Psi_1(\xi)$ *and* $\Psi_2(\xi)$ *for the Cauchy problem (3.4.6) such that the asymptotics is true*

$$\varphi_j(t) = \psi_j(t) + O\left(\varepsilon^2 t^{-\beta}\right), \ j = 1, 2$$

for large time $t \to \infty$ *uniformly with respect to* $\xi \in \mathbb{R}^2$.

PROOF. Let $(\varphi_1(t), \varphi_2(t))$ be a given solution of System (3.4.5). Denote $\psi_j^{(0)}(t) \equiv \varphi_j(t)$ and define $\psi_j^{(n)}(t)$ for $n \geq 1$ as a solution of the linearized version of System (3.4.6)

$$\left\{ \begin{array}{l} \frac{d}{dt}\psi_1^{(n)} = t^{-1}\overline{\psi_1^{(n-1)}}\psi_2^{(n-1)}, \ t > 1, \\ \frac{d}{dt}\psi_2^{(n)} = -t^{-1}\left(\psi_1^{(n-1)}\right)^2, \ t > 1, \end{array} \right. \tag{3.4.7}$$

with a final state condition

$$\lim_{t\to\infty} \left\| \varphi_j(t) - \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} = 0.$$

Let us prove by induction that

$$\sup_{t\geq 1} t^\beta \sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2 \tag{3.4.8}$$

for all $n \geq 0$. For $n = 0$ the estimate (3.4.8) is true. Next by induction we assume that (3.4.8) is fulfilled for all $0 \leq n \leq k$ and consider System (3.4.7) for $n = k+1$. We already know by Lemma 2.3.6 that

$$\sup_{t\geq 1} \sum_{j=1}^{2} \left\| \varphi_j(t) \right\|_{\mathbf{L}^\infty} \leq \varepsilon.$$

Therefore also we have $\left\| \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon$ for all $0 \leq n \leq k$.
By System (3.4.5) and System (3.4.7) we find for $n = k+1$ and $t \geq 1$

$$\begin{aligned} & \sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j^{(n)}(t) \right\|_{\mathbf{L}^\infty} \\ \leq \ & C\varepsilon \sum_{j=1}^{2} \int_t^\infty \left\| \varphi_j(\tau) - \psi_j^{(n-1)}(\tau) \right\|_{\mathbf{L}^\infty} \frac{d\tau}{\tau} + O\left(\varepsilon^2 t^{-\beta}\right) \\ \leq \ & C\varepsilon^2 t^{-\beta} \end{aligned}$$

from which the estimate (3.4.8) with $n = k+1$ follows. Thus by induction the estimate (3.4.8) is valid for all $n \geq 0$.
In the same way as above we find the estimate

$$\sup_{t\geq 1} t^\beta \sum_{j=1}^{2} \left\| \psi_j^{(n)}(t) - \psi_j^{(n-1)}(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2$$

for all $n \geq 1$, which implies by the usual contraction mapping principle that there exists a unique solution $(\psi_1(t), \psi_2(t)) \in \mathbf{C}^1([1,\infty); \mathbf{L}^\infty) \cap \mathbf{C}([1,\infty); \mathbf{L}^\infty)$ of the follwing system

$$\left\{ \begin{array}{ll} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, & t > 1, \\ \frac{d}{dt}\psi_2 = -t^{-1}(\psi_1)^2, & t > 1 \end{array} \right.$$

satisfying

$$\sup_{t\geq 1} t^\beta \sum_{j=1}^{2} \left\| \varphi_j(t) - \psi_j(t) \right\|_{\mathbf{L}^\infty} \leq C\varepsilon^2.$$

Lemma 3.4.2 is proved. ∎

Application of Lemma 3.4.2 to System (3.4.3) yields the result of Theorem 3.4.1.

3.5. Appendix

This is a joint work with Professor Nakao Hayashi and Professor Pavel I. Naumkin. We consider the following Cauchy problem

$$\left\{\begin{array}{l} \frac{d}{dt}\psi_1 = t^{-1}\overline{\psi_1}\psi_2, \quad t>1, \\ \frac{d}{dt}\psi_2 = -t^{-1}\psi_1^2, \quad t>1, \\ \psi_1(1)=\Psi_1, \quad \psi_2(1)=\Psi_2, \end{array}\right. \tag{3.5.1}$$

$(t,x)\in\mathbb{R}\times\mathbb{R}^2$. We change $t=e^{t'}$ to exclude the explicit dependence on t in System (3.5.1) (we omit the prime of t'). Then we substitute $\psi_1(t)=r(t)e^{i\phi(t)}$ and $\psi_2(t)=\rho(t)e^{i\theta(t)}$, to get from (3.5.1)

$$\left\{\begin{array}{l} r' + ir\phi' = r\rho e^{i(\theta-2\phi)}, \quad t>0, \\ \rho' + i\rho\theta' = -r^2 e^{i(2\phi-\theta)}, \quad t>0. \end{array}\right.$$

We also denote $\alpha=\theta-2\phi$, then

$$\left\{\begin{array}{l} r' = r\rho\cos\alpha, \\ \rho' = -r^2\cos\alpha, \\ \alpha' = \frac{r^2-2\rho^2}{\rho}\sin\alpha, \\ \phi' = \rho\sin\alpha. \end{array}\right. \tag{3.5.2}$$

By the first two equations of System (3.5.2) we have $\frac{d}{dt}\left(r^2+\rho^2\right)=0$, therefore

$$r^2(t)+\rho^2(t) = |\Psi_1|^2+|\Psi_2|^2$$

for all $t>0$. Then by the first three equations of System (3.5.2) we find $\frac{d}{dt}\left(\rho r^2\sin\alpha\right)=0$. Therefore

$$\rho(t)r^2(t)\sin\alpha(t) = |\Psi_2||\Psi_1|^2\sin(\arg\Psi_2-2\arg\Psi_1)$$

for all $t>0$. Denote $z=r^2-2\rho^2=3r^2-2b$, $b=|\Psi_1|^2+|\Psi_2|^2$. Then by System (3.5.2) we see that $z'=6\rho r^2\cos\alpha$ and $z''=-6r^2z=-2(z+2b)z$. We can exclude the constant b from the equation if we make a change $z(t)=b\widetilde{z}\left(\widetilde{t}\right), r(t)=\sqrt{b}\widetilde{r}\left(\widetilde{t}\right), \rho(t)=\sqrt{b}\widetilde{z}\left(\widetilde{t}\right), \sqrt{b}t=\widetilde{t}$. Hence we get (we omit the tilde of $\widetilde{t}$ and $\widetilde{z}$)

$$\left\{\begin{array}{l} \frac{d^2z}{dt^2} = -2(z+2)z, \\ z(0) = r^2(0)-2\rho^2(0), \\ z'(0) = 6\rho(0)r^2(0)\cos\alpha(0). \end{array}\right. \tag{3.5.3}$$

Note that $-2\le z(t)\le 1$. Multiplying (3.5.3) by $\frac{dz}{dt}$ and integrating we get

$$\left(\frac{dz}{dt}\right)^2+4z^2+\frac{4}{3}z^3 = C.$$

Since $\rho=\sqrt{\frac{1-z}{3}}$, $r^2=\frac{z+2}{3}$, and $z'=6\rho r^2\cos\alpha$, we have $C=\frac{4}{3}(1-z)(z+2)^2\cos^2\alpha+4z^2+\frac{4}{3}z^3$. Therefore we can see that $0\le C\le\frac{16}{3}$ for $-2\le z\le 1$. Denote the function $f(z)=C-4z^2-\frac{4}{3}z^3$ for $-2\le z\le 1$. Then we have the equation

$$\left(\frac{dz}{dt}\right)^2 = f(z).$$

Consider the function $f(z) = C - 4z^2 - \frac{4}{3}z^3$ for $-2 \leq z \leq 1$. It has the local maximum $f(z) = C$ at $z = 0$ and the local minimum $f(z) = C - \frac{16}{3}$ at $z = -2$. Therefore if $0 < C < \frac{16}{3}$, then there are three roots $z_1 < -2 < z_2 < 0 < z_3 < 1$ of equation $f(z) = 0$ such that $f(z) \geq 0$ for all $z_2 \leq z \leq z_3$. Note that the initial data satisfies the inequality $z_2 \leq z(0) \leq z_3$ since $f(z(0)) = (z'(0))^2 \geq 0$. Therefore the solution $z(t)$ is always periodic $z_2 \leq z(t) \leq z_3$ with a period $T = \int_{z_2}^{z_3} \frac{dz}{\sqrt{f(z)}}$. For the exceptional cases $C = 0$ and $C = \frac{16}{3}$ there are two equilibrium points $z = -2$ for $C = 0$ (i.e. $\psi_1 = \Psi_1 = 0,\ \psi_2 = \Psi_2$ in System (3.5.1)) and $z = 0$ for $C = \frac{16}{3}$ (i.e. $\psi_1 = \Psi_1 e^{\frac{i}{\sqrt{2}}|\Psi_1| \log(1+t)},\ \psi_2 = \frac{i}{\sqrt{2}|\Psi_1|}\Psi_1^2 e^{\frac{2i}{\sqrt{2}}|\Psi_1| \log(1+t)}$ in System (3.5.1).

CHAPTER 4

Modified wave operators and more about wave operators

4.1. Introduction

This chapter is a joint work with Professor Nakao Hayashi and Professor Pavel I. Naumkin. We consider the system of nonlinear Schrödinger equations as follows:

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \gamma \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \end{cases} \tag{4.1.1}$$

in $\mathbb{R}\times\mathbb{R}^2$, where $\Delta = \sum_{j=1}^{2}\partial_j^2$ is the laplacian, $\partial_j = \partial/\partial x_j$, m_1, m_2 are the masses of particles and $\gamma \in \mathbb{C}$, $|\gamma| = 1$. A little bit more general system

$$\begin{cases} i\partial_t v_1 + \frac{1}{2m_1}\Delta v_1 = \lambda \overline{v_1} v_2, \\ i\partial_t v_2 + \frac{1}{2m_2}\Delta v_2 = \mu v_1^2, \end{cases}$$

with $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ can be easily reduced to System (4.1.1) with $\gamma = \frac{\lambda\mu}{|\lambda\mu|}$ by virtue of the scaling $v_1 = \frac{1}{\sqrt{|\lambda\mu|}}u_1$ and $v_2 = \frac{\mu}{|\lambda\mu|}u_2$. When $\gamma = 1$, System (4.1.1) obeys the $\mathbf{L}^2$ conservation law

$$\partial_t \left(\|u_1\|^2 + \|u_2\|^2 \right) = 0.$$

System (4.1.1) can be considered as the non relativistic version of the system of nonlinear Klein-Gordon equations

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 v_1 - \frac{1}{2m_1}\Delta v_1 + \frac{m_1 c^2}{2} v_1 = -\gamma \overline{v_1} v_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 v_2 - \frac{1}{2m_2}\Delta v_2 + \frac{m_2 c^2}{2} v_2 = -v_1^2, \end{cases} \tag{4.1.2}$$

in $\mathbb{R}\times\mathbb{R}^2$, where c is the speed of light. Indeed we change $v_j = e^{-itm_j c^2} u_j$ to get the following equations

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 u_1 - i\partial_t u_1 - \frac{1}{2m_1}\Delta u_1 = -\gamma e^{itc^2(2m_1 - m_2)} \overline{u_1} u_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 u_2 - i\partial_t u_2 - \frac{1}{2m_2}\Delta u_2 = -e^{itc^2(m_2 - 2m_1)} u_1^2. \end{cases}$$

Under the resonance condition

$$2m_1 = m_2, \tag{4.1.3}$$

we find the system

$$\begin{cases} \frac{1}{2c^2 m_1}\partial_t^2 u_1 - i\partial_t u_1 - \frac{1}{2m_1}\Delta u_1 = -\gamma \overline{u_1} u_2, \\ \frac{1}{2c^2 m_2}\partial_t^2 u_2 - i\partial_t u_2 - \frac{1}{2m_2}\Delta u_2 = -u_1^2, \end{cases}$$

which converts into System (4.1.1) in the limit $c \to \infty$. Equation (4.1.2) is closely related to a system of nonlinear Klein-Gordon equations studied in [**6**], [**16**], [**21**] and a system of Dirac-Klein-Gordon equations studied in [**1**], [**9**]. In these papers, the case of three space dimensions was studied. On the other hand, these systems in two space dimensions are critical in general. For example, small solutions of the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|\, u, \ \ (t,x) \in \mathbb{R} \times \mathbb{R}^2$$

do not behave like free solutions as $t \to \infty$ (see [**2**]). Furthermore, the exact asymptotic behavior of small solutions was obtained in [**13**] for the initial value problem and in [**7**] for the final state problem.

Global existence of small solutions to the Cauchy problem for System (4.1.2) with a non resonance mass condition

$$2m_1 \neq m_2, m_1 \neq m_2, \tag{4.1.4}$$

was proved in [**38**] for the case of the initial data in a weighted Sobolev space. Also it was shown in [**38**] that solutions behave asymptotically like free solutions. Global existence and time decay estimates were obtained in [**23**] for the case of small solutions of the Cauchy problem to System (4.1.2) with the resonance mass condition (4.1.3), when the data belongs to a Sobolev space and has a compact support. As far as we know the large time asymptotic behavior of solutions for System (4.1.1) or System (4.1.2) under Condition (4.1.3) is not known.

In Chapter 2, the time decay of solutions of the Cauchy problem of System (4.1.1) with $\gamma = 1$ is studied under the resonance mass condition (4.1.3). In the case of higher dimensions the small data scattering problem for System (4.1.1) is considered by Professor Nakao Hayashi, Professor Tohru Ozawa and myself (see [**12**]). However, the large time asymptotic behavior of solutions of System (4.1.1) to the Cauchy problems is not known as stated before.

To study the wave operator for the system of nonlinear Schrödinger equations (4.1.1) we consider the final state problem

$$\begin{cases} i\partial_t u_1 + \frac{1}{2m_1}\Delta u_1 = \gamma \overline{u_1} u_2, \\ i\partial_t u_2 + \frac{1}{2m_2}\Delta u_2 = u_1^2, \\ \|u_1(t) - F_{1,S}(t)\| \to 0 \text{ as } t \to \infty, \\ \|u_2(t) - F_{2,S}(t)\| \to 0 \text{ as } t \to \infty, \end{cases}$$

where the final state $(F_{1,S}(t), F_{2,S}(t))$ is defined by some final data $\left(\phi_{1+}, \phi_{2+}\right)$. If the final state $(F_{1,S}(t), F_{2,S}(t))$ can be taken in the form $(F_{1,S}(t), F_{2,S}(t)) = \left(U_{\frac{1}{m_1}}(t)\phi_{1+}, U_{\frac{1}{m_2}}(t)\phi_{2+}\right)$, where $U_{\frac{1}{m_j}}(t) = e^{\frac{it}{2m_j}\Delta}$ for $1 \leq j \leq 2$ are free Schrödinger evolution group, and the final state problem has a nontrivial solution, then we say that there exists a usual wave operator. However, when the nonlinearity is critical and it is impossible to find a solution in the neighborhood of the free final state $\left(U_{\frac{1}{m_1}}(t)\phi_{1+}, U_{\frac{1}{m_2}}(t)\phi_{2+}\right)$, then we need to modify the time dependence of the final state. Note that the modified wave operator for nonlinear dispersive equation was first constructed in [**31**] for the cubic nonlinear Schrödinger equations and then in [**19**] for the derivative nonlinear Schrödinger equation, by changing it via a suitable transformation (see [**8**]) to a system of cubic nonlinear Schrödinger equations without derivatives of unknown function.

In this chapter, we construct the modified wave operators for the final state problem for System (4.1.1) with $\gamma = \pm 1$ under the resonance mass condition (4.1.3). Moreover we will prove existence of the usual wave operators under the non resonance mass condition (4.1.4) and also under the resonance mass condition $m_1 = m_2$.

The non resonance condition (4.1.4) corresponds to non self-conjugate condition for the power nonlinearity in a single nonlinear Schrödinger equation (see [**14**]). We follow here the method of [**14**] (for the cubic nonlinear Klein-Gordon equation, see [**15**]).

We use the following factorization formula (see [**18**]) for the free Schrödinger evolution group

$$U_{\frac{1}{m}}(t) = M^{-m} D_{\frac{t}{m}} \mathcal{M}_{-m} \mathcal{F} \tag{4.1.5}$$

with $M = e^{-\frac{i}{2t}|x|^2}$, the dilation operator $(D_t\phi)(x) = \frac{1}{it}\phi\left(\frac{x}{t}\right)$ and $\mathcal{M}_m = \mathcal{F} M^m \mathcal{F}^{-1}$. We also have

$$\mathcal{F} U_{\frac{1}{m}}(-t) = -\mathcal{M}_m E^{\frac{1}{m}} D_{\frac{m}{t}}, \tag{4.1.6}$$

where $E = e^{-\frac{it}{2}|\xi|^2}$. Here we have used the commutation identity

$$(D_{\frac{m}{t}} M^m \phi)(x) = (E^{\frac{1}{m}} D_{\frac{m}{t}} \phi)(x).$$

Now we state the main results of this chapter. First we consider System (4.1.1) with $\gamma = \pm 1$ in the case of the resonance mass condition (4.1.3). Consider the system of ordinary differential equations depending on $\xi \in \mathbb{R}^2$ as a parameter

$$\begin{cases} i\partial_t \varphi_{1\gamma} = \gamma t^{-1} \overline{\varphi_{1\gamma}} \varphi_{2\gamma}, \\ i\partial_t \varphi_{2\gamma} = t^{-1} \varphi_{1\gamma}^2. \end{cases} \tag{4.1.7}$$

It is known from Section 3.4 that the solutions of the system of ordinary differential equations (4.1.7) define the asymptotic behavior of solutions of System (4.1.1) under the resonance condition (4.1.3). Indeed if we multiply both sides of (4.1.7) by

$$\left(-U_{\frac{1}{m_1}}(t)\,\mathcal{F}^{-1} D_{m_1}, -U_{\frac{1}{m_2}}(t)\,\mathcal{F}^{-1} D_{m_2}\right),$$

then we can find that

$$\left(-U_{\frac{1}{m_1}}(t)\,\mathcal{F}^{-1} D_{m_1}\varphi_{1\gamma}, -U_{\frac{1}{m_2}}(t)\,\mathcal{F}^{-1} D_{m_2}\varphi_{2\gamma}\right)$$

is an approximate solution of System (4.1.1) under the resonance condition (4.1.3). If we replace the Schrödinger group by the Klein-Gordon group, the wave group, the Airy group, etc., we could expect that our method is also applicable to other types of dispersive equations (see Remark 4.1.6 below). By a direct calculation we can see System (4.1.7) has the following particular solutions

$$\varphi_{1\gamma}(t,\xi) = -\frac{i\omega(\xi)\, e^{i\theta(\xi)}}{1+\omega(\xi)\log t},\ \varphi_{2\gamma}(t,\xi) = -\frac{i\omega(\xi)\, e^{2i\theta(\xi)}}{1+\omega(\xi)\log t} \tag{4.1.8}$$

for the case of $\gamma = -1$ and

$$\begin{aligned} \varphi_{1\gamma}(t,\xi) &= \omega(\xi)\, e^{i\theta(\xi) + \frac{i}{\sqrt{2}}\omega(\xi)\log t} \\ \varphi_{2\gamma}(t,\xi) &= -\frac{1}{\sqrt{2}}\omega(\xi)\, e^{2i\theta(\xi) + i\sqrt{2}\omega(\xi)\log t} \end{aligned} \tag{4.1.9}$$

for the case of $\gamma = 1$, where $\omega(\xi) > 0$ and $\theta(\xi)$ is a real valued given function. Also we note that the time independent functions

$$\varphi_{1\gamma}(\xi) = 0, \varphi_{2\gamma}(\xi) = \omega(\xi)\, e^{i\theta(\xi)}$$

satisfy System (4.1.7) for both cases $\gamma = \pm 1$. This special solution implies the necessity of the condition $\varphi_{1\gamma}(\xi) \neq 0$ in the proof of nonexistence of the usual scattering states in [**10**].

Theorem 4.1.1. *Let $2m_1 = m_2$ and $\gamma = \pm 1$. Then there exists an $\varepsilon > 0$ such that for any $\omega, \theta \in \mathbf{H}^2$ with norm $\|\omega\|_{\mathbf{H}^2} \leq \varepsilon$, System (4.1.1) has a unique global solution*

$$(u_1, u_2) \in \mathbf{C}\left([1, \infty); \mathbf{L}^2\right).$$

Moreover, the following estimate

$$\sum_{j=1}^{2} \left\| u_j(t) + U_{\frac{1}{m_j}}(t) \mathcal{F}^{-1} D_{m_j} \varphi_{j\gamma}(t) \right\| \leq Ct^{-b}$$

holds for all $t \geq 1$, where $\frac{1}{2} < b < 1$.

Remark 4.1.2. *By using the identity $U_{\frac{1}{m}}(t) = M^{-m} D_{\frac{t}{m}} \mathcal{F} M^{-m}$ and the estimates of Theorem 4.1.1 we can write the estimate of the above theorem as follows:*

$$\sum_{j=1}^{2} \left\| u_j(t) - iM^{-m_j} D_t \mathcal{M}_{-\frac{1}{m_j}} \varphi_{j\gamma}(t) \right\| \leq Ct^{-b}.$$

Remark 4.1.3. *In the case of $\varphi_{1\gamma}(t, \xi) = 0, \varphi_{2\gamma}(t, \xi) = \omega(\xi) e^{i\theta(\xi)}$ we do not need any phase modification. For this case the solutions can be found directly from the integral equations associated to System (4.1.1) without any mass condition.*

We now consider System (4.1.1) in the case of the non resonance condition $2m_1 \neq m_2$ and $m_1 \neq m_2$.

Theorem 4.1.4. *Let $2m_1 \neq m_2$ and $m_1 \neq m_2$. Then there exists an $\varepsilon > 0$ such that for any $\phi_{1+} \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}$, $\phi_{2+} \in \mathbf{H}^{0,2}$ with the norm*

$$\left\|\phi_{1+}\right\|_{\mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}} + \left\|\phi_{2+}\right\|_{\mathbf{H}^{0,2}} \leq \varepsilon,$$

System (4.1.1) has a unique global solution

$$(u_1, u_2) \in \mathbf{C}\left([1, \infty); \mathbf{L}^2\right).$$

Moreover, the following estimate

$$\left\| u_1(t) - U_{\frac{1}{m_1}}(t) \phi_{1+} \right\| + \left\| u_2(t) - U_{\frac{1}{m_2}}(t) \phi_{2+} \right\| \leq Ct^{-b}$$

holds for all $t \geq 1$, where $\frac{1}{2} < b < 1$.

Finally we consider System (4.1.1) in the resonance case $m_1 = m_2$ under the support conditions on the data.

Theorem 4.1.5. *Let $m_1 = m_2$. Assume that $\phi_{1+} \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}$, $\phi_{2+} \in \mathbf{H}^{\frac{1}{2},2}$ and*

$$\text{supp } \widehat{\phi_{1+}} \cap \text{supp } \widehat{\phi_{2+}} \text{ is empty.}$$

Then there exists an $\varepsilon > 0$ such that for any $\left(\phi_{1+}, \phi_{2+}\right)$ with the norm

$$\left\|\phi_{1+}\right\|_{\mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}} + \left\|\phi_{2+}\right\|_{\mathbf{H}^{0,2}} \leq \varepsilon,$$

there exists a unique solution

$$(u_1, u_2) \in \mathbf{C}\left([1, \infty); \mathbf{L}^2\right)$$

for System (4.1.1) satisfying the estimate

$$\left\| u_1(t) - U_{\frac{1}{m_1}}(t) \phi_{1+} \right\| + \left\| u_2(t) - U_{\frac{1}{m_2}}(t)\phi_{2+} \right\| \leq Ct^{-b}$$

for all $t \geq 1$, where $\frac{1}{2} < b < \frac{3}{4}$.

We will give proofs of these theorems in Section 4.3.

REMARK 4.1.6. *We expect that similar results hold for a system of nonlinear Klein-Gordon equations*

$$\begin{cases} \left(i\partial_t - c\left\langle i\nabla\right\rangle_{m_1c}\right) u_1 = \gamma \overline{u_1} u_2, \\ \left(i\partial_t - c\left\langle i\nabla\right\rangle_{m_2c}\right) u_2 = u_1^2, \end{cases} \tag{4.1.10}$$

where $\left\langle i\nabla\right\rangle_{m_jc} = \left(m_j^2c^2 - \Delta\right)^{\frac{1}{2}}, j = 1,2$. *We denote* $\mathcal{U}_{KG,m_jc}(t) = e^{-ict\left\langle i\nabla\right\rangle_{m_jc}}, j = 1,2$. *Note that the solutions of the linear Klein-Gordon equations* $\mathcal{U}_{KG,m_jc}(t)\phi$ *for* $j = 1,2$ *asymptotically behave as*

$$\frac{m_j}{it\left(1-\left(\frac{x}{tc}\right)^2\right)} e^{-itm_jc^2\sqrt{1-\left(\frac{x}{tc}\right)^2}} \widehat{\phi}\left(\frac{m_jx}{t\sqrt{1-\left(\frac{x}{tc}\right)^2}}\right)$$

inside of the light cone $|x| < tc$. *Therefore from our results the following asymptotic behavior could be conjectured*

$$\begin{aligned} &\left\| u_1(t) - \frac{e^{-itm_1c^2\sqrt{1-\left(\frac{x}{tc}\right)^2}}}{it\left(1-\left(\frac{x}{tc}\right)^2\right)} \varphi_{1\gamma}\left(t, \frac{x}{t}\left(1-\left(\frac{x}{tc}\right)^2\right)^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^2(|x|<ct)} \\ &+ \left\| u_2(t) - \frac{e^{-itm_2c^2\sqrt{1-\left(\frac{x}{tc}\right)^2}}}{it\left(1-\left(\frac{x}{tc}\right)^2\right)} \varphi_{2\gamma}\left(t, \frac{x}{t}\left(1-\left(\frac{x}{tc}\right)^2\right)^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^2(|x|<ct)} \\ &+ \left\| u_1(t)\right\|_{\mathbf{L}^2(|x|\geq ct)} + \left\| u_2(t)\right\|_{\mathbf{L}^2(|x|\geq ct)} \leq Ct^{-b}. \end{aligned}$$

under the mass condition (4.1.3).

4.2. Preliminary estimates

For Theorems 4.1.1, 4.1.4 and 4.1.5 we denote $v_{j\gamma}(t) = -U_{\frac{1}{m_j}}(t)\mathcal{F}^{-1}D_{m_j}\varphi_{j\gamma}(t)$, where $\varphi_{j\gamma}$ are defined by (4.1.8) - (4.1.9), $j = 1,2$ and $\gamma = \pm 1$. Multiplying both sides of System (4.1.7) by

$$\left(-U_{\frac{1}{m_1}}(t)\mathcal{F}^{-1}D_{m_1}, -U_{\frac{1}{m_2}}(t)\mathcal{F}^{-1}D_{m_2}\right)$$

via the identity $U_{\frac{1}{m_j}}(t)i\partial_t = \mathcal{L}_{m_j}U_{\frac{1}{m_j}}(t)$ with $\mathcal{L}_{m_j} = i\partial_t + \frac{1}{2m_j}\Delta$, we obtain

$$\begin{cases} \mathcal{L}_{m_1}v_{1\gamma} = \gamma\overline{v_{1\gamma}}v_{2\gamma} + \gamma R_1, \\ \mathcal{L}_{m_2}v_{2\gamma} = v_{1\gamma}^2 + R_2, \end{cases} \tag{4.2.1}$$

where we denote the remainder terms

$$\begin{aligned} R_1 &= -t^{-1}U_{\frac{1}{m_1}}(t)\mathcal{F}^{-1}D_{m_1}\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right) - \overline{v_{1\gamma}}v_{2\gamma}, \\ R_2 &= -t^{-1}U_{\frac{1}{m_2}}(t)\mathcal{F}^{-1}D_{m_2}\left(\varphi_{1\gamma}^2\right) - v_{1\gamma}^2. \end{aligned}$$

Next we estimate the remainder terms R_1 and R_2.

Lemma 4.2.1. *Let $j=1,2$ and $2m_1=m_2$. If $\omega,\theta\in\mathbf{H}^2$, then the following estimates are true for all $t\geq 1$*

$$\|R_j(t)\|\leq Ct^{-2}\left(1+\|\omega\|_{\mathbf{H}^2}^2\log^2 t\right)^2\|\omega\|_{\mathbf{H}^2}^2\left(1+\|\theta\|_{\mathbf{H}^2}^2\right)^2.$$

Proof. By the factorization formula (4.1.5), the identities $D_\alpha\mathcal{F}^{-1}=\mathcal{F}^{-1}D_{\frac{1}{\alpha}}$ and $D_\alpha\mathcal{M}_{-\frac{1}{m}}=\mathcal{M}_{-\frac{\alpha^2}{m}}D_\alpha$ we get

$$\begin{aligned}
&t^{-1}U_{\frac{1}{m_1}}(t)\,\mathcal{F}^{-1}D_{m_1}\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)\\
=\;&t^{-1}M^{-m_1}D_{\frac{t}{m_1}}\mathcal{M}_{-m_1}D_{m_1}\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)\\
=\;&\frac{1}{it}M^{-m_1}D_t\mathcal{M}_{-\frac{1}{m_1}}\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)=\frac{1}{it}M^{-m_1}D_t\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)+R_3,
\end{aligned}$$

where the remainder term

$$R_3=\frac{1}{it}M^{-m_1}D_t\left(\mathcal{M}_{-\frac{1}{m_1}}-1\right)\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right).$$

Also by the factorization formula (4.1.5) we have

$$v_{j\gamma}(t)=-U_{\frac{1}{m_j}}(t)\,\mathcal{F}^{-1}D_{m_j}\varphi_{j\gamma}(t)=iM^{-m_j}D_t\mathcal{M}_{-\frac{1}{m_j}}\varphi_{j\gamma}(t)$$

for $j=1,2$ and $\gamma=\pm 1$. Then since $2m_1=m_2$ we find

$$\begin{aligned}
&\frac{1}{it}M^{-m_1}D_t\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)=-M^{-m_1}\left(\overline{D_t\varphi_{1\gamma}}\right)\left(D_t\varphi_{2\gamma}\right)\\
=\;&-\left(\overline{M^{-m_1}D_t\varphi_{1\gamma}}\right)\left(M^{-m_2}D_t\varphi_{2\gamma}\right)\\
=\;&-\left(\overline{M^{-m_1}D_t\mathcal{M}_{-\frac{1}{m_1}}\varphi_{1\gamma}}\right)\left(M^{-m_2}D_t\mathcal{M}_{-\frac{1}{m_2}}\varphi_{2\gamma}\right)+R_4\\
=\;&-\overline{v_{1\gamma}}v_{2\gamma}+R_4,
\end{aligned}$$

where the remainder term

$$\begin{aligned}
R_4\;=\;&\left(\overline{M^{-m_1}D_t\mathcal{M}_{-\frac{1}{m_1}}\varphi_{1\gamma}}\right)\left(M^{-m_2}D_t\mathcal{M}_{-\frac{1}{m_2}}\varphi_{2\gamma}\right)\\
&-\left(\overline{M^{-m_1}D_t\varphi_{1\gamma}}\right)\left(M^{-m_2}D_t\varphi_{2\gamma}\right).
\end{aligned}$$

Thus we obtain the representation

$$\begin{aligned}
R_1\;=\;&-t^{-1}U_{\frac{1}{m_1}}(t)\,\mathcal{F}^{-1}D_{m_1}\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)-\overline{v_{1\gamma}}v_{2\gamma}\\
=\;&-\frac{1}{it}M^{-m_1}D_t\left(\overline{\varphi_{1\gamma}}\varphi_{2\gamma}\right)-\overline{v_{1\gamma}}v_{2\gamma}-R_3=-R_3-R_4.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&t^{-1}U_{\frac{1}{m_2}}(t)\,\mathcal{F}^{-1}D_{m_2}\left(\varphi_{1\gamma}^2\right)\\
=\;&t^{-1}M^{-m_2}D_{\frac{t}{m_2}}\mathcal{M}_{-m_2}D_{m_2}\left(\varphi_{1\gamma}^2\right)\\
=\;&\frac{1}{it}M^{-m_2}D_t\mathcal{M}_{-\frac{1}{m_2}}\left(\varphi_{1\gamma}^2\right)=\frac{1}{it}M^{-m_2}D_t\left(\varphi_{1\gamma}^2\right)+R_5,
\end{aligned}$$

where the remainder term

$$R_5=\frac{1}{it}M^{-m_2}D_t\left(\mathcal{M}_{-\frac{1}{m_2}}-1\right)\left(\varphi_{1\gamma}^2\right).$$

Then since $2m_1 = m_2$ we find

$$\begin{aligned} & \frac{1}{it} M^{-m_2} D_t \left(\varphi_{1\gamma}^2\right) = M^{-m_2} \left(D_t \varphi_{1\gamma}\right)^2 \\ = & \left(M^{-m_1} D_t \varphi_{1\gamma}\right)^2 \\ = & \left(M^{-m_1} D_t \mathcal{M}_{-\frac{1}{m_1}} \varphi_{1\gamma}\right)^2 + R_6 = -v_{1\gamma}^2 + R_6, \end{aligned}$$

where the remainder term

$$R_6 = \left(M^{-m_1} D_t \varphi_{1\gamma}\right)^2 - \left(M^{-m_1} D_t \mathcal{M}_{-\frac{1}{m_1}} \varphi_{1\gamma}\right)^2.$$

Thus we get the following representation

$$\begin{aligned} R_2 &= -t^{-1} U_{\frac{1}{m_2}}(t) \mathcal{F}^{-1} D_{m_2} \left(\varphi_{1\gamma}^2\right) - v_{1\gamma}^2 \\ &= -\frac{1}{it} M^{-m_2} D_t \left(\varphi_{1\gamma}^2\right) - v_{1\gamma}^2 - R_5 = -R_5 - R_6. \end{aligned}$$

We now are in a position to estimate the remainder terms R_1 and R_2. By the definition of the dilation operator D_t we have

$$\|D_t \psi\| = \|\psi\|$$

and

$$\|\partial_j D_t \psi\| \le C t^{-1} \|\partial_j \psi\|.$$

Therefore we obtain by the Sobolev imbedding theorem

$$\begin{aligned} & \|R_3(t)\| \\ \le & C t^{-1} \left\| \left(\mathcal{M}_{-\frac{1}{m_1}} - 1\right) \left(\overline{\varphi_{1\gamma}} \varphi_{2\gamma}\right) \right\| \\ = & C t^{-1} \left\| \left(M^{-\frac{1}{m_1}} - 1\right) \mathcal{F}^{-1} \left(\overline{\varphi_{1\gamma}} \varphi_{2\gamma}\right) \right\| \\ \le & C t^{-2} \left\| \Delta \left(\overline{\varphi_{1\gamma}} \varphi_{2\gamma}\right) \right\| \\ \le & C t^{-2} \left\|\varphi_{1\gamma}\right\|_{\mathbf{H}^2} \left\|\varphi_{2\gamma}\right\|_{\mathbf{H}^2} \\ \le & C t^{-2} \left(1 + \|\omega\|_{\mathbf{H}^2}^2 \log^2 t\right)^2 \|\omega\|_{\mathbf{H}^2}^2 \left(1 + \|\theta\|_{\mathbf{H}^2}^2\right)^2 \end{aligned}$$

for all $t \ge 1$.
In the same manner

$$\begin{aligned} & \|R_4(t)\| \\ \le & C t^{-1} \left\| \left(\overline{\mathcal{M}_{-\frac{1}{m_1}} \varphi_{1\gamma}}\right) \left(\mathcal{M}_{-\frac{1}{m_2}} \varphi_{2\gamma}\right) - \overline{\varphi_{1\gamma}} \varphi_{2\gamma} \right\| \\ \le & C t^{-1} \left\| \left(\mathcal{M}_{-\frac{1}{m_1}} - 1\right) \varphi_{1\gamma} \right\| \left\| \mathcal{M}_{-\frac{1}{m_2}} \varphi_{2\gamma} \right\|_{\mathbf{L}^\infty} \\ & + C t^{-1} \left\| \left(\mathcal{M}_{-\frac{1}{m_2}} - 1\right) \varphi_{2\gamma} \right\| \left\|\varphi_{1\gamma}\right\|_{\mathbf{L}^\infty} \\ \le & C t^{-2} \left\|\varphi_{1\gamma}\right\|_{\mathbf{H}^2} \left\|\varphi_{2\gamma}\right\|_{\mathbf{H}^2} \\ \le & C t^{-2} \left(1 + \|\omega\|_{\mathbf{H}^2}^2 \log^2 t\right)^2 \|\omega\|_{\mathbf{H}^2}^2 \left(1 + \|\theta\|_{\mathbf{H}^2}^2\right)^2 \end{aligned}$$

for all $t \geq 1$.
Similarly,

$$\begin{aligned}
& \|R_5(t)\| \\
\leq \; & Ct^{-1}\left\|\left(\mathcal{M}_{-\frac{1}{m_2}}-1\right)\left(\varphi_{1\gamma}^2\right)\right\| \\
= \; & Ct^{-1}\left\|\left(M^{-\frac{1}{m_2}}-1\right)\mathcal{F}^{-1}\left(\varphi_{1\gamma}^2\right)\right\| \\
\leq \; & Ct^{-2}\left\|\Delta\left(\varphi_{1\gamma}^2\right)\right\| \\
\leq \; & Ct^{-2}\left(1+\|\omega\|_{\mathbf{H}^2}^2\log^2 t\right)^2\|\omega\|_{\mathbf{H}^2}^2\left(1+\|\theta\|_{\mathbf{H}^2}^2\right)^2
\end{aligned}$$

for all $t \geq 1$.
Analogously,

$$\begin{aligned}
& \|R_6(t)\| \\
\leq \; & Ct^{-1}\left\|\left(\mathcal{M}_{-\frac{1}{m_1}}\varphi_{1\gamma}\right)^2-\varphi_{1\gamma}^2\right\| \\
\leq \; & Ct^{-1}\left\|\left(\mathcal{M}_{-\frac{1}{m_1}}-1\right)\varphi_{1\gamma}\right\|\left(\left\|\mathcal{M}_{-\frac{1}{m_1}}\varphi_{1\gamma}\right\|_{\mathbf{L}^\infty}+\|\varphi_{1\gamma}\|_{\mathbf{L}^\infty}\right) \\
\leq \; & Ct^{-2}\|\varphi_{1\gamma}\|_{\mathbf{H}^2}^2 \\
\leq \; & Ct^{-2}\left(1+\|\omega\|_{\mathbf{H}^2}^2\log^2 t\right)^2\|\omega\|_{\mathbf{H}^2}^2\left(1+\|\theta\|_{\mathbf{H}^2}^2\right)^2
\end{aligned}$$

for all $t \geq 1$.
Therefore we have the result of the lemma. ∎

We denote $v_{j+} = U_{\frac{1}{m_j}}(t)\phi_{j+}$ with some given functions ϕ_{j+} for $j = 1, 2$. The following estimates will be used in the proof of Theorem 4.1.4.

Lemma 4.2.2. *Let $2m_1 \neq m_2$ and $m_1 \neq m_2$. We assume $\phi_{1+} \in \mathbf{H}^{0,2} \cap \dot{\mathbf{H}}^{-2b}$ and $\phi_{2+} \in \mathbf{H}^{0,2}$, where $b \in \left(\frac{1}{2}, 1\right)$. Then the estimates are true*

$$\left\|\int_t^\infty U_{\frac{1}{m_1}}(-\tau)\left(\overline{v_{1+}}v_{2+}\right)d\tau\right\| \leq Ct^{-b}\|\phi_{1+}\|_{\mathbf{H}^{0,2}\cap\dot{\mathbf{H}}^{-2b}}\|\phi_{2+}\|_{\mathbf{H}^{0,2}}$$

and

$$\left\|\int_t^\infty U_{\frac{1}{m_2}}(-\tau)\left(v_{1+}^2\right)d\tau\right\| \leq Ct^{-b}\|\phi_{1+}\|_{\mathbf{H}^{0,2}\cap\dot{\mathbf{H}}^{-2b}}\|\phi_{1+}\|_{\mathbf{H}^{0,2}}$$

for all $t \geq 1$.

Proof. By the factorization formula (4.1.5) we have $v_{j+} = M^{-m_j}D_{\frac{t}{m_j}}\mathcal{M}_{-m_j}\widehat{\phi_{j+}}$ for $j = 1, 2$. Then by the identities $D_\alpha\mathcal{M}_{-\frac{1}{m}} = \mathcal{M}_{-\frac{\alpha^2}{m}}D_\alpha$ and $D_\alpha D_\beta = -iD_{\alpha\beta}$, we find

$$\begin{aligned}
& D_{\frac{1}{m_1}}\mathcal{F}U_{\frac{1}{m_1}}(-t)\left(\overline{v_{1+}}v_{2+}\right) \\
= \; & i\mathcal{M}_{\frac{1}{m_1}}E^{m_1}(t)D_{\frac{1}{t}}\left(\overline{v_{1+}}v_{2+}\right) \\
= \; & t^{-1}\mathcal{M}_{\frac{1}{m_1}}E^{2m_1-m_2}\left(\overline{\mathcal{M}_{-\frac{1}{m_1}}D_{\frac{1}{m_1}}\widehat{\phi_{1+}}}\right)\left(\mathcal{M}_{-\frac{1}{m_2}}\mathcal{D}_{\frac{1}{m_2}}\widehat{\phi_{2+}}\right) \\
= \; & t^{-1}\mathcal{M}_{\frac{1}{m_1}}E^{2m_1-m_2}\left(\overline{D_{\frac{1}{m_1}}\widehat{\phi_{1+}}}\right)\left(D_{\frac{1}{m_2}}\widehat{\phi_{2+}}\right)-R_7, \qquad (4.2.2)
\end{aligned}$$

where

$$\begin{aligned} R_7 &= t^{-1}\mathcal{M}_{\frac{1}{m_1}}E^{2m_1-m_2}\left(\left(\overline{D_{\frac{1}{m_1}}\widehat{\phi_{1+}}}\right)\left(D_{\frac{1}{m_2}}\widehat{\phi_{2+}}\right)\right. \\ &\quad \left.-\left(\overline{\mathcal{M}_{-\frac{1}{m_1}}D_{\frac{1}{m_1}}\widehat{\phi_{1+}}}\right)\left(\mathcal{M}_{-\frac{1}{m_2}}D_{\frac{1}{m_2}}\widehat{\phi_{2+}}\right)\right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)\left(v_{1+}^2\right) \\ &= i\mathcal{M}_{\frac{1}{m_2}}E^{m_2}(t)\,D_{\frac{1}{t}}\left(v_{1+}^2\right) \\ &= t^{-1}\mathcal{M}_{\frac{1}{m_2}}E^{m_2-2m_1}\left(\mathcal{M}_{-\frac{1}{m_1}}D_{\frac{1}{m_1}}\widehat{\phi_{1+}}\right)^2 \\ &= t^{-1}\mathcal{M}_{\frac{1}{m_2}}E^{m_2-2m_1}\left(D_{\frac{1}{m_1}}\widehat{\phi_{1+}}\right)^2 - R_8, \end{aligned} \tag{4.2.3}$$

where

$$R_8 = t^{-1}\mathcal{M}_{\frac{1}{m_2}}E^{m_2-2m_1}\left(\left(D_{\frac{1}{m_1}}\widehat{\phi_{1+}}\right)^2-\left(\mathcal{M}_{-\frac{1}{m_1}}D_{\frac{1}{m_1}}\widehat{\phi_{1+}}\right)^2\right).$$

To study the non resonant nonlinear terms we now compute the commutator of $\mathcal{M}_{\frac{1}{m}}$ and E^{α} (see [**14**])

$$\begin{aligned} \mathcal{M}_{\frac{1}{m}}E^{m-\rho}\phi &= \frac{mt}{2\pi i}\int_{\mathbb{R}^2} e^{\frac{imt}{2}|\eta-\xi|^2}e^{\frac{it}{2}(\rho-m)|\eta|^2}\phi(\eta)\,d\eta \\ &= iE^{m\left(\frac{m}{\rho}-1\right)}\mathcal{M}_{\frac{\rho}{m^2}}D_{\frac{\rho}{m}}\phi \end{aligned}$$

for $\rho \neq 0$.
Note that in the resonant case of $\rho = 0$

$$\mathcal{M}_{\frac{1}{m}}E^{m}\phi = \frac{mt}{2\pi i}E^{-m}\int_{\mathbf{R}^2} e^{-imt(\xi\cdot\eta)}\phi(\eta)\,d\eta = E^{-m}D_{\frac{1}{mt}}\mathcal{F}\phi.$$

Hence in the case $2m_1-m_2 \neq 0$ and $m_1-m_2\neq 0$ we get

$$\mathcal{M}_{\frac{1}{m_1}}E^{2m_1-m_2} = iE^{m_1\frac{2m_1-m_2}{m_2-m_1}}\mathcal{M}_{\frac{m_2-m_1}{m_1^2}}D_{\frac{m_2-m_1}{m_1}} \tag{4.2.4}$$

and

$$\mathcal{M}_{\frac{1}{m_2}}E^{m_2-2m_1} = iE^{m_2\frac{m_2-2m_1}{2m_1}}\mathcal{M}_{\frac{2m_1}{m_2^2}}D_{\frac{2m_1}{m_2}}. \tag{4.2.5}$$

From (4.2.4) and (4.2.5) we obtain

$$\begin{aligned} & D_{\frac{1}{m_1}}\mathcal{F}U_{\frac{1}{m_1}}(-t)\left(\overline{v_{1+}}v_{2+}\right) \\ &= it^{-1}E^{m_1\frac{2m_1-m_2}{m_2-m_1}}\mathcal{M}_{\frac{m_2-m_1}{m_1^2}}\psi_1 - R_7 \\ &= it^{-1}E^{m_1\frac{2m_1-m_2}{m_2-m_1}}\psi_1 - R_9 - R_7, \end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned} & D_{\frac{1}{m_2}}\mathcal{F}U_{\frac{1}{m_2}}(-t)\left(v_{1+}^2\right) \\ &= it^{-1}E^{m_2\frac{m_2-2m_1}{2m_1}}\mathcal{M}_{\frac{2m_1}{m_2^2}}\psi_2 - R_8 \\ &= it^{-1}E^{m_2\frac{m_2-2m_1}{2m_1}}\psi_2 - R_{10} - R_8, \end{aligned} \tag{4.2.7}$$

where we denote

$$\psi_1(\xi) = D_{\frac{m_2-m_1}{m_1}} \left(\overline{D_{\frac{1}{m_1}} \widehat{\phi_{1+}}}\right) \left(D_{\frac{1}{m_2}} \widehat{\phi_{2+}}\right),$$
$$\psi_2(\xi) = D_{\frac{2m_1}{m_2}} \left(D_{\frac{1}{m_1}} \widehat{\phi_{1+}}\right)^2$$

and

$$R_9 = it^{-1} E^{m_1 \frac{2m_1-m_2}{m_2-m_1}} \left(1 - \mathcal{M}_{\frac{m_2-m_1}{m_1^2}}\right) \psi_1,$$
$$R_{10} = it^{-1} E^{m_2 \frac{m_2-2m_1}{2m_1}} \left(1 - \mathcal{M}_{\frac{2m_1}{m_2^2}}\right) \psi_2.$$

By (4.2.6) and (4.2.7) we obtain

$$\begin{aligned} & \left\| \int_t^\infty U_{\frac{1}{m_1}}(-\tau) \left(\overline{v_{1+}} v_{2+}\right) d\tau \right\| \\ = & \left\| \int_t^\infty D_{\frac{1}{m_1}} \mathcal{F} U_{\frac{1}{m_1}}(-\tau) \left(\overline{v_{1+}} v_{2+}\right) d\tau \right\| \\ \le & \ C \left\| \int_t^\infty E^{m_1 \frac{2m_1-m_2}{m_2-m_1}} \psi_1 \frac{d\tau}{\tau} \right\| + \left\| \int_t^\infty R_7 d\tau \right\| + \left\| \int_t^\infty R_9 d\tau \right\| \end{aligned} \tag{4.2.8}$$

and

$$\begin{aligned} & \left\| \int_t^\infty U_{\frac{1}{m_2}}(-\tau) \left(v_{1+}^2\right) d\tau \right\| \\ = & \left\| \int_t^\infty D_{\frac{1}{m_2}} \mathcal{F} U_{\frac{1}{m_2}}(-\tau) \left(v_{1+}^2\right) d\tau \right\| \\ \le & \ C \left\| \int_t^\infty E^{m_2 \frac{m_2-2m_1}{2m_1}} \psi_2 \frac{d\tau}{\tau} \right\| + \left\| \int_t^\infty R_8 d\tau \right\| + \left\| \int_t^\infty R_{10} d\tau \right\|. \end{aligned} \tag{4.2.9}$$

Using the identity

$$\partial_t \left(t E^\alpha\right) = \left(1 - \frac{\alpha i t}{2} |\xi|^2\right) E^\alpha$$

we integrate by parts

$$\left| \int_t^\infty E^\alpha \frac{d\tau}{\tau} \right| = \left| - \frac{E^\alpha}{1 - \frac{\alpha i t}{2} |\xi|^2} + \int_t^\infty \frac{\left(1 - \alpha i \tau |\xi|^2\right) E^\alpha}{\tau \left(1 - \frac{\alpha i \tau}{2} |\xi|^2\right)^2} d\tau \right| \le C t^{-b} |\xi|^{-2b}$$

for all $t \ge 1$, where $\alpha \ne 0$ and $b \in (\frac{1}{2}, 1)$. Hence we can estimate

$$\begin{aligned} & \left\| \int_t^\infty E^{m_1 \frac{2m_1-m_2}{m_2-m_1}} \psi_1 \frac{d\tau}{\tau} \right\| \le \left\| \left| \int_t^\infty E^{m_1 \frac{2m_1-m_2}{m_2-m_1}} \frac{d\tau}{\tau} \right| |\psi_1| \right\| \\ \le & \ C t^{-b} \left\| |\xi|^{-2b} \psi_1 \right\| \\ \le & \ C t^{-b} \left\| |\xi|^{-2b} D_{\frac{m_2-m_1}{m_1}} \left(\overline{D_{\frac{1}{m_1}} \widehat{\phi_{1+}}}\right) \left(D_{\frac{1}{m_2}} \widehat{\phi_{2+}}\right) \right\| \\ \le & \ C t^{-b} \left\| \widehat{\phi}_{1+} \right\|_{\dot{\mathbf{H}}^{0,-2b}} \left\| \widehat{\phi}_{2+} \right\|_{\mathbf{L}^\infty} \\ \le & \ C t^{-b} \left\| \phi_{1+} \right\|_{\dot{\mathbf{H}}^{-2b}} \left\| \phi_{2+} \right\|_{\mathbf{H}^{0,2}} \end{aligned} \tag{4.2.10}$$

for all $t \geq 1$,where $b \in (\frac{1}{2}, 1)$. In the same way we have

$$\begin{aligned} & \left\| \int_t^\infty E^{m_2 \frac{m_2-2m_1}{2m_1}} \psi_2 \frac{d\tau}{\tau} \right\| \\ \leq & \left\| \left| \int_t^\infty E^{m_2 \frac{m_2-2m_1}{2m_1}} \frac{d\tau}{\tau} \right| |\psi_2| \right\| \\ \leq & \; Ct^{-b} \left\| |\xi|^{-2b} \psi_2 \right\| \\ \leq & \; Ct^{-b} \left\| |\xi|^{-2b} D_{\frac{2m_1}{m_2}} \left(D_{\frac{1}{m_1}} \widehat{\phi_{1+}} \right)^2 \right\| \\ \leq & \; Ct^{-b} \left\| \widehat{\phi_{1+}} \right\|_{\dot{\mathbf{H}}^{0,-2b}} \left\| \widehat{\phi_{1+}} \right\|_{\mathbf{L}^\infty} \\ \leq & \; Ct^{-b} \left\| \phi_{1+} \right\|_{\dot{\mathbf{H}}^{-2b}} \left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} \end{aligned} \tag{4.2.11}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$.
We can estimate the remainder terms

$$\begin{aligned} & \|R_7\| \leq Ct^{-1} \left\| \overline{\left(D_{\frac{1}{m_1}} \widehat{\phi_{1+}} \right)} \left(D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right) \right. \\ & \left. - \overline{\left(\mathcal{M}_{-\frac{1}{m_1}} D_{\frac{1}{m_1}} \widehat{\phi_{1+}} \right)} \left(\mathcal{M}_{-\frac{1}{m_2}} D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right) \right\| \\ \leq & \; Ct^{-1} \left\| \left(\mathcal{M}_{-\frac{1}{m_1}} - 1 \right) D_{\frac{1}{m_1}} \widehat{\phi_{1+}} \right\|_{\mathbf{L}^\infty} \left\| \left(\mathcal{M}_{-\frac{1}{m_2}} - 1 \right) D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right\| \\ & + Ct^{-1} \left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} \left\| \left(\mathcal{M}_{-\frac{1}{m_2}} - 1 \right) D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right\| \\ & + Ct^{-1} \left\| \phi_{2+} \right\|_{\mathbf{H}^{0,2}} \left\| \left(\mathcal{M}_{-\frac{1}{m_1}} - 1 \right) D_{\frac{1}{m_1}} \widehat{\phi_{1+}} \right\| \\ \leq & \; Ct^{-2} \left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} \left\| \phi_{2+} \right\|_{\mathbf{H}^{0,2}} \end{aligned} \tag{4.2.12}$$

for all $t \geq 1$.
Similarly, we have

$$\|R_j\| \leq Ct^{-2} \left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} \left(\left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} + \left\| \phi_{2+} \right\|_{\mathbf{H}^{0,2}} \right)$$

for all $t \geq 1$ and $j = 8, 9, 10$.
Therefore

$$\left\| \int_t^\infty R_j d\tau \right\| \leq Ct^{-1} \left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} \left(\left\| \phi_{1+} \right\|_{\mathbf{H}^{0,2}} + \left\| \phi_{2+} \right\|_{\mathbf{H}^{0,2}} \right) \tag{4.2.13}$$

for all $t \geq 1$ and $j = 7, 8, 9, 10$.
Substitution of the estimates (4.2.10), (4.2.11) and (4.2.13) into (4.2.8) and (4.2.9) yield the result of the lemma. ∎

4.3. Existence of modified wave operators or wave operators

In this section, we give proofs of existence of modified wave operators or wave operators for System (4.1.1) under different mass conditions as follows:

Proof of Theorem 4.1.1. We define the following function space

$$\mathbf{X} = \left\{ (f_1, f_2) \in \mathbf{C}\left([1, \infty) ; \mathbf{L}^2 \right) ; \|(f_1, f_2)\|_{\mathbf{X}} < \infty \right\},$$

with the norm

$$\|(f_1,f_2)\|_{\mathbf{X}}=\sum_{j=1}^{2}\sup_{t\in[1,\infty)}t^{b}\left(\|f_j-v_{j\gamma}\|_{\mathbf{L}_t^{\infty}([t,\infty);\mathbf{L}^2)}+\|f_j-v_{j\gamma}\|_{\mathbf{L}_t^{4}([t,\infty);\mathbf{L}^4)}\right),$$

where $\frac{1}{2}< b<1$, $v_{j\gamma}(t)$ are defined in the previous section as a particular solution of System (4.2.1). We also denote the closed ball $\mathbf{X}_\rho=\{(f_1,f_2)\in\mathbf{X};\|(f_1,f_2)\|_{\mathbf{X}}\le\rho\}$. We now find a solution of System (4.1.1) in the neighborhood of $(v_{1\gamma},v_{2\gamma})$. By System (4.1.1) and System (4.2.1) we obtain the following system of integral equations

$$\left\{\begin{array}{c} u_1(t)-v_{1\gamma}(t)=i\int_t^{\infty}U_{\frac{1}{m_1}}(t-\tau)\left(F_1(u_1,u_2)-\gamma R_1\right)d\tau,\\ u_2(t)-v_{2\gamma}(t)=i\int_t^{\infty}U_{\frac{1}{m_2}}(t-\tau)\left(F_2(u_1,u_2)-R_2\right)d\tau,\end{array}\right. \tag{4.3.1}$$

where

$$F_1(u_1,u_2)=\gamma\left(\overline{u_1}u_2-\overline{v_{1\gamma}}v_{2\gamma}\right),\ F_2(u_1,u_2)=u_1^2-v_{1\gamma}^2.$$

Linearized version of System (4.3.1) is written as

$$\left\{\begin{array}{c} u_1(t)-v_{1\gamma}(t)=i\int_t^{\infty}U_{\frac{1}{m_1}}(t-\tau)\left(F_1(w_1,w_2)-\gamma R_1\right)d\tau,\\ u_2(t)-v_{2\gamma}(t)=i\int_t^{\infty}U_{\frac{1}{m_2}}(t-\tau)\left(F_2(w_1,w_2)-R_2\right)d\tau,\end{array}\right. \tag{4.3.2}$$

where $(w_1,w_2)\in\mathbf{X}_\rho$. Note that

$$F_1(w_1,w_2)=\gamma\left(\overline{w_1}-\overline{v_{1\gamma}}\right)(w_2-v_{2\gamma})+\gamma\overline{v_{1\gamma}}(w_2-v_{2\gamma})+\gamma v_{2\gamma}\left(\overline{w_1}-\overline{v_{1\gamma}}\right).$$

Hence by the Strichartz estimate (see Lemma 2.6.1) we find with $\mathbf{I}=(t,\infty)$

$$\begin{array}{rl} & \|u_1-v_{1\gamma}\|_{\mathbf{L}_t^{\infty}(\mathbf{I};\mathbf{L}^2)}+\|u_1-v_{1\gamma}\|_{\mathbf{L}_t^{4}(\mathbf{I};\mathbf{L}^4)}\\ \le & C\left\|\left(\overline{w_1}-\overline{v_{1\gamma}}\right)(w_2-v_{2\gamma})\right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)}+C\left\|\overline{v_{1\gamma}}(w_2-v_{2\gamma})\right\|_{\mathbf{L}_t^{1}(\mathbf{I};\mathbf{L}^2)}\\ & +C\left\|v_{2\gamma}\left(\overline{w_1}-\overline{v_{1\gamma}}\right)\right\|_{\mathbf{L}_t^{1}(\mathbf{I};\mathbf{L}^2)}+C\|R_1\|_{\mathbf{L}_t^{1}(\mathbf{I};\mathbf{L}^2)}.\end{array}$$

Since

$$\begin{array}{rl} & \|v_{j\gamma}\|_{\mathbf{L}^\infty}\\ \le & Ct^{-1}\left\|\varphi_{j\gamma}(t)\right\|_{\mathbf{L}^\infty}+Ct^{-1}\left\|\left(\mathcal{M}_{-\frac{1}{m_j}}-1\right)\varphi_{j\gamma}(t)\right\|_{\mathbf{L}^\infty}\\ \le & C\varepsilon t^{-1},\end{array}$$

$$\begin{array}{rl} & \int_t^{\infty}\|w_j-v_{j\gamma}\|_{\mathbf{L}_t^{\infty}((t,\infty);\mathbf{L}^2)}t^{-1}dt\\ \le & \|(w_1,w_2)\|_{\mathbf{X}}\int_t^{\infty}t^{-b-1}dt\le Ct^{-b}\|(w_1,w_2)\|_{\mathbf{X}}\end{array}$$

and

$$\begin{array}{rl} & \left(\int_t^{\infty}\|w_1-v_{1\gamma}\|^2_{\mathbf{L}_t^{\infty}((t,\infty);\mathbf{L}^2)}dt\right)^{\frac{1}{2}}\\ \le & \|(w_1,w_2)\|_{\mathbf{X}}\left(\int_t^{\infty}t^{-2b}dt\right)^{\frac{1}{2}}\le Ct^{\frac{1}{2}-b}\|(w_1,w_2)\|_{\mathbf{X}},\end{array}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$, then by the Hölder inequality and Lemma 4.2.1 we find

$$\begin{aligned} & \left\|u_1 - v_{1\gamma}\right\|_{\mathbf{L}_t^{\infty}(\mathbf{I};\mathbf{L}^2)} + \left\|u_1 - v_{1\gamma}\right\|_{\mathbf{L}_t^{4}(\mathbf{I};\mathbf{L}^4)} \\ \leq\; & C\left\|w_1 - v_{1\gamma}\right\|_{\mathbf{L}_t^{2}(\mathbf{I};\mathbf{L}^2)} \left\|w_2 - v_{2\gamma}\right\|_{\mathbf{L}_t^{4}(\mathbf{I};\mathbf{L}^4)} \\ & + C\varepsilon t^{-b}\left\|(w_1, w_2)\right\|_{\mathbf{X}} + C\varepsilon^2 t^{-b} \\ \leq\; & Ct^{\frac{1}{2}-2b}\left\|(w_1, w_2)\right\|_{\mathbf{X}}^2 + C\varepsilon t^{-b}\left\|(w_1, w_2)\right\|_{\mathbf{X}} + C\varepsilon^2 t^{-b} \end{aligned}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$.
In the same manner we obtain

$$\begin{aligned} & \left\|u_2 - v_{2\gamma}\right\|_{\mathbf{L}_t^{\infty}(\mathbf{I};\mathbf{L}^2)} + \left\|u_2 - v_{2\gamma}\right\|_{\mathbf{L}_t^{4}(\mathbf{I};\mathbf{L}^4)} \\ \leq\; & C\left\|(w_1 - v_{1\gamma})^2\right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)} + C\left\|v_{1\gamma}(w_1 - v_{1\gamma})\right\|_{\mathbf{L}_t^{1}(\mathbf{I};\mathbf{L}^2)} \\ & + \left\|R_2\right\|_{\mathbf{L}_t^{1}(\mathbf{I};\mathbf{L}^2)} \\ \leq\; & C\left\|w_1 - v_{1\gamma}\right\|_{\mathbf{L}_t^{2}(\mathbf{I};\mathbf{L}^2)} \left\|w_1 - v_{1\gamma}\right\|_{\mathbf{L}_t^{4}(\mathbf{I};\mathbf{L}^4)} \\ & + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \\ \leq\; & C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \end{aligned}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$, since $(w_1, w_2) \in \mathbf{X}_\rho$.
Thus we obtain

$$\left\|(u_1, u_2)\right\|_{\mathbf{X}} \leq C\rho^2 t^{\frac{1}{2}-b} + C\varepsilon\rho + C\varepsilon^2 \tag{4.3.3}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$.
Therefore there exist ε and ρ such that $\left\|(u_1, u_2)\right\|_{\mathbf{X}} \leq \rho$.

We denote

$$\begin{cases} \widetilde{u}_1(t) - v_{1\gamma}(t) = i\int_t^{\infty} U_{\frac{1}{m_1}}(t-\tau)\left(F_1(\widetilde{w}_1, \widetilde{w}_2) - \gamma R_1\right)d\tau, \\ \widetilde{u}_2(t) - v_{2\gamma}(t) = i\int_t^{\infty} U_{\frac{1}{m_2}}(t-\tau)\left(F_2(\widetilde{w}_1, \widetilde{w}_2) - R_2\right)d\tau, \end{cases} \tag{4.3.4}$$

where $(\widetilde{w}_1, \widetilde{w}_2) \in \mathbf{X}_\rho$.

We now consider the difference between System (4.3.1) and System (4.3.4)

$$\begin{cases} (u_1 - \widetilde{u}_1)(t) = i\int_t^{\infty} U_{\frac{1}{m_1}}(t-\tau)\left(F_1(w_1, w_2) - F_1(\widetilde{w}_1, \widetilde{w}_2)\right)d\tau, \\ (u_2 - \widetilde{u}_2)(t) = i\int_t^{\infty} U_{\frac{1}{m_2}}(t-\tau)\left(F_2(w_1, w_2) - F_2(\widetilde{w}_1, \widetilde{w}_2)\right)d\tau. \end{cases}$$

In the same way as in the proof of (4.3.3) we have

$$\left\|(u_1, u_2) - (\widetilde{u}_1, \widetilde{u}_2)\right\|_{\mathbf{X}} \leq \frac{1}{2}\left\|(w_1, w_2) - (\widetilde{w}_1, \widetilde{w}_2)\right\|_{\mathbf{X}}.$$

Thus we have the desired result by the contraction mapping principle. ∎

Proof of Theorem 4.1.4. We now define $(v_{1+}, v_{2+}) = \left(U_{\frac{1}{m_1}}(t)\phi_{1+}, U_{\frac{1}{m_2}}(t)\phi_{2+}\right)$. Then the integral equations associated with System (4.1.1) can be written as follows:

$$\begin{cases} u_1(t) - v_{1+}(t) = i\gamma\int_t^{\infty} U_{\frac{1}{m_1}}(t-\tau)\left(\left(\overline{u_1}u_2 - \overline{v_{1+}}v_{2+}\right) + \overline{v_{1+}}v_{2+}\right)d\tau, \\ u_2(t) - v_{2+}(t) = i\int_t^{\infty} U_{\frac{1}{m_2}}(t-\tau)\left(\left(u_1^2 - v_{1+}^2\right) + v_{1+}^2\right)d\tau. \end{cases} \tag{4.3.5}$$

The linearized version of System (4.3.5) has the form

$$(4.3.6)\quad \begin{cases} u_1(t) - v_{1+}(t) = i\gamma \int_t^\infty U_{\frac{1}{m_1}}(t-\tau)\left(\left(\overline{w_1}w_2 - \overline{v_{1+}}v_{2+}\right) + \overline{v_{1+}}v_{2+}\right)d\tau, \\ u_2(t) - v_{2+}(t) = i\int_t^\infty U_{\frac{1}{m_2}}(t-\tau)\left(\left(w_1^2 - v_{1+}^2\right) + v_{1+}^2\right)d\tau, \end{cases}$$

where $(w_1, w_2) \in \mathbf{X}_\rho$, the functional spaces $\mathbf{X}$ and $\mathbf{X}_\rho$ are defined similarly to the proof of Theorem 4.1.1 with the norm

$$\|(f_1, f_2)\|_{\mathbf{X}} = \sum_{j=1}^{2} \sup_{t\in[1,\infty)} t^b \left(\|f_j - v_{j+}\|_{\mathbf{L}_t^\infty([t,\infty);\mathbf{L}^2)} + \|f_j - v_{j+}\|_{\mathbf{L}_t^4([t,\infty);\mathbf{L}^4)} \right),$$

where $\frac{1}{2} < b < 1$.

By the Strichartz type estimates (Lemma 2.6.1) with $\mathbf{I} = (t, \infty)$ we get from System (4.3.6)

$$\begin{aligned} & \|u_1 - v_{1+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_1 - v_{1+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\ \le\ & C\left\|\left(\overline{w_1} - \overline{v_{1+}}\right)\left(w_2 - v_{2+}\right)\right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)} \\ & + C\left\|\overline{v_{1+}}\left(w_2 - v_{2+}\right)\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\ & + C\left\|v_{2+}\left(\overline{w_1} - \overline{v_{1+}}\right)\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\ & + C\left\|U_{\frac{1}{m_1}}(t)\int_t^\infty U_{\frac{1}{m_1}}(-\tau)\,\overline{v_{1+}}v_{2+}d\tau\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} \\ & + C\left\|U_{\frac{1}{m_1}}(t)\int_t^\infty U_{\frac{1}{m_1}}(-\tau)\,\overline{v_{1+}}v_{2+}d\tau\right\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)}. \end{aligned}$$

Since $\|v_{j+}\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-1}$,

$$\int_t^\infty \|w_j - v_{j+}\|_{\mathbf{L}_t^\infty((t,\infty);\mathbf{L}^2)} t^{-1}dt \le C\rho \int_t^\infty t^{-b-1}dt \le C\rho t^{-b}$$

and

$$\left(\int_t^\infty \|w_1 - v_{1+}\|^2_{\mathbf{L}_t^\infty((t,\infty);\mathbf{L}^2)} dt\right)^{\frac{1}{2}} \le C\rho\left(\int_t^\infty t^{-2b}dt\right)^{\frac{1}{2}} \le C\rho t^{\frac{1}{2}-b}$$

for all $t \ge 1$ and $j = 1, 2$, where $b \in (\frac{1}{2}, 1)$, then by the Hölder inequality and estimates of Lemma 4.2.2 we find

$$\begin{aligned} & \|u_1 - v_{1+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_1 - v_{1+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\ \le\ & C\|w_1 - v_{1+}\|_{\mathbf{L}_t^2(\mathbf{I};\mathbf{L}^2)}\|w_2 - v_{2+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} + C\rho\varepsilon t^{-b} + C\varepsilon^2 t^{-b} \\ \le\ & C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \end{aligned}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$.
In the same way as above we obtain

$$\begin{aligned}
& \|u_2 - v_{2+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_2 - v_{2+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq\ & C \left\| (w_1 - v_{1+})^2 \right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)} \\
& + C \left\| v_{1+} (w_1 - v_{1+}) \right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\
& + C \left\| U_{\frac{1}{m_2}}(t) \int_t^\infty U_{\frac{1}{m_2}}(-\tau) v_{1+}^2 d\tau \right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} \\
& + C \left\| U_{\frac{1}{m_2}}(t) \int_t^\infty U_{\frac{1}{m_2}}(-\tau) v_{1+}^2 d\tau \right\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq\ & C \|w_1 - v_{1+}\|_{\mathbf{L}_t^2(\mathbf{I};\mathbf{L}^2)} \|w_1 - v_{1+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \\
\leq\ & C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b}
\end{aligned}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, 1)$.
Thus we find

$$\|u_j - v_{j+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_j - v_{j+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \leq C t^{-b} \left(\rho^2 t^{\frac{1}{2}-b} + \varepsilon\rho + \varepsilon^2 \right)$$

for all $t \geq 1$ and $j = 1, 2$, where $b \in (\frac{1}{2}, 1)$.
Therefore there exist ε and ρ such that $\|(u_1, u_2)\|_{\mathbf{X}} \leq \rho$.

We denote

$$\begin{cases} \widetilde{u}_1(t) - v_{1+}(t) = i\gamma \int_t^\infty U_{\frac{1}{m_1}}(t-\tau) \left(\left(\overline{\widetilde{w}_1} \widetilde{w}_2 - \overline{v_{1+}} v_{2+} \right) + \overline{v_{1+}} v_{2+} \right) d\tau, \\ \widetilde{u}_2(t) - v_{2+}(t) = i \int_t^\infty U_{\frac{1}{m_2}}(t-\tau) \left(\left(\widetilde{w}_1^2 - v_{1+}^2 \right) + v_{1+}^2 \right) d\tau, \end{cases}$$

where $(\widetilde{w}_1, \widetilde{w}_2) \in \mathbf{X}_\rho$.
In the same way we get for the differences

$$\|(u_1, u_2) - (\widetilde{u}_1, \widetilde{u}_2)\|_{\mathbf{X}} \leq \frac{1}{2} \|(w_1, w_2) - (\widetilde{w}_1, \widetilde{w}_2)\|_{\mathbf{X}}.$$

Thus we have the result of Theorem 4.1.4 by the contraction mapping principle.
∎

Proof of Theorem 4.1.5. We define $(v_{1+}, v_{2+}) = \left(U_{\frac{1}{m_1}}(t) \phi_{1+}, U_{\frac{1}{m_2}}(t) \phi_{2+} \right)$ and write the integral equations (4.3.5) associated with System (4.1.1) and the linearized version (4.3.6) with $(w_1, w_2) \in \mathbf{X}_\rho$. By (4.2.2) we have for the resonant case of $m_1 = m_2$

$$\begin{aligned}
& D_{\frac{1}{m_1}} \mathcal{F} U_{\frac{1}{m_1}}(-t) \left(\overline{v_{1+}} v_{2+} \right) \\
=\ & t^{-1} \mathcal{M}_{\frac{1}{m_1}} E^{m_1} \left(\overline{D_{\frac{1}{m_1}} \widehat{\phi_{1+}}} \right) \left(D_{\frac{1}{m_1}} \widehat{\phi_{2+}} \right) - R_7, \qquad (4.3.7)
\end{aligned}$$

where

$$\begin{aligned}
R_7 = \ & t^{-1} \mathcal{M}_{\frac{1}{m_1}} E^{m_1} \left(\left(\overline{D_{\frac{1}{m_1}} \widehat{\phi_{1+}}} \right) \left(D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right) \right. \\
& \left. - \left(\overline{\mathcal{M}_{-\frac{1}{m_1}} D_{\frac{1}{m_1}} \widehat{\phi_{1+}}} \right) \left(\mathcal{M}_{-\frac{1}{m_2}} D_{\frac{1}{m_2}} \widehat{\phi_{2+}} \right) \right).
\end{aligned}$$

Therefore by (4.3.7), (4.2.12) and by support assumption on the data of the theorem for the resonant case of $m_1 = m_2$ we get

$$\begin{aligned}
& \left\| U_{\frac{1}{m_1}}(-t)\left(\overline{v_{1+}}v_{2+}\right)\right\| \\
= & \left\| D_{\frac{1}{m_1}}\mathcal{F}U_{\frac{1}{m_1}}(-t)\left(\overline{v_{1+}}v_{2+}\right)\right\| \\
\leq & \ t^{-1}\left\|\left(\overline{D_{\frac{1}{m_1}}\widehat{\phi_{1+}}}\right)\left(D_{\frac{1}{m_1}}\widehat{\phi_{2+}}\right)\right\| + \|R_7\| \\
\leq & \ Ct^{-1}\left\|\overline{\widehat{\phi_{1+}}}\widehat{\phi_{2+}}\right\| + Ct^{-2}\left\|\phi_{1+}\right\|_{\mathbf{H}^{0,2}}\left\|\phi_{2+}\right\|_{\mathbf{H}^{0,2}} \\
\leq & \ Ct^{-2}\left\|\phi_{1+}\right\|_{\mathbf{H}^{0,2}}\left\|\phi_{2+}\right\|_{\mathbf{H}^{0,2}}
\end{aligned} \tag{4.3.8}$$

for all $t \geq 1$.

Note that the second estimate of Lemma 4.2.2 is true for the resonant case of $m_1 = m_2$. By the estimate $\|\mathcal{F}^{-1}D_{m_1}R_7\|_{\mathbf{L}^{\frac{4}{3}}} \leq C\varepsilon^2 t^{-\frac{3}{2}}$ which is obtained in the same way as (4.2.12), we get from (4.3.6)

$$\begin{aligned}
& \|u_1 - v_{1+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_1 - v_{1+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq & \ C\left\|\left(\overline{w_1} - \overline{v_{1+}}\right)(w_2 - v_{2+})\right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)} \\
& +C\left\|\overline{v_{1+}}(w_2 - v_{2+})\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\
& +C\left\|v_{2+}\left(\overline{w_1} - \overline{v_{1+}}\right)\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\
& +C\left\|U_{\frac{1}{m_1}}(t)\int_t^\infty U_{\frac{1}{m_1}}(-\tau)\,\overline{v_{1+}}v_{2+}d\tau\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} \\
& +C\left\|U_{\frac{1}{m_1}}(t)\int_t^\infty U_{\frac{1}{m_1}}(-\tau)\,\overline{v_{1+}}v_{2+}d\tau\right\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq & \ C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \\
& +C\left\|t^{-\frac{1}{2}}\int_t^\infty \left\|U_{\frac{1}{m_1}}(-\tau)\,\overline{v_{1+}}v_{2+}\right\|_{\mathbf{L}^{\frac{4}{3}}} d\tau\right\|_{\mathbf{L}_t^4(\mathbf{I})} \\
\leq & \ C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b}
\end{aligned}$$

and by the second estimate of Lemma 4.2.2

$$\begin{aligned}
& \|u_2 - v_{2+}\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_2 - v_{2+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq & \ C\left\|(w_1 - v_{1+})^2\right\|_{\mathbf{L}_t^{\frac{4}{3}}\left(\mathbf{I};\mathbf{L}^{\frac{4}{3}}\right)} \\
& +C\left\|v_{1+}(w_1 - v_{1+})\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{L}^2)} \\
& +C\left\|U_{\frac{1}{m_2}}(t)\int_t^\infty U_{\frac{1}{m_2}}(-\tau)\,v_{1+}^2 d\tau\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}^2)} \\
& +C\left\|U_{\frac{1}{m_2}}(t)\int_t^\infty U_{\frac{1}{m_2}}(-\tau)\,v_{1+}^2 d\tau\right\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} \\
\leq & \ C\|w_1 - v_{1+}\|_{\mathbf{L}_t^2(\mathbf{I};\mathbf{L}^2)}\|w_1 - v_{1+}\|_{\mathbf{L}_t^4(\mathbf{I};\mathbf{L}^4)} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b} \\
\leq & \ C\rho^2 t^{\frac{1}{2}-2b} + C\varepsilon\rho t^{-b} + C\varepsilon^2 t^{-b}.
\end{aligned}$$

for all $t \geq 1$, where $b \in (\frac{1}{2}, \frac{3}{4})$.

Therefore there exist ε and ρ such that $\|(u_1, u_2)\|_{\mathbf{X}} \leq \rho$.

We denote

$$\begin{cases} \widetilde{u}_1(t) - v_{1+}(t) = i\gamma \int_t^\infty U_{\frac{1}{m_1}}(t-\tau)\left(\left(\overline{\widetilde{w}_1}\widetilde{w}_2 - \overline{v_{1+}}v_{2+}\right) + \overline{v_{1+}}v_{2+}\right) d\tau, \\ \widetilde{u}_2(t) - v_{2+}(t) = i \int_t^\infty U_{\frac{1}{m_2}}(t-\tau)\left(\left(\widetilde{w}_1^2 - v_{1+}^2\right) + v_{1+}^2\right) d\tau, \end{cases}$$

where $(\widetilde{w}_1, \widetilde{w}_2) \in \mathbf{X}_\rho$.

In the same way we have for the differences

$$\|(u_1, u_2) - (\widetilde{u}_1, \widetilde{u}_2)\|_{\mathbf{X}} \le \frac{1}{2} \|(w_1, w_2) - (\widetilde{w}_1, \widetilde{w}_2)\|_{\mathbf{X}}.$$

Thus the result of Theorem 4.1.5 follows by the contraction mapping principle. ∎

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