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Introduction

On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let E be a vector bundle over a quaternionic Kähler manifold M, and $p: Z \longrightarrow M$ the corresponding twistor space defined by Salamon [S1]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between E's with such connections and the set of Einstein-Hermitian vector bundles over Z.

Let $\mathbb{H}$ be the skew field of quaternions. Then the $Sp(n)\cdot Sp(1)$-module $\Lambda^2\mathbb{H}^n$ is a direct sum $N_2' \oplus N_2'' \oplus L_2$ of its irreducible submodules N_2', N_2'', L_2, where N_2' (resp. L_2) is the submodule of the elements fixed by $Sp(n)$ (resp. $Sp(1)$)

and for $n = 1$, we have $N_2'' = \{0\}$. Hence, the vector bundle $\wedge^2 T^*M$ is written as a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its holonomy-invariant subbundles in such a way that A_2', A_2'', B_2 correspond respectively to N_2', N_2'', L_2. Now, a connection for E is called an A_2'-connection (resp. B_2-connection) if the corresponding curvature is an $\mathrm{End}(E)$-valued A_2'-form (resp. B_2-form). Then we have :

Theorem (0.1). All A_2'-connections and also all B_2-connections are Yang-Mills connections.

Furthermore, for E with a B_2-connection we can associate an E-valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of B_2-connections (see Theorem (3.5)).

For our quaternionic Kähler manifold M, a pair (E, D_E) of a vector bundle E over M and a B_2-connection D_E on E is called a Hermitian pair on M if D_E is a Hermitian connection on E. On the other hand, a pair (F, D_F) of a holomorphic vector bundle over Z and a Hermitian $(1,0)$-connection D_F on F is called an excellent pair on Z if the following conditions are satisfied:

(a) F with the corresponding Hermitian metric h_F restricts to a flat bundle on each fibre of $p : Z \longrightarrow M$. (Hence the real structure $\tau : Z \longrightarrow Z$ (cf. Nitta and Takeuchi [N-T]) naturally lifts to a bundle automorphism $\tau' : F \longrightarrow F$.)

(b) Let $\sigma : F \longrightarrow F^*$ be the bundle map defined by $F_z \ni f \longmapsto \sigma(f) \in F^*_{\tau(z)}$ $(z \in Z)$, where $\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an antiholomorphic bundle automorphism. We then have the following generalization of a result of Penrose's type (cf. Atiyah, Hitchin and Singer [A-H-S] ; see also Salamon [S2], Berard-Bergery and Ochiai [B-O]):

Theorem (0.2). *Let $\mathcal{H}$ (resp. $\widetilde{\mathcal{H}}$) be the set of all Hermitian pairs (resp. all excellent pairs) on M (resp. Z). Then*

$$\mathcal{H} \ni (E, D_E) \longmapsto (p^*E, p^*D_E) \in \widetilde{\mathcal{H}}$$

defines a bijective correspondence between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$.

In paticular, if M has positive scalar curvature, then every excellent pair (F, D_F) on Z is a Ricci-flat Einstein-Hermitian vector bundle.

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1. Notation, convention and preliminaries

In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [S1], Nitta and Takeuchi [N-T]).

(1.1) Let $H^{(m)}$ denote the standard $Sp(m)$-module#) $\mathbb{H}^m$ $(= \mathbb{C}^{2m})$ of complex dimension $2m$, where $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ $(= \mathbb{C} + j\mathbb{C})$. Then the multiplication on $\mathbb{H}^m$ by j from the right naturally induces a $Sp(m)$-equivariant anti-linear map $j^{(m)} : H^{(m)} \longrightarrow H^{(m)}$ with $(j^{(m)})^2 = -\mathrm{id}$. We now define a non-degenerate skew-symmetric bilinear form $\omega^{(m)}$ on $\mathbb{H}^m$ by

$$\omega^{(m)}(h,h') := -\langle h, j^{(m)}h' \rangle \qquad (h,h' \in \mathbb{H}^m),$$

where $\langle\ ,\ \rangle$ is the standard Hermitian inner product on $\mathbb{C}^{2m}$ $(= \mathbb{H}^m)$. This $\omega^{(m)}$ can be regarded as an $Sp(m)$-invariant bilinear form on $H^{(m)}$ such that

$$(1.1.1) \qquad \omega^{(m)}(j^{(m)}h, j^{(m)}h') = (\omega^{(m)}(h,h'))^{-} \qquad (h,h' \in H^{(m)}).$$

Let $Sp(n)\cdot Sp(1) = Sp(n) \times Sp(1) \big/ \mathbb{Z}_2$. Then $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ is naturally a $Sp(n)\cdot Sp(1)$-module of complex dimension $4n$ with a real structure $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \ni a \longmapsto \bar{a} \in H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ defined by

$$(1.1.2) \qquad (h \otimes h')^{-} := j^{(m)}h \otimes j^{(1)}h' \qquad (h \in H^{(n)},\ h' \in H^{(1)}).$$

#) $Sp(m) = \{S \in GL(m,\mathbb{H}) \mid S\cdot {}^t\bar{S} = I\}$ is imbedded in $GL(2m,\mathbb{C})$ by $Sp(m) \ni A + jB \longmapsto \begin{pmatrix} A, & -\bar{B} \\ B, & \bar{A} \end{pmatrix} \in GL(2m,\mathbb{C})$ where $A, B \in GL(m,\mathbb{C})$.

We consider the corresponding real form $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$. Then the symmetric bilinear form $\omega^{(n)} \otimes \omega^{(1)}$ $\in S^2((H^{(n)})^* \otimes (H^{(1)})^*)$ induces an inner product $\langle\langle\ ,\ \rangle\rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$.

(1.2) Recall that a 4n-dimensional Riemannian manifold (M, g_M) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in $Sp(n) \cdot Sp(1)$ ($\subset SO(4n)$) with the additional condition for $n = 1$ that g_M is a self-dual Einstein metric. Throughout this paper, we fix once for all a quaternionic Kähler manifold (M, g_M). By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle TM is reduced to a principal $Sp(n) \cdot Sp(1)$-bundle P. Then TM can be regarded as the vector bundle

$$(1.2.1) \qquad P \times_{Sp(n) \cdot Sp(1)} (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$$

associated to the $Sp(n) \cdot Sp(1)$-module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$. The inner product $\langle\langle\ ,\ \rangle\rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ induces a Riemannian metric g on TM, which coincides with g_M up to constant multiple. Without loss of generality, we may assume $g = g_M$.

(1.3) Let $Sp(n)$ act trivially on $\mathbb{C}^2$. Then the standard $Sp(1)$-action on $\mathbb{C}^2$ naturally induces an $Sp(n) \times Sp(1)$-action (resp. $Sp(n) \cdot Sp(1)$-action) on $\mathbb{C}^2$ (resp. $\mathbb{P}^1\mathbb{C}$). Associated to these actions, we have:

$$\hat{p} : V(:= P \times_{Sp(n)\times Sp(1)} \mathbb{C}^2) \longrightarrow M$$
$$(\text{resp. } p : Z(:= P \times_{Sp(n)\cdot Sp(1)} \mathbb{P}^1\mathbb{C}) \longrightarrow M),$$

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle Z is nothing but $\mathbb{P}(V) := V - \{\text{zero section}\} \big/ \mathbb{C}^*$, and is called the twistor space of M (see Salamon [Sl; p.147]). Then Z is a complex manifold with a natural real structure τ as follows:

(1.3.1) By the connection on V induced from that of P, we have a decomposition of $T(V - \{\text{zero section}\})$ into the subbundles S^h and S^v corresponding respectively to horizontal and vertical distributions. Let y be an arbitrary point of $V - \{\text{zero section}\}$, and put $x := \hat{p}(y)$. Via the projection $\hat{p}$, the fibre $(S^h)_y$ of S^h over y is regarded as the tangent space T_xM at x. Then by the identification of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ with $T_xM^{\mathbb{C}}$ (cf. (1.2.1)), the space $H^{(n)} \otimes \mathbb{C}y$ defines a $\mathbb{C}$-linear subspace of $(T_xM)^{\mathbb{C}}$, denoted also by $H^{(n)} \otimes \mathbb{C}y$. Furthermore, let $(H^{(n)} \otimes \mathbb{C}y)'$ be the subspace of $(T_x^*M)^{\mathbb{C}}$ corresponing to $H^{(n)} \otimes \mathbb{C}y$ via the natural isomorphism $(T_x^*M)^{\mathbb{C}} \cong (T_xM)^{\mathbb{C}}$ induced by g_M. Now we define complex structure of T_yV by specifying the subspace $\Lambda_y^{1,0}$ of (1,0)-forms in $(T_y^*V)^{\mathbb{C}}$ as follows:

$$\Lambda_y^{1,0} = (\Lambda_y^{1,0})^h \oplus (\Lambda_y^{1,0})^v,$$

where $(\Lambda_y^{1,0})^h := \hat{p}^*((H^{(n)} \otimes \mathbb{C}y)')$, and $(\Lambda_y^{1,0})^v$ is the subspace of (1,0)-forms in $T_y\mathbb{C}^2$ by the identification of V_x with $\mathbb{C}^2$. Then this induces a complex structure on Z.

(1.3.2) The map $j^{(1)} : H^{(1)} \longrightarrow H^{(1)}$ naturally defines an antilinear bundle automorphism $\hat{\tau} : V \longrightarrow V$, which induces a real structure τ on Z.

(1.3.3) Recall that M always has a constant scalar curvature (denoted by t). Let g_F be the Fubini-Study metric for $\mathbb{P}^1\mathbb{C}$ $(= (\mathbb{C} + j\mathbb{C} - \{0\})/\mathbb{C}^*)$. If $t \neq 0$, then for some nonzero real constant c_t,

$$g_Z := p^*g_M + c_t g_F$$

defines a pseudo-Kählerian metric on Z, i.e., the corresponding (1,1)-form on Z is a nondegenerate d-closed (1,1)-form.

2. A'_2-connections and B_2-connections

We shall here give fundamental properties of the A'_2-connections and B_2-connections defined in the Introduction.

(2.1) Let $(H^{(m)})^*$ be the dual $Sp(m)$-module of $H^{(m)}$. Then in view of $\Lambda^2(H^{(1)})^* = \mathbb{C}\omega^{(1)}$, we have

$$\Lambda^2((H^{(n)})^*\otimes_{\mathbb{C}}(H^{(1)})^*) = (\Lambda^2(H^{(n)})^*\otimes_{\mathbb{C}}S^2(H^{(1)})^*) \oplus (S^2(H^{(n)}{}^*)\otimes_{\mathbb{C}}\mathbb{C}\omega^{(1)}).$$

Furthermore, the $Sp(n)$-module $\Lambda^2(H^{(n)})^*$ is written as a direct sum $\mathbb{C}\omega^{(n)} + \Lambda^2_0(H^{(n)})^*$ of its submodules, where $\Lambda^2_0(H^{(n)})^*$ is the orthogonal complement of $\mathbb{C}\omega^{(n)}$ in $\Lambda^2(H^{(n)})^*$. Hence,

$$(2.1.1)\qquad \Lambda^2((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = N'^{\mathbb{C}}_2 \oplus N''^{\mathbb{C}}_2 \oplus L_2^{\mathbb{C}},$$

where $N'^{\mathbb{C}}_2 := \mathbb{C}\omega^{(n)} \otimes_{\mathbb{C}} S^2(H^{(1)})^*$, $N''^{\mathbb{C}}_2 := \Lambda^2_0(H^{(n)})^* \otimes_{\mathbb{C}} S^2(H^{(1)})^*$ and $L_2^{\mathbb{C}} := S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)}$. Note that the $Sp(n)\cdot Sp(1)$-modules $N'^{\mathbb{C}}_2$, $N''^{\mathbb{C}}_2$, $L^{\mathbb{C}}_2$ respectively admit real forms N'_2, N''_2, L_2 fixed by the real structure induced from the one in (1.1.2). We have the identification $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong (H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*$ by the metric $\langle\langle\ ,\ \rangle\rangle$ (cf. (1.1)). Together with $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong \mathbb{H}^n \otimes_{\mathbb{R}} \mathbb{C}$, the above (2.1.1) induces the decomposition of its real form:

$$\wedge^2 \mathbb{H}^n = N_2' \oplus N_2'' \oplus L_2,$$

which is nothing but the decomposition in the Introduction now for our principal $Sp(n)\cdot Sp(1)$-bundle P, the bundle T^*M is regarded as the vector bundle associated to the $Sp(n)\cdot Sp(1)$-module $((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)*}))_{\mathbb{R}} = \mathbb{H}^n$. Hence, $\wedge^2 T^*M$ is a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its subbundles A_2', A_2'', B_2 corresponding respectively to the $Sp(n)\cdot Sp(1)$-modules N_2', N_2'', L_2 (cf. Introduction).

(2.2) Fix an arbitrary point x of M. Note that each point z on the fibre Z_x defines an almost complex structure J_z on T_x^*M (cf. (1.3.1)). We then have the corresponding space $\wedge^{1,1}(T_x^*M, J_z)$ of (1,1)-forms of (T_x^*M, J_z). Choose a point $y (\neq 0)$ of V such that its natural image (denoted by $[y]$) is z. In view of (1.3.1), the space $\wedge^{1,1}(T_x^*M, J_z)$ in $\wedge^2(T_x^*M)^{\mathbb{C}}$ is associated to the $\mathbb{C}$-linear subspace $(H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)' \wedge ((H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)')^-$ in the $Sp(n)\cdot Sp(1)$-module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^* \wedge (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*$. Since $j^{(n)}$ preserves $H^{(n)}$, we have (cf. (1.1.2)):

$$\begin{aligned}(H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}y) \wedge ((H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}y)^-) &= (H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}y) \wedge (H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}j^{(1)}y) \\ &= (\wedge^2 H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}(y\otimes j^{(1)}y + j^{(1)}y\otimes y)) \oplus (S^2 H^{(n)}\otimes_{\mathbb{C}}\mathbb{C}(y\wedge j^{(1)}y)).\end{aligned}$$

The space $\mathbb{C}(y\wedge j^{(1)}y))$ (where $y\wedge j^{(1)}y = (y\otimes j^{(1)}y - j^{(1)}y\otimes y)/2)$ in $H^{(1)}\otimes_{\mathbb{C}}H^{(1)}$ corresponds to $\mathbb{C}\omega^{(1)}$ in $(H^{(1)})^*\otimes_{\mathbb{C}}(H^{(1)})^*$ via the natural isomorphism $H^{(1)}\otimes_{\mathbb{C}}H^{(1)} \cong (H^{(1)})^*\otimes_{\mathbb{C}}(H^{(1)})^*$ induced by the nondegenerate bilinear form $\omega^{(1)}$. Furthermore,

$$\bigcap_y \mathbb{C}(y\otimes j^{(1)}y + j^{(1)}y\otimes y) = \{0\},$$

where $\bigcap_y$ always denotes the intersection taken over all y in $V_x - \{0\}$. Thus,

$$\bigcap_y (H^{(n)}\otimes\mathbb{C}y)'\wedge\overline{(H^{(n)}\otimes\mathbb{C}y)'} = S^2(H^{(n)})^*\otimes_{\mathbb{C}}\mathbb{C}\omega^{(1)}$$
$$= L_2 \qquad \text{(cf. Introduction)},$$

and we obtain:

Lemma (2.3). The fibre $(B_2)_x$ of B_2 over x is given by

$$(B_2)_x = \bigcap_y \Lambda^{1,1}(T_x^*M, J_{[y]}).$$

We next give a typical example of an A_2'-connection and also a B_2-connection.

Example (2.4). If $n \geqq 2$, the induced connection on the locally defined vector bundle

$$V := P \times_{Sp(n)\times Sp(1)} H^{(1)} \qquad (\text{resp. } W := P \times_{Sp(n)\times Sp(1)} H^{(n)})$$

is an A_2'-connection (resp. B_2-connection). See Salamon [Sl;p.150] for related computations of cuvatures.

Recall that a connection ∇ is called a <u>Yang-Mills connection</u> if the corresponding curvature R^∇ satisfies $d^\nabla_* R^\nabla = 0$. We shall finally show:

Theorem (2.5). <u>All A_2'-connections and also all B_2-connections are Yang-Mills connections.</u>

Corollary (2.6). <u>The Riemannian connection on</u> TM <u>is a Yang-Mills connection.</u>

Proof of (2.6): By (1.2), (2.4) and (2.5), we obtain (2.6).

Proof of (2.5): Fix an arbitrary point x_0 of M. It then suffices to show $(d^{\nabla} * R^{\nabla})(x_0) = 0$. We may take a local section s to P over a neighbourhood U of x_0 such that the corresponding differential at the point x_0 transforms the tangent space $T_{x_0}M$ to a horizontal space at $s(x_0)$ in the tangent space $T_{s(x_0)}P$. Let $(u^1, \cdots, u^{4n})$ be the local frame of $T^*M_{|U}$ associated to s. Then all covariant derivatives of u^i's $(1 \leqq i \leqq 4n)$ at the point x_0 is zero. Moreover in terms of the frame $(u^1, \cdots, u^{4n})$, we can identify $T^*M_{|U}$ with $U \times \mathbb{R}^{4n}$ $(U \times \mathbb{H}^n)$. Note that ∇ on E naturally induces a connection (denoted by the same ∇) on End(E).

(i) We first assume that ∇ is an A_2'-connection on E. Recall that the rank 3 subbundle A_2' of $\Lambda^2 T^*M$ corresponds to the $Sp(n)\cdot Sp(1)$-submodule N_2' of $\Lambda^2 H^n$, where N_2' is the irreducible submodule of the elements fixed by Sp(n) (cf. Introduction). Let I,J and K be

$$I = \Sigma_{k=0}^{n-1}(u^{4k+1} \wedge u^{4k+2} + u^{4k+3} \wedge u^{4k+4}),$$
$$J = \Sigma_{k=0}^{n-1}(u^{4k+1} \wedge u^{4k+3} + u^{4k+4} \wedge u^{4k+2}),$$
$$K = \Sigma_{k=0}^{n-1}(u^{4k+1} \wedge u^{4k+4} + u^{4k+2} \wedge u^{4k+3}).$$

Then it is easy to check that $A_{2|U}'$ is spanned by the sections I,J and K. Therefore, the curvature form R^{∇} is written on U as

$$R^{\nabla} = a \otimes I + b \otimes J + c \otimes K,$$

where a, b and c are smooth sections to End(E) over U. Let $(u_1, \cdots, u_{4n})$ be the base for $TM_{|U}$ dual to $(u^1, \cdots, u^{4n})$ defined by $u^i(u_j) = \delta_{ij}$. Then by the first Bianchi identity,

$$\begin{aligned} 0 &= d^{\nabla}(R^{\nabla})(x_0) \\ &= \Sigma_{i=1}^{4n}\{(\nabla_i a)u^i(x_0)\wedge I(x_0)+(\nabla_i b)u^i(x_0)\wedge J(x_0)+(\nabla_i c)u^i(x_0)\wedge K(x_0)\}, \end{aligned}$$

where ∇_i denotes $\nabla_{u_i(x_0)}$. Consequently,

$$\nabla_i a = \nabla_i b = \nabla_i c = 0, \quad \text{for} \quad 1 \leqq i \leqq 4n \text{ if } n \geqq 2.$$

Therefore, $(d^{\nabla} * R^{\nabla})(x_0) = 0$.

(ii) We next assume that ∇ is a B_2-connection on E. Since the vector subbundle B_2 (of rank $n(2n+1)$) of $\Lambda^2 T^* M$ corresponds to the irreducible $Sp(n)\cdot Sp(1)$-submodule L_2 of the elements in $\Lambda^2 \mathbb{H}^n$ fixed by $Sp(1)$, the subbundle $B_{2|U}$ is spanned by

$$I_s, J_s, K_s, D_{pq}, E_{pq}, F_{pq}, G_{pq}, \quad (0 \leqq s \leqq n-1, \quad 0 \leqq p < q \leqq n-1).$$

where

$$
\begin{aligned}
I_s &= u^{4s+1} \wedge u^{4s+2} - u^{4s+3} \wedge u^{4s+4}, \\
J_s &= u^{4s+1} \wedge u^{4s+3} - u^{4s+4} \wedge u^{4s+2}, \\
K_s &= u^{4s+1} \wedge u^{4s+4} - u^{4s+2} \wedge u^{4s+3}, \\
D_{pq} &= u^{4p+1} \wedge u^{4q+1} + u^{4p+2} \wedge u^{4q+2} + u^{4p+3} \wedge u^{4q+3} + u^{4p+4} \wedge u^{4q+4}, \\
E_{pq} &= u^{4p+1} \wedge u^{4q+2} - u^{4p+2} \wedge u^{4q+1} - u^{4p+3} \wedge u^{4q+4} + u^{4p+4} \wedge u^{4q+3}, \\
F_{pq} &= u^{4p+1} \wedge u^{4q+3} + u^{4p+2} \wedge u^{4q+4} - u^{4p+3} \wedge u^{4q+1} - u^{4p+4} \wedge u^{4q+2}, \\
G_{pq} &= u^{4p+1} \wedge u^{4q+4} - u^{4p+2} \wedge u^{4q+3} + u^{4p+3} \wedge u^{4q+2} - u^{4p+4} \wedge u^{4q+1}.
\end{aligned}
$$

Let ∇ be a B_2-connection on E. Then over U, the curvature form R^∇ is written in the form

$$
\begin{aligned}
R^\nabla = {} & \Sigma_{0 \leqq s \leqq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s) \\
& + \Sigma_{0 \leqq p < q \leqq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}),
\end{aligned}
$$

where $i_s, j_s, k_s, d_{pq}, e_{pq}, f_{pq}$ and g_{pq} are smooth sections to End(E) over U. In view of the first Bianchi identity $d^\nabla R^\nabla = 0$, we have

$$
\begin{aligned}
-\nabla_{4s+3} i_s + \nabla_{4s+2} j_s + \nabla_{4s+1} k_s &= 0, \\
\nabla_{4s+1} i_s - \nabla_{4s+4} j_s + \nabla_{4s+3} k_s &= 0, \\
\nabla_{4s+4} i_s + \nabla_{4s+1} j_s - \nabla_{4s+2} k_s &= 0, \\
\nabla_{4s+2} i_s + \nabla_{4s+3} j_s + \nabla_{4s+4} k_s &= 0,
\end{aligned}
$$

for s with $0 \leqq s \leqq n-1$. Furthermore, if ℓ is either p or q, the identity $d^{\nabla}R^{\nabla} = 0$ implies

$$
\begin{aligned}
(-1)^{\varepsilon(\ell)}\nabla_{4\ell+1}d_{pq} - \nabla_{4\ell+2}e_{pq} - \nabla_{4\ell+3}f_{pq} - \nabla_{4\ell+4}g_{pq} &= 0,\\
(-1)^{\varepsilon(\ell)}\nabla_{4\ell+1}d_{pq} - \nabla_{4\ell+3}e_{pq} + \nabla_{4\ell+2}f_{pq} + \nabla_{4\ell+1}g_{pq} &= 0,\\
(-1)^{\varepsilon(\ell)}\nabla_{4\ell+2}d_{pq} + \nabla_{4\ell+1}e_{pq} - \nabla_{4\ell+4}f_{pq} + \nabla_{4\ell+3}g_{pq} &= 0,\\
(-1)^{\varepsilon(\ell)}\nabla_{4\ell+3}d_{pq} + \nabla_{4\ell+4}e_{pq} + \nabla_{4\ell+1}f_{pq} - \nabla_{4\ell+2}g_{pq} &= 0,
\end{aligned}
$$

for all p,q with $0 \leqq p < q \leqq n-1$, where $\varepsilon(p):=0$ and $\varepsilon(q):=1$. Then a straightforward computation shows that $(d^{\nabla}*R^{\nabla})(x_0)=0$, as required.

3. Deformations of B_2-connections

In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of B_2-connections on M (see Salamon [S2] for a similar complex).

(3.1) Let r be an integer with $r \geqq 2$. By setting $N_r^{\mathbb{C}} := \wedge^r (H^{(n)})^* \otimes_{\mathbb{C}} S^r (H^{(1)})^*$ (cf. (2.1)), we can express the $Sp(n)\cdot Sp(1)$-module $\wedge^r (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*$ as a direct sum $N_r^{\mathbb{C}} \oplus L_r^{\mathbb{C}}$, where $L_r^{\mathbb{C}}$ is the orthogonal complement of $N_r^{\mathbb{C}}$ in $\wedge^r (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*$. As in (2.1), the $Sp(n)\cdot Sp(1)$-modules $N_r^{\mathbb{C}}$ and $L_r^{\mathbb{C}}$ respectively admit real forms N_r and L_r fixed by the natural real structure (cf. (1.1.2)). Since T^*M is associated to the $Sp(n)\cdot Sp(1)$-module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*_{\mathbb{R}}$ (see (1.2.1)), the vector bundle $\wedge^r T^*M$ is a direct sum $A_r \oplus B_r$ of its subbundles A_r, B_r corresponding respectively to N_r, L_r. Let $\pi^r : \wedge^r T^*M \; (= A_r \oplus B_r) \longrightarrow A_r$ be the projection to the first factor. Then we have:

Theorem (3.2). <u>For a B_2-connection ∇ on E, the following is an elliptic complex:</u>

$$(3.2.1) \quad 0 \longrightarrow \mathcal{E}(E) \xrightarrow{\nabla} \mathcal{E}(E \otimes T^*M) \xrightarrow{d_1} \mathcal{E}(E \otimes A_2)$$

$$\xrightarrow{d_2} \mathcal{E}(E \otimes A_3) \xrightarrow{d_3} \cdots \xrightarrow{d_{2n-1}} \mathcal{E}(E \otimes A_{2n}) \longrightarrow 0,$$

<u>where</u> $d_i := (\mathrm{id} \otimes \pi^{i+1}) \circ d^{\nabla}$ <u>and for every vector bundle</u> E'

on M, we denote by $\mathcal{E}(E')$ the sheaf of germs of C^∞-sections of E'.

Proof.

(i) Fix a section $s \in \Gamma(M, E\otimes A_i)$ $(i \geqq 1)$ and define a section $t \in \Gamma(M, E\otimes B_{i+1})$ by

$$d^\nabla s = d_i s + t.$$

Then from $(d^\nabla \circ d^\nabla)s = (d^\nabla \circ d_i)s + d^\nabla t$, we obtain

$$((id\otimes\pi_{i+2}) \circ d^\nabla \circ d^\nabla)s = (d_{i+1} \circ d_i)s + ((id\otimes\pi_{i+2}) \circ d^\nabla)t.$$

Since ∇ is a B_2-connection, the A_{i+2}-component of $(d^\nabla \circ d^\nabla)s$ is zero, i.e.,

$$0 = (d_{i+1} \circ d_i)s + ((id\otimes\pi_{i+2}) \circ d^\nabla)t.$$

Write t as $t = \Sigma_k v_k \otimes b_k$ locally, where v_k, b_k is a local section of V^*, B_{i+1}, respectively. The $S^{i+1}(V^*)$-component of b_k is zero, and hence the $S^{i+2}(V^*)$-component of $\nabla(v_k) \wedge b_k$ is zero. Therefore,

$$((id\otimes\pi_{i+2}) \circ d^\nabla)t = \Sigma_k v_k \otimes db_k .$$

Since d is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$-component of db_k is zero. Thus, $(d_{i+1} \circ d_i)s = 0$, as required.

(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_i,u)$ $(u \in T_x^*M - \{0\})$. Fix a point of M and an element s of $E_x \otimes A_{ix}$. All computations below are taken at the point x.

$$\sigma(d_i,u)s := (d/dt)(e^{-tq}d_i(e^{tq}s)|_{t=0} = (id\otimes\pi_{i+1})(u\wedge s),$$

where q is a locally defined function such that $dq_x=u$. We next show that the following sequence is exact for every u:

$$(3.2.2) \quad E\otimes A_{i-1} \xrightarrow{\sigma(d_{i-1},u)} E\otimes A_i \xrightarrow{\sigma(d_i,u)} E\otimes A_{i+1}.$$

Without loss of generality, we may assume

$$u = e_1\otimes h_1 + (e_1\otimes h_1)^- \ (= e_1\otimes h_1 + e_2\otimes h_2),$$

where $\langle e_1,\cdots,e_{2n}\rangle$ (resp. $\langle h_1,h_2\rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)}e_{2j+1} = e_{2j+2}$ (resp. $j^{(1)}h_1 = h_2$). Let $s \in E\otimes A_i$ be such that $\sigma(d_{i+1},u)s = 0$. Note that $S^iV^* = \mathrm{Span}(h_1{}^k\cdot h_2{}^{i-k};\ 0 \leq k \leq i)$, where $h_1{}^k\cdot h_2{}^{i-k}$ denotes the symmetric component of $h_1{}^k\otimes h_2{}^{i-k}$.

Hence, there are local sections $s_0,\cdots,s_i$ of $E\otimes\Lambda^i W^*$ such that

$$s = \Sigma_{k=0}^{i}\ s_k\otimes h_1{}^k\cdot h_2{}^{i-k}.$$

We can now write $\sigma(d_{i+1},s) = 0$ as follows:

$$0 = (id\otimes\pi_{i+1})(u\wedge s) = (id\otimes\pi_{i+1})((e_1\otimes h_1+e_2\otimes h_2)\wedge\Sigma s_k\otimes h_1{}^k\cdot h_2{}^{i-k})$$

$$= \Sigma_{k=0}^{i}((e_1\wedge s_k)\otimes h_1{}^{k+1}\cdot h_2{}^{i-k} + (e_2\wedge s_k)\otimes h_1{}^k\cdot h_2{}^{i+1-k}).$$

Since the coefficient of the right-hand side in $h_1{}^k\cdot h_2{}^{i+1-k}$ is zero, we have:

$$\begin{array}{lr} (0) & e_2\wedge s_0 = 0, \\ (1) & e_1\wedge s_0 + e_2\wedge s_1 = 0, \\ & \cdot \\ & \cdot \\ & \cdot \\ (i) & e_1\wedge s_{i-1} + e_2\wedge s_i = 0, \\ (i+1) & e_1\wedge s_i \qquad\qquad = 0. \end{array}$$

By (0), there exists $r_0 \in \Lambda^{i-1}W^*$ such that

$$s_0 = e_2\wedge r_0.$$

Plugging this into (1), we obtain

$$e_2\wedge(-e_1\wedge r_0 + s_1) = 0.$$

Hence there exists $r_1 \in \Lambda^{i-1}W^*$ such that

$$s_1 = e_1\wedge r_0 + e_2\wedge r_1.$$

Repeating this process inductively, we obtain $r_k \in \Lambda^{i-1}W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k$, $1 \leqq k \leqq i$. Now by (i+1), the identity $e_1 \wedge e_2 \wedge r_i = 0$ holds. It then follows that there exists $r_i' \in \Lambda^{i-2}W^*$ such that $e_2 \wedge r_i = e_1 \wedge e_2 \wedge r_i'$. Since $e_2 \wedge (r_{i-1} + e_2 \wedge r_i') = e_2 \wedge r_{i-1}$, we may replace r_{i-1} by $r_{i-1} + e_2 \wedge r_i'$. Therefore,

$$
\begin{aligned}
s_0 &= \qquad\qquad e_2 \wedge r_0, \\
s_1 &= e_1 \wedge r_0 + e_2 \wedge r_1, \\
&\;\;\vdots \\
s_i &= e_1 \wedge r_{i-1} \qquad\qquad .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
s &= \Sigma_{k=0}^{i}\, s_k \otimes h_1{}^k \cdot h_2{}^{i-k} \\
&= \sigma(d_{i-1}, u)\,(\Sigma_{k=0}^{i-1}\, r_k \otimes h_1{}^k \cdot h_2{}^{i-1-k}),
\end{aligned}
$$

i.e., the sequence (3.2.2) is exact, as required.

Definition (3.3). Let $\mathcal{C}$ be the set of all B_2-connections on E with holonomy groups contained in a compact semisimple Lie group G. Assume that $\mathcal{C} \neq \emptyset$ and let $\nabla \in \mathcal{C}$. Then the frame bundle Q of E can be regarded as a principal G-bundle. Put $G_Q := Q \times_\theta G$ and $\mathfrak{g}_Q := Q \times_{Ad} \mathfrak{g}$, where θ is the group conjugation and $Ad : G \longrightarrow GL(\mathfrak{g})$ is the adjoint representation of G. Now, a C^∞-section to G_Q over M is called a <u>gauge transformation</u> of Q. Let $\mathcal{G}$ be the set of all gauge transformations of Q. Then $\mathcal{G}$ naturally acts on $\mathcal{C}$ (see Atiyah-Hitchin-Singer [A-H-S]). We call $\mathcal{M}$ $(:= \mathcal{C}/\mathcal{G})$ the moduli space of the B_2-connections on E with holonomy groups in G.

(3.4) Let $\nabla \in \mathcal{C}$ be irreducible in the sense that $\mathfrak{g}_Q$ admits no nonzero parallel section over M. Fix a smooth one-parameter family ∇^t ($|t| < \varepsilon$) of connections in $\mathcal{C}$ such that $\nabla^0 = \nabla$. Put $S = (d/dt)\nabla^t|_{t=0}$. We write the curvature form R^{∇^t} of ∇^t as

$$R^{\nabla^t} = R^{\nabla} + td^{\nabla'}S + \text{higher order terms in } t,$$

where ∇' is the connection on $\mathfrak{g}_Q$ naturally induced by ∇. Since R^{∇^t} is a $\mathfrak{g}_Q$-valued B_2-form, the corresponding derivative $d^{\nabla'}S$ at $t=0$ also satisfies

$$((id\otimes\pi^2) \circ d^{\nabla'})S = 0.$$

Let f^t ($|t| < \varepsilon$) be a one-parameter family of gauge transformations such that $f^0 = id$. Then,

$$\frac{d}{dt}(f^t(\nabla))|_{t=0} = \nabla'(\dot{f}) \quad ,$$

where $\dot{f} := (d/dt)(f^t)|_{t=0}$. Since $f^t(\nabla) \in \mathcal{C}$ for all t, the same argument as above shows that the $\mathfrak{g}_Q$-valued 1-form $\nabla'(\dot{f})$ satisfies

$$((id\otimes\pi^2)\circ d^{\nabla'})(\nabla'(\dot{f})) = 0.$$

For each $A \in \Gamma(\mathfrak{g}_Q)$, there exists a one-parameter family $f^t = \exp(tA)$ such that $(d/dt)f^t|_{t=0} = A$. Then together with (3.2), we immediately obtain the following:

Theorem (3.5). Assume that $\mathcal{C} \neq \emptyset$ and let $\nabla \in \mathcal{C}$ be irreducible. Then the space of infinitesimal (essential) deformations at ∇ of connections in $\mathcal{C}$, that is, the tangent space of $\mathcal{M}$ at ∇ is a linear subspace of the first cohomology group of the elliptic complex

$$0 \longrightarrow \mathcal{E}(\mathfrak{g}_Q) \xrightarrow{\nabla'} \mathcal{E}(\mathfrak{g}_Q\otimes T^*M) \xrightarrow{d_1'} \mathcal{E}(\mathfrak{g}_Q\otimes A_2)$$
$$\xrightarrow{d_2'} \mathcal{E}(\mathfrak{g}_Q\otimes A_3) \xrightarrow{d_3'} \cdots \xrightarrow{d_{2n-1}'} \mathcal{E}(\mathfrak{g}_Q\otimes A_{2n}) \longrightarrow 0,$$

where $d_i' := (id\otimes\pi^{i+1})\circ d^{\nabla'}$.

4. Einstein-Hermitian connections associated with B_2-connections

In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between B_2-connections and the corresponding Einstein-Hermitian connections.

Proof of (0.2) : (i) Let (E, D_E) be a Hermitian pair. Then by the definition of B_2-connections, the curvature form corresponding to the connection D_E is an End(E)-valued B_2-form, and by Lemma (2.3) the curvature form corresponding to the connection p^*D_E on p^*E is an End(p^*E)-valued (1,1)-form. Hence the connection p^*D_E induces naturally an integrable complex structure on p^*E as follows : Put $\ell := \operatorname{rank}(E)$ and denote by $q: p^*E \longrightarrow Z$ the natural projection. Let $(s_1, \cdots, s_\ell)$ (resp. $(y^1, \cdots, y^\ell)$) be a local unitary frame for p^*E (resp. the dual frame corresponding to $(s_1, \cdots, s_\ell)$). Then the vector subbundle $\Lambda^{1,0}T^*(p^*E)$ of type (1,0) in the complexification $T^*(p^*E)^{\mathbb{C}}$ of the cotangent bundle $T^*(p^*E)$ is defined as the direct sum of the pull-back $q^*(\Lambda^{1,0}T^*Z)$ and the space spanned by $\{dy^j + \Sigma_{i=1}^{\ell} y^i q^*\theta_{ji},\ 1 \leqq j \leqq \ell\}$, where (θ_{ij}) is the connection matrix for p^*D_E with respect to the frame $(s_1, \cdots, s_\ell)$ (i.e., $(p^*D_E)s_j = \Sigma_{i=1}^{\ell} s_i \theta_{ij}$). Now, we may take the frame $(s_1, \cdots, s_\ell)$ as the pull-back $(p^*t_1, \cdots, p^*t_\ell)$ of a local unitary frame $(t_1, \cdots, t_\ell)$ on E. Then the 1-forms θ_{ij}, $1 \leqq i, j \leqq \ell$, are written as $p^*\psi_{ij}$, where (ψ_{ij}) denotes the connection matrix for D_E with respect to the frame $(t_1,\cdots,t_\ell)$. Let $q':(p^*E)^* \longrightarrow Z$ be the projection naturally induced from $q:p^*E \longrightarrow Z$. Since the real structure $\tau:Z \longrightarrow Z$ is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping $q'\circ\sigma : p^*E \longrightarrow Z$ is equal to $\tau\circ q$, the mapping $\sigma:p^*E \longrightarrow (p^*E)^*$ is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on p^*E and $(p^*E)^*$.

(ii) We next fix an arbitrary excellent pair (F,D_F) on Z. Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover $\{U_\lambda\}$ of M, and a local unitary frame $(f_1^\lambda,\cdots,f_r^\lambda)$ (r = rank of F) of $F_{|p^{-1}(U_\lambda)}$ such that each restriction $(f^\lambda_{1|p^{-1}(x)},\cdots,f^\lambda_{r|p^{-1}(x)})$ over $p^{-1}(x)$ $(x \in U_\lambda)$ forms a holomorphic frame for $F_{|p^{-1}(x)}$. When $U_\lambda \cap U_\mu \neq \emptyset$, the transition matrix for F in terms of the frames $(f_1^\lambda,\cdots,f_r^\lambda)$, $(f_1^\mu,\cdots,f_r^\mu)$ is holomorphic (and hence constant) along each fibre $p^{-1}(x)$ $(x \in U_\lambda \cap U_\mu)$. Hence there exists a Hermitian vector bundle E on M such that, including metrics, we have $p^*E = F$. In paticular, we obtain a local unitary frame $(f_1'^\lambda,\cdots,f_r'^\lambda)$ for $E_{|U_\lambda}$ such that $(p^*f_1'^\lambda,\cdots,p^*f_r'^\lambda)$ coincides with the previous $(f_1^\lambda,\cdots,f_r^\lambda)$ over $p^{-1}(U_\lambda)$. Fix an arbitrary λ. If there is no fear of confusion, we shall omit the suffix λ and denote U_λ, $(f_1^\lambda,\cdots,f_r^\lambda)$, $\cdots$ simply by U, $(f_1,\cdots,f_r),\cdots,$

respectively. Let (ω_{ij}) be the connection matrix of D_F with respect to the frame $(f_1, \cdots, f_r)$, i.e., $D_F f_j = \Sigma_{i=1}^{r} f_i \omega_{ij}$. Furthermore, we choose a local symplectic basis $(e_1, \cdots, e_{2n})$ (resp. (h_1, h_2)) for $W^*_{|U}$ (resp. $V^*_{|U}$) (see Section 3). Now, since D_F is a Hermitian connection, we have:

$$(1) \qquad \omega_{ij} + \overline{\omega_{ji}} = 0, \qquad \text{for} \quad 1 \leqq i,j \leqq r.$$

Then the construction of D_E is reduced to showing that there exist 1-forms ω'_{ij} $(1 \leqq i,j \leqq r)$ on U satisfying $\omega_{ij} = p^*\omega'_{ij}$. In fact, once we can find such 1-forms ω'_{ij}, they define a Hermitian connection on E, such that the corresponding curvature form is pulled back by p to an $\mathrm{End}(F)$-valued (1,1)-form on Z, which together with Lemma (2.3) implies that our connection on E is a B_2-connection. Recall that, for each $x \in U$, the frame $(f_{1|p^{-1}(x)}, \cdots, f_{r|p^{-1}(x)})$ for $F_{|p^{-1}(x)}$ is trivial. Hence,

$$(2) \qquad \omega_{ij}(v) = 0 , \qquad 1 \leqq i,j \leqq r,$$

for every vector v tangent to $p^{-1}(x)$ $(\cong \mathbb{P}^1\mathbb{C})$. Since $(e_1 \otimes h_1, e_1 \otimes h_2, \cdots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)$ is a frame for $T^*M^{\mathbb{C}}_{|U} = W^*_{|U} \otimes V^*_{|U}$, there exist by (2) C^∞-functions a^k_{ij}, b^k_{ij} $(1 \leqq i,j \leqq r,$ $1 \leqq k \leqq 2n)$ on $p^{-1}(U)$ such that

$$(3) \qquad \omega_{ij} = \Sigma_{k=1}^{2n}(a_{ij}^{k}p^*(e_k \otimes h_1) + b_{ij}^{k}p^*(e_k \otimes h_2)) \ , \qquad 1 \leqq i,j \leqq r \ .$$

For every form η on $Z_{|U}$, we denote by $\hat{\eta}$ the pull-back of η to $(V - \{\text{zero section}\})_{|U}$. Then by (3), we have :

$$\begin{aligned} \hat{R}_{ij} &= d\hat{\omega}_{ij} + \Sigma_{t=1}^{r}\hat{\omega}_{it} \wedge \hat{\omega}_{tj} \\ &= \Sigma_{k=1}^{2n} \, d(\hat{a}_{ij}^{k}\hat{p}^*(e_k \otimes h_1)) + d(\hat{b}_{ij}^{k}\hat{p}^*(e_k \otimes h_2)) \ + \Sigma_{t=1}^{r}\hat{\omega}_{it} \wedge \hat{\omega}_{tj} . \end{aligned}$$

Fix an arbitrary point x on U. Choosing an appropriate $(e_1,\cdots,e_{2n})$ (resp. (h_1,h_2)), we may assume that $(\nabla^{V^*}e_k)(x) = 0$, $k = 1,2,\cdots,2n$ (resp. $(\nabla^{W^*}h_i)(x) = 0$, $i = 1,2)$, where ∇^{V^*} (resp. ∇^{W^*}) denotes the connection of V^* (resp. W^*) canonically induced by that of P (cf. Example (2.4)). Then, on $\hat{p}^{-1}(x)$,

$$\begin{aligned} \hat{R}_{ij} = \Sigma_{k=1}^{2n}\{d(\hat{a}_{ij}^{k}) \wedge \hat{p}^*(e_k \otimes h_1) + d(\hat{b}_{ij}^{k}) \wedge \hat{p}^*(e_k \otimes h_2)\} \\ + \Sigma_{t=1}^{r}\hat{\omega}_{it} \wedge \hat{\omega}_{tj} \ . \end{aligned}$$

Recall that the complex structure on the twistor space Z $(= (V - \{\text{zero section}\}) \, / \, \mathbb{C}^*)$ is induced by the complex structure on $V - \{\text{zero section}\}$ (see Section 1). Since $\hat{R}_{ij}$ is of type (1,1), we have :

$$\begin{aligned} (4) \quad & \Sigma_{k=1}^{2n}\{\partial(\hat{a}_{ij}^{k}) \wedge (\hat{p}^*(e_k \otimes h_1))^{(1,0)} + \partial(\hat{b}_{ij}^{k}) \wedge (\hat{p}^*(e_k \otimes h_2))^{(1,0)}\} \\ & + \Sigma_{t=1}^{r}\hat{\omega}_{it}^{(1,0)} \wedge \hat{\omega}_{tj}^{(1,0)} = 0 \qquad \text{on } \ p^{-1}(x) \ ; \end{aligned}$$

(5) $\Sigma_{k=1}^{2n}\{\bar\partial(\hat a_{ij}^{k}) \wedge (\hat p^*(e_k\otimes h_1))^{(0,1)} + \bar\partial(\hat b_{ij}^{k}) \wedge (\hat p^*(e_k\otimes h_2))^{(0,1)}\}$
$+ \Sigma_{t=1}^{r}\hat\omega_{it}^{(0,1)} \wedge \hat\omega_{tj}^{(0,1)} = 0 \quad \text{on} \quad p^{-1}(x),$

where for every 1-forms ζ on $(V - \{\text{zero section}\})_{|U}$, $\zeta^{(1,0)}$ (resp. $\zeta^{(0,1)}$) always denotes the $(1,0)$-component (resp. $(0,1)$-component) of ζ. Let (z^1, z^2) be the local triviality for $V_{|U}$ corresponding to (h_1, h_2). Then, by the definition of the complex structure of $(V - \{\text{zero section}\})$, we obtain from (4) and (5) the following :

$$(4')\quad \Sigma_{k=1}^{2n}\{(\frac{\partial}{\partial z^1}\hat a_{ij}^{k}dz^1+\frac{\partial}{\partial z^2}\hat a_{ij}^{k}dz^2)\wedge\bar z^1(z^1\hat p^*(e_k\otimes h_1)+z^2\hat p^*(e_k\otimes h_2))$$
$$+ (\frac{\partial}{\partial z^1}\hat b_{ij}^{k}dz^1+\frac{\partial}{\partial z^2}\hat b_{ij}^{k}dz^2)\wedge\bar z^2(z^1\hat p^*(e_k\otimes h_1)+z^2\hat p^*(e_k\otimes h_2))\}$$
$$= 0 \qquad \text{on} \quad \hat p^{-1}(x) \quad ;$$

$$(5')\quad \Sigma_{k=1}^{2n}\{(\frac{\partial}{\partial \bar z^1}\hat a_{ij}^{k}d\bar z^1+\frac{\partial}{\partial \bar z^2}\hat a_{ij}^{k}d\bar z^2)\wedge(-z^2)(\bar z^1\hat p^*(e_k\otimes h_2)-\bar z^2\hat p^*(e_k\otimes h_1))$$
$$+ (\frac{\partial}{\partial \bar z^1}\hat b_{ij}^{k}d\bar z^1+\frac{\partial}{\partial \bar z^2}\hat b_{ij}^{k}d\bar z^2)\wedge z^1(\bar z^1\hat p^*(e_k\otimes h_2)-\bar z^2\hat p^*(e_k\otimes h_1))\}$$
$$= 0 \qquad \text{on} \quad \hat p^{-1}(x) \quad .$$

Since both $z^1{}_{|p^{-1}(x)}$ and $z^2{}_{|p^{-1}(x)}$ are holomorphic on $\hat p^{-1}(x) \cong \mathbb{C}^2 - \{0\}$, we have

$$\frac{\partial}{\partial \bar{z}^i}(z^1\bar{\hat{a}}_{ij}^{\,k} + z^2\bar{\hat{b}}_{ij}^{\,k}) = \frac{\partial}{\partial \bar{z}^i}(-z^2\hat{a}_{ij}^{\,k} + z^1\hat{b}_{ij}^{\,k}) = 0 \quad (i = 1,2),$$

on $p^{-1}(x)$, i.e., both $f_1(z^1,z^2) := z^1\bar{\hat{a}}_{ij}^{\,k} + z^2\bar{\hat{b}}_{ij}^{\,k}$ and $f_2(z^1,z^2) := -z^2\hat{a}_{ij}^{\,k} + z^1\hat{b}_{ij}^{\,k}$ are holomorphic on $\mathbb{C}^2 - \{0\}$. By Hartogs' theorem, both f_1 and f_2 extend further to holomorphic functions on $\mathbb{C}^2$. Since $f_i(cz^1,cz^2) = cf_i(z^1,z^2)$ for all $z = (z^1,z^2) \in \mathbb{C}^2$ and $c \in \mathbb{C}^*$ $(i = 1,2)$, there exist constants $\alpha_{ij}^{\,k}$, $\beta_{ij}^{\,k}$, $\gamma_{ij}^{\,k}$, $\delta_{ij}^{\,k} \in \mathbb{C}$ independent of z such that

$$(6) \qquad z^1\bar{\hat{a}}_{ij}^{\,k} + z^2\bar{\hat{b}}_{ij}^{\,k} = z^1\bar{\alpha}_{ij}^{\,k} + z^2\bar{\beta}_{ij}^{\,k},$$

$$(7) \qquad -z^2\hat{a}_{ij}^{\,k} + z^1\hat{b}_{ij}^{\,k} = -z^2\gamma_{ij}^{\,k} + z^1\delta_{ij}^{\,k}, \quad (1 \leqq k \leqq 2n).$$

Let $\Gamma(Z,F^*)$ (resp. $\Gamma(Z,F^* \otimes T^*Z^{\mathbb{C}})$) be the space of global C^∞-sections over Z to F^* (resp. $F^* \otimes T^*Z^{\mathbb{C}}$). Let $\psi : \Gamma(Z,F^*) \longrightarrow \Gamma(Z,F^* \otimes T^*Z^{\mathbb{C}})$ be the $\mathbb{C}$-linear map sending each $s \in \Gamma(Z,F^*)$ to an element $\psi(s)$ of $\Gamma(Z,F^* \otimes T^*Z^{\mathbb{C}})$ defined by

$$\psi(s)(X) := \sigma((D_F)_{\overline{\tau_*(X)}}(\sigma^{-1}s)) \in F^*_z,$$

for $X \in T_zZ^{\mathbb{C}}$ $(z \in Z)$.

Then by the condition (b) in the Introduction, this ψ defines a Hermitian (1,0)-connection on the holomorphic vector bundle F^*. The corresponding connection matrix with respect to the frame $(\sigma f_1, \cdots, \sigma f_r)$ for $F^*|_{p^{-1}(U)}$ is written as $(\tau^* \overline{\omega_{ij}})$. By the definition of σ, it is easy to check that the frame $(\sigma f_1, \cdots, \sigma f_r)$ is dual to our previous $(f_1, \cdots, f_r)$. Hence the uniqueness of the (1,0)-connection on the Hermitian vector bundle F^* implies the equality $\overline{(\tau^*\omega_{ij})^{-}} = \omega_{ij}{}^*$, where $\omega_{ij}{}^* := -\omega_{ji}$. In view of (1), we have $\tau^*\omega_{ij} = \omega_{ij}$ and $\hat{\tau}^*\hat{\omega}_{ij} = \hat{\omega}_{ij}$. By (3) and $\hat{p} \circ \hat{\tau} = \hat{p}$, we obtain :

$$(8) \qquad \hat{\tau}^*\hat{a}_{ij}{}^k = \hat{a}_{ij}{}^k \quad \text{and} \quad \hat{\tau}^*\hat{b}_{ij}{}^k = \hat{b}_{ij}{}^k \qquad (1 \leqq k \leqq 2n).$$

Therefore,

$$-\bar{z}^2\hat{\tau}^*\bar{\hat{a}}_{ij}{}^k + \bar{z}^1\hat{\tau}^*\bar{\hat{b}}_{ij}{}^k = -\bar{z}^2\bar{\alpha}_{ij}{}^k + \bar{z}^1\bar{\beta}_{ij}{}^k \qquad (1 \leqq k \leqq 2n).$$

Moreover by (6),

$$(9) \qquad -z^2\hat{a}_{ij}{}^k + z^1\hat{b}_{ij}{}^k = -z^2\alpha_{ij}{}^k + z^1\beta_{ij}{}^k \qquad (1 \leqq k \leqq 2n).$$

Hence by (7) and (9), we obtain :

$$(10) \qquad \alpha_{ij}{}^k = \gamma_{ij}{}^k \quad \text{and} \quad \beta_{ij}{}^k = \delta_{ij}{}^k \qquad (1 \leqq k \leqq 2n).$$

Now, in view of (6), (7) and (10), we see that

$$\begin{pmatrix} \bar{z}^1, & \bar{z}^2 \\ -z^2, & z^1 \end{pmatrix} \begin{pmatrix} \hat{a}_{ij}^{\,k} - \alpha_{ij}^{\,k} \\ \hat{b}_{ij}^{\,k} - \beta_{ij}^{\,k} \end{pmatrix} = 0 \qquad (1 \leqq k \leqq 2n),$$

where $(z^1, z^2) \in \mathbb{C}^2 - \{0\}$ $(= \hat{p}^{-1}(x))$. Thus, $\hat{a}_{ij}^{\,k} = \alpha_{ij}^{\,k}$ and $\hat{b}_{ij}^{\,k} = \beta_{ij}^{\,k}$ $(1 \leqq k \leqq 2n)$, i.e., both $a_{ij}^{\,k}$ and $b_{ij}^{\,k}$ are constant along $p^{-1}(x)$, as required.

Remark (4.1). In some sense, our Theorem (0.2) completely clarifies the following result by Salamon [S2] (see Berard Bergery and Ochiai [B-O] for another generalization) :

For a Hermitian pair (E, D_E) *on* M, *the pull-back* (p^*E, p^*D_E) *to* Z *is a Hermitian holomorphic vector bundle over* Z.

Corollary (4.2). *Let* (F, D_F) *be an excellent pair on* Z. *If the quaternionic Kähler manifold* M *has positive scalar curvature, then* F *with* D_F *is a Ricci-flat Einstein Hermitian vector bundle over* Z.

Proof. Consider the twistor space $p: Z \longrightarrow M$. Then the horizontal component of the Kähler form on Z is a p^*A_2'-form (cf. (1.2), (1.3)). Recall that the curvature of D_F is an $\mathrm{End}(F)$-valued p^*B_2-form. Hence the Hermitian vector bundle F with D_F is Ricci-flat.

Remark (4.3). We have the decomposition of $TZ = T^h \oplus T^v$, where T^h(resp. T^v) is the horizontal (resp. vertical) distribution in terms of the connection on Z induced by that of P. Since the complex structure on TZ is a direct sum of complex structures on T^h and T^v, the holomorphic part $TZ^{(1,0)}$ admits the corresponding decomposition $TZ^{(1,0)} = T^{h(1,0)} \oplus T^{v(1,0)}$, where $T^{h(1,0)}$(resp. $T^{v(1,0)}$) denotes $T^{h\mathbb{C}} \cap TZ^{(1,0)}$ (resp. $T^{v\mathbb{C}} \cap TZ^{(1,0)}$). Recently, Zandi [Z] obtained the following:

The vector bundle $(T^{h(1,0)}, D^h)$ is an Einstein-Hermitian vector bundle, where D^h is the connection on $T^{h(1,0)}$ obtained as the restriction of the Riemannian connection on TZ to $T^{h(1,0)}$.

This result can be regarded as a straightforward consequence of our (4.2). We denote by L a locally defined (line) subbundle of p^*W (cf. (2.4)) such that, along each fibre $p^{-1}(x) = \mathbb{P}^1\mathbb{C}$ $(x \in M)$, it restricts to a universal bundle over $\mathbb{P}^1\mathbb{C}$. Let ∇^V (resp. ∇^W) denote the connection of V (resp. W) canonically induced by that of P and ∇^L the restriction of

$p^*\nabla^W$ to L. Then the vector bundle $(T^{h(1,0)}, D^h)$ is nothing but $(p^*W \otimes L^*, p^*\nabla^W \otimes (\nabla^L)^*)$, where $(L^*, (\nabla^L)^*)$ is dual to (L, ∇^L) (see Salamon [S1]). Since L^* is a locally defined line bundle and since ∇^W is a B_2-connection on W, Corollary (4.2) clearly implies Zandi's result.

ADDED IN PROOF. After the completion of this paper, I received a preprint : M.M.Capria and S.M.Salamon "Yang-Mills fields on quaternionic Kähler spaces" , which gives (i) for (2.6),a slightly stronger result and (ii) a statement similar to (3.2).

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