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Vector bundles over quaternionic Kähler manifolds

Takashi Nitta
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Takashi Nitta

Introduction

On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let $E$ be a vector bundle over a quaternionic Kähler manifold $M$, and $p: Z \rightarrow M$ the corresponding twistor space defined by Salamon [Sl]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between $E$'s with such connections and the set of Einstein-Hermitian vector bundles over $Z$.

Let $\mathbb{H}$ be the skew field of quaternions. Then the $\text{Sp}(n) \cdot \text{Sp}(1)$-module $\Lambda^2 \mathbb{H}^n$ is a direct sum $N^1_2 \oplus N^2_2 \oplus L_2$ of its irreducible submodules $N^1_2, N^2_2, L_2$, where $N^1_2$ (resp. $L_2$) is the submodule of the elements fixed by $\text{Sp}(n)$ (resp. $\text{Sp}(1)$).
and for \( n = 1 \), we have \( N_2^n = \{0\} \). Hence, the vector bundle \( \Lambda^2 T^* M \) is written as a direct sum \( A_2' \oplus A_2'' \oplus B_2 \) of its holonomy-invariant subbundles in such a way that \( A_2', A_2'', B_2 \) correspond respectively to \( N_2', N_2'', L_2 \). Now, a connection for \( E \) is called an \( A_2' \)-connection (resp. \( B_2 \)-connection) if the corresponding curvature is an \( \text{End}(E) \)-valued \( A_2' \)-form (resp. \( B_2 \)-form). Then we have:

**Theorem (0.1).** All \( A_2' \)-connections and also all \( B_2 \)-connections are Yang-Mills connections.

Furthermore, for \( E \) with a \( B_2 \)-connection we can associate an \( E \)-valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of \( B_2 \)-connections (see Theorem (3.5)).

For our quaternionic Kähler manifold \( M \), a pair \((E, D_E)\) of a vector bundle \( E \) over \( M \) and a \( B_2 \)-connection \( D_E \) on \( E \) is called a **Hermitian pair** on \( M \) if \( D_E \) is a Hermitian connection on \( E \). On the other hand, a pair \((F, D_F)\) of a holomorphic vector bundle over \( Z \) and a Hermitian \((1, 0)\)-connection \( D_F \) on \( F \) is called an **excellent pair** on \( Z \) if the following conditions are satisfied:

(a) \( F \) with the corresponding Hermitian metric \( h_F \) restricts to a flat bundle.
on each fibre of $p: Z \rightarrow M$. (Hence the real structure
\[ \tau : Z \rightarrow Z \] (cf. Nitta and Takeuchi [N-T]) naturally lifts to
a bundle automorphism $\tau' : F \rightarrow F$.)

(b) Let $\sigma : F \rightarrow F^*$ be the bundle map defined
by $F_z \ni f \mapsto \sigma(f) \in F_{\tau(z)}^*(z \in Z)$, where
$\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then $\sigma$ is an
antiholomorphic bundle automorphism. We then have the following
generalization of a result of Penrose's type (cf. Atiyah, Hitchin
and Singer [A-H-S]; see also Salamon [S2], Berard-Bergery
and Ochiai [B-O]):

Theorem (0.2). Let $\mathcal{H}$ (resp. $\mathcal{H}$) be the set of all
Hermitian pairs (resp. all excellent pairs) on $M$ (resp. $Z$).
Then

\[ \mathcal{H} \ni (E, D_E) \quad \mapsto \quad (p^*E, p^*D_E) \in \mathcal{H} \]

defines a bijective correspondence between $\mathcal{H}$ and $\mathcal{H}$.

In particular, if $M$ has positive scalar curvature, then every
excellent pair $(F, D_F)$ on $Z$ is a Ricci-flat Einstein-Hermitian
vector bundle.

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1. Notation, convention and preliminaries

In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [Sl], Nitta and Takeuchi [N-T]).

(1.1) Let $H^m$ denote the standard $Sp(m)$-module $^\#$ $H^m (= \mathbb{C}^{2m})$ of complex dimension $2m$, where $H = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R} (= \mathbb{C} + j\mathbb{C})$. Then the multiplication on $H^m$ by $j$ from the right naturally induces a $Sp(m)$-equivariant anti-linear map $j(m) : H^m \rightarrow H^m$ with $(j(m))^2 = -id$. We now define a non-degenerate skew-symmetric bilinear form $\omega^m$ on $H^m$ by

$$\omega^m (h, h') := -<h, j(m) h'> \quad (h, h' \in H^m),$$

where $< , >$ is the standard Hermitian inner product on $\mathbb{C}^{2m}$ ($= H^m$). This $\omega^m$ can be regarded as an $Sp(m)$-invariant bilinear form on $H^m$ such that

(1.1.1) $\omega^m (j(m) h, j(m) h') = (\omega^m (h, h'))^* \quad (h, h' \in H^m)$.

Let $Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1) / \mathbb{Z}_2$. Then $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ is naturally a $Sp(n) \cdot Sp(1)$-module of complex dimension $4n$ with a real structure $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \ni a \mapsto a \in H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ defined by

(1.1.2) $(h \otimes h')^* := j^{(n)} h \otimes j^{(1)} h' \quad (h \in H^{(n)}, h' \in H^{(1)})$.

$^\#$ $Sp(m) = \{ S \in GL(m, H) \mid S \cdot \overline{S} = I \}$ is imbedded in $GL(2m, \mathbb{C})$ by $Sp(m) \ni A + jB \mapsto (A, -B, B, A) \in GL(2m, \mathbb{C})$ where $A, B \in GL(m, \mathbb{C})$. 
We consider the corresponding real form \((H^{(n)} \otimes \mathbb{C} H^{(1)})_{\mathbb{R}}\) of \(H^{(n)} \otimes \mathbb{C} H^{(1)}\). Then the symmetric bilinear form \(\omega^{(n)} \otimes \omega^{(1)}\) \(\in \text{S}^{2}(\mathbb{H}^{(n)})^* \otimes (\mathbb{H}^{(1)})^*\) induces an inner product \(\langle , \rangle\) on \((H^{(n)} \otimes \mathbb{C} H^{(1)})_{\mathbb{R}}\).

(1.2) Recall that a 4n-dimensional Riemannian manifold \((M, g_M)\) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in \(\text{Sp}(n) \cdot \text{Sp}(l)\) \((\subset \text{SO}(4n))\) with the additional condition for \(n = l\) that \(g_M\) is a self-dual Einstein metric. Throughout this paper, we fix once for all a quaternionic Kähler manifold \((M, g_M)\).

By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle \(TM\) is reduced to a principal \(\text{Sp}(n) \cdot \text{Sp}(l)\)-bundle \(P\). Then \(TM\) can be regarded as the vector bundle

\[
P \times_{\text{Sp}(n) \cdot \text{Sp}(l)} (H^{(n)} \otimes \mathbb{C} H^{(1)})_{\mathbb{R}}
\]

associated to the \(\text{Sp}(n) \cdot \text{Sp}(l)\)-module \((H^{(n)} \otimes \mathbb{C} H^{(1)})_{\mathbb{R}}\).

The inner product \(\langle , \rangle\) on \((H^{(n)} \otimes \mathbb{C} H^{(1)})_{\mathbb{R}}\) induces a Riemannian metric \(g\) on \(TM\), which coincides with \(g_M\) up to constant multiple. Without loss of generality, we may assume \(g = g_M\).

(1.3) Let \(\text{Sp}(n)\) act trivially on \(\mathbb{C}^2\). Then the standard \(\text{Sp}(l)\)-action on \(\mathbb{C}^2\) naturally induces an \(\text{Sp}(n) \times \text{Sp}(l)\)-action (resp. \(\text{Sp}(n) \cdot \text{Sp}(l)\)-action) on \(\mathbb{C}^2\) (resp. \(\mathbb{P}^1\mathbb{C}\)). Associated to these actions, we have:
\[ \hat{\rho} : V := \mathbb{P} \times \text{Sp}(n) \times \text{Sp}(l) \mathbb{C}^2 \longrightarrow M \]
\[ \text{resp.} \quad p : Z := \mathbb{P} \times \text{Sp}(n) \cdot \text{Sp}(l) \mathbb{P}^1 \mathbb{C} \longrightarrow M, \]

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle \( Z \) is nothing but \( \mathbb{P}(V) := V - \{ \text{zero section} \} / \mathbb{C}^* \), and is called the twistor space of \( M \) (see Salamon [Sl; p.147]). Then \( Z \) is a complex manifold with a natural real structure \( \tau \) as follows:

\[ (1.3.1) \quad \text{By the connection on} \quad V \quad \text{induced from that of} \quad P, \quad \text{we have a decomposition of} \quad T(V - \{ \text{zero section} \}) \quad \text{into the subbundles} \quad S^h \quad \text{and} \quad S^v \quad \text{corresponding respectively to horizontal and vertical distributions. Let} \quad y \quad \text{be an arbitrary point of} \quad V - \{ \text{zero section} \}, \quad \text{and put} \quad x := \hat{\rho}(y). \quad \text{Via the projection} \quad \hat{\rho}, \quad \text{the fibre} \quad (S^h)_y \quad \text{of} \quad S^h \quad \text{over} \quad y \quad \text{is regarded as the tangent space} \quad T_x M \quad \text{at} \quad x. \quad \text{Then by the identification of} \quad H^{(n)} \otimes_C H^{(l)} \quad \text{with} \quad T_x M^C \quad (\text{cf. (1.2.1)}), \quad \text{the space} \quad H^{(n)} \otimes_C C y \quad \text{defines a} \quad C \text{-linear subspace of} \quad (T_x M)^C, \quad \text{denoted also by} \quad H^{(n)} \otimes_C C y. \quad \text{Furthermore, let} \quad (H^{(n)} \otimes C y)^* \quad \text{be the subspace of} \quad (T^*x M)^C \quad \text{corresponding to} \quad H^{(n)} \otimes_C C y \quad \text{via the natural isomorphism} \quad (T^*x M)^C \cong (T_x M)^C \quad \text{induced by} \quad g_M. \quad \text{Now we define complex structure of} \quad T_y V \quad \text{by specifying the subspace} \quad \Lambda^1,0_y \quad \text{of} \quad (1,0)\text{-forms in} \quad (T^*y V)^C \quad \text{as follows:}
\]
\[ \Lambda^1,0_y = (\Lambda^1,0_y^h \oplus (\Lambda^1,0_y)^v, \]

where \((\Lambda_{y}^{1,0})_{h}:=\tilde{\rho}^{*}((\Lambda^{n} (\otimes \nu_{y})')\)), and \((\Lambda_{y}^{1,0})_{V}\) is the subspace of \((1,0)\)-forms in \(T_{y}t^{2}\) by the identification of \(V_{x}\) with \(\mathbb{C}^{2}\). Then this induces a complex structure on \(Z\).

(1.3.2) The map \(j^{(1)}: H^{(1)} \to H^{(1)}\) naturally defines an antilinear bundle automorphism \(\tau: V \to V\), which induces a real structure \(\tau\) on \(Z\).

(1.3.3) Recall that \(M\) always has a constant scalar curvature (denoted by \(t\)). Let \(g_{F}\) be the Fubini-Study metric for \(p^{1}\mathbb{C}\) (= \((\mathbb{C} + j\mathbb{C} - \{0\})/\mathbb{C}^{*}\)). If \(t \neq 0\), then for some nonzero real constant \(c_{t}\),

\[g_{Z}:=p^{*}g_{M} + c_{t}g_{F}\]

defines a pseudo-Kählerian metric on \(Z\), i.e., the corresponding \((1,1)\)-form on \(Z\) is a nondegenerate \(d\)-closed \((1,1)\)-form.
2. \( A_2' \)-connections and \( B_2 \)-connections

We shall here give fundamental properties of the \( A_2' \)-connections and \( B_2 \)-connections defined in the Introduction.

\[(2.1) \quad \text{Let} \ (H^{(m)})^* \text{ be the dual } \text{Sp}(m) \text{-module of } H^{(m)} .\]

Then in view of \( \Lambda^2 (H^{(1)})^* = \mathcal{C} \omega^{(1)} \), we have

\[
\Lambda^2 ((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = (\Lambda^2 (H^{(n)})^* \otimes_{\mathbb{C}} S^2 (H^{(1)})^*) \oplus (S^2 (H^{(n)})^* \otimes_{\mathbb{C}} \mathcal{C} \omega^{(1)}) .
\]

Furthermore, the \( \text{Sp}(n) \)-module \( \Lambda^2 (H^{(n)})^* \) is written as a direct sum \( \mathcal{C} \omega^{(n)} + \Lambda^2_0 (H^{(n)})^* \) of its submodules, where \( \Lambda^2_0 (H^{(n)})^* \) is the orthogonal complement of \( \mathcal{C} \omega^{(n)} \) in \( \Lambda^2 (H^{(n)})^* \). Hence,

\[(2.1.1) \quad \Lambda^2 ((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = N_2^1 \mathbb{C} \oplus N_2^2 \mathbb{C} \oplus L_2 \mathbb{C} ,
\]

where \( N_2^1 \mathbb{C} := \mathcal{C} \omega^{(n)} \otimes_{\mathbb{C}} S^2 (H^{(1)})^* , \ N_2^2 \mathbb{C} := \Lambda^2_0 (H^{(n)})^* \otimes_{\mathbb{C}} S^2 (H^{(1)})^* \) and \( L_2 \mathbb{C} := S^2 (H^{(n)})^* \otimes_{\mathbb{C}} \mathcal{C} \omega^{(1)} . \) Note that the \( \text{Sp}(n) \cdot \text{Sp}(1) \)-modules \( N_2^1 \mathbb{C} , N_2^2 \mathbb{C} , L_2 \mathbb{C} \) respectively admit real forms \( N_2^1 , N_2^2 , L_2 \) fixed by the real structure induced from the one in \( (1.1.2) \). We have the identification \( H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong (H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^* \) by the metric \( \langle \cdot , \cdot \rangle \) (cf. \( (1.1) \)). Together with \( H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong H^n \otimes_{\mathbb{R}} \mathbb{C} \), the above \( (2.1.1) \) induces the decomposition of its real form:
\[ \Lambda^2 H^n = N_2' \oplus N_2'' \oplus L_2, \]

which is nothing but the decomposition in the Introduction now for our principal $\text{Sp}(n) \cdot \text{Sp}(l)$-bundle $P$, the bundle $T^*M$ is regarded as the vector bundle associated to the $\text{Sp}(n) \cdot \text{Sp}(l)$-module \(((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(l)})^*))_R = H^n$. Hence, $\Lambda^2 T^*M$ is a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its subbundles $A_2', A_2'', B_2$ corresponding respectively to the $\text{Sp}(n) \cdot \text{Sp}(l)$-modules $N_2', N_2'', L_2$ (cf. Introduction).

(2.2) Fix an arbitrary point $x$ of $M$. Note that each point $z$ on the fibre $Z_x$ defines an almost complex structure $J_z$ on $T^*_x M$ (cf. (1.3.1)). We then have the corresponding space $\Lambda^{1,1}(T^*_x M, J_z)$ of $(1,1)$-forms of $(T^*_x M, J_z)$. Choose a point $y(\neq 0)$ of $V$ such that its natural image (denoted by $[y]$) is $z$. In view of (1.3.1), the space $\Lambda^{1,1}(T^*_x M, J_z)$ in $\Lambda^2(T^*_x M)_x$ is associated to the $\mathcal{C}$-linear subspace

\[ (H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} y)' \sim ((H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} y)')' \text{ in the } \text{Sp}(n) \cdot \text{Sp}(l)-\text{module} \]

\[ (H^{(n)} \otimes_{\mathbb{C}} H^{(l)})^* \sim (H^{(n)} \otimes_{\mathbb{C}} H^{(l)})^*. \]

Since $j^{(n)}$ preserves $H^{(n)}$, we have (cf. (1.1.2)):

\[
(H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} y)^* (H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} y)^{-1} = (H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} y)^* (H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} j^{(l)} y)
\]

\[ = (\Lambda^2 H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} (y \otimes j^{(l)} y + j^{(l)} y \otimes y)) \otimes (S^2 H^{(n)} \otimes_{\mathbb{C}} \mathcal{C} (y \otimes j^{(l)} y)). \]
The space $\mathfrak{C}(\gamma \cdot j^{(1)}y)$ (where $\gamma \cdot j^{(1)}y = (\gamma \otimes j^{(1)}y - j^{(1)}y \otimes y)/2$) in $H^{(1)} \otimes \mathfrak{C}H^{(1)}$ corresponds to $\mathfrak{C} \omega^{(1)}$ in $(H^{(1)} \otimes \mathfrak{C}(H^{(1)}))^*$ via the natural isomorphism $H^{(1)} \otimes \mathfrak{C}H^{(1)} \cong (H^{(1)} \otimes \mathfrak{C}(H^{(1)}))^*$ induced by the nondegenerate bilinear form $\omega^{(1)}$. Furthermore,

$$\bigcap_y \mathfrak{C}(\gamma \otimes j^{(1)}y + j^{(1)}y \otimes y) = \{0\},$$

where $\bigcap_y$ always denotes the intersection taken over all $y$ in $\mathcal{V}_x - \{0\}$. Thus,

$$\bigcap_y (H^{(n)} \otimes \mathfrak{C}y)' \cdot (H^{(n)} \otimes \mathfrak{C}y)' = S^2(H^{(n)} \otimes \mathfrak{C} \omega^{(1)}) = L_2 \quad \text{(cf. Introduction)},$$

and we obtain:

**Lemma (2.3).** The fibre $(B_2)_x$ of $B_2$ over $x$ is given by

$$(B_2)_x = \bigcap_y \Lambda^{1,1}(T_x^*M, J[y]).$$
We next give a typical example of an $A'_2$-connection and also a $B_2$-connection.

Example (2.4). If $n \geq 2$, the induced connection on the locally defined vector bundle

$$V := P \times_{\text{Sp}(n) \times \text{Sp}(1)} H^{(1)}$$

(resp. $W := P \times_{\text{Sp}(n) \times \text{Sp}(1)} H^{(n)}$)

is an $A'_2$-connection (resp. $B_2$-connection). See Salamon [Sl;p.150] for related computations of curvatures.

Recall that a connection $V$ is called a Yang-Mills connection if the corresponding curvature $R^V$ satisfies $d^V R^V = 0$.

We shall finally show:

Theorem (2.5). All $A'_2$-connections and also all $B_2$-connections are Yang-Mills connections.

Corollary (2.6). The Riemannian connection on $TM$ is a Yang-Mills connection.

Proof of (2.6): By (1.2), (2.4) and (2.5), we obtain (2.6).
Proof of (2.5): Fix an arbitrary point $x_0$ of $M$. It then suffices to show $(d^\nabla \cdot R^\nabla)^\nabla(x_0) = 0$. We may take a local section $s$ to $\mathcal{P}$ over a neighbourhood $U$ of $x_0$ such that the corresponding differential at the point $x_0$ transforms the tangent space $T_{x_0}M$ to a horizontal space at $s(x_0)$ in the tangent space $T_s(x_0)^\nabla \mathcal{P}$. Let $(u^1, \cdots, u^{4n})$ be the local frame of $T^*M|_U$ associated to $s$. Then all covariant derivatives of $u^i$'s ($1 \leq i \leq 4n$) at the point $x_0$ is zero. Moreover in terms of the frame $(u^1, \cdots, u^{4n})$, we can identify $T^*M|_U$ with $U \times \mathbb{R}^{4n}$ ($U \times H^n$). Note that $\nabla$ on $E$ naturally induces a connection (denoted by the same $\nabla$) on $\text{End}(E)$.

(i) We first assume that $\nabla$ is an $A_2'$-connection on $E$. Recall that the rank 3 subbundle $A_2'$ of $\wedge^2 T^*M$ corresponds to the $\text{Sp}(n) \cdot \text{Sp}(1)$-submodule $N_2'$ of $\wedge^2 H^n$, where $N_2'$ is the irreducible submodule of the elements fixed by $\text{Sp}(n)$ (cf. Introduction). Let $I, J$ and $K$ be

\[
I = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+2} + u^{4k+3} \wedge u^{4k+4}),
\]
\[
J = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+3} + u^{4k+4} \wedge u^{4k+2}),
\]
\[
K = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+4} + u^{4k+2} \wedge u^{4k+3}).
\]

Then it is easy to check that $A_2'|_U$ is spanned by the sections $I, J$ and $K$. Therefore, the curvature form $R^\nabla$ is written on $U$ as

\[
R^\nabla = a \cdot I + b \cdot J + c \cdot K,
\]
where \( a, b \) and \( c \) are smooth sections to \( \text{End}(E) \) over \( U \).

Let \( (u_1, \ldots, u_{4n}) \) be the base for \( T_M \mid U \) dual to \( (u^1, \ldots, u^{4n}) \)
defined by \( u^i(u_j) = \delta_{ij} \). Then by the first Bianchi identity,

\[
0 = d^V(R^V)(x_0)
= \sum_{i=1}^{4n} \left( (\nabla_i a) u^i(x_0) \wedge (x_0) + (\nabla_i b) u^i(x_0) \wedge (x_0) \wedge (\nabla_i c) u^i(x_0) \wedge (x_0) \right),
\]

where \( \nabla_i \) denotes \( \nabla_{u_i}(x_0) \). Consequently,

\[\nabla_i a = \nabla_i b = \nabla_i c = 0, \quad \text{for} \quad 1 \leq i \leq 4n \quad \text{if} \quad n \geq 2.\]

Therefore, \( (d^V \star R^V)(x_0) = 0. \)

(ii) We next assume that \( \nabla \) is a \( B_2 \)-connection on \( E \).

Since the vector subbundle \( B_2 \) (of rank \( n(2n+1) \)) of \( \Lambda^2 T^* M \)
corresponds to the irreducible \( \text{Sp}(n) \cdot \text{Sp}(1) \)-submodule \( L_2 \)
of the elements in \( \Lambda^2 H^n \) fixed by \( \text{Sp}(1) \), the subbundle \( B_2 \mid U \)
is spanned by

\[I_s, J_s, K_s, D_{pq}, E_{pq}, F_{pq}, G_{pq}, \quad (0 \leq s \leq n-1, \quad 0 \leq p < q \leq n-1).\]

where
\[ I_s = u^{4s+1} u^{4s+2} - u^{4s+3} u^{4s+4}, \]
\[ J_s = u^{4s+1} u^{4s+3} - u^{4s+4} u^{4s+2}, \]
\[ K_s = u^{4s+1} u^{4s+4} - u^{4s+2} u^{4s+3}, \]
\[ D_{pq} = u^{4p+1} u^{4q+1} + u^{4p+2} u^{4q+2} + u^{4p+3} u^{4q+3} + u^{4p+4} u^{4q+4}, \]
\[ E_{pq} = u^{4p+1} u^{4q+2} - u^{4p+2} u^{4q+1} - u^{4p+3} u^{4q+4} + u^{4p+4} u^{4q+3}, \]
\[ F_{pq} = u^{4p+1} u^{4q+3} + u^{4p+2} u^{4q+4} - u^{4p+3} u^{4q+1} - u^{4p+4} u^{4q+2}, \]
\[ G_{pq} = u^{4p+1} u^{4q+4} - u^{4p+2} u^{4q+3} + u^{4p+3} u^{4q+2} - u^{4p+4} u^{4q+1}. \]

Let \( V \) be a \( B_2 \)-connection on \( E \). Then over \( U \), the curvature form \( R^V \) is written in the form

\[ R^V = \sum_{0 \leq s \leq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s) \]
\[ + \sum_{0 \leq p < q \leq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}), \]

where \( i_s, j_s, k_s, d_{pq}, e_{pq}, f_{pq}, \) and \( g_{pq} \) are smooth sections to \( \text{End}(E) \) over \( U \). In view of the first Bianchi identity \( d^V R^V = 0 \), we have

\[ \nabla_{4s+1} i_s + \nabla_{4s+2} j_s + \nabla_{4s+3} k_s = 0, \]
\[ \nabla_{4s+1} i_s - \nabla_{4s+4} j_s + \nabla_{4s+3} k_s = 0, \]
\[ \nabla_{4s+4} i_s + \nabla_{4s+1} j_s - \nabla_{4s+2} k_s = 0, \]
\[ \nabla_{4s+2} i_s + \nabla_{4s+3} j_s + \nabla_{4s+4} k_s = 0, \]
for $s$ with $0 \leq s \leq n-1$. Furthermore, if $\ell$ is either $p$ or $q$, the identity $d^\nabla R^\nabla = 0$ implies

\begin{align*}
(-1)^\ell & \varepsilon(\ell) \nabla_{4\ell+1} d_{pq} - \nabla_{4\ell+2} e_{pq} - \nabla_{4\ell+3} f_{pq} - \nabla_{4\ell+4} g_{pq} = 0, \\
(-1)^\ell & \varepsilon(\ell) \nabla_{4\ell+1} d_{pq} - \nabla_{4\ell+3} e_{pq} + \nabla_{4\ell+2} f_{pq} + \nabla_{4\ell+1} g_{pq} = 0, \\
(-1)^\ell & \varepsilon(\ell) \nabla_{4\ell+2} d_{pq} + \nabla_{4\ell+1} e_{pq} - \nabla_{4\ell+4} f_{pq} + \nabla_{4\ell+3} g_{pq} = 0, \\
(-1)^\ell & \varepsilon(\ell) \nabla_{4\ell+3} d_{pq} + \nabla_{4\ell+4} e_{pq} + \nabla_{4\ell+1} f_{pq} - \nabla_{4\ell+2} g_{pq} = 0,
\end{align*}

for all $p, q$ with $0 \leq p < q \leq n-1$, where $\varepsilon(p) = 0$ and $\varepsilon(q) = 1$.

Then a straightforward computation shows that $(d^\nabla * R^\nabla)(x_0) = 0$, as required.
3. Deformations of $B_2$-connections

In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of $B_2$-connections on $M$ (see Salamon [S2] for a similar complex).

(3.1) Let $r$ be an integer with $r \geq 2$. By setting
\[ N_r^C := \wedge^r (H^{(n)})^* \otimes C^r (H^{(1)})^* \] (cf. (2.1)), we can express the $\text{Sp}(n) \cdot \text{Sp}(1)$-module $\wedge^r (H^{(n)})^* \otimes C^r (H^{(1)})^*$ as a direct sum
\[ N_r^C \oplus L_r^C, \] where $L_r^C$ is the orthogonal complement of $N_r^C$ in $\wedge^r (H^{(n)})^* \otimes C^r (H^{(1)})^*$. As in (2.1), the $\text{Sp}(n) \cdot \text{Sp}(1)$-modules $N_r^C$ and $L_r^C$ respectively admit real forms $N_r$ and $L_r$ fixed by the natural real structure (cf. (1.1.2)). Since $T^*M$ is associated to the $\text{Sp}(n) \cdot \text{Sp}(1)$-module $(H^{(n)})^* \otimes C (H^{(1)})^*_R$ (see (1.2.1)), the vector bundle $\wedge^r T^*M$ is a direct sum
\[ A_r \oplus B_r \] of its subbundles $A_r$, $B_r$ corresponding respectively to $N_r$, $L_r$. Let $\pi^r : \wedge^r T^*M (= A_r \oplus B_r) \longrightarrow A_r$ be the projection to the first factor. Then we have:

**Theorem (3.2).** For a $B_2$-connection $\nabla$ on $E$, the following is an elliptic complex:

\begin{align*}
0 \longrightarrow \mathcal{E}(E) & \xrightarrow{\nabla} \mathcal{E}(E \otimes T^*M) \xrightarrow{d_1} \mathcal{E}(E \otimes A_2) \\
& \xrightarrow{d_2} \mathcal{E}(E \otimes A_3) \xrightarrow{d_3} \cdots \xrightarrow{d_{2n-1}} \mathcal{E}(E \otimes A_{2n}) \longrightarrow 0,
\end{align*}

where $d_i := (\text{id} \otimes \pi^{i+1}) \circ d^\nabla$ and for every vector bundle $E'$. 

on $M$, we denote by $\mathcal{E}(E')$ the sheaf of germs of $C^\infty$-sections of $E'$.

Proof.

(i) Fix a section $s \in \Gamma(M, E \otimes A_i)$ ($i \geq 1$) and define a section $t \in \Gamma(M, E \otimes B_{i+1})$ by

$$d^\nabla s = d_i s + t.$$ 

Then from $(d^\nabla \circ d^\nabla)s = (d^\nabla \circ d_i)s + d^\nabla t$, we obtain

$$(id \otimes_{i+2}) \circ d^\nabla \circ d^\nabla)s = (d_{i+1} \circ d_i)s + ((id \otimes_{i+2}) \circ d^\nabla)t.$$ 

Since $\nabla$ is a $B_2$-connection, the $A_{i+2}$-component of $(d^\nabla \circ d^\nabla)s$ is zero, i.e.,

$$0 = (d_{i+1} \circ d_i)s + ((id \otimes_{i+2}) \circ d^\nabla)t.$$ 

Write $t$ as $t = \Sigma_k v_k \otimes b_k$ locally, where $v_k$, $b_k$ is a local section of $V^*, B_{i+1}$ respectively. The $S^{i+1}(V^*)$-component of $b_k$ is zero, and hence the $S^{i+2}(V^*)$-component of $V(v_k) \cdot b_k$ is zero. Therefore,

$$(id \otimes_{i+2}) \circ d^\nabla)t = \Sigma_k v_k \otimes db_k.$$ 

Since $d$ is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$-component of $db_k$ is zero. Thus, $(d_{i+1} \circ d_i)s = 0$, as required.
(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_i, u)$ ($u \in T^*_x M - \{0\}$). Fix a point of $M$ and an element $s$ of $E_x \sigma A_{i-1}$. All computations below are taken at the point $x$.

\[ \sigma(d_i, u)s := (d/dt)(e^{-tq}d_i(e^{tq} s)|_{t=0} = (i.d \otimes \tau_{i+1})(u, s), \]

where $q$ is a locally defined function such that $dq_x = u$.

We next show that the following sequence is exact for every $u$:

\[ (3.2.2) \quad E \sigma A_{i-1} \xrightarrow{\sigma(d_{i-1}, u)} E \sigma A_i \xrightarrow{\sigma(d_i, u)} E \sigma A_{i+1}. \]

Without loss of generality, we may assume

\[ u = e_1 \otimes h_1 + (e_1 \otimes h_1)^- (= e_1 \otimes h_1 + e_2 \otimes h_2), \]

where $\langle e_1, \ldots, e_{2n} \rangle$ (resp. $\langle h_1, h_2 \rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)} e_{2j+1} = e_{2j+2}$ (resp. $j^{(1)} h_1 = h_2$). Let $s \in E \sigma A_i$ be such that $\sigma(d_{i+1}, u)s = 0$.

Note that $s^i^* V^* = \text{Span}(h_1^k \cdot h_2^{i-k}; 0 \leq k \leq i)$, where $h_1^k \cdot h_2^{i-k}$ denotes the symmetric component of $h_1^k \otimes h_2^{i-k}$.

Hence, there are local sections $s_0, \ldots, s_i$ of $E \sigma^i W^*$ such that

\[ s = \sum_{k=0}^{i} s_k \otimes h_1^k \cdot h_2^{i-k}. \]

We can now write $\sigma(d_{i+1}, s) = 0$ as follows:
\[ 0 = (\text{id} \otimes \pi_{i+1})(\mathbf{u} \otimes s) = (\text{id} \otimes \pi_{i+1})(e_1 \otimes h_1 + e_2 \otimes h_2) \otimes s \otimes h_1^k \cdot h_2^{i-k} \]

\[ = \sum_{k=0}^i (e_1 \otimes s_k) \otimes h_1^{k+1} \cdot h_2^{-i-k} + (e_2 \otimes s_k) \otimes h_1^k \cdot h_2^{i+1-k}. \]

Since the coefficient of the right-hand side in \( h_1^k \cdot h_2^{i+1-k} \) is zero, we have:

\[ (0) \quad e_2 \otimes s_0 = 0, \]

\[ (1) \quad e_1 \otimes s_0 + e_2 \otimes s_1 = 0, \]

\[ \vdots \]

\[ (i) \quad e_1 \otimes s_{i-1} + e_2 \otimes s_i = 0, \]

\[ \vdots \]

\[ (i+1) \quad e_1 \otimes s_i = 0. \]

By (0), there exists \( r_0 \in \Lambda^{i-1}W^* \) such that

\[ s_0 = e_2 \otimes r_0. \]

Plugging this into (1), we obtain

\[ e_2 \otimes (-e_1 \otimes r_0 + s_1) = 0. \]

Hence there exists \( r_1 \in \Lambda^{i-1}W^* \) such that

\[ s_1 = e_1 \otimes r_0 + e_2 \otimes r_1. \]
Repeating this process inductively, we obtain $r_k \in \Lambda^{i-1} W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k$, $1 \leq k \leq i$. Now by (i+1), the identity $e_1 \wedge e_2 \wedge r_i = 0$ holds. It then follows that there exists $r'_i \in \Lambda^{i-2} W^*$ such that $e_2 \wedge r_i = e_1 \wedge e_2 \wedge r'_i$. Since $e_2 \wedge (r_{i-1} + e_2 \wedge r'_i) = e_2 \wedge r_{i-1}$, we may replace $r_{i-1}$ by $r_{i-1} + e_2 \wedge r'_i$. Therefore,

$$
\begin{align*}
    s_0 &= e_2 \wedge r_0', \\
    s_1 &= e_1 \wedge r_0 + e_2 \wedge r_1', \\
    &\vdots \\
    &\vdots \\
    s_i &= e_1 \wedge r_{i-1}
\end{align*}
$$

Thus,

$$
\begin{align*}
    s &= \sum_{k=0}^{i} s_k \otimes h_1^k \cdot h_2^{i-k} \\
    &= \sigma(d_{i-1}, u)(\sum_{k=0}^{i-1} s_k \otimes h_1^k \cdot h_2^{i-1-k}),
\end{align*}
$$

i.e., the sequence (3.2.2) is exact, as required.

Definition (3.3). Let $\mathcal{C}$ be the set of all $B_2$-connections on $E$ with holonomy groups contained in a compact semisimple Lie group $G$. Assume that $\mathcal{C} \neq \emptyset$ and let $\nabla \in \mathcal{C}$. Then the frame bundle $Q$ of $E$ can be regarded as a principal $G$-bundle. Put $G_Q := Q \times_\emptyset G$ and $\mathcal{F}_Q := Q \times_{\text{Ad}} G$, where $\emptyset$ is the group conjugation and $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ is the adjoint representation of $G$. Now, a $C^\infty$-section to $G_Q$ over $M$ is called a gauge transformation of $Q$. Let $\mathcal{G}$ be the set of all gauge transformations of $Q$. Then $\mathcal{G}$ naturally acts on $\mathcal{C}$ (see Atiyah-Hitchin-Singer [A-H-S]). We call $\mathcal{M} := \mathcal{C}/\mathcal{G}$ the
moduli space of the $B_2$-connections on $E$ with holonomy groups in $G$.

(3.4) Let $\nabla \in \mathcal{C}$ be irreducible in the sense that $\mathfrak{g}_Q$ admits no nonzero parallel section over $M$. Fix a smooth one-parameter family $\nabla^t (|t| < \epsilon)$ of connections in $\mathcal{C}$ such that $\nabla^0 = \nabla$. Put $S = (d/dt)\nabla^t |_{t=0}$. We write the curvature form $R^\nabla^t$ of $\nabla^t$ as

$$R^\nabla^t = R^\nabla + t d^\nabla' S + \text{higher order terms in } t,$$

where $\nabla'$ is the connection on $\mathfrak{g}_Q$ naturally induced by $\nabla$. Since $R^\nabla$ is a $\mathfrak{g}_Q$-valued $B_2$-form, the corresponding derivative $d^\nabla' S$ at $t=0$ also satisfies

$$((id \otimes \theta^2) \circ d^\nabla') S = 0.$$

Let $f^t (|t| < \epsilon)$ be a one-parameter family of gauge transformations such that $f^0 = id$. Then,

$$\left. \frac{d}{dt}(f^t(\nabla)) \right|_{t=0} = \nabla'(\dot{f}) ,$$

where $\dot{f} := (d/dt)(f^t) |_{t=0}$. Since $f^t(\nabla) \in \mathcal{C}$ for all $t$, the same argument as above shows that the $\mathfrak{g}_Q$-valued $1$-form $\nabla'(\dot{f})$ satisfies
For each $A \in \Gamma(\mathcal{F}_Q)$, there exists a one-parameter family $f^t = \exp(tA)$ such that $(d/dt)f^t|_{t=0} = A$. Then together with (3.2), we immediately obtain the following:

**Theorem (3.5).** Assume that $\mathcal{C} \neq \emptyset$ and let $\mathcal{V} \in \mathcal{C}$ be irreducible. Then the space of infinitesimal (essential) deformations at $\mathcal{V}$ of connections in $\mathcal{C}$, that is, the tangent space of $\mathcal{M}$ at $\mathcal{V}$ is a linear subspace of the first cohomology group of the elliptic complex

$$
0 \longrightarrow \mathcal{E}(\mathcal{F}_Q) \overset{\mathcal{V}'}{\longrightarrow} \mathcal{E}(\mathcal{F}_Q \otimes \Omega^1 M) \overset{d_1'}{\longrightarrow} \mathcal{E}(\mathcal{F}_Q \otimes \mathcal{A}_2) \overset{d_2'}{\longrightarrow} \mathcal{E}(\mathcal{F}_Q \otimes \mathcal{A}_3) \overset{d_3'}{\longrightarrow} \cdots \overset{d_{2n-1}'}{\longrightarrow} \mathcal{E}(\mathcal{F}_Q \otimes \mathcal{A}_{2n}) \longrightarrow 0,
$$

where $d_1' := (\text{id} \otimes \pi^{i+1}) \circ \mathcal{V}'$. 

4. **Einstein-Hermitian connections associated with** $B_2$-connections

In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between $B_2$-connections and the corresponding Einstein-Hermitian connections.

Proof of (0.2) : (i) Let $(E, D_E)$ be a Hermitian pair. Then by the definition of $B_2$-connections, the curvature form corresponding to the connection $D_E$ is an $\text{End}(E)$-valued $B_2$-form, and by Lemma (2.3) the curvature form corresponding to the connection $p^*D_E$ on $p^*E$ is an $\text{End}(p^*E)$-valued $(1,1)$-form. Hence the connection $p^*D_E$ induces naturally an integrable complex structure on $p^*E$ as follows: Put $\ell := \text{rank}(E)$ and denote by $q : p^*E \to \mathbb{Z}$ the natural projection. Let $(s_1, \cdots, s_\ell)$ (resp. $(y^1, \cdots, y^\ell)$) be a local unitary frame for $p^*E$ (resp. the dual frame corresponding to $(s_1, \cdots, s_\ell)$). Then the vector subbundle $\mathfrak{A}^{1,0}_{T^*}(p^*E)$ of type $(1,0)$ in the complexification $T^*(p^*E)^c$ of the cotangent bundle $T^*(p^*E)$ is defined as the direct sum of the pull-back $q^*(\mathfrak{A}^{1,0}_{T^*} \mathbb{Z})$ and the space spanned by $\{ d\theta^j_i + \Sigma_{l=1}^\ell q^j_l \theta_{lji}, \ 1 \leq j \leq \ell \}$, where $(\theta_{lji})$ is the connection matrix for $p^*D_E$ with respect to the frame $(s_1, \cdots, s_\ell)$ (i.e., $(p^*D_E) s_j = \Sigma_{l=1}^\ell s_l \theta_{lji}$). Now, we may take the frame $(s_1, \cdots, s_\ell)$ as the pull-back $(p^*t_1, \cdots, p^*t_\ell)$ of a local unitary frame $(t_1, \cdots, t_\ell)$ on $E$. Then the 1-forms $\theta_{lji}, \ 1 \leq i, j \leq \ell$, are written
as $p^*\psi_{ij}$, where $(\psi_{ij})$ denotes the connection matrix for $D_E$ with respect to the frame $(t_1, \ldots, t_N)$. Let $q'(p^*E)^* \to Z$ be the projection naturally induced from $q : p^*E \to Z$. Since the real structure $\tau : Z \to Z$ is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping $q' \circ \sigma : p^*E \to Z$ is equal to $\tau \circ q$, the mapping $\sigma : p^*E \to (p^*E)^*$ is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on $p^*E$ and $(p^*E)^*$.

(ii) We next fix an arbitrary excellent pair $(F, D_F)$ on $Z$. Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover $\{U_\lambda\}$ of $M$, and a local unitary frame $(f^\lambda_1, \ldots, f^\lambda_N)$ ($N = \text{rank of } F$) of $F|_{p^{-1}(U_\lambda)}$ such that each restriction $(f^\lambda_1|_{p^{-1}(x)}, \ldots, f^\lambda_N|_{p^{-1}(x)})$ over $p^{-1}(x)$ ($x \in U_\lambda$) forms a holomorphic frame for $F|_{p^{-1}(x)}$. When $U_\lambda \cap U_\mu \neq \emptyset$, the transition matrix for $F$ in terms of the frames $(f^\lambda_1, \ldots, f^\mu_N)$, $(f^\mu_1, \ldots, f^\mu_N)$ is holomorphic (and hence constant) along each fibre $p^{-1}(x)$ ($x \in U_\lambda \cap U_\mu$). Hence there exists a Hermitian vector bundle $E$ on $M$ such that, including metrics, we have $p^*E = F$. In particular, we obtain a local unitary frame $(f^\lambda_1, \ldots, f^\lambda_N)$ for $E|_{U_\lambda}$ such that $(p^*f^\lambda_1, \ldots, p^*f^\lambda_N)$ coincides with the previous $(f^\lambda_1, \ldots, f^\lambda_N)$ over $p^{-1}(U_\lambda)$. Fix an arbitrary $\lambda$. If there is no fear of confusion, we shall omit the suffix $\lambda$ and denote $U_\lambda$, $(f^\lambda_1, \ldots, f^\lambda_N)$, $\cdots$ simply by $U$, $(f_1, \ldots, f_N)$, $\cdots$. 
respectively. Let \((\omega_{ij})\) be the connection matrix of \(D_F\) with respect to the frame \((f_1, \cdots, f_r)\), i.e., \(D_F f_j = \sum_{i=1}^{r} f_i \omega_{ij}\).

Furthermore, we choose a local symplectic basis \((e_1, \cdots, e_{2n})\) (resp. \((h_1, h_2)\)) for \(W^*_U\) (resp. \(V^*_U\)) (see Section 3). Now, since \(D_F\) is a Hermitian connection, we have:

\[(1) \quad \omega_{ij} + \omega_{ji} = 0, \quad \text{for } 1 \leq i, j \leq r.\]

Then the construction of \(D_E\) is reduced to showing that there exist 1-forms \(\omega'_{ij}\) (\(1 \leq i, j \leq r\)) on \(U\) satisfying \(\omega_{ij} = p^* \omega'_{ij}\). In fact, once we can find such 1-forms \(\omega'_{ij}\), they define a Hermitian connection on \(E\), such that the corresponding curvature form is pulled back by \(p\) to an \(\text{End}(F)\)-valued \((1, 1)\)-form on \(Z\), which together with Lemma (2.3) implies that our connection on \(E\) is a \(B_2\)-connection. Recall that, for each \(x \in U\), the frame \((f_1|_{p^{-1}(x)}, \cdots, f_r|_{p^{-1}(x)})\) for \(F|_{p^{-1}(x)}\) is trivial. Hence,

\[(2) \quad \omega_{ij}(v) = 0, \quad 1 \leq i, j \leq r,\]

for every vector \(v\) tangent to \(p^{-1}(x)\) (\(\cong \mathbb{P}^1 \mathbb{C}\)). Since \((e_1 \otimes h_1, e_1 \otimes h_2, \cdots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)\) is a frame for \(T^*_M^\mathbb{C}|_U = W^*|_U \otimes V^*|_U\), there exist by (2) \(C^\infty\)-functions \(a_{ij}^k, b_{ij}^k\) (\(1 \leq i, j \leq r, 1 \leq k \leq 2n\)) on \(p^{-1}(U)\) such that
\[ (3) \quad \omega_{ij} = \sum_{k=1}^{2n} \left( a_{ij}^k p^*(e_k \otimes h_1) + b_{ij}^k p^*(e_k \otimes h_2) \right), \quad 1 \leq i, j \leq r. \]

For every form \( \eta \) on \( Z|_U \), we denote by \( \hat{\eta} \) the pull-back of \( \eta \) to \( (V - \{ \text{zero section} \})|_U \). Then by (3), we have:

\[
\hat{\omega}_{ij} = d\hat{\omega}_{ij} + \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj} \\
= \sum_{k=1}^{2n} \left( \hat{a}_{ij}^k p^*(e_k \otimes h_1) + \hat{b}_{ij}^k p^*(e_k \otimes h_2) \right) + \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj}.
\]

Fix an arbitrary point \( x \) on \( U \). Choosing an appropriate \( (e_1, \cdots, e_{2n}) \) (resp. \( (h_1, h_2) \)), we may assume that \( (V^* e_k)(x) = 0 \), \( k = 1, 2, \cdots, 2n \) (resp. \( (V^* h_i)(x) = 0 \), \( i = 1, 2 \)), where \( V^* \) (resp. \( W^* \)) denotes the connection of \( V^* \) (resp. \( W^* \)) canonically induced by that of \( P \) (cf. Example (2.4)). Then, on \( \hat{p}^{-1}(x) \),

\[
\hat{R}_{ij} = \sum_{k=1}^{2n} \left\{ \hat{a}_{ij}^k \wedge \hat{p}^*(e_k \otimes h_1) + \hat{b}_{ij}^k \wedge \hat{p}^*(e_k \otimes h_2) \right\} \\
+ \sum_{t=1}^{r} \hat{\omega}_{it} \wedge \hat{\omega}_{tj}.
\]

Recall that the complex structure on the twistor space \( Z (= (V - \{ \text{zero section} \}) / \mathbb{C}^*) \) is induced by the complex structure on \( V - \{ \text{zero section} \} \) (see Section 1). Since \( \hat{R}_{ij} \) is of type \((1,1)\), we have:

\[
(4) \quad \sum_{k=1}^{2n} \left\{ \hat{a}_{ij}^k \wedge (\hat{p}^*(e_k \otimes h_1))(1,0) + \hat{b}_{ij}^k \wedge (\hat{p}^*(e_k \otimes h_2))(1,0) \right\} \\
+ \sum_{t=1}^{r} \hat{\omega}_{it} \wedge (1,0) \wedge \hat{\omega}_{tj} \wedge (1,0) = 0 \quad \text{on } \hat{p}^{-1}(x). 
\]
\[(5) \sum_{k=1}^{2n} \left( \frac{2}{2} \left( a_{ij}^{k} \right) + \left( p^\ast(e_k \otimes h_1) \right) (0,1) + \left( b_{ij}^{k} \right) + \left( p^\ast(e_k \otimes h_2) \right) (0,1) \right) \right. \\
\left. + \sum_{t=1}^{\omega^\ast_{W_{\text{mit}}} \omega_{W_{\text{tj}}}} (0,1) \right) = 0 \quad \text{on} \quad p^{-1}(x), \]

where for every 1-forms \( \xi \) on \((V \setminus \{\text{zero section}\}) \mid U \), \( \xi^{(1,0)} \) (resp. \( \xi^{(0,1)} \)) always denotes the \((1,0)\)-component (resp. \((0,1)\)-component) of \( \xi \). Let \((z^1, z^2)\) be the local triviality for \( V \mid U \) corresponding to \((h_1, h_2)\). Then, by the definition of the complex structure of \((V \setminus \{\text{zero section}\})\), we obtain from \((4)\) and \((5)\) the following:

\[(4') \sum_{k=1}^{2n} \left( \frac{2}{2} \left( a_{ij}^{k} \right) \otimes d\bar{z}^1 + \frac{2}{2} \left( a_{ij}^{k} \right) \otimes d\bar{z}^2 \right) \otimes z^{-1} \left( z^1 p^\ast(e_k \otimes h_1) + z^2 p^\ast(e_k \otimes h_2) \right) \right. \\
\left. + \left( \frac{2}{2} \left( b_{ij}^{k} \right) \otimes d\bar{z}^1 + \frac{2}{2} \left( b_{ij}^{k} \right) \otimes d\bar{z}^2 \right) \otimes z^{-2} \left( z^1 p^\ast(e_k \otimes h_1) + z^2 p^\ast(e_k \otimes h_2) \right) \right) \right. \\
\left. = 0 \quad \text{on} \quad p^{-1}(x) \right); \\

\[(5') \sum_{k=1}^{2n} \left( \frac{2}{2} \left( a_{ij}^{k} \right) \otimes d\bar{z}^1 + \frac{2}{2} \left( a_{ij}^{k} \right) \otimes d\bar{z}^2 \right) \otimes (-z^1) \left( z^1 p^\ast(e_k \otimes h_1) - z^2 p^\ast(e_k \otimes h_2) \right) \right. \\
\left. + \left( \frac{2}{2} \left( b_{ij}^{k} \right) \otimes d\bar{z}^1 + \frac{2}{2} \left( b_{ij}^{k} \right) \otimes d\bar{z}^2 \right) \otimes z^{-1} \left( z^1 p^\ast(e_k \otimes h_1) - z^2 p^\ast(e_k \otimes h_1) \right) \right) \right. \\
\left. = 0 \quad \text{on} \quad p^{-1}(x) \right). \\

Since both \( z^1 \mid p^{-1}(x) \) and \( z^2 \mid p^{-1}(x) \) are holomorphic on \( p^{-1}(x) \cong \mathbb{C}^2 - \{0\} \), we have
\[
\frac{3}{\partial z_i}(z^{1A} i_j + z^{2B} i_j) = \frac{3}{\partial z_i}(-z^{1A} i_j + z^{1B} i_j) = 0 \quad (i = 1,2),
\]

on \( p^{-1}(x) \), i.e., both \( f_1(z^1, z^2) := z^{1A} i_j + z^{2B} i_j \) and \( f_2(z^1, z^2) := -z^{1A} i_j + z^{1B} i_j \) are holomorphic on \( \mathbb{C}^2 - \{0\} \).

By Hartogs' theorem, both \( f_1 \) and \( f_2 \) extend further to holomorphic functions on \( \mathbb{C}^2 \). Since \( f_i(cz^1, cz^2) = cf_i(z^1, z^2) \) for all \( z = (z^1, z^2) \in \mathbb{C}^2 \) and \( c \in \mathbb{C}^* \) (\( i = 1, 2 \)), there exist constants \( a_{ij}^k, b_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k \in \mathbb{C} \) independent of \( z \) such that

\[
\begin{align*}
(6) \quad & z^{1A} i_j + z^{2B} i_j = z^{1A} i_j + z^{2B} i_j, \\
(7) \quad & -z^{1A} i_j + z^{1B} i_j = -z^{1A} i_j + z^{1B} i_j, \quad (1 \leq k \leq 2n).
\end{align*}
\]

Let \( \Gamma(F^*) \) (resp. \( \Gamma(F^* \otimes T^* Z^C) \)) be the space of global \( C^\infty \)-sections over \( F \) to \( F^* \) (resp. \( F^* \otimes T^* Z^C \)).

Let \( \psi : \Gamma(F^*) \longrightarrow \Gamma(F^* \otimes T^* Z^C) \) be the \( \mathbb{C} \)-linear map sending each \( s \in \Gamma(F^*) \) to an element \( \psi(s) \) of \( \Gamma(F^* \otimes T^* Z^C) \) defined by

\[
\psi(s)(X) := \sigma((D_F^* \tau_X)(\sigma^{-1}s)) \in F_{\tau_X} Z^C,
\]

for \( X \in T^* Z^C \) (\( z \in Z \)).
Then by the condition (b) in the Introduction, this $\psi$ defines a Hermitian $(1,0)$-connection on the holomorphic vector bundle $F^*$. The corresponding connection matrix with respect to the frame $(\sigma_{f_1}, \cdots, \sigma_{f_r})$ for $F^*|_{P^{-1}(U)}$ is written as

$$(\tau^*\omega_{ij})^* = \omega_{ij}^*.$$ By the definition of $\sigma$, it is easy to check that the frame $(\sigma_{f_1}, \cdots, \sigma_{f_r})$ is dual to our previous $(f_1, \cdots, f_r)$. Hence the uniqueness of the $(1,0)$-connection on the Hermitian vector bundle $F^*$ implies the equality $(\tau^*\omega_{ij})^* = \omega_{ij}^*$, where $\omega_{ij}^* := -\omega_{ji}$. In view of (1), we have $\tau^*\omega_{ij} = \omega_{ij}$ and $\hat{\tau}^*\hat{\omega}_{ij} = \hat{\omega}_{ij}$. By (3) and $\hat{\rho} \circ \hat{\gamma} = \hat{\rho}$, we obtain:

$$(8) \quad \hat{\tau}^*\hat{\alpha}^k_{ij} = \hat{\alpha}^k_{ij} \text{ and } \hat{\tau}^*\hat{\beta}^k_{ij} = \hat{\beta}^k_{ij} \quad (1 \leq k \leq 2n).$$

Therefore,

$$-\frac{2}{z} \hat{\tau}^*\hat{\alpha}^k_{ij} + \frac{1}{z} \hat{\tau}^*\hat{\beta}^k_{ij} = -\frac{2}{z} \hat{\alpha}^k_{ij} + \frac{1}{z} \hat{\beta}^k_{ij} \quad (1 \leq k \leq 2n).$$

Moreover by (6),

$$-\frac{2}{z} \hat{\alpha}^k_{ij} + \frac{1}{z} \hat{\beta}^k_{ij} = -\frac{2}{z} \alpha^k_{ij} + \frac{1}{z} \beta^k_{ij} \quad (1 \leq k \leq 2n).$$

Hence by (7) and (9), we obtain:

$$(10) \quad \alpha^k_{ij} = \gamma^k_{ij} \text{ and } \beta^k_{ij} = \delta^k_{ij} \quad (1 \leq k \leq 2n).$$
Now, in view of (6), (7) and (10), we see that

\[
\begin{pmatrix}
-z^1, & z^2 \\
-z^2, & z^1
\end{pmatrix}
\begin{pmatrix}
\hat{a}_{ij}^k - a_{ij}^k \\
\beta_{ij}^k - \beta_{ij}^k
\end{pmatrix} = 0 \quad (1 \leq k \leq 2n),
\]

where \((z^1, z^2) \in C^2 - \{0\} (= \tilde{p}^{-1}(x)). Thus, \hat{a}_{ij}^k = \hat{a}_{ij}^k \text{ and } \\
\beta_{ij}^k = \beta_{ij}^k \quad (1 \leq k \leq 2n), i.e., both \ a_{ij}^k \text{ and } \beta_{ij}^k \text{ are constant}
\]

along \(p^{-1}(x), \text{ as required.}\)

Remark (4.1). In some sense, our Theorem (0.2) completely clarifies the following result by Salamon [S2] (see Berard Bergery and Chialai [B-O] for another generalization):

For a Hermitian pair \((E, D_E)\) on \(M\), the pull-back \(p^*E, p^*D_E\) to \(Z\) is a Hermitian holomorphic vector bundle over \(Z\).

Corollary (4.2). Let \((F, D_F)\) be an excellent pair on \(Z\). If the quaternionic Kähler manifold \(M\) has positive scalar curvature, then \(F\) with \(D_F\) is a Ricci-flat Einstein Hermitian vector bundle over \(Z\).
Proof. Consider the twistor space \( p: Z \rightarrow M \).
Then the horizontal component of the Kähler form on \( Z \) is
a \( p^*A^*_2 \)-form (cf. (1.2), (1.3)). Recall that the
curvature of \( D_F \) is an \( \text{End}(F) \)-valued \( p^*B^*_2 \)-form. Hence the
Hermitian vector bundle \( F \) with \( D_F \) is Ricci-flat.

Remark (4.3). We have the decomposition of \( TZ = T^h \oplus T^v \),
where \( T^h \) (resp. \( T^v \)) is the horizontal (resp. vertical) distribution
in terms of the connection on \( Z \) induced by that of \( P \).
Since the complex structure on \( TZ \) is a direct sum of complex
structures on \( T^h \) and \( T^v \), the holomorphic part \( TZ^{(1,0)} \)
admits the corresponding decomposition \( TZ^{(1,0)} = T^h(1,0) \oplus T^v(1,0) \),
where \( T^h(1,0) \) (resp. \( T^v(1,0) \)) denotes \( T^h \cap TZ^{(1,0)} \)
(resp. \( T^v \cap TZ^{(1,0)} \)). Recently, Zandi [2] obtained the following:

The vector bundle \( (T^{(1,0)}h, D^h) \) is an Einstein-Hermitian
vector bundle, where \( D^h \) is the connection on \( T^{(1,0)}h \)
obtained as the restriction of the Riemannian connection on
\( TZ \) to \( T^{(1,0)}h \).

This result can be regarded as a straightforward consequence
of our (4.2). We denote by \( L \) a locally defined (line)
subbundle of \( p^*W \) (cf. (2.4)) such that, along each fibre
\( p^{-1}(x) = P^1_C (x \in M) \), it restricts to a universal bundle over
\( P^1_C \). Let \( V^L \) (resp. \( W^L \)) denote the connection of \( V \) (resp. \( W \))
canonicaliy induced by that of \( P \) and \( V^L \) the restriction of
\( p^*\psi^W \) to \( L \). Then the vector bundle \( (T^h(1,0),D^h) \) is nothing but \( (p^*W \otimes L^*, p^*\psi^W \otimes (\psi^L)^*) \), where \( (L^*, (\psi^L)^*) \) is dual to \( (L, \psi^L) \) (see Salamon [S1]). Since \( L^* \) is a locally defined line bundle and since \( \psi^W \) is a \( B_2 \)-connection on \( W \), Corollary (4.2) clearly implies Zandi's result.

ADDED IN PROOF. After the completion of this paper, I received a preprint: M.M. Capria and S.M. Salamon "Yang-Mills fields on quaternionic Kähler spaces", which gives (i) for (2.6), a slightly stronger result and (ii) a statement similar to (3.2).
References


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