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Takashi Nitta

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Introduction

On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let E be a vector bundle over a quaternionic Kähler manifold M , and $p: Z \rightarrow M$ the corresponding twistor space defined by Salamon [S1]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between E 's with such connections and the set of Einstein-Hermitian vector bundles over Z .

Let \mathbb{H} be the skew field of quaternions. Then the $\text{Sp}(n) \cdot \text{Sp}(1)$ -module $\Lambda^2 \mathbb{H}^n$ is a direct sum $N_2' \oplus N_2'' \oplus L_2$ of its irreducible submodules N_2', N_2'', L_2 , where N_2' (resp. L_2) is the submodule of the elements fixed by $\text{Sp}(n)$ (resp. $\text{Sp}(1)$)

and for $n = 1$, we have $N_2'' = \{0\}$. Hence, the vector bundle $\Lambda^2 T^*M$ is written as a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its holonomy-invariant subbundles in such a way that A_2' , A_2'' , B_2 correspond respectively to N_2' , N_2'' , L_2 . Now, a connection for E is called an A_2' -connection (resp. B_2 -connection) if the corresponding curvature is an $\text{End}(E)$ -valued A_2' -form (resp. B_2 -form). Then we have :

Theorem (0.1). All A_2' -connections and also all B_2 -connections are Yang-Mills connections.

Furthermore, for E with a B_2 -connection we can associate an E -valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of B_2 -connections (see Theorem (3.5)).

For our quaternionic Kähler manifold M , a pair (E, D_E) of a vector bundle E over M and a B_2 -connection D_E on E is called a Hermitian pair on M if D_E is a Hermitian connection on E . On the other hand, a pair (F, D_F) of a holomorphic vector bundle over Z and a Hermitian $(1,0)$ -connection D_F on F is called an excellent pair on Z if the following conditions are satisfied:.

- (a) F with the corresponding Hermitian metric h_F restricts to a flat bundle

on each fibre of $p : Z \longrightarrow M$. (Hence the real structure $\tau : Z \longrightarrow Z$ (cf. Nitta and Takeuchi [N-T]) naturally lifts to a bundle automorphism $\tau' : F \longrightarrow F$.)

(b) Let $\sigma : F \longrightarrow F^*$ be the bundle map defined by $F_z \ni f \longmapsto \sigma(f) \in F_{\tau(z)}^*$ ($z \in Z$), where $\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an antiholomorphic bundle automorphism. We then have the following generalization of a result of Penrose's type (cf. Atiyah, Hitchin and Singer [A-H-S] ; see also Salamon [S2], Berard-Bergery and Ochiai [B-O]):

Theorem (0.2). Let \mathcal{H} (resp. $\widetilde{\mathcal{H}}$) be the set of all Hermitian pairs (resp. all excellent pairs) on M (resp. Z).

Then

$$\mathcal{H} \ni (E, D_E) \longmapsto (p^*E, p^*D_E) \in \widetilde{\mathcal{H}}$$

defines a bijective correspondence between \mathcal{H} and $\widetilde{\mathcal{H}}$.

In particular, if M has positive scalar curvature, then every excellent pair (F, D_F) on Z is a Ricci-flat Einstein-Hermitian vector bundle.

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1. Notation, convention and preliminaries

In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [S1], Nitta and Takeuchi [N-T]).

(1.1) Let $H^{(m)}$ denote the standard $Sp(m)$ -module^{#)} $H^m (= \mathbb{C}^{2m})$ of complex dimension $2m$, where $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R} (= \mathbb{C} + j\mathbb{C})$. Then the multiplication on H^m by j from the right naturally induces a $Sp(m)$ -equivariant anti-linear map $j^{(m)} : H^{(m)} \longrightarrow H^{(m)}$ with $(j^{(m)})^2 = -id$. We now define a non-degenerate skew-symmetric bilinear form $\omega^{(m)}$ on H^m by

$$\omega^{(m)}(h, h') := -\langle h, j^{(m)} h' \rangle \quad (h, h' \in H^m),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on \mathbb{C}^{2m} ($= H^m$). This $\omega^{(m)}$ can be regarded as an $Sp(m)$ -invariant bilinear form on $H^{(m)}$ such that

$$(1.1.1) \quad \omega^{(m)}(j^{(m)} h, j^{(m)} h') = (\omega^{(m)}(h, h'))^- \quad (h, h' \in H^{(m)}).$$

Let $Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1) / \mathbb{Z}_2$. Then $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ is naturally a $Sp(n) \cdot Sp(1)$ -module of complex dimension $4n$ with a real structure $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \ni a \longmapsto \bar{a} \in H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ defined by

$$(1.1.2) \quad (h \otimes h')^- := j^{(n)} h \otimes j^{(1)} h' \quad (h \in H^{(n)}, h' \in H^{(1)}).$$

#) $Sp(m) = \{S \in GL(m, \mathbb{H}) \mid S \cdot {}^t \bar{S} = I\}$ is imbedded in $GL(2m, \mathbb{C})$ by $Sp(m) \ni A + jB \longmapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \in GL(2m, \mathbb{C})$ where $A, B \in GL(m, \mathbb{C})$.

We consider the corresponding real form $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$. Then the symmetric bilinear form $\omega^{(n)} \otimes \omega^{(1)}$ ($\in S^2((H^{(n)})^* \otimes (H^{(1)})^*)$) induces an inner product $\langle \langle \cdot, \cdot \rangle \rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$.

(1.2) Recall that a $4n$ -dimensional Riemannian manifold (M, g_M) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in $Sp(n) \cdot Sp(1)$ ($\subset SO(4n)$) with the additional condition for $n = 1$ that g_M is a self-dual Einstein metric. Throughout this paper, we fix once for all a quaternionic Kähler manifold (M, g_M) . By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle TM is reduced to a principal $Sp(n) \cdot Sp(1)$ -bundle P . Then TM can be regarded as the vector bundle

$$(1.2.1) \quad P \times_{Sp(n) \cdot Sp(1)} (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$$

associated to the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$. The inner product $\langle \langle \cdot, \cdot \rangle \rangle$ on $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})_{\mathbb{R}}$ induces a Riemannian metric g on TM , which coincides with g_M up to constant multiple. Without loss of generality, we may assume $g = g_M$.

(1.3) Let $Sp(n)$ act trivially on \mathbb{C}^2 . Then the standard $Sp(1)$ -action on \mathbb{C}^2 naturally induces an $Sp(n) \times Sp(1)$ -action (resp. $Sp(n) \cdot Sp(1)$ -action) on \mathbb{C}^2 (resp. $P^1\mathbb{C}$). Associated to these actions, we have:

$$\hat{p} : V(:= P \times_{Sp(n) \times Sp(1)} \mathbb{C}^2) \longrightarrow M$$

$$(\text{resp. } p : Z(:= P \times_{Sp(n) \cdot Sp(1)} \mathbb{P}^1 \mathbb{C}) \longrightarrow M),$$

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle Z is nothing but $\mathbb{P}(V) := V - \{\text{zero section}\} / \mathbb{C}^*$, and is called the twistor space of M (see Salamon [S1; p.147]). Then Z is a complex manifold with a natural real structure τ as follows:

(1.3.1) By the connection on V induced from that of P , we have a decomposition of $T(V - \{\text{zero section}\})$ into the subbundles S^h and S^v corresponding respectively to horizontal and vertical distributions. Let y be an arbitrary point of $V - \{\text{zero section}\}$, and put $x := \hat{p}(y)$. Via the projection \hat{p} , the fibre $(S^h)_y$ of S^h over y is regarded as the tangent space $T_x M$ at x . Then by the identification of $H^{(n)} \otimes_{\mathbb{C}} H^{(1)}$ with $T_x M^{\mathbb{C}}$ (cf. (1.2.1)), the space $H^{(n)} \otimes \mathbb{C}y$ defines a \mathbb{C} -linear subspace of $(T_x M)^{\mathbb{C}}$, denoted also by $H^{(n)} \otimes \mathbb{C}y$. Furthermore, let $(H^{(n)} \otimes \mathbb{C}y)'$ be the subspace of $(T_x^* M)^{\mathbb{C}}$ corresponding to $H^{(n)} \otimes \mathbb{C}y$ via the natural isomorphism $(T_x^* M)^{\mathbb{C}} \cong (T_x M)^{\mathbb{C}}$ induced by g_M . Now we define complex structure of $T_y V$ by specifying the subspace $\Lambda_Y^{1,0}$ of $(1,0)$ -forms in $(T_y^* V)^{\mathbb{C}}$ as follows:

$$\Lambda_Y^{1,0} = (\Lambda_Y^{1,0})^h \oplus (\Lambda_Y^{1,0})^v,$$

where $(\wedge_Y^{1,0})^h := \hat{p}^*((H^{(n)} \otimes \mathbb{C}y)')$, and $(\wedge_Y^{1,0})^v$ is the subspace of $(1,0)$ -forms in $T_Y \mathbb{C}^2$ by the identification of V_x with \mathbb{C}^2 . Then this induces a complex structure on Z .

(1.3.2) The map $j^{(1)} : H^{(1)} \longrightarrow H^{(1)}$ naturally defines an antilinear bundle automorphism $\hat{\tau} : V \longrightarrow V$, which induces a real structure τ on Z .

(1.3.3) Recall that M always has a constant scalar curvature (denoted by t). Let g_F be the Fubini-Study metric for $\mathbb{P}^1 \mathbb{C}$ ($= (\mathbb{C} + j\mathbb{C} - \{0\}) / \mathbb{C}^*$). If $t \neq 0$, then for some nonzero real constant c_t ,

$$g_Z := p^*g_M + c_t g_F$$

defines a pseudo-Kählerian metric on Z , i.e., the corresponding $(1,1)$ -form on Z is a nondegenerate d -closed $(1,1)$ -form.

2. A_2^1 -connections and B_2 -connections

We shall here give fundamental properties of the A_2^1 -connections and B_2 -connections defined in the Introduction.

(2.1) Let $(H^{(m)})^*$ be the dual $Sp(m)$ -module of $H^{(m)}$. Then in view of $\Lambda^2(H^{(1)})^* = \mathbb{C}\omega^{(1)}$, we have

$$\Lambda^2((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = (\Lambda^2(H^{(n)})^* \otimes_{\mathbb{C}} S^2(H^{(1)})^*) \oplus (S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)}).$$

Furthermore, the $Sp(n)$ -module $\Lambda^2(H^{(n)})^*$ is written as a direct sum $\mathbb{C}\omega^{(n)} + \Lambda_0^2(H^{(n)})^*$ of its submodules, where $\Lambda_0^2(H^{(n)})^*$ is the orthogonal complement of $\mathbb{C}\omega^{(n)}$ in $\Lambda^2(H^{(n)})^*$. Hence,

$$(2.1.1) \quad \Lambda^2((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*) = N_2^{\prime\mathbb{C}} \oplus N_2^{\prime\prime\mathbb{C}} \oplus L_2^{\mathbb{C}},$$

where $N_2^{\prime\mathbb{C}} := \mathbb{C}\omega^{(n)} \otimes_{\mathbb{C}} S^2(H^{(1)})^*$, $N_2^{\prime\prime\mathbb{C}} := \Lambda_0^2(H^{(n)})^* \otimes_{\mathbb{C}} S^2(H^{(1)})^*$ and $L_2^{\mathbb{C}} := S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)}$. Note that the $Sp(n) \cdot Sp(1)$ -modules $N_2^{\prime\mathbb{C}}$, $N_2^{\prime\prime\mathbb{C}}$, $L_2^{\mathbb{C}}$ respectively admit real forms

N_2^{\prime} , $N_2^{\prime\prime}$, L_2 fixed by the real structure induced from the one in (1.1.2). We have the identification

$$H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong (H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^* \text{ by the metric } \langle \langle \cdot, \cdot \rangle \rangle$$

(cf. (1.1)). Together with $H^{(n)} \otimes_{\mathbb{C}} H^{(1)} \cong \mathbb{H}^n \otimes_{\mathbb{R}} \mathbb{C}$, the above (2.1.1) induces the decomposition of its real form:

$$\wedge^2 H^n = N_2' \oplus N_2'' \oplus L_2,$$

which is nothing but the decomposition in the Introduction now for our principal $Sp(n) \cdot Sp(1)$ -bundle P , the bundle T^*M is regarded as the vector bundle associated to the $Sp(n) \cdot Sp(1)$ -module $((H^{(n)})^* \otimes_{\mathbb{C}} (H^{(1)})^*)_{\mathbb{R}} = \mathbb{H}^n$. Hence, $\wedge^2 T^*M$ is a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its subbundles A_2', A_2'', B_2 corresponding respectively to the $Sp(n) \cdot Sp(1)$ -modules N_2', N_2'', L_2 (cf. Introduction).

(2.2) Fix an arbitrary point x of M . Note that each point z on the fibre Z_x defines an almost complex structure J_z on T_x^*M (cf. (1.3.1)). We then have the corresponding space $\wedge^{1,1}(T_x^*M, J_z)$ of $(1,1)$ -forms of (T_x^*M, J_z) . Choose a point $y (\neq 0)$ of V such that its natural image (denoted by $[y]$) is z . In view of (1.3.1), the space $\wedge^{1,1}(T_x^*M, J_z)$ in $\wedge^2(T_x^*M)^{\mathbb{C}}$ is associated to the \mathbb{C} -linear subspace $(H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)' \wedge ((H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)')^{-}$ in the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^* \wedge (H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*$. Since $j^{(n)}$ preserves $H^{(n)}$, we have (cf. (1.1.2)):

$$\begin{aligned} (H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)' \wedge ((H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)')^{-} &= (H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}y)' \wedge (H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}j^{(1)}y) \\ &= (\wedge^2 H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}(y \otimes j^{(1)}y + j^{(1)}y \otimes y)) \oplus (S^2 H^{(n)} \otimes_{\mathbb{C}} \mathbb{C}(y \wedge j^{(1)}y)). \end{aligned}$$

The space $\mathbb{C}(y \wedge j^{(1)}_y)$ (where $y \wedge j^{(1)}_y = (y \otimes j^{(1)}_y - j^{(1)}_y \otimes y)/2$) in $H^{(1)} \otimes_{\mathbb{C}} H^{(1)}$ corresponds to $\mathbb{C}\omega^{(1)}$ in $(H^{(1)})^* \otimes_{\mathbb{C}} (H^{(1)})^*$ via the natural isomorphism $H^{(1)} \otimes_{\mathbb{C}} H^{(1)} \cong (H^{(1)})^* \otimes_{\mathbb{C}} (H^{(1)})^*$ induced by the nondegenerate bilinear form $\omega^{(1)}$. Furthermore,

$$\bigcap_y \mathbb{C}(y \otimes j^{(1)}_y + j^{(1)}_y \otimes y) = \{0\},$$

where \bigcap_y always denotes the intersection taken over all y in $V_x - \{0\}$. Thus,

$$\begin{aligned} \bigcap_y (H^{(n)} \otimes_{\mathbb{C}} y) \wedge \overline{(H^{(n)} \otimes_{\mathbb{C}} y)} &= S^2(H^{(n)})^* \otimes_{\mathbb{C}} \mathbb{C}\omega^{(1)} \\ &= L_2 \quad (\text{cf. Introduction}), \end{aligned}$$

and we obtain:

Lemma (2.3). The fibre $(B_2)_x$ of B_2 over x
is given by

$$(B_2)_x = \bigcap_y \Lambda^{1,1}(T^*_{x^*M, J}[Y]).$$

We next give a typical example of an A_2' -connection and also a B_2 -connection.

Example (2.4). If $n \geq 2$, the induced connection on the locally defined vector bundle

$$V := P \times_{\text{Sp}(n) \times \text{Sp}(1)} H^{(1)} \quad (\text{resp. } W := P \times_{\text{Sp}(n) \times \text{Sp}(1)} H^{(n)})$$

is an A_2' -connection (resp. B_2 -connection). See Salamon [S1;p.150] for related computations of curvatures.

Recall that a connection ∇ is called a Yang-Mills connection if the corresponding curvature R^∇ satisfies $d_*^\nabla R^\nabla = 0$.

We shall finally show:

Theorem (2.5). All A_2' -connections and also all B_2 -connections are Yang-Mills connections.

Corollary (2.6). The Riemannian connection on TM is a Yang-Mills connection.

Proof of (2.6): By (1.2), (2.4) and (2.5), we obtain (2.6).

Proof of (2.5): Fix an arbitrary point x_0 of M . It then suffices to show $(d^\nabla * R^\nabla)(x_0) = 0$. We may take a local section s to P over a neighbourhood U of x_0 such that the corresponding differential at the point x_0 transforms the tangent space $T_{x_0} M$ to a horizontal space at $s(x_0)$ in the tangent space $T_{s(x_0)} P$. Let (u^1, \dots, u^{4n}) be the local frame of $T^*M|_U$ associated to s . Then all covariant derivatives of u^i 's ($1 \leq i \leq 4n$) at the point x_0 is zero. Moreover in terms of the frame (u^1, \dots, u^{4n}) , we can identify $T^*M|_U$ with $U \times \mathbb{R}^{4n}$ ($U \times \mathbb{H}^n$). Note that ∇ on E naturally induces a connection (denoted by the same ∇) on $\text{End}(E)$.

(i) We first assume that ∇ is an A_2^1 -connection on E . Recall that the rank 3 subbundle A_2^1 of $\wedge^2 T^*M$ corresponds to the $\text{Sp}(n) \cdot \text{Sp}(1)$ -submodule N_2^1 of $\wedge^2 \mathbb{H}^n$, where N_2^1 is the irreducible submodule of the elements fixed by $\text{Sp}(n)$ (cf. Introduction). Let I, J and K be

$$\begin{aligned} I &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+2} + u^{4k+3} \wedge u^{4k+4}), \\ J &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+3} + u^{4k+4} \wedge u^{4k+2}), \\ K &= \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+4} + u^{4k+2} \wedge u^{4k+3}). \end{aligned}$$

Then it is easy to check that $A_2^1|_U$ is spanned by the sections I, J and K . Therefore, the curvature form R^∇ is written on U as

$$R^\nabla = a \otimes I + b \otimes J + c \otimes K,$$

where a , b and c are smooth sections to $\text{End}(E)$ over U .

Let (u_1, \dots, u_{4n}) be the base for $\text{TM}|_U$ dual to (u^1, \dots, u^{4n}) defined by $u^i(u_j) = \delta_{ij}$. Then by the first Bianchi identity,

$$\begin{aligned} 0 &= d^\nabla(R^\nabla)(x_0) \\ &= \sum_{i=1}^{4n} \{(\nabla_i a) u^i(x_0) \wedge I(x_0) + (\nabla_i b) u^i(x_0) \wedge J(x_0) + (\nabla_i c) u^i(x_0) \wedge K(x_0)\}, \end{aligned}$$

where ∇_i denotes $\nabla_{u_i}(x_0)$. Consequently,

$$\nabla_i a = \nabla_i b = \nabla_i c = 0, \quad \text{for } 1 \leq i \leq 4n \text{ if } n \geq 2.$$

Therefore, $(d^\nabla * R^\nabla)(x_0) = 0$.

(ii) We next assume that ∇ is a B_2 -connection on E . Since the vector subbundle B_2 (of rank $n(2n+1)$) of $\Lambda^2 T^*M$ corresponds to the irreducible $\text{Sp}(n) \cdot \text{Sp}(1)$ -submodule L_2 of the elements in $\Lambda^2 \mathbb{H}^n$ fixed by $\text{Sp}(1)$, the subbundle $B_2|_U$ is spanned by

$$I_s, J_s, K_s, D_{pq}, E_{pq}, F_{pq}, G_{pq}, \quad (0 \leq s \leq n-1, \quad 0 \leq p < q \leq n-1).$$

where

$$\begin{aligned}
I_s &= u^{4s+1} \wedge u^{4s+2} - u^{4s+3} \wedge u^{4s+4}, \\
J_s &= u^{4s+1} \wedge u^{4s+3} - u^{4s+4} \wedge u^{4s+2}, \\
K_s &= u^{4s+1} \wedge u^{4s+4} - u^{4s+2} \wedge u^{4s+3}, \\
D_{pq} &= u^{4p+1} \wedge u^{4q+1} + u^{4p+2} \wedge u^{4q+2} + u^{4p+3} \wedge u^{4q+3} + u^{4p+4} \wedge u^{4q+4}, \\
E_{pq} &= u^{4p+1} \wedge u^{4q+2} - u^{4p+2} \wedge u^{4q+1} - u^{4p+3} \wedge u^{4q+4} + u^{4p+4} \wedge u^{4q+3}, \\
F_{pq} &= u^{4p+1} \wedge u^{4q+3} + u^{4p+2} \wedge u^{4q+4} - u^{4p+3} \wedge u^{4q+1} - u^{4p+4} \wedge u^{4q+2}, \\
G_{pq} &= u^{4p+1} \wedge u^{4q+4} - u^{4p+2} \wedge u^{4q+3} + u^{4p+3} \wedge u^{4q+2} - u^{4p+4} \wedge u^{4q+1}.
\end{aligned}$$

Let ∇ be a B_2 -connection on E . Then over U , the curvature form R^∇ is written in the form

$$\begin{aligned}
R^\nabla &= \sum_{0 \leq s \leq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s) \\
&\quad + \sum_{0 \leq p < q \leq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}),
\end{aligned}$$

where $i_s, j_s, k_s, d_{pq}, e_{pq}, f_{pq}$ and g_{pq} are smooth sections to $\text{End}(E)$ over U . In view of the first Bianchi identity $d^\nabla R^\nabla = 0$, we have

$$\begin{aligned}
-\nabla_{4s+3} i_s + \nabla_{4s+2} j_s + \nabla_{4s+1} k_s &= 0, \\
\nabla_{4s+1} i_s - \nabla_{4s+4} j_s + \nabla_{4s+3} k_s &= 0, \\
\nabla_{4s+4} i_s + \nabla_{4s+1} j_s - \nabla_{4s+2} k_s &= 0, \\
\nabla_{4s+2} i_s + \nabla_{4s+3} j_s + \nabla_{4s+4} k_s &= 0,
\end{aligned}$$

for s with $0 \leq s \leq n-1$. Furthermore, if l is either p or q , the identity $d^{\nabla} R^{\nabla} = 0$ implies

$$\begin{aligned} (-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+2} e_{pq} - \nabla_{4l+3} f_{pq} - \nabla_{4l+4} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+1} d_{pq} - \nabla_{4l+3} e_{pq} + \nabla_{4l+2} f_{pq} + \nabla_{4l+1} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+2} d_{pq} + \nabla_{4l+1} e_{pq} - \nabla_{4l+4} f_{pq} + \nabla_{4l+3} g_{pq} &= 0, \\ (-1)^{\varepsilon(l)} \nabla_{4l+3} d_{pq} + \nabla_{4l+4} e_{pq} + \nabla_{4l+1} f_{pq} - \nabla_{4l+2} g_{pq} &= 0, \end{aligned}$$

for all p, q with $0 \leq p < q \leq n-1$, where $\varepsilon(p) := 0$ and $\varepsilon(q) := 1$.

Then a straightforward computation shows that $(d^{\nabla} * R^{\nabla})(x_0) = 0$, as required.

3. Deformations of B_2 -connections

In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of B_2 -connections on M (see Salamon [S2] for a similar complex).

(3.1) Let r be an integer with $r \geq 2$. By setting $N_r^{\mathbb{C}} := \Lambda^r(H^{(n)})^* \otimes_{\mathbb{C}} S^r(H^{(1)})^*$ (cf. (2.1)), we can express the $Sp(n) \cdot Sp(1)$ -module $\Lambda^r(H^{(n)}) \otimes_{\mathbb{C}} H^{(1)}$ as a direct sum $N_r^{\mathbb{C}} \oplus L_r^{\mathbb{C}}$, where $L_r^{\mathbb{C}}$ is the orthogonal complement of $N_r^{\mathbb{C}}$ in $\Lambda^r(H^{(n)}) \otimes_{\mathbb{C}} H^{(1)}$. As in (2.1), the $Sp(n) \cdot Sp(1)$ -modules $N_r^{\mathbb{C}}$ and $L_r^{\mathbb{C}}$ respectively admit real forms N_r and L_r fixed by the natural real structure (cf. (1.1.2)). Since T^*M is associated to the $Sp(n) \cdot Sp(1)$ -module $(H^{(n)} \otimes_{\mathbb{C}} H^{(1)})^*_{\mathbb{R}}$ (see (1.2.1)), the vector bundle $\Lambda^r T^*M$ is a direct sum $A_r \oplus B_r$ of its subbundles A_r, B_r corresponding respectively to N_r, L_r . Let $\pi^r : \Lambda^r T^*M (= A_r \oplus B_r) \longrightarrow A_r$ be the projection to the first factor. Then we have:

Theorem (3.2). For a B_2 -connection ∇ on E , the following is an elliptic complex:

$$(3.2.1) \quad 0 \longrightarrow \mathcal{E}(E) \xrightarrow{\nabla} \mathcal{E}(E \otimes T^*M) \xrightarrow{d_1} \mathcal{E}(E \otimes A_2) \\ \xrightarrow{d_2} \mathcal{E}(E \otimes A_3) \xrightarrow{d_3} \dots \xrightarrow{d_{2n-1}} \mathcal{E}(E \otimes A_{2n}) \longrightarrow 0,$$

where $d_i := (\text{id} \otimes \pi^{i+1}) \circ d^{\nabla}$ and for every vector bundle E'

on M , we denote by $\mathcal{E}(E')$ the sheaf of germs of C^∞ -sections of E' .

Proof.

(i) Fix a section $s \in \Gamma(M, E \otimes A_i)$ ($i \geq 1$) and define a section $t \in \Gamma(M, E \otimes B_{i+1})$ by

$$d^\nabla s = d_i s + t.$$

Then from $(d^\nabla \circ d^\nabla)s = (d^\nabla \circ d_i)s + d^\nabla t$, we obtain

$$((\text{id} \otimes \pi_{i+2}) \circ d^\nabla \circ d^\nabla)s = (d_{i+1} \circ d_i)s + ((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t.$$

Since ∇ is a B_2 -connection, the A_{i+2} -component of $(d^\nabla \circ d^\nabla)s$ is zero, i.e.,

$$0 = (d_{i+1} \circ d_i)s + ((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t.$$

Write t as $t = \sum_k v_k \otimes b_k$ locally, where v_k, b_k is a local section of V^*, B_{i+1} , respectively. The $S^{i+1}(V^*)$ -component of b_k is zero, and hence the $S^{i+2}(V^*)$ -component of $\nabla(v_k) \wedge b_k$ is zero. Therefore,

$$((\text{id} \otimes \pi_{i+2}) \circ d^\nabla)t = \sum_k v_k \otimes db_k.$$

Since d is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$ -component of db_k is zero. Thus, $(d_{i+1} \circ d_i)s = 0$, as required.

(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_i, u)$ ($u \in T_x^*M - \{0\}$). Fix a point of M and an element s of $E_x \otimes A_{ix}$. All computations below are taken at the point x .

$$\sigma(d_i, u)s := (d/dt) (e^{-tq} d_i (e^{tq} s)) \Big|_{t=0} = (id \otimes \pi_{i+1})(u \wedge s),$$

where q is a locally defined function such that $dq_x = u$.

We next show that the following sequence is exact for every u :

$$(3.2.2) \quad E \otimes A_{i-1} \xrightarrow{\sigma(d_{i-1}, u)} E \otimes A_i \xrightarrow{\sigma(d_i, u)} E \otimes A_{i+1}.$$

Without loss of generality, we may assume

$$u = e_1 \otimes h_1 + (e_1 \otimes h_1)^{\perp} (= e_1 \otimes h_1 + e_2 \otimes h_2),$$

where $\langle e_1, \dots, e_{2n} \rangle$ (resp. $\langle h_1, h_2 \rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)} e_{2j+1} = e_{2j+2}$ (resp. $j^{(1)} h_1 = h_2$). Let $s \in E \otimes A_i$ be such that $\sigma(d_{i+1}, u)s = 0$. Note that $S^i V^* = \text{Span}(h_1^k \cdot h_2^{i-k}; 0 \leq k \leq i)$, where $h_1^k \cdot h_2^{i-k}$

denotes the symmetric component of $h_1^k \otimes h_2^{i-k}$.

Hence, there are local sections s_0, \dots, s_i of $E \otimes \Lambda^i W^*$ such that

$$s = \sum_{k=0}^i s_k \otimes h_1^k \cdot h_2^{i-k}.$$

We can now write $\sigma(d_{i+1}, s) = 0$ as follows:

$$\begin{aligned}
0 &= (\text{id} \otimes \pi_{i+1})(u \wedge s) = (\text{id} \otimes \pi_{i+1})((e_1 \otimes h_1 + e_2 \otimes h_2) \wedge \sum_{k=0}^i s_k \otimes h_1^k \cdot h_2^{i-k}) \\
&= \sum_{k=0}^i ((e_1 \wedge s_k) \otimes h_1^{k+1} \cdot h_2^{i-k} + (e_2 \wedge s_k) \otimes h_1^k \cdot h_2^{i+1-k}).
\end{aligned}$$

Since the coefficient of the right-hand side in $h_1^k \cdot h_2^{i+1-k}$ is zero, we have:

$$(0) \quad e_2 \wedge s_0 = 0,$$

$$(1) \quad e_1 \wedge s_0 + e_2 \wedge s_1 = 0,$$

.

.

.

$$(i) \quad e_1 \wedge s_{i-1} + e_2 \wedge s_i = 0,$$

$$(i+1) \quad e_1 \wedge s_i = 0.$$

By (0), there exists $r_0 \in \wedge^{i-1} W^*$ such that

$$s_0 = e_2 \wedge r_0.$$

Plugging this into (1), we obtain

$$e_2 \wedge (-e_1 \wedge r_0 + s_1) = 0.$$

Hence there exists $r_1 \in \wedge^{i-1} W^*$ such that

$$s_1 = e_1 \wedge r_0 + e_2 \wedge r_1.$$

Repeating this process inductively, we obtain $r_k \in \Lambda^{i-1} W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k$, $1 \leq k \leq i$. Now by (i+1), the identity $e_1 \wedge e_2 \wedge r_i = 0$ holds. It then follows that there exists $r_i' \in \Lambda^{i-2} W^*$ such that $e_2 \wedge r_i = e_1 \wedge e_2 \wedge r_i'$. Since $e_2 \wedge (r_{i-1} + e_2 \wedge r_i') = e_2 \wedge r_{i-1}$, we may replace r_{i-1} by $r_{i-1} + e_2 \wedge r_i'$. Therefore,

$$s_0 = e_2 \wedge r_0,$$

$$s_1 = e_1 \wedge r_0 + e_2 \wedge r_1,$$

.

.

.

$$s_i = e_1 \wedge r_{i-1}$$

Thus,

$$\begin{aligned} s &= \sum_{k=0}^i s_k \otimes h_1^k \cdot h_2^{i-k} \\ &= \sigma(d_{i-1}, u) \left(\sum_{k=0}^{i-1} r_k \otimes h_1^k \cdot h_2^{i-1-k} \right), \end{aligned}$$

i.e., the sequence (3.2.2) is exact, as required.

Definition (3.3). Let \mathcal{C} be the set of all B_2 -connections on E with holonomy groups contained in a compact semisimple Lie group G . Assume that $\mathcal{C} \neq \emptyset$ and let $\nabla \in \mathcal{C}$. Then the frame bundle Q of E can be regarded as a principal G -bundle. Put $G_Q := Q \times_{\theta} G$ and $\mathcal{G}_Q := Q \times_{\text{Ad}} \mathcal{G}$, where θ is the group conjugation and $\text{Ad} : G \rightarrow \text{GL}(\mathcal{G})$ is the adjoint representation of G . Now, a C^∞ -section to G_Q over M is called a gauge transformation of Q . Let \mathcal{G} be the set of all gauge transformations of Q . Then \mathcal{G} naturally acts on \mathcal{C} (see Atiyah-Hitchin-Singer [A-H-S]). We call $\mathcal{M} (:= \mathcal{C}/\mathcal{G})$ the

moduli space of the B_2 -connections on E with holonomy groups in G .

(3.4) Let $\nabla \in \mathcal{C}$ be irreducible in the sense that \mathcal{G}_Q admits no nonzero parallel section over M . Fix a smooth one-parameter family ∇^t ($|t| < \varepsilon$) of connections in \mathcal{C} such that $\nabla^0 = \nabla$. Put $S = (d/dt)\nabla^t|_{t=0}$. We write the curvature form R^{∇^t} of ∇^t as

$$R^{\nabla^t} = R^\nabla + t d^{\nabla'} S + \text{higher order terms in } t,$$

where ∇' is the connection on \mathcal{G}_Q naturally induced by ∇ . Since R^{∇^t} is a \mathcal{G}_Q -valued B_2 -form, the corresponding derivative $d^{\nabla'} S$ at $t=0$ also satisfies

$$((\text{id} \otimes \pi^2) \circ d^{\nabla'}) S = 0.$$

Let f^t ($|t| < \varepsilon$) be a one-parameter family of gauge transformations such that $f^0 = \text{id}$. Then,

$$\frac{d}{dt}(f^t(\nabla))|_{t=0} = \nabla'(\dot{f}),$$

where $\dot{f} := (d/dt)(f^t)|_{t=0}$. Since $f^t(\nabla) \in \mathcal{C}$ for all t , the same argument as above shows that the \mathcal{G}_Q -valued 1-form $\nabla'(\dot{f})$ satisfies

$$((\text{id} \otimes \pi^2) \circ d^{\nabla'}) (\nabla'(\dot{f})) = 0.$$

For each $A \in \Gamma(\mathcal{A}_Q)$, there exists a one-parameter family $f^t = \exp(tA)$ such that $(d/dt)f^t|_{t=0} = A$. Then together with (3.2), we immediately obtain the following:

Theorem (3.5). Assume that $\mathcal{C} \neq \emptyset$ and let $\nabla \in \mathcal{C}$ be irreducible. Then the space of infinitesimal (essential) deformations at ∇ of connections in \mathcal{C} , that is, the tangent space of \mathcal{M} at ∇ is a linear subspace of the first cohomology group of the elliptic complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}(\mathcal{A}_Q) & \xrightarrow{\nabla'} & \mathcal{E}(\mathcal{A}_Q \otimes T^*M) & \xrightarrow{d'_1} & \mathcal{E}(\mathcal{A}_Q \otimes A_2) \\ & & & & & & \\ & & \xrightarrow{d'_2} & \mathcal{E}(\mathcal{A}_Q \otimes A_3) & \xrightarrow{d'_3} & \dots & \xrightarrow{d'_{2n-1}} \mathcal{E}(\mathcal{A}_Q \otimes A_{2n}) \longrightarrow 0, \end{array}$$

where $d'_i := (\text{id} \otimes \pi^{i+1}) \circ d^{\nabla'}$.

4. Einstein-Hermitian connections associated
with B_2 -connections

In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between B_2 -connections and the corresponding Einstein-Hermitian connections.

Proof of (0.2) : (i) Let (E, D_E) be a Hermitian pair. Then by the definition of B_2 -connections, the curvature form corresponding to the connection D_E is an $\text{End}(E)$ -valued B_2 -form, and by Lemma (2.3) the curvature form corresponding to the connection p^*D_E on p^*E is an $\text{End}(p^*E)$ -valued $(1,1)$ -form. Hence the connection p^*D_E induces naturally an integrable complex structure on p^*E as follows : Put $\ell := \text{rank}(E)$ and denote by $q: p^*E \rightarrow Z$ the natural projection. Let (s_1, \dots, s_ℓ) (resp. (y^1, \dots, y^ℓ)) be a local unitary frame for p^*E (resp. the dual frame corresponding to (s_1, \dots, s_ℓ)). Then the vector subbundle $\Lambda^{1,0} T^*(p^*E)$ of type $(1,0)$ in the complexification $T^*(p^*E)^\mathbb{C}$ of the cotangent bundle $T^*(p^*E)$ is defined as the direct sum of the pull-back $q^*(\Lambda^{1,0} T^*Z)$ and the space spanned by $\{\bar{\alpha}y^j + \sum_{i=1}^{\ell} y^i q^* \theta_{ji}, 1 \leq j \leq \ell\}$, where (θ_{ij}) is the connection matrix for p^*D_E with respect to the frame (s_1, \dots, s_ℓ) (i.e., $(p^*D_E)s_j = \sum_{i=1}^{\ell} s_i \theta_{ij}$). Now, we may take the frame (s_1, \dots, s_ℓ) as the pull-back $(p^*t_1, \dots, p^*t_\ell)$ of a local unitary frame (t_1, \dots, t_ℓ) on E . Then the 1-forms $\theta_{ij}, 1 \leq i, j \leq \ell$, are written

as $p^*\psi_{ij}$, where (ψ_{ij}) denotes the connection matrix for D_E with respect to the frame (t_1, \dots, t_ℓ) . Let $q': (p^*E)^* \rightarrow Z$ be the projection naturally induced from $q: p^*E \rightarrow Z$. Since the real structure $\tau: Z \rightarrow Z$ is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping $q' \circ \sigma: p^*E \rightarrow Z$ is equal to $\tau \circ q$, the mapping $\sigma: p^*E \rightarrow (p^*E)^*$ is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on p^*E and $(p^*E)^*$.

(ii) We next fix an arbitrary excellent pair (F, D_F) on Z . Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover $\{U_\lambda\}$ of M , and a local unitary frame $(f_1^\lambda, \dots, f_r^\lambda)$ ($r = \text{rank of } F$) of $F|_{p^{-1}(U_\lambda)}$ such that each restriction $(f_1^\lambda|_{p^{-1}(x)}, \dots, f_r^\lambda|_{p^{-1}(x)})$ over $p^{-1}(x)$ ($x \in U_\lambda$) forms a holomorphic frame for $F|_{p^{-1}(x)}$. When $U_\lambda \cap U_\mu \neq \emptyset$, the transition matrix for F in terms of the frames $(f_1^\lambda, \dots, f_r^\lambda)$, $(f_1^\mu, \dots, f_r^\mu)$ is holomorphic (and hence constant) along each fibre $p^{-1}(x)$ ($x \in U_\lambda \cap U_\mu$). Hence there exists a Hermitian vector bundle E on M such that, including metrics, we have $p^*E = F$. In particular, we obtain a local unitary frame $(f_1^{\prime\lambda}, \dots, f_r^{\prime\lambda})$ for $E|_{U_\lambda}$ such that $(p^*f_1^{\prime\lambda}, \dots, p^*f_r^{\prime\lambda})$ coincides with the previous $(f_1^\lambda, \dots, f_r^\lambda)$ over $p^{-1}(U_\lambda)$. Fix an arbitrary λ . If there is no fear of confusion, we shall omit the suffix λ and denote $U_\lambda, (f_1^\lambda, \dots, f_r^\lambda), \dots$ simply by $U, (f_1, \dots, f_r), \dots$,

respectively. Let (ω_{ij}) be the connection matrix of D_F with respect to the frame (f_1, \dots, f_r) , i.e., $D_F f_j = \sum_{i=1}^r f_i \omega_{ij}$. Furthermore, we choose a local symplectic basis (e_1, \dots, e_{2n}) (resp. (h_1, h_2)) for $W^*|_U$ (resp. $V^*|_U$) (see Section 3). Now, since D_F is a Hermitian connection, we have:

$$(1) \quad \omega_{ij} + \overline{\omega_{ji}} = 0, \quad \text{for } 1 \leq i, j \leq r.$$

Then the construction of D_E is reduced to showing that there exist 1-forms $\omega_{ij}^!$ ($1 \leq i, j \leq r$) on U satisfying $\omega_{ij} = p^* \omega_{ij}^!$. In fact, once we can find such 1-forms $\omega_{ij}^!$, they define a Hermitian connection on E , such that the corresponding curvature form is pulled back by p to an $\text{End}(F)$ -valued $(1,1)$ -form on Z , which together with Lemma (2.3) implies that our connection on E is a B_2 -connection. Recall that, for each $x \in U$, the frame $(f_1|_{p^{-1}(x)}, \dots, f_r|_{p^{-1}(x)})$ for $F|_{p^{-1}(x)}$ is trivial. Hence,

$$(2) \quad \omega_{ij}(v) = 0, \quad 1 \leq i, j \leq r,$$

for every vector v tangent to $p^{-1}(x) (\cong \mathbb{P}^1\mathbb{C})$. Since $(e_1 \otimes h_1, e_1 \otimes h_2, \dots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)$ is a frame for $T^*M^{\mathbb{C}}|_U = W^*|_U \otimes V^*|_U$, there exist by (2) C^∞ -functions a_{ij}^k, b_{ij}^k ($1 \leq i, j \leq r, 1 \leq k \leq 2n$) on $p^{-1}(U)$ such that

$$(3) \quad \omega_{ij} = \sum_{k=1}^{2n} (a_{ij}^k p^*(e_k \otimes h_1) + b_{ij}^k p^*(e_k \otimes h_2)) , \quad 1 \leq i, j \leq r .$$

For every form η on $Z|_U$, we denote by $\hat{\eta}$ the pull-back of η to $(V - \{\text{zero section}\})|_U$. Then by (3), we have :

$$\begin{aligned} \hat{R}_{ij} &= d\hat{\omega}_{ij} + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj} \\ &= \sum_{k=1}^{2n} d(\hat{a}_{ij}^k p^*(e_k \otimes h_1)) + d(\hat{b}_{ij}^k p^*(e_k \otimes h_2)) + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj} . \end{aligned}$$

Fix an arbitrary point x on U . Choosing an appropriate (e_1, \dots, e_{2n}) (resp. (h_1, h_2)), we may assume that $(\nabla^{V^*} e_k)(x) = 0$, $k = 1, 2, \dots, 2n$ (resp. $(\nabla^{W^*} h_i)(x) = 0$, $i = 1, 2$), where ∇^{V^*} (resp. ∇^{W^*}) denotes the connection of V^* (resp. W^*) canonically induced by that of P (cf. Example (2.4)). Then, on $\hat{p}^{-1}(x)$,

$$\begin{aligned} \hat{R}_{ij} &= \sum_{k=1}^{2n} \{d(\hat{a}_{ij}^k) \wedge \hat{p}^*(e_k \otimes h_1) + d(\hat{b}_{ij}^k) \wedge \hat{p}^*(e_k \otimes h_2)\} \\ &\quad + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj} . \end{aligned}$$

Recall that the complex structure on the twistor space Z

$(= (V - \{\text{zero section}\}) / \mathbb{C}^*)$ is induced by the complex structure on $V - \{\text{zero section}\}$ (see Section 1). Since \hat{R}_{ij} is of type $(1,1)$, we have :

$$(4) \quad \sum_{k=1}^{2n} \{\partial(\hat{a}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_1))^{(1,0)} + \partial(\hat{b}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_2))^{(1,0)}\} + \sum_{t=1}^r \hat{\omega}_{it}^{(1,0)} \wedge \hat{\omega}_{tj}^{(1,0)} = 0 \quad \text{on } \hat{p}^{-1}(x) ;$$

$$(5) \quad \sum_{k=1}^{2n} \{ \bar{\partial}(\hat{a}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_1))^{(0,1)} + \bar{\partial}(\hat{b}_{ij}^k) \wedge (\hat{p}^*(e_k \otimes h_2))^{(0,1)} \} \\ + \sum_{t=1}^r \hat{\omega}_{it}^{(0,1)} \wedge \hat{\omega}_{tj}^{(0,1)} = 0 \quad \text{on } \hat{p}^{-1}(x),$$

where for every 1-forms ζ on $(V - \{\text{zero section}\})|_U$, $\zeta^{(1,0)}$ (resp. $\zeta^{(0,1)}$) always denotes the $(1,0)$ -component (resp. $(0,1)$ -component) of ζ . Let (z^1, z^2) be the local triviality for $V|_U$ corresponding to (h_1, h_2) . Then, by the definition of the complex structure of $(V - \{\text{zero section}\})$, we obtain from (4) and (5) the following :

$$(4') \quad \sum_{k=1}^{2n} \{ (\frac{\partial}{\partial z^1} \hat{a}_{ij}^k dz^1 + \frac{\partial}{\partial z^2} \hat{a}_{ij}^k dz^2) \wedge \bar{z}^{-1} (z^1 \hat{p}^*(e_k \otimes h_1) + z^2 \hat{p}^*(e_k \otimes h_2)) \\ + (\frac{\partial}{\partial z^1} \hat{b}_{ij}^k dz^1 + \frac{\partial}{\partial z^2} \hat{b}_{ij}^k dz^2) \wedge \bar{z}^{-2} (z^1 \hat{p}^*(e_k \otimes h_1) + z^2 \hat{p}^*(e_k \otimes h_2)) \} \\ = 0 \quad \text{on } \hat{p}^{-1}(x) ;$$

$$(5') \quad \sum_{k=1}^{2n} \{ (\frac{\partial}{\partial \bar{z}^1} \hat{a}_{ij}^k d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2} \hat{a}_{ij}^k d\bar{z}^2) \wedge (-z^2) (\bar{z}^{-1} \hat{p}^*(e_k \otimes h_2) - \bar{z}^{-2} \hat{p}^*(e_k \otimes h_1)) \\ + (\frac{\partial}{\partial \bar{z}^1} \hat{b}_{ij}^k d\bar{z}^1 + \frac{\partial}{\partial \bar{z}^2} \hat{b}_{ij}^k d\bar{z}^2) \wedge z^1 (\bar{z}^{-1} \hat{p}^*(e_k \otimes h_2) - \bar{z}^{-2} \hat{p}^*(e_k \otimes h_1)) \} \\ = 0 \quad \text{on } \hat{p}^{-1}(x) .$$

Since both $z^1|_{\hat{p}^{-1}(x)}$ and $z^2|_{\hat{p}^{-1}(x)}$ are holomorphic on $\hat{p}^{-1}(x) \cong \mathbb{C}^2 - \{0\}$, we have

$$\frac{\partial}{\partial \bar{z}^i} (z^1 \bar{a}_{ij}^k + z^2 \bar{b}_{ij}^k) = \frac{\partial}{\partial \bar{z}^i} (-z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k) = 0 \quad (i = 1, 2),$$

on $p^{-1}(x)$, i.e., both $f_1(z^1, z^2) := z^1 \bar{a}_{ij}^k + z^2 \bar{b}_{ij}^k$ and $f_2(z^1, z^2) := -z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k$ are holomorphic on $\mathbb{C}^2 - \{0\}$.

By Hartogs' theorem, both f_1 and f_2 extend further to holomorphic functions on \mathbb{C}^2 . Since $f_i(cz^1, cz^2) = cf_i(z^1, z^2)$ for all $z = (z^1, z^2) \in \mathbb{C}^2$ and $c \in \mathbb{C}^*$ ($i = 1, 2$), there exist constants $\alpha_{ij}^k, \beta_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k \in \mathbb{C}$ independent of z such that

$$(6) \quad z^1 \bar{a}_{ij}^k + z^2 \bar{b}_{ij}^k = z^1 \bar{c}_{ij}^k + z^2 \bar{\beta}_{ij}^k,$$

$$(7) \quad -z^2 \hat{a}_{ij}^k + z^1 \hat{b}_{ij}^k = -z^2 \gamma_{ij}^k + z^1 \delta_{ij}^k, \quad (1 \leq k \leq 2n).$$

Let $\Gamma(Z, F^*)$ (resp. $\Gamma(Z, F^* \otimes T^*Z^{\mathbb{C}})$) be the space of global C^∞ -sections over Z to F^* (resp. $F^* \otimes T^*Z^{\mathbb{C}}$).

Let $\psi : \Gamma(Z, F^*) \longrightarrow \Gamma(Z, F^* \otimes T^*Z^{\mathbb{C}})$ be the \mathbb{C} -linear map sending each $s \in \Gamma(Z, F^*)$ to an element $\psi(s)$ of $\Gamma(Z, F^* \otimes T^*Z^{\mathbb{C}})$ defined by

$$\psi(s)(X) := \sigma \left((D_F) \frac{1}{T_*(X)} (\sigma^{-1} s) \right) \in F^*_{z \downarrow}$$

for $X \in T_z Z^{\mathbb{C}}$ ($z \in Z$).

Then by the condition (b) in the Introduction, this ψ defines a Hermitian $(1,0)$ -connection on the holomorphic vector bundle F^* . The corresponding connection matrix with respect to the frame $(\sigma f_1, \dots, \sigma f_r)$ for $F^*|_{p^{-1}(U)}$ is written as $(\tau^* \overline{\omega_{ij}})$. By the definition of σ , it is easy to check that the frame $(\bar{\sigma} f_1, \dots, \bar{\sigma} f_r)$ is dual to our previous (f_1, \dots, f_r) . Hence the uniqueness of the $(1,0)$ -connection on the Hermitian vector bundle F^* implies the equality $(\tau^* \omega_{ij})^{-\bar{-}} = \omega_{ij}^*$, where $\omega_{ij}^* := -\omega_{ji}$. In view of (1), we have $\tau^* \omega_{ij} = \omega_{ij}$ and $\hat{\tau}^* \hat{\omega}_{ij} = \hat{\omega}_{ij}$. By (3) and $\hat{p} \circ \hat{\tau} = \hat{p}$, we obtain :

$$(8) \quad \hat{\tau}^* \hat{\alpha}_{ij}^k = \hat{\alpha}_{ij}^k \quad \text{and} \quad \hat{\tau}^* \hat{\beta}_{ij}^k = \hat{\beta}_{ij}^k \quad (1 \leq k \leq 2n).$$

Therefore,

$$-\frac{2}{z} \hat{\tau}^* \hat{\alpha}_{ij}^k + \frac{1}{z} \hat{\tau}^* \hat{\beta}_{ij}^k = -\frac{2}{z} \alpha_{ij}^k + \frac{1}{z} \beta_{ij}^k \quad (1 \leq k \leq 2n).$$

Moreover by (6),

$$(9) \quad -z^2 \hat{\alpha}_{ij}^k + z \hat{\beta}_{ij}^k = -z^2 \alpha_{ij}^k + z \beta_{ij}^k \quad (1 \leq k \leq 2n).$$

Hence by (7) and (9), we obtain :

$$(10) \quad \alpha_{ij}^k = \gamma_{ij}^k \quad \text{and} \quad \beta_{ij}^k = \delta_{ij}^k \quad (1 \leq k \leq 2n).$$

Now, in view of (6), (7) and (10), we see that

$$\begin{pmatrix} \bar{z}^1, \bar{z}^2 \\ -z^2, z^1 \end{pmatrix} \begin{pmatrix} \hat{a}_{ij}^k - \alpha_{ij}^k \\ \hat{b}_{ij}^k - \beta_{ij}^k \end{pmatrix} = 0 \quad (1 \leq k \leq 2n),$$

where $(z^1, z^2) \in \mathbb{C}^2 - \{0\}$ ($= \hat{p}^{-1}(x)$). Thus, $\hat{a}_{ij}^k = \alpha_{ij}^k$ and $\hat{b}_{ij}^k = \beta_{ij}^k$ ($1 \leq k \leq 2n$), i.e., both a_{ij}^k and b_{ij}^k are constant along $p^{-1}(x)$, as required.

Remark (4.1). In some sense, our Theorem (0.2) completely clarifies the following result by Salamon [S2] (see Berard Bergery and Ochiai [B-O] for another generalization) :

For a Hermitian pair (E, D_E) on M , the pull-back (p^*E, p^*D_E) to Z is a Hermitian holomorphic vector bundle over Z .

Corollary (4.2). Let (F, D_F) be an excellent pair on Z . If the quaternionic Kähler manifold M has positive scalar curvature, then F with D_F is a Ricci-flat Einstein Hermitian vector bundle over Z .

Proof. Consider the twistor space $p: Z \rightarrow M$. Then the horizontal component of the Kähler form on Z is a $p^*A_2^h$ -form (cf. (1.2), (1.3)). Recall that the curvature of D_F is an $\text{End}(F)$ -valued p^*B_2 -form. Hence the Hermitian vector bundle F with D_F is Ricci-flat.

Remark (4.3). We have the decomposition of $TZ = T^h \oplus T^v$, where T^h (resp. T^v) is the horizontal (resp. vertical) distribution in terms of the connection on Z induced by that of P . Since the complex structure on TZ is a direct sum of complex structures on T^h and T^v , the holomorphic part $TZ^{(1,0)}$ admits the corresponding decomposition $TZ^{(1,0)} = T^{h(1,0)} \oplus T^{v(1,0)}$, where $T^{h(1,0)}$ (resp. $T^{v(1,0)}$) denotes $T^{h\mathbb{C}} \cap TZ^{(1,0)}$ (resp. $T^{v\mathbb{C}} \cap TZ^{(1,0)}$). Recently, Zandi [Z] obtained the following:

The vector bundle $(T^{h(1,0)}, D^h)$ is an Einstein-Hermitian vector bundle, where D^h is the connection on $T^{h(1,0)}$ obtained as the restriction of the Riemannian connection on TZ to $T^{h(1,0)}$.

This result can be regarded as a straightforward consequence of our (4.2). We denote by L a locally defined (line) subbundle of p^*W (cf. (2.4)) such that, along each fibre $p^{-1}(x) = \mathbb{P}^1\mathbb{C}$ ($x \in M$), it restricts to a universal bundle over $\mathbb{P}^1\mathbb{C}$. Let ∇^V (resp. ∇^W) denote the connection of V (resp. W) canonically induced by that of P and ∇^L the restriction of

$p^*\nabla^W$ to L . Then the vector bundle $(T^{h(1,0)}, D^h)$ is nothing but $(p^*W \otimes L^*, p^*\nabla^W \otimes (\nabla^L)^*)$, where $(L^*, (\nabla^L)^*)$ is dual to (L, ∇^L) (see Salamon [S1]). Since L^* is a locally defined line bundle and since ∇^W is a B_2 -connection on W , Corollary (4.2) clearly implies Zandi's result.

ADDED IN PROOF. After the completion of this paper, I received a preprint : M.M.Capria and S.M.Salamon "Yang-Mills fields on quaternionic Kähler spaces" , which gives (i) for (2.6), a slightly stronger result and (ii) a statement similar to (3.2).

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