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Canonical Quantization of Witten's String Field Theory in Mid-point Time Formalism

MASAHIRO MAENO

Department of Physics, Osaka University
Toyonaka, Osaka 560, JAPAN

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Osaka University

ABSTRACT

We carry out canonical quantization of Witten's string field theory in a mid-point time formalism. In this formalism, the string interaction is represented as a local interaction, and one can apply the canonical quantization procedure. A path integral with respect to the momentum in phase space can be performed after the Batalin-Vilkovsky gauge fixing procedure to get a naive Lagrangian path integral. The kinetic term of the string, when rewritten in the mid-point time coordinates, contains an apparently divergent expression. A prescription is given to regularize it by discretizing the string. We calculate the equal time commutation relation of string fields, and the theory is shown to coincide with the one which is expected in the formal Lagrangian path integral quantization that has been conventionally used.

CONTENTS

1. Introduction

The string or the superstring theory is expected as a candidate of unified theory which governs all the matter and geometry. Each of excitation modes of a string corresponds as an elementary particle, and all the interactions of elementary particles are simply represented by processes of joining and splitting of strings. Particularly, the gravity is included in the theory of closed string which contains a massless spin-2 particle in the spectrum. Further, the spectrum of an open string contains the Yang-Mills field in its massless sector. Green and Schwarz have shown the superstring theory with the gauge group $SO(32)$ or $E_8 \times E_8$ is free from chiral gauge and gravitational anomalies.[1] For these reasons, the superstring theory is considered to be the most promising candidate of a unified theory including renormalizable quantum gravity.

There have been many developments in string theory in recent years. Most of them, however, are performed with the first quantized approach, for example, the Polyakov path-integral formalism[2] and the first quantized operator formalism which is studied as two-dimensional conformal field theory. In the Polyakov formalism, S-matrix is represented as a path integral where one sums up all possible trajectories(weighted by an exponentiated action) of a string(worldsheet) in space-time with a given boundary condition. In this summation we have to sum over all possible worldsheet topologies as well. However, within the first quantized approach alone, the weight of each topology cannot be determined *a priori*. While, a theory of second quantized string(string field theory) can handle this problem of relative weights. The string field theory must be also useful for studying non-perturbative effects of strings. The bosonic string theory is known to be consistent only in critical dimension $D = 26$. For a superstring, the critical dimension is $D = 10$. Since our real world has space-time dimensions four, the extra 22 (or 6) dimensions must be compactified on the Plank length scale. The first quantized treatment gives us only little information on non-perturbative effects such as compactification.

The second quantization of string was first carried out by Kaku and Kikkawa in the light-cone gauge.[3] The interactions of strings are introduced by constructing five types of vertices which represent joining and splitting process of strings. In this gauge, quantization can be carried out by the canonical formalism, and the theory is a consistent field theory which satisfies unitarity. However, Lorentz covariance is not manifest in the light-cone gauge. A manifestly covariant theory of a free string was first given by Siegel[4] by use of the BRST[5] method. The interactions of covariant string field theories have been later introduced in two fashions.

(i) A natural extension of the light-cone gauge interaction, in other words, a joining and splitting type interaction, has been introduced by Hata, Itoh, Kugo, Kunitomo and Ogawa. Theories of both open[6] and closed[7] strings are constructed. Their theories involve an unphysical string width parameter α, which plays a role similar to the momentum p_+ in the light-cone gauge string field theory. This unphysical parameter, however, causes some troubles in the theory, *e.g.*, introduction of α gives rise to a divergence in loop amplitudes and causes the breakdown of unitarity at the loop level in closed string theory.

(ii) An interaction vertex on which three strings shares their mid-points was introduced by Witten.[8] There is no need to bring any unphysical parameter because all strings have the same width π in this theory. At first the theory of an open string was constructed. Recently, the field theory of a closed string was constructed by introducing a non-polynomial interaction.[9–11] However, the non-polynomial theory cannot reproduce a correct loop amplitude.

As mentioned above, covariant string field theory appears to be plagued with some problems about quantization in an essential way . One source of these problems is absence of the canonical formalism. In most works on the string field theory, the quantization procedures are based on a formal Lagrangian path integral formalism, in which, one simply mimics the perturbation method by regarding the kinetic energy as an unperturbed term and the interaction as the perturbed one. In such an approach one can not ensure consistency of theories with the canonical formalism. There is, in general, no guarantee that this leads to the correct Feynman rules. In some cases even in ordinary local field theories, such a naive prescription fails to yield the correct canonical representation of the path integral. A well-known example is the Lee-Yang term in a non-linear σ-model. Without the Lee-Yang term, the unitarity for the non-linear σ-model is broken at loop levels. Similarly, the problem of unitarity breaking of closed string theory might have an origin in the absence of the canonical formalism. Hata[12] resolved the problem by adding terms which recover the BRST invariance of path integral of string field by using the Batalin-Vilkovisky methods.[13] He has obtained unitary amplitudes of a closed bosonic string at the loop-level.

Some attempts in the canonical formalism are performed for free string.[14-15] However, the canonical quantization of an interacting string has not been carried out in a satisfactory fashion. The main difficulty lies in the non-locality of interaction. The non-locality arises from the displacement of the center of mass coordinates of three interacting strings. However, the mid-points of three strings are connected one another. The purpose of this paper is to show that, the difficulty for Witten's string field theory can be overcome by employing the mid-point time formalism. After rewriting the interaction term by using the mid-point coordinates, the interaction of strings become local. Hence we can carry out canonical quantization. As a by-product, it turns out that the Batalin-Vilkovisky gauge-fixing procedure is greatly simplified in the mid-point time formalism as compared with the standard approach. In particular, the quantum correction which is necessary in the center of mass coordinates does not arise. When rewritten in the mid-point coordinates, the kinetic term of the string contains an apparently divergent expression. This divergence is regularized by discretizing a string. Under the regularization, we can show that the equal time commutation relation of the string fields in the mid-point coordinates is equivalent to the one which is expected in the formal Lagrangian path integral.

The rest of this paper is organized as follows. In section 2, we review the first quantization of a string and Witten's string field theory. In section 3, the mid-point coordinate of a string is introduced. We show that Witten's interaction of open strings is local in the mid-point coordinates. We shall find a divergence in the kinetic term of the string field, which is regularized later by discretizing the string. In section 4, we proceed canonical quantization by constructing the canonical representation of the path integrals and find that the result coincides with the formal configuration-space path integrals which have been conventionally used without justification. In this section we find that the gauge-fixed action does not need no quantum correction when we carry out the Batalin-Vilkovsky gauge-fixing procedure. Further we study the canonical commutation relation of the string with discretization of the string coordinates. We thereby clarify the physical meaning of the divergence in the string kinetic term. The last section is devoted to conclusions. Some appendices are added.

2. Review of Witten's String Field Theory

2.1 First Quantization

In this section, we briefly review the first quantization of an open string. The action describing a propagation of an open string is

$$S = -\frac{1}{4\pi\alpha'}\int_{-\infty}^{\infty} d\tau \int_{o}^{\pi} d\sigma \sqrt{-g} g^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu}$ =diag$(-,+,\cdots,+)$ is the metric of 26-dimensional space-time and g^{ab} is the metric of the two-dimensional worldsheet. The symbols τ and $\sigma(0 \leq \sigma \leq \pi)$ denote time and space coordinates of two-dimensional worldsheet, respectively. In the present paper, we set the parameter $\alpha' = \frac{1}{2}$. The solution to the equation of motion following from (2.1) represented as

$$X^{\mu}(\tau,\sigma) = x_0^{\mu} + p_0^{\mu}\tau + i\sum_{n\neq 0}\frac{\alpha_n^{\mu}}{n}e^{-in\tau}\cos n\sigma. \tag{2.2}$$

provided with the boundary condition

$$\frac{\partial X^\mu(\tau,\sigma)}{\partial\sigma}\Big|_{\sigma=0,\pi}=0 \tag{2.3}$$

where $x_0^\mu = \frac{1}{\pi}\int_0^\pi d\sigma X^\mu(0,\sigma)$ is the center of mass coordinate of the string at $\tau = 0$ and p_0^μ is the total momentum of the string. In the following, we suppress space-time Lorentz indices whenever no confusion occurs.

The momentum conjugate to $X(\tau,\sigma)$ is

$$\begin{aligned} P(\tau,\sigma) =& \frac{\delta S}{\delta\frac{\partial X}{\partial\tau}} \\ =& \frac{1}{\pi}\left[p_0 + \sum_{n\neq 0}\alpha_n e^{-in\tau}\cos n\sigma\right]. \end{aligned} \tag{2.4}$$

From now on, we only consider the fields on the $\tau = 0$ slice of worldsheet.

$$\begin{aligned} X(\sigma) =& x_0 + \sqrt{2}\sum_{n>0} x_n \cos n\sigma, \\ P(\sigma) =& \frac{1}{\pi}\left[p_0 + \sqrt{2}\sum_{n>0} p_n \cos n\sigma\right], \end{aligned} \tag{2.5}$$

where

$$x_n = \frac{i}{\sqrt{2}n}(\alpha_n - \alpha_{-n}), \qquad p_n = \frac{1}{\sqrt{2}}(\alpha_n + \alpha_{-n}). \tag{2.6}$$

After the first quantization, the commutation relations are given by

$$[X^\mu(\sigma), P_\nu(\sigma')] = i\eta^\mu{}_\nu\delta(\sigma-\sigma'), \tag{2.7}$$

which leads

$$[x_m, p_n] = i\delta_{mn}, \qquad [\alpha_m, \alpha_n] = m\delta_{m+n,0}. \tag{2.8}$$

To fix the reparametrization freedom of the action, we choose the so-called conformal gauge $g_{ab} = e^\phi\eta_{ab}(\eta_{ab} = \mathrm{diag}(-1,1))$. A covariant first quantization with the gauge-fixing has been carried out by Kato and Ogawa[16] by using the BRST formalism. For gauge-fixing, we introduce Faddeev-Popov ghosts and anti-ghosts:

$$
\begin{aligned}
c^0(\sigma) &= c_0 + \sum_{n\neq 0} c_n \cos n\sigma, \\
c^1(\sigma) &= -i \sum_{n\neq 0} c_n \sin n\sigma, \\
b_0(\sigma) &= \frac{-i}{\pi} \sum_{n\neq 0} b_n \sin n\sigma, \\
b_1(\sigma) &= \frac{1}{\pi}\left[b_0 + \sum_{n\neq 0} b_n \cos n\sigma\right],
\end{aligned}
\tag{2.9}
$$

which obey the anti-commutation relations

$$
\begin{aligned}
\{b_1(\sigma), c^0(\sigma')\} &= \delta(\sigma - \sigma'), \\
\{b_0(\sigma), c^1(\sigma')\} &= \delta(\sigma - \sigma'), \\
\{b_m, c_n\} &= \delta_{m+n,0}.
\end{aligned}
\tag{2.10}
$$

Because of the relation $\{b_0, c_0\} = 1$, the Fock vacuum of the ghost and anti-ghost system is doubly degenerate with the vacua $|+\rangle$ and $|-\rangle$ defined as

$$
\begin{aligned}
&c_n|+\rangle = 0, \qquad b_n|+\rangle = 0, \qquad (n > 0) \\
&c_0|+\rangle = 0, \qquad b_0|+\rangle = |-\rangle
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&c_n|-\rangle = 0, \qquad b_n|-\rangle = 0, \qquad (n > 0) \\
&b_0|-\rangle = 0, \qquad c_0|-\rangle = |+\rangle.
\end{aligned}
\tag{2.12}
$$

Their inner products are given by

$$\langle + | + \rangle = \langle - | - \rangle = 0, \qquad \langle + | - \rangle = \langle - | + \rangle = 1. \tag{2.13}$$

We assign ghost numbers to all state and operators. The ghost number is defined by the eigenvalue of operator

$$N_g = \frac{1}{2}[c_0, b_0] + \sum_{n>0}(c_{-n}b_n - b_{-n}c_n). \tag{2.14}$$

The ghost $c^a(\sigma)$ has $N_g = 1$ and the anti-ghost $b_a(\sigma)$ has $N_g = -1$. The states $|+\rangle$ and $|-\rangle$ have $N_g = +\frac{1}{2}$ and $N_g = -\frac{1}{2}$, respectively. The generator of the BRST transformation is given by

$$Q_B = -\frac{1}{2}\sum_{m,n} : \alpha_m \alpha_n c_{-n-m} : + \frac{1}{2}\sum_{m,n}(n-m) : c_{-n}c_{-m}b_{n+m} : + c_0, \tag{2.15}$$

Because of the invariance of theory, part of the Fock space is redundant. Hence we must reduce the Fock space into the subspace spanned by all physical states satisfying the condition:

$$Q_B|\Phi_{phys.}\rangle = 0. \tag{2.16}$$

This equation expresses a quantum version of reparametrization invariance of the string worldsheet. The subspace of $|\Phi_{phys.}\rangle$ in the Fock space is shown to contain no negative norm state.

2.2 String Field and the Action

In this subsection, we briefly review the construction of string field theory. In second quantized theory, a wave functional of the first quantized theory becomes an operator which creates or annihilates a string. Since a wave functional of string is a functional of $X(\sigma), b_0(\sigma)$ and $c^0(\sigma)$, the string field is a functional of them also, which we express as $\Phi(X(\sigma), b_0(\sigma), c^0(\sigma))$. A convenient expression of the string field is given by a state vector in the first quantized Fock space, which is related with the functional representation by

$$\Phi(X(\sigma), b_0(\sigma), c^0(\sigma)) = \langle X(\sigma), b_0(\sigma), c^0(\sigma)|\Phi\rangle. \tag{2.17}$$

The state $\langle X(\sigma), b_0(\sigma), c^0(\sigma)|$ is an eigenstate of $X(\sigma), b_0(\sigma)$ and $c^0(\sigma)$. The open string fields have two indices of the Lie algebra($O(n)$ or $Sp(2n)$) attached at two edges of a string, *i.e.*, open string fields are matrix valued. In free theory, the string field satisfies the equation of motion

$$Q_B|\Phi\rangle = 0. \tag{2.18}$$

The action of the free string field theory is constructed to provide (2.18). The Lagrangian density of the free string field theory is given by

$$\mathcal{L} = \Phi\big(X(\pi-\sigma), -c(\pi-\sigma), b(\pi-\sigma)\big) \times Q_B\Phi\big(X(\sigma), c(\sigma), b(\sigma)\big) \tag{2.19}$$

in the coordinate representation. Reversing of orientation ($\sigma \to \pi - \sigma$) defines charge conjugation, or, transposition of matrix indices of string field. Our string field is "real" in the sense $\Phi^\dagger\big(X(\sigma), c(\sigma), b(\sigma)\big) = \Phi\big(X(\pi-\sigma), -c(\pi-\sigma), b(\pi-\sigma)\big)$. The Lagrangian density (2.19) can be written in the Fock representation as

$$\mathcal{L} = \langle V_2|\Phi\rangle Q_B|\Phi\rangle. \tag{2.20}$$

where $|\Phi\rangle$ is assumed to have ghost number $N_g = -\frac{1}{2}$. The two string vertex $\langle V_2|$ is defined by the following connection condition:

$$\begin{aligned}
&\langle V_2|\big(X^{(1)}(\sigma) - X^{(2)}(\pi-\sigma)\big) = 0,\\
&\langle V_2|\big(c^{(1)}(\sigma) + c^{(2)}(\pi-\sigma)\big) = 0,\\
&\langle V_2|\big(b^{(1)}(\sigma) - b^{(2)}(\pi-\sigma)\big) = 0.
\end{aligned} \tag{2.21}$$

In the Fock representation, the expression of $\langle V_2|$ is given by

$$\begin{aligned}\langle V_2| =& {}_1\langle -|_2\langle -|(c_0^{(1)}+c_0^{(2)})\\ &\times \exp\left[-\sum_{n>0}(-)^n[\frac{1}{n}\alpha_n^{(1)}\alpha_n^{(2)}+nc_n^{(1)}b_n^{(2)}-nb_n^{(1)}c_n^{(2)}]\right]\end{aligned} \tag{2.22}$$

As is easily confirmed, $\langle V_2|$ has no ghost number. Some other properties of two string vertex are listed in appendix C. The reality condition is expressed in the Fock representation

$$|\Phi\rangle^\dagger = \langle V_2|\Phi\rangle \tag{2.23}$$

Now we can write down the gauge invariant action of string field theory. In the following, we expand string field as $|\Phi\rangle = b_0|\phi^{(0)}\rangle + |\psi^{(0)}\rangle$. Further, the fields $|\phi^{(0)}\rangle$ and $|\psi^{(0)}\rangle$ are expanded as follows:

$$|\phi^{(0)}\rangle = \sum_A |A\rangle\phi_A^{(0)} \tag{2.24}$$

and

$$|\psi^{(0)}\rangle = \sum_A |A\rangle\psi_A^{(0)}, \tag{2.25}$$

where the summations cover all possible particle modes signified with A. We set the grade of a state $|A\rangle$ as

$$(-)^A = \begin{cases} -1, (|A\rangle \text{ is fermionic}) & \text{if } N_g(A) = \frac{3}{2} \bmod 2 \\ 1, (|A\rangle \text{ is bosonic}) & \text{if } N_g(A) = \frac{1}{2} \bmod 2 \end{cases} \tag{2.26}$$

The component fields $\phi_A^{(0)}$ and $\psi_A^{(0)}$ have no ghost number N_g, which is indicated with the superscript (0). Since total fields $|\Phi\rangle$ have $N_g = -\frac{1}{2}$, the state $|A\rangle$ in (2.24) has $N_g = +\frac{1}{2}$. Similarly, the state $|A\rangle$ in (2.25) has $N_g = -\frac{1}{2}$. After second quantization, the component fields become creation-annihilation operators of the excited particle modes. The local mode expansions of the first few lower mass states for ϕ-field and ψ-field are as follows:

$$
\begin{aligned}
|\phi^{(0)}\rangle &= |+\rangle\phi(x) + \alpha^{\mu}_{-1}|+\rangle A_\mu(x) + b_{-1}c_{-1}|+\rangle D(x) + \cdots, \\
|\psi^{(0)}\rangle &= ib_{-1}|+\rangle B(x) + \cdots.
\end{aligned}
\tag{2.27}
$$

Obeying the definition by Kato and Ogawa,[16] the BRST charge can be expanded in powers of ghost zero modes as

$$
Q_B = c_0 L^{KO} + M^{KO} b_0 + \tilde{Q}_B^{KO} \tag{2.28}
$$

where

$$
\begin{aligned}
L^{KO} &= -\frac{1}{2}p^2 - \frac{1}{2}\sum_{n\neq 0} :\alpha_{-n}\alpha_n: - \sum_n :nc_{-n}b_n: +1, \\
M^{KO} &= \sum_n :nc_{-n}c_n:, \\
\tilde{Q}_B^{KO} &= -\frac{1}{2}\sum_{n\neq 0}\sum_{\substack{m\neq 0 \\ m\neq n}} :\alpha_{n-m}\alpha_m c_{-n}: \\
&\quad + \frac{1}{2}\sum_{n\neq 0}\sum_{\substack{m\neq 0 \\ n+m\neq 0}} (n-m) :c_{-n}c_{-m}b_{n+m}: .
\end{aligned}
\tag{2.29}
$$

By using this expansion, the Lagrangian density turns out to be

$$
\begin{aligned}
\mathcal{L}_{inv.} =& \langle V_2|b_0|\phi^{(0)}\rangle L^{KO}|\phi^{(0)}\rangle - \langle V_2|b_0|\psi^{(0)}\rangle M^{KO}|\psi^{(0)}\rangle \\
&+ 2\langle V_2|b_0|\psi^{(0)}\rangle \tilde{Q}_B^{KO}|\phi^{(0)}\rangle.
\end{aligned}
\tag{2.30}
$$

In the above expression, we have carried out some partial integrations. (See appendix C.) By using (2.27), we can obtain local field expression of the Lagrangian density as

$$\mathcal{L}_{inv.} =\phi(x)(\frac{1}{2}\partial^2+1)\phi(x)+\frac{1}{2}A^\mu(x)\partial^2 A_\mu(x)-D(x)(\frac{1}{2}\partial^2-1)D(x)+\cdots$$
$$+B(x)^2-2B(x)\partial_\mu A^\mu(x)+\cdots. \tag{2.31}$$

where the fields $\phi(x)$,$A^\mu(x)$,$D(x)$ and $B(x)$ are tachyon, massless vector field, massive scalar field and auxiliary field, respectively.

The first term in (2.30) can be written as $\langle V_2|b_0|A\rangle L^{KO}|B\rangle\phi^{(0)}_A\phi^{(0)}_B$. In the following, we use matrix notation as $(L^{KO})_{AB}=\langle V_2|b_0|A\rangle L^{KO}|B\rangle$. For instance, $(L^{KO})_{AB}=\frac{1}{2}(\partial^2-m^2)I_{AB}$ if A is an index associated with the state $|A>$ whose mass is m^2. For the detail of this notation, see the appendix A. By using this matrix representation, (2.30) can be written as

$$\mathcal{L}_{inv.} =\phi^{(0)}_A(L^{KO})_{AB}\phi^{(0)}_B-\psi^{(0)}_A(M^{KO})_{AB}\psi^{(0)}_B+2\phi^{(0)}_A(\tilde{Q}^{KO}_B)_{AB}\psi^{(0)}_B \tag{2.32}$$

The action has a gauge invariance under the transformation $\delta|\Phi\rangle=Q_B|\Lambda\rangle$, where $|\Lambda\rangle$ is a gauge parameter which has ghost number $N_g=-\frac{3}{2}$. The transformation can be written in terms of component fields as

$$\begin{aligned}\delta\phi^{(0)}_A &=I^{-1}_{AB}((M^{KO})_{BC}\rho_C-(\tilde{Q}^{KO}_B)_{BC}\lambda_C)\\ \delta\psi^{(0)}_A &=I^{-1}_{AB}((L^{KO})_{BC}\lambda_C+(\tilde{Q}^{KO}_B)_{BC}\rho_C),\end{aligned} \tag{2.33}$$

where the gauge parameter $|\Lambda\rangle$ is expanded as $|\Lambda\rangle=b_0|A\rangle\lambda_A+|A\rangle\rho_A$. The coefficients λ_A and ρ_A are c-number parameters. Insertion of I^{-1} is necessary for the indefinite metric of the first quantized Fock space. (For the definition of I^{-1} and justification of this insertion, see Appendix A.) If we expand

$$|\Lambda\rangle=b_0b_{-1}|+\rangle\lambda(x)+\cdots, \tag{2.34}$$

the gauge transformation (2.33) can be expressed as

$$\begin{aligned} \delta A_\mu(x) &= \partial_\mu \lambda(x) + \cdots \\ \delta B(x) &= \frac{1}{2}\partial^2 \lambda(x) + \cdots \end{aligned} \tag{2.35}$$

whose massless sector coincides with Yang-Mills gauge transformation of particle gauge field theory.

The invariance can be fixed by using the Siegel gauge condition $b_0|\Phi\rangle = 0$.[17] In terms of component fields, this condition can be reduced to $\psi_A^{(0)} = 0$.

In free case ($g = 0$), the gauge fixing procedure can be performed as was shown in Refs. [18][19]. In the following, we briefly describe the procedure. First, we replace the gauge parameters with the Faddeev-Popov ghosts. Then, the transformation (2.33) can be replaced by the BRST transformation

$$\begin{aligned} \delta\phi_A^{(0)} &= I_{AB}^{-1}\left[(M^{KO})_{BC}\psi_C^{(1)} - (\tilde{Q}_B^{KO})_{BC}\phi_C^{(1)}\right]\epsilon, \\ \delta\psi_A^{(0)} &= I_{AB}^{-1}\left[(L^{KO})_{BC}\phi_C^{(1)} + (\tilde{Q}_B^{KO})_{BC}\psi_C^{(1)}\right]\epsilon. \end{aligned} \tag{2.36}$$

where ϵ is a Grassmann odd parameter which has $N_g = -1$. $\phi_A^{(1)}$ and $\psi_A^{(1)}$ are the components of Faddeev-Popov ghosts which have ghost number $N_g = 1$ since the total ghost number of $b_0|A\rangle\phi_A^{(1)} + |A\rangle\psi_A^{(1)}$ is $-\frac{1}{2}$. According to the ordinary BRST-gauge fixing procedure, we add the gauge fixing term $\mathcal{L}_{GF}$ and the Faddeev-Popov ghost term $\mathcal{L}_{FP}$ to the gauge invariant action. The additional terms are

$$\begin{aligned} \mathcal{L}_{FP} + \mathcal{L}_{GF} =& \delta(-2\phi_A^{(-1)} I_{AB}\psi_B^{(0)}) \\ =& -2B_A^{(0)} I_{AB}\psi_B^{(0)} \\ & -2\phi_A^{(-1)}\left((L^{KO})_{AB}\phi_B^{(1)} + (\tilde{Q}_B^{KO})_{AB}\psi_B^{(1)}\right) \end{aligned} \tag{2.37}$$

where $\phi_A^{(-1)}$ and $B_A^{(0)}$ are anti-ghosts($N_g = -1$) and Nakanishi-Lautrap fields($N_g = 0$), respectively. The coefficient -2 is a matter of convention. Their BRST transformations are

$$\begin{aligned} \delta\phi_A^{(-1)} &= B_A^{(0)}\epsilon, \\ \delta B_A^{(0)} &= 0. \end{aligned} \tag{2.38}$$

We set even ghost number component field to be bosonic and odd ghost number component field to be fermionic.

This procedure fixes the gauge freedom under the transformation (2.33). However, the added action has the same type gauge invariance, *i.e.*,

$$\begin{aligned} \delta\phi_A^{(1)} &= I_{AB}^{-1}\left[(M^{KO})_{BC}\rho'_C - (\tilde{Q}_B^{KO})_{BC}\lambda'_C\right], \\ \delta\psi_A^{(1)} &= I_{AB}^{-1}\left[(L^{KO})_{BC}\lambda'_C + (\tilde{Q}_B^{KO})_{BC}\rho'_C\right]. \end{aligned} \tag{2.39}$$

This indicates the gauge transformation (2.33) has zero modes, in other words, the gauge symmetry is reducible. To fix this invariance, we have to introduce new ghosts $\phi_A^{(2)}$ and $\psi_A^{(2)}$. Then, we repeat the same procedure as above. Eventually, it is necessary to introduce an infinite series of the BRST auxiliary fields listed below.

comp. field	N_g of $\lvert A\rangle$	BRST transformation	
$\phi_A^{(n)}$ $(n \geq 0)$	$\frac{1}{2} - n$	$I_{AB}^{-1}\left[(M^{KO})_{BC}\psi_C^{(n+1)} - (\tilde{Q}_B^{KO})_{BC}\phi_C^{(n+1)}\right]$	
$\psi_A^{(n)}$ $(n \geq 0)$	$-\frac{1}{2} - n$	$I_{AB}^{-1}\left[(L^{KO})_{BC}\phi_C^{(n+1)} + (\tilde{Q}_B^{KO})_{BC}\psi_C^{(n+1)}\right]$	(2.40)
$\phi_A^{(n)}$ $(n < 0)$	$\frac{1}{2} - n$	$B_A^{(n+1)}$	
$B_A^{(n)}$ $(n < 0)$	$\frac{3}{2} - n$	0	

As the result, we add to the action the terms $\delta(-2(-)^n\phi_A^{(-n-1)}I_{AB}\psi_B^{(n)})$. By this gauge fixing procedure, we can obtain a gauge fixed action for a free string theory such as

$$
\begin{aligned}
\mathcal{L}_{fixed} =& \phi_A^{(0)}(L^{KO})_{AB}\phi_B^{(0)} + 2\sum_{n>0}(-)^n\phi_A^{(-n)}(L^{KO})_{AB}\phi_B^{(n)} \\
& - \psi_A^{(0)}(M^{KO})_{AB}\psi_B^{(0)} + 2\sum_{n\geq 0}(-)^n\phi_A^{(-n)}(\tilde{Q}_B^{KO})_{AB}\psi_B^{(n)} - 2\sum_{n\geq 0}B_A^{(-n)}I_{AB}\psi_B^{(n)},
\end{aligned}
\tag{2.41}
$$

or, in a more compact form,

$$
\begin{aligned}
\mathcal{L}_{fixed} =& \langle V_2|b_0|\phi\rangle L^{KO}|\phi\rangle - \langle V_2|b_0|\psi^{(0)}\rangle M^{KO}|\psi^{(0)}\rangle \\
& + 2\langle V_2|b_0|\psi\rangle \tilde{Q}_B^{KO}|\phi\rangle - 2\langle V_2|b_0|B\rangle|\psi\rangle
\end{aligned}
\tag{2.42}
$$

where

$$
|\phi\rangle = \sum_{n=-\infty}^{\infty}|\phi^{(n)}\rangle, |\psi\rangle = \sum_{n=0}^{\infty}|\psi^{(n)}\rangle, |B\rangle = \sum_{n=0}^{\infty}|B^{(-n)}\rangle. \tag{2.43}
$$

If we substitute the equation of motion of B-fields, (2.41) and (2.42) reduce to

$$
\mathcal{L}_{fixed} = \sum_n (-)^n\phi_A^{(-n)}(L^{KO})_{AB}\phi_B^{(n)} = \langle V_2|b_0|\phi\rangle L^{KO}|\phi\rangle \tag{2.44}
$$

Consequently, a gauge fixed action can be obtained from gauge-invariant action by setting $\psi = 0$ and removing the constraint for ghost number of component fields.

We introduce an interaction of string fields by adding a term to (2.20). As mentioned in the introduction, there are two ways to introduce interactions. In Witten's string field theory, open string interaction can be described by the connection of three strings as shown in Fig. 1. This connection condition is realized by the three string vertex $\langle V_3|$ which satisfies

$$
\begin{aligned}
&\langle V_3|(X^{(r)}(\sigma) - X^{(r+1)}(\pi-\sigma)) = 0, \\
&\langle V_3|(c^{(r)}(\sigma) + c^{(r+1)}(\pi-\sigma)) = 0, \\
&\langle V_3|(b^{(r)}(\sigma) - b^{(r+1)}(\pi-\sigma)) = 0, \qquad (0 \leq \sigma \leq \frac{\pi}{2})
\end{aligned}
\tag{2.45}
$$

where $r(=1,2,3)$ denotes the channel of participating strings and $X^{(4)}$ represents $X^{(1)}$. The three string vertex can be expressed in a Fock space representation:

$$\langle V_3| =_1 \langle +|_2\langle +|_3\langle +| \exp\,[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n\geq 0}^{\infty} \alpha_m^r N_{mn}^{rs}\alpha_n^s]\,\exp\,[\sum_{r,s=1}^{3}\sum_{\substack{m\geq 0\\ n>0}}^{\infty} b_m^r N_{cmn}^{rs} c_n^s] \tag{2.46}$$

where N_{mn}^{rs} and N_{cmn}^{rs} are constant coefficients (for the definition, see the next section). If we assume that the rule for the gauge-fixing procedure explained above can be applied to the interacting case, we can guess a gauge-fixed action

$$\mathcal{L}_{fixed} = \langle V_2|b_0|\phi\rangle L|\phi\rangle + \frac{2g}{3}\langle V_3|b_0|\phi\rangle b_0|\phi\rangle b_0|\phi\rangle, \tag{2.47}$$

which is obtained from the gauge invariant action by setting $\psi = 0$ and removing the constraint of N_g. In interacting case, it is not clear whether the gauge-fixing procedure works well. We discuss the problem later in section 4.

2.3 Formal Lagrangian Path Integral Quantization

In the former works,[6–8] a generating functional of string scattering amplitudes is evaluated by using the formal Lagrangian path integral

$$W[J] = \int d\phi e^{i\int \mathcal{L}_{fixed} + i\int dx J\phi}, \tag{2.48}$$

where J is an external source of string field. Shifting the string field ϕ to $\phi - \frac{1}{2}L^{-1}J$ and integrating out ϕ from the path integral, we can obtain a generating functional of string field theory as

$$W[J] = \mathcal{N} e^{i\int dx \frac{2g}{3} f_{ABC}(-i\frac{\delta}{\delta J_A})(-i\frac{\delta}{\delta J_B})(-i\frac{\delta}{\delta J_C})} \times e^{-i\int dx \frac{1}{4} J L^{-1} J}, \tag{2.49}$$

where $\mathcal{N}$ is a constant and f_{ABC} is a matrix representation of three string vertex(defined in Appendix A). We can obtain Green functions or scattering amplitudes from the functional (2.49). For example, the propagator of string field is shown to

$$\langle T\phi_A^{(m)}\phi_B^{(n)}\rangle = (-i\frac{\delta}{\delta J_1})(-i\frac{\delta}{\delta J_2})W[J]\big|_{J=0}$$
$$= \frac{i}{2}\delta_{m+n,0}(L^{-1})_{AB} + (\text{loop effect}). \tag{2.50}$$

Similarly, scattering amplitudes of strings can be obtained.

Now we comment on the problem of non-locality of vertex. Originally, the path integral is formulated based on canonical formalism. Path integral must be defined in phase space as

$$W[J] = \int d\Pi d\phi e^{i\int(\Pi\dot{\phi} - \mathcal{H}) + i\int J\phi}, \tag{2.51}$$

where Π is the momentum conjugate to ϕ and $\mathcal{H}$ is the hamiltonian density. In most cases, we can obtain (2.48) from (2.51) after integrating out Π. In such cases, these two formalisms are shown to be equivalent. However it is known that, if the interaction term contains time derivatives, Π-integration generates some non-trivial terms. A well-known example is the Lee-Yang term in non-linear σ model. In string field theory, the situation is worse than the non-linear σ model by the following reason. The three string vertex (2.46) contains p_0 in the exponent. The zero mode p_0 of string momentum is a derivative with respect to center of mass of the string, and it is regarded as a generator of translation of the string. This interaction is non-local because the centers of mass of three strings sit on different places from one another. As a result, $\langle V_3|$ contains infinite orders of the derivatives, especially of the time derivatives. Clearly, in so far as we stick at the center of mass time formalism, we cannot apply the canonical quantization procedure to the string field theory. However, this non-locality is spurious, because all points $X_\mu^r(\sigma)(r = 1,2,3)$ of three strings are connected locally in Witten's three string vertex. Especially, all mid-points $X(\sigma = \frac{\pi}{2})$ of three strings are connected one another. (See Fig. 1.) We expect that, if we express the three string vertex in the mid-point coordinates instead of the center of mass coordinates, the vertex no longer contains any derivative with respect to the mid-point coordinates.

Usefulness of the mid-point coordinates was first pointed out by Witten himself in his paper with respect to superstring field theory.[20] Explicit rewriting of the vertex was first carried out by Morris,[21] and further studied by Mañes[22] and the present author.[23] Difficulty concerning non-locality can be avoided by employing the mid-point coordinates. In the next section, we develop the mid-point coordinate formalism.

3. Mid-point Coordinates

3.1 Mid-point Coordinates

Usually, coordinates and momenta of first-quantized string are expanded around the center of mass as reviewed in section 2. In this representation, an independent set of canonical variables of string coordinates and momenta is given by

$$\begin{aligned} &\text{coordinates:} \qquad x_0 \text{ and } x_n = \frac{i}{\sqrt{2}n}(\alpha_n - \alpha_{-n}), \\ &\text{momenta:} \qquad p_0 \text{ and } p_n = \frac{1}{\sqrt{2}}(\alpha_n + \alpha_{-n}), \end{aligned} \tag{3.1}$$

where n runs over positive integers. Now we change the zero modes of coordinates from x_0 to $x_I = X(\frac{\pi}{2})$. This change is realized by the unitary operator U

$$\begin{aligned} U &= \exp\left[ip_0(x_0 - x_I)\right] \\ &= \exp\Big[-i\sqrt{2}p_0 \sum_{n>0} x_n \cos n\frac{\pi}{2}\Big] \\ &= \exp\Big[p_0 \sum_{n\neq 0} \frac{\alpha_n}{n} \cos n\frac{\pi}{2}\Big]. \end{aligned} \tag{3.2}$$

then

$$U^\dagger x_0 U = x_I,$$
$$U^\dagger x_n U = x_n. \tag{3.3}$$

According to this transformation of coordinates, we must transform momentum as $\tilde{p}_n = U^\dagger p_n U$. We can consider that this transformation is a canonical one and that x_I,x_n and p_0,$\tilde{p}_n$ are new canonical variables. In the same manner, we can define creation-annihilation operators associated with the new set of coordinate and momentum, i.e.

$$\tilde{\alpha}_m \equiv U^\dagger \alpha_m U = \alpha_m - p_0 \cos m\frac{\pi}{2} \tag{3.4}$$

which commute with $x_I = X(\frac{\pi}{2})$. Now, an independent set of coordinates and momenta is

$$\text{coordinates:} \quad x_I \text{ and } x_n = \frac{i}{\sqrt{2}n}(\tilde{\alpha}_n - \tilde{\alpha}_{-n}),$$
$$\text{momenta:} \quad p_0 \text{ and } \tilde{p}_n = \frac{1}{\sqrt{2}}(\tilde{\alpha}_n + \tilde{\alpha}_{-n}). \tag{3.5}$$

Their commutation relations have the same form as the ones for the center of mass coordinates, *i.e.*,

$$[\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta_{m+n,0} \quad , \quad [x_I, p_0] = i \tag{3.6}$$

In terms of the new coordinates, the normal mode expansions of string coordinates become

$$X(\sigma) = x_I + \sqrt{2}\sum_{n>0} x_n(\cos n\sigma - \cos n\frac{\pi}{2}),$$
$$P(\sigma) = \frac{1}{\pi}\left[\pi p_0 \delta(\sigma - \frac{\pi}{2}) + \sqrt{2}\sum_{n>0} \tilde{p}_n \cos n\sigma\right]. \tag{3.7}$$

The Fock vacuum which is annihilated by $\tilde{\alpha}$ is related to the Fock vacuum which is annihilated by α as

$$\begin{aligned}\langle 0| &= \langle\tilde{0}|U^\dagger \\ &= \langle\tilde{0}|\exp\big[-\frac{1}{2}(p_0)^2\sum_{n>0}\frac{1}{n}\cos^2 n\frac{\pi}{2}\big]\times\exp\big[-p_0\sum_{n>0}\frac{1}{n}\tilde{\alpha}_n\cos n\frac{\pi}{2}\big].\end{aligned} \tag{3.8}$$

The p^2 term in the first exponent arises from normal ordering of U and the coefficient diverges when the summation is carried out. However this divergence is regularized by using the method in the next subsection.

Similarly, ghosts and anti-ghosts can be rewritten as

$$\begin{aligned}c^0(\sigma) =& c_I + \sum_{n\neq 0} c_n(\cos n\sigma - \cos n\frac{\pi}{2}),\\ c^1(\sigma) =& -i\sum_{n\neq 0} c_n \sin n\sigma,\\ b_0(\sigma) =& \frac{-i}{\pi}\sum_{n\neq 0}\tilde{b}_n \sin n\sigma,\\ b_1(\sigma) =& \frac{1}{\pi}\left[\pi b_0\delta(\sigma-\frac{\pi}{2}) + \sum_{n\neq 0}\tilde{b}_n\cos n\sigma\right],\\ &(\{\tilde{b}_m, c_n\} = \delta_{mn} \quad , \quad \{b_0, c_I\} = 1)\end{aligned} \tag{3.9}$$

where $\tilde{b}_m = b_m - b_0\cos m\frac{\pi}{2}$. This rewriting is realized by the unitary operator

$$\bar{U} \equiv e^{-b_0(c_0 - c_I)} = \exp\big[b_0\sum_n c_n\cos n\frac{\pi}{2}\big], \tag{3.10}$$

and

$$\begin{aligned}\bar{U}^\dagger c_0\bar{U} &= c_I,\\ \bar{U}^\dagger c_n\bar{U} &= c_n,\\ \bar{U}^\dagger b_n\bar{U} &= \tilde{b}_n.\end{aligned} \tag{3.11}$$

The Fock vacuum is related to the one in the center of mass coordinates as

$$
\begin{aligned}
\langle + | &= \langle \tilde{+} | \bar{U}^{\dagger} \\
&= \langle \tilde{+} | \exp \big[- b_0 \sum_{n>0} c_n \cos n\frac{\pi}{2} \big].
\end{aligned}
\tag{3.12}
$$

3.2 Three String Vertex in Mid-point Coordinates

In this subsection, we rewrite the three string vertex by using the mid-point coordinates. First we will consider the bosonic coordinates part of the vertex. To this end, we substitute (3.4) and (3.8) to (2.46). After substituting, we can obtain

$$
\langle V_3 | =_1 \langle \tilde{+} |_2 \langle \tilde{+} |_3 \langle \tilde{+} | e^{V(\tilde{\alpha})}, \tag{3.13}
$$

where $V(\tilde{\alpha})$ is given as the sum of following three terms,

$$
\begin{aligned}
\mathcal{O}(p^0): &\quad \frac{1}{2} \sum_{m>0} \sum_{n>0} \sum_{r,s=1}^{3} \tilde{\alpha}_m^r N_{mn}^{rs} \tilde{\alpha}_n^s \\
\mathcal{O}(p^1): &\quad \sum_{r,s=1}^{3} \sum_{n>0} p_0^r \big[N_{0n}^{rs} + \sum_{m>0} \cos m\frac{\pi}{2} N_{mn}^{rs} - \delta^{rs} \frac{1}{n} \cos n\frac{\pi}{2} \big] \tilde{\alpha}_n^s \\
\mathcal{O}(p^2): &\quad \frac{1}{2} \sum_{r,s=1}^{3} p_0^r p_0^s \big[N_{00}^{rs} + 2 \sum_{n>0} N_{0n}^{rs} \cos n\frac{\pi}{2} + \sum_{m>0} \sum_{n>0} \cos m\frac{\pi}{2} N_{mn}^{rs} \cos n\frac{\pi}{2} \\
&\qquad - \delta^{rs} \sum_{n>0} \frac{1}{n} \cos^2 n\frac{\pi}{2} \big]
\end{aligned}
\tag{3.14}
$$

We will demonstrate that both $\mathcal{O}(p^1)$ term and $\mathcal{O}(p^2)$ terms vanish.

We start from the definition of N^{rs}_{mn}

$$\ln|z-\tilde{z}| + \ln|z-\tilde{z}^*| - \frac{2}{3}\ln|z(z-1)| - \frac{2}{3}\ln|\tilde{z}(\tilde{z}-1)|$$
$$= -\delta^{rs}\Big\{\sum_{n\geq 1}\frac{2}{n}e^{-n|\xi_r(z)-\xi_s(\tilde{z})|}\cos(n\sigma_r(z))\cos(n\sigma_s(\tilde{z})) - 2\max(\xi_r(z),\xi_s(\tilde{z}))\Big\}$$
$$+2\sum_{n,m\geq 0} N^{rs}_{mn}e^{n\xi_r(z)+m\xi_s(\tilde{z})}\cos(n\sigma_r(z))\cos(m\sigma_s(\tilde{z})) - \frac{2}{3}\xi_r(z) - \frac{2}{3}\xi_s(\tilde{z}), \tag{3.15}$$

where we use the definition of Witten's vertices by Suehiro.[24] The symbols $\xi_r(z)$ and $\sigma_r(z)$ are two dimensional space- and time- coordinates of the r-th string. The indices r, s denote channels of three strings. They take values 1,2,3 corresponding to the position of z and $\tilde{z}$. (See Fig. 2.) The complex coordinate z is related to the coordinate on complex ρ-plane as

$$\begin{gathered}\rho(z) = \ln\left[\frac{2}{3\sqrt{3}}\frac{(z+1)(z+2)(z-\frac{1}{2}) + (z^2-z+1)^{3/2}}{z(z-1)}\right], \\ \frac{d\rho(z)}{dz} = \frac{\sqrt{(z-z_0)(z-z_0^*)}}{z(z-1)}, \qquad z_0 = e^{i\pi/3}.\end{gathered} \tag{3.16}$$

The solid line of Fig. 2. denotes the branch cut of (3.16). The coordinates ξ_r and σ_r are related to ρ as

$$\rho(z) = \begin{cases}\alpha_r\zeta_r + i\pi\delta_{r3}, & \text{if } (\mathcal{I}m(\rho)\geq 0), \quad 0\geq\sigma_r\geq\pi, \\ \alpha_r\zeta_r - i\pi\delta_{r3}, & \text{if } (\mathcal{I}m(\rho)< 0), \quad -\pi\geq\sigma_r\geq 0,\end{cases} \tag{3.17}$$

where

$$\zeta_r = \xi_r + i\sigma_r, \qquad \alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -1. \tag{3.18}$$

We choose the points Z_r on z-plane which correspond $\xi_r = -\infty (r = 1, 2, \text{ and } 3)$ as $Z_1 = 1, Z_2 = 0$, and $Z_3 = -\infty$, respectively. First we check $\mathcal{O}(p^1)$ term. We will prove

$$\sum_{r=1}^{3}\sum_{n>0} p_0^r\,[N_{0n}^{rs} + \sum_{m>0} \cos m\frac{\pi}{2} N_{mn}^{rs} - \delta^{rs}\frac{1}{n}\cos n\frac{\pi}{2}] = 0. \tag{3.19}$$

From (3.15), we can obtain

$$\begin{aligned} &\ln|z_0 - \tilde{z}| + \ln|z_0 - \tilde{z}^*| - \frac{2}{3}\ln|z_0(z_0-1)| - \frac{2}{3}\ln|\tilde{z}(\tilde{z}-1)| \\ =&2[\sum_{m,n\geq 0} \cos m\frac{\pi}{2} N_{mn}^{rs} e^{n\xi_s(\tilde{z})}\cos n\sigma_s(\tilde{z}) \\ &- \delta^{rs}\sum_{n>0}\frac{1}{n}e^{n\xi_s(\tilde{z})}\cos n\frac{\pi}{2}\cos n\sigma_s(\tilde{z}) - \frac{2}{3}\xi_s(\tilde{z})] \end{aligned} \tag{3.20}$$

by $z \to z_0$. The symbol z_0 is the interacting point of three strings defined as $\zeta_r(z_0) = \frac{\pi}{2}i$ for $r = 1, 2, 3$. Because the LHS does not depend on the index r, if we multiply it by p_0^r and sum up with respect to r, the LHS of (3.20) vanishes owing to momentum conservation $\sum_r p_0^r = 0$, *i.e.*,

$$\begin{aligned} 0 = \sum_r p_0^r [&\sum_{m,n\geq 0} \cos m\frac{\pi}{2} N_{mn}^{rs} e^{n\xi_s(\tilde{z})}\cos n\sigma_s(\tilde{z}) \\ &- \delta^{rs}\sum_{n>0}\frac{1}{n}e^{n\xi_s(\tilde{z})}\cos n\frac{\pi}{2}\cos n\sigma_s(\tilde{z}) - \frac{1}{3}\xi_s(\tilde{z})] \end{aligned} \tag{3.21}$$

Picking up the terms proportional to $\cos n\sigma_s(\tilde{z})$ in (3.21), we can confirm the $\mathcal{O}(p^1)$ term vanishing.

In $\mathcal{O}(p^2)$ term, we find a divergence. In order to evaluate the divergences, we regularize the commutation relations of first quantized operator by the point-splitting method as follows.

$$[A(\tau,\sigma), B(\tau',\sigma')] \to \begin{cases} [A(\tau,\sigma), B(\tau'-\delta,\sigma')] & \text{if } \tau > \tau' \\ [A(\tau-\delta,\sigma), B(\tau',\sigma')] & \text{if } \tau < \tau' \end{cases} \tag{3.22}$$

In the following, we work in Wick-rotated two-dimensional space where τ is replaced with $i\tau$. Under the regularization, the $\mathcal{O}(p^2)$ divergence coming from normal ordering of U amounts to

$$-\frac{1}{2}\sum_{n>0}\frac{1}{n}\cos^2 n\frac{\pi}{2} \to -\frac{1}{2}\sum_{n>0}\frac{1}{n}\cos^2 n\frac{\pi}{2}e^{-n\delta}, \tag{3.23}$$

where δ is the small number introduced in (3.22). The definition of $\tilde{\alpha}$ also should be replaced as

$$\alpha_n - p_0 \cos n\frac{\pi}{2} \to \alpha_n - p_0 \cos n\frac{\pi}{2}e^{-n\delta}. \tag{3.24}$$

Now we are prepared to the evaluation of $\mathcal{O}(p^2)$ term.

$$\begin{aligned}&\frac{1}{2}\sum_{r,s=1}^{3} p_0^r p_0^s [N_{00}^{rs} + 2\sum_{n>0} N_{0n}^{rs}\cos n\frac{\pi}{2}e^{-n\delta}\\ &\qquad + \sum_{m>0}\sum_{n>0}\cos m\frac{\pi}{2}e^{-m\delta}N_{mn}^{rs}\cos n\frac{\pi}{2}e^{-n\delta} - \delta^{rs}\sum_{n>0}\frac{1}{n}\cos^2 n\frac{\pi}{2}e^{-n\delta}]\end{aligned} \tag{3.25}$$

From the definition of N_{mn}^{rs}, we can obtain

$$\begin{aligned}0 = &\sum_r p_0^r [N_{00}^{rs} + \sum_{n>0} N_{0n}^{rs}\cos n\frac{\pi}{2} + \sum_{n>0} N_{0n}^{rs}\cos n\frac{\pi}{2}e^{-n\delta}\\ &+ \sum_{m>0}\sum_{n>0}\cos m\frac{\pi}{2}N_{mn}^{rs}\cos n\frac{\pi}{2}e^{-n\delta} - \delta^{rs}\sum_{n>0}\frac{1}{n}\cos^2 n\frac{\pi}{2}e^{-n\delta} + \frac{2}{3}\delta]\end{aligned} \tag{3.26}$$

by $z \to z_0$ and $\tilde{z} \to Z(z_0,\delta)$, where $Z(z_0,\delta)$ is defined by $\zeta(Z(z_0,\delta)) = \frac{i\pi}{2} - \delta$ The LHS of (3.26) vanishes by the same reasoning for the case of (3.19).

The vanishing of zero mode for the ghost part of the vertex can be proved by the same manner as above. After substituting $b_m = \tilde{b}_m + b_0 \cos m\frac{\pi}{2}$ and (3.12) to (2.46), the ghost part of the vertex is given by

$$\begin{aligned}&\langle\tilde{+}|\exp[\sum_{r,s=1}^{3}\sum_{m,n>0}^{\infty}\tilde{b}_m^r N_{cmn}^{rs}c_n^s]\\ &\times \exp[\sum_{r,s=1}^{3}\sum_{\substack{m\geq 0\\ n>0}} b_0^r \cos m\frac{\pi}{2}N_{cmn}^{rs}c_n^s - \sum_{r=1}^{3} b_0^r \sum_{n>0} c_n^r \cos n\frac{\pi}{2}].\end{aligned} \tag{3.27}$$

The calculation is easier than that in coordinate part because the exponent contains at most the linear term in the zero mode b_0. From now on, we will demonstrate that

$$0 = \sum_{m\geq 0} N^{rs}_{cmn} \cos m\frac{\pi}{2} - \delta_{rs} \cos n\frac{\pi}{2}. \tag{3.28}$$

From the definition of N^{rs}_{cmn}

$$\begin{aligned}
&-\frac{1}{z-\tilde{z}}\frac{z(z-1)}{\tilde{z}(\tilde{z}-1)}\left(\frac{d\rho(z)}{dz}\right)\left(\frac{d\rho(\tilde{z})}{d\tilde{z}}\right)^{-2} \\
&=\frac{\delta_{rs}}{\alpha_r}\left\{\theta\big(\xi_s(\tilde{z})-\xi_r(z)\big)\sum_{n\geq 0} e^{-n(\zeta_s(\tilde{z})-\zeta_r(z))} - \theta\big(\xi_r(z)-\xi_s(\tilde{z})\big)\sum_{n\geq 0} e^{-n(\zeta_r(z)-\zeta_s(\tilde{z}))}\right\} \\
&\quad -\frac{\alpha_r}{\alpha_s^2}\sum_{\substack{m\geq 0\\ n\geq 1}} N^{rs}_{cmn} e^{n\zeta_r(z)+m\zeta_s(\tilde{z})},
\end{aligned} \tag{3.29}$$

we can obtain

$$0 = \frac{\delta_{rs}}{\alpha_r}\sum_{n\geq 0} e^{-in\pi/2} e^{\zeta_s(n\tilde{z}))} - \frac{\alpha_r}{\alpha_s^2}\sum_{\substack{m\geq 0\\ n>0}} N^{rs}_{cmn} e^{im\pi/2} e^{n\zeta_s(\tilde{z})} \tag{3.30}$$

by setting $z \to z_0$. The LHS of (3.30) vanishes due to $\frac{d\rho}{dz}(z_0) = 0$. Picking up the terms proportional to $e^{n\zeta_s(\tilde{z})}$, we get

$$0 = \frac{\delta_{rs}}{\alpha_r} e^{-in\pi/2} - \frac{\alpha_r}{\alpha_s^2}\sum_{m\geq 0} N^{rs}_{cmn} e^{im\pi/2}. \tag{3.31}$$

We can confirm (3.28) from the real part of (3.31) and $\alpha_r^2 = 1$.

Now the three string vertex in the mid-point coordinate representation no longer contains any derivative, *i.e.*,

$$\langle V_3| ={}_1\langle\tilde{+}|_2\langle\tilde{+}|_3\langle\tilde{+}| \exp\Big[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n>0}^{\infty} \tilde{\alpha}^r_m N^{rs}_{mn}\tilde{\alpha}^s_n\Big] \exp\Big[\sum_{r,s=1}^{3}\sum_{m,n>0}^{\infty} \tilde{b}^r_m N^{rs}_{cmn} c^s_n\Big] \tag{3.32}$$

We have obtained a local vertex of strings.

3.3 BRST Charge in Mid-point Coordinates

Let us turn our eyes to the BRST charge (2.15) in the kinetic term in the mid-point coordinates. It can be rewritten as

$$Q_B = c_I L + M b_0 + K + c_I b_0 J_- - b_0 c_I J_+, \tag{3.33}$$

where

$$\begin{aligned} L =& \frac{\pi}{2}\delta(0)\partial^2 + \ell^\mu \partial_\mu + L', \\ \ell^\mu =& i \sum_{n\neq 0} \tilde{\alpha}_n^\mu \cos n\frac{\pi}{2}, \\ L' =& -\frac{1}{2}\sum_{n\neq 0} : \tilde{\alpha}_{-n}\tilde{\alpha}_n : - \sum_{n\neq 0} : n c_{-n}\tilde{b}_n : +1, \end{aligned} \tag{3.34}$$

$$\begin{aligned} K =& k^\mu \partial_\mu + K', \\ k^\mu =& -i \sum_{n\neq 0} \tilde{\alpha}_n^\mu \sin n\frac{\pi}{2} \sum_{m\neq 0} c_m \sin m\frac{\pi}{2}, \\ K' =& -\frac{1}{2}\sum_{n\neq 0}\sum_{\substack{m\neq 0 \\ n\neq m}} \tilde{\alpha}_{n-m}\tilde{\alpha}_m c_{-n} \\ & + \frac{1}{2}\sum_{n\neq 0}\sum_{\substack{m\neq 0 \\ n+m\neq 0}} (n-m) : c_{-n}c_{-m}\tilde{b}_{n+m} : \\ & - : \sum_{n\neq 0} c_n \cos n\frac{\pi}{2} L' :, \end{aligned} \tag{3.35}$$

$$M = \sum_n n c_n \sin n\frac{\pi}{2} \sum_m c_m \sin m\frac{\pi}{2}, \tag{3.36}$$

$$\begin{aligned} J_+ = -\sum_{n>0} n c_n \cos n\frac{\pi}{2} \quad , \quad J_- =& -\sum_{n<0} n c_n \cos n\frac{\pi}{2}, \\ J =& J_+ + J_-, \end{aligned} \tag{3.37}$$

where $\partial = ip$ denotes the derivative with respect to the mid-point coordinate. The coefficient $\delta(0)$ in L means a divergence of one-dimensional δ-function, *i.e.*,

$$\sum_n \cos^2 n\frac{\pi}{2} = \sum_n \sin^2 n\frac{\pi}{2} = \pi\delta(0). \tag{3.38}$$

This divergence of the kinetic term comes from the δ-function in $P(\sigma)$. Note that Q_B contains the term $\pi \int_0^\pi d\sigma c^0(\sigma) P(\sigma)^2$.

4. Quantization of String Field in Mid-Point Time Formalism

4.1 Gauge Invariant Action and Gauge Fixing

In the following, we use matrix representation based on the Fock space of the mid-point coordinates. The variables used in this section are written by those of the mid-point representation. By using them, we can write down the gauge invariant action of string field theory in the mid-point formalism. If we expand string field as in section 2, the Lagrangian density turns out to be

$$\begin{aligned}\mathcal{L}_{inv.} =& \phi_A^{(0)} L_{AB} \phi_B^{(0)} - \psi_A^{(0)} M_{AB} \psi_B^{(0)} + 2\psi_A^{(0)} (K - J_+)_{AB} \phi_B^{(0)} \\ &+ \frac{2g}{3} f_{ABC} \phi_A^{(0)} \phi_B^{(0)} \phi_C^{(0)}.\end{aligned} \tag{4.1}$$

The interaction in (2.32) is local with regard to the mid-point, in particular, to the mid-point time coordinate. Therefore, the string field can be considered as a set of infinite component Klein-Gordon type fields.

As same as in section 2, the action is invariant under the gauge transformation

$$\begin{aligned}\delta\phi_A^{(0)} =& I_{AB}^{-1} \left[M_{BC}\rho_C - (K - J_+)_{BC}\lambda_C \right], \\ \delta\psi_A^{(0)} =& I_{AB}^{-1} \left[L_{BC}\lambda_C + (K + J_-)_{BC}\rho_C + g f_{BCD}(\phi_C^{(0)}\lambda_D + \lambda_C \phi_D^{(0)}) \right].\end{aligned} \tag{4.2}$$

The invariance can be fixed by using Siegel-like gauge condition $b_0|\Phi\rangle = 0$, or, $\psi_A^{(0)} = 0$. Although this equation is same expression as the ordinary Siegel gauge, the meaning of b_0 is not the derivative with respect to c_0 but the derivative with respect to c_I. In other words, ψ-components in our expansion are proportional to c_I while those in the center of mass expansion are proportional to c_0. Remarkably, owing to the vanishing of "mid-point of ghost" c_I on three string vertex, the ψ-component of string field does not present in the interaction term. Owing to this situation, the problem of gauge fixing in interacting case is easier than the center of mass representation. We will discuss it in the next subsection.

In free case ($g = 0$), the gauge fixing procedure can be performed in the similar way as in section 2. As the result of the gauge fixing procedure, we can obtain a gauge fixed action for a free theory. If we assume that the procedure can be applied in the interacting case, we can obtain a gauge-fixed Lagrangian density in the following compact form:

$$\begin{aligned} \mathcal{L}_{fixed} =& \langle V_2|b_0|\phi\rangle L|\phi\rangle - \langle V_2|b_0|\psi^{(0)}\rangle M|\psi^{(0)}\rangle \\ &+ 2\langle V_2|b_0|\psi\rangle(K - J_+)|\phi\rangle + 2\langle V_2|b_0|B\rangle|\psi\rangle \\ &+ \frac{2g}{3}\langle V_3|b_0|\phi\rangle b_0|\phi\rangle b_0|\phi\rangle, \end{aligned} \tag{4.3}$$

where

$$|\phi\rangle = \sum_{n=-\infty}^{\infty} |\phi^{(n)}\rangle, |\psi\rangle = \sum_{n=0}^{\infty} |\psi^{(n)}\rangle, |B\rangle = \sum_{n=0}^{\infty} |B^{(-n)}\rangle. \tag{4.4}$$

By using the matrix notation, we (4.3) is expressed as

$$\begin{aligned} \mathcal{L}_{fixed} =& \sum_n (-)^n \phi_A^{(n)} L_{AB} \phi_B^{(-n)} - \psi_A^{(0)} M_{AB} \psi_B^{(0)} \\ &+ 2\sum_{n\geq 0} \psi_A^{(n)} (K - J_+)_{AB} \phi_B^{(-n)} - 2\sum_{n\geq 0} B_A^{(-n)} I_{AB} \psi_B^{(n)} \\ &+ \frac{2g}{3} f_{ABC} \sum_{l,m,n} \phi_A^{(l)} \phi_B^{(m)} \phi_C^{(n)} \delta_{l+m+n,0}. \end{aligned} \tag{4.5}$$

Here we point out that the Fock vacuum in the mid-point coordinates is not the solution of the equation of motion

$$L|\phi\rangle = 0. \tag{4.6}$$

The vacuum in the center of mass representation $|0\rangle$ is a solution of (4.6) if it has a momentum $p^2 = 2$. Any basis vector in the Fock space

$$\alpha_{-n_1}\alpha_{-n_2}\cdots\alpha_{-n_I}|0\rangle \tag{4.7}$$

can be a solution if an appropriate mass-shell-condition is satisfied. As is easily understood, our vacuum $|\tilde{0}\rangle$ can not be a solution of (4.6), neither any other state

$$\tilde{\alpha}_{-n_1}\tilde{\alpha}_{-n_2}\cdots\tilde{\alpha}_{-n_I}|0\rangle \tag{4.8}$$

can. If we express (4.7) by using the mid-point coordinates, we shall find an infinite order of derivatives. This fact indicates the solution of the equation of motion in the mid-point coordinates representation is a non-local configuration.

4.2 Gauge Fixing in Interacting Case

In this subsection, we carry out the gauge-fixing of interacting string field theory. First, we comment on the difficulty of gauge-fixing in interacting case. For the case of $g \neq 0$, first we introduce the BRST transformation of component fields in the gauge invariant action as

$$\begin{aligned}
\delta\phi^{(0)}_A &= I^{-1}_{AB}\big[M_{BC}\psi^{(1)}_C - (K - J_+)_{BC}\phi^{(1)}_C\big]\epsilon,\\
\delta\psi^{(0)}_A &= I^{-1}_{AB}\big[L_{BC}\phi^{(1)}_C + (K + J_-)_{BC}\psi^{(1)}_C + gf_{BCD}(\phi^{(0)}_C\phi^{(1)}_D + \phi^{(1)}_C\phi^{(0)}_D)\big]\epsilon,\\
\delta\psi^{(1)}_A &= gI^{-1}_{AB}f_{BCD}\phi^{(1)}_C\phi^{(1)}_D\epsilon,\\
\delta\phi^{(-1)}_A &= B^{(0)}_A\epsilon,\\
\delta\phi^{(1)}_A &= \delta B^{(0)}_A = 0.
\end{aligned} \tag{4.9}$$

The additional terms in the first stage of gauge-fixing for the Lagrangian density are easily obtained as

$$
\begin{aligned}
\mathcal{L}_{FP}+\mathcal{L}_{GF} =&\delta(-2\phi_A^{(-1)} I_{AB}\psi_B^{(0)}) \\
=&-2B_A^{(0)} I_{AB}\psi_B^{(0)} \\
&-2\phi_A^{(-1)}(L_{AB}\phi_B^{(1)}+(K+J_-)_{AB}\psi_B^{(1)}) \\
&+2gf_{ABC}(\phi_A^{(-1)}\phi_B^{(0)}\phi_C^{(1)}+\phi_A^{(-1)}\phi_B^{(1)}\phi_C^{(0)}).
\end{aligned}
\tag{4.10}
$$

The first stage Lagrangian density $\mathcal{L}^{(1)}=\mathcal{L}_{inv.}+\mathcal{L}_{FP}+\mathcal{L}_{GF}$ is a part of the full Lagrangian density $\mathcal{L}_{fixed.}$, which is given by setting all fields having superscript $(n)(|n|>1)$ to zero. Naively we guess that $\mathcal{L}^{(1)}$ has the second stage BRST transformation as follows:

$$
\begin{aligned}
\delta'\phi_A^{(1)} &= I_{AB}^{-1}[M_{BC}\psi_C^{(2)}-(K-J_+)_{BC}\phi_C^{(2)}]\epsilon \\
\delta'\psi_A^{(1)} &= I_{AB}^{-1}[L_{BC}\phi_C^{(2)}+(K+J_-)_{BC}\psi_C^{(2)}+gf_{BCD}(\phi_C^{(2)}\phi_D^{(0)}+\phi_C^{(0)}\phi_D^{(2)})]\epsilon. \\
\delta'\psi_A^{(0)} &= gI_{AB}^{-1}f_{BCD}(\phi_C^{(2)}\phi_D^{(-1)}+\phi_C^{(-1)}\phi_D^{(2)})\epsilon. \\
\delta'\phi_A^{(-2)} &= B_A^{(-1)}\epsilon \\
\delta'\phi_A^{(0)} &= \delta'\phi_A^{(1)}=\delta'\phi_A^{(2)}=\delta' B_A^{(0)}=\delta' B_A^{(-1)}=0
\end{aligned}
\tag{4.11}
$$

A straight-forward calculation gives

$$
\delta'\mathcal{L}^{(1)}=-2gf_{ABC}B_A^{(0)}\phi_B^{(2)}\phi_C^{(-1)}-2gf_{ABC}B_A^{(0)}\phi_B^{(-1)}\phi_C^{(2)}. \tag{4.12}
$$

Clearly, $\mathcal{L}_{(1)}$ is not invariant under the transformation (4.11). To recover the invariance, we must add the term

$$
\mathcal{L}_{add.}=2gf_{ABC}\phi_A^{(-1)}\phi_B^{(-1)}\phi_C^{(2)}. \tag{4.13}
$$

After adding (4.13), $\mathcal{L}^{(1)}$ becomes invariant when we carry out the transformations (4.9) and (4.11) simultaneously, *i.e.*,

$$(\delta + \delta')(\mathcal{L}^{(1)} + \mathcal{L}_{add.}) = 0. \tag{4.14}$$

Repeating this procedure, we can obtain (4.5). The additional term (4.13) is interpreted as an anti-ghost — anti-ghost — ghost interaction. Such a term can not be generated in the procedure discussed in the previous subsection.

Some works about the gauge-fixings of string field theory[25–27] are performed by the Batalin-Vilkovisky methods (A short review is given in Appendix D). Hereafter, we apply this procedure to our mid-point formalism.

At the first step, we construct an extended action which contains all fields, ghosts, and their anti-fields which are assigned for each field. The fields and anti-fields in string field theory are as follows:

$$\begin{aligned} &\text{field } \phi_A^{(n)} \qquad \text{anti-field } (\phi_A^{(n)})^* \equiv I_{AB}\psi_B^{(-n-1)}, \\ &\text{field } \psi_A^{(n)} \qquad \text{anti-field } (\psi_A^{(n)})^* \equiv I_{AB}\phi_B^{(-n-1)}. \end{aligned} \tag{4.15}$$

The field and its anti-field have the same indices of the Fock space and the opposite statistics of each other. The anti-fields $\phi_A^{(-n)}$ and $\psi_A^{(-n)} (n > 0)$ are different from the anti-ghost introduced in the above discussion. The anti-fields will be replaced by anti-ghosts as seen below.

The string field theory is an example of infinite order reducible theory. The gauge generators of all stages have the same form which can be written as

$$Z_n = \begin{array}{c} \\ \phi_A^{(n)} \\ \psi_A^{(n)} \end{array} \begin{array}{c} \phi_B^{(n+1)} \qquad\qquad\qquad\qquad \psi_B^{(n+1)} \\ \begin{pmatrix} I_{AC}^{-1}(K - J_+)_{CB} & I_{AC}^{-1}M_{CB} \\ I_{AC}^{-1}\left[L_{CB} + gf_{CDB}\phi_D^{(0)} + gf_{CBD}\phi_D^{(0)}\right] & I_{AC}^{-1}(K + J_-)_{CB} \end{pmatrix} \end{array}. \tag{4.16}$$

As is easily confirmed, the action (4.1) is invariant under the transformation

$$\begin{pmatrix} \delta\phi_A^{(0)} \\ \delta\psi_A^{(0)} \end{pmatrix} = Z_0 \begin{pmatrix} \phi_A^{(1)} \\ \psi_A^{(1)} \end{pmatrix}, \tag{4.17}$$

and the generators Z_n have zero eigenvalue vectors such as

$$Z_0 Z_1 = Z_1 Z_2 = \cdots = Z_n Z_{n+1} = \cdots = 0. \tag{4.18}$$

The gauge-fixed action must satisfy the conditions listed in Appendix D. The naive extended action

$$\begin{aligned} S = \int dx \sum_n \Big[\phi_A^{(-n)} L_{AB} \phi_B^{(n)} - \psi_A^{(-n)} M_{AB} \psi_B^{(n)} + 2\phi_A^{(-n)} (K + J_-)_{AB} \psi_B^{(n)} \Big] \\ + \sum_{l,m,n} \frac{2g}{3} f_{ABC} \phi_A^{(l)} \phi_B^{(m)} \phi_C^{(n)} \delta_{l+m+n,0}, \end{aligned} \tag{4.19}$$

satisfy all the conditions listed below.

(i) If we set all anti-fields($\phi_A^{(n)}$ and $\psi_A^{(n)}$ $(n < 0)$) to be zero, it reduces to classical gauge-invariant action (4.19).

(ii) It satisfies the master equation which is represented as

$$\frac{1}{2}(S,S) = \frac{\delta_r S}{\delta \Phi_A} \frac{\delta_l S}{\delta \Phi_A^*} = \sum_{n \geq 0} \{ \frac{\delta_r S}{\delta \phi_A^{(n)}} I_{AB}^{-1} \frac{\delta_l S}{\delta \psi_B^{(-n-1)}} + \frac{\delta_r S}{\delta \psi_A^{(n)}} I_{AB}^{-1} \frac{\delta_l S}{\delta \phi_B^{(-n-1)}} \} = 0. \tag{4.20}$$

The proof of (4.20) is straightforward due to the nilpotency of Q_B and the property of three string vertex.

(iii) The second derivatives of S are gauge generators as written symbolically,

$$\frac{\delta_l \delta_r S}{\delta \Phi_A \delta \Phi_B^*} = Z_n{}^A{}_B. \tag{4.21}$$

Remarkably, the action needs no quantum correction because of the equality

$$\frac{\delta_l \delta_r S}{\delta \Phi_A \delta \Phi_A^*} = \sum_{n \geq 0} I_{AB}^{-1} \{ \frac{\delta_r \delta_l S}{\delta \phi_A^{(n)} \delta \psi_B^{(-n-1)}} + \frac{\delta_r \delta_l S}{\delta \psi_A^{(n)} \delta \phi_B^{(-n-1)}} \} = 0. \tag{4.22}$$

This is just the RHS of $\mathcal{O}(\hbar)$ of the equation (D10). As is easily confirmed, we can set all quantum corrections $W_n (n > 0)$ to be zero. As is pointed out by Thorn[19] , in the center of mass coordinate, (4.22) is not zero because the interaction term contains the terms proportional to $\phi_A^{(n)}\psi_B^{(-n-1)} \times (\phi_C^{(1)}$ or $\psi_C^{(1)})$. Hence, the quantum correction is necessary in the center of mass coordinate. The mid-point coordinate is useful also in this point.

In the next, we introduce the auxiliary fields listed in Fig. 3. The ghost numbers of the associated states with each component field are as follows:

$$
\begin{array}{llll}
 & \text{comp. field} & & N_g \text{of the state } |A\rangle \\
\text{anti-ghost} & \bar{C}_{\phi_A}^{(-n)} & (n>0) & \frac{1}{2}+n \\
 & \bar{C}_{\psi_A}^{(-n)} & (n>0) & \frac{3}{2}+n \\
 & & & \\
\text{extra-ghost} & C_{\phi_A}^{<s>(n)} & (n\geq 0) & -\frac{1}{2}-s-n \\
 & C_{\psi_A}^{<s>(n)} & (n\geq 0) & -\frac{3}{2}-s-n \\
 & & & (s=1,3,5,\cdots) \\
 & & & \\
\text{extra-anti-ghost} & \bar{C}_{\phi_A}^{<s>(-n)} & (n>0) & \frac{1}{2}+s+n \\
 & \bar{C}_{\psi_A}^{<s>(-n)} & (n>0) & \frac{3}{2}+s+n \\
 & & & (s=2,4,6,\cdots)
\end{array}
\tag{4.23}
$$

where $< s >$ denotes the number of primes. Adding to it, we introduce the Lagrange multipliers for each auxiliary fields. Each of the component fields of the Lagrange multiplier has the same index A and the opposite sign ghost number with the associated auxiliary field.

For the gauge-fixing, we choose so called *gauge fermion.* The gauge fermion Ψ must satisfy the conditions

$$
\operatorname{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_A^{(-n-1)}\delta C_B^{(n)}}\bigg|_{\Phi=\Phi_0} = m_n - m_{n+1} + m_{n+2} - \cdots, \tag{4.24}
$$

where m_n is the number of gauge parameter at the n-th stage, and $\bar{C}_A^{(-n)} = \{\bar{C}_{\phi_A}^{(-n)}, \bar{C}_{\psi_A}^{(-n)}\}, C_B^{(n)} = \{\phi_A^{(n)}, \psi_A^{(n)}\}$.

The RHS of (4.24) is an infinite number, however, which can be evaluated as follows. m_n is equal to the number of $\phi_A^{(n+1)}$ and $\psi_A^{(n+1)}$. The number of $\psi_A^{(n+1)}$ is equal to the one of $\phi_A^{(n+2)}$ because both are number of the states $|A\rangle$ which have ghost number $N_g = n + \frac{1}{2}$. Hence the number of $\psi_A^{(n+1)}$ in m_n cancels with the number of $\phi_A^{(n+2)}$ in m_{n+1}. Such subtractions are repeated successively. As the result, the RHS of (4.24) is the number of $\phi_A^{(n+1)}$, or the one of $\psi_A^{(n)}$. The simplest structure of the gauge fermion which satisfies (4.24) is the product of $\psi_A^{(n-1)}$ and $\bar{C}_{\phi_A}^{(-n)}$ for any positive integers n. The above discussion is applicable for the condition for extra-ghost and extra-anti-ghost. Hence, we can choose the structure of the gauge fermion Ψ as

$$
\begin{aligned}
\Psi = & \sum_{n>0} \psi_A^{(n-1)} I_{AB} \bar{C}_{\phi_B}^{(-n)} + \sum_{n>0} \bar{C}_{\psi_A}^{(-n)} I_{AB} C_{\phi_B}^{\prime(n-1)} + \sum_{n>0} C_{\psi_A}^{\prime(n-1)} I_{AB} \bar{C}_{\phi_B}^{\prime\prime(-n)} \\
& + \sum_{n>0} \bar{C}_{\psi_A}^{\prime\prime(-n)} I_{AB} C_{\phi_B}^{\prime\prime\prime(n-1)} + \sum_{n>0} C_{\psi_A}^{\prime\prime\prime(n-1} I_{AB} \bar{C}_{\phi_B}^{\prime\prime\prime\prime(-n)} + \cdots
\end{aligned}
\tag{4.25}
$$

The gauge-fixing condition is of the form

$$
\begin{aligned}
\frac{\delta\Psi}{\delta\bar{C}} = \frac{\delta\Psi}{\delta\bar{C}''} = \frac{\delta\Psi}{\delta\bar{C}''''} = \cdots = 0, \\
\frac{\delta\Psi}{\delta C'} = \frac{\delta\Psi}{\delta C'''} = \frac{\delta\Psi}{\delta C'''''} = \cdots = 0.
\end{aligned}
\tag{4.26}
$$

From (4.25) and (4.26), we can confirm that all extra-ghosts and extra-anti-ghosts are eliminated by the gauge-fixing conditions. The conditions eliminate $\bar{C}_{\psi_A}^{(n)}$, similarly. For example, the conditions $\frac{\delta\Psi}{\delta\bar{C}} = 0$ eliminate $\psi_A^{(n)}$ and $C_{\phi_A}^{\prime(n)}$, and $\frac{\delta\Psi}{\delta C'} = 0$ eliminate $\bar{C}_{\psi_A}^{(-n)}$ and $\bar{C}_{\phi_A}^{\prime\prime(-n)}$. Under the choice of the gauge fermion (4.25), the anti-fields are replaced as

$$
\begin{aligned}
\left(\phi_A^{(n)}\right)^* = I_{AB}\psi_B^{(-n-1)} &= \frac{\delta\Psi}{\delta\phi_A^{(n)}} = 0 \\
\left(\psi_A^{(n)}\right)^* = I_{AB}\phi_B^{(-n-1)} &= \frac{\delta\Psi}{\delta\psi_A^{(n)}} = I_{AB}\bar{C}_{\phi_A}^{(-n)}
\end{aligned}
\tag{4.27}
$$

As a consequence, we obtain the gauge-fixed action (4.5) after we replace $\phi_A^{(-n)}$ with $\bar{C}_{\phi_A}^{(-n)}$.

constraints) as follows:

$$
\begin{aligned}
\mathcal{H} =&\frac{1}{2\pi\delta(0)}\sum_{n}\Pi_{\phi_A}^{(n)} I^{-1}_{AB}\Pi_{\phi_B}^{(-n)}\\
&+\frac{1}{\pi\delta(0)}\sum_{n\geq 0}\Pi_{\phi_A}^{(n)} I^{-1}_{AB}(\ell^0)_{BC}\phi_C^{(-n)}\\
&-\phi_A^{(0)}(\frac{\pi\delta(0)}{2}\overrightarrow{\partial}^2+\overrightarrow{\ell}\cdot\overrightarrow{\partial}+L'+\frac{1}{2\pi\delta(0)}(\ell^0)^2)_{AB}\phi_B^{(0)}\\
&-2\sum_{n>0}(-)^n\phi_A^{(-n)}(\frac{\pi\delta(0)}{2}\overrightarrow{\partial}^2+\overrightarrow{\ell}\cdot\overrightarrow{\partial}+L')_{AB}\phi_B^{(n)}\\
&-\sum_{n\geq 0}\psi_A^{(n)}\mathcal{C}_A^{(-n)}\\
&-\frac{2g}{3}\sum_{l,m,n} f_{ABC}\phi_A^{(l)}\phi_B^{(m)}\phi_C^{(n)}\delta_{l+m+n,0}
\end{aligned}
\tag{4.32}
$$

where

$$
\begin{aligned}
\mathcal{C}_A^{(-n)} \equiv&\frac{2}{\pi\delta(0)}(k^0)_{AB}I^{-1}_{BC}\Pi_{\phi C}^{(-n)}\\
&+2(K'+\overrightarrow{k}\cdot\overrightarrow{\partial}-J_+ +\frac{1}{\pi\delta(0)}k^0\ell^0)_{AB}\phi_B^{(-n)}\\
&+2(-)^n I_{AB}B_B^{(-n)}.
\end{aligned}
\tag{4.33}
$$

In the above calculation, we have used the commutation relations in the Appendix B.

From now on, we calculate the constraint algebra by using the Poisson bracket

$$
\{A,B\}_{P.B.}\equiv\sum_i\frac{\delta_r A}{\delta\Phi_i}\frac{\delta_\ell B}{\delta\Pi_i}-(-)^{AB}\frac{\delta_r B}{\delta\Phi_i}\frac{\delta_\ell A}{\delta\Pi_i}. \tag{4.34}
$$

where δ_ℓ and δ_r represent a left- and right-derivative, respectively. The symbols Π_i and Φ_i denote general momenta and coordinates, respectively. The primary constraints (4.31) generate two series of secondary constraints such as

$$
\begin{aligned}
C_{3A}^{(n)} &\equiv \psi_A^{(n)} \approx 0,\\
C_{4A}^{(-n)} &\equiv \mathcal{C}_A^{(-n)} \approx 0.
\end{aligned}
\tag{4.35}
$$

All of the above four series of constraints are second-class. Poisson brackets between them are

$$
\begin{aligned}
\{C1_A^{(-m)}(x), C3_B^{(n)}(y)\}_{P.B.} =& (-)^n \delta_{mn}\delta_{AB}\delta(x-y), \\
\{C2_A^{(m)}(x), C4_B^{(-n)}(y)\}_{P.B.} =& 2(-)^n \delta_{mn} I_{AB}\delta(x-y), \\
\text{any others } =& 0.
\end{aligned}
\tag{4.36}
$$

Since the determinant of their Poisson bracket matrix is constant, it decouples from path-integral. The canonical path integral then can be written as

$$
\int d\mu \ \exp\left[i\int dx(\Pi_\phi\dot{\phi}+\Pi_\psi\dot{\psi}+\Pi_B\dot{B}-\mathcal{H})\right]\times\delta(\Pi_\psi)\delta(\psi)\delta(\Pi_B)\delta(\mathcal{C}) \tag{4.37}
$$

where

$$
d\mu = \mathcal{D}\phi\mathcal{D}\Pi_\phi\mathcal{D}\psi\mathcal{D}\Pi_\psi\mathcal{D}B\mathcal{D}\Pi_B. \tag{4.38}
$$

The integrations for Π_ϕ,ψ,Π_ψ,B and Π_B can be performed and the result amounts to

$$
\int \mathcal{D}\phi \exp\left[\sum_n(-)^n\phi_A^{(n)}L_{AB}\phi_B^{(-n)}+\frac{2g}{3}\sum_{l,m,n}f_{ABC}\phi_A^{(l)}\phi_B^{(m)}\phi_C^{(n)}\delta_{l+m+n,0}\right]. \tag{4.39}
$$

Except for the treatment of $\delta(0)$ in L_{AB}, for which we see below, the formula (4.39) is a well defined path integral, namely the measure is defined according to the canonical rule.

Once the path integrals are written we can transform back to the representation of the center of mass coordinates from the mid-point coordinates. If we substitute $\tilde{\alpha}_m = \alpha_m - p_0\cos m\frac{\pi}{2}$, the kinetic operator L can be written as

$$
L = -\frac{1}{2}p^2 - \frac{1}{2}\sum_{n\neq 0} :\alpha_{-n}\alpha_n: - \sum_n :nc_{-n}\tilde{b}_n: +1. \tag{4.40}
$$

In the above calculation we use a formula $\pi\delta(0)-\sum_{n\neq 0}\cos^2 n\frac{\pi}{2} = 1$ which is derived from (3.38). If we replace $\tilde{b}_n$ with b_n, (4.40) turns out to be the same form as the kinetic operator L^{KO} in the conventional Lagrangian quantization approach of string field theory.[6–8] The vertex can be rewritten similarly in term of usual center of mass coordinate. Here we do not rewrite the ghost part of the vertex, however, the form of it is same as the one in the center of mass coordinate if we replace $\tilde{b}_n$ with b_n after the gauge-fixing. The Feynman rule then coincides with the one in the formal Lagrangian path integral method. The above result is plausible if we consider that the divergent coefficient of the kinetic term can be properly regularized. If we are permitted to be optimistic, we can conclude that the canonical quantization of Witten's string field theory reproduces the same results as the naive Lagrangian path integral method. However, from a more severe point of view, we need to know the physical interpretation of the divergent coefficient of the kinetic term. In the next subsection, we will come to this question.

4.4 Discretized String

Let us consider a divergent coefficient of the kinetic term which is written as

$$-\sum_{n} \frac{\pi\delta(0)}{2}(-)^n \partial\phi_A^{(n)} I_{AB} \partial\phi_B^{(-n)}. \tag{4.41}$$

Potting, Taylor and Velikson[28] regularize this divergence by ζ-function method and set $\delta(0) = 0$. Then, only first order derivatives are present in their theory. However, it is unsatisfactory since their regularization changes the dynamics of string field theory from Klein-Gordon type to Dirac type.

In order to regularize this divergence, we apply the discretized approach of strings.[29] [30] A discretized string becomes a one-dimensional lattice with $N+1$ points. (See Fig. 4.) Integrations are replaced to summations, and δ-functions to Kronecker-δ symbols, *i.e.*,

$$\frac{1}{\pi}\int d\sigma \;\rightarrow\; \frac{1}{N+1}\sum_{n=0}^{N}$$
$$\pi\delta(\frac{n}{N}\pi - \frac{n'}{N}\pi) \;\rightarrow\; (N+1)\delta_{nn'}. \tag{4.42}$$

Accordingly, the divergence of kinetic term is regularized as

$$-\frac{\pi}{2}\delta(0)\partial\phi\partial\phi \;\rightarrow\; -(N+1)\frac{1}{2}\partial\phi\partial\phi. \tag{4.43}$$

In this discretized approach, the string field is expressed as a function having $N+1$ arguments, *i.e.*, $\phi = \phi(X(0), X(\frac{1}{N}\pi), X(\frac{2}{N}\pi), \cdots X(\pi))$. We postulate that the integration measure of string coordinates is expressed as

$$\mathcal{D}X = \prod_{n=0}^{N} dX(\frac{n}{N}\pi). \tag{4.44}$$

We have two choices of zero mode of string coordinates. We use the term "zero mode" for the coordinate which indicates the location of whole string. In the first choice, we can choose the mid-point $x_I = X(\frac{\pi}{2})$ as the zero mode. The fluctuation around the mid-point is described by $X'(\frac{n}{N}\pi) = X(\frac{n}{N}\pi) - X(\frac{\pi}{2})$. The transformation of integral variables affects the integration measure through the Jacobian factor

$$\frac{\partial(X(\frac{\pi}{2}),X'(0),\overbrace{\cdots\cdots}^{\text{exclude } X'(\frac{\pi}{2})},X'(\pi))}{\partial(X(0),X(\frac{1}{N}\pi),\cdots,X(\frac{\pi}{2}),\cdots X(\pi))} =$$

$$\det \begin{array}{c} \\ X(0) \\ X(\frac{1}{N}\pi) \\ \vdots \\ X(\frac{\pi}{2}) \\ \vdots \\ X(\frac{N-1}{N}\pi) \\ X(\pi) \end{array} \begin{array}{c} \begin{array}{cccccc} X(\frac{\pi}{2}) & X'(0) & X'(\frac{1}{N}\pi) & \cdots & X'(\frac{N-1}{N}\pi) & X'(\pi) \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array} \tag{4.45}$$

$$= 1.$$

Then the integration measure is equal to

$$\mathcal{D}X = dx_I \mathcal{D}X'. \tag{4.46}$$

where

$$\mathcal{D}X' = \prod_{\substack{n=0 \\ n\neq\frac{N}{2}}}^{N} dX'(\frac{n}{N}\pi). \tag{4.47}$$

In the second choice, we can choose the center of mass $x_0 = \frac{1}{N+1}\sum X(\frac{n}{N}\pi)$ as the zero mode. The fluctuation around the center of mass is described by $\check{X}(\frac{n}{N}\pi) = X(\frac{n}{N}\pi) - x_0$. Since all of $\check{X}(\frac{n}{N}\pi)$ $(0 \leq n \leq N)$ are not independent, we can exclude $\check{X}(\frac{\pi}{2})$ from $\check{X}$s. The Jacobian factor becomes

$$\frac{\partial(X(\frac{\pi}{2}),\check{X}(0),\overbrace{\cdots\cdots\cdots}^{\text{exclude }\check{X}(\frac{\pi}{2})},\check{X}(\pi))}{\partial(X(0),X(\frac{1}{N}\pi),\cdots,X(\frac{\pi}{2}),\cdots X(\pi))} =$$

$$\det \begin{array}{c} \\ X(0) \\ X(\frac{1}{N}\pi) \\ \vdots \\ X(\frac{N-1}{N}\pi) \\ X(\pi) \end{array} \begin{array}{c} \begin{array}{cccccc} x_0 & \check{X}(0) & \check{X}(\frac{1}{N}\pi) & \cdots & \check{X}(\frac{N-1}{N}\pi) & \check{X}(\pi) \end{array} \\ \begin{pmatrix} \frac{1}{N+1} & \frac{N}{N+1} & -\frac{1}{N+1} & \cdots & -\frac{1}{N+1} & -\frac{1}{N+1} \\ \frac{1}{N+1} & -\frac{1}{N+1} & \frac{N}{N+1} & \cdots & -\frac{1}{N+1} & -\frac{1}{N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N+1} & -\frac{1}{N+1} & -\frac{1}{N+1} & \cdots & \frac{N}{N+1} & -\frac{1}{N+1} \\ \frac{1}{N+1} & -\frac{1}{N+1} & -\frac{1}{N+1} & \cdots & -\frac{1}{N+1} & \frac{N}{N+1} \end{pmatrix} \end{array} \quad (4.48)$$

$$= \frac{1}{N+1}.$$

Then (4.44) can be written as

$$\mathcal{D}X = (N+1)^D dx_0 \mathcal{D}\check{X} \quad (4.49)$$

where $D = 26$ is the space-time dimension and

$$\mathcal{D}\check{X} = \prod_{\substack{n=0 \\ n\neq\frac{N}{2}}}^{N} d\check{X}(\frac{n}{N}\pi). \quad (4.50)$$

Since we can easily confirm $\mathcal{D}X' = \mathcal{D}\check{X}$, these two integration measures (dx_0 and dx_I) differ by the factor $(N+1)^D$ which arises from the Jacobian. The factor $N+1$ becomes $\pi\delta(0)$ in continuous limit.

We consider what follows from the difference of the two measures in coordinate space representation. Due to the factor obtained above, the second order derivative term in the action

$$\int dx_0 \mathcal{D}X' \left[-\frac{N+1}{2} \partial\phi(x_I, X')\partial\phi(x_I, \tilde{X}')\right] \tag{4.51}$$

in the center of mass coordinates turns out to be

$$\int dx_I \mathcal{D}X' \left[-\frac{1}{(N+1)^{D-1}}\frac{1}{2} \partial\phi(x_I, X')\partial\phi(x_I, \tilde{X}')\right]. \tag{4.52}$$

The tildes mean change of orientation of string. In the Fock space representation, the integration of nonzero mode $\mathcal{D}X'$ is replaced by the summation of the Fock space indices. By assuming the time-component of x_I as the canonical time, we can introduce the momentum conjugate to ϕ as

$$\Pi(x_I, X') = \frac{1}{(N+1)^{D-1}} \partial_0\phi(x_I, \tilde{X}') + (\text{ no derivative term}). \tag{4.53}$$

Calculating canonical equal time commutation relations with use of Dirac bracket as in the previous section, we obtain

$$[\phi(x_I, X'), \partial_0\phi(y_I, Y')]\big|_{x_I^0=y_I^0} = i(N+1)^{D-1}\delta(\overrightarrow{x_I} - \overrightarrow{y_I})\delta(X' - \tilde{Y}'), \tag{4.54}$$

where the LHS contains a divergent factor. However, this factor can be attributed to the Jacobian factor due to the transformation from the mid-point δ-functional to the center of mass δ-functional, *i.e.*,

$$(N+1)^{D-1}\delta(\overrightarrow{x_I} - \overrightarrow{y_I})\delta(X' - \tilde{Y}') = \delta(\overrightarrow{x_0} - \overrightarrow{y_0})\delta(\check{X} - \tilde{\check{Y}}). \tag{4.55}$$

Eventually, we obtain a commutation relation of string field in the center of mass coordinates as

$$[\phi(x_0, \check{X}), \partial_0\phi(y_0, \check{Y})]\big|_{x_0^0=y_0^0} = i\delta(\overrightarrow{x_0} - \overrightarrow{y_0})\delta(\check{X} - \tilde{\check{Y}}). \tag{4.56}$$

The origin of the divergent factor is given as follows. The center of mass is the coordinate which is obtained from integration through a whole string. In contrast, the mid-point coordinate $X(\frac{\pi}{2})$ is a measure zero coordinate. The ratio of the weights between two coordinates is the origin of the divergence.

Now let us come back to the continuous theory. We have the commutation relation (4.56) in the center of mass coordinate. It can be written in the Fock state representation as

$$[\phi_A^{(m)}(x_0), \partial_0 \phi_B^{(n)}(y_0)]|_{x_0^0=y_0^0} = i\delta_{m+n,0} I_{AB}^{-1} \delta(\vec{x_0} - \vec{y_0}) \tag{4.57}$$

The above commutation relation leads to the free propagator

$$\langle T\phi_A^{(m)}(x)\phi_B^{(n)}(y)\rangle = I_{BA}^{-1}\delta_{m+n,0} \int \frac{d^D p}{(2\pi)^D} \frac{-i}{p^2 + m^2(A)} e^{-ip(x-y)}, \tag{4.58}$$

where $m(A)$ is the mass of the mode $|A\rangle$. Since $L_{AB} = -\frac{1}{2}(p^2 + m^2(A))I_{AB}$, the above result coincides (2.50). The discussion given at the end of the previous subsection has been now justified.

5. Conclusions

We have quantized Witten's string field theory in the canonical formalism by using the mid-point time variable and discretization of the string coordinates, and clarified that the formal path integral approach leads to correct Feynman rules. We emphasize here that the ordinary center of mass time formalism, due to non-locality of the vertex, does not define a simple canonical momentum conjugate to the string field Φ, while in our formalism the canonical momentum can be defined as (4.30) because the vertex is local. The path integral measure for string field Φ is then well defined. Starting with this (mid-point time) canonical formalism and changing the representation from the mid-point back to the center of mass normal mode expression, we have reproduced the perturbative string theory which have been conventionally used. Another advantage of the mid-point time formalism is that, the gauge-fixing procedure become simple even for interacting strings since the interaction term contains only the component fields which survive after gauge-fixing. In particular, unlike to the center of mass coordinates, there is no quantum correction to the action with our definition of fields and anti-fields.

To apply our mid-point formalism to other string field theory is a highly non-trivial problem. A closed strings have no special point like a mid-point. The interaction of closed strings may occur at any point on a string. It is not clear which zero mode of a string one should use to make the string interaction local. The situation is the same with HIKKO's open string field theory. In particular, a recently proposed non-polynomial theory for closed string[9–11] is inevitavly non-local since the number of interacting points of a string on the n-string vertices is not one. However, the absence of canonical quantization of closed string field theory is a more serious problem which should be resolved, because none of manifestly covariant closed string field theory gives us a correct amplitude at loop-level. The work by Hata[12] produces a correct loop-amplitude. The additional terms in his work can be considered as a generalization of the Lee-Yang term. These terms may come from the path-integration over string field momenta if one finds a suitable canonical formalism of closed string field theory. This problem seems to be a worthwhile issue to investigate.

Finally we comment on Witten's superstring field theory. Because the vertex of the interaction of Ramond — Ramond — Neveu-Schwarz superstring is necessary to multiply a pre-factor which contains a derivative, the interaction of superstring is a derivative coupling interaction after rewriting in the mid-point coordinates. Hence the canonical momentum contains a non-linear term arising from the interaction term even in the mid-point coordinates. Does the term generate any non-trivial result such as the Lee-Yang term? This will not be a problem. The derivative is only first order, and the integration of momenta may be performed without any trouble. This subject is now in study.

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APPENDIX A

Matrix Representation of First Quantized Operator

We express the matrix element of an operator $\mathcal{O}$ as

$$\mathcal{O}_{AB} = \langle V_2|b_0|A\rangle\mathcal{O}|B\rangle. \tag{A1}$$

The symbol A runs through all Fock space vectors. We select the basis of Fock state vectors as

$$|A\rangle = \lambda(A)\alpha_{\ell_1}\cdots\alpha_{\ell_I}b_{m_1}\cdots b_{m_J}c_{n_1}\cdots c_{n_K}|+\rangle \tag{A2}$$

where the state $|A\rangle$ has ghost number $N_g(A) = -J + K + \frac{1}{2}$.

The normalization constant $\lambda(A)$ satisfies the following conditions.

(i) The reality of λ is decided to satisfy $\langle V_2|A\rangle = (|A\rangle)^\dagger = \langle A|$ For this condition, λ takes a real or pure imaginary value.

(ii) We choose $\mathcal{R}e(\lambda(A)) > 0, \mathcal{I}m(\lambda(A)) > 0$, as a convention.

(iii) If we define a state $|\bar{A}\rangle$ as

$$|\bar{A}\rangle = \lambda(\bar{A})\alpha_{\ell_1}\cdots\alpha_{\ell_I}b_{n_1}\cdots b_{n_K}c_{m_1}\cdots c_{m_J}|+\rangle, \tag{A3}$$

only a nonzero inner product with $|A\rangle$ is $\langle A|b_0|\bar{A}\rangle$. We decide the absolute value of λ in such a way as $|\langle V_2|b_0|A\rangle|\bar{A}\rangle|$ becomes one and $|\lambda(A)| = |\lambda(\bar{A})|$. As the result, $\langle V_2|b_0|A\rangle|\bar{A}\rangle = \pm 1$ or $\pm i$.

Under this definition, the matrix element $\mathcal{O}_{AB}$ can be written in more natural form

$$\mathcal{O}_{AB} = \langle -A|\mathcal{O}|B\rangle. \tag{A4}$$

where $|-A\rangle$ is obtained from $|A\rangle$ by replacing $|+\rangle$ with $|-\rangle$. Matrix element has a nonzero value only when $N_g(\mathcal{O}) + N_g(A) + N_g(B) - 1 = 0$ or $(-)^{\mathcal{O}+A+B} = 1$.

The general inner product of string fields φ_A and φ_B is expressed by matrix representation as

$$\langle V_2|b_0|A\rangle\varphi_A\mathcal{O}|B\rangle\varphi_B = \langle V_2|b_0|A\rangle\mathcal{O}|B\rangle\varphi_A\varphi_B(-)^{\varphi_A(B+\mathcal{O})} \tag{A5}$$

If φ is ψ-field (then, $(-)^{\psi_A} = -(-)^A$), the sign factor of (A5) is 1. In the case of ϕ-field($(-)^{\phi_A} = (-)^A$), it is $(-)^A$.

The matrix element for $\mathcal{O} = 1$ in (A5) becomes

$$I_{AB} \equiv \langle V_2|b_0|A\rangle|B\rangle = \langle V_2|b_0|A\rangle|\bar{A}\rangle\delta_{A\bar{B}}. \tag{A6}$$

The inverse matrix of I_{AB} is

$$I^{-1}_{AB} \equiv \frac{1}{\langle V_2|b_0|\bar{A}\rangle|A\rangle}\delta_{A\bar{B}} \tag{A7}$$

which satisfies

$$I_{AB}I^{-1}_{BC} = \delta_{AC}, \qquad I^{-1}_{AB}I_{BC} = \delta_{AC}. \tag{A8}$$

Through this paper, repeated indices A, B are summed up. The inverse matrix I^{-1}_{AB} satisfies a very useful formula:

$$\mathcal{O}_{AB}I^{-1}_{BC}\mathcal{O}'_{CD} = (\mathcal{O}\mathcal{O}')_{AD}. \tag{A9}$$

The proof of (A9) is obvious once we notice

$$\sum_A |A\rangle\frac{1}{\langle V_2|b_0|\bar{A}\rangle|A\rangle}\langle V_2|b_0|\bar{A}\rangle = 1. \tag{A10}$$

in the space spanned by the state excited from $|+\rangle$. Multiplying (A10) by $\langle V_2|b_0|B\rangle\mathcal{O}$ from left, and by $\mathcal{O}'|C\rangle$ from right, we can obtain (A9). As is easily understood, I^{-1} is an indefinite metric matrix of the first quantized Fock space.

We express three string vertex with the same manner.

$$
\begin{aligned}
&\langle V_3|b_0|A\rangle\varphi_{A}b_0|B\rangle\varphi_{B}b_0|C\rangle\varphi_{C} \\
=&\langle V_3|b_0|A\rangle b_0|B\rangle b_0|C\rangle\varphi_{A}\varphi_{B}\varphi_{C}(-)^{\varphi_A(B+C)}(-)^{\varphi_B(C+1)}.
\end{aligned}
\tag{A11}
$$

If all φ's are ϕ-fields, (A11) reduces to

$$
f_{ABC}\varphi_A\varphi_B\varphi_C \tag{A12}
$$

where

$$
f_{ABC} = \langle V_3|A\rangle|B\rangle|C\rangle(-)^{AC}. \tag{A13}
$$

In the above we have used $(-)^{\varphi_A} = (-)^A$ and $(-)^{A+B+C} = 1$.

APPENDIX B

Commutation Relations in Mid-Point Coordinates

In this appendix, we will calculate some commutation relations between the operators in the mid-point coordinates. The BRST charge in the mid-point coordinate can be expanded as (3.33). In order to satisfy the nilpotency of the BRST charge, we have to show the following formulas:

(1) $[L,K] = LJ_+ + J_-L,$

(2) $K^2 + ML = \{J_+, K\},$

(3) $[K,M] = MJ,$

(4) $[L,M] = -\{K,J\}.$

In the following, we confirm these commutation relations. The calculations are easily performed by using the well-known relation

$$
\begin{aligned}
[L^{KO}, \tilde{Q}_B^{KO}] &= 0, \\
[M^{KO}, \tilde{Q}_B^{KO}] &= 0, \\
(\tilde{Q}_B^{KO})^2 &= -M^{KO} L^{KO}.
\end{aligned} \tag{B1}
$$

Note that our operators can be expressed as

$$
\begin{aligned}
L' &= L^{KO}\Big|_{\substack{p=0 \\ \alpha,b\to\tilde{\alpha},\tilde{b}}}, \\
K' &= \tilde{Q}_B^{KO}\Big|_{\substack{p=0 \\ \alpha,b\to\tilde{\alpha},\tilde{b}}} + :\sum_n c_n \cos n\frac{\pi}{2} L' : .
\end{aligned} \tag{B2}
$$

The equation(1) can be expanded in powers of ∂^μ as

$(1-a)$ $$[L', K'] = L' J_+ + J_- L',$$

$(1-b)$ $$[L', k^\mu] + [\ell^\mu, K'] = J\ell^\mu,$$

$(1-c)$ $$[\ell^\mu, k^\nu] = \frac{\pi\delta(0)}{2} J\eta^{\mu\nu}.$$

By using (B2), we can easily obtain $(1-a)$. The proof of $(1-b)$ can be performed straightforwardly. In order to obtain $(1-c)$, we need a regularization. The LHS of $(1-c)$ can be written as

$$
\left[i\int d\sigma \sum_n \tilde{\alpha}_l^\mu \cos l\sigma\delta(\sigma-\frac{\pi}{2}), -i\int d\sigma' \sum_m \tilde{\alpha}_m^\nu \sin m\sigma' \sum_n c_n \sin n\sigma'\delta(\sigma'-\frac{\pi}{2})\right], \tag{B3}
$$

which becomes

$$
\int d\sigma \int d\sigma' \sum_n [-n\cos n\sigma \sin n\sigma'] \sum_n c_n \sin n\sigma' \delta(\sigma-\frac{\pi}{2})\delta(\sigma'-\frac{\pi}{2})\eta^{\mu\nu}. \tag{B4}
$$

After making partial integration with respect to σ', we obtain

$$
-\frac{1}{2}\int d\sigma \int d\sigma' \sum_n nc_n \cos n\sigma' \delta(\sigma-\sigma')\delta(\sigma-\frac{\pi}{2})\pi\delta(\sigma'-\frac{\pi}{2})\eta^{\mu\nu}. \tag{B5}
$$

The above result reduces to $\frac{\pi\delta(0)}{2}\eta^{\mu\nu}J$.

We can prove the other commutation relations

$$(2-a) \qquad (K')^2 + ML' = \{J_+, K'\},$$
$$(2-b) \qquad \{K', k^\mu\} + M\ell^\mu = 0$$
$$(2-c) \qquad \{k^\mu, k^\nu\} + \pi\delta(0)\eta^{\mu\nu}M = 0$$
$$(3-a) \qquad [K', M] = MJ,$$
$$(4-a) \qquad [L', M] = -\{K', J\}.$$
$$(4-b) \qquad [\ell^\mu, M] = -\{k^\mu, J\}.$$

in the same manner. In the calculation, we use (B1). Since the formula (B1)works only in $D = 26$, our BRST charge is nilpotent only in a critical dimension, too.

APPENDIX C

Properties of Two String Vertex

The first quantized operators have the following partial integration laws on the two string vertex $\langle V_2|$

$$\begin{aligned} \langle V_2|(\tilde{\alpha}^{(1)}_{-n} + \tilde{\alpha}^{(2)}_n(-)^n) &= 0, \\ \langle V_2|(\tilde{b}^{(1)}_{-n} - \tilde{b}^{(2)}_n(-)^n) &= 0, \\ \langle V_2|(c^{(1)}_{-n} + c^{(2)}_n(-)^n) &= 0, \\ \langle V_2|(p^{(1)} + p^{(2)}) &= 0, \end{aligned} \qquad \text{(C1)}$$

which lead the equalities:

$$\begin{aligned} \langle V_2|(L^{(1)} - L^{(2)}) &= 0, \\ \langle V_2|(M^{(1)} + M^{(2)}) &= 0, \\ \langle V_2|(J^{(1)}_+ - J^{(2)}_-) &= 0, \\ \langle V_2|(K^{(1)} + K^{(2)}) &= 0, \\ \langle V_2|(\ell^{(1)} + \ell^{(2)}) &= 0, \\ \langle V_2|(k^{(1)} - k^{(2)}) &= 0. \end{aligned} \qquad \text{(C2)}$$

By using (C2), we can calculate the transposition of matrices in appendix A as

$$\mathcal{O}_{AB} = \tilde{\mathcal{O}}_{BA}(-)^{\mathcal{O}\cdot B+\mathcal{O}+B} \tag{C3}$$

where

$$\langle V_2|(\mathcal{O}^{(1)} - \tilde{\mathcal{O}}^{(2)}) = 0. \tag{C4}$$

APPENDIX D

Review of the Batalin-Vilkovisky Method

In this appendix, we shortly review the method to obtain the gauge fixed quantum action developed by Batalin and Vilkovisky.[13]

1. Irreducible Theories

We consider a classical action $S(\phi)$ of the field $\phi_A(A = 1, 2, \cdots, N)$, which is gauge invariant under the transformation

$$\delta\phi_A = R_i^A\lambda_i, \tag{D1}$$

where the coefficients $R_i^A(i = 1, 2, \cdots, m)$ may be functions of ϕ, and λ_i are arbitrary parameters. The invariance generates the Noether identities

$$\frac{\delta_r S(\phi)}{\delta\phi_A} R_i^A = 0. \tag{D2}$$

First we consider the case of irreducible theory(R_i^A are linearly independent one another, hence, $\mathrm{rank} R_i^A|_{\Phi^*=0} = 0$. Because of this invariance, the rank of the Hessian of $S(\phi)$ ($\mathrm{rank}\left.\frac{\delta_r\delta_l S(\phi)}{\delta\phi_A\delta\phi_B^*}\right|_{\phi_0}$) is not N but $N - m$, where ϕ_0 is a solution of the equation of motion derived from $\mathcal{L}(\phi)$.

We enlarge the space of fields from ϕ_A to $\Phi_A = \{\phi_A, C_i\}$ adding the ghost field C_i. Further we assign the anti-field Φ^*_A for each field Φ_A. The anti-field Φ^*_A has an opposite grade against Φ_A. The classical limit of the action is realized by the limit $\Phi^* = 0$. Hence, the action S must satisfy the boundary condition $S(\Phi, \Phi^* = 0) = S(\phi)$. One more boundary condition for S is

$$\left.\frac{\delta_l \delta_r S(\Phi, \Phi^*)}{\delta \phi^*_A \delta C_i}\right|_{\Phi^*=0} = R_i^A, \tag{D3}$$

which guarantees the rank of the Hessian of S ($\text{rank}\left.\frac{\delta_r \delta_l S}{\delta \Phi_A \delta \Phi^*_B}\right|_{\Phi_0}$) to be equal to $N + m$ (where Φ_0 is a solution of the equation of motion derived from S). The essential condition for the action is so called *master equation*, *i.e.*, $(S, S) = 0$. The symbol (,) is the anti-bracket defined by

$$(A, B) \equiv \sum_i \frac{\delta_r A}{\delta \Phi_i}\frac{\delta_\ell B}{\delta \Phi^*_i} - \frac{\delta_r A}{\delta \Phi^*_i}\frac{\delta_\ell B}{\delta \Phi_i}. \tag{D4}$$

Further we introduce some auxiliary fields, namely anti-ghost $\bar{C}_i$ and Lagrange multipliers π_i and add the terms $\bar{C}^*_i \pi_i$ to the action. Even after adding the terms, the action satisfies the master equation. The additional term $\bar{C}^*_i \pi_i$ will become a gauge fixing term ($\pi_i \times$(gauge fixing condition)) as seen below. For the gauge-fixing, we introduced so called *gauge fermion* Ψ which has ghost number -1. The gauge-fixed partition function is obtained by

$$Z_\Psi = \int \mathcal{D}\Phi \exp\left[\frac{i}{\hbar} W(\Phi, \Phi^* = \frac{\delta \Psi}{\delta \Phi})\right], \tag{D5}$$

where W is the quantum action which is expanded as

$$W = S + \sum_{n>0} \hbar^n W_n. \tag{D6}$$

When we change the gauge fermion $\Psi \to \Psi + \delta\Psi$, the partition function (D5) changes $Z_\Psi \to Z_\Psi + \delta Z$ as

$$\delta Z = \int \mathcal{D}\Phi \delta\Psi \frac{\delta_r}{\delta\Phi_A}\frac{\delta_l}{\delta\Phi^*_A} \exp\left[\frac{i}{\hbar}W(\Phi,\Phi^*)\right]\Bigg|_{\Phi^*=\frac{\delta\Psi}{\delta\Phi}}. \tag{D7}$$

Hence, the path integral (D5) is invariant under the change of Ψ if W satisfies the equation

$$\frac{\delta_r}{\delta\Phi_A}\frac{\delta_l}{\delta\Phi^*_A} \exp\left[\frac{i}{\hbar}W(\Phi,\Phi^*)\right]\Bigg|_{\Phi^*=\frac{\delta\Psi}{\delta\Phi}} = 0, \tag{D8}$$

or,

$$\frac{1}{2}(W,W) = i\hbar\frac{\delta_r\delta_l W}{\delta\Phi_A\delta\Phi^*_A}. \tag{D9}$$

The equation (D9) is expanded in powers of $\hbar$ as

$$\begin{aligned}
(S,S) =& 0\\
(W_1,S) =& i\frac{\delta_r\delta_l S}{\delta\Phi_A\delta\Phi^*_A},\\
(W_2,S) =& i\frac{\delta_r\delta_l W_1}{\delta\Phi_A\delta\Phi^*_A} - \frac{1}{2}(W_1,W_1),\\
&\vdots\\
(W_n,S) =& i\frac{\delta_r\delta_l W_{n-1}}{\delta\Phi_A\delta\Phi^*_A} - \frac{1}{2}\sum_{m=1}^{n-1}(W_m,W_{n-m}),\\
&\vdots
\end{aligned} \tag{D10}$$

The classical part($\mathcal{O}(\hbar^0)$) of (D10) is just the master equation. By solving the equation(D10) order by order, and choosing the gauge fermion Ψ, we can obtain the correct quantum action. We choose Ψ as $\frac{\delta\Psi}{\delta \bar{C}}$ to be gauge conditions. Namely, the gauge fixing term is given by

$$\frac{\delta\Psi}{\delta\bar{C}_i}\pi_i. \tag{D11}$$

The kinetic term of ghost—anti-ghost system is given by

$$\bar{C}_i \left[\frac{\delta_r \delta_l \Psi}{\delta \bar{C}_i \delta \phi_A} R_j^A \right] \bigg|_{\Phi^*=0} C_j, \tag{D12}$$

which is obtained from $\phi_A^* R_i^A C_i$ by substituting $\phi_A^* = \frac{\delta \Psi}{\delta \phi}$. Since the ghost—anti-ghost system has no gauge invariance, the condition

$$\text{rank} \left[\frac{\delta_r \delta_l \Psi}{\delta \bar{C}_i \delta \phi_A} R_j^A \right] \bigg|_{\Phi^*=0} = m \tag{D13}$$

must be satisfied. Note that m is the number of $C_i, \bar{C}_i$ pairs.

2. Reducible Theories

Hereafter we consider the case when R_i^A obeys some linear relations,*i.e.*,

$$R_{i_0}^A Z_{1,i_1}^{i_0} = 0. \qquad (1 \leq i_0 \leq m_0, 1 \leq i_1 \leq m_1) \tag{D14}$$

for the zero eigenvalue vector Z_1. If Z_1 are linearly independent, the theory is called first-stage reducible. In such cases, we have to enlarge the space of fields as $\Phi = \{\phi, C_0, C_1\}$ because the ghost C_0 is now a gauge field. C_1 is the second-stage ghost which has ghost number 2. Similarly to (D3), the boundary condition for the action is

$$\frac{\delta_l \delta_r S(\Phi, \Phi^*)}{\delta C_{0,i_0}^* \delta C_{1,i_1}} \bigg|_{\Phi^*=0} = Z_{1,i_1}^{i_0}. \tag{D15}$$

For the gauge-fixing, we need three pairs of auxiliary fields,

$$\bar{C}_{0,i_0}, \pi_{0,i_0}, \bar{C}_{1,i_1}, \pi_{1,i_1}, C'_{1,i_1}, \pi'_{1,i_1}, \tag{D16}$$

where $\bar{C}_i$ are anti-ghosts for the ghosts C_i. The field C'_i is the extra-ghost which is necessary because the first stage anti-ghost $\bar{C}_0$ is also a gauge field. Now the condition for the gauge fermion (D13) is extended as

$$\begin{aligned}
\text{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_{0,i_0}\delta\phi_A}R^A_{j_0}\Big|_{\Phi_0} &= m_0 - m_1, \\
\text{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_{1,i_1}\delta C_{0,i_0}}Z^{i_0}_{1,j_1}\Big|_{\Phi_0} &= m_1, \\
\text{rank}\bar{Z}^{i_0}_{1,j_1}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_{0,i_0}\delta C'_{1,i_1}}\Big|_{\Phi_0} &= m_1,
\end{aligned} \tag{D17}$$

where $\bar{Z}_1$ is left zero eigenvalue vector of the kinetic operator of ghost, *i.e.*,

$$\bar{Z}^{i_0}_{1,i_1}\frac{\delta_r\delta_l\Psi}{\delta\bar{C}_{i_0}\delta\phi_A}R^A_{j_0}\Big|_{\Phi^*=0} = 0. \tag{D18}$$

The another condition comes from counting of the degree of freedom of gauge field. Although the field ϕ_A has the degree of freedom N, its $m_0 - m_1$(not m_0) components are redundant. Namely, the gauge fixing condition $\frac{\delta\Psi}{\delta\bar{C}} = 0$ should fix $m_0 - m_1$ gauge freedom of ϕ. This condition can be written as

$$\text{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_{0,i_0}\delta\phi_A} = m_0 - m_1. \tag{D19}$$

The theory is called L-th order reducible if there are the zero eigenvalue vectors such as

$$Z1^{i_0}_{i_1}Z2^{i_1}_{i_2} = Z2^{i_1}_{i_2}Z3^{i_2}_{i_3} = \cdots = Z_{L-1}{}^{i_{L-2}}_{i_{L-1}}Z_L{}^{i_{L-1}}_{i_L} = 0. \tag{D20}$$

In such a case, we must introduce a series of ghosts $C_0, C_1, \cdots, C_L$ and their anti-field. The master equation and the boundary conditions are the same with irreducible ones. Moreover, the boundary conditions for ghost fields are as follows

$$\frac{\delta_l\delta_r S(\Phi,\Phi^*)}{\delta C_{n-1}{}^*_{i_{n-1}}\delta C_{n,i_n}}\Big|_{\Phi^*=0} = Z^{i_{n-1}}_{i_n}. \qquad (1 \le n \le L) \tag{D21}$$

For the gauge-fixing, we must introduce auxiliary fields listed below.

(i) anti-ghosts and extra-ghosts.

$$
\begin{aligned}
&\bar{C}_0,\\
&\bar{C}_1, C_1',\\
&\bar{C}_2, C_2', \bar{C}_2'',\\
&\bar{C}_3, C_3', \bar{C}_3'', C_3'''\\
&\vdots
\end{aligned}
\tag{D22}
$$

(ii) Lagrange multipliers

$$
\begin{aligned}
&\pi_0,\\
&\pi_1, \pi_1',\\
&\pi_2, \pi_2', \pi_2'',\\
&\pi_3, \pi_3', \pi_3'', \pi_3'''\\
&\vdots
\end{aligned}
\tag{D23}
$$

The additional terms are the products of an anti-field of a ghost in the list (i) and the associated Lagangean multiplier in the list (ii), namely,

$$
\bar{C}_0^*\pi_0 + \bar{C}_1^*\pi_1 + C_1'^*\pi' + \bar{C}_2^*\pi_2 + C_2'^*\pi_2' + \bar{C}_2''^*\pi_2'' + \cdots . \tag{D24}
$$

The conditions for gauge fermion are of the form

$$
\text{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_n^{<s>}\delta C_{n-1}^{<s-1>}} = m_n - m_{n-1} + m_{n-2} - m_{n-3} + \cdots , \tag{D25}
$$

and

$$
\text{rank}\frac{\delta_l\delta_r\Psi}{\delta\bar{C}_{n-1}^{<s-1>}\delta C_n^{<s>}} = m_n - m_{n-1} + m_{n-2} - m_{n-3} + \cdots , \tag{D26}
$$

where the superscripts $< s >$ denote the number of primes.

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FIGURE CAPTION

*Figure*1. The connection of three string on Witten's three string vertex.

*Figure*2. z-plane of the conformal mapping of three string vertex. The symbols Z_1, Z_2, Z_3 are places of strings at the infinite past. z_0 and z_0^* are interacting points($\sigma = \frac{\pi}{2}$).

*Figure*3. A Table for ghosts and anti-ghosts. The fields introduced at the same stage(ghost, anti-ghost, extra-ghost, and extra-anti-ghost) are listed in a horizontal line. The numbers under the component fields denote the ghost number of the state $|A\rangle$. The sum of the ghost number of two component fields connected by an arrow is -1.

*Figure*4. Descretized string. The area $0 \leq \sigma \leq \pi$ divides to $N + 1$ points.

Fig. 1

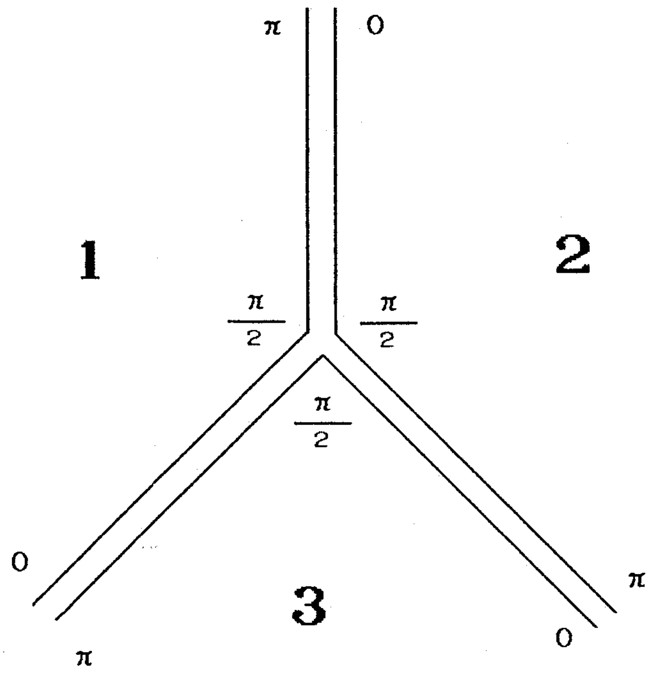


Fig. 2

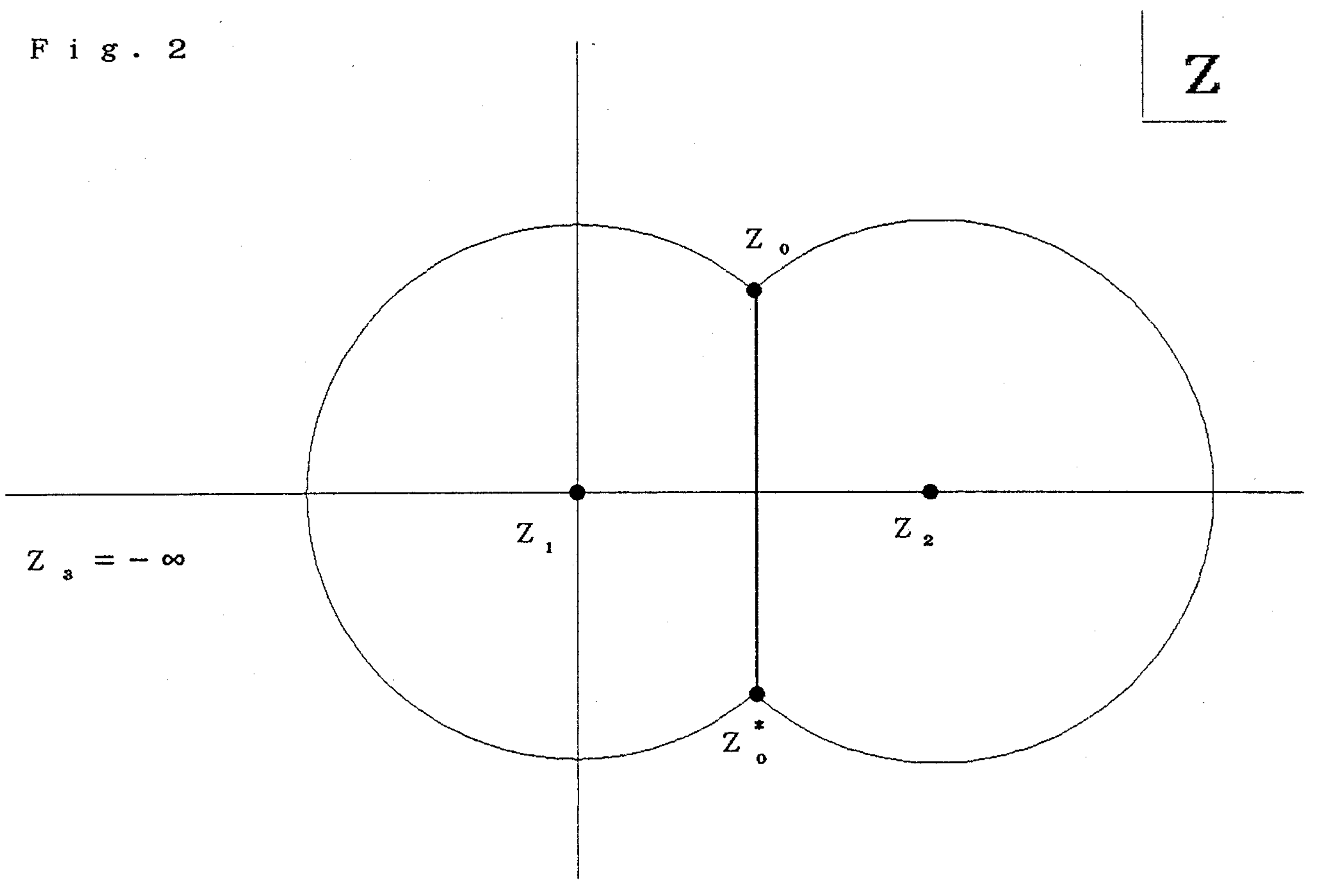
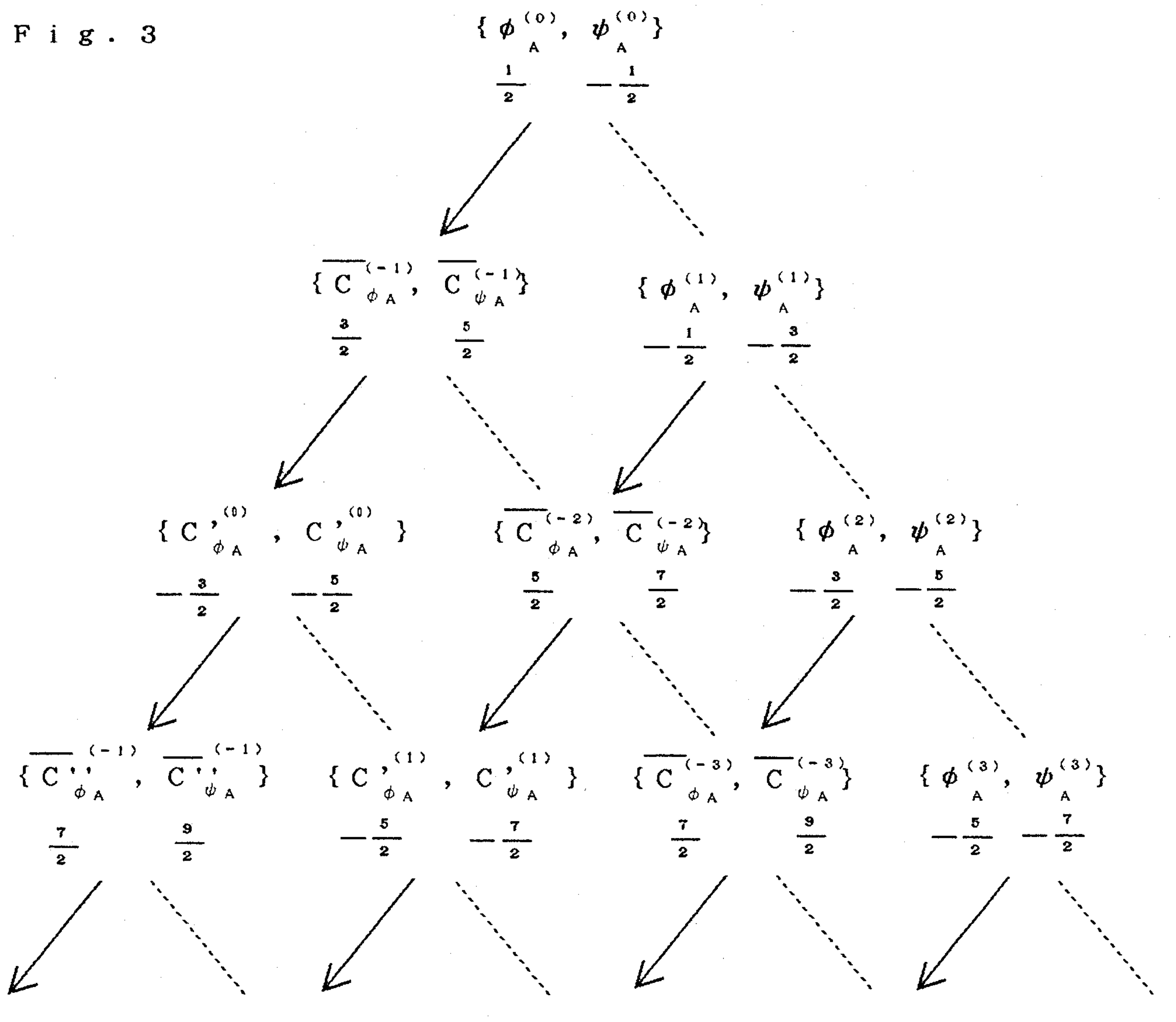
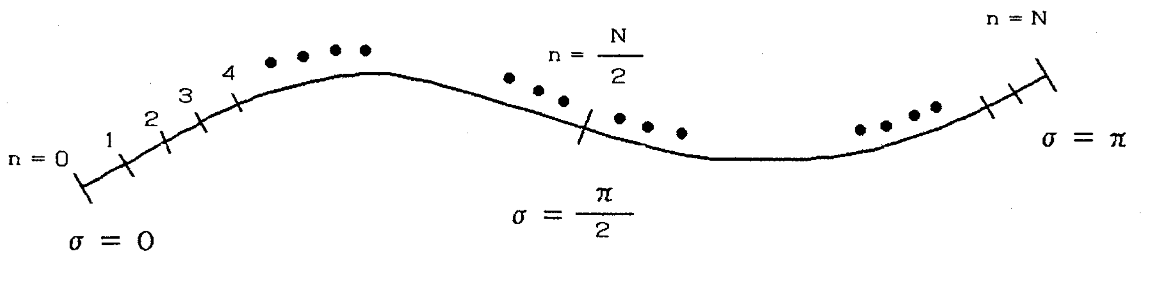


Fig. 3



$\{\phi_A^{(0)}, \psi_A^{(0)}\}$
$\frac{1}{2}$ $-\frac{1}{2}$
$\{\overline{C}_{\phi A}^{(-1)}, \overline{C}_{\psi A}^{(-1)}\}$
$\frac{3}{2}$ $\frac{5}{2}$
$\{\phi_A^{(1)}, \psi_A^{(1)}\}$
$-\frac{1}{2}$ $-\frac{3}{2}$
$\{C'^{(0)}_{\phi A}, C'^{(0)}_{\psi A}\}$
$-\frac{3}{2}$ $-\frac{5}{2}$
$\{\overline{C}_{\phi A}^{(-2)}, \overline{C}_{\psi A}^{(-2)}\}$
$\frac{5}{2}$ $\frac{7}{2}$
$\{\phi_A^{(2)}, \psi_A^{(2)}\}$
$-\frac{3}{2}$ $-\frac{5}{2}$
$\{\overline{C''}_{\phi A}^{(-1)}, \overline{C''}_{\psi A}^{(-1)}\}$
$\frac{7}{2}$ $\frac{9}{2}$
$\{C'^{(1)}_{\phi A}, C'^{(1)}_{\psi A}\}$
$-\frac{5}{2}$ $-\frac{7}{2}$
$\{\overline{C}_{\phi A}^{(-3)}, \overline{C}_{\psi A}^{(-3)}\}$
$\frac{7}{2}$ $\frac{9}{2}$
$\{\phi_A^{(3)}, \psi_A^{(3)}\}$
$-\frac{5}{2}$ $-\frac{7}{2}$

Fig. 4



n = 0
1
2
3
4
$n = \frac{N}{2}$
n = N
$\sigma = 0$
$\sigma = \frac{\pi}{2}$
$\sigma = \pi$