IRREDUCIBLE DECOMPOSITIONS
OF
NON-TYPE I REPRESENTATIONS

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Errata Sheet

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To my wife

Keiko

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Abstract

In this thesis, we study the variaty of irreducible decompositions of non-type I representations. It is known that irreducible decompositions into direct integrals of non-type I representations are not unique in general. Long before examples are known which demonstrate this fact and these are about regular representations of non-type I groups.

In this thesis, we show many other new examples of this non-uniqueness phenomenon of decompositions not only for regular representations but also for some factor representations and will make it clear that this phenomenon has some close relation with ergodic measures and cohomology groups of transformation groups. To do this, we investigate cohomology theory for double transformation groups and generalize the Mackey's theory of induced representations. Furthermore, we describe explicitly the interrelations between decompositions and maximal abelian subalgebras from a view point of operator algebras. As an application of our theory, we obtain some new families of irreducible representations of certain non-type I groups.

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Introduction

In this thesis, we study the variety of irreducible decompositions of non-type I representations. It is known that irreducible decompositions into direct integrals of non-type I representations are not unique in general. Long before, examples are known which demonstrate this fact, and these are about regular representations of non-type I groups. In this thesis, we show many other new examples of this non-uniqueness phenomenon of decompositions not only for regular representations but also for some factor representations. We then construct a theory which makes it clear that this phenomenon has some close relation with ergodic measures and cohomology groups of transformation groups.

This thesis consists of five chapters. In chapter I, under rather systematic versions, we arrange examples obtained before by several authors. In chapters II and III, we investigate cohomology theory for double transformation groups and generalize the Mackey's theory of induced representations. These will culminate in the description of our theory of irreducible decompositions of non-type I representations in chapter IV. In chapter V, we extend these results from a view point of operator algebras. Now we will explain briefly the contents of each chapter.

In chapter I, we study the phenomenon that regular representations of some non-type I groups may be decomposed into direct integrals of irreducible representations in completely different ways.

In 1951, H.Yoshizawa first pointed it out about the free group on two generators [54]. In the same year, G.W. Mackey independently showed similar results about some discrete semi-direct product groups by applying the method of induced representations [29]. In 1974, M.Saito studied the classification of cojugacy classes of Cartan subgroups of SL(2, $\mathbb{Z}$) and found that the regular representation of SL(2, $\mathbb{Z}$) was possible to be decomposed into irreducible components in infinitely many completely differnt ways [46]. Furthermore, A.A.Kirillov reported in [28] that the phenomenon occurs even in the case of the Mautner group which is not discrete but a simply connected solvable Lie group.

In this chapter, we first introduce these four examples under rather systematic versions and offer new other examples. In section 1.1, we describe fundamental techniques which are used to give different decompositions of regular representations, by using Mackey's method of induced representations [30]. In section 1.2, we state the criteria of irreducibility and equivalency of monomial representations of discrete groups according to [29]. Using these results, we review the non-uniqueness phenomenon about individual discrete groups. In section 1.3, we generalize the results about the Mautner group obtained by A.A.Kirillov [28] and consider the occurrence of the phenomenon in search about semi-direct product groups, which include Mackey's examples in some sense [24].

In chapter II, we explain the notion of cohomology of transformation groups [20]. This notion appeared in the Mackey's works [33], [34], [35] and its study has been developed by several authors. K.Schmidt has studied it related with ergodic theory [45]. Another way of the development was pursued by A.Guichardet [17] and C.C.Moore [37], [38] who considered this cohomology as the one cohomology of locally compact groups. Further, there is a way followed by G.W.Mackey and A.Ramsay. They have investigated it as a family of similarity classes of homomorphisms of a measure groupoid or a virtual group [34], [43], [44].

In section 2.1, we describe elementary properties of the cohomology of topological transformation groups. Propositions 2.1.1 and 2.1.2 are fundamental and may follow from the results in some of the works by C.C.Moore and A.Ramsay. However, we will add the proofs for completeness.

In section 2.2, we introduce double transformation groups and their cohomology. These replace certain non-smooth topological transformation groups and their cohomology, and play a principal role in our considerations.

In section 2.3, we state the cohomology subordinate to measures. We often use this cohomology in later arguments.

In section 2.4, we study the notion of weak cohomology. This notion is important as an index showing the variety of decompositions of representations, which is one of our main subjects [27].

In section 2.5, subgroups of cohomology groups and weak cohomology groups are found in some concrete cases.

In section 2.6, we argue again decompositions of the regular representation of semi-direct product groups related with cohomology, as an application of this chapter [24].

In chapter III, we investigate generalized induced representations for double transformation groups, related with cohomology and we construct families of non-Mackey representations of certain non-regular semi-direct product groups as a generalization of Mackey's method [20]. Applying this construction to the Mautner group, we obtain a new parametrized family of non-Mackey representations. The representations found by L.Baggett form a part of this family.

In 1978, L.Baggett found a family of non-Mackey irreducible representations of the Mautner group via the decompositions of a generalizd tensor product of some concrete representations [4]. In order to elucidate the mechanism of his family, we develop a theory of generalized induced representations in this chapter. In 1976, A.Ramsay turned the Mackey's theory into a representation theory of measure groupoids [44] and obtained a generalization of induced representations. Our notion is close to his but there are some differences. These differences will be seen to be crucial in the decomposition theory in later chapters. It is known that, for a connected and simply connected solvable Lie group G, there exists an algebraic solvable Lie group $\tilde{G}$ which contains G such that $[\tilde{G},\tilde{G}] = [G,G] = N$ and $\tilde{G}$ acts on $\hat{N}$ (the dual of N) smoothly. L.Pukanszky made an extensive use of this fact in [41], [42]. We impose a similar assumption (*) for nonregular semi-direct product groups, which will be used effectively as a substitute of the fact above-mentioned.

In section 3.1, for a double transformation group, we define unitary representations in relation to cohomology, which appears as a generalization of the Mackey's induced representations [30], [33].

In section 3.2, following the construction in section 3.1, we have families of non-Mackey representations of non-regular semi-direct product groups satisfying our condition (*). In Theorem 3.2.6, we show when such representations are mutually equivalent, and in Theorem 3.2.7, we give a criterion of the irreducibility. In Proposition 3.2.9, we mention a property which characterizes these representations. The results obtained are akin to the results in [33] or [44] but ours are more precise according to the strong conditions imposed. Moreover, the techniques employed by L.Baggett [4] will be better understood from our points of view.

In section 3.3, we apply our general results to the discrete Mautner group and the Mautner group.

In chapter IV, we consider the irreducible decompositions of type II factor representations of some non-regular semi-direct product groups [21]. Taking a certain factor representation of such a group, we show it can be decomposed in many different ways into direct integrals of irreducible representations, while the diagonal algebras are spatially isomorphic with each other. The explicit form of the diagonal algebra is also given.

The theory of irreducible decompositions is based on the following general result of F.I. Mautner [36]. Let G be a locally compact group and π be a unitary representation of G on a separable Hilbert sapce $\mathfrak{h}$. Suppose that $\mathfrak{A}$ is an abelian von Neumann subalgebra in $\pi(G)'$. Then, there exists a standard measure space (Y, ν) such that $\mathfrak{A}$ is algebraically isomorphic with $L^\infty(Y, \nu)$ and π is decomposed as follows.

$$\pi \cong \int_Y^\oplus \pi^\eta \, d\nu(\eta) .$$

Moreover, π^η is irreducible for ν-almost all $\eta \in Y$ if and only if $\mathfrak{A}$ is maximal abelian in $\pi(G)'$.

In chapter IV, we consider the irreducible decompositions of type II factor representations of some non-regular semi-direct product group. In Theorem 4.1.3, a certain representation π^χ of such a group G will be decomposed in an explicit way to a direct integral of irreducible representations, each component having a definite form. The corresponding maximal abelian von Neumann subalgebra in $\pi(G)'$ is also described in a concrete form.

It is known that the non-type I'ness of a locally compact group or a C^*-algebra is closely related to the non-smoothness of topological transformation groups [12], [15], [16]. In non-smooth topological transformation groups, there are various kinds of quasi-orbits. Furthermore, the cohomology group for each non-transitive quasi-orbit seems to be huge, at least it is known to be non-trivial under some conditions [38]. The non-uniqueness of decompositions of a non-type I representation seems to depend deeply on these two facts. The results in [14] and [50] are certainly connected with the former, viz. the existence of various quasi-orbits and the examples in [28], [29], and section 1.3 also seem to be so intrincically. The present chapter is an attempt to describe the relation of the non-uniqueness of decompositions with the latter phenomenon viz. the non-triviality of cohomology groups.

The decomposition in Theorem 4.1.3 is done by using a cocycle of the cohomology theory studied in chapter II, and it is shown in Proposition 4.2.1 that two decompositions are completely different when only the used cocycles are not weakly cohomologous, even the diagonal algebras are spatially isomorphic with each other. Thus we may get a large number of different decompositions of a given representation into irreducible components. To illustrate various possibilities, we will give two examples in section 4.3.

Chapter V is devoted to study representations of certain C^*-crossed products [26]. From a view point of operator algebras, we will extend the results investigated in the previous chapters.

For two closed subgroups H and K of a locally compact abelian group G, we get a C*-crossed product $A = C_0(G/H)\times_\gamma K$. We investigate decompositions of a certain representation π^0 of A. In section 5.1, we study two families of abelian von Neumann subalgebras $\{\mathcal{O}\!\mathcal{L}^a\}$ and $\{\mathcal{B}^b\}$ in the commuting algebra $\pi^0(A)'$, $\mathcal{O}\!\mathcal{L}^a$ being associated with the automorphism α^a of $\pi^0(A)'$, where $a \in Z(K; G; H)$, and $\mathcal{B}^b$ with β^b, $b \in Z(H^\perp; \hat{G}; K^\perp)$. We will have also some necessary and sufficient conditions of the maximality of $\mathcal{O}\!\mathcal{L}^a$ and $\mathcal{B}^b$ in $\pi^0(A)'$. In section 5.2, we study decompositions of π^0 corresponding to $\mathcal{O}\!\mathcal{L}^a$. To do this, we study generalized induced representations of C^*-crossed products following to chapter III. In section 5.3, using the concept of Heisenberg representations, we write down decompositions of π^0 corresponding to the above $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in explicit forms. In section 5.4, we show some examples and an application to unitary representations of a certain locally compact group.

Preliminaries and notations

In the thesis, a representation means a continuous unitary representation of a locally compact group or a bounded *-representation of a C*-algebra. For a representation π of a locally compact group G, π is called to be of type I (resp. type II, type III, and non-type I) if the von Neumann algebra generated by $\pi(G)$ is of type I (resp. type II, type III, and non-type I). Similar definitions are done for a repreaentation of a C*-algebra.

Through the thesis, when we use a terminology of a locally compact group or a locally compact space, they are assumed to satisfy the second axiom of countability. Furthermore, a Hilbert space and a C*-algebra are also assumed to be separable.

We often treat some semi-direct product groups given as follows. Let N and K be locally compact abelian groups. K acts on N as an automorphism group of N and the action is denoted by $N \ni z \longrightarrow k \cdot z \in N$ for each $k \in K$. Let G be a locally compact group which is $N \times K$ as a topological space and whose multiplication is given by

$$(z, k)(z', k') = (z + k \cdot z', k + k')$$

for $z, z' \in N$ and $k, k' \in K$. This group G is called a semi-direct product of N with K and denoted by $N \times_s K$. We note that G is a unimodular group. We identify the subgroup $\{(z,0); z \in N\}$ of G with N and the subgroup $\{(0,k); k \in K\}$ of G with K. We often consider the topological transformation group $(K;\hat{N})$, canonically obtained for $G = N\times_s K$, where $\hat{N}$ is the dual group of N and the action of K on the space $\hat{N}$ is given by, for each $k\in K$ and $\chi\in\hat{N}$,

$$\langle z, k\cdot\chi\rangle = \langle k\cdot z, \chi\rangle$$

for $z\in N$.

We denote each abelian group of integers, rational numbers, real numbers, and complex numbers by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ respectively. The positve parts of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are written by $\mathbb{N}$, $\mathbb{Q}^*$, and $\mathbb{R}^+$ respectively. One-dimensional torus group, namely, the abelian group of complex numbers with absolute value 1 is denoted by $\mathbb{T}$.

Other terminologies and notations mainly follow from J.Dixmier's books [8],[9], G.W.Mackey's works [31],[35] and the monograph [2] by L.Auslander and C.C.Moore.

Chapter I. Decompositions of regular representations

In this chapter, we study the phenomenon that regular representations of some non-type I groups may be decomposed into direct integrals of irreducible representations in completely different ways.

In 1951, H.Yoshizawa first pointed it out about the free group on two generators [54]. In the same year, G.W. Mackey independently showed similar results about some discrete semi-direct product groups by applying his method of induced representations [29]. In 1974, M.Saito studied the classification of cojugacy classes of Cartan subgroups of $SL(2,\mathbb{Z})$ and found that the regular representation of $SL(2,\mathbb{Z})$ was possible to be decomposed into irreducible components in infinitely many completely differnt ways [46]. Furthermore, A.A.Kirillov reported in [28] that the phenomenon occurs even in the case of the Mautner group which is not discrete but a simply connected solvable Lie group.

In this chapter, we introduce these four examples under rather systematic versions and offer new other examples. In section 1.1, we describe fundamental techniques which are used to give different decompositions of regular representations, by using the Mackey's method of induced represenations [30]. In section 1.2, we state the criteria of irreducibility and equivalency of monomial representations of discrete groups according to [29]. Using these results, we review the non-uniqueness phenomenon about individual discrete groups. In section 1.3, we generalize the results about the Mautner group obtained by A.A.Kirillov [28] and consider the phenomenon about semi-direct product groups, which also includes Mackey's examples in some sense.

1.1. Fundamental techniques by induced representations

Let G be a locally compact group. Take a closed abelian subgroup H of G and denote the dual group of H by $\hat{H}$. For a unitary character χ of H, we get a unitary representation U^{χ} of G by $U^{\chi} = \mathrm{Ind}_H^G \chi$ which is the induced representation of χ from H to G, developed by G.W.Mackey [29] [30]. Let λ be the right regular representation of G and ι be the trivial representation of $\{e\}$. Then, by general considerations about induced representations, we see that

$$\lambda \cong \int_{\hat{H}}^{\oplus} U^{\chi}\, d\mu(\chi)$$

where μ is a Haar measure of $\hat{H}$. Indeed,

$$\begin{aligned}
\lambda &\cong \mathrm{Ind}_{\{e\}}^G \iota \\
&\cong \mathrm{Ind}_H^G \mathrm{Ind}_{\{e\}}^H \iota && \text{(by stage theorem)} \\
&\cong \mathrm{Ind}_H^G \int_{\hat{H}}^{\oplus} \chi\, d\mu(\chi) && \text{(by Fourier transform)} \\
&\cong \int_{\hat{H}}^{\oplus} \mathrm{Ind}_H^G \chi\, d\mu(\chi) && \text{(by Theorem 10.1 in [30])} \\
&= \int_{\hat{H}}^{\oplus} U^{\chi} d\mu(\chi).
\end{aligned}$$

Next, Take another closed abelian subgroup K of G and denote by V^{η} the unitary representation of G given by $V^{\eta} = \mathrm{Ind}_K^G \eta$ for $\eta \in \hat{K}$. Then, by similar arguments, we see that

$$\lambda \cong \int_{\hat{K}}^{\oplus} V^{\eta} d\nu(\eta)$$

where ν is a Haar measure of $\hat{K}$.

Therefore, for two closed abelian subgroups H and K of G, we get two decompositions of the right regular representation λ of G as

$$\lambda \cong \int_{\hat{H}}^{\oplus} U^{\chi} d\mu(\chi) \cong \int_{\hat{K}}^{\oplus} V^{\eta} d\nu(\eta). \tag{1.1.1}$$

Here we must consider the following problems (A) and (B).

Problem

(A) Irreducibility: When the representations U^{χ} and V^{η} $(\chi \in \hat{H},\ \eta \in \hat{K})$ are irreducible ?

(B) Inequivalency: Which are unitarily equivalent or inequivalent among U^{χ} and V^{η} $(\chi \in \hat{H},\ \eta \in \hat{K})$?

In the next sections, we study these problems (A) and (B) in indivisual cases and show that the formula (1.1.1) gives indeed different decompositions into irreducible representations under some situations imposed.

1.2. Discrete groups

For discrete groups, the results obtained by G.W.Mackey [29] are valuable as criteria for (A) and (B).

Proposition 1.2.1. (Theorem 6' and 7' in [29])

let G be a locally compact group.

(A) Let H be an open subgroup of G and U^{χ} denote the representation $\mathrm{Ind}_H^G \chi$ of G for a unitary character χ of H. Then U^{χ} is irreducible if and only if for each $g \notin H$ one of the following statements is true.

(i) $\chi \neq g\cdot\chi$ on $g^{-1}Hg\cap H$ where $(g\cdot\chi)(h)=\chi(ghg^{-1})$ for $h\in g^{-1}Hg\cap H$.

(ii) $[g^{-1}Hg : g^{-1}Hg\cap H]=\infty$ or $[H : g^{-1}Hg\cap H]=\infty$, where [:] means the index.

(B) Let H_i (i=1,2) be two open subgroups of G and U^{χ_i} be the representations $\mathrm{Ind}_{H_i}^G \chi_i$ of G for unitary characters χ_i of H_i (i=1,2). Assume that U^{χ_i} (i=1,2) are irreducible. Then U^{χ_1} is unitarily equivalent to U^{χ_2} if and only if there exists $g\in G$ such that

(i) $\chi_2 = g\cdot\chi_1$ on $g^{-1}H_1g\cap H_2$,

(ii) $[g^{-1}H_1g : g^{-1}H_1g\cap H_2]<\infty$ and $[H_2 : g^{-1}H_1g\cap H_2]<\infty$.

Example I (G.W.Mackey)

Let G be a semi-direct product group $N\times_s K$ where N and K are infinite "discrete" abelian groups and K acts on N as an automorphism group of N.

Taking two closed subgroups N and K, we have two decompositions of the right regular representation λ of G according to (1.1.1) as follows.

$$\lambda \cong \int_{\hat{N}}^{\oplus} U^{\chi}d\mu(\chi) \cong \int_{\hat{K}}^{\oplus} V^{\eta}d\nu(\chi) \tag{1.2.1}$$

where $U^{\chi}=\mathrm{Ind}_N^G \chi$ for $\chi\in\hat{N}$, $V^{\eta}=\mathrm{Ind}_K^G$ for $\eta\in\hat{K}$, and μ (resp. ν) is a Haar measure of $\hat{N}$ (resp. $\hat{K}$).

G.W.Mackey considered the following condition(a) about the action of K on N.

Condition(a) All non-trivial orbits of N under the action of K are infinite.

Proposition 1.2.2. (Lemma1,2,3 in §3 of [29])

(A)(i) U^{χ} is irreducible if and only if $k\cdot\chi \neq \chi$ for $k \in K$ distinct from the unit 0.

(ii) Under the condition(a), V^{η} is irreducible for each $\eta \in \hat{K}$.

(B)(i) For $\chi_1, \chi_2 \in \hat{N}$, U^{χ_1} is unitarily equivalent to U^{χ_2} if and only if there exists $k \in K$ such that $\chi_2 = k\cdot\chi_1$.

(ii) Under the condition(a), for $\eta_1, \eta_2 \in \hat{N}$, V^{η_1} is unitarily equivalent to V^{η_2} if and only if $\eta_1 = \eta_2$.

(iii) U^{χ} is never unitarily equivalent to V^{η} for arbitray choice of $\chi \in \hat{N}$ and $\eta \in \hat{K}$.

Proof. This follows directly from Proposition 1.2.1. For the detail, see [29]. [Q.E.D.]

Let $\mathbb{Q}^*$ denote the multiplicative abelian group of all positive rational numbers. Then $\mathbb{Q}^*$ acts on $\mathbb{Q}$ by the multiplication as an automorphism group of $\mathbb{Q}$ and we have a semi-direct product group $G = \mathbb{Q} \times_s \mathbb{Q}^*$. This group G satisfy the condition(a) and the decompositions (1.2.1) means that the regular representation of G may be decomposed into irreducible parts in two entirely different ways.

Example II (H.Yoshizawa)

Let F_2 denote the free group on two generators a and b. Let us now choose as H the abelian subgroup of F_2 generated by a and as K the abelian subgroup of F_2 generated by b. Then, both H and K are isomorphic with $\mathbb{Z}$ of integers so that their dual groups H and K are isomorphic with the one-dimensional torus group $\mathbb{T}$. By general consideratios in section 1.1, the right regular representation λ of F_2 is decomposed as

$$\lambda \cong \int_{\mathbb{T}}^{\oplus} U^{\chi} d\mu(\chi) \cong \int_{\mathbb{T}}^{\oplus} V^{\eta} d\nu(\eta)$$

where μ is the normalized Haar measure of $\mathbb{T}$ and $U^{\chi} = \mathrm{Ind}_K^G \chi$ for $\chi \in \hat{H}$ and $\eta \in \hat{K}$ parametrized with each element of $\mathbb{T}$.

Proposition 1.2.3. ([54])

(A) U^{χ} and V^{η} are irreducible.

(B)(i) For $\chi_1, \chi_2 \in \hat{H}$, U^{χ_1} is unitarily equivalent to U^{χ_2} if and only if $\chi_1 = \chi_2$.

(ii) For $\eta_1, \eta_2 \in \hat{K}$, V^{η_1} is unitarily equivalent to V^{η_2} if and only if $\eta_1 = \eta_2$.

(iii) U^{χ} is never unitarily equivalent to V^{η} for arbitrary choice of $\chi \in \hat{H}$ and $\eta \in \hat{K}$.

Proof. We give the proof according to Proposition 1.2.1 different from H.Yoshizawa's method.

(A) For each $g \notin H$, $g^{-1}Hg \cap H = \{e\}$ so that $[H : g^{-1}Hg \cap H] = \infty$. Therefore, by (A) in Proposition 1.2.1, U^{χ} is irreducible. Similarly, V^{η} is irreducible.

(B) Suppose U^{χ_1} is unitarily equivalent to U^{χ_2}. Then, by (B) in Proposition 1.2.1, there should exist $g \in G$ such that $[H : g^{-1}Hg \cap H] < \infty$ and $\chi_2 = g \cdot \chi_1$ on $g^{-1}Hg \cap H$. Since $g^{-1}Hg\ H = \{e\}$ for any $g \notin H$, the condition $[H : g^{-1}Hg \cap H] < \infty$ implies that $g \in H$ and so $\chi_2 = \chi_1$ on H. The coverse is clear.

(ii) is shown similarly as (i).

(iii) follows immediately from the fact that $g^{-1}Hg \cap K = \{e\}$ for any $g \in G$. [Q.E.D.]

Therefore, we see that the regular representation of F_2 is also decomposed into irreducible parts in two entirely different ways.

Example III (M.Siato)

Let G be a locally compact group. M.Saito considered a family $\mathcal{A}$ of open subgroups of G satisfying the following condition (b).

Condition (b)

(i) For H_1, $H_2 \in \mathcal{A}$ and $g \in G$, if $[H_2 : g^{-1}H_1 g \cap H_2] < \infty$, then $H_2 \subset g^{-1}H_1 g$.

(ii) For $H \in \mathcal{A}$ and $g \in G$, $gHg^{-1} \subset H$ yields $g \in N(H)$ where N(H) is the normalizer of H.

For a unitary character χ of H in $\mathcal{A}$, the representation $\mathrm{Ind}_H^G \chi$ of G is denoted by U^{χ}. He stated the irreducibility and equivalency of such representations by his original method [46]. Here we show them by using Proposition 1.2.1 but under some restrictions.

Let $\mathcal{A}^+$ denote the subfamily of satisfying that N(H)=H for $H \in \mathcal{A}^+$. Then, we get the following.

Proposition 1.2.4.

(A) For $H \in \mathcal{A}^+$, U^{χ} is irreducible for every unitary character χ of H.

(B) For unitary characters χ_i of H_i in $\mathcal{A}^+$ (i=1,2), U^{χ_1} is unitarily equivalent to U^{χ_2} if and only if there exists $g \in G$ such that $H_2 = g^{-1}H_1 g$ and $\chi_2 = g \cdot \chi_1$ on H_2.

proof. (A) Suppose $[H : H\cap g^{-1}Hg]<\infty$ for $g\in G$. Then, by (i) of the condition(b), $H\subset g^{-1}Hg$ and so $gHg^{-1}\subset H$. By (ii) of the condition (b), g must be in N(H). Since N(H)=H, we get $g\in H$. Therefore, for any $g\notin H$, $[H : H\cap g^{-1}Hg] = \infty$ holds. This implies that U^{χ} is irreducible by (A) of Proposition 1.2.1.

(B) Suppose U^{χ_1} is unitarily equivalent to U^{χ_2}. Then, by (B) of Proposition 1.2.1, there exists $g\in G$ such that (i) $\chi_2 = g\cdot\chi_1$ on $g^{-1}H_1 g\cap H_2$ and (ii) $[g^{-1}H_2 g : g^{-1}H_1 g\cap H_2]<\infty$ and $[H_2 : g^{-1}H_1 g\cap H_2]<\infty$. The first condition in (ii) is equivalent to the condition $[H_2 : H_1\cap gH_2g^{-1}]<\infty$, which implies that $H_1\subset gH_2g^{-1}$ by (i) of the condition (b). Similarly, by the second condition in (ii), we get $H_2\subset g^{-1}H_1g$. Therfore, we see that $H_2 = g^{-1}H_1g$. [Q.E.D.]

M.Saito [46] studied the conjugacy classes of Cartan subgroups of $SL(2,\mathbb{Z})$ and showed that the family of Cartan subgroups satisfies the condition (b). Furtheremore, he pointed out that a Cartan subgroup H satysfying N(H)=H is an abelian group isomorphic with $\mathbb{Z}\times\{\pm1\}$ and that the conjugacy classes of such groups are infinite.

For such a group H, according to section 1.1, we get a decomposition of the regular representation λ of $SL(2,\mathbb{Z})$ as

$$\lambda \cong \int_{\hat{H}}^{\oplus} U^{\chi}d\mu(\chi).$$

Therefore, by Proposition 1.2.4, it is clear that the regular representation of $SL(2,\mathbb{Z})$ may be decomposed into irreducible constituents in infinitely many entirely different ways.

Example IV

Observing the above three examples, we can also give infinitely many entirely different decompositions of the regular represntation of the free group F_2 on two generators as a generalization of Example II.

We consider a family $\mathcal{B}^+$ of subgroups H of F_2 satisfying that

(i) H is generated by one element of F_2.

(ii) N(H) = H.

Then, we get the following.

Proposition 1.2.5. $\mathcal{B}^+$ satisfies the condition (b).

For the proof of Proposition 1.2.5, we need the following lemma.

Lemma 1.2.6. For g_1, $g_2 \in F_2$, if $g_1^m = g_2^n$ for some $m,n \in \mathbb{Z}$, then there exists $k \in F_2$ and $m',n' \in \mathbb{Z}$ such that $g_1 = k^{m'}$ and $g_2 = k^{n'}$.

Proof. Suppose $g_1 g_2 \neq g_2 g_1$. Then, the equation $g_1^m = g_2^n$ would give a relation in F_2, which contradicts with the freeness of F_2. Therefore, g_1 and g_2 generate an abelian subgroup of F_2. Since an abelian subgroup of F_2 is generated by one element, there exists $k \in F_2$ such that $g_1 = k^{m'}$ and $g_2 = k^{n'}$ for some m', $n' \in \mathbb{Z}$. [Q.E.D.]

Proof of Proposition 1.2.5.

At first we show that $\mathcal{B}^+$ satisfies (i) of the condition (b). For H_1, $H_2 \in \mathcal{B}^+$ and $g \in F_2$, suppose $[H_2 : g^{-1}H_1 g \cap H_2] < \infty$.

Then, $g^{-1}H_1g \cap H_2 \neq \{e\}$. Hence, there exist non-zero integers m, n such that $g^{-1}h_1^m g = h_2^n$ where h_i are generators of $H_i (i=1,2)$. Therefore, we get $(g^{-1}h_1g)^m = h_2^n$. By Lemma 1.2.6, there exists $k \in F_2$ such that $g^{-1}h_1g = k^{m'}$ and $h_2 = k^{m'}$ for some $m', n' \in \mathbb{Z}$. By the condition $N(H_2) = H_2$, we see that $k = h_2$ or h_2^{-1}. Then, we get $g^{-1}H_1g \subset H_2$.

Next we check (ii) of the condition (b). For $H \in \mathcal{B}^+$ and $g \in F_2$, suppose $gHg^{-1} \subset H$. Let h be a generator of H. Then, $ghg^{-1} = h^n$ for some integer n. By the condition $N(H) = H$, n must be ± 1. Hence, we see that $gHg^{-1} = H$, which implies $g \in N(H)$. [Q.E.D.]

Each subgroup H in $\mathcal{B}^+$ is an abelian group isomorphic with $\mathbb{Z}$ and it is easily verified that the cojugacy classes of $\mathcal{B}^+$ are infinite. Therefore, by similar arguments as in Example III, we know that the regular representation of F_2 is decomposed into irreducible components in infinitely many entirely different ways.

1.3. Semi-direct product groups

Let G be a semi-direct product group $N \times_s K$ of N with K where N and K are both locally compact abelian groups and K acts on N as an automorphism group. In this section, we consider irreducible decompositions of the regular representation λ of G.

Applying the general consideration in section 1.1, we get

$$\lambda \cong \int_{\hat{N}}^{\oplus} U^{\chi} d\mu(\chi) \cong \int_{\hat{K}}^{\oplus} V^{\eta} d\nu(\eta)$$

where $U^{\chi} = \mathrm{Ind}_N^G \chi$ $(\chi \in \hat{N})$, $V^{\eta} = \mathrm{Ind}_K^G \eta$ $(\eta \in \hat{K})$, and μ (resp. ν) is a Haar measure of $\hat{N}$ (resp. $\hat{K}$). For the topological transformation group $(K;\hat{N})$, let H_{χ} denote the stability group of K at $\chi \in \hat{N}$. Then, we get the following criteria.

Proposition 1.3.1.

(A)(i) U^{χ} is irreducible if and only if $H_{\chi} = \{0\}$.

(ii) V^{η} is irreducible if and only if the Haar measure μ of $\hat{N}$ is ergodic under the action of K.

(B)(i) If μ is a non-transitive measure, U^{χ} is never unitarily equivalent to V^{η} for arbitrary choice of $\chi \in \hat{N}$ and $\eta \in \hat{K}$.

Proof. The proof will be given later in a more general situation (see Theorem 1.3.3). [Q.E.D.]

In the case that N and K are discrete abelian groups, we can give a sufficient condition of ergodicity of the measure μ on $\hat{N}$ under K.

Lemma 1.3.2. Let N and K be discrete abelian groups. If the action of K on N satisfies the condition (a) in section 1.2, the Haar measure of $\hat{N}$ is ergodic under the action of K.

Proof. Since N is a discrete abelian group, the dual $\hat{N}$ of N is a compact group. We denote by μ the normalized Haar measure of $\hat{N}$. Let δ_z be a continuous function on $\hat{N}$

given by $\delta_z(\chi) = \langle z, \chi \rangle$ ($\chi \in \hat{N}$) for each $z \in N$. Then, the family $\{\delta_z;\ z \in N\}$ is a orthonormal basis of the Hilbert space $L^2(\hat{N}, \mu)$ of all square integrable functions on $\hat{N}$ with respect to μ. Let $L^\infty(\hat{N}, \mu)$ be the space of all essentially bounded measurable functions with respect to μ. For f in $L^\infty(\hat{N}, \mu)$, we define the action of K by $(k \cdot f)(\chi) = f(k \cdot \chi)$.

Suppose that f in $L^\infty(\hat{N}, \mu)$ satisfies that $k \cdot f = f$ (μ-a.a.). Then, we show that f = constant (μ-a.a.). This will establish the ergodicity of the measure μ under the action of K. Since μ is a finite measure on $\hat{N}$, f belongs to $L^2(\hat{N}, \mu)$ and $k \cdot f = f$ in L^2-norm. Let the Fourier expansion of f be

$$f = \sum_{z \in N} a_z \delta_z \qquad \text{in } L^2\text{-norm}$$

where

$$a_z \in \mathbb{C} \quad \text{and} \quad \sum_{z\ N} |a_z|^2 < \infty \qquad (1.3.1)$$

By simple calculations, we get, for each $k \in K$,

$$k \cdot f = \sum_{z \in N} a_{(-k) \cdot z} \delta_z \qquad \text{in } L^2\text{-norm.}$$

By the assumption that $k \cdot f = f$ in L^2-norm and the uniqueness of the coefficients of the Forier expansion, we see that for each $k \in K$,

$$a_z = a_{(-k) \cdot z} \qquad (1.3.2)$$

Suppose $a_z \neq 0$ for $z \neq 0$. Then, the equality (1.3.2) and the condition (a) stand in contraction with the fact (1.3.1). Therefore, a_z must be 0 for $z \neq 0$ and so we get

$$f = a_0 \delta_0 \qquad \text{in } L^2\text{-norm,}$$

which yields

$$f = a_0 \qquad\qquad (\mu\text{-a.a.}). \qquad\qquad [\text{Q.E.D.}]$$

By Lemma 1.3.2, we see that the criterion of irreducibility of V^η ($\eta\in\hat{K}$) in Proposition 1.3.1 covers the discrete cases described in section 1.2 and that Mackey's example $G = \mathbb{Q}\times_s \mathbb{Q}^*$ satisfies the conditions in Proposition 1.3.1.

Now, we consider more general cases that H_χ is not necessarily trivial or μ is not necessarily ergodic.

When $H_\chi \neq \{0\}$, for each $\omega\in\hat{H}_\chi$, we put

$$L^{(\chi,\omega)}_{(z,h)} = \langle z,\chi\rangle\langle h,\omega\rangle$$

for $(z,h)\in N\times_s H_\chi = G_\chi$. Then, $L^{(\chi,\omega)}$ is a unitary character of $G_\chi = N\times_s H_\chi$ and we get a unitary representation $U^{(\chi,\omega)}$ of G by

$$U^{(\chi,\omega)} = \mathrm{Ind}\,{}^{G}_{G_\chi} L^{(\chi,\omega)}.$$

By general considerations of induced representations [30], we see that U^χ is decomposed as

$$U^\chi \cong \int^{\oplus}_{\hat{H}_\chi} U^{(\chi,\omega)}\, d\tau_\chi(\omega)$$

where τ_χ is a Haar measure of $\hat{H}_\chi$.

When μ is not ergodic under K, we decompose into ergodic measures as

$$\mu = \int^{\oplus}_{Z} \mu_\zeta\, d\sigma(\zeta)$$

where (Z,σ) is a standard measure space and μ_ζ $(\zeta\in Z)$ are quasi-invariant ergodic measures on $\hat{N}$ under the action of K. According to this decomposition of μ, we get a decompsition of the Hilbert space $L^2(\hat{N},\mu)$ as

$$L^2(\hat{N},\mu) \cong \int_Z^{\oplus} L^2(\hat{N},\mu_\zeta)d\sigma(\zeta)$$

and a decomposition of the representation V^η of G as

$$V^\eta \cong \int_Z^{\oplus} V^{(\eta,\zeta)}\ d\sigma(\zeta).$$

We can give an explicit form of the component representation $V^{(\eta,\zeta)}$ on $L^2(\hat{N},\mu_\zeta)$ as follows. Take a $\mathbb{R}^+$-valued Borel function $\rho_\zeta(k,\chi)$ on $K\times\hat{N}$ satisfying for each $k\in K$,

$$\rho_\zeta(k,\chi) = \frac{d(k\cdot\mu_\zeta)}{d\mu_\zeta}(\chi) \qquad (\mu_\zeta\text{-a.a.}\chi\ )$$

where $(k\cdot\mu_\zeta)(E) = \mu_\zeta(k\cdot E)$ for each Borel set E in $\hat{N}$. Then, for each $\xi(\chi)\in L^2(\hat{N},\mu_\zeta)$,

$$(V^{(\eta,\zeta)}_{(z,k)}\xi)(\chi) = \rho_\zeta(k,\chi)\langle z,\chi\rangle\langle k,\eta\rangle\xi(k\cdot\chi)$$

for $(z,k)\in N\times_s K = G$.

Here, we get irreducible decompositions of the regular representation λ of a semi-direct product group $G = N\times_s K$ of locally compact abelian groups N and K as follows.

Theorem 1.3.3. The regular representation λ of G is decomposed as

$$\lambda \cong \int_{\hat{N}}^{\oplus}\int_{\hat{H}_\chi}^{\oplus} U^{(\chi,\omega)}\ d\tau_\chi(\omega)d\mu(\chi)$$

$$\cong \int_K^{\oplus}\int_Z^{\oplus} V^{(\eta,\zeta)}\ d\sigma(\zeta)d\nu(\eta).$$

(A) $U^{(\chi,\omega)}$ and $V^{(\eta,\zeta)}$ are irreducible representations of G.

(B) When μ_ζ is a non-transitive measure, $V^{(\eta,\zeta)}$ is never unitarily equivalent to $U^{(\chi,\omega)}$ for arbitrary choice of $\eta\in\hat{K}$, $\chi\in\hat{N}$, and $\omega\in\hat{H}_\chi$.

Proof. The irreducibility of $U^{(\chi,\omega)}$ follows from general considerations of induced representations [30]. We show the irreducibility of $V^{(\eta,\zeta)}$. Suppose $WV^{(\eta,\zeta)}_{(z,k)} = V^{(\eta,\zeta)}_{(z,k)}W$ for some bounded operator W on $L^2(\hat{N},\mu_\zeta)$. Since the set of operators $V^{(\eta,\zeta)}_{(z,0)}$ $(z\in N)$ generates the maximal abelian von Neumann algebra $L^\infty(\hat{N},\mu_\zeta)$ of all multiplication operatos in $\mathcal{L}(L^2(\hat{N},\mu_\zeta))$, the algebra of all bounded operators on $L^2(\hat{N},\mu_\zeta)$, the equality $WV^{(\eta,\zeta)}_{(z,o)} = V^{(\eta,\zeta)}_{(z,0)}W$ for each $z\in N$ implies that W must be a multiplication operator $\rho(f)$ $(f\in L^\infty(\hat{N},\mu_\zeta))$. By simple calculations, we see that

$$V^{(\eta,\zeta)*}_{(0,k)}\rho(f)V^{(\eta,\zeta)}_{(0,k)} = \rho(k\cdot f)$$

where $(k\cdot f)(\chi) = f(k\cdot\chi)$. On the other hand, by the assumption, for each $k\in K$,

$$V^{(\eta,\zeta)*}_{(0,k)}\rho(f)V^{(\eta,\zeta)}_{(0,k)} = \rho(f).$$

Therfore, we get, for each $k\in K$,

$$k\cdot f = f \qquad (\mu_\zeta\text{-a.a.}).$$

Since μ_ζ is an ergodic measure on $\hat{N}$, we see that

$$f = \text{constant} \qquad (\mu_\zeta\text{-a.a.}).$$

Hence W must be a constant operator on $L^2(\hat{N},\mu_\zeta)$. This implies that $V^{(\eta,\zeta)}$ is an irreducible representation of G.

(B) Let $_N|U^{(\chi,\omega)}$ and $_N|V^{(\eta,\zeta)}$ denote the representations of the subgroup N of G, given by the restrictions to N of $U^{(\chi,\omega)}$ and $V^{(\eta,\zeta)}$ respectively. These representations of N are decomposed as follows.

$$_N|U^{(\chi,\omega)} \cong \int_{\hat{N}}^{\oplus} \gamma \, d\nu_\chi(\gamma) \ ,$$

$$_N|V^{(\eta,\zeta)} \cong \int_{\hat{N}}^{\oplus} \gamma d\mu_\zeta(\gamma) \ ,$$

where ν_χ is the canonical transitive quasi-invariant measure concentrated on $\mathrm{Orb}_K\chi$ on $\hat{N}$. By the assumption that the measure μ_ζ is non-transitive, μ_ζ is never equivalent to ν_χ so that $_N|V^{(\eta,\zeta)}$ is never unitarily equivalent to $_N|U^{(\chi,\omega)}$ for any $\eta\in\hat{K}$, $\chi\in\hat{N}$, and $\omega\in\hat{H}_\chi$ [32]. Therfore, we get the desired conclusion. [Q.E.D.]

Remark 1.3.4. The Mautner group is given as a semi-direct product group $\mathbb{C}^2 \times_s \mathbb{R}$ of two dimensional vector group $\mathbb{C}^2$ over $\mathbb{C}$ with $\mathbb{R}$. Applying our result to this group, we get the example obtained by A.A.Kirillov ([28]). Furthermore, our result is applicable to the discrete Mautner group, the discrete Heisenberg group, and the Dixmier group.

Remark 1.3.5. We can give other irreducible decompositions of the regular representation of $G = N \times_s K$ different from those in Theorem 1.3.3. These are given with related to cohomology groups, which will be described later (see section 2.6).

Chapter II. Cohomology of transformation groups

In this chapter, we explain the notion of cohomology of transformation groups. This notion appeared in the Mackey's works [33], [34], [35] and its study has been developed by several authors. K.Schmidt has studied it related with ergodic theory [45]. Another way of the development was pursued by A.Guichardet [17] and C.C.Moore [37], [38] who considered this cohomology as the one cohomology of locally compact groups. Further, there is a way followed by G.W.Mackey and A.Ramsay. They have investigated it as a family of similarity classes of homomorphisms of a measure groupoid or a virtual group [34], [43], [44].

In section 2.1, we describe elementary properties of cohomology of topological transformation groups. Propositions 2.1.1 and 2.1.2 are fundamental and may follow from the results in some of the works by C.C.Moore and A.Ramsay. However, we add the proofs for completeness.

In section 2.2, we introduce double transformation groups and their cohomology. These replace certain non-smooth topological transformation groups and their cohomology, and play a principal role in our considerations.

In section 2.3, we state the cohomology subordinate to measures. We often use this cohomology in later arguments.

In section 2.4, we study the notion of weak cohomology. This notion is important as an index showing the variety of decompositions of representations, which is one of our main subjects.

In section 2.5, subgroups of cohomology groups and weak cohomology groups are found in some concrete cases. It is known that $H^C_\mu(\mathbb{Z};\mathbb{T})$ and $H^C_\nu(\mathbb{R};\mathbb{T}^2)$ are isomorphic as cohomology groups as a general statement in [43]. We give here a concrete imbedding of $H^C_\mu(\mathbb{Z};\mathbb{T})$ to $H^C_\nu(\mathbb{R};\mathbb{T}^2)$. This makes us possible to get some concrete cocycles easily.

In section 2.6, we argue again decompositions of the regular representation of a semi-direct product group related with cohomology,as an application of this chapter.

2.1. Elementary properties

Let (G;X) be a topological transformation group. The action of the group G on the topological space X is denoted by $(g,x) \longrightarrow g\cdot x$, where $x \longrightarrow g\cdot x$ is a homeomorphism of X, and we suppose it satisfies $g_2\cdot(g_1\cdot x) = (g_1g_2)\cdot x$. Let $\mathfrak{A}$ be a von Neumann algebra on a separable Hilbert space. Then we can define the cohomology of (G;X) as follows (see [28], [35], [45]).

Let $\mathfrak{A}^u$ denote the set of unitary oparators of $\mathfrak{A}$ equipped with the Borel structure generated by the weak oparator topology. G and X have the canonical Borel structures induced by their topologies. A Borel function C of G×X into $\mathfrak{A}^u$ is said to be an $\mathfrak{A}^u$-valued cocycle, if it satisfies the condition

$$C(g_1g_2,x) = C(g_1,x)C(g_2,g_1\cdot x)$$

for all g_1, $g_2 \in G$ and all $x \in X$.

Such cocycles C_1 and C_2 are said to be cohomologous when there exists an $\mathfrak{A}^u$-valued Borel function A on X such that

$$C_2(g,x) = A(x)^* C_1(g,x) A(g \cdot x)$$

for all $g \in G$ and all $x \in X$.

It is clear that the above relation of "cohomologous" is an equivalence relation. If a cocycle C is cohomologous to the one which equals identically the identity oparator of $\mathfrak{A}$, C is said to be an $\mathfrak{A}^u$-valued coboundary. We denote the set of all $\mathfrak{A}^u$-valued cocycles of (G;X) by $Z^{\mathfrak{A}}(G;X)$, and the set of all $\mathfrak{A}^u$-valued coboundaries of (G;X) by $B^{\mathfrak{A}}(G;X)$. Moreover, let us denote $H^{\mathfrak{A}}(G;X)$ the quotient set of $Z^{\mathfrak{A}}(G;X)$ by the above equivalence relation. This is called the $\mathfrak{A}^u$-valued cohomology set of (G;X). Suppose $\mathfrak{A}$ abelian. Then, if C_1 and C_2 are in $Z^{\mathfrak{A}}(G;X)$, so is C_1C_2 (pointwise product) as well as C_1^{-1}, so that $Z^{\mathfrak{A}}(G;X)$ has an abelian group structure and moreover $B^{\mathfrak{A}}(G;X)$ is a subgroup of $Z^{\mathfrak{A}}(G;X)$. In this case, $H^{\mathfrak{A}}(G;X)$ is regarded as the quotient group of $Z^{\mathfrak{A}}(G;X)$ by $B^{\mathfrak{A}}(G;X)$, and it is called an $\mathfrak{A}^u$-valued cohomology group of (G;X).

A topological transformation group (G;X) is said to be smooth if every orbit is locally closed in X (see [12]. [16]), and effective if each stability group is trivial.

Proposition 2.1.1. If a topological transformation group (G;X) is smooth and effective, then $Z^{\mathfrak{A}}(G;X) = B^{\mathfrak{A}}(G;X)$ i.e. $H^{\mathfrak{A}}(G;X)$ is trivial.

Proof. Since (G;X) is smooth, there exists a Borel cross section c from the orbit space $G\backslash X$ to X (see [12]).

Then the map $\psi :(g,y) \longrightarrow g\cdot c(y)$ from $G\times(G\backslash X)$ to X is Borel and bijective as $(G;X)$ is effective. By the smoothness of $(G;X)$, $G\backslash X$ is a standard space and so is $G\times(G\backslash X)$. Hence the inverse of ψ is Borel. Thus we have a Borel map a from X to G and a Borel map b from X to a cross section of X under G such that $\psi^{-1}(x)=(a(x),b(x))$ i.e. $x = a(x)\cdot b(x)$ for each $x\in X$. For an arbitrary cocycle $C(g,x)$, we put

$$A(x) = C(a(x),b(x)) \qquad \text{for } x\in X.$$

Then A is an $\mathfrak{A}^u$-valued Borel function on X satisfying

$$C(g,x) = A(x)^* A(g\cdot x) \quad \text{for all } g\in G \text{ and all } x\in X. \tag{2.1.1}$$

Indeed, observing $g\cdot x$ in two ways, i.e.

$$g\cdot x = a(g\cdot x)\cdot b(g\cdot x) = a(g\cdot x)\cdot b(x)$$
$$g\cdot x = g\ (a(x)\cdot b(x)) = (a(x)g)\cdot b(x),$$

we have, by the fact that $(G;X)$ is effective,

$$a(g\cdot x) = a(x)g \text{ and } b(g\cdot x) = b(x).$$

Therefore, we get

$$\begin{aligned} &A(g\cdot x) \\ &= C(a(g\cdot x),b(g\cdot x)) \\ &= C(a(x)g,b(x)) \\ &= C(a(x),b(x))C(g,a(x)\cdot b(x)) \\ &= A(x)C(g,x). \end{aligned}$$

This implies (2.1.1), so that C is a coboundary of $(G;X)$.

[Q.E.D.]

When (G;X) is not effective, it is hard to get general results. Here we only mention the following proposition.

Proposition 2.1.2. If (G;X) is transitive and $\mathfrak{a}$ is one-dimensional, then the cohomology group $H^{\mathfrak{a}}(G;X)$ is isomorphic with $X(G_0)$ which is the group of all continuous unitary characters of some stability group G_0.

Proof. Fix an $x_0 \in X$ and let G_0 be the stability group of G at x_0. The map $G \ni g \longrightarrow g \cdot x_0$ gives a Borel isomorphism from $G_0 \backslash G$ onto X. For each cocycle C(g,x),

$$C(g_1 g_2, x_0) = C(g_1, x_0) C(g_2, x_0) \qquad \text{for all } g_1, g_2 \in G_0$$

holds. Then the map $g \longrightarrow C(g, x_0)$ from G_0 to $\mathbf{T}$ is Borel homomorphic, therefore continuous, where $\mathbf{T}$ is the one-dimensional torus and equals $\mathfrak{a}^u$. So there exists a continuous unitary character χ_C of G_0 such that

$$C(g, x_0) = \chi_C(g) \qquad \text{for all } g \in G_0.$$

In this way we get a map $\psi : C \longrightarrow \chi_C$ from $Z^{\mathbb{C}}(G;X)$ to $X(G_0)$. It is verified with no trouble that this ψ is homomorphic and surjective. Rest to show is that $\mathrm{Ker}\,\psi = B^{\mathbb{C}}(G;X)$. Suppose that

$$C(g_0, x_0) = 1 \qquad \text{for all } g_0 \in G_0.$$

Then we get

$$C(g_0 g, x_0) = C(g, x_0) \qquad \text{for all } g_0 \in G_0 \text{ and all } g \in G,$$

and it follows from this that there exists a Borel function

A on X such that

$$C(g,x_0) = A(g\cdot x_0) \qquad \text{for all } g\in G.$$

Since for each $x\in X$ there exists $g_1\in G$ such that $x = g_1\cdot x_0$, we have

$$\begin{aligned} & C(g,x) \\ &= C(g,g_1\cdot x_0) \\ &= C(g_1,x_0)^* C(g_1 g,x_0) \\ &= A(g_1\cdot x_0)^* A(g\cdot(g_1\cdot x_0)) \\ &= A(x)^* A(g\cdot x). \end{aligned}$$

This implies that C is a coboundary of (G;X) and $\mathrm{Ker}\psi \subset B^{\mathbb{C}}(G;X)$. As the inverse inclusion relation is clear, we obtain that $\mathrm{Ker}\psi = B^{\mathbb{C}}(G;X)$. [Q.E.D.]

2.2. Double transformation groups

When (G;X) is not smooth, instead of considering (G;X), we take a double transformation group (G;Y;H) such that (G;Y) is smooth and (G;X) can be looked at as the same with (G;Y/H) as topological transformation groups.

Definition 2.2.1. We call (G;X;H) a double transformation group if groups G and H act on the same space X as topological transformation groups, where the actions of G and H on X are denoted by $(g,x) \longrightarrow g\cdot x$ and $(h,x) \longrightarrow x\cdot h$, and the following conditions are satisfied.

(1) $(g\cdot x)\cdot h = g\cdot(x\cdot h)$ for all $g\in G$, all $h\in H$, and all $x\in X$,

(2) the map: $(g,x,h) \longrightarrow g\cdot x\cdot h$ from $G\times X\times H$ to X is continuous.

Given a double transformation group $(G;X;H)$ and a von Neumann algebra $\mathfrak{A}$, we will define an $\mathfrak{A}^u$-valued cocycle, coboundary, and cohomology of $(G;X;H)$ as follows.

Definition 2.2.2. We call an $\mathfrak{A}^u$-valued Borel function A on X an $\mathfrak{A}^u$-valued cocycle of $(G;X;H)$ if

$$A(g\ x\ h) = A(g\cdot x)A(x)^{*}A(x\cdot h) \quad \text{for all } g\in G, \text{ all } h\in H, \text{ and all } x\in X \tag{2.2.1}$$

is satisfied. Such cocycles A_1 and A_2 are said to be cohomologous if there exist an H-invariant cocycle B_1 and a G-invariant cocycle B_2 such that

$$A_2(x) = B_1(x)A_1(x)B_2(x) \quad \text{for all } x\in X. \tag{2.2.2}$$

If a cocycle A is cohomologous to the one which equals identically the identity operator of $\mathfrak{A}$, we say that A is an $\mathfrak{A}^u$-valued coboundary of $(G;X;H)$.

We denote the set of cocycles by $Z^{\mathfrak{A}}(G;X;H)$, the set of cohomology classes by $H^{\mathfrak{A}}(G;X;H)$, and the set of coboundaries by $B^{\mathfrak{A}}(G;X;H)$. If $\mathfrak{A}$ is abelian, they all have abelian group structures and $H^{\mathfrak{A}}(G;X;H) \cong Z^{\mathfrak{A}}(G;X;H)/B^{\mathfrak{A}}(G;X;H)$.

Proposition 2.2.3. Let $(G;X;H)$ be a double transformation group and $\mathfrak{A}$ be an abelian von Neumann algebra. $(G;X)$ and $(H;X)$ are supposed to be smooth and effective. Then the following three abelian groups are isomorphic to each other.

(1) $H^{\mathfrak{A}}(G;X;H)$

(2) $H^{\mathfrak{A}}(G;X/H)$

(3) $H^{\mathfrak{A}}(H;G\backslash X)$

Proof. In general, the orbit spaces X/H and $G\backslash X$ may not be Hausdorff. However, $H^{\mathfrak{A}}(G;X/H)$ and $H^{\mathfrak{A}}(H;G\backslash X)$ are well-defined because the definition of a cohomology group depends only on the Borel structure, and we remark that the Borel structures of X/H and $G\backslash X$ induced by their topologies coincide with the quotient Borel structures by the smoothness of $(H;X)$ and $(G;X)$ (see [12]).

Let A be an $\mathfrak{A}^u$-valued cocycle of $(G;X;H)$. Using this A, we define C and D by

$$C(g,x) = A(x)A(g\cdot x)^* \quad \text{for } g\in G \text{ and } x\in X, \qquad (2.2.3)$$

$$D(h,x) = A(x)^*A(x\cdot h) \quad \text{for } h\in H \text{ and } x\in X. \qquad (2.2.4)$$

Then, the equality (2.2.1) implies that $C(g,x)$ is H-invariant with respect to the variable $x\in X$ and $D(h,x)$ is G-invariant with respect to $x\in X$. Hence we may regard C as a cocycle of $(G;X/H)$ and D as a cocycle of $(H;G\backslash X)$ because the cocycle conditions about C and D follow immediately from their definitions. The corespondences $A \longrightarrow C$ and $A \longrightarrow D$ induce the isomorphism from $H^{\mathfrak{A}}(G;X;H)$ onto $H^{\mathfrak{A}}(G;X/H)$ and from $H^{\mathfrak{A}}(G;X;H)$ onto $H^{\mathfrak{A}}(H;G\backslash X)$. In fact, let ψ be the map $A \longrightarrow C$ from $Z^{\mathfrak{A}}(G;X;H)$ to $Z^{\mathfrak{A}}(G;X/H)$. Then it is not hard to see that ψ is homomorphic and $\psi^{-1}(B^{\mathfrak{A}}(G;X/H)) = B^{\mathfrak{A}}(G;X;H)$. Moreover, Proposition 2.1.1 together with the assumptions imply that ψ is surjective. [Q.E.D.]

Remark 2.3.4. Let $\tilde{G}$ be a group, and G and H be closed subgroups of $\tilde{G}$. In this case, we define the actions of G and H on $\tilde{G}$ by

$$G\times\tilde{G}\times H \ni (g,x,h) \longrightarrow h^{-1}\cdot x\cdot g\in\tilde{G}.$$

Then, $(G;\tilde{G};H)$ is a double trasformation group satisfying the assumptions of Proposition 2.2.3.

2.3. Cohomology subordinate to measures

When a measure is put on the space X, we shall consider the cohomology groups of (G;X) and (G;X;H) subordinate to this measure. Let μ be a positive Radon measure on a topological space X, and $\mathfrak{A}$ be a von Neumann algebra. In the case of a topological transformation group (G;X), we shall change the former definitions as follows.

Let C_1 and C_2 be in $Z^{\mathfrak{A}}(G;X)$. Then we say that C_1 is μ-cohomologous to C_2 if there exists an $\mathfrak{A}^u$-valued Borel function A on X such that for each $g\in G$

$$C_2(g,x) = A(x)^*C_1(g,x)A(g\cdot x) \quad \text{for } \mu\text{-a.a. } x\in X.$$

We denote the set of all μ-cohomology classes of $Z^{\mathfrak{A}}(G;X)$ by $H_\mu^{\mathfrak{A}}(G;X)$. A cocycle C is said to be a μ-coboundary if C is μ-cohomologous to the one which equals identically the identity operatorof $\mathfrak{A}$, and we denote the set of all μ-cobounbaries of (G;X) by $B_\mu^{\mathfrak{A}}(G;X)$.

Next, in the case of a double transformation group

$(G;X;H)$, for A_1 and A_2 in $Z^{\mathfrak{A}}(G;X;H)$, we say that A_1 is μ-cohomologous to A_2 if there exist an H-invariant cocycle B_1 and a G-invariant cocycle B_2 such that

$$A_2(x) = B_1(x)A_1(x)B_2(x) \qquad \text{for } \mu\text{-a.a. } x \in X.$$

The set of all μ-cohomology classes of $Z^{\mathfrak{A}}(G;X;H)$ is denoted by $H^{\mathfrak{A}}_{\mu}(G;X;H)$. A μ-coboundary is defined in the same way as above, and we denote the set of all μ-coboundaries of $(G;X;H)$ by $B^{\mathfrak{A}}_{\mu}(G;X;H)$.

Assuming $\mathfrak{A}$ abelian, $Z^{\mathfrak{A}}$, $B^{\mathfrak{A}}_{\mu}$, and $H^{\mathfrak{A}}_{\mu}$ have abelian group structures and we have $H^{\mathfrak{A}}_{\mu} \cong Z^{\mathfrak{A}}/B^{\mathfrak{A}}_{\mu}$ as groups in either case.

Note that the above definitions depend only on the measure class $C(\mu)$ of μ and not on μ itself. Therefore we write sometimes $B^{\mathfrak{A}}_{C(\mu)}$ and $H^{\mathfrak{A}}_{C(\mu)}$. A measure class $C(\mu)$ is said to be a quasi-orbit if μ is quasi-invariant and ergodic under the action of G on X.

Proposition 2.3.1. If a topological transformation group $(G;X)$ is smooth and $C(\mu)$ is a quasi-orbit on X, then $H^{\mathbb{C}}_{C(\mu)}(G;X)$ is isomorphic to $X(G_0)$ as abelian groups where G_0 is some closed subgroup of G.

Proof. By the smoothness of $(G;X)$, $C(\mu)$ must be a transitive quasi-orbit. Therefore there exists an $x_0 \in X$ such that $C(\mu)$ is concentrated on the orbit $G \cdot x_0$ which is isomorphic to $G_0 \backslash G$ as topological transformation groups where G_0 is the stability group of G at x_0, and such $C(\mu)$ is the unique measure class which corresponds to the canonical class on $G_0 \backslash G$.

Let C be in $B^{\mathbb{C}}_{C(\mu)}(G;X)$. Then there exists a Borel function A on X such that for each $g \in G$

$$C(g,x) = A(x)^* A(g\cdot x) \quad \text{for } \mu\text{-a.a. } x \in X. \tag{2.3.1}$$

Now we define cocycles C_1 and C_2 by

$$C_1(g,x) = A(x)^* A(g\cdot x) \tag{2.3.2}$$

$$C_2(g,x) = C_1(g,x)^* C(g,x) \text{ for } g\in G \text{ and } x\in X. \tag{2.3.3}$$

Then, we have, for each $g \in G$

$$C_2(g,x) = 1 \qquad \text{for } \mu\text{-a.a. } x\in X. \tag{2.3.4}$$

Suppose that there exists a $g_0 \in G_0$ such that

$$C_2(g_0,x_0) \neq 1. \tag{2.3.5}$$

The cocycle condition implies that

$$C_2(g_2,g_1\cdot x_0) = C_2(g_1,x_0)^* C_2(g_1 g_2,x_0) \text{ for all } g_1,g_2 \in G.$$

If we define a Borel function B on G by $B(g) = C_2(g,x_0)$, we have

$$C_2(g_2,g_1\cdot x_0) = B(g_1)^* B(g_1 g_2)$$
$$\text{for all } g_1,g_2 \in G. \tag{2.3.6}$$

According to (2.3.4), for each $g_2 \in G$

$$B(g_1) = B(g_1 g_2) \qquad \text{for a.a. } g_1 \in G$$

holds because $C(\mu)$ may be considered as the canonical class on $G_0 \backslash G$. By Fubini's Theorem, we get, for almost all $g_1 \in G$

$$B(g_1) = B(g_1 g_2) \qquad \text{for a.a. } g_2 \in G.$$

Therefore we have

$$B(g) = K \text{ (constant)} \quad \text{for a.a. } g \in G. \qquad (2.3.7)$$

However, by (2.3.6)

$$B(g_0 g) = B(g_0)B(g) \qquad \text{for all } g \in G$$

holds and $B(g_0) \neq 1$ by (2.3.5), so that we have

$$B(g) \neq K \qquad \text{for a.a. } g \in G.$$

This fact contradicts with (2.3.7). So we get

$$C_2(g, x_0) = 1 \qquad \text{for all } g \in G_0.$$

We have already shown in the proof of Proposition 2.1.2 that this fact implies $C_2 \in B^{\mathbb{C}}(G;X)$. Since C_1 is in $B^{\mathbb{C}}(G;X)$ and $B^{\mathbb{C}}(G;X)$ is an abelian group, we get $C \in B^{\mathbb{C}}(G;X)$ by (2.3.3) and so $B^{\mathbb{C}}_{C(\mu)}(G;X) \subset B^{\mathbb{C}}(G;X)$ has been shown. It is clear that the inverse inclusion relation holds, so that we have $B^{\mathbb{C}}_{C(\mu)}(G;X) = B^{\mathbb{C}}(G;X)$. Hence we get $H^{\mathbb{C}}_{C(\mu)}(G;X) = H^{\mathbb{C}}(G;X)$ $= X(G_0)$ by Proposition 2.1.2. [Q.E.D.]

Next we shall consider the case where (G;X) is not necessarily smooth. For a quasi-orbit $C(\mu)$ on X, we often find a large group $\tilde{G}$ and its closed subgroup H such that $C(\mu)$ can be identified with the $\tilde{G}$-quasi-invariant measure class on $H\backslash\tilde{G}$. In this case we have the following theorem.

Theorem 2.3.2. Let $\mathfrak{A}$ be an abelian von Neumann algebra and let $(G;\tilde{G};H)$ be a double transformation group where G and

H are closed subgroup of a group $\tilde{G}$ and their actions are defined as in Remark 2.2.4. If σ is a Haar measure of $\tilde{G}$ and μ (resp. ν) is a canonical quasi-invariant measure on $H\backslash\tilde{G}$ (resp. $\tilde{G}/G$), then the following three abelian groups are isomorphic to each other.

(1) $H^{\mathfrak{A}}_{\sigma}(G;\tilde{G};H)$

(2) $H^{\mathfrak{A}}_{\mu}(G;H\backslash\tilde{G})$

(3) $H^{\mathfrak{A}}_{\nu}(H;\tilde{G}/G)$

Proof. This follows from Proposition 2.2.3 combined with some measure theoretic arguments. We omit the details.

[Q.E.D.]

2.4. Weak cohomology

Let (G;X) be a topological transformation group and $\mathfrak{A}$ be an abelian von Neumann algebra. Then, we define $\mathfrak{A}^u$-valued weak cohomology of (G;X) as follows.

Definition 2.4.1 For two $\mathfrak{A}^u$-valued cocycles C_1 and C_2 of (G;X), we call that C_1 is weakly cohomologous to C_2 if $C_1C_2^*$ is cohomologous to some continuous homomorphism from G to $\mathfrak{A}^u$. We denote all $\mathfrak{A}^u$-valued weakly cohomologous classes of (G;X) by $\hat{H}^{\mathfrak{A}}(G;X)$, which has also an abelian group structure. We call $\tilde{H}^{\mathfrak{A}}(G;X)$ $\mathfrak{A}^u$-valued weak cohomology group.

Let $Z^{\mathfrak{A}}_0(G;X)$ denote all continuous homomorphisms from G to $\mathfrak{A}^u$ and $H^{\mathfrak{A}}_0(G;X)$ be the factor group of $Z^{\mathfrak{A}}_0(G;X)$ by $Z^{\mathfrak{A}}_0(G;X)\cap B^{\mathfrak{A}}(G;X)$. Then, we see that

$$\tilde{H}^{\mathfrak{A}}(G;X) = H^{\mathfrak{A}}(G;X)/H_0^{\mathfrak{A}}(G;X).$$

When a positive Radon measure μ is put on X, we can also define $\mathfrak{A}^u$-valued weak μ-cohomology group $\tilde{H}_\mu^{\mathfrak{A}}(G;X)$ of (G;X) by routine arguments as in section 2.3.

Then, we get immediately the following propositions, according to Proposition 2.1.2 and Proposition 2.3.1.

Proposition 2.4.2. Let (G;X) be a transitive topological transformation group where G is supposed to be a loclly compact abelian group. Then, $\tilde{H}^{\mathbb{C}}(G;X)$ is trivial.

Proposition 2.4.3. Let (G;X) be a smooth topological transformation group where G is supposed to be abelian. Then, for any quasi-orbit $C(\mu)$ on X, $\tilde{H}_{C(\mu)}^{\mathbb{C}}(G;X)$ is trivial.

Now, let G be a locally compact group. Taking two closed subgroups H and K of G, we consider a double transformation group (K;G;H). Let $\mathfrak{A}$ be an abelian von Neumann algebra. Then we can also define $\mathfrak{A}^u$-valued weak cohomology group $\tilde{H}^{\mathfrak{A}}(K;G;H)$ of (K;G;H) as follows.

Dfinition 2.4.4. For two $\mathfrak{A}^u$-valued cocycles A_1 and A_2 of (K;G;H), A_1 is called to be weakly cohomologous to A_2 if $A_1A_2^*$ is σ-cohomologous to some continuous homomorphism from G to $\mathfrak{A}^u$ where σ is a Haar measure of G. We denote all $\mathfrak{A}^u$-valued weakly cohomologous classes by $\tilde{H}^{\mathfrak{A}}(K;G;H)$ and we call it $\mathfrak{A}^u$-valued weak cohomology group of (K;G;H).

Proposition 2.4.5. Let G be a locally compact abelian group and μ (resp. ν) denote a Haar measure of G/H (resp. G/K). then, the following three abelian groups are isomorphic with each other.

(1) $\tilde{H}^{\mathfrak{N}}(K;G;H)$

(2) $\tilde{H}^{\mathfrak{N}}_{\mu}(K;G/H)$

(3) $\tilde{H}^{\mathfrak{N}}_{\nu}(h;G/K)$

Proof. This follows from the definition and Theorem 2.3.2. [Q.E.D.]

Our considerations go on in the situation that G is abelian and $\mathfrak{N}$ is one-dimensional. Let $\hat{G}$ be the dual group of G and $H^{\perp}$ (resp. $K^{\perp}$) denote the annihilator of the subgroup H (resp. K) of G in $\hat{G}$. We denote by $H_0^{\mathbb{C}}(K;G;H)$ the factor group of $\hat{G}$ by $\hat{G} \cap B_{\sigma}^{\mathbb{C}}(K;G;H)$. Then, we see that

$$\tilde{H}^{\mathbb{C}}(K;G;H) \cong H_{\sigma}^{\mathbb{C}}(K;G;H)/H_0^{\mathbb{C}}(K;G;H).$$

Furthermore, we get the following.

Proposition 2.4.6. If K + H is dense in G,

$$H_0^{\mathbb{C}}(K;G;H) \cong \hat{G}/(K^{\perp}+H^{\perp}).$$

Proof. If $\chi \in \hat{G}$ is written as $\chi(t) = \chi_1(t)\ \chi_2(t)$ for some $\chi_1 \in K^{\perp}$ and $\chi_2 \in H^{\perp}$, it is clear that χ is a coboundary by definition. So we shall show the converse.

Suppose that for $\chi \in \hat{G}$, $\chi(t) = E(t)F(t)$ for almost all $t \in G$, where E is an H-invariant cocycle and F is an K-invariant cocycle. Since χ satisfies $\chi(t_1+t_2)\overline{\chi(t_1)}\,\overline{\chi(t_2)} = 1$ for all $(t_1, t_2)\in G^2$, we get

$$E(t_1+t_2)\overline{E(t_1)}\,\overline{E(t_2)} = \overline{F(t_1+t_2)}F(t_1)F(t_2)$$

for almost all $(t_1,t_2)\in G^2$. Put

$$\phi(t_1,t_2) = E(t_1+t_2)\overline{E(t_1)}\,\overline{E(t_2)}$$

for $(t_1,t_2)\in G^2$. Then, by the property of E and F, ϕ is $(K+H)^2$-invariant. Since K+H is dense in G, $(K+H)^2$ is also dense in G^2. Hence, $(K+H)^2$ acts on G^2 ergodically. Therefore, we get

$$\phi(t_1,t_2) = c \text{ (constant)} \quad \text{for a.a. } (t_1,t_2)\in G^2.$$

When we put $E' = cE$ and $F' = \bar{c}F$, we see that

$$E'(t_1+t_2) = E'(t_1)E'(t_2),$$

$$F'(t_1+t_2) = F'(t_1)F'(t_2) \quad \text{for a.a. } (t_1,t_2)\in G^2$$

and

$$\chi(t) = E'(t)\,F'(t) \quad \text{for a.a. } t\in G.$$

By Theorem 5.1 in [41], there exists χ_1, $\chi_2\in\hat{G}$ such that $\chi_1(t) = E'(t)$ and $\chi_2(t) = F'(t)$ for a.a. $t\in G$. Moreover, by the continuity of χ_1 and χ_2, χ_1 must be H-invariant (i.e. $\chi_1\in H^\perp$), χ_2 must be K-invariant (i.e. $\chi_2\in K^\perp$), and $\chi(t) = \chi_1(t)\chi_2(t)$ for all $t\in G$. Therefore, χ must be in $K^\perp + H^\perp$. [Q.E.D.]

The proof of this lemma was suggested by Professor M. Takesaki.

Remark 2.4.7. Assume that $K^{\perp} + H^{\perp}$ is not closed in $\hat{G}$. Then, the induced Borel structure of $\hat{G}/(K^{\perp}+H^{\perp})$ is not standard so that the cardinal number of $H_0^{\mathbb{C}}(K;G;H)$ must be uncountable infinity. Therefore, in such a case, we can conclude that the cardinal number of $H_\sigma^{\mathbb{C}}(K;G;H)$ is also uncountable infinity. In some indivisual cases, we know that the cardinal number of $\widetilde{H}^{\mathbb{C}}(K;G;H)$ is uncountable infinity (see [27]) but general considerations about the weak cohomology group have not yet been obtained.

2.5. Examples and some calculations

Here we shall treat the following two transformation groups.

(a) $(\mathbf{Z};\mathbf{T})$ where $\mathbf{Z}$ is the additve group of integers and $\mathbf{T}$ is the one-dimensional torus. The action of $\mathbf{Z}$ on $\mathbf{T}$ is defined by

$$n\cdot\xi = e^{in}\xi \qquad \text{for } n\in\mathbf{Z} \text{ and } \xi\in\mathbf{T}.$$

(b) $(\mathbf{R};\mathbf{T}^2)$ where $\mathbf{R}$ is the additive group of real numbers and $\mathbf{T}^2$ is the two-dimensional torus. The action of $\mathbf{R}$ on $\mathbf{T}^2$ is defined by

$$t\cdot(\xi,\eta) = (e^{it}\xi, e^{2\pi it}\eta) \qquad \text{for } t\in\mathbf{R} \text{ and } (\xi,\eta)\in\mathbf{T}^2.$$

Now we find the following double transformation groups corresponding to the cases (a) and (b).

(a-1) $(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$. The actions of $\mathbf{Z}$ and $2\pi\mathbf{Z}$ on $\mathbf{R}$ are defined by

$n \cdot z = z+n$ for $n \in \mathbf{Z}$ and $z \in \mathbf{R}$,

$z \cdot (2\pi m) = z+2\pi m$ for $2\pi m \in 2\pi\mathbf{Z}$ and $z \in \mathbf{R}$.

$(\mathbf{Z};\mathbf{T}) \cong (\mathbf{Z};\mathbf{R}/2\pi\mathbf{Z})$ as topological transformation groups.

(b-1) $(\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$. The actions of $\mathbf{R}$ and $(2\pi\mathbf{Z})^2$ on $\mathbf{R}^2$ are defined by

$t \cdot (x,y) = (x+t, y+2\pi t)$ for $t \in \mathbf{R}$ and $(x,y) \in \mathbf{R}^2$,

$(x,y) \cdot (2\pi m, 2\pi n) = (x+2\pi m, y+2\pi n)$
for $(2\pi m, 2\pi n) \in (2\pi\mathbf{Z})^2$ and $(x,y) \in \mathbf{R}^2$.

$(\mathbf{R};\mathbf{T}^2) \cong (\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$ as topological transformation groups.

Let μ, ν, α, and β be Haar measures of $\mathbf{T}$, $\mathbf{T}^2$, $\mathbf{R}$, and $\mathbf{R}^2$ respectively. According to Theorem 2.3.2, we get

(a-2) $H_\mu^{\mathbf{C}}(\mathbf{Z};\mathbf{T}) \cong H_\alpha^{\mathbf{C}}(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$,

(b-2) $H_\nu^{\mathbf{C}}(\mathbf{R};\mathbf{T}^2) \cong H_\beta^{\mathbf{C}}(\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$.

We shall determine a part of these cohomology groups. Let us define the abelian group Z^0 and its subgroup B_μ^0 by

$Z^0 = \{$all $\mathbf{T}$-vlued Borel functions on $\mathbf{T}\}$

$B_\mu^0 = \{b(\xi) \in Z^0;$ there exists an $a(\xi) \in Z^0$ such that
$b(\xi) = a(\xi)^* a(e^i \xi)$ for μ-almost all $\xi \in \mathbf{T}\}$

We denote the quotient group Z^0/B_μ^0 by H_μ^0.

Lemma 2.5.1. $H_\mu^{\mathbf{C}}(\mathbf{Z};\mathbf{T})$ is isomorphic with H_μ^0 as abelian groups.

Proof. For $C(n,\xi) \in Z^{\mathbf{C}}(\mathbf{Z};\mathbf{T})$, we put $a(\xi) = C(1,\xi) \in Z^0$. Then we have a map $\psi : C \longrightarrow a$ from $Z^{\mathbf{C}}(\mathbf{Z};\mathbf{T})$ to Z^0. It is easily checked that this ψ is an injective homomorphism and $\psi(B^{\mathbb{C}}_{\mu}(\mathbf{Z};\mathbf{T})) \subset B^0_{\mu}$. We show that ψ is surjective and $\psi(B^{\mathbb{C}}_{\mu}(\mathbf{Z};\mathbf{T})) = B^0_{\mu}$. Let $a \in Z^0$, we construct $C(n,\xi)$ as follows.

$$C(n,\xi) = \begin{cases} \prod_{k=0}^{n-1} a(e^{ik}\xi) & \text{if } n \geqq 1 \\ 1 & \text{if } n = 0 \\ \prod_{k=0}^{-(n+1)} a(e^{i(n+k)}\xi)^{*} & \text{if } n \leqq -1. \end{cases}$$

Then $C(n,\xi)$ is in $Z^{\mathbb{C}}(\mathbf{Z};\mathbf{T})$ and $\psi(C) = a$, so ψ is surjective. If b is in B^0, then this construction gives a $B(n,\xi)$ in $B^{\mathbb{C}}_{\mu}(\mathbf{Z};\mathbf{T})$. Indeed, if $b(\xi) = a(\xi)^{*}a(e^{i}\xi)$ for μ-almost all $\xi \in \mathbf{T}$, then, for example, when $n \geqq 1$ we see that

$$\begin{aligned} B(n,\xi) &= \prod_{k=0}^{n-1} b(e^{ik}\xi) \\ &= \prod_{k=0}^{n-1} a(e^{ik}\xi)^{*}a(e^{i(k+1)}\xi) \\ &= a(\xi)^{*}a(e^{in}\xi) \qquad \text{for } \mu\text{-a.a. } \xi \in \mathbf{T}. \end{aligned}$$

Therefore ψ induces an isomorphism ψ_{*} from $H^{\mathbb{C}}_{\mu}(\mathbf{Z};\mathbf{T})$ onto H^0_{μ}. [Q.E.D.]

We also take the following abelian groups Z^1, B^1_{α}, and H^1_{α} defined by

$$Z^1 = \{\text{all } \mathbf{T}\text{-valued Borel functions on } \mathbf{R} \text{ with period } 2\pi\},$$

$$B^1_{\alpha} = \{a(z) \in Z^1 \;;\; \text{there exists } b(z) \in Z^1 \text{ such that } a(z) = b(z)^{*}b(z+1) \text{ for } \alpha\text{-almost all } z \in \mathbf{R}\},$$

$$H^1_{\alpha} = Z^1/B^1_{\alpha}.$$

Then it is clear that $Z^0 \cong Z^1$, $B^0_{\mu} \cong B^1_{\alpha}$, and $H^0_{\mu} \cong H^1_{\alpha}$.

Lemma 2.5.2. $(\mathbb{R}/\mathbb{Z}+2\pi\mathbb{Z})+\mathbb{Z}$ is a subgroup of $H^{\mathbb{C}}_{\mu}(\mathbb{Z};\mathbb{T})$.

Proof. For $\lambda\in\mathbb{R}$ and $d\in\mathbb{Z}$, we define $a^{(\lambda,d)}\in Z^1$ by

$$a^{(\lambda,d)}(z) = e^{i(dz+\lambda)} \qquad \text{for } z\in\mathbb{R}.$$

The set of all $a^{(\lambda,d)}$ with $\lambda\in\mathbb{R}$ and $d\in\mathbb{Z}$ forms a subgroup of Z^1 and we will determine which ones fall in B^1_α among these $a^{(\lambda,d)}$'s.

Suppose that $a^{(\lambda,d)}\in B^1_\alpha$. Then it implies that there exists a $b(z)\in Z^1$ such that

$$b(z)a^{(\lambda,d)}(z) = b(z+1) \quad \text{for } \alpha\text{-a.a. } z\in\mathbb{R}. \qquad (2.5.1)$$

For Borel functions $f(z)$ on $\mathbb{R}$ with period 2π, we adopt L^2-norm $\|\cdot\|_2$ defined by

$$\|f\|_2^2 = \int_0^{2\pi}|f(z)|^2\,dz.$$

Then, (2.5.1) is equivalent to

$$b(z)a^{(\lambda,d)}(z) = b(z+1) \quad \text{in } L^2\text{-norm}. \qquad (2.5.2)$$

We have the Fourier expansion of $b(z)$.

$$b(z) = \sum_{n\in\mathbb{Z}} b_n e^{inz} \qquad \text{in } L^2\text{-norm}. \qquad (2.5.3)$$

Then, by (2.5.2) we get

$$\sum_{n\ Z} b_n e^{i\lambda}e^{i(n+d)z} = \sum_{n\in\mathbb{Z}} b_n e^{in}e^{inz} \quad \text{in } L^2\text{-norm}. \ (2.5.4)$$

By the uniqueness of the coefficients of a Fourier expansion,

$$b_n e^{i\lambda} = b_{n+d}e^{i(n+d)} \qquad \text{for all } n\in\mathbb{Z}. \qquad (2.5.5)$$

Hence we have

$$|b_n| = |b_{n+d}| \qquad \text{for all } n \in \mathbf{Z}. \qquad (2.5.6)$$

On the other hand,

$$\sum_{n \in Z} |b_n|^2 = \|b\|_2^2 < \infty.$$

If $d \neq 0$, this fact contradicts (2.5.6). Therefore d must be 0, and in this case, by (2.5.5), we get

$$b_n e^{i\lambda} = b_n e^{in}.$$

By the fact $\sum_{n \in \mathbf{Z}} |b_n|^2 \neq 0$, there exists an $n_0 \in \mathbf{Z}$ such that $b_{n_0} \neq 0$. Hence we have $e^{i\lambda} = e^{in_0}$, which implies that $\lambda \in \mathbf{Z}+2\pi\mathbf{Z}$. Conversely, we can check easily that $a^{(\lambda,d)} \in B^1_\alpha$ if $d = 0$ and $\lambda \in \mathbf{Z}+2\pi\mathbf{Z}$. [Q.E.D.]

Here we note that using this lemma we have the family of representations which L.Baggett has got in [4], and the argument in the above is parallel with his in some sense. But the next proposition will give rise to an essentially new parametrized family of irreducible representations of the Mautner group. Let $\mathbb{Q}$ be the additive group of rational numbers.

Proposition 2.5.3. $(\mathbb{R}/\mathbf{Z}+2\pi\mathbf{Z})+\mathbb{Q}$ is a subgroup of $H^{\mathbf{C}}_\mu(\mathbf{Z};\mathbf{T})$.

Proof. For $p \in \mathbb{N}$ (positive integers), $q \in \mathbf{Z}$, and $\lambda \in \mathbb{R}$, we define $\tilde{a}^{(\lambda,q/p)}$ by

$$\tilde{a}^{(\lambda,q/p)}(z) = e^{i((q/p)z+\lambda)} \qquad \text{for } 0 \leq z < 2\pi.$$

Then we are able to extend it to $a^{(\lambda,q/p)} \in Z^1$. By definition,

$$(a^{(\lambda,q/p)})^p = a^{(p\lambda,q)}.$$

If $a^{(\lambda,q/p)} \in B^1_\alpha$, then $(a^{(\lambda,q/p)})^p$ is also in B^1_α, because B^1_α is a group, so that $a^{(p\lambda,q)} \in B^1_\alpha$. This fact implies $q = 0$. (see the proof of Lemma 2.5.2). In the same way, we have $\lambda \in \mathbf{Z}+2\pi\mathbf{Z}$. The converse is trivial. [Q.E.D.]

There exists a relation between $H^{\mathbf{C}}_\mu(\mathbf{Z};\mathbf{T})$ and $H^{\mathbf{C}}_\nu(\mathbf{R};\mathbf{T}^2)$, shown in the next lemma, so that we can get an information about $H^{\mathbf{C}}_\nu(\mathbf{R};\mathbf{T}^2)$ from that of $H^{\mathbf{C}}_\mu(\mathbf{Z};\mathbf{T})$.

Lemma 2.5.4. $H^{\mathbf{C}}_\mu(\mathbf{Z};\mathbf{T})$ can be imbedded isomorphically into $H^{\mathbf{C}}_\nu(\mathbf{R};\mathbf{T}^2)$ as an abelian group.

Proof. Instead of $H^{\mathbf{C}}_\mu(\mathbf{Z};\mathbf{T})$ and $H^{\mathbf{C}}_\nu(\mathbf{R};\mathbf{T}^2)$, we take $H^{\mathbf{C}}_\alpha(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$ and $H^{\mathbf{C}}_\beta(\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$(see (a-2) and (b-2)). For $A(z) \in Z^{\mathbf{C}}(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$, we define a $\mathbf{T}$-valued Borel function $\tilde{A}(x,y)$ on $\mathbf{R}^2$ by

$$\tilde{A}(x,y) = A(x-\frac{1}{2\pi}\overline{y})$$

where for $y \in \mathbf{R}$, $y = \overline{y} + [y]$, $0 \leqq \overline{y} < 2\pi$, and $[y] \in 2\pi\mathbf{Z}$. Then we have $A(x,y) \in Z^C(R;R^2;(2\pi\mathbf{Z})^2)$. In fact, for $t \in R$, $(x,y) \in R^2$, and $(2\pi m, 2\pi n) \in (2\pi\mathbf{Z})^2$,

$$\begin{aligned}
&\tilde{A}(t\cdot(x,y)\cdot(2\pi m,2\pi n)) \\
&= \tilde{A}((x+2\pi m+t, y+2\pi n+2\pi t)) \\
&= A(x+2\pi m+t-\frac{1}{2\pi}\overline{y+2\pi n+2\pi t}\ ) \\
&= A(x+2\pi m+t-\ \frac{1}{2\pi}(y+2\pi t)+\frac{1}{2\pi}[y+2\pi y]) \\
&= A(\frac{1}{2\pi}(-[y]+[y+2\pi t])+(x-\frac{1}{2\pi}\overline{y})+2\pi m))
\end{aligned}$$

$$= A((x-\tfrac{1}{2\pi}\overline{y})+\tfrac{1}{2\pi}(-[y]+[y+2\pi t]))A(x-\tfrac{1}{2\pi}y)^{*}A(x-\tfrac{1}{2\pi}y+2\pi m)$$

$$= A(x+t-\tfrac{1}{2\pi}\overline{y+2\pi t})A(x-\tfrac{1}{2\pi}\overline{y})^{*}A(x+2\pi m-\tfrac{1}{2\pi}\overline{y+2\pi n})$$

$$= \tilde{A}(x+t,y+2\pi t)\tilde{A}(x,y)^{*}\tilde{A}(x+2\pi m,y+2\pi n)$$

$$= \tilde{A}(t\cdot(x,y))\tilde{A}(x,y)^{*}\tilde{A}((x,y)\cdot(2\pi m,2\pi n)).$$

Therefore a map $\psi : A \longrightarrow \tilde{A}$ from $Z^{\mathbb{C}}(\mathbb{Z};\mathbb{R};2\pi\mathbb{Z})$ into $Z^{\mathbb{C}}(\mathbb{Z};\mathbb{R};(2\pi\mathbb{Z})^{2})$ is obtained and it is easily verified that this map is an injective homomorphism.

Suppose that A_1 is cohomologous to A_2, in other words, there exists a $\mathbb{Z}$-invariant cocycle B and a $2\pi\mathbb{Z}$-invariant cocycle C such that

$$A_2(z)A_1(z)^{*} = B(z)C(z) \qquad \text{for all } z\in\mathbb{R}.$$

We will see that $\tilde{B}(x,y)$ is $\mathbb{R}$-invariant and $\tilde{C}(x,y)$ is $(2\pi\mathbb{Z})^2$-invariant and moreover

$$\tilde{A}_2(x,y)\tilde{A}_1(x,y)^{*} = \tilde{B}(x,y)\tilde{C}(x,y) \quad \text{for all } (x,y)\in\mathbb{R}^2,$$

which imply that A_1 is cohomologous to A_2. Indeed, for $t\in\mathbb{R}$ and $(x,y)\in\mathbb{R}^2$, we get

$$\tilde{B}(t\cdot(x,y))$$

$$= \tilde{B}(x+y,y+2\pi t)$$

$$= B(x+t-\tfrac{1}{2\pi}\overline{y+2\pi t})$$

$$= B(x+t-\tfrac{1}{2\pi}(y+2\pi t)+\tfrac{1}{2\pi}[y+2\pi t])$$

$$= B(x-\tfrac{1}{2\pi}y)$$

$$= B(x-\tfrac{1}{2\pi}\overline{y})$$

$$= \tilde{B}(x,y),$$

and for $(2\pi m, 2\pi n) \in (2\pi\mathbf{Z})^2$ and $(x,y) \in \mathbf{R}^2$,

$$\begin{aligned}
&\tilde{C}((x,y)\cdot(2\pi m, 2\pi n)) \\
&= \tilde{C}(x+2\pi m, y+2\pi n) \\
&= C(x+2\pi m - \tfrac{1}{2\pi}\overline{y+2\pi n}) \\
&= C(x - \tfrac{1}{2\pi}\bar{y} + 2\pi m) \\
&= C(x - \tfrac{1}{2\pi}\bar{y}) \\
&= \tilde{C}(x,y).
\end{aligned}$$

Conversely, suppose that $\tilde{A}_1$ is cohomologous to $\tilde{A}_2$. Let $B'(x,y)$ be an $\mathbf{R}$-invariant cocycle of $(\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$ and $C'(x,y)$ be a $(2\pi\mathbf{Z})^2$-invariant one, satisfying

$$\tilde{A}_2(x,y)\tilde{A}_1(x,y)^* = B'(x,y)C'(x,y) \quad \text{for all } (x,y) \in \mathbf{R}^2.$$

We define cocycle $B(z)$ and $C(z)$ of $(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$ by

$$B(z) = B'(z,0) \quad \text{and} \quad C(z) = C'(z,0) \qquad \text{for all } z \in \mathbf{R}.$$

Then we have

$$\begin{aligned}
A_2(z)A_1(z)^* &= \tilde{A}_2(z,0)\tilde{A}_1(z,0)^* \\
&= B(z)C(z) \qquad \text{for all } z \in \mathbf{R},
\end{aligned}$$

where $B(z)$ is $\mathbf{Z}$-invariant and $C(z)$ is $2\pi\mathbf{Z}$-invariant. In fact, observing that $B'(x,y)$ is $2\pi\mathbf{Z}$-invariant with respect to the second variable, as A_1, A_2, and C' are so, we have

$$\begin{aligned}
B(z+n) &= B'(z+n,0) \\
&= B'(z,-2\pi n) \\
&= B'(z,0)
\end{aligned}$$

$$= B(z)$$

and likely

$$C(z+2\pi m) = C(z) \qquad \text{for all } z\in\mathbf{R}.$$

Therefore A_1 is cohomologous to A_2.

At last, we shall show that A_1 is α-cohomologous to A_2 if and only if $\tilde{A}_1$ is β-cohomologous to $\tilde{A}_2$. To this end, by the fact shown above, it is sufficient to show that $A(z) = 1$ for α-almost all $z\in\mathbf{R}$ if and only if $\tilde{A}(x,y) = 1$ for β-almost all $(x,y)\in\mathbf{R}^2$. Let us define N, $\tilde{N}$, and N_y by

$$N = \{\, z\in\mathbf{R} \; ; \; A(z) \neq 1\} ,$$

$$\tilde{N} = \{\, (x,y)\in\mathbf{R}^2 \; ; \; A(x,y) \neq 1\} ,$$

$$N_y = \{\, x\in\mathbf{R} \; ; \; (x,y)\in\tilde{N}\}, \qquad \text{for } \; y\in\mathbf{R}.$$

Then it is verified with no trouble that

$$\alpha(N) = \alpha(N_y) \qquad \text{for all } y \; R,$$

$$\beta(N) = \int_{\mathbf{R}} \alpha(N_y)\, d\alpha(y) ,$$

so that $\alpha(N) = 0$ if and only if $\beta(\tilde{N}) = 0$.

Therefore ψ induces an isomorphism ψ_* from $H_\alpha^{\mathbf{C}}(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$ into $H_\beta^{\mathbf{C}}(\mathbf{R};\mathbf{R}^2;(2\pi\mathbf{Z})^2)$. [Q.E.D.]

By the above Lemma 2.5.4 and Proposition 2.5.3, we have the following proposition.

Proposition 2.5.5. $(\mathbf{R}/\mathbf{Z}+2\pi\mathbf{Z})+\mathbf{Q})$ is a subgroup of $H_\nu^{\mathbf{C}}(\mathbf{R};\mathbf{T}^2)$.

2.5.6. Consequences and examples

(a-3) $(\mathbb{R}/\mathbb{Z}+2\pi\mathbb{Z})+\mathbb{Q}$ is a subgroup of $H^{\mathbb{C}}_{\alpha}(\mathbb{Z};\mathbb{R};2\pi\mathbb{Z})$.

For example, if we take $A^q(z) = e^{-(i/2)qz^2}$ $(q\in\mathbb{Z})$, we get

$$C^q(n,z) = e^{(iq/2)n^2}e^{iqnz} \qquad (q\in\mathbb{Z}).$$

(b-3) $(\mathbb{R}/\mathbb{Z}+2\pi\mathbb{Z})+\mathbb{Q}$ is a subgroup of $H^{\mathbb{C}}_{\beta}(\mathbb{R};\mathbb{R}^2;(2\pi\mathbb{Z})^2)$.
If we take $\tilde{A}^q(x,y) = \tilde{A}^q(x-(1/2\pi)\bar{y}) = e^{-(i/2)q(x-(1/2\pi)\bar{y})^2}$ induced from the above A^q (q Z), we get

$$\tilde{C}^q(t,(x,y)) = \tilde{A}^q(x,y)\tilde{A}^q(t\ (x,y))^*$$
$$= e^{(q/2\pi)ix[\bar{y}+2\pi t]}e^{(q/8\pi^2)i([\bar{y}+2\pi t]-2\bar{y}[\bar{y}+2\pi t]^2)}$$

for $t\in\mathbb{R}$, and $(x,y)\in\mathbb{R}^2$. $(q\in\mathbb{Z})$

(a-4) $\mathbb{Q}$ is a subgroup of $\tilde{H}^{\mathbb{C}}(\mathbb{Z};\mathbb{R};2\pi\mathbb{Z})$.

(b-4) $\mathbb{Q}$ is a subgroup of $\tilde{H}^{\mathbb{C}}(\mathbb{R};\mathbb{R}^2;(2\pi\mathbb{Z})^2)$.

Remark 2.5.7. Furthermore, we note that the weak cohomology groups $H^{\mathbb{C}}(\mathbb{Z};\mathbb{R};2\pi\mathbb{Z})$ and $H^{\mathbb{C}}(\mathbb{R};\mathbb{R}^2;(2\pi\mathbb{Z})^2)$ have the cardinal number of uncountable infinity (see [27].).

2.6. An application to decompositions of regular representations

In this section, we consider again decompositions of regular representations of semi-direct product groups related with the cohomology groups. We assume that all notaions and situations are similar as described in section 1.3.

Let λ be the right regular representation of a semi-direct product group $G = N\times_s K$ where N and K are locally compact abelian groups. In this section, for simplicity, we assume that the Haar measure μ of $\hat{N}$ is invariant under the action of K. Then, the regular representation λ of G may be realized on the Hilbert space $L^2(\hat{N}\times K)$ as follows.

Lemma 2.6.1. For $\xi(\chi,t)\in L^2(\hat{N}\times K)$,

$$(\lambda_{(z,k)}\xi)(\chi,t) = \langle z,\chi\rangle\xi(k\cdot\chi,t-k)$$

for $(z,k)\in N\times_s K = G$.

Proof. For $\xi(g)\in L^2(G)$, put $(W_1\xi)(g) = \xi(g^{-1})$. Then, W_1 is a unitary operator on $L^2(G)$ because G is unimodular. Let F denote the Fourier transformation of $L^2(N)$ onto $L^2(\hat{N})$. Take a unitary operator W_2 from $L^2(G)$ onto $L^2(\hat{N}\times K)$ defined by $W_2 = F\otimes I$ where I is the identity operator on $L^2(K)$ and we identify $L^2(G)$ with $L^2(N)\otimes L^2(K)$ and $L^2(\hat{N}\times K)$ with $L^2(\hat{N})\otimes L^2(K)$. Then, we see that $W_2W_1\lambda_{(z,k)}W_1^*W_2^*$ is the desired one by simple calculations. [Q.E.D.]

For a cocycle $C(k,\chi)\in Z^C(K;\hat{N})$, we define a unitary representation λ^C of G by, for $\xi(\chi,t)\in L^2(\hat{N}\times K)$,

$$(\lambda^C_{(z,k)}\xi)(\chi,t) = C(k,\chi)\langle z,\chi\rangle\, \xi(k\cdot\chi,t-k)$$

for $(z,k)\in N\times_s K = G$. Then, we get the following lemma.

Lemma 2.6.2. λ is unitarily equivalent to λ^C.

Proof. The cocycle $C(k,\chi)$ may be regarded as an element of $Z^{\mathbf{C}}(K;\hat{N}\times K)$. Since the action of K on the space $\hat{N}\times K$ is smooth and effective, we see that $C(k,\chi)$ becomes a coboundary in $Z^{\mathbf{C}}(K;N\times K)$ by Theorem 2.1.1. Then, there exists an $\mathbf{T}$-valued Borel function $B(\chi,t)$ on $\hat{N}\times K$ such that, for each $k\in K$,

$$C(k,\chi) = \overline{B(\chi,t)}B(k\cdot\chi,t-k) \qquad (2.6.1)$$

for all $(\chi,t)\in\hat{N}\times K$. Take a unitary operator W given by, for $\xi(\chi,t)\in L^2(\hat{N}\times K)$,

$$(W\xi)(\chi,t) = B(\chi,t)\xi(\chi,t).$$

Then, it is easy to check that $W^*\lambda_{(z,k)}W = \lambda^C_{(z,k)}$ by the equation (2.6.1) and Lemma 2.6.1. [Q.E.D.]

In section 1.3, we gave two kinds of entirely different decompositions of λ related with ergodic measures. Here we can give other many decompositions of λ related with the cohomology group.

The Haar measure μ on $\hat{N}$ was decomposed into ergodic measures as

$$\mu = \int_Z^{\oplus} \mu_\zeta \, d\sigma(\zeta).$$

For simplicity of our arguments, we also assume that all components μ_ζ ($\zeta\in Z$) are invariant under the action of K. Then, for a cocycle $C(k,\chi)\in Z^{\mathbf{C}}(K;\hat{N})$ and $\eta\in K$, we can define a unitary representation $V^{(C,\eta,\zeta)}$ of G as follows.

For $\xi(\chi)\in L^2(\hat{N},\mu_\zeta)$,

$$(V^{(C,\eta,\zeta)}_{(z,k)}\xi)(\chi) = C(k,\chi)\langle k,\eta\rangle\langle z,\chi\rangle\xi(k\,\chi).$$

Theorem 2.6.3. The right regular representation λ of

$G = N\times_s K$ is decomposed as

$$\lambda \cong \int_{\hat{K}}^{\oplus}\int_{Z}^{\oplus} V^{(C,\eta,\zeta)}\, d\sigma(\zeta)d\nu(\eta).$$

(A) $V^{(C,\eta,\zeta)}$ $(C\in Z^{\mathbb{C}}(K;\hat{N}), \eta\in\hat{K}, \zeta\in Z)$ are irreducible.

(B) $V^{(C,\eta,\zeta)}$ is unitarily equivalent to $V^{(C',\eta',\zeta')}$ if and only if $\zeta' = \zeta$ and $C + \eta$ is μ_ζ-cohomologous to $C' + \eta'$.

Proof. We realize λ^C on the Hilbert space $L^2(\hat{N}\times\hat{K})$ as, for $\xi(\chi,\eta)\in L^2(\hat{N}\times\hat{K})$,

$$(\lambda^C_{(z,k)}\xi)(\chi,\eta) = C(k,\chi)\langle k,\eta\rangle\langle z,\chi\rangle\xi(k\cdot\chi,\eta).$$

Then, by similar arguments as in section 1.3, we get

$$\lambda^C \cong \int_{\hat{K}}^{\oplus}\int_{Z}^{\oplus} V^{(C,\eta,\zeta)}\, d\sigma(\zeta)d\nu(\eta).$$

Since $\lambda \cong \lambda^C$ by Lemma 2.6.2,

$$\lambda \cong \int_{\hat{K}}^{\oplus}\int_{Z}^{\oplus} V^{(C,\eta,\zeta)}\, d\sigma(\zeta)d\nu(\eta).$$

The properties will be obtained by the modification of the proof in Theorem 1.3.3. We omit the detail. [Q.E.D.]

By Theorem 2.6.3, we see that regular representations of some concrete non-type I groups, for example, the discrete Mautner group, the discrete Heisenberg group, the Mautner group, and the Dixmier group, have infinitely many completely different irreducible decompositions. For the detail, see [24] and [27].

Chapter III. Generalized induced representations

In this chapter, we investigate generalized induced representations for double transformation groups, related with cohomology and we construct families of non-Mackey representations of certain non-regular semi-direct product groups as a generalization of Mackey's method. Applying this construction to the Mautner group, we obtain a new parametrized family of non-Mackey representations. The representations found by L.Baggett [4] form a part of this family.

In 1978, L.Baggett found a family of non-Mackey irreducible representations of the Mautner group via the decompositions of a generalizd tensor product of some concrete representations [4]. In order to elucidate the mechanism of his family, we develop a theory of generalized induced representations in this chapter. In 1976, A.Ramsay turned the Mackey's theory into a representation theory of measure groupoids [44] and obtained a generalization of induced representations. Our notion is close to his but there are some differences. These differences will be seen to be crucial in the decomposition theory in later chapters. It is known that, for a connected and simply connected solvable Lie group G, there exists an algebraic solvable Lie group $\tilde{G}$ which contains G such that $[\tilde{G},\tilde{G}] = [G,G] = N$ and $\tilde{G}$ acts on $\hat{N}$ (the dual of N) smoothly. L.Pukanszky made an extensive use of this fact in [41], [42]. We impose similar assumption (*) for non-regular semi-direct product groups, which will be used effectively as a substitute of this fact.

In section 3.1, for a double transformation group, we define unitary representations in relation to cohomology, which appears as a generalization of the Mackey's induced representations [30], [33].

In section 3.2, following the construction in section 3.1, we have families of non-Mackey representations of non-regular semi-direct product groups satisfying a certain condition (*). In Theorem 3.2.6, we show when such representations are mutually equivalent, and in Theorem 3.2.7, we give a criterion of the irreducibility. In Proposition 3.2.9, we mention a property which characterizes such representations. The results obtained are akin to the results in [33] or [44] but ours are more precise according to the strong conditions imposed. Moreover, the techniques employed by L.Baggett [4] will be better understood from our points of view.

In section 3.3, we apply our general results to the discrete Mautner group and the Mautner group.

3.1. A generalization of induced representations

Let (G;X;H) be a double transformation group. When a continuous unitary representation L of H on a separable Hilbert space $\mathfrak{h}(L)$ is given, we construct a unitary representation of G in the following way. Let $\mathfrak{A}$ denote the commuting algebra of L, in other words, the set of all bounded oparators on $\mathfrak{h}(L)$ which commute with all L_h for $h \in H$. We take an $\mathfrak{A}^u$-valued cocycle A(x) of (G;X;H) and denote $D(h,x) = A(x)^* A(x \cdot h)$. Then D(h,x) is an $\mathfrak{A}^u$-valued cocycle of (H;X) which is G-invariant with respect to the variable $x \in X$.

We assume that $(H;X)$ is smooth. Let μ be a quasi-invariant Radon measure on X/H under the action of G. We put $\mu_g(E) = \mu(g \cdot E)$ for Borel sets E of X/H and let $\sigma(g,\dot{x})$ be a Radon-Nikodym derivative of μ_g by μ for each $g \in G$. We assume that there exists a Borel function $\tilde{\sigma}(g,\dot{x})$ on $G \times X/H$ such that for each $g \in G$, $\tilde{\sigma}(g,\dot{x}) = \sigma(g,\dot{x})$ for μ-almost all $\dot{x} \in X/H$. Put $\rho(g,x) = \tilde{\sigma}(g,\dot{x})$, then $\rho(g,x)$ is a Borel function on $G \times X$ and H-invariant with respect to the variable $x \in X$.

Let $\mathfrak{h}^A$ denote the set of all f's satisfying the following conditions.

(1) f is a weakly Borel function on X with values in $\mathfrak{h}(L)$,

(2) $f(x \cdot h) = D(h,x)^* L_h^* f(x)$ for all $x \in X$ and all $h \in H$,

(3) $\int_{X/H} \|f(x)\|^2 d\mu(\dot{x})$ is finite.

We define the inner product of $\mathfrak{h}^A$ by

$$(f;f') = \int_{X/H} (f(x),f'(x))\, d\mu(\dot{x}) \quad \text{for } f \text{ and } f' \in \mathfrak{h}^A.$$

Then it is verified by usual arguments that $\mathfrak{h}^A$ is a Hilbert space with $(\; ; \;)$. For each $g \in G$, if $f(x) \in \mathfrak{h}^A$, then $\rho(g,x)^{1/2} f(g \cdot x) \in \mathfrak{h}^A$, and they have the same norm. So we have a unitary oparator $U_g : f(x) \longrightarrow \rho(g,x)^{1/2} f(g \cdot x)$ on $\mathfrak{h}^A$.

Proposition 3.1.1. U $(g \longrightarrow U_g)$ is a continuous unitary representation of G.

Proof. This follows by routine arguments. [Q.E.D.]

We note that if $(H;X)$ is effective, $\mathfrak{h}^A$ is not empty and moreover isomorphic to $L^2(X/H;\mathfrak{h}(L);\mu)$. In fact, by the assumption that (H;X) is smooth and effective, we can decompose $x \in X$ such that $x = b(x)\cdot a(x)$ where a is a Borel function from X to H and b is a Borel function from X to a cross section of X under the action of H. Let f(x) be an $\mathfrak{h}(L)$-valued weakly Borel function on X/H such that $\int_{X/H} \|f(x)\|^2 \, d\mu(x)$ is finite. Put

$$\tilde{f}(x) = A(x)^* L^*_{a(x)} f(x) \qquad \text{for } x \in X,$$

then we get $f \in \mathfrak{h}^A$. The correspondence $f \longrightarrow \tilde{f}$ induces an isomorphism from $L^2(X/H;\mathfrak{h}(L);\mu)$ onto $\mathfrak{h}^A$.

Let us consider the case where X is a group $\tilde{G}$, H is a closed subgroup of $\tilde{G}$, and G is taken to be equal to $\tilde{G}$. Then the cocycles A are all trivial. Let μ be the canonical quasi-invariant measure on $H\backslash\tilde{G}$. Under these situations, the representation U defined in the above reduces to the ordinary induced representation of G from L (see [30]). If we take G as a closed subgroup of $\tilde{G}$, and the cocycle A to be trivial, then the above U reduces to the restriction of $\mathrm{Ind}_H^{\tilde{G}} L$ to G. Therefore we can regard the above representation U as a "generalized induced representation of G from L through (G;X;H) twisted by the cocycle A", and we denote U by $\mathrm{Ind}_H^G(\mu.A,L)$.

For A_i in $Z^{\mathfrak{A}}(G;X;H)$ we denote $\mathrm{Ind}_H^G(\mu,A_i,L)$ by U^{A_i} (i=1,2). Then we have the folowing proposition.

Proposition 3.1.2. If A_1 is cohomologous to A_2, U^{A_1} is unitarily equivalent to U^{A_2}.

Proof. From A_i, we get $\mathfrak{A}^u$-valued cocycle D_i of (H;X) by

$$D_i(h,x) = A_i(x)^* A_i(x \cdot h) \qquad \text{for } h \in H \text{ and } x \in X \quad (i=1,2).$$

Since A_1 is cohomologous to A_2, there exists an $\mathfrak{A}^u$-valued G-invariant Borel function B on X such that

$$D_2(h,x) = B(x)^* D_1(h,x) B(x \cdot h) \text{ for all } h \in H \text{ and all } x \in X.$$

Let $\mathfrak{h}^{A_i}$ be the Hilbert space corresponding to A_i. If $f(x) \in \mathfrak{h}^{A_1}$, then $\tilde{f}(x) = B(x)^* f(x)$ is in $\mathfrak{h}^{A_2}$ and $\|f\| = \|\tilde{f}\|$. Indeed, for example,

$$\begin{aligned}
&\tilde{f}(x \cdot h) \\
&= B(x \cdot h)^* f(x \cdot h) \\
&= B(x \cdot h)^* D_1(h,x)^* L_h^* f(x) \\
&= B(x \cdot h)^* D_1(h,x)^* B(x) L_h^* B(x)^* f(x) \\
&= D_2(h,x)^* L_h^* f(x) \quad \text{for all } h \in H \text{ and all } x \in X.
\end{aligned}$$

Hence we get a unitary operator $W : f(x) \longrightarrow B(x)^* f(x)$ from $\mathfrak{h}^{A_1}$ to $\mathfrak{h}^{A_2}$, and it is easy to see that

$$U_g^{A_1} = W^* U_g^{A_2} W \qquad \text{for all } g \in G.$$

This implies that U^{A_1} is unitarily equivalent to U^{A_2}. [Q.E.D.]

As a result of the above proposition, we see that U^A is defined essentialy by the cohomology class [A] in $Z^{\mathfrak{A}}(G;X;H)$. Suppose that L is a multiplicity free representation of H and a double transformation group (G;X;H) satisfies the assumption of Proposition 2.2.3. Then the elements of the three groups $H^{\mathfrak{A}}(G;X;H)$, $H^{\mathfrak{A}}(G;X/H)$, and $H^{\mathfrak{A}}(H;G\backslash X)$, which correspond one another by Proposition 2.2.3, determine a generalized induced representation of G up to equivalence.

At this moment, we shall look at the representations corresponding to the elements of $H^{\mathfrak{A}}(G;X/H)$ somewhat in detail. Take an $\mathfrak{A}^u$-valued H-invariant cocycle C of (G;X) derived from a cocycle A by $C(g,x) = A(x)A(g\cdot x)^*$. We define a representation V^C of G as follows. In place of the condition (2) in the definition of $\mathfrak{H}^A$, we put

$$(2)' \quad f(x\cdot h) = L_h^* f(x) \qquad \text{for } h\in H \text{ and } x\in X, \text{ and}$$

leave the other conditions behind. We denote the set of all such f's by $\mathfrak{H}$. Define V_g^C by, for $g\in G$,

$$V_g^C : \mathfrak{H} \ni f(x) \longrightarrow \rho(g,x)^{1/2} C(g,x) f(g\cdot x) \in \mathfrak{H}.$$

It is easily seen that $V^C(g \longrightarrow V_g^C)$ is also a unitary representation of G and it is unitarily equivalent to U^A. Moreover, it is verified with no trouble that V^{C_1} is unitarily equivalent to V^{C_2} even if C_1 is "μ"-cohomologous to C_2 as elements in $Z^{\mathfrak{A}}(G;X/H)$. In the case of Remark 2.2.4, generalized induced representations are determined up to equivalence by the mutually corresponding elements of the cohomology groups $H_\alpha^{\mathfrak{A}}(G;G;H)$, $H_\mu^{\mathfrak{A}}(G;H\ G)$, and $H_\nu^{\mathfrak{A}}(H;G/G)$ by Theorem 2.3.2.

3.2. Irreducible representations of semi-direct product groups

Let G be a semi-direct product group $N\times_s K$, where K acts on N as an automorphism group. We assume that N and K are abelian groups. The action is denoted by $N \ni z \longrightarrow k\cdot z \in N$. The element of G is written as (z,k) ($z\in N$, $k\in K$) and the multiplication is given by

$$(z,k)(z',k') = (z+k\,z',\ k+k').$$

Then an action of K on the topological space $\hat{N}$, the dual of N, is defined, for $k\in K$ and $\chi\in\hat{N}$, by

$$\langle z,\ k\cdot\chi\rangle = \langle k\cdot z,\ \chi\rangle \qquad \text{for all } z\in N.$$

So we get a topological transformation group $(K;\hat{N})$. G.W.Mackey called $G = N\times_s K$ a "regular" semi-direct product group when $(K;\hat{N})$ is smooth, and determined all irreducible representations of such a group. We shall treat mainly the case where G is not a regular semi-direct product group, and try to construct a family of non-Mackey irreducible representations of G. To do this, our main assumption is this.

(*) There is an abelian group $\tilde{K}$ containing K as closed subgroup. The group $\tilde{K}$ acts on $\hat{N}$ as an automorphism group and, as such, it is an extention of K. $\tilde{G} = N\times_s\tilde{K}$ is a regular semi-direct product group.

3.2.1. A construction of representations of G

First, we take $\chi\in\hat{N}$ and denote by H_χ, the stability group of $\tilde{K}$ at χ, i.e. the set of $t\in\tilde{K}$ such that $t\cdot\chi = \chi$. H_χ is a closed subgroup of $\tilde{K}$. Put $G_\chi = N\times_s H_\chi$. We define L^χ by

$$L^\chi_{(z,h)} = \langle z,\ \chi\rangle \qquad \text{for all } (z,h)\in G_\chi.$$

Then L^{χ} is a unitary character of G_{χ}. K and H_{χ} are closed subgroups of $\tilde{K}$, G and G_{χ} are closed subgroups of $\tilde{G}$, and so we get double transformation groups $(K;\tilde{K};H_{\chi})$ and $(G;\tilde{G};G_{\chi})$ as in Remark 2.2.4. Next, we take a $\mathbb{T}$-valued cocycle A of $(K;\tilde{K};H_{\chi})$ where $\mathbb{T}$ is the one-dimensional torus which equals $\mathbb{C}^u = \{z \in \mathbb{C};\ |z| = 1\}$. If we put

$$\tilde{A}(z,t) = A(t) \qquad \text{for all } (z,t) \in \tilde{G},$$

then $\tilde{A}$ is a $\mathbb{T}$-valued cocycle of $(G;\tilde{G};G_{\chi})$. Let μ be a Haar measure of the abelian group $H_{\chi}\backslash\tilde{K} \cong G_{\chi}\backslash\tilde{G}$. Under these preparations, we define a unitary representation $U^{(\chi,A)} = \mathrm{Ind}_{G_{\chi}}^{G}(\mu,\tilde{A},L^{\chi})$ of G, as described in section 3.1.1.

Remark 3.2.2. Since a unitary character ϕ of $\tilde{K}$ is a cocycle of $(K;\tilde{K};H_{\chi})$, we get $U^{(\chi,\phi)}$. This is a typical example, and, in the case that $G = \tilde{G}$ and therefore $K = \tilde{K}$, $U^{(\chi,\phi)}$ coincides with the representation obtained by the Mackey's method.

3.2.3. Realization of $U^{(\chi,A)}$ on $L^2(H_{\chi}\backslash\tilde{K},\mu)$

Let $C(k,t)$ be a $\mathbb{T}$-valued cocycle of $(K;\tilde{K})$ defined by $C(k,t) = A(t)A(t+k)^*$. As $C(k,t)$ is H_{χ}-invariant with respect to $t \in \tilde{K}$, we can regard it as a cocycle of $(K;H_{\chi}\backslash\tilde{K})$ and when it is considered in such a way, it is written as $C(k,x)$. Since H_{χ} is a closed subgroup of $\tilde{K}$, there exists a Borel cross section $c : x \longrightarrow c(x)$ from $H_{\chi}\backslash\tilde{K}$ to $\tilde{K}$. Then, by routine arguments, we have the following result.

For $f(x) \in L^2(H_\chi \backslash \tilde{K}, \mu)$

$$(U^{(\chi,A)}_{(z,k)}f)(x) = C(k,x)\langle z, c(x)\cdot k\rangle f(x\cdot k)$$

for all $(z,k) \in G$.

3.2.4. Realization of $U^{(\chi,A)}$ on $L^2(\hat{N},\tilde{\mu})$

The Borel map $\psi : H_\chi + t \longrightarrow t\cdot\chi$ from $H_\chi \backslash \tilde{K}$ to $\hat{N}$ induces a Radon measure $\tilde{\mu} = \psi_*(\mu)$ on $\hat{N}$ concentrated on $\mathrm{Orb}_{\tilde{K}}(\chi)$, the set$\{ t\cdot\chi \in \hat{N} \ ; \ t \in \tilde{K}\}$. We define a cocycle $C(k,\omega)$ of $(K;\hat{N})$ by

$$C(k,\omega) = \begin{cases} C(k,t) & \text{if } \omega \in \mathrm{Orb}_{\tilde{K}}(\chi) \text{ and } \omega = t\cdot\chi \\ 1 & \text{if } \omega \notin \mathrm{Orb}_{\tilde{K}}(\chi). \end{cases}$$

Then we can also realize $U^{(\chi,A)}$ on $L^2(\hat{N},\tilde{\mu})$ in the following form.

For $f(\omega) \in L^2(\hat{N},\tilde{\mu})$

$$(U^{(\chi,A)}_{(z,k)}f)(\omega) = C(k,\omega)\langle z,\omega\rangle f(k\cdot\omega) \quad \text{for all } (z,k)\in G.$$

Define an action of $\tilde{K}$ on $Z^{\mathbb{C}}(K;\tilde{K};H_\chi)$ by

$$(t\cdot A)(t') = A(t+t') \qquad \text{for } t\in\tilde{K} \text{ and all } t'\in\tilde{K},$$

and transfer this action to C in $Z^{\mathbb{C}}(K;N)$. Then we get

$$(t\cdot C)(k,\omega) = C(k,t\cdot\omega)$$
for $t\in\tilde{K}$ and all $(k,\omega)\in K\times\hat{N}$.

Let V be a unitary representation of $\tilde{K}$ on $L^2(\hat{N},\tilde{\mu})$ obtained by putting, for $f(\omega)\in L^2(\hat{N},\tilde{\mu})$,

$$(V_t f)(\omega) = f(t\cdot\omega) \qquad \text{for } t\in\tilde{K}.$$

Then we have, for $f(\omega)\in L^2(\hat{N},\tilde{\mu})$,

$$(V_t U^{(\chi,A)}_{(z,k)} V_t^* f)(\omega) = C(k,t\cdot\omega)\langle z,t\cdot\omega\rangle f(k\cdot\omega)$$
$$= (U^{(t\cdot\chi,t\cdot A)}_{(z,k)} f)(\omega) \quad \text{for all } (z,k)\in G.$$

This fact shows the following lemma.

Lemma 3.2.5. $U^{(\chi,A)}$ is unitarily equivalent to $U^{(t\cdot\chi,t\cdot A)}$ for for all $t\in\tilde{K}$.

Let α be a Haar measure of $\tilde{K}$. Then we have the following theorem.

Theorem 3.2.6. $U^{(\chi_1,A_1)}$ is unitarily equivalent to $U^{(\chi_2,A_2)}$ if and only if $\chi_2\in \mathrm{Orb}_{\tilde{K}}(\chi_1)$ and A_2 is α-cohomologous to $t_0\cdot A_1$ where $t_0\in\tilde{K}$ satisfies $\chi_2 = t_0\cdot\chi_1$.

Proof. The "if" part is easily verified by Lemma 3.2.5 and the last assertion in section 3.1. We show the "only if" part.

Suppose that $U^{(\chi_1,A_1)} = U^{(\chi_2,A_2)}$. Then, We have ${}_N|U^{(\chi_1,A_1)} = {}_N|U^{(\chi_2,A_2)}$ which are restrictions to N of $U^{(\chi_1,A_1)}$ and $U^{(\chi_2,A_2)}$. Let

$${}_N|U^{(\chi_i,A_i)} = \int_{\hat{N}}^{\oplus} \omega d\tilde{\mu}_i(\omega) \qquad (i = 1,2)$$

be the irreducible decompositions of ${}_N|U^{(\chi_i,A_i)}$, the measures μ_i being concentrated on $\mathrm{Orb}_{\tilde{K}}(\chi_i)$ by the definition of $U^{(\chi_i,A_i)}$. As such a decomposition of an abelian group is unique, we have first $\mathrm{Orb}_{\tilde{K}}(\chi_1) = \mathrm{Orb}_{\tilde{K}}(\chi_2)$. Hence there exists a $t_0\in\tilde{K}$ such that $\chi_2 = t_0\cdot\chi_1$. By Lemma 3.2.5,

$U^{(\chi_1,A_1)} \cong U^{(t_\sigma \chi_1, t_0\cdot A_1)} \cong U^{(\chi_2, t_0\cdot A_1)}$ holds, so that we have $U^{(\chi_2,t_0\cdot A_2)} \cong U^{(\chi_2, A_2)}$. Therefore if we show that $U^{(\chi,A_1)} \cong U^{(\chi,A_2)}$ implies that A_2 is α-cohomologous to A_1, the proof will be complete.

Let $U^{(\chi,A_1)}$ and $U^{(\chi,A_2)}$ be realized on $L^2(\hat{N},\tilde{\mu})$ and let W be a unitary operator on $L^2(\hat{N},\tilde{\mu})$ such that

$$U^{(\chi,A_2)}_{(z,k)} = W^* U^{(\chi,A_1)}_{(z,k)} W \qquad \text{for all } (z,k)\in G.$$

From the expressions of the operators $U^{(\chi,A)}_{(z,k)}$ in 3.2.4, we have

$$(U^{(\chi,A)}_{(z,0)} f)(\omega) = \langle z,\omega\rangle f(\omega) \qquad f(\omega)\in L^2(\hat{N},\tilde{\mu}),$$

therefore $U^{(\chi,A_1)}_{(z,0)} = U^{(\chi,A_2)}_{(z,0)}$ for all $z\in N$, and these operators generate a maximal abelian von Neumann algebra $L^\infty(\hat{N},\tilde{\mu})$ on $L^2(\hat{N},\tilde{\mu})$. Then by the condition $U^{(\chi,A_1)}_{(z,0)} W = W U^{(\chi,A_2)}_{(z,0)}$ for all $z\in N$, W must be in $L^\infty(\hat{N},\tilde{\mu})$, i.e. W is equal to a multiplicative operator $\tilde{B}(\omega)$ such that $|\tilde{B}(\omega)| = 1$. Then we see that, for $f(\omega)\in L^2(\hat{N},\tilde{\mu})$, on the one hand we get

$$(W^* U^{(\chi,A_1)}_{(0,k)} W f)(\omega) = B(\omega)^* \tilde{C}_1(k,\omega)\tilde{B}(k\cdot\omega) f(k\cdot\omega),$$

and on the other hand we get

$$(U^{(\chi,A_2)}_{(0,k)} f)(\omega) = \tilde{C}_2(k,\omega) f(k\cdot\omega),$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are the canonical cocycle of $(K;\hat{N})$ corresponding to A_1 and A_2. So we have, for each $k\in K$,

$$\tilde{C}_2(k,\omega) = \tilde{B}(\omega)^* \tilde{C}_1(k,\omega)\tilde{B}(k\cdot\omega) \quad \text{for } \mu\text{-a.a. } \omega\in\hat{N}.$$

Take a $\mathbb{T}$-valued Borel function B_0 on $H_\chi \backslash \tilde{K}$ such that

$\tilde{B}_0(\omega) = \tilde{B}(\omega)$ for $\tilde{\mu}$-almost all $\omega \in \hat{N}$, where $\tilde{B}_0$ is defined by

$$\tilde{B}_0(\omega) = \begin{cases} B_0(\dot{t}) & \text{if } \omega \in Orb_{\tilde{K}}(X) \text{ and } \omega = t \cdot X, \\ 1 & \text{if } \omega \notin Orb_{\tilde{K}}(X). \end{cases}$$

Then we see that for each $k \in K$

$$C_2(k,x) = B_0(x)^* C_1(k,x) B_0(x\ k) \text{ for } \mu\text{-a.a. } x \in H_X \backslash K,$$

in other words, C_2 is μ-cohomologous to C_1. Therefore it follows that A_2 is α-cohomologous to A_1 by Theorem 2.3.2.

[Q.E.D.]

Next we give a necessary and sufficient condition that $U^{(X,A)}$ is irreducible. Let $Orb_K(X)$ be the set $\{k \cdot X \in \hat{N} : k \in K\}$ and $Orb_{\tilde{K}}(X)$ be the set $\{t \cdot X \in \hat{N} : t \in \tilde{K}\}$, both of them are considered as a topological space with the topology induced from $\hat{N}$.

Theorem 3.2.7. $U^{(X,A)}$ is an irreducible representation of G if and only if $Orb_K(X)$ is dense in $Orb_{\tilde{K}}(X)$.

Proof. First, we show the "if" part. Since $(\tilde{K};\hat{N})$ is smooth, we see that the map $\psi : H_X + t \longrightarrow t \cdot X$ from $H_X \backslash \tilde{K}$ to $Orb_{\tilde{K}}(X)$ is homeomorphic (see[12]). The set $\psi^{-1}(Orb_K(X))$ is equal to the orbit of K on $H_X \backslash \tilde{K}$ passing the unit element of $H_X \backslash \tilde{K}$, and it is dense in $H_X \backslash \tilde{K}$ by the assumption. Moreover the action of K on $H_X \backslash \tilde{K}$ is equal to the action of $H_X \backslash H_X + K$ on $H_X \backslash \tilde{K}$ as under a subgroup. Therefore the Haar measure μ of $H_X \backslash \tilde{K}$ is invariant and ergodic under the action of K, and so is $\tilde{\mu}$ on $\hat{N}$.

Suppose that an operator W on $L^2(\hat{N},\tilde{\mu})$ satisfies

$$U^{(\chi,A)}_{(z,0)}W = WU^{(\chi,A)}_{(z,k)} \qquad \text{for all } (z,k)\in G.$$

Then we get

$$U^{(\chi,A)}_{(z,0)}W = WU^{(\chi,A)}_{(z,0)} \qquad \text{for all } z\in N,$$

so that W must be a multiplicative operator $\tilde{B}(\omega)$ in the same way as in the above proof. We observe that, for $f(\omega)\in L^2(\hat{N},\tilde{\mu})$,

$$(U^{(\chi,A)}_{(0,k)}Wf)(\omega) = \tilde{C}(k,\omega)\tilde{B}(k\ \omega)f(k\cdot\omega)$$

$$(WU^{(\chi,A)}_{(0,k)}f)(\omega) = \tilde{B}(\omega)\tilde{C}(k,\omega)f(k\cdot\omega) \qquad \text{for } k\in K.$$

Hence we get, for each $k\in K$,

$$\tilde{B}(k\cdot\omega) = \tilde{B}(\omega) \qquad \text{for } \tilde{\mu}\text{-a.a. } \omega\in\hat{N}.$$

Then

$$\tilde{B}(\omega) = \text{constant} \qquad \text{for } \tilde{\mu}\text{-a.a. } \omega\in\hat{N},$$

by the ergodicity of $\tilde{\mu}$, so that W must be a scalar operator on $L^2(\hat{N},\tilde{\mu})$. This fact implies that $U^{(\chi,A)}$ is irreducible.

The "only if" part is shown in the following way. If $U^{(\chi,A)}$ is irreducible, the measure $\tilde{\mu}$ on $\hat{N}$ corresponding to $U^{(\chi,A)}$ must be ergodic and so is μ. Moreover, the ergodicity of μ implies that $H_\chi\backslash H_\chi+K$ is a dense subgroup of $H_\chi\backslash K$ and so $Orb_K(\chi)$ is dense in $Orb_{\tilde{K}}(\chi)$ through the homeomorphism ψ.

Remark 3.2.8. The action of $\tilde{K}$ on $Z^{\mathbb{C}}(K;\tilde{K};H_\chi)$ induces naturally an action of $\tilde{K}$ on $H^{\mathbb{C}}_{\alpha}(K;\tilde{K};H_\chi)$. Let Ω be a $\tilde{K}$-invariant Borel subset of $\hat{N}$ such that $(K;\Omega\ )$ is essentially free i.e. all stability groups H_X of $\tilde{K}$ at $X \in \Omega$ are equal to H. Moreover we assume that K+H is dense in $\tilde{K}$. Then the above two theorems assert that the irreducible representations $U^{(X,A)}$ of G are parametrized by the orbits of $\Omega \times H_\alpha^{\mathbf{C}}(K;\tilde{K};H_X)$ under the action of $\tilde{K}$.

Let $C(\tilde{\mu})$ be a quasi-orbit for the action of G on $\hat{N}$ concentrated on $\mathrm{Orb}_{\tilde{K}}(X)$ for some $X \in \hat{N}$, where $\tilde{\mu} = \psi_*(\mu)$ for a Haar measure μ of $H_X \backslash \tilde{K}$ through the above map ψ from $H_X \backslash \tilde{K}$ to $\hat{N}$. If a $\mathbf{T}$-valued Borel function A on $\tilde{K}$ satisfies the condition that, for each $k \in K$ and $h \in H_X$,

$$A(k+t+h) = A(k+t)A(t)^* A(t+h) \quad \text{for } \alpha\text{-a.a. } t \in \tilde{K},$$

we call it an α-cocycle of $(K;\tilde{K};H_X)$. For an α-cocycle A, we can also define a unitary representation $U^{(X,A)}$ of G in the same way as above. Such $U^{(X,A)}$ is an irreducible representation of G restricting to $C(\tilde{\mu})$ with multiplicity one if it satisfies the assumption of Theorem 3.2.7. Conversely, we have the following proposition.

Proposition 3.2.9. If U is an irreducible representation of G restricting to $C(\tilde{\mu})$ with multiplicity one, then there exists an α-cocycle A of $(K;\tilde{K};H_X)$ such that U is unitarily equivalent to $U^{(X,A)}$.

Proof. By general results [33] about semi-direct product groups, there exists a $\mathbf{T}$-valued Borel function $C(k,\omega)$ on $K \times \hat{N}$ such that

(1) for $f(\omega)\in L^2(\hat{N},\tilde{\mu})$,

$$(U_{(z,k)}f)(\omega) = C(k,\omega)\langle z,\omega\rangle f(k\cdot\omega) \quad \text{for } (z,k)\in G,$$

(2) for each $k_1,k_2\in K$,

$$C(k_1+k_2,\omega) = C(k_1,\omega)C(k_2,k_1\cdot\omega) \quad \text{for } \tilde{\mu}\text{-a.a. } \omega\in\hat{N}.$$

Take a Borel function $D(k,t)$ on $K\times\tilde{K}$ such that $D(k,t) = C(k,\psi(\dot{t}))$. Then $D(k,t)$ is H_X-invariant with respect to $t\in\tilde{K}$ and satisfies, for each k_1,k_2 K,

$$D(k_1+k_2,t) = D(k_1,t)D(k_2,k_1+t) \quad \text{for } \alpha\text{-a.a. } t\in\tilde{K}.$$

We choose a Borel cross section $c_1: K\backslash\tilde{K} \longrightarrow \tilde{K}$ and replace t by $k+c_1(x)$ $(k\in K, x\in K\backslash\tilde{K})$ in the above expression. Then, using Fubini's theorem, we see the existence of an element k_0 in K for which, rewriting $c(x)$ instead of $k_0+c_1(x)$, we can claim

for almost all $x\in K\backslash\tilde{K}$,

$$D(k_1+k_2,c(x)) = D(k_1,c(x))D(k_2,c(x)+k_1) \qquad (3.2.1)$$

for almost all $k_1,k_2\in K$.

Put $b(t) = c(\dot{t})$ and $a(t) = t-b(t)$ for $t\in K$. We define a $\mathbf{T}$-valued Borl function $A(t)$ on K by

$$A(t) = D(a(t),b(t)).$$

Let $\tilde{D}(k,t)$ be a cocycle of $(K;\tilde{K})$ defined by $\tilde{D}(k,t) = A(t)^*A(t+k)$. Then, (3.2.1) implies that

$$\tilde{D}(k,t) = D(k,t) \qquad \text{a.a. } (k,t)\in K\times\tilde{K}.$$

However, by the fact that D and $\tilde{D}$ define the same "continuous" unitary representation of K on $L^2(\tilde{K})$, we get, for each $k \in K$

$$\tilde{D}(k,t) = D(k,t) \qquad \alpha\text{-a.a. } t \in \tilde{K}.$$

Therefore we see that, for each $k \in K$ and each $h \in H_\chi$

$$\tilde{D}(k,t+h) = D(k,t) \qquad \alpha\text{-a.a. } t \in \tilde{K}.$$

This implies that A is an α-cocycle of $(K;\tilde{K};H_\chi)$ and it is verified with no trouble that U is unitarily equivalent to $U^{(X,A)}$. [Q.E.D.]

Remark 3.2.10. By applying general considerations about cohomology [43], we may get a stronger result. Under the same assumption of Theorem 3.2.9, we can take a "cocycle" A in $Z^{\mathbb{C}}(K;\tilde{K};H_\chi)$ instead of a " α-cocycle" A as in Theorem 3.2.9.

3.3. Applications and Examples

In this section, following section 3.2 and using the results in section 2.5, we shall give new families of irreducible representations, which are non-Mackey representations, of the discrete Mautner group and the Mautner group.

Case (a) ; the discrete Mautner group.

Let G be the discrete Mautner group defined to be the semi-direct product $\mathbb{C} \times_s \mathbb{Z}$ of the additive group $\mathbb{C}$ of complex numbers with the additive group $\mathbb{Z}$ of integers, where the multiplication is given by

$$(z,n)(z',n') = (z+e^{in}z', n+n').$$

Corresponding to this group, we take the universal covering group $\tilde{G}$ of the motion group, which is defined to be the semi-direct product $\mathbf{C}\times_s \mathbf{R}$, where $\mathbf{R}$ is the additive group of real numbers and the multiplication is given by

$$(z,t)(z',t') = (z+e^{it}z',t+t').$$

We regard G as a closed subgroup of $\tilde{G}$.

At first, we take $\chi^r \in \hat{\mathbf{C}}$ such that $r\in\mathbf{R}^+$(positive real numbers) and

$$\langle z, \chi^r \rangle = e^{i(r,z)} \qquad \text{for all } z\in\mathbf{C},$$

where (,) means the real inner product in $\mathbf{C}$. Then the stability group of $\mathbf{R}$ at χ^r is equal to $2\pi\mathbf{Z}$ for all $r\in\mathbf{R}^+$. Let G_0 denote the semi-direct product $\mathbf{C}\times_s 2\pi\mathbf{Z}$, and we take a unitary character L^r of G_0 defined by

$$L^r(z,2\pi n) = \langle z, \chi^r\rangle \qquad \text{for all } (z,2\pi n)\in G_0.$$

Next, we take $A^{(\lambda,q)}\in Z^{\mathbf{C}}(\mathbf{Z};\mathbf{R};2\pi\mathbf{Z})$ where $\lambda\in\mathbf{R}$, $q\in\mathbf{Q}$ according to (a-3) of section 2.5.

Under these preparations, we get a unitary representation $U^{(r,\lambda,q)}$ of G by section 3.2 as

$$U^{(r,\lambda,q)} = \mathrm{Ind}_{G_0}^{G}(A^{(\lambda,q)};L^r) \qquad (r\in\mathbf{R},\lambda\in\mathbf{R},q\in\mathbf{Q}).$$

As $\mathbf{Z}+2\pi\mathbf{Z}$ is dense in $\mathbf{R}$, $U^{(r,\lambda,q)}$ are irreducible by Theorem 3.2.7 or Remark 3.2.8. Moreover we see that $U^{(r,\lambda,q)}$ is unitarily equivalent to $U^{(r',\lambda',q')}$ if and only if $r=r'$, $q=q'$, and $\lambda-\lambda'\in\mathbf{Z}+2\pi\mathbf{Z}$ by Theorem 3.2.6 and (a-3) of 2.5.6.

It is clear that these $U^{(r,\lambda,q)}$ are non-Mackey representations. When $q\in\mathbf{Z}$, they are reduced to those obtained by L.Baggett [4]. However, when $q\notin\mathbf{Z}$, they are "new".

Case (b) ; the Mautner group

This time, Let G denote the Mautner group defined as a semi-direct product $\mathbf{C}^2\times_s\mathbf{R}$ of the two-dimensional vecter group $\mathbf{C}^2$ on complex numbers with $\mathbf{R}$, where the multiplication is given by

$$(z,w,t)(z',w',t') = (z+e^{it}z',w+e^{2\pi it}w',t+t').$$

Corresponding to this group G, we pick up the connected and simply connected 6-dimensional algebraic solvable Lie group $\tilde{G}$, which is defined as the semi-direct product $\mathbf{C}^2\times_s\mathbf{R}^2$, where the multiplication is given by

$$(z,w,t,u)(z',w',t',u') = (z+e^{it}z',w+e^{iu}w',t+t',u+u').$$

We imbed G in $\tilde{G}$ as a closed subgroup by the following injection.

$$G\ni(z,w,t) \longrightarrow (z,w,t,2\pi t)\in\tilde{G}.$$

As in the case (a), we take at first $\chi^{(r,s)}\in\hat{\mathbf{C}}^2$ such that $r,\ s\in\mathbf{R}^+$ and

$$\langle(z,w)\ \ \chi^{(r,s)}\rangle = e^{i(r,z)}e^{i(s,w)} \quad \text{for all } (z,w)\in\mathbf{C}^2.$$

Then the stability group of $\mathbf{R}^2$ at $\chi^{(r,s)}$ is equal to $(2\pi\mathbf{Z})^2$ for any $r,\ s\in\mathbf{R}^+$. Let G_0 be the semi-direct product $\mathbf{C}^2\times_s(2\pi\mathbf{Z})^2$ of $\mathbf{C}^2$ with $(2\pi\mathbf{Z})^2$ and $L^{(r,s)}$ be a unitary character of G_0 defined by

$$L^{(r,s)}_{(z,w,2\pi m,2\pi n)} = \langle(z,w),\ \chi^{(r,s)}\rangle \quad \text{for } (z,w,2\pi m,2\pi n)\in G_0.$$

Next, we take $A^{(\lambda,q)} \in Z^{\mathbb{C}}(\mathbb{R};\mathbb{R}^2;(2\pi\mathbb{Z})^2)$ by (b-3) of 2.5.6 where $\lambda \in \mathbb{R}$ and $q \in \mathbb{Q}$.

Under these preparations, we get the following unitary representation of G (see section 3.2).

$$U^{(r,s,\lambda,q)} = \mathrm{Ind}_{G_0}^{G} (A^{(\lambda,q)}; L^{(r,s)}) ,$$

where $r,s \in \mathbb{R}^+$, $\lambda \in \mathbb{R}$, $q \in \mathbb{Q}$.
The $U^{(r,s,\lambda,q)}$ are irreducible by Theorem 3.2.7 and they are non-Mackey representations. Moreover, $U^{(r,s,\lambda,q)}$ is unitarily equivalent to $U^{(r',s',\lambda',q')}$ if and only if $r=r'$, $s=s'$ $q=q'$, and $\lambda'\lambda \in \mathbb{Z}+2\pi\mathbb{Z}$ by Theorem 3.2.6 and (b-3) of 2.5.6. In case $q \in \mathbf{Z}$, we can easily write them in a concrete form as follows.

For $f(x,y) \in L^2([0,2\pi)\times[0,2\pi))$

$$(U^{(r,s,\lambda,q)}_{(z,w,t)} f)(x,y)$$
$$= e^{i(q/2\pi)x[y+2\pi t]} e^{i(q/8\pi^2)([y+2\pi t]-2y[y+2\pi t]^2)}$$
$$e^{i\lambda t} e^{i(re^{-ix},z)} e^{i(se^{-iy},w)} f(\overline{x+t},\overline{y+2\pi t}) \quad \text{for } (z,w,t) \in G.$$

It is verified that these are unitarily equivalent to those found by L. Baggett[4]. In other cases, namely, $q \notin \mathbf{Z}$, the $U^{(r,s,\lambda,q)}$ $(r,s \in \mathbb{R}^+, \lambda \in \mathbb{R}, q \in \mathbb{Q})$ are "new" representations. According to 3.2.3 and section 2.5, it is not hard to write down these representations but they are much complicated than those shown in the above in the case $q \in \mathbf{Z}$.

Chapter IV. Decompositions of some factor representations

In this chapter, we consider the irreducible decompositions of type II factor representations of some non-regular semi-direct product groups. Taking a certain factor representation of such a group, we show it can be decomposed in many different ways into direct integrals of irreducible representations, while the diagonal algebras are spatially isomorphic each other. The explicit form of the diagonal algebra is also given.

It is well-known that irreducible decompositions of a non-type I representation are not unique in general. There are some examples which demonstrate this fact. There are about regular representations of certain groups, due to H. Yoshizawa [46], G.W. Mackey [29], A.A. Kirillov [28], and M. Saito [46], as introduced in chapter I. Moreover, M. Takesaki [50] and S. Funakoshi [14] studied decompositions of representations, related to ergodic measures.

The theory of irreducible decompositions is based on the following general result of F.I. Mautner [36]. Let G be a locally compact group and π be a unitary representation of G on a separable Hilbert sapce $\mathfrak{h}$. Suppose that $\mathfrak{A}$ is an abelian von Neumann subalgebra in $\pi(G)'$. Then, there exists a standard measure space (Y, ν) such that $\mathfrak{A}$ is algebraically isomorphic with $L^{\infty}(Y, \nu)$ and π is decomposed as follows.

$$\pi \cong \int_Y^{\oplus} \pi^{\eta} d\nu(\eta) .$$

Moreover, π^{η} is irreducible for ν-almost all $\eta \in Y$ if and only if $\mathcal{O}$ is maximal abelian in $\pi(G)'$.

In the chapter, we consider the irreducible decompositions of type II factor representations of some non-regular semi-direct product group. In Theorem 4.1.3, a certain representation π^{χ} of such a group G will be decomposed in an explicit way to a direct integral of irreducible representations, each component having a definite form. The corresponding to maximal abelian von Neumann subalgebra in $\pi^{\chi}(G)'$ is also described in a concrete form.

It is known that the non-type I'ness of a locally compact group or a C^*-algebra is closely related to the non-smoothness of topological transformation groups [12], [15], [16]. In non-smooth topological transformation groups, there are various kinds of quasi-orbits and the cohomology group for each non-transitive quasi-orbit seems to be huge, at least it is known to be non-trivial under some conditions [38]. The non-uniqueness of decompositions of a non-type I representation seems to depend deeply on these two facts. The results in [14] and [50] are certainly connected with the former and the examples in [28], [29], and section 1.3 also seem to be so intrincically. The present chapter is an attempt to describe the relation of the non-uniqueness of decompositions with the latter phenomenon.

In chapter II, we studied the cohomology groups of double transformation groups, related with irreducible representations of some non-regular semi-direct product groups. The decomposition in Theorem 4.1.3 is done by using a cocycle of this cohomology theory, and it is shown in Proposition 4.2.1 that two decompositions are completely different when the used cocycles are not weakly cohomologous, whereas the diagonal algebras are spatially isomorphic each other. Thus we may get a large number of different decompositions of a given representation into irreducible components, an observation which will be new. To illustrate various possibilities, we give two examples in section 4.3.

4.1. Decompositions of π^X

Let G be a semi-direct product group $N \times_s K$, where K acts on N as an automorphism group. We assume that N and K are locally compact abelian groups which satisfy the axiom of second countability. The action is denoted by $N \ni z \to k \cdot z \in N$ for $k \in K$. The element of G is written as (z, k) $(z \in N,\ k \in K)$ and the multiplication is given by $(z, k)(z', k') = (z+k\cdot z', k+k')$. Then, an action of K on the topological space $\hat{N}$ (the dual of N) is defined, for $k \in K$ and $\omega \in \hat{N}$, by $\langle z, k\cdot\omega\rangle = \langle k\cdot z, \omega\rangle$ for all $z \in N$. So we get a topological transformation group $(K; \hat{N})$ which satisfy $k_2\cdot(k_1\cdot\omega) = (k_1 k_2)\cdot\omega$ for k_1, $k_2 \in K$ and $\omega \in \hat{N}$. $G = N \times_s K$ is called a "regular" semi-direct product group when $(K; \hat{N})$ is smooth, namely, when each orbit of $\hat{N}$ under the action of K is locally closed (see [12]). We treat mainly the case where G is not a regular semi-direct product group. However, we assume the following conditon (*).

(*) There is a locally compact abelian group $\tilde{K}$ containing K as a closed subgroup. The group $\tilde{K}$ acts on N as an automorphism group and, as such, it is an extension of K. $\tilde{G} = N \times_s \tilde{K}$ is a regular semi-direct product group.

For a unitary character χ of N, we get a unitary representation π^χ of G, defined by $\pi^\chi = {}_G|\mathrm{Ind}_N^{\tilde{G}}\,\chi$ which is the restriction to G of the representation of $\tilde{G}$ induced by χ from N. Let H_χ denote the stability group of $\tilde{K}$ at χ and let $\mathbf{T}$ be the one-dimensional torus group. When a $\mathbf{T}$-valued Borel function A on $\tilde{K}$ satisfies $A(k+t+h) = A(k+t)\overline{A(t)}A(t+h)$ for all $k \in K$, $t \in \tilde{K}$, and $h \in H_\chi$, we call it a cocycle of the double transformation group $(K;\ \tilde{K};\ H_\chi)$ (see section 2.2). Using this cocycle A, we get a cocycle C^A of $(H_\chi;\ \tilde{K})$ and a cocycle D^A of $(K;\ \tilde{K})$ by

$$C^A(h,\ t) = \overline{A(t)}A(t+h)$$

$$D^A(k,\ t) = A(t)\overline{A(t+k)}$$

for $h \in H_\chi$, $k \in K$, and $t \in \tilde{K}$. We note that $C^A(h,t)$ is K-invariant and $D^A(k,t)$ is H_χ-nvariant with respect to $t \in \tilde{K}$.

In this chapter, we consider the representation π^χ of G and, corresponding to each cocycle A, we give decompositions of π^χ and the abelian von Neumann algebra in $\pi^\chi(G)'$.

According to the Mackey's theory of induced representations [30], we get a canonical decomposition of π^χ as follows.

$$\pi^\chi = {}_G|\mathrm{Ind}_N^{\tilde{G}}\,\chi$$

$$\cong {}_G|\mathrm{Ind}_{G_\chi}^{\tilde{G}}(\mathrm{Ind}_N^{G_\chi}\,\chi) \quad ,\text{ where } G_\chi = N \times_s H_\chi,$$

$$\cong {}_G\Big| \mathrm{Ind}_{G_\chi}^{\tilde{G}} \Big(\int_{\hat{H}_\chi}^{\oplus} L^{(\chi,\eta)} \, d\mu(\eta) \Big)$$

$$\cong \int_{H_\chi}^{\oplus} \Big({}_G\Big| \mathrm{Ind}_{G_\chi}^{\tilde{G}} L^{(\chi,\eta)} \Big) \, d\mu(\eta) ,$$

where μ is a Haar measure of $\hat{H}_\chi$ and $L^{(\chi,\eta)}$ ($\eta \in \hat{H}_\chi$) is a unitary character of G_χ , defined by $L^{(\chi,\eta)}_{(z,h)} = \langle z,\chi\rangle\langle h,\eta\rangle$ for $(z,h) \in G_\chi$.

Now, we consider the following problems.

(a) How do we get decompositions of π^χ which are completely different from the above one ?

(b) What is the abelian von Neumann algebra which gives rise to each decomposition ?

At first, we will have a realization of π^χ in the following way.

Lemma 4.1.1. π^χ is realized on $L^2(\tilde{K})$ by

$$(\pi^\chi_{(z,k)} \xi)(t) = \langle z, t\cdot\chi\rangle \, \xi(t+k)$$

for $\xi(t) \in L^2(\tilde{K})$ and $(z,k) \in G$.

Proof. This follows from simple calculations (see [30]). [Q.E.D.]

Next, for an arbitrary cocycle A of the double transformation group $(K; \tilde{K}; H_\chi)$, a unitary operator λ^A_h on $L^2(\tilde{K})$ for $h \in H_\chi$ is defined as follows. For $\xi(t) \in L^2(\tilde{K})$, $(\lambda^A_h \, \xi)(t) = c^A(h,t)\xi(t+h)$. λ^A is a unitary representation of H_χ . We denote by $\mathfrak{A}^A$ the von Neumann algebra generated by λ^A_h for all $h \in H_\chi$.

Lemma 4.1.2. $\mathfrak{A}^A$ is an abelian von Neumann algebra in $\pi^\chi(G)'$ on $L^2(\tilde{K})$.

Proof. Since H_χ is an abelian subgroup of $\tilde{K}$, it is clear that $\mathfrak{A}^A$ is abelian. Therefore, it is sufficient to show that

$$\pi^\chi(z,k)\lambda_h^A = \lambda_h^A \pi^\chi(z,k)$$

for all $(z,k) \in G$ and all $h \in H_\chi$. For $\xi(t) \in L^2(\tilde{K})$, we have, on the one hand,

$$\begin{aligned}
&(\pi^\chi_{(z,k)}\lambda_h^A \xi)(t) \\
&= \langle z,\ t\cdot\chi\rangle\ (\lambda_h^A\xi)(t+k) \\
&= \langle z,\ t\cdot\chi\rangle C^A(h,\ t+k)\xi(t+k+h) \\
&= \langle z,\ t\cdot\chi\rangle C^A(h,t)\xi(t+h+k),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(\lambda_h^A\pi^\chi_{(z,k)}\xi)(t) \\
&= C^A(h,t)(\pi^\chi_{(z,k)}\xi)(t+h) \\
&= C^A(h,t)\ \langle z,\ (t+h)\cdot\chi\rangle\xi(t+h+k) \\
&= C^A(h,t)\ \langle z,\ t\cdot\chi\rangle\ \xi(t+h+k).
\end{aligned}$$

[Q.E.D.]

Now, to practice a decomposition of π^χ according to $\mathfrak{A}^A$, we prepare a family of non-Mackey representations $U^{(\chi,A,\eta)}$ ($\eta \in \hat{H}_\chi$) of $G = N \times_s K$ following the way in chapter III. Let $L^{(\chi,\eta)}$ ($\eta \in \hat{H}_\chi$) be unitary characters of $G_\chi = N \times_s H_\chi$, defined by

$$L^{(\chi,\eta)}_{(z,h)} = \langle z,\chi\rangle \langle h,\eta\rangle$$

for $(z,h) \in G_\chi$. Then we can construct a unitary representation $U^{(\chi,A,\eta)}$ of G by

$$U^{(\chi,A,\eta)} = \mathrm{Ind}_{G_\chi}^{G}(A; L^{(\chi,\eta)})$$

When A is trivial,

$$U^{(\chi,A,\eta)} = {}_G\big|\,\mathrm{Ind}_{G_\chi}^{\tilde{G}} L^{(\chi,\eta)} .$$

Our main theorem is the following. It shows an explicit decomposition of π^χ corresponding to $\mathfrak{A}^A$ in $\pi^\chi(G)'$, which will offer one answer of (a) and (b).

Theorem 4.1.3. Corresponding to $\mathfrak{A}^A$, the unitary representation π^χ of $G=N\times_s K$ is decomposed as follows.

$$\pi^\chi \cong \int_{\hat{H}_\chi}^{\oplus} U^{(\chi,A,\eta)}\, d\mu(\eta)$$

where μ is a Haar measure of $\hat{H}_\chi$.

Before going into the proof of the theorem, we will say about $U^{(\chi,A,\eta)}$ in more detail. Define the action of K on $\tilde{K}/H_\chi$ by $k\cdot x = \overline{k+t}$ for $k \in K$ and $x = \bar{t} \in \tilde{K}/H_\chi$. Then, $t\cdot\chi$ in $\hat{N}$ may be written $x\cdot\chi$ for $x = \bar{t} \in \tilde{K}/H_\chi$ because H_χ is the stability group of $\tilde{K}$ at $\chi \in \hat{N}$. Since a cocycle $D^A(k,t) = A(t)\overline{A(t+k)}$ of $(K; \tilde{K})$ is H_χ-invariant, $\tilde{D}^A(k,\bar{t}) = D^A(k, t)$ is a cocycle of $(K; \tilde{K}/H_\chi)$. We note that an element of $\hat{\tilde{K}}$ can be regarded as a cocycle of $(K; \tilde{K}; H_\chi)$.

Lemma 4.1.4. $U^{(\chi,A,\eta)}$ is unitarily equivalent to $U^{(\chi,A+\tilde{\eta},0)}$. It can be realized on $L^2(\tilde{K}/H_\chi)$ as follows.

For $\xi(x) \in L^2(\tilde{K}/H_\chi)$,

$$(U^{(\chi,A,\eta)}_{(z,k)}\xi)(x) = \langle z, x\cdot\chi\rangle\langle k, \eta\rangle \tilde{D}^A(k,x)\xi(k,x) \quad \text{for}$$

$(z,k) \in G$, where $\tilde{\eta}$ is an extension of $\eta \in \hat{H}_\chi$ to $\tilde{K}$.

Proof. First we identify the representation space of $U^{(\chi,A,\eta)}$ with the space of $U^{(\chi,A+\tilde{\eta},0)}$. To do this, we have only to check that the conditions of a complex valued Borel function f on $\tilde{G}$ to belong to the sapce of $U^{(\chi,A,\eta)}$ and the space of $U^{(\chi,A+\tilde{\eta},\, 0)}$ are the same (see section 3.2). Indeed, for $(z',h) \in G_\chi$ and $(z,t) \in \tilde{G}$,

$$\begin{aligned} & f((z',h)(z,t)) \\ &= L^{(\chi,\eta)*}_{(z',h)} A(t)\overline{A(t+h)} f((z,t)) \\ &= \overline{\langle z', \chi\rangle}\ \overline{\langle h, \eta\rangle}\, A(t)\overline{A(t+h)} f((z,t)) \\ &= L^{(\chi,0)*}_{(z',h)} (A+\tilde{\eta})(t)\overline{(A+\tilde{\eta})(t+h)} f((z,t)). \end{aligned}$$

By section 3.1, it is easy to see that $U^{(\chi,A+\tilde{\eta},0)}$ is realized on $L^2(\tilde{K}/H_\chi)$ as the above form. [Q.E.D]

Proof of Theorem 4.1.3. In order to prove Theorem 4.1.3, we shall take four unitary operators W_1, W_2, W_3, and W_4 in order and transform the representation space $L^2(\tilde{K})$ to the suitable one.

In the first place, we put $(W_1\xi)(t) = A(t)\xi(t)$ for $\xi(t) \in L^2(\tilde{K})$.

Next, take a Borel cross section c from $\tilde{K}/H_\chi$ to $\tilde{K}$. Then, by choosing suitable Haar measures of H_χ and $\tilde{K}/H_\chi$, we can define a unitary operator W_2 from $L^2(\hat{K})$ to $L^2(H_\chi)\otimes$

$L^2(\widetilde{K}/H_\chi)$ satisfying, for $h \in H_\chi$ and $x \in \widetilde{K}/H_\chi$,

$$(W_2\xi)(h, x) = \xi(h + c(x))$$

for $\xi(t) \in L^2(\widetilde{K})$.

Further, we denote by W_3 a unitary operator $F \otimes I$ from $L^2(H_\chi) \otimes L^2(\widetilde{K}/H_\chi)$ to $L^2(\widehat{H}_\chi) \otimes L^2(\widetilde{K}/H_\chi)$ where F is the Fourier transformation from $L^2(H_\chi)$ to $L^2(\widehat{H}_\chi)$ and I is the identity operator on $L^2(\widetilde{K}/H_\chi)$.

Let $H_\chi^\perp$ denote the annihilator of H_χ in $\widetilde{K}$. Then, by Pontrjiagin's duality, $H_\chi^\perp$ is isomorphic with $\widehat{\widetilde{K}}/H_\chi^\perp$. Via a Borel cross section from $\widehat{\widetilde{K}}/H_\chi^\perp$ to $\widehat{K}$, we can define a Borel extension $\tilde{\eta}$ $(\eta \in \widehat{H}_\chi)$ from $H_\chi^\perp$ to $\widehat{\widetilde{K}}$. So, we can define a unitary operator W_4 on $L^2(\widehat{H}_\chi) \otimes L^2(\widetilde{K}/H_\chi)$ by

$$(W_4\xi)(\eta,x) = \langle c(x), \tilde{\eta}\rangle\, \xi(\eta,x)$$

for $\xi(\eta,x) \in L^2(\widehat{H}_\chi) \otimes L^2(\widetilde{K}/H_\chi)$.

Now, we denote by W the operator $W_4 \circ W_3 \circ W_2 \circ W_1$, which is a unitary operator from $L^2(\widetilde{K})$ to $L^2(\widehat{H}_\chi) \otimes L^2(\widetilde{K}/H_\chi)$.

After some calculations, we get, for $\tau(\eta) \otimes \xi(x) \in L^2(\widehat{H}_\chi) \otimes L^2(\widetilde{K}/H_\chi)$,

$$W\lambda_h^A W^* \ ; \quad \tau(\eta)\otimes\xi(x)$$

$$\longrightarrow \langle h,\eta\rangle\, \tau(\eta)\otimes\xi(x)$$

and

$$W\pi^\chi_{(z,k)} W^* : \tau(\eta)\otimes\xi(x)$$

$$\longrightarrow \tau(\eta)\otimes\langle k,\tilde{\eta}\rangle\langle z, x\cdot\chi\rangle \widetilde{D}^A(k,x)\,\xi(k\ x)$$

$$= \tau(\eta)\otimes(U^{(\chi,A,\eta)}_{(z,k)}\,\xi)(x) \quad \text{(see Lemma 4.1.4)}.$$

It is clear that $\{ W\lambda_h^A W^* \ ; \ h \in H_\chi \}$ generates an abelian von Neumann algebra $L^\infty(\hat{H}_\chi) \otimes \mathbb{C}_{L^2(\tilde{K}/H_\chi)}$ which is spatially isomorphic with $\mathfrak{A}^A$. Hence, corresponding to this algebra, we can decompose the Hilbert space $L^2(\hat{H}_\chi) \otimes L^2(\tilde{K}/H_\chi)$ to $\int_{\hat{H}_\chi}^{\oplus} \mathfrak{H}^\eta \, d\mu(\eta)$ where μ is the Haar measure of $\hat{H}_\chi$ and $\mathfrak{H}^\eta$ is constantly equal to $L^2(\tilde{K}/H_\chi)$. By this decomposition, we get,

$$\lambda_h^A \cong \int_{\hat{H}_\chi}^{\oplus} \langle h, \eta \rangle \, d\mu(\eta)$$

and

$$\pi_{(z,k)}^\chi \cong \int_{\hat{H}_\chi}^{\oplus} U_{(z,k)}^{(\chi,A,\eta)} \, d\mu(\eta) . \qquad \text{[Q.E.D.]}$$

4.2. Properties of π^χ and $U^{(\chi,A,\eta)}$

In this section, we investigate the properties of the representations π^χ and $U^{(\chi,A,\eta)}$ of $G = N \times_s K$. In chapter II, we studied the cohomology group of double transformation groups. For the double transformation group $(K; \tilde{K}; H_\chi)$, we denote the abelian group of all $\mathbf{T}$-valued cocycles of $(K; \tilde{K}; H_\chi)$ by $Z(K; \tilde{K}; H_\chi)$. A cocycle A is called a coboundary if there exist an H_χ-invariant cocycle E and K-invariant cocycle F such that $A(t)=E(t)F(t)$ for almost all $t \in \tilde{K}$. The subgroup of all coboundaries in $Z(K; \tilde{K}; H_\chi)$ is denoted by $B(K; \tilde{K}; H_\chi)$ Then, we define the cohomology group $H(K; \tilde{K}; H_\chi)$, as the quotient group of $Z(K; \tilde{K}; H_\chi)$ by $B(K; \tilde{K}; H_\chi)$. Since the group of unitary characters of $\tilde{K}$, namely $\hat{\tilde{K}}$, is considered as a subgroup of $Z(K; \tilde{K}; H_\chi)$, we may make the factor group $\hat{\tilde{K}}/(\hat{\tilde{K}} \cap B(K; \tilde{K}; H_\chi))$, denoyed by $H_0(K; \tilde{K}; H_\chi)$. We denote by

$\tilde{H}(K;\ \tilde{K};\ H_\chi)$ the $\mathbb{T}$-valued weak cohomology group of $(K;\ \tilde{K};\ H_\chi)$ which is defined by $H(K;\ \tilde{K};\ H_\chi)/H_0(K;\ \tilde{K};\ H_\chi)$. For the detail, see chapter II.

By Theorem 3.2.6 and Lemma 4.1.4, we get immediately the following proposition.

Proposition 4.2.1. If a cocycle A is not weakly cohomologous to a cocycle A', then, for any choice of η and η' in $\hat{H}_\chi$, $U^{(\chi,A,\eta)}$ is never unitarily equivalent to $U^{(\chi,A',\eta')}$.

Thus, we see that Theorem 4.1.3. gives at least as many completely different decompositions of π^χ as the elements of $\tilde{H}(K;\ \tilde{K};\ H_\chi)$. Note that the abelian von Neumann algebras $\mathfrak{A}^A$ are mutually spatially isomorphic. The emphasis may be put on this fact, as this possibility has never been pointed out before.

In section 3.2, we studied the irreducibility of $U^{(\chi,A,\eta)}$.

Proposition 4.2.2. If $K + H_\chi$ is dense in $\tilde{K}$, $U^{(\chi,A,\eta)}$ $(\eta \in \hat{H}_\chi)$ are irreducible representations of G.

Therefore, we get the following proposition.

Proposition 4.2.3. If $K + H_\chi$ is dense in $\tilde{K}$, $\mathfrak{A}^A$ is a maximal abelian subalgebra in $\pi^\chi(G)'$ for each $A \in Z(K;\ \tilde{K};\ H_\chi)$.

When we take a unitary character ζ of $\tilde{K}$ as a cocycle of

$(K; \tilde{K}; H_\chi)$ and we consider the representation $U^{(\chi,\zeta,0)}$, simply denoted by $U^{(\chi,\zeta)}$, we see the following lemma by Theorem 3.2.6 and Proposition 2.4.6.

Lemma 4.2.4. $U^{(\chi,\zeta)}$ is unitarily equivalent to $U^{(\chi,\zeta')}$ if and only if $\zeta' - \zeta \in K^\perp + H_\chi^\perp$.

Note that $U^{(\chi,\tilde{\eta})} = U^{(\chi,0,\eta)}$ for $\eta \in \hat{H}_\chi$ where $\hat{H}_\chi \ni \eta \to \tilde{\eta} \in \hat{\tilde{K}}$ is a Borel extension map. Then, Theorem 4.1.3 asserts that

$$\pi^\chi \cong \int_{\hat{H}_\chi}^{\oplus} U^{(\chi,\tilde{\eta})}\, d\mu(\eta)$$

by considering the case that a cocycle A is trivial.

Next, we define unitary representations $\tilde{\pi}^\chi$ and $\tilde{\tilde{\pi}}^\chi$ of $G = N \times_s K$ as follows.

$$\tilde{\pi}^\chi = \int_{\hat{K}}^{\oplus} U^{(\chi,\tilde{\sigma})}\, d\nu(\sigma)$$

where ν is a Haar measure of $\hat{K}$ and $\hat{K} \ni \sigma \longrightarrow \tilde{\sigma} \in \hat{\tilde{K}}$ is a Borel extension map.

$$\tilde{\tilde{\pi}}^\chi = \int_{\hat{\tilde{K}}}^{\oplus} U^{(\chi,\zeta)}\, d\gamma(\zeta)$$

where γ is a Haar measure of $\hat{\tilde{K}}$.

Proposition 4.2.5. π^χ , $\tilde{\pi}^\chi$, and $\tilde{\tilde{\pi}}^\chi$ are mutually quasi-equivalent. Therefore, $\pi^\chi(G)''$, $\tilde{\pi}^\chi(G)''$, and $\tilde{\tilde{\pi}}^\chi(G)''$ are algebraically isomorphic each other.

Proof. This follows immedictely from Proposition 2.4.6

[Q.E.D]

We denote by $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ a von Neumann algebra obtained as a crossed product of $L^\infty(\tilde{K}/H_\chi)$ with K by the canonical action α of K on $L^\infty(\tilde{K}/H_\chi)$. Then, we have the following.

Lemma 4.2.6. $\tilde{\pi}^\chi(G)''$ is spatially isomorphic with $L^\infty(\hat{K}/H_\chi) \times_\alpha K$.

Proof. π^χ is realized on $L^2(K) \quad L^2(K/H_\chi)$ as follows. For $\tau(\sigma)\otimes\xi(x) \in L^2(\hat{K}) \otimes L^2(\tilde{K}/H_\chi)$,

$$\tilde{\pi}^\chi_{(z,k)} : \tau(\sigma) \otimes \xi(x)$$
$$\longrightarrow \langle k,\sigma\rangle \tau(\sigma) \otimes \langle z, x\cdot\chi\rangle \xi(k\cdot x)$$

for $(z,k) \in G$. Let W be a unitary operator $F \otimes I$ from $L^2(\hat{K})\otimes L^2(\tilde{K}/H_\chi)$ to $L^2(K) \otimes L^2(\tilde{K}/H_\chi)$ where F is the Fourier transformation from $L^2(\hat{K})$ to $L^2(K)$. Then, by simple calculations, we get, for $\rho(s) \otimes \xi(x) \in L^2(K)\otimes L^2(\tilde{K}/H_\chi)$,

$$W \tilde{\pi}^\chi_{(z,k)} W^* : \rho(s) \otimes \xi(x)$$
$$\longrightarrow \rho(s+k)\otimes \langle z, x\cdot\chi\rangle \xi(k\cdot x).$$

It is clear that the set of $\psi_z : x \longrightarrow \langle z, x\cdot\chi\rangle (z \in N)$ generates an abelian von Neumann algebra $L^\infty(\tilde{K}/H_\chi)$ on $L^2(K/H_\chi)$. Therefore, we see that $W \tilde{\pi}^\chi(G)W^*$ generates $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$.

[Q.E.D.]

Lemma 4.2.7. If $K + H_\chi$ is dense in $\tilde{K}$ and $K \cap H_\chi = \{0\}$, $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ is a factor. Under these assumptions, $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ is an injective type II factor if and only if

$K + H_\chi \neq \tilde{K}$.

Proof. The assumptions that $K + H_\chi$ is dense in $\tilde{K}$ and $K \cap H_\chi = \{0\}$ means that the action α of K on $\tilde{K}/H_\chi$ is ergodic and free. Then, it is clear that $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ is a factor under these assumptions [18]. Further, $K + H_\chi = \tilde{K}$ if and only if the action α of K on $\tilde{K}/H_\chi$ is transitive. Therefore, $K + H_\chi \neq \tilde{K}$ if and only if $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ is a non-type I factor. Since the measure on $\tilde{K}/H_\chi$ is K-invariant, $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$ must be an injective type II factor (see [6] and [18]).

[Q.E.D.]

Combining Proposition 4.2.5 with Lemma 4.2.6 and Lemma 4.2.7, we get the following theorem.

Theorem 4.2.8. $\pi^\chi(G)''$ is algebraically isomorphic with $L^\infty(\tilde{K}/H_\chi) \times_\alpha K$. If $K + H_\chi$ is dense in $\tilde{K}$ and $K \cap H_\chi = \{0\}$, π^χ is a factor representation. Under these assumptions, π^χ is an injective type II factor representation if and only if $K + H_\chi \neq \tilde{K}$.

Remark 4.2.9. We are interested in the case that $K + H_\chi$ is dense in $\tilde{K}$ and not equal to $\tilde{K}$. Under this situation, $U^{(\chi, A, \eta)}$ is a non-Mackey irreducible representation of G and π^χ is a non-type I representation so that G is not a type I group. Moreover, we have got the result $H(K; \tilde{K}; H_\chi) \supset H_0(K; \tilde{K}; H_\chi) \cong \hat{\tilde{K}}/(K^\perp + H_\chi^\perp)$ in Proposition 2.4.6, which is stronger than the general result in [38] that $H(K; \tilde{K}; H_\chi) \neq \{0\}$. In some cases, we know that $\tilde{H}(K; \tilde{K}; H_\chi) \supset \mathbb{Q}$ where $\mathbb{Q}$ is the set of rational numbers (see (a-4) and (b-4) in section 2.5).

Remark 4.2.10. By using Theorem 12.1 in [30], we can get another decomposition of π^{χ}. Let ψ be the Borel map from $\tilde{K}$ onto $\mathrm{Orb}_{\tilde{K}}(\chi)$ in $\hat{N}$, defined by $\psi(t) = t\cdot\chi$. Put $V^{(\chi,t)} = \mathrm{Ind}_N^G\psi(t)$ $(t\in\tilde{K})$. Then, it is clear that $V^{(\chi,t)}$ $V^{(\chi,t')}$ if and only if $t'-t\in K + H_\chi$. Take a Borel cross section c from $\tilde{K}/K$ to $\tilde{K}$. Then, $\tilde{K}/K \ni y \longrightarrow V^{(\chi,c(y))}$ is measurable and it is seen that

$$\pi^{\chi} \cong \int_{\tilde{K}/K}^{\oplus} V^{(\chi,c(y))}\, d\nu(y) ,$$

where ν is a Haar measure of $\tilde{K}/K$. If $K\cap H_\chi = \{0\}$, $V^{(\chi,t)} = \mathrm{Ind}_N^G\psi(t)$ is Mackey irreducible representation of G. If $K + H_\chi$ is dense in $\tilde{K}$ and $K + H_\chi \neq \tilde{K}$, $V^{(\chi,t)}$ $(t\in\tilde{K})$ are never unitarily equivalent to $U^{(\chi,A,\eta)}$ for whatever $A\in Z(K;\tilde{K};H_\chi)$ and $\eta\in\hat{H}_\chi$. When, we observe these phenomena from the view point of group C^*- algbera of G, it is easier to understand. We will show it in the subsequent chapter.

4.3. Applications and examples

Here, we consider the discrete Mautner group and the Mautner group, both of which are non-regular semi-direct product groups and satisfy the condition (*) in section 4.1. We denote the additive groups of complex numbers, real numbers, rational numbers, and integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, respectively.

Example 4.3.1: the discrete Mautner group

Let $\tilde{G}$ be the semi-direct product $\mathbb{C} \times_s \mathbb{R}$, where the multiplications is defined by $(z,t)(z',t') = (z+e^{it}z', t+t')$ for $z, z' \in \mathbb{C}$ and $t, t' \in \mathbb{R}$. This group $\tilde{G}$ is the universal covering group of 3-dimensional motion group and a regular semi-direct product group. We take a closed subgroup G of $\tilde{G}$, given by

$$G = \{(z,n) \in \tilde{G}; \quad z \in \mathbb{C} \text{ and } n \in \mathbf{Z}\}.$$

$G = \mathbb{C} \times_s \mathbf{Z}$ is the discrete Mautner group.

We take a unitary character χ^r of $\mathbb{C}$ $(r \in \mathbb{R}^+)$, defined by $\langle z, \chi^r \rangle = e^{i(r,z)}$ for $z \in \mathbb{C}$, where $(\ , \ )$ means the real inner product in $\mathbb{C}$. The stability group H_r at χ^r is $2\pi\mathbf{Z}$ for all $r \in \mathbb{R}^+$. Put $G_0 = \mathbb{C} \times_s 2\pi\mathbf{Z} = \mathbb{C} \times 2\pi\mathbf{Z}$. Then, the unitary representations $\pi^r (r \in \mathbb{R}^+)$ and $U^{(r,q,\lambda)}$ $(r \in \mathbb{R}^+, q \in \mathbb{Q}, \lambda \in [0,1))$ are as follows.

$$\pi^r = {}_G\Big|\operatorname{Ind}_{\mathbb{C}}^{\tilde{G}} \chi^r$$

$$U^{(r,q,\lambda)} = \operatorname{Ind}_{G_0}^{G}(A^q; \chi^r \times \eta^\lambda) \quad \text{(See chapter III)}$$

where $\eta^\lambda (\lambda \in [0, 1))$ are unitary characters of $2\pi\mathbf{Z}$, defined by $\eta^\lambda(2\pi n) = e^{2\pi i\lambda n}$ and A^q $(q \in \mathbb{Q})$ are cocycles of $(\mathbf{Z};\mathbb{R};2\pi\mathbf{Z})$ given in 2.5.6. We know in section 2.5 that A^q is weakly cohomologous to $A^{q'}$ if and only if $q = q'$.

Now, we get, by Theorem 4.1.3,

$$\pi^r \cong \int_0^{1} \oplus \, U^{(r,q,\lambda)} \, d\mu(\lambda) \qquad \text{for each } q \in \mathbb{Q},$$

where μ is the Lebesgue measure of $[0, 1)$. Since $\mathbf{Z} + 2\pi\mathbf{Z}$

is dense in $\mathbf{R}$, $U^{(r,q,\lambda)}$ are irreducible representations of G by Theorem 3.2.6. If $q \neq q'$, $U^{(r,q,\lambda)}$ is never unitarily equivalent to $U^{(r,q',\lambda')}$ for any $\lambda, \lambda' \in [0, 1)$ by Proposition 4.2.1. Moreover, $\pi^r(G)''$ is algebraically isomorphic with $L^\infty(\mathbf{T}) \times_\alpha \mathbf{Z}$, so that π^r is an injective type II_1 factor representation by Theorem 4.2.8.

Example 4.3.2: the Mautner group

Let G denote the Mautner group which is the semi-direct product group $\mathbf{C}^2 \times_s \mathbf{R}$ with the multiplication

$$(z,w,t)(z',w',t') = (z+e^{it}z', \ w+e^{2\pi it}w', \ t+t')$$

for $z,z',w,w' \in \mathbf{C}$ and $t,t' \in \mathbf{R}$.

Associated with this G, we take a 6-dimensional algebraic solvable Lie group $\tilde{G}$ which is the semi-direct product $\mathbf{C}^2 \times_s \mathbf{R}^2$ with the multiplication

$$(z,w,u,v)(z',w',u',v') = (z+e^{iu}z', \ w+e^{iv}w, \ u+u', \ v+v').$$

G is regarded as a closed subgroup of $\tilde{G}$ by the imbedding

$$G \ni (z,w,t) \longrightarrow (z,w,t, 2\pi t) \in \tilde{G}.$$

We take a unitary character $\chi^{(r,s)}$ of $\mathbf{C}^2$ $(r,s \in \mathbf{R}^+)$, defined by

$$\langle (z,w), \chi^{(r,s)} \rangle = e^{i(r,z)} e^{i(s,w)}$$

for $(z,w) \in \mathbf{C}^2$. The stability group $H^{(r,s)}$ of $\mathbf{R}^2$ at $\chi^{(r,s)}$ is $(2\pi\mathbf{Z})^2$ for all $r, s \in \mathbf{R}^+$. Put $G_0 = \mathbf{C}^2 \times_s (2\pi\mathbf{Z})^2 = \mathbf{C}^2 \times (2\pi\mathbf{Z})^2$. Then the unitary representations $\pi^{(r,s)}$ and $U^{(r,s,q,\lambda,\omega)}$

$(r,s \in \mathbf{R}^+,\ q \in \mathbf{Q},\ \lambda,\ \omega \in \mathbf{R})$ are as follows.

$$\pi^{(r,s)} = {}_G\Big|\mathrm{Ind}_{\mathbf{C}^2}^{\tilde{G}}\ \chi^{(r,s)}$$

$$U^{(r,s,q,\lambda,\omega)} = \mathrm{Ind}_{G_0}^{G}(A^q;\chi^{(r,s)}\times\eta^{(\lambda,\omega)})$$

where unitary characters $\eta^{(\lambda,\omega)}$ of $(2\pi\mathbf{Z})^2$ $(\lambda,\omega \in \mathbf{R})$ are defined by

$$\langle(2\pi m,\ 2\pi n),\ \eta^{(\lambda,\omega)}\rangle = e^{2\pi i\lambda m}\ e^{2\pi i\omega n}$$

for $(2\pi m,\ 2\pi n) \in (2\pi\mathbf{Z})^2$ and A^q $(q \in \mathbf{Q})$ are cocycles of $(\mathbf{R},\ \mathbf{R}^2;(2\pi\mathbf{Z})^2)$ constructed in section 2.5. We note that A^q is weakly cohomologous to $A^{q'}$ if and only if $q=q'$.

Theorem 4.1.3 asserts that

$$\pi^{(r,s)} \cong \int_0^1\!\!\oplus \int_0^1\!\!\oplus\ U^{(r,s,q,\lambda,\omega)}\ d\mu(\lambda)\ d\mu(\omega)$$

for each $q \in \mathbf{Q}$, where μ is the Lebesgue measure of the interval $[0,\ 1]$. Since $\mathbf{R} + (2\pi\mathbf{Z})^2$ is dense in $\mathbf{R}^2$, identifying the subgroup $\{(t,\ 2\pi t) \in \mathbf{R}^2;\ t \in \mathbf{R}\}$ with $\mathbf{R}$, $U^{(r,s,q,\lambda,\omega)}$ are irreducible representations of G by Theorem 3.2.6. If $q \neq q'$, $U^{(r,s,q,\lambda,\omega)}$ are never unitarily equivalent to $U^{(r,s,q',\lambda',\omega')}$ for arbitrary choice of λ,ω,λ', and $\omega' \in \mathbf{R}$ by Proposition 4.2.1. Therefore, the above decomposition of $\pi^{(r,s)}$ means that there are at least as many completely different irreducible decompositions of $\pi^{(r,s)}$ as the elements of $\mathbf{Q}$. Moreover, $\pi^{(r,s)}(G)''$ is algebraically isomorphic with $L^\infty(\mathbf{T}^2) \times_\beta \mathbf{R}$ and so $\pi^{(r,s)}$ is an injective type II_∞ factor representation of G by Theorem 4.2.8.

Chapter V. Representations of C^*-crossed products

This chapter is devoted to study representations of certain C^*-crossed products. In the previous chapters, we investigated decompositions of representations of some non-regular semi-direct product groups, related with a kind of cohomology group. In this chapter, we extend those results from a view point of operator algebras and show that the variety of decompositions of a non-type I representation is connected not only with ergodic measures but also with the cohomology group.

For two closed subgroups H and K of a locally compact abelian group G, we get a C*-crossed product $A = C_0(G/H) \times_\gamma K$. We investigate decompositions of a certain representation π^0 of A. In section 5.1, we find abelian von Neumann subalgebras $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in the commuting algebra $\pi^0(A)'$ associated with the automorphisms α^a and β^b of $\pi^0(A)'$ where $a \in Z(K; G; H)$ and $b \in Z(H^\perp; \hat{G}; K^\perp)$. We will have also some necessary and sufficient conditions of the maximality of $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in $\pi^0(A)'$. In section 5.2, we study decompositions of π^0 corresponding to $\mathfrak{A}^a$. To do this, we study generalized induced representations of C^*-crossed products following to chapter III. In section 5.3, using the concept of Heisenberg representations, we write down decompositions of π^0 corresponding to the above $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in explicit forms. In section 5.4, we show some examples and an application to unitary representations of a certain locally compact group.

5.1. Abelian subalgebras in $\pi^0(A)'$

Let G be a locally compact second countable "abelian" group. For a closed subgroup H of G, we can consider two von Neumann algebras $\mathfrak{A}(H)$ and $\mathfrak{B}(G/H)$ on the Hilbert space $L^2(G)$ of all square summable measurable functions with respect to a Haar measure of G. $\mathfrak{A}(H)$ is the von Neumann algebra generated by $\{U_h ; h \in H\}$ where U is the regular representation of G. $\mathfrak{B}(G/H)$ is the von Neumann algebra generated by the multiplication operators $\rho(f)$ on $L^2(G)$ defined by functions of $L^\infty(G)^H$, the set of all H-invariant essentially bounded measurable functions of G.

For a family of closed subgroups $\{H_i ; i \in I\}$ of G, we denote by $\bigvee_{i \in I} H_i$ the closed subgroup of G generated by $\bigcup_{i\ I} H_i$. Similarly, for a family of von Neumann algebras $\{\mathfrak{M}_i ; i \in I\}$ on $L^2(G)$, we denote by $\bigvee_{i \in I} \mathfrak{M}_i$ the von Neumann algebra generated by $\bigcup_{i \in I} \mathfrak{M}_i$. The following fact of lattice correspondence about $\mathfrak{A}(H)$ and $\mathfrak{B}(G/H)$ was obtained by M. Takesaki and N. Tatsuuma.

Lemma 5.1.1. (Theorem 4 in [53])

If $\{H_i ; i \in I\}$ is a family of closed subgroups of G, then

(i) $\bigvee_{i \in I} \mathfrak{A}(H_i) = \mathfrak{A}(\bigvee_{i \in I} H_i)$

(ii) $\bigcap_{i \in I} \mathfrak{A}(H_i) = \mathfrak{A}(\bigcap_{i \in I} H_i)$

(i)' $\bigvee_{i \in I} \mathfrak{B}(G/H_i) = \mathfrak{B}(G/\bigcap_{i \in I} H_i)$

(ii)' $\bigcap_{i \in I} \mathfrak{B}(G/H_i) = \mathfrak{B}(G/\bigvee_{i \in I} H_i)$.

For two arbitrary closed subgroups H and K of G, we denote by $\mathfrak{M}(G/K, H)$ the von Neumann algebra generated by $\mathfrak{A}(H)$ and $\mathfrak{B}(G/K)$. Then, the following generalized commutation relation holds.

Lemma 5.1.2. (M. Takesaki [52])

$\mathfrak{M}(G/K, H)' = \mathfrak{M}(G/H, K)$.

We can generalize the result of Lemma 5.1.1 as follows.

Proposition 5.1.3. Let $\{H_i ; i \in I\}$ and $\{K_i ; i \in I\}$ be families of closed subgroups of G, then

(i) $\bigvee_{i \in I} \mathfrak{M}(G/K_i, H_i) = \mathfrak{M}(G/\bigcap_{i \in I} K_i, \bigvee_{i \in I} H_i)$

(ii) $\bigcap_{i \in I} \mathfrak{M}(G/K_i, H_i) = \mathfrak{M}(G/\bigvee_{i \in I} K_i, \bigcap_{i \in I} H_i)$

Proof. This follows immediately from Lemma 5.1.1 and Lemma 5.1.2 by simple calculations. [Q.E.D.]

Corollary 5.1.4. Let H and K be closed subgroups of G. Then,

(i) $\mathfrak{M}(G/K, H)$ is a factor if and only if $K \cap H = \{0\}$ and $K + H$ is dense in G.

(ii) $\mathfrak{A}(H)$ is a maximal abelian von Neumann subalgebra in $\mathfrak{M}(G/K, H)$ if and only if $K + H$ is dense in G.

(iii) $\mathfrak{B}(G/K)$ is a maximal abelian von Neumann subalgebra in $\mathfrak{M}(G/K, H)$ if and only if $K \cap H = \{0\}$.

Proof.

(i) $\mathfrak{M}(G/K, H)$ is a factor.

$\Longleftrightarrow \mathfrak{M}(G/K,H) \cap \mathfrak{M}(G/K,H)' = \mathbb{C}_{L^2(G)}$

$\Longleftrightarrow \mathfrak{M}(G/K,H) \cap \mathfrak{M}(G/H,K) = \mathbb{C}_{L^2(G)}$

$\Longleftrightarrow \mathfrak{M}(G/H \vee K, H \cap K) = \mathbb{C}_{L^2(G)}$

$\Longleftrightarrow H \vee K = G$ and $H \cap K = \{0\}$.

(ii) $\mathfrak{A}(H)$ is maximal abelian in $\mathfrak{M}(G/K, H)$

$\Longleftrightarrow \mathfrak{A}(H)' \cap \mathfrak{M}(G/K,H) = \mathfrak{A}(H)$

$\Longleftrightarrow \mathfrak{M}(G/G,H)' \cap \mathfrak{M}(G/K,H) = \mathfrak{M}(G/G,H)$

$\Longleftrightarrow \mathfrak{M}(G/H,G) \cap \mathfrak{M}(G/K,H) = \mathfrak{M}(G/G, H)$

$\Longleftrightarrow \mathfrak{M}(G/H \vee K,H) = \mathfrak{M}(G/G,H)$

$\Longleftrightarrow H \vee K = G$

(iii) $\mathfrak{B}(G/K)$ is maximal abelian in $\mathfrak{M}(G/K,H)$

$\Longleftrightarrow \mathfrak{B}(G/K)' \cap \mathfrak{M}(G/K,H) = \mathfrak{B}(G/K)$

$\Longleftrightarrow \mathfrak{M}(G/K, \{0\})' \cap \mathfrak{M}(G/K,H) = \mathfrak{M}(G/K,\{0\})$

$\Longleftrightarrow \mathfrak{M}(G/\{0\},K) \cap \mathfrak{M}(G/K,H) = \mathfrak{M}(G/K,\{0\})$

$\Longleftrightarrow \mathfrak{M}(G/K,K \cap H) = \mathfrak{M}(G/K,\{0\})$

$\Longleftrightarrow K \cap H = \{0\}$ [Q.E.D.]

For two closed subgroups H and K of G, we are interested in the following case (*).

(*) $H \cap K = \{0\}$ and $H + K$ is dense in G.

Under the condition (*), we see that $\mathfrak{M}(G/K,H)$ is a factor and that $\mathfrak{A}(H)$ and $\mathfrak{B}(G/K)$ are maximal abelian von Neumann subalgebras in $\mathfrak{M}(G/K,H)$. Moreover, under the condition (*), we note that $\mathfrak{M}(G/K,H)$ is an injective type II factor if and only if $K + H \neq G$. The next examples satisfy such conditions.

Example 5.1.5. Let $\mathbf{Z}$ and $\mathbf{R}$ denote the additive groups of integers and real numbers respectively and let θ be a positive irrational number.

(i) $G = \mathbf{R}$, $H = \theta\mathbf{Z}$, and $K = \mathbf{Z}$.

(ii) $G = \mathbf{R}^2$, $H = \mathbf{Z}^2$, and $K = \{(x, y) \in \mathbf{R}^2 \ ; \ y = \theta x\}$.

For two closed subgroups H and K of G, we get a double transformation group (K;G;H) as defined in section 2.2. Let $\mathbf{T}$ be the torus group of the complex numbers with absolute value 1. When a $\mathbf{T}$-valued Borel function a(g) on G satisfies the cocycle condition;

$$a(k + g + h) = a(k + g)\overline{a(g)}\, a(g + h)$$

for each $k \in K$, $g \in G$, and $h \in H$, a(g) is called a cocycle of (K; G; H) and the abelian group of all such cocycles is denoted by Z(K; G; H). A cocycle a_1 is said to be cohomologous to a cocycle a_2 if there exist a K-invariant cocycle e_1 and an H-invariant cocycle e_2 such that $a_1(g)\overline{a_2(g)} = e_1(g)e_2(g)$ for almost all $g \in G$. Further, a_1 is said to be weakly cohomologous to a_2 if $a_1\overline{a_2}$ is cohomologous to some unitary character of G. We denote by H(K; G; H) the cohomology group of all cohomologous classes of cocycles of (K;G;H) and by $\tilde{H}$(K; G; H) the factor group of all weakly cohomologous classes of such cocycles.

For a cocycle a in Z(K; G; H), we define an operator $\rho(a)$ on $L^2(G)$ by

$$(\rho(a)\xi)(g) = a(g)\xi(g) \quad \text{for } \xi(g) \in L^2(G).$$

For a bounded operator S on $L^2(G)$, we put

$$\alpha^a(S) = \rho(a)^* S \rho(a).$$

Then, α^a is an automorphism of the full operator algebra $\mathcal{L}(L^2(G))$. For an operator $\rho(f)$ of $\mathcal{B}(G/K)$, we get

$$\begin{aligned}\alpha^a(\rho(f)) &= \rho(a)^*\rho(f)\rho(a) \\ &= \rho(\bar{a}\ f\ a) \\ &= \rho(f).\end{aligned}$$

Thus α^a leaves each operator of $\mathcal{B}(G/K)$ invariant and $\alpha^a(\mathcal{B}(G/K)) \subset \mathfrak{M}(G/K,H)$.

Next, we show $\alpha^a(\mathfrak{U}(H)) \subset \mathfrak{M}(G/K,H)$. Let U_h $(h \in H)$ be a generator of $\mathfrak{U}(H)$. Then, $\alpha^a(U_h) \in \mathcal{B}(G/H)'$ and $\alpha^a(U_h) \in \mathfrak{U}(K)'$. Indeed, on the one hand, for an operator $\rho(f) \in \mathcal{B}(G/H)$ and $\xi(g) \in L^2(G)$,

$$\begin{aligned}&(\alpha^a(U_h)\rho(f)\xi)(g) \\ =\ &(\rho(a)^*U_h\rho(a)\rho(f)\xi)(g) \\ =\ &\overline{a(g)}a(g + h)f(g + h)\xi(g + h) \\ =\ &f(g)\overline{a(g)}a(g + h)\xi(g + h) \\ =\ &(\rho(f)\rho(a)^*U_h\rho(a)\xi)(g) \\ =\ &(\rho(f)\alpha^a(U_h)\xi)(g).\end{aligned}$$

On the other hand, for a generator U_k $(k \in K)$ of $\mathfrak{U}(K)$ and an arbitrary $\xi(g) \in L^2(G)$,

$$\begin{aligned}&(U_k\alpha^a(U_h)\xi)(g) \\ =\ &\overline{a(g+k)}a(g + k + h)\xi(g + k + h)\end{aligned}$$

$$= \overline{a(g)}a(g + h)\xi(g + k + h)$$

$$= (\alpha^a(U_h)U_k\xi)(g).$$

Therefore, we get

$$\begin{aligned} \alpha^a(U_h) \in \mathcal{B}(G/H)' \cap \mathfrak{A}(K)' \\ &= (\mathcal{B}(G/H) \vee \mathfrak{A}(K))' \\ &= \mathfrak{M}(G/H, K)' \\ &= \mathfrak{M}(G/K, H). \end{aligned}$$

Hence, $\alpha^a(\mathfrak{A}(H)) \subset \mathfrak{M}(G/K, H)$. We see now $\mathfrak{M}(G/K,H)$ is α^a-invariant and so α^a gives an automorphism of $\mathfrak{M}(G/K,H)$.

Now, we define an automorphism β^b of $\mathfrak{M}(G/K,H)$ for $b \in Z(H^\perp; \hat{G}; K^\perp)$ as follows, where $\hat{G}$ is the dual group of G and $H^\perp$ and $K^\perp$ are the annihilators of H and K in $\hat{G}$. Take the Fourier transformation F from $L^2(G)$ onto $L^2(\hat{G})$ and, for a bounded operator S on $L^2(G)$, put

$$\tilde{F}(S) = F \circ S \circ F .$$

Then, $\tilde{F}(S)$ is a bounded operator on $L^2(\hat{G})$. Through this $\tilde{F}$, $\mathfrak{A}(H)$ is transformed onto $\mathcal{B}(\hat{G}/H^\perp)$ and $\mathcal{B}(G/K)$ is transformed onto $\mathfrak{A}(K^\perp)$ so that $\mathfrak{M}(G/K,H)$ is transformed onto $\mathfrak{M}(\hat{G}/H^\perp, K^\perp)$. Similarly as in the case α^a, a cocycle b in $Z(H^\perp; \hat{G}; K^\perp)$ gives rise to an automorphism $\tilde{\beta}^b$ of $\mathfrak{M}(\hat{G}/H^\perp, K^\perp)$ which leaves each operator of $\mathcal{B}(\hat{G}/H^\perp)$ invariant. Put

$$\beta^b = \tilde{F}^{-1} \circ \tilde{\beta}^b \circ \tilde{F}.$$

Then, β^b is an automorphism of $\mathfrak{M}(G/K,H)$ which leaves each operator of $\mathfrak{A}(H)$ invariant. Then we get the following theorem.

Theorem 5.1.6. For a cocycle a in Z(K; G; H), α^a is an automorphism of $\mathfrak{M}(G/K, H)$ which leaves each operator of $\mathcal{B}(G/K)$ invariant. A cocycle b in $Z(H^\perp; \hat{G}; K^\perp)$ also gives rise to an automorphism β^b of $\mathfrak{M}(G/K, H)$ which leaves each operator of $\mathfrak{A}(H)$ invariant. The correspondences $a \to \alpha^a$ and $b \to \beta^b$ are isomorphisms from Z(K; G; H) and $Z(H^\perp; \hat{G}; K^\perp)$ into Aut $\mathfrak{M}(G/K,H)$ of all automorphisms of $\mathfrak{M}(G/K,H)$ and satisfy $\alpha^a \circ \beta^b = \beta^b \circ \alpha^a$.

Proof. The latter properties about α^a and β^b follow immediately from their definitions. [Q.E.D.]

Put $\mathfrak{A}^a = \alpha^a(\mathfrak{A}(H))$ and $\mathcal{B}^b = \beta^b(\mathcal{B}(G/K))$ for the automorphisms α^a and β^b ($a \in Z(K; G; H)$) and $b \in Z(H^\perp; \hat{G}; K^\perp)$) of $\mathfrak{M}(G/K, H)$. Then, we get abelian von Neumann subalgebras $\mathfrak{A}^a$ and $\mathcal{B}^b$ in $\mathfrak{M}(G/K,H)$ and if $a_1 \neq a_2$ ($b_1 \neq b_2$), $\mathfrak{A}^{a_1} \neq \mathfrak{A}^{a_2}$ ($\mathcal{B}^{b_1} \neq \mathcal{B}^{b_2}$).

Corollary 5.1.7. If K + H is dense in G, each $\mathfrak{A}^a$ ($a \in Z(K; G; H)$) is a maximal abelian subalgebra in $\mathfrak{M}(G/K,H)$. If $K \cap H = \{0\}$, each $\mathcal{B}^b$ ($b \in Z(H^\perp; \hat{G}; K^\perp)$) is a maximal abelian subalgebra in $\mathfrak{M}(G/K,H)$.

Proof. This follows immediately from Corollary 5.1.4 and Theorem 5.1.6. [Q.E.D.]

For a locally compact abelian group G, take and fix two closed subgroups H and K of G. Let $C_0(G/H)$ denote the abelian C*-algebra of all continuous functions on the locally compact homogeneous space G/H vanishing at infinity. Then, the canonical action of K on G/H induces an action of K on the C*-algebra $C_0(G/H)$ as automorphisms and we denote such an action by γ. From the C*-dynamical system $(C_0(G/H),K,\gamma)$, we get the C*-crossed product $C_0(G/H)\times_\gamma K$ (see [40]) which we will denote by A(G/H,K) or abbreviatedly by A, hereafter.

Let ρ^0 be a representation of $C_0(G/H)$ on $L^2(G)$ defined by

$$(\rho^0(f)\xi)(g) = f(\dot{g})\xi(g)$$

for $f \in C_0(G/H)$ and $\xi \in L^2(G)$, where $G \ni g \longrightarrow \dot{g} \in G/H$ is the canonical projection. Next, we define a unitary representation U^0 of K on $L^2(G)$ by

$$(U^0_k\xi)(g) = \xi(g + k)$$

for k K.

Then, it is clear that (ρ^0, U^0) is a convariant representation of the C^*-dynamical system $(C_0(G/H), K, \gamma)$. We denote by π^0 the representation of the C^*-algebra A naturally defined from this convariant representation (ρ^0, U^0) [11]. Then, we get the following.

Proposition 5.1.8.

(i) $\pi^0(A)' = \mathfrak{M}(G/K,H)$ and $\pi^0(A)'' = \mathfrak{M}(G/H,K)$

(ii) Under the condition (*), π^0 is a factor representation of A and there exist maximal abelian subalgebras $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in $\pi^0(A)'$ parametrized by $a \in Z(K;G;H)$ and $b \in Z(H^\perp;\hat{G};k^\perp)$.

(iii) $\mathfrak{A}^a$ are spatially isomorphic with each other and algebraically isomorphic with $L^\infty(\hat{H}, \mu)$, where μ is a Haar measure of $\hat{H}$. $\mathfrak{B}^b$ are spatially isomorphic with each other and algebraically isomorphic with $L^\infty(G/K, \nu)$, where ν is a Haar measure of the factor group G/K.

Due to Proposition 5.1.8, we know that under the condition (*) the representation π^0 of A is decomposed into a direct integral of irreducible representations corresponding to each maximal abelian subalgebra $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in $\pi^0(A)'$. In section 5.2, we shall describe decompositions of π^0 corresponding to $\mathfrak{A}^a$. In section 5. 3, we shall give an explicit form of the irreducible representations of A which appear as components of decompositions of π^0 corresponding to $\mathfrak{A}^a$ and $\mathfrak{B}^b$.

5.2. Decompositions of π^0

In this section, we study generalized induced representations of C^*-crossed products following to chapter III, in order to state decompositions of π^0. The results described here were obtained by suggestions of T. Kajiwara [27].

Let G be a locally compact second countable abelian group and (B, G, α) be a C*-dynamical system. We take two closed subgroups K and H of G and denote by α^K and α^H of the automorphism α considered for the elements of K and H.

Let (λ, V) be a convariant representation on $\mathfrak{h}_0$ of the C*-dynamical system (B, H, α^H). Put $\mathcal{C} = \lambda(B)' \cap V(H)'$ and we denote by $\mathcal{C}^u$ the unitary group of $\mathcal{C}$. Let $a(g)$ be a $\mathcal{C}^u$-valued cocycle of the double transformation group $(K; G; H)$. The definition of cocycles and the cohomologous relation among them are quite similar to that we have given for T-valued case. For the precise definition, see chapter II. We define a covariant representation of the C*-dynamical system (B, K, α^K) starting from (λ, V) and $a(g)$. This definition is analogous to the group case described in chapter III.

We denote by $\mathfrak{h}$ the set of all ξ's satisfying the following conditions.

(1) ξ is an $\mathfrak{h}_0$-valued Borel function on G

(2) $\xi(g + h) = V_h^* \xi(g)$ for every $g \in G$ and $h \in H$.

(3) $\int_{G/H} \| \xi(g) \|^2 d\nu(\dot{g}) < \infty$ where ν is a Haar measure of the factor group G/H.

Then, $\mathfrak{h}$ has a Hilbert space structure with the norm defined by

$$|||\xi||| = \left(\int_{G/H} \|\xi(g)\|^2 d\nu(\dot{g})\right)^{1/2}.$$

For each $k \in K$, we define an operator U_k on $\mathfrak{h}$ by

$$(U_k \xi)(g) = a(g)a(g + k)^* \xi(g + k)$$

for each $\xi \in \mathfrak{h}$ and $g \in G$. By the fact that

$$\begin{aligned}(U_k \xi)(g + h) &= a(g + h)a(g + k + h)^* \xi(g + k + h) \\ &= a(g)a(g + k)^* V_h^* \xi(g + k) \\ &= V_h^* a(g)a(g + k)^* \xi(g + k) \\ &= V_h^*(U_k \xi)(g)\end{aligned}$$

for every $\xi \in \mathfrak{h}$ and $h \in H$, $U_k \xi$ belongs to $\mathfrak{h}$. It is clear that

U_k is a unitary operator on $\mathfrak{h}$. Next, for each $y \in B$, we define an operator $\rho(y)$ on $\mathfrak{h}$ by

$$(\rho(y)\xi)(g) = \lambda(\alpha_g(y))\xi(g)$$

for each $\xi \in \mathfrak{h}$ and $g \in G$. By the fact that

$$\begin{aligned}(\rho(y)\xi)(g+h) &= \lambda(\alpha_{g+h}(y))\xi(g+h) \\ &= V_h^*\lambda(\alpha_g(y))\, V_h V_h^* \xi(g) \\ &= V_h^*\, (\rho(y)\xi)(g)\end{aligned}$$

for every $\xi \in \mathfrak{h}$ and $h \in H$, $\rho(y)\xi$ belongs to $\mathfrak{h}$. Clearly, $\rho(y)$ is a bounded operator on $\mathfrak{h}$.

Proposition 5.2.1. $(\rho\ ,U)$ is a convariant representation on $\mathfrak{h}$ of the C*-dynamical system (B, K, α^k).

Proof. This follows immediately by routine arguments and simple calculations. [Q.E.D.]

This construction is a generalization of Takesaki's induced convariant representation studied in [51]. We denote the above (ρ, U) by $\mathrm{Ind}_H^K\ (a; \lambda, V)$. When $K = G$ and a is trivial, $\mathrm{Ind}_H^K\ (a; \lambda, V)$ coincides with the ordinary induced representation $\mathrm{Ind}_H^G\ (\lambda, V)$.

Remark 5.2.2. Our construction is closely related to Ramsay's induced representation [44] but in some cases ours affords finer informations.

We shall develop some general theory of these representations.

Lemma 5.2.3. For two cocycles a_1 and a_2 of (K; G; H), if a_1 is cohomologous to a_2, $\mathrm{Ind}_H^K(a_1; \lambda, V) \cong \mathrm{Ind}_H^K(a_2; \lambda, V)$.

Proof. This follows immediately from the definition.

[Q.E.D]

Let H_1 and H_2 be two closed subgroups of H such that $H_1 \subset H_2$. We assume henceforth that cocycles of (K; G; H) are all **T**-valued. Take a **T**-valued cocycle a, then we can regard this a as a cocycle of (K; G; H_i) (i = 1, 2). Let (λ, V) be a covariant representation of (B, H_1, α^{H_1}). Then, we get the following stage theorem.

Proposition 5.2.4. (Stage Theorem)

$\mathrm{Ind}_{H_1}^K(a; \lambda, V) \cong \mathrm{Ind}_{H_2}^K(a; \mathrm{Ind}_{H_1}^{H_2}(\lambda, V))$.

Proof. The proof is carried out by an analogy of the proof of the stage theorem about ordinary induced representations by G.W. Mackey (Theorem 4.1 in [30]). We omit the detail. [Q.E.D.]

Remark 5.2.5. This is not the same with the stage theorem of A. Ramsay [44], because H, H_1, and H_2 are not necessarily subobjects of K.

Next, we shall describe the subgroup theorem. Let K_1 be a closed subgroup of K. We denote by G_1 the closed subgroup generated by K_1 + H in G. A **T**-valued cocycle a of (K; G; H) can be regarded as a cocycle of (K_1; G_1; H) by the restriction to G_1. Let (λ, V) be a covariant representation of (B, H, α^H). There exists a canonical action of G on $Z(K_1; G_1; H)$ of all $\mathbf{T}$-valued cocycles of $(K_1; G_1; H)$ and on Rep B of all representations of B defined, for each $g \in G$, by

$$(g\cdot a)(g') = a(g + g') \qquad \text{for all } g' \in G_1$$

and

$$(g\cdot\lambda)(y) = \lambda(\alpha_g(y)) \qquad \text{for all } y \in B.$$

Let $G/G_1 \ni t \longrightarrow \tilde{t} \in G$ be a Borel cross section from G/G_1 to G and ν be a Haar measure of G/G_1. Then, we get the following.

Proposition 5.2.6. (Subgroup Theorem)

$$(B, K_1, \alpha^{K_1}) \,\Big|\, \mathrm{Ind}_H^K(a;\lambda,V) \cong \int_{G/G_1}^{\oplus} \mathrm{Ind}_H^{K_1}(\tilde{t}\cdot a;\tilde{t}\cdot\lambda,V)\,d\nu(t)$$

Proof. This also follows by the modifications of the proof of the subgroup theorem of G.W. Mackey in [30].

[Q.E.D.]

Let a be a cocycle of (K; G; H) again. Suppose that a covariant representation (λ, V) of (B, H, α^H) is decomposed as follows.

$$(\lambda, V) \cong \int_Z^{\oplus} (\lambda^\zeta, V^\zeta)\, d\nu(\zeta)$$

on some measure space (Z, ν). Then, we have the following proposition by routine arguments.

Proposition 5.2.7. The field $\{Z \ni \zeta \longrightarrow \mathrm{Ind}_H^K(a; \lambda^\zeta, V^\zeta)$

is ν-measurable and

$$\mathrm{Ind}_H^K(a;\ \lambda,\ V) \cong \int_Z^{\oplus} \mathrm{Ind}_H^K(a;\ \lambda^{\zeta},\ V^{\zeta})\ d\nu(\zeta).$$

Now, we shall consider the decompositions of the representation π^0 given in section 5.1. For a locally compact second countable abelian group G, take and fix two closed subgroups H and K of G. Recall that we have denoted by A the C^*-crossed product $C_0(G/H)\times_\gamma K$ for the C^*-dynamical system $(C_0(G/H),\ K,\ \gamma)$.

Let $(\iota,\ I)$ be the covariant representation of the C*-dynamical system $(C_0(G/H),\ H,\ \gamma^H)$ given by $\iota(f) = f(0)$, $I(h) = 1$, for every f in $C_0(G/H)$ and every h in H. For $a \in Z(K;\ G;\ H)$, We get the covariant representation $(\rho^a,\ U^a)$ of $(C_0(G/H),\ K, \gamma)$ by

$$(\rho^a,\ U^a) = \mathrm{Ind}_{\{0\}}^K(a;\ \iota,\ I)$$

where we regarded a as a cocycle of $(K;\ G;\ \{0\})$. It is clear that all cocycles of $(K;\ G;\ \{0\})$ are cohomologous to the trivial one. Therefore, by Lemma 5.2.3, we see that

$$(\rho^a,\ U^a) \cong (\rho^0,\ U^0)$$

for all $a \in Z(K;\ G;\ H)$, here the superscript 0 stands for the trivial cocycle. These covariant representations define the class of representations of π^0 of A which was just defined in section 5.1.

Let (ι, ζ) be the one-dimensional covariant representation of $(C_0(G/H),\ H,\ \gamma^H)$ where ζ is in $\hat{H}$ (the dual group of H). For $a \in Z(K;\ G;\ H)$, we have a covariant representaion

$(\rho^{(a,\zeta)}, U^{(a,\zeta)}) = \mathrm{Ind}_H^K(a; \iota, \zeta)$ of $(C_0(G/H), K, \gamma)$. This $(\rho^{(a,\zeta)}, U^{(a,\zeta)})$ defines the representation $\pi^{(a,\zeta)}$ of A [11].

For $a \in Z(K; G; H)$, we have got the abelian subalgebra $\mathfrak{A}^a$ in $\pi^0(A)'$ described in section 5.1. Let μ be a Haar measure of $\hat{H}$. Under these preparations, we get the following main theorem.

Theorem 5.2.8. Corresponding to each $\mathfrak{A}^a$, the representation π^0 of A is decomposed as follows.

$$\pi^0 \cong \int_{\hat{H}}^{\oplus} \pi^{(a,\zeta)} \, d\mu(\zeta).$$

Proof. We consider the covariant representations (ρ^0, U^0) and $(\rho^{(a,\zeta)}, U^{(a,\zeta)})$ of $(C_0(G/H), K, \gamma)$ which defined π^0 and $\pi^{(a,\zeta)}$ respectively.

$$\begin{aligned}
&(\rho^0, U^0) \\
&\cong (\rho^a, U^a) \\
&= \mathrm{Ind}_{\{0\}}^K(a; \iota, I) \\
&\cong \mathrm{Ind}_H^K(a; \mathrm{Ind}_{\{0\}}^H(\iota, I)) \quad \text{by Proposition 5.2.4} \\
&\cong \mathrm{Ind}_H^K(a; \int_{\hat{H}}^{\oplus}(\iota, \zeta)\, d\mu(\zeta)) \\
&\cong \int_{\hat{H}}^{\oplus} \mathrm{Ind}_H^K(a; \iota, \zeta)\, d\mu(\zeta) \quad \text{by Proposition 5.2.7} \\
&= \int_{H}^{\oplus} (\rho^{(a,\zeta)}, U^{(a,\zeta)})\, d\mu(\zeta).
\end{aligned}$$

Hence we get

$$\pi^0 \cong \int_H^{\oplus} \pi^{(a,\zeta)} \, d\mu(\zeta) .$$

It is not difficult to verify that the subalgebra $\mathfrak{A}^a$ is transformed to $\mathfrak{M}(H)$ by the unitary operator which gives the unitary equivalence between (ρ^0, U^0) and (ρ^a, U^a) so that $\mathfrak{M}(H)$ becomes the diagonal algebra of the decomposition.

[Q.E.D.]

Now, we shall describe the properties of the family of $\{\pi^{(a,\zeta)};\ a \in Z(K;\ G;\ H)$ and $\zeta \in \hat{H}\}$. We note that a unitary character of G is a cocycle of (K; G; H). Let $\tilde{\zeta}$ denote a character which extends ζ to G.

Proposition 5.2.9. The representations $\pi^{(a,\zeta)}$ $(a \in Z(K;G;H)$ and $\zeta \in \hat{H})$ of A have the following properties.

(i) $\pi^{(a,\zeta)}$ is unitarily equivalent to $\pi^{(a+\tilde{\zeta},0)}$.

(ii) $\pi^{(a,0)}$ is unitarily equivalent to $\pi^{(a',0)}$ if and only if a is cohomologous to a'.

(iii) If a cocycle a is not weakly cohomologous to a cocycle a', $\pi^{(a,\zeta)}$ is never unitarily equivalent to $\pi^{(a',\zeta')}$ for arbitrary choice of $\zeta,\ \zeta' \in \hat{H}$.

(iv) $\pi^{(a,\zeta)}$ is irreducible if and only if H + K is dense in G.

Proof. The proof goes on by a modification of the techniques described in section 3.2. So we omit the detail.

[Q.E.D.]

We see that the representation π^0 of A has been decomposed into a direct integral of $\pi^{(a,\zeta)}$ in as many ways as the cardinal number of the weak cohomology group $\tilde{H}(K;\ G;\ H)$. We note that this fact does not depend on whether the representation π^0 is type I or not. When π^0 is a factor representation of A, all $\pi^{(a, \zeta)}$ are irreducible representations of A (see Corollary 5.1.4).

5.3. Further discussions on the decomposition of π^0

We denote by Rep G the set of all unitary representations of a locally compact group G and by Rep A the set of all non-degenerate *-representations of a C^*-algebra A.

Suppose G be a locally compact second countable abelian group. Let K be a closed subgroup of G and H' be a closed subgroup of the dual $\hat{G}$ of G.

Definition 5.3.1 If a unitary representation U of K on $\mathfrak{h}$ and a unitary representation V of H' on the same space are so related that the Heisenberg commutation relation

$$U_k V_\omega = \langle k,\omega \rangle \; V_\omega U_k$$

holds for each $k \in K$ and $\omega \in H'$, we call the pair (V, U) a Heisenberg representation of (H', K). We denote by H-Rep(H', K) the set of all Heisenberg representations of (H', K). For two Heisenberg representations (V^1, U^1) and (V^2, U^2) of (H', K), we say that (V^1, U^1) is unitarily equivalent to (V^2, U^2) if there exists a unitary operator W from the representation space of (V^1, U^1) to the space of (V^2, U^2) such that $V^2 = W V^1 W^*$ and $U^2 = W U^1 W^*$.

Take and fix two closed subgroups H and K of G. We construct the C^*-dynamical system $(C_0(G/H), K, \gamma)$ and the C^*-algebra $A = C_0(G/H) \times_\gamma K$, as before.

Let $H^\perp$ be the annihilator of H in $\hat{G}$ and $C^*(H^\perp)$ be the group C^*-algebra of $H^\perp$. Then, it is well-known that there is a bijective correspondence $V \longrightarrow \tilde{V}$ from Rep $H^\perp$ to Rep $C^*(H^\perp)$ such that

$$\tilde{V}(f) = \int_{H^\perp} f(\omega)V(\omega)\, d\nu(\omega)$$

for $f \in L^1(H^\perp, \nu)$ where ν is a Haar measure of $H^\perp$ [10]. Since we can regard the dual of $H^\perp$ as G/H by Pontryagin's duality, we see that $C^*(H^\perp) \cong C_0(G/H)$ by the Gelfand transformation. Hence we get a bijection ϕ ; $V \longrightarrow \phi(V)$ from Rep $H^\perp$ to Rep $C_0(G/H)$. For a Heisenberg representation (V, U) of $(H^\perp, K)$, $(\phi(V), U)$ becomes a covariant representation of $(C_0(G/H), K, \gamma)$. It is clear that the correspondence (V, U) $(\phi(V), U)$ is bijective from H-Rep$(H^\perp, K)$ to C-Rep$(C_0(G/H), K, \gamma)$ of all covariant representations of $(C_0(G/H), K, \gamma)$. Further, we know that there exists a canonical correspondence between C-Rep$(C_0(G/H), K, \gamma)$ and Rep A [40]. Then, we get the following lemma.

Lemma 5.3.2. There is a bijective correspondence $(V, U) \longrightarrow \pi^{(V,U)}$ from H-Rep$(H^\perp, K)$ to Rep A such that $(\phi(V), U)$ is the covariant representation of $(C_0(G/H), K, \gamma)$ which defines $\pi^{(V,U)}$. The correspondence $(V, U) \longrightarrow \pi^{(V,U)}$ is a one to one map from the set of equivalence classes of H-Rep$(H^\perp, K)$ onto the set of equivalence classes of Rep A.

We shall say that a Heisenberg representation (V, U) of $(H^\perp, K)$ is associated with a representation π of A if $\pi^{(V,U)}$

is unitarily equivalent to π.

Example 5.3.3.

(i) A Heisenberg representation (V^0, U^0) of $(H^\perp, K)$ associated with the representation π^0 of A described in section 5.1 and 5.2 is given as follows.

For $\xi(g) \in L^2(G)$,

$$(V^0 \xi)(g) = \langle g, \omega \rangle \xi(g) \quad \text{for each} \quad \omega \in H^\perp$$

and

$$(U_k^0 \xi)(g) = \xi(g + k) \quad \text{for each } k \in K.$$

(ii) The following Heisenberg representation $(\hat{V}^0, \hat{U}^0)$ of $(H^\perp, K)$ is also associated with the same representation π^0.

For $\eta(\chi) \in L^2(\hat{G})$,

$$(\hat{V}_\omega^0 \eta)(\chi) = \eta(\chi + \omega) \quad \text{for each} \quad \omega \in H^\perp$$

and

$$(\hat{U}_k^0 \eta)(\chi) = \langle k, \chi \rangle \eta(\chi) \quad \text{for each } k \in K.$$

In section 5.1, we found abelian subalgebras $\mathfrak{A}^a$ $(a \in Z(K; G; H))$ and $\mathfrak{B}^b$ $(b \in Z(H^\perp; \hat{G}; K^\perp))$ in $\pi^0(A)'$. In section 5.2, we got the decomposition of π^0 corresponding to $\mathfrak{A}^a$ as

$$\pi^0 \cong \int_{\hat{H}}^{\oplus} \pi(a, \zeta) \, d\mu(\zeta).$$

$\pi^{(a,\zeta)}$ in here was defined as a covariant representation

$$(\rho^{(a,\zeta)}, {}^0U^{(a,\zeta)}) = \mathrm{Ind}_H^K(a; \iota, \zeta).$$

Now we will give an explicit form for this component representation using Heisenberg representations. Let $\tilde{\zeta}$ be a character of G which is an extension of ζ . The canonical action of K on the space G/H is denoted by $G/H \ni x \longrightarrow k\cdot x \in G/H$ for each $k \in K$. For a cocycle a(g) of (K; G; H), we have a cocycle $c^a(k, x)$ of (K; G/H) by

$$c^a(k, \dot{g}) = a(g)\overline{a(g + k)}$$

for every $k \in K$ and $g \in G$.

Proposition 5.3.4. The Heisenberg representation $(V^{(a,\zeta)}, U^{(a,\zeta)})$ of $(H^{\perp}, K)$ associated with $\pi^{(a,\zeta)}$ for $a \in Z(K; G; H)$ and $\zeta \in \hat{H}$ is given as follows.

For $\xi(x) \in L^2(G/H)$,

$$(V_{\omega}^{(a,\zeta)}\xi)(x) = \langle x, \omega\rangle \xi(x) \quad \text{for each} \quad \omega \in H^{\perp}$$

and

$$(U_k^{(a,\zeta)}\xi)(x) = \langle k, \zeta\rangle c^a(k, x)\xi(k\cdot x) \quad \text{for each } k \in K.$$

Proof. Let $\mathfrak{H}^{(a,\zeta)}$ be the representation space of the covariant representation $(\rho^{(a,\zeta)}, {}^{0}U^{(a,\zeta)})$ which defines $\pi^{(a,\zeta)}$. Then, for each $\xi \in \mathfrak{H}^{(a,\zeta)}$, $\tilde{\zeta}\xi$ becomes H-invariant and square summable on G/H. Therfore, the correspondence $\xi \longrightarrow \tilde{\zeta}\xi$ defines a unitary operator W from $\mathfrak{H}^{(a,\zeta)}$ onto $L^2(G/H)$. By simple calculations it is easy to check that

$$W\rho^{(a,\zeta)}W^* = \phi(V^{(a,\zeta)})$$

and

$$W\,{}^{0}U(a,\zeta)W^{*} = U(a,\zeta).$$ [Q.E.D]

Next, we will see the decomposition of π^0 corresponding to $\mathcal{B}^b$. Let $G/K \ni z \longrightarrow \tilde{z} \in G$ be a Borel cross section. We denote the canonical action of $H^\perp$ on the space $\hat{G}/K^\perp$ by $\hat{G}/K^\perp \ni \sigma \longrightarrow \omega\cdot\sigma \in \hat{G}/K^\perp$ for each $\omega \in H^\perp$. A cocycle $b(\chi)$ of $(H^\perp;\hat{G};K^\perp)$ defines a cocycle $d^b(\omega,\sigma)$ of $(H^\perp;\ \hat{G}/K^\perp)$ by

$$d^b(\omega,\ \dot{\chi}) = b(\chi)\overline{b(\chi+\omega)}$$

for all $\omega \in H^\perp$ and $\chi \in \hat{G}$. Then, we get the following theorem.

Theorem 5.3.5. Corresponding to each of the abelian subalgebras $\mathcal{B}^b$ in $\pi^0(A)'$ $(b \in Z(H^\perp;\ \hat{G};\ K^\perp))$, the representation π^0 of A is decomposed as follows.

$$\pi^0 \cong \int_{G/K}^{\oplus} \hat{\pi}^{(b,z)}\, d\nu(z)$$

where ν is a Haar measure of the factor group G/K and the Heisenberg representations $(\hat{V}^{(b,z)},\ \hat{U}^{(b,z)})$ of $(H^\perp,\ K)$ associated with the above $\hat{\pi}^{(b,z)}$ $(z \in G/K)$ are given for $\eta(\sigma) \in L^2(\hat{G}/K^\perp)$ by

$$(\hat{V}_\omega^{(b,z)}\,\eta)(\sigma) = \langle \tilde{z},\omega\rangle\ d^b(\omega,\sigma)\eta(\omega\cdot\sigma) \text{ for each } \omega\in H^\perp$$

and

$$(\hat{U}_k^{(b,z)}\,\eta)(\sigma) = \langle k,\sigma\rangle\eta(\sigma) \text{ for each } k \in K.$$

Proof. Note that the Heisenberg representation $(V^0,\ U^0)$ associated with π^0 in (i) of Example 5.3.3 was decomposed into a direct integral of the Heisenberg representations as was shown in Proposition 5.3.4 corresponding to $\mathfrak{A}^a$. If we go over to the dual $\hat{G}$ of G and take the Heisenberg representation ($\hat{V}^0$, $\hat{U}^0$) in (ii) of Example 5.3.3 instead of (V^0, U^0), the desired assertions are immediately obtained. [Q.E.D.]

Now, we shall describe the properties of $\hat{\pi}^{(b,z)}$. We note that we can regard an element of G as a cocycle of ($H^\perp$; $\hat{G}$; $K^\perp$) by the fact that $\hat{\hat{G}} \cong G$ as locally compact groups.

Proposition 5.3.6. The representations $\hat{\pi}^{(b,z)}$ of A ($b \in Z(H^\perp; \hat{G}; K^\perp)$ and $z \in G/K$) have the following properties.

(i) $\hat{\pi}^{(b,z)} \cong \hat{\pi}^{(b+\tilde{z},0)}$, where $G/K \ni z \longrightarrow \tilde{z} \in G$ is a Borel cross section.

(ii) $\hat{\pi}^{(b,0)}$ is unitarily equivalent to $\hat{\pi}^{(b',0)}$ if and only if b is cohomologous to b'.

(iii) Suppose a cocycle b is not weakly cohomologous to a cocycle b'. Then, $\hat{\pi}^{(b,z)}$ is never unitarily equivalent to $\hat{\pi}^{(b',z')}$ for arbitrary choice of z and $z' \in G/K$.

(iv) $\hat{\pi}^{(b,z)}$ is irreducible if and only if $K \cap H = \{0\}$.

Proof. Take the Heisenberg representations ($\hat{V}^{(b,z)}$, $\hat{U}^{(b,z)}$) associated with $\hat{\pi}^{(b,z)}$ and compare them with ($V^{(a,\zeta)}$, $U^{(a,\zeta)}$) in Proposition 5.3.4. Then, we see that the former becomes the latter if we exchange the role of $\hat{G}$ and G. Hence, the above statements about $\hat{\pi}^{(b,z)}$ are immediate from the properties of $\pi^{(a,\zeta)}$ and the duality of G. [Q.E.D]

Thus, we see that Theorem 5.3.5 gives as many completely different decompositions of π^0 as the cardinal number of

$\tilde{H}(H^{\perp}; \hat{G}; K^{\perp})$. However, it is difficult to find a general relation between the family of $\pi^{(a,\zeta)}$ and the family of $\hat{\pi}^{(b,z)}$. In some concrete cases, we can show that almost all $\hat{\pi}^{(b,z)}$ are not unitarily equivalent to any $\pi^{(a,\zeta)}$. This will be noticed in the later part of this chapter.

When π^0 is a non-type I factor representation of A, π^0 must be an injective type II factor representation. In this case, by Corollary 5.1.4, H + K is not equal to G but is dense in G and $H \cap K = \{0\}$. Therefore, all $\pi^{(a,\zeta)}$ and $\hat{\pi}^{(b,z)}$ are irreducible representations. Moreover, it will be seen easily that $\pi^{(0,\zeta)}$ (resp. $\hat{\pi}^{(0,z)}$) are never unitarily equivalent to any $\hat{\pi}^{(b,z)}$ (resp. $\pi^{(a,\zeta)}$).

When π^0 is a type I factor representation, all $\pi^{(a,\zeta)}$ and $\hat{\pi}^{(b,z)}$ are still irreducible representations but in this case $H(K; G; H) = \{0\}$ and $H(H^{\perp}; \hat{G}; K^{\perp}) = \{0\}$ as we have seen in chapter II. Therefore, it is no trouble to verify that all component representations are unitarily equivalent and that the decompositions of π^0 as given in Theorem 5.2.8 and Theorem 5.3.5 become all same.

When π^0 is a type I but not factor represenation, H + K is not dense or $H \cap K = \{0\}$. In the first case, none of $\pi^{(a,\zeta)}$ are irreducible representations and in the second case, none of $\hat{\pi}^{(b,z)}$ are irreducible. $\tilde{H}(K;G;H)$ and $\tilde{H}(H^{\perp}; \hat{G}; K^{\perp})$ may or may not be trivial.

Remark 5.3.7. When a cocycle b of $(H^{\perp}; \hat{G}; K^{\perp})$ is the trivial one, the assertion of Theorem 5.3.5 coincides with the result obtained by applying the subgroup theorem

(Proposition 5.2.6) to the covariant representation which defines π^0.

5.4. Applications and examples

First we argue the decomposition of representations of the pair of groups in Example 5.1.5.

Example 5.4.1. The case of $G = \mathbb{R}$, $H = \theta\mathbb{Z}$, and $K = \mathbb{Z}$.

The C^*-crossed product $A = C(\mathbb{R}/\theta\mathbb{Z})\times_\gamma \mathbb{Z}$ is the irrational rotation algebra. Let χ^u ($u \in \mathbb{R}$) be unitary characters of $\mathbb{R}$ defined by $\chi^u(t) = e^{2\pi iut}$ for each $t \in \mathbb{R}$. Then, the dual group $\hat{G} = \{\chi^u;\ u \in \mathbb{R}\}$ can be identified with $\mathbb{R}$ as a locally compact abelian group. Through this identification we can regard the annihilator $K^\perp$ of K as $\mathbb{Z}$ and the annihilator $H^\perp$ of H as $(1/\theta)\mathbb{Z}$.

For p, q $\in \mathbb{Z}$, put

$$a^p(t) = e^{(i\pi p/\theta)t^2} \quad \text{for each } t \in \mathbb{R} = G$$

and

$$b^q(u) = e^{i\pi\theta qu^2} \quad \text{for each } u \in \mathbb{R} = \hat{G}.$$

Then, it is not difficult to check that $a^p \in Z(K;\ G;\ H)$ and $b^q \in Z(H^\perp;\ \hat{G};\ K^\perp)$ for every p and q $\in \mathbb{Z}$. Further, by similar techniques as in chapter II, we see that a^p(resp. b^q) is weakly cohomologous to $a^{p'}$ (resp. $b^{q'}$) if and only if $p = p'$ (resp. $q = q'$).

Let π^0 be the representation of A on $L^2(\mathbb{R})$ as defined in section 5.1. Then, π^0 is decomposed as follows.

$$\pi^0 \cong \int_0^{1/\theta\,\oplus} \pi^{(p,u)}\,du \qquad (p \in \mathbf{Z}) \qquad \text{(I)}$$

$$\cong \int_0^{1\,\oplus} \hat{\pi}^{(q,s)}\,ds \qquad (q \in \mathbf{Z}) \qquad \text{(II)}$$

(I) The Heisenberg representations $(V^{(p,u)}, U^{(p,u)})$ of $(H^{\perp}, K) = ((1/\theta)\mathbf{Z}, \mathbf{Z})$ associated with $\pi^{(p,u)}$ are given by, for $\xi(x) \in L^2([0, \theta))$,

$$(V_{\omega}^{(p,u)}\xi)(x) = e^{2\pi i\omega x}\xi(x) \quad \text{for each } \omega \in (1/\theta)\mathbf{Z}$$

and

$$(U_m^{(p,u)}\xi)(x) = e^{2\pi ium}\, e^{(2\pi ip/\theta)mx}\, e^{(\pi ip/\theta)m^2}\,\xi(\overline{x+m})$$

for each $m \in \mathbf{Z}$, where $x = [x]+\bar{x}$ ($[x] \in \theta\mathbf{Z}$, $\bar{x}$ $[0,\theta]$).

(II) The Heisenberg representations of $(\hat{V}^{(q,s)}, \hat{U}^{(q,s)})$ of $(H^{\perp}, K) = ((1/\theta)\mathbf{Z}, \mathbf{Z})$ associated with $\hat{\pi}^{(q,s)}$ are given by, for $\eta(y) \in L^2([0, 1))$,

$$(\hat{V}_{\omega}^{(q,s)}\eta)(y) = e^{2\pi is\omega}\, e^{2\pi i\theta q\omega y}\, e^{\pi i\theta q\omega^2}\,\eta(\overline{y+\omega})$$

for each $\omega \in (1/\theta)\mathbf{Z}$, and

$$(\hat{U}_m^{(q,s)}\eta)(y) = e^{2\pi imy}\eta(y) \quad \text{for each } m \in \mathbf{Z}.$$

By simple calculations, we can see the following facts. Let χ^v ($v \in [0, \theta)$) be unitary characters of $(1/\theta)\mathbf{Z}$ given by $\langle \omega, \chi^v \rangle = e^{2\pi iv\omega}$ for $\omega \in (1/\theta)\mathbf{Z}$. Then,

(a) $V^{(p,u)} \cong \int_0^{\theta\,\oplus} \chi^v\,dv.$

(b) $\hat{V}^{(0,s)} \cong \sum_{n\in Z}^{\oplus} \chi^{\overline{s+n}}$.

(c) $\hat{V}^{(q,s)} \cong |q| \int_0^{\theta\,\oplus} \chi^v\,dv$ if $q \neq 0$,

$|q|$ appears here as the multiplicity of the representation.

Therefore, we can conclude that $\hat{\pi}^{(q,s)}$ is never unitarily equivalent to $\pi^{(p,u)}$ for every $p \in \mathbf{Z}$, $u \in [0,1/\theta)$ and $s \in [0, 1)$ if $q \neq \pm 1$. For $q = 1$ or -1, we know that $\hat{\pi}^{(q,s)}$ is unitarily equivalent to $\pi^{(a,u)}$ for some $a \in Z(H;G;K)$ and $u \in [0,1/\theta)$. Note that π^0 is an injective type II_1 factor representation of A and all $\pi^{(p,u)}$ and $\hat{\pi}^{(q,s)}$ are irreducible representations of A.

Example 5.4.2. The case of $G = \mathbf{R}^2$, $H = \mathbf{Z}^2$, and $K = \{(x,y) \in \mathbf{R}^2;\ y = \theta x\}$.

In this case, $A = C_0(G/H) \times_\gamma K = C(\mathbf{T}^2) \times_\delta \mathbf{R}$ where the action δ of $\mathbf{R}$ is defined by

$$\delta(t) \cdot (x, y) = (\overline{x + t}, \overline{y + \theta t})$$

for $t \in \mathbf{R}$ and $(x, y) \in [0, 1) \times [0, 1) \cong \mathbf{T}^2$.

Let $\chi^{(u,v)}$ $(u, v \in \mathbf{R})$ be unitary characters of $\mathbf{R}^2$ given by

$$\langle (x, y), \chi^{(u,v)} \rangle = e^{2\pi i(ux+vy)}$$

for $(x, y) \in \mathbf{R}^2$. Then, $\hat{G} = \{\chi^{(u,v)};\ u, v \in \mathbf{R}\}$ can be indentified with $\mathbf{R}^2$ as a locally compact group. Through this identification,

$$K^{\perp} = \{(u, v) \in \mathbf{R}^2;\ v = -(1/\theta)u\} \cong \mathbf{R}$$

$$H^{\perp} = \{(u, v) \in \mathbf{R}^2;\ u, v \in \mathbf{Z}\} \cong \mathbf{Z}^2.$$

For p and $q \in \mathbf{Z}$, put

$$a^p((x, y)) = e^{-\pi ip(\theta y-\bar{x})^2}$$

for $(x, y)\in \mathbf{R}^2$ where $0 \leq \bar{x} < 1$, and

$$b^q((x, y)) = e^{-\pi iq(\theta y+\bar{x})^2}$$

for $(x, y)\in \mathbf{R}^2$. Then, it is no trouble to verify that $a^p \in Z(K; G; H)$ and $b^q \in Z(H^\perp: \hat{G}; K^\perp)$ for every p and $q \in \mathbf{Z}$.

Let π^0 be the representation of A as defined in section 5.1. Then, π^0 is decomposed as follows.

$$\pi^0 \cong \int_0^1\int_0^1 \oplus\, \pi^{(p,u,v)}\, dudv \qquad \text{(I)}$$

$$\cong \int_{-\infty}^{\infty} \oplus\, \hat{\pi}^{(q,s)}\, ds \qquad \text{(II)}$$

The diagonal algebra of the decomposition (I) is $\mathfrak{A}^{a^p}$ and the diagonal algebra of (II) is $\mathfrak{B}^{b^q}$.

(I) The Heisenberg representation $(V^{(p,u,v)}, U^{(p,u,v)})$ of $(H^\perp,K) = (\mathbf{Z}^2, \mathbf{R})$ associated with $\pi^{(p,u,v)}$ is given as follows.

For $\xi(x, y)\in L^2([0, 1)\times[0, 1))$,

$$(V^{(p,u,v)}_{(m,n)}\xi)(x, y) = e^{2\pi i\theta mx}\, e^{2\pi iny}\, \xi(x, y)$$

for $(m, n) \in \mathbf{Z}^2$, and

$$(U^{(p,u,v)}_t\xi)(x,y) = e^{2\pi iut}\, e^{2\pi iv\theta t} e^{ip\theta\{2[x+t](y-\theta t)+ \theta[x+t]^2\}}\, \xi(\overline{x + t}, \overline{y + \theta t})$$

for $t \in \mathbf{R}$ where $x + t = \overline{x+t} + [x+t]$, $0\leq\overline{x+t}<1$ and $[x+t] \in \mathbf{Z}$.

(II) The Heisenberg representation $(\hat{V}^{(q,s)}, \hat{U}^{(q,s)})$ of $(H^\perp,K) = (\mathbf{Z}^2,\mathbf{R})$ associated with $\hat{\pi}^{(q,s)}$ is given as follows.

For $\eta(z)\in L^2(\mathbf{R})$,

$$(\hat{V}^{(q,s)}_{(m,n)}\eta)(z) = e^{2\pi ims}\ e^{\pi i\theta q(2nz+n^2)}\eta(z+n+(1/\theta)m)$$

for $(m, n) \in \mathbf{Z}^2$ and

$$(\hat{U}^{(q,s)}_t\eta)(z) = e^{2\pi izt}\ \eta(z)$$

for $t \in \mathbf{R}$.

Take the representation $U^{(p,u)}$ ($p\in\mathbf{Z}$ and $u\in[0,1/\theta)$) of $\mathbf{Z}$ in Example 5.4.1. Then we get

$$U^{(p,u,v)} \cong \mathrm{Ind}_{\mathbf{Z}}^{\mathbf{R}}\ U^{(p,\overline{u+(1/\theta)v})}$$

where $0 \leqq \overline{u+(1/\theta)v} < 1/\theta$. Hence, we see the following facts. Let χ^u ($u \in \mathbf{R}$) be unitary characters of $\mathbf{R}$ given by $\chi^u(t) = e^{2\pi int}$ for $t \in \mathbf{R}$.

(a) $\hat{U}^{(q,s)} \cong \int_{-\infty}^{\infty} \oplus\ \chi^u\ du$

(b) $U^{(0,u,v)} \cong \sum_{m,n\in Z}^{\oplus} \chi^{u+n+\theta(v+m)}$

(c) $U^{(p,u,v)} \cong |p| \int_{-\infty}^{\infty} \oplus\ \chi^u\ du$ if $p \neq 0$

where $|p|$ appears as the multiplicity of the representation.

Therefore, we get the fact that for each u, $v\in[0, 1)$, $\pi^{(p,u,v)}$ is never unitarily equivalent to $\hat{\pi}^{(q,s)}$ for arbitrary choice of $q \in \mathbf{Z}$ and $s \in \mathbf{R}$ if $p \neq \pm 1$.

We note that the representation π^0 of A is an injective type II_∞ factor representation and all $\pi^{(p,u,v)}$ and $\hat{\pi}^{(q,s)}$ are irreducible.

Now, we show an application of our results to unitary representations of certain locally compact groups.

Let $\tilde{G} = N \times_s \tilde{K}$ be a regular semi-direct product of N with $\tilde{K}$ where N and $\tilde{K}$ are locally compact second countable abelian groups and $\tilde{K}$ acts on N as an automorphism group. The action of $\tilde{K}$ on N induces the action of $\tilde{K}$ on $\hat{N}$ (the dual of N) as a topological transformation group and the regularity of $\tilde{G}$ means that the topological transformation group $(\tilde{K}; \hat{N})$ is smooth ([12], [16], [32]). Taking a closed subgroup K of $\tilde{K}$, we shall consider the closed subgroup $G = N \times_s K$ of $\tilde{G} = N \times_s \tilde{K}$.

For a unitary character χ of N, we get a unitary representation π^χ of G given by

$$\pi^\chi = {}_G\left|\mathrm{Ind}_N^{\tilde{G}} \chi \right. .$$

We denote by $\tilde{\pi}^\chi$ the representation of the group C*-algebra $C^*(G)$ of G corresponding to the above representation π^χ of G [40]. Let H_χ denote the stability group of $\tilde{K}$ at $\chi \in \hat{N}$. Then, we have a C^*-crossed product $A^\chi = C_0(\tilde{K}/H_\chi) \times_\gamma K$ where γ is the canonical action of $\tilde{K}$ on $C_0(\tilde{K}/H_\chi)$. Let π^0 be the representation of A^χ as defined in section 5.1. Then, we get the following proposition.

Proposition 5.4.3. There exists a homomorphism ψ^χ from $C^*(G)$ onto A^χ such that $\mathrm{Ker}\, \psi^\chi = \mathrm{Ker}\, \tilde{\pi}^\chi$ and $\tilde{\pi}^\chi = \pi^0 \circ \psi^\chi$. This ψ^χ induces a natural correspondence Ψ^χ from Rep A^χ to Rep G which has the following properties.

(i) For each $\pi \in \mathrm{Rep}\, A^\chi$, $\pi(A^\chi)'' = \Psi^\chi(\pi)(G)''$

(ii) For $\pi, \pi' \in \mathrm{Rep}\, A^\chi$, $\pi \cong \pi'$ if and only if $\Psi^\chi(\pi) \cong \Psi^\chi(\pi')$.

(iii) Suppose $\pi^0 \cong \int_Z^\oplus \pi^\zeta d\mu(\zeta)$.

Then, we get
$$\pi^{\chi} \cong \int_{Z}^{\oplus} \Psi^{\chi}(\pi^{\zeta})\, d\mu(\zeta).$$

Proof. It is not difficult to see that $C^*(G)/\mathrm{Ker}\, \tilde{\pi}^{\chi}$ is isomorphic with the C^*-crossed product $C_0(\mathrm{Orb}_{\tilde{K}}(\chi)) \times_{\delta} K$ where δ is the action of $\tilde{K}$ on N and $\mathrm{Orb}_{\tilde{K}}(\chi) = \{\delta(t)\cdot\chi;\ t \in \tilde{K}\}$. Put $\phi^{\chi}(t) = \delta(t)\cdot\chi$ for every $t \in \tilde{K}$. Then, ϕ^{χ} induces a map $\tilde{\phi}^{\chi}$ from $\tilde{K}/H_{\chi}$ to $\mathrm{Orb}_{\tilde{K}}(\chi)$ which is a homeomorphism due to the regularity of $\tilde{G}$ [12]. Through this $\tilde{\phi}^{\chi}$, the action δ of K is transformed into the action γ of K and $C_0(\mathrm{Orb}_{\tilde{K}}(\chi)) \times_{\delta} K$ is isomorphically transformed to $C_0(\tilde{K}/H_{\chi}) \times_{\gamma} K = A^{\chi}$. Then, we get a homomorphism ψ^{χ} from $C^*(G)$ onto A^{χ} such that $\mathrm{Ker}\, \psi^{\chi} = \mathrm{Ker}\, \tilde{\pi}^{\chi}$ by composing these maps.

Since there exists a canonical correspondence between Rep G and Rep $C^*(G)$, the correspondence $\pi \rightarrow \pi\circ\psi^{\chi}$ from Rep A^{χ} to Rep $C^*(G)$ induces the correspondence Ψ^{χ} from Rep A^{χ} into Rep G.

Other properties follow immediately from the above definitions of ψ^{χ} and Ψ^{χ}. We omit the detail. [Q.E.D.]

The following corollary is easily obtained.

Corollary 5.4.4. The unitary representation π^{χ} of G have the following properties.

(i) $\pi^{\chi}(G)'' = \mathfrak{M}(\tilde{K}/H_{\chi}, K)$ and $\pi^{\chi}(G)' = \mathfrak{M}(\tilde{K}/K, H_{\chi})$.

(ii) There exist abelian von Neumann subalgebras $\mathfrak{A}^a$ and $\mathfrak{B}^b$ in $\pi^{\chi}(G)'$ parametrized with $a \in Z(K;\ \tilde{K};\ H_{\chi})$ and $b \in Z(H_{\chi}^{\perp};\ \hat{\tilde{K}};\ K^{\perp})$.

(iii) Corresponding to $\mathfrak{A}^a$, π^{χ} is decomposed as follows.

$$\pi^{\chi} \cong \int_{\hat{H}_{\chi}}^{\oplus} \Psi^{\chi}(\pi^{(a,\zeta)})\, d\mu(\zeta).$$

If $K + H_{\chi}$ is dense in $\tilde{K}$, $\mathfrak{A}^a$ is maximal abelian in $\pi^{\chi}(G)'$ and $\Psi^{\chi}(\pi^{(a,\zeta)})$ are all irreducible representations of G.

(iv) Corresponding to $\mathfrak{B}^b$, π^{χ} is decomposed as follows.

$$\pi^{\chi} \cong \int_{\tilde{K}/K}^{\oplus} \Psi^{\chi}(\hat{\pi}^{(b,z)})\, d\nu(z).$$

If $K \cap H_{\chi} = \{0\}$, $\mathfrak{B}^b$ is maximal abelian in $\pi^{\chi}(G)'$ and $\Psi^{\chi}(\hat{\pi}^{(b,z)})$ are all irreducible representations of G.

We note that the assertion (iii) coincides with the result described in chapter III and that the decomposition in the case of b = 0 at (iv) coincides with the result obtained from the subgroup theorem of induced representations by G.W. Mackey [30]. Corollary 5.4.4 shows that there are other possibilities of decompositions of the representation π^{χ} of G.

Example 5.4.5. The discrete Mautner group

Let $\tilde{G}$ be the 3-dimensional solvable Lie group given as a semi-direct product $\mathbf{C} \times_s R$ of the additive group $\mathbf{C}$ of all complex numbers with R with the multiplication;

$$(z, t)(z', t') = (z + e^{it}z', t + t')$$

for $z, z' \in \mathbf{C}$ and $t, t' \in R$. Take the closed subgroup $G = \mathbf{C} \times_s Z$ of $\tilde{G} = \mathbf{C} \times_s R$. This group G is the discrete Mautner group.

Let χ^{ζ} ($\zeta \in \mathbf{C}$) be unitary characters of $\mathbf{C}$ given by $\chi^{\zeta}(z) = e^{i(z,\zeta)}$, where (,) means the real inner product of $\mathbf{C}$. For $r \in \mathbf{R}^+$, put

$$\pi^r = {}_G\Big| \mathrm{Ind}_{\mathbf{C}}^{\tilde{G}} \ \chi^r.$$

Then, π^r is an injective type II_1 factor representation of G for each $r \in \mathbf{R}^+$.

By Example 5.4.1 and Corollary 5.4.4, we get the following decompositions of π^r.

$$\pi^r \cong \int_0^{1/2 \,\oplus} \pi^{(r,p,u)}\, du \qquad (p \in \mathbf{Z}) \qquad \text{(I)}$$

$$\cong \int_0^{1 \,\oplus} \hat{\pi}^{(r,q,s)}\, ds \qquad (q \in \mathbf{Z}) \qquad \text{(II)}$$

Observe that the $\pi^{(r,p,u)}$ and $\hat{\pi}^{(r,q,s)}$ are irreducible representations of G. We have moreover the following properties.

(i) If $p \neq p'$ (resp. $q \neq q'$), each $\pi^{(r,p,u)}$ (resp. $\hat{\pi}^{(r,q,s)}$) is never unitarily equivalent to $\pi^{(r,p',s)}$ (resp. $\hat{\pi}^{(r,q',s')}$) for arbitrary choice of $u' \in [0,1/2\pi)$ (resp. $s' \in [0,1)$).

(ii) (a) $\quad {}_{\mathbf{C}}\big|\pi^{(r,p,u)} \cong \int_{|\zeta|=r}^{\oplus} \chi^{\zeta}\, d\zeta.$

(b) $\quad {}_{\mathbf{C}}\big|\hat{\pi}^{(r,0,s)} \cong \sum_{n \in \mathbf{Z}}^{\oplus} \chi^{re^{i(n+2\pi s)}}.$

(c) $\quad {}_{\mathbf{C}}\big|\hat{\pi}^{(r,q,s)} \cong |q| \int_{|\zeta|=r}^{\oplus} \chi^{\zeta}\, d\zeta \qquad \text{if } q \neq 0.$

Hence we see that every $\hat{\pi}^{(r,q,s)}$ except the case of $q = \pm 1$ is never unitarily equivalent to each $\pi^{(r,p,u)}$.

We note that the decomposition (II) of π^r is a newly obtained one. It was not described in chapter III and it shows that G has the irreducible representations such as $\hat{\pi}^{(r,q,s)}$ with the property (c) of (ii). We couldn't give an explicit form of $\hat{\pi}^{(r,q,s)}$, but we know the existence of such irreducible representations.

Example 5.4.6. the discrete Heisenberg group

Next, let G be the discrete Heisenberg group, defined as a group of matrices :

$$G = \left\{ \begin{pmatrix} 1 & k & u \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} ;\ k, \ell \in \mathbf{Z} \text{ and } u \in \mathbf{R} \right\}.$$

We take a larger group G given as

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & t & u \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} ;\ \ell \in \mathbf{Z} \text{ and } t,\ u \in \mathbf{R} \right. .$$

Then, G is a closed subgroup of $\tilde{G}$ which is a type I group. We denote an element $\begin{pmatrix} 1 & t & u \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix}$ of $\tilde{G}$ as (ℓ, u, t).

Put

$$N = \{(\ell, u, 0)\ ;\ \ell \in \mathbf{Z} \text{ and } u \in \mathbf{R}\}.$$

Then, N is a normal subgroup of $\tilde{G}$. Let χ^σ $(\sigma \in \mathbf{R})$ be a unitary character of N given by

$$\chi^\sigma((\ell, u, 0)) = e^{i\sigma u}.$$

We define a unitary representation π^σ of G by

$$\pi^\sigma = {}_G\middle|\operatorname{Ind}_N^{\tilde{G}} \chi^\sigma.$$

Then, by Corollary 5.4.4, we get the following decompositions of π^σ.

$$\pi^{\sigma} \cong \int_{0}^{\sigma}{}^{\oplus} U^{(\sigma,p,\lambda)}\, d\lambda \qquad (\, p \in \mathbf{Z}\,) \qquad (\mathrm{I})$$

$$\cong \int_{0}^{1}{}^{\oplus} V^{(\sigma,q,r)}\, dr \qquad (\, q \in \mathbf{Z}\,) \qquad (\mathrm{II})$$

(i) If $\sigma \in \mathbf{R}\backslash(1/2\pi)\mathbf{Q}$, π^{σ} is an injective type II_1 factor representation of G and $U^{(\sigma,p,\lambda)}$ and $V^{(\sigma,q,r)}$ are irreducible representations of G.

(ii) $U^{(\sigma,p,\lambda)}$ is written in an explicit form as follows.

For $\xi(x) \in L^2([0,2\pi/\sigma))$,

$$(U^{(\sigma,p,\lambda)}_{(\ell,u,k)}\xi)(x) = e^{i\sigma u}e^{i\ell x}e^{i\lambda k}e^{i\sigma pkx}e^{i\sigma pk^2/2}\xi(\overline{x+k}),$$

where $0 \leq \overline{x+k} < 2\pi/\sigma$. If $p \neq p'$, $U^{(\sigma,p,\lambda)}$ is never unitarily equivalent to $U^{(\sigma',p',\lambda')}$ for arbitrary choice of $\sigma' \in \mathbf{R}$, $\lambda' \in [0,\sigma)$.

(iii) $V^{(\sigma,q,r)}$ is also written in an explicit form as follows.

For $\zeta(y) \in L^2([0,2\pi))$,

$$(V^{(\sigma,q,r)}_{(\ell,u,k)}\zeta)(y) = e^{i\sigma u}e^{iyk}e^{ir\sigma}\, e^{iq\ell y}e^{i\sigma q\ell^2/2}\zeta(\overline{y+\sigma\ell}),$$

where $0 \leq \overline{y+\sigma\ell} < 2\pi$. If $q \neq q'$, $V^{(\sigma,q,r)}$ is never unitarily equivalent to $V^{(\sigma',q',r')}$ for arbitrary choice of $\sigma' \in \mathbf{R}$, $r' \in [0,\, 1)$.

(iv) If $p \neq \pm 1$ (resp. $q \neq \pm 1$), $U^{(\sigma,p,\lambda)}$ (resp. $V^{(\sigma,q,r)}$) is never unitarily equivalent to $V^{(\sigma',q',r')}$ (resp. $U^{(\sigma',p',\lambda')}$ for any $\sigma' \in \mathbf{R}$, $q' \in \mathbf{Z}$, and $r' \in [0,\, 1)$ (resp. $\sigma' \in \mathbf{R}$, $p' \in \mathbf{Z}$, and $\lambda' \in [0,\, \sigma)$).

For the detail, see [22].

Example 5.4.7. A subgroup of the discrete Heisenberg group

Let $\mathbf{H}$ denote the subgroup of the discrete Heisenberg group, defined by

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & k & m \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} \; ; \quad k, \ell, m \in \mathbf{Z} \right\}.$$

Put

$$1\pi^{\sigma} = {}_{\mathbf{H}}| \pi^{\sigma} \qquad (\sigma \in \mathbf{R}),$$

$$1_U(\sigma,p,\lambda) = {}_{\mathbf{H}}| U(\sigma,p,\lambda) \qquad (p \in \mathbf{Z}, \quad \lambda \in \mathbf{R}),$$

$$1_V(\sigma,q,r) = {}_{\mathbf{H}}| V(\sigma,q,r) \qquad (q \in \mathbf{Z}, \ r \in \mathbf{R}).$$

Then, we get the following results about the unitary representations of $\mathbf{H}$.

$$1_{\pi}{}^{\sigma} \cong \int_0^{\sigma} \oplus \ 1_U(\sigma,p,\lambda) \, d\lambda \qquad \text{for each } p \in \mathbf{Z}$$

$$\cong \int_0^{1} \oplus \ 1_V(\sigma,q,r) \, dr \qquad \text{for each } q \in \mathbf{Z}.$$

(i) If $\sigma \in \mathbf{R} \setminus (1/2\pi)\mathbf{Q}$, $1_{\pi}{}^{\sigma}$ is an injective type II_1 factor representation of H and $1_U(\sigma,p,\lambda)$ and $1_V(\sigma,q,r)$ are irreducible representations of $\mathbf{H}$.

(ii) $1_U(\sigma,p,\lambda) \cong 1_U(\sigma',p',\lambda')$ if and only if $\sigma' - \sigma \in 2\pi Z$, $p' = p$, and $\lambda' - \lambda \in 2\pi\mathbf{Z} + \sigma\mathbf{Z}$. $1_V(\sigma,q,r) \cong 1_V(\sigma',q',r')$ if and only if $\sigma' - \sigma \in 2\pi\mathbf{Z}$, $q' = q$, and $r' - r \in \mathbf{Z} + (2\pi/\sigma)\mathbf{Z}$. Moreover, if $q \neq \pm 1$, each $1_V(\sigma,q,r)$ is never unitarily equivalent to any of $1_U(\sigma,p,\lambda)$.

Example 5.4.8. The free group F_2 on two generators

Let F_2 be the free group generated by two elements a and b. Then, there exists a homomorphism ψ from F_2 onto $\mathbb{H}$ such that

$$\psi(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\psi(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Put

$$ {}^2\pi^\sigma = \pi^\sigma \circ \psi \qquad (\sigma \in \mathbb{R})$$

$$ {}^2U(\sigma, p, \lambda) = {}^1U(\sigma, p, \lambda) \circ \psi \qquad (p \in \mathbb{Z},\ \lambda \in \mathbb{R})$$

$$ {}^2V(\sigma, q, r) = {}^1V(\sigma, q, r) \circ \psi \qquad (q \in \mathbb{Z},\ r \in \mathbb{R}).$$

Then, we can interpret the results in Example 5.4.7 as they concern the representations of F_2

H. Yoshizawa [54] showed, the non-uniqueness of irreducible decompositions of a representation, and it was about the regular representation of F_2 (see section 1.2). Our results are entirely different from his case. We note that ${}^2\pi^\sigma$ is an "injective" type II_1 factor representation of F_2, while the regular representation of F_2 is not injective but type II_1 factor representation.

5.4.9. The Dixmier group $\mathbb{D}$

Let $\mathbb{D}$ be a 7-dimensional simply connected solvable Lie group with the multiplication

$$(u,v,x,y,z)(u',v',x',y',z')$$
$$= (u+e^{2\pi i x}u', v+e^{2\pi i y}v', x+x', y+y', z+z'+xy')$$

for u, $v \in \mathbf{C}$ and x, y, $z \in \mathbf{R}$. $\mathbf{D}$ is known to be a non-type I group among simply connected solvable Lie groups as pointed out by J. Dixmier [8] and so we shall call $\mathbf{D}$ the Dixmier group. Using the Mackey's machine and the results concerning the discrete Heisenberg group G, we shall show that we obtain a large number of irreducible representations of $\mathbf{D}$.

Put

$$M = \{(u, v, 0, 0, 0) , u, v \in \mathbf{C}\}$$

and

$$H = \{(0, 0, x, y, z) \; ; \; x, y, z \in \mathbf{R}\}.$$

Then, M is an abelian group isomorphic with $\mathbf{C}^2$ and H is the Heisenberg group. H acts on M by the inner automorphism and D can be regarded as $M \times_s H$ which is a regular semi-direct product of M with H.

Let $\omega^{(s,t)}$ $(s, t \in R^+)$ be unitary characters of M given by

$$\langle (u, v), \omega^{(s,t)} \rangle = e^{i(s,u)} e^{i(t,v)}$$

for $(u,v) \in \mathbf{C}^2 = N$ where (,) means the real inner product of $\mathbf{C}$. Then, the stability group of H at $\omega^{(s,t)}$ in $\hat{M}$ (the dual of M) is just the discrete Heisenberg group G. Put $\mathbf{D}_0 = M \times_s G$ and define a representation ${}^0W^{(s,t,L)}$ of D_0 for

$\omega^{(s,t)} \in \hat{M}$ and some representation L of G by

$$0_{W(u,v;q)}^{(s,t,L)} = \omega_{(u,v)}^{(s,t)} L_g .$$

Then, we get a unitary representation $W^{(s,t,L)}$ of $\mathbb{D}$ by

$$W^{(s,t,L)} = \mathrm{Ind}_{D_0}^{D}\, 0_W^{(s,t,L)} .$$

For s, $t \in \mathbb{R}^+$ and $\sigma \in \mathbb{R}$, Put

$$3_\pi(s,t,\sigma) = W^{(s,t,L)} \qquad \text{When } L = \pi^\sigma,$$

$$3_U(s,t,\sigma,p,\lambda) = W^{(s,t,L)} \qquad \text{When } L = U^{(\sigma,p,\lambda)},$$

$$3_V(s,t,\sigma,q,r) = W^{(s,t,L)} \qquad \text{When } L = V^{(\sigma,q,r)},$$

where $p \in \mathbb{Z}$, $\lambda \in \mathbb{R}$, $q \in \mathbb{Z}$, and $r \in \mathbb{R}$.

Then, we get the following results immediately.

$$3_\pi(s,t,\sigma)$$

$$\cong \int_0^{\sigma\,\oplus} 3_U(s,t,\sigma,p,\lambda)\, d\lambda \qquad \text{for each } p \in Z$$

$$\cong \int_0^{1\,\oplus} 3_V(s,t,\sigma,q,r)\, dr \qquad \text{for each } q \in Z.$$

(i) If $\sigma \in \mathbb{R} \setminus (1/2\pi)\mathbb{Q}$, $3_\pi(s,t,\sigma)$ is an injective type II_∞ factor representation of $\mathbb{D}$ and $3_U(s,t,\sigma,p,\lambda)$ and $3_V(s,t,\sigma,p,r)$ are irreducible representations of $\mathbb{D}$.

(ii) $3_U(s,t,\sigma,p,\lambda) \cong 3_U(s',t',\sigma',p',\lambda')$ if and only if $s'=s$, $t'=t$, $\sigma'=\sigma$, $p'=p$, and $\lambda'-\lambda \in 2\pi\mathbb{Z} + \sigma\mathbb{Z}$. $3_V(s,t,\sigma,q,r) \cong 3_V(s',t',\sigma',q',r')$ if and only if $s'=s$, $t'=t$, $\sigma'=\sigma$, $q'=q$, and $r'-r \in \mathbb{Z} + (2\pi/\sigma)\mathbb{Z}$. Moreover, if $q \neq \pm 1$, $3_V(s,t,\sigma,q,r)$ is never unitarily equivalent to each $3_U(s',t',\sigma',p,\lambda)$.

Here, we note that ${}^{3}U^{(s,t,\sigma,p,\lambda)}$ ($s,t \in \mathbb{R}^{+}$, $\sigma \in \mathbb{R} \setminus (1/2\pi)\mathbb{Q}$, $p \in \mathbb{Z}$, $\lambda \in \mathbb{R}$) and $V^{(s,t,\sigma,q,r)}$ ($s,t \in \mathbb{R}^{+}$, $\sigma \in \mathbb{R} \setminus (1/2\pi)\mathbb{Q}$, $q \in \mathbb{Z} \setminus \{0, \pm 1\}$, $r \in \mathbb{R}$) are new parametrized families of irreducible representations of the Dixmier group $\mathbb{D}$ so far as we know.

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