The Cauchy problem for Schrödinger type equations with variable coefficients

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0. Introduction. In this paper we study the Cauchy problem for Schrödinger type equations with variable coefficients

$$(0.1)\quad \begin{cases} Lu(t,x) \equiv \frac{1}{i}\partial_t u(t,x) - \frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j}(g^{jk}(x)\partial_{x_k} u) + \\ \qquad\qquad \sum_{j=1}^{n} b^j(x)\partial_{x_j} u + c(x)u = f(t,x) \quad (x \in R^n), \\ u(0,x) = u_0(x), \end{cases}$$

where $g^{jk}(x)$, $b^j(x)$ and $c(x)$ are in $\mathcal{B}^\infty(R^n)$. We suppose that

(0.2) $g^{jk}(x)$ $(j,k = 1,2,\cdots,n)$ are real valued and satisfy

$$g^{jk}(x) = g^{kj}(x)$$

and that the uniform ellipticity

$$(0.3)\qquad \delta^{-1}|p|^2 \le \left| \sum_{j,k=1}^{n} g^{jk}(x)p_j p_k \right| \le \delta|p|^2$$

with a positive constant δ. First of all, remark that it is impossible to consider the well posedness of (0.1) in $C^\infty(R^n)$ space, because (0.1) has an infinite propagation speed (see [10]). Therefore, in the present paper we shall consider the well posedness of (0.1) in the sense of L^2. We denote the set of all L^2 valued continuous functions in $t \in [0,T]$ by $\mathcal{E}_t^0([0,T];L^2)$.

We adapt the following definition.

Defnition 0.1. We say that the Cauchy problem (0.1) is L^2 well posed on $[0,T_0]$ ($T_0 > 0$) (resp. $[T_0,0]$ ($T_0 < 0$)), if the following is valid for each $T \in (0,T_0]$ (resp. $[T_0,0)$). For any $u_0(x) \in L^2$ and any $f(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ (resp. $\mathcal{E}^0_t([T,0];L^2)$) there exists one and only one solution $u(t,x)$ of (0.1) in $\mathcal{E}^0_t([0,T];L^2)$ (resp. $\mathcal{E}^0_t([T,0];L^2)$).

In order to make clear the character of the Cauchy problem (0.1) we compare the following three types of the equations on R^1_x :

$$(0.4)_\varepsilon \qquad \partial_t u(t,x) - \frac{1}{2}\varepsilon\,\partial_x^2 u + ib(x)\partial_x u + ic(x)u = f(t,x)$$
$$(\varepsilon = 1,0,i),$$

that is, parabolic equations ($\varepsilon = 1$), kowalewskian type equations ($\varepsilon = 0$) and Schrödinger type equations ($\varepsilon = i$), where $b(x) \in \mathcal{B}^\infty(R^1)$ and $c(x) \in \mathcal{B}^\infty(R^1)$. For the parabolic equations ($\varepsilon = 1$) it is well known that for any $b(x)$ and $c(x)$ the Cauchy problem is L^2 well posed for positive direction in t, but never L^2 well posed for negative direction. As to kowalewskian type equations ($\varepsilon = 0$) it is well posed for positive and also negative directions in t, if and only if Re $b(x)$ is identically zero. For Schrödinger type equations ($\varepsilon = i$) the characterization of the L^2 well posedness for positive (or negative) direction in t is that $|\int_0^\rho \mathrm{Re}\, b(\theta)d\theta|$ remains

bounded for all $\rho \in R^1$ ([11],[15],[16]). Thus, we would like to remark that, in contrast to the parabolic and kowalewskian types, the characterization of the well posedness of the Cauchy problem for Schrödinger type equations can not be given in a local property, but in a global property of the coefficients.

The results on the above special Schrödinger type equations can be extended to the equations whose $g^{jk}(x)$ are all constants ([4],[5],[11],[16],[17]). In the present paper we study the general equations with variable coefficients $g^{jk}(x)$. In order to state our theorem we introduce the classical orbit of (0.1). Set

$$(0.5) \qquad H(x,p) = \frac{1}{2} \sum_{j,k=1}^{n} g^{jk}(x) p_j p_k$$

and let $(X(t,x,p),P(t,x,p))$ be the solutions of

$$(0.6) \quad \begin{cases} \dfrac{d}{dt} X_j = \dfrac{\partial H}{\partial p_j}(X,P), \ \dfrac{d}{dt} P_k = - \dfrac{\partial H}{\partial x_k}(X,P) \ (j,k = 1,2,\cdots,n). \\ (X,P)\big|_{t=0} = (x,p). \end{cases}$$

$H(x,p)$ and $(X(t,x,p),P(t,x,p))$ are called the Hamiltonian function of (0.1) and the classical orbit of (0.1) respectively. Our theorem is as follows :

Theorem. It is necessary for (0.1) to be L^2 well posed on $[0,T_0]$ or $[T_0,0]$ for a $T_0 \neq 0$ that the inequality

$$(0.7) \quad \sup_{(x,p)\in R^{2n},\rho\in R} \Big| \sum_{j=1}^{n} \int_0^{\rho} \mathrm{Re}\ b^j(X(\theta,x,p))P_j(\theta,x,p)d\theta \Big| < \infty$$

holds.

Remark 0.1. If all $g^{jk}(x)$ are constant, Theorem accords

with results which had been obtained before.

Now, the inequality (0.7) has a geometrical interpretation as follows. We shall introduce in R^n_x a Riemannian metric

$$g_x\langle Y,Z\rangle = \sum_{i,j=1}^{n} g_{ij}(x)Y^iZ^j \quad (Y = \sum_i Y^i\partial_{x_i},\ Z = \sum_j Z^j\partial_{x_j} \in T_xR^n),$$

where $(g_{ij}(x);\ {}^{i\downarrow}_{j\rightarrow}1,2,\cdots,n)$ is the inverse matrix of $(g^{ij}(x);\ {}^{i\downarrow}_{j\rightarrow}1,2,\cdots,n)$. We denote this Riemannian manifold by M. Let ω be a one form on M defined by

$$(0.8) \qquad \omega(Y) = g_x\langle Y, \sum_{j=1}^{n} \mathrm{Re}\ b^j(x)\partial_{x_j}\rangle \quad (Y \in T_xM),$$

that is,

$$(0.8)' \qquad \omega = \sum_{j,k=1}^{n} \mathrm{Re}\ b^j(x)g_{jk}(x)dx_k.$$

If we use the above notations, Theorem may be rewritten as follows:

Theorem'. It is necessary for (0.1) to be L^2 well posed on $[0,T_0]$ or $[T_0,0]$ for a $T_0 \neq 0$ that the inequality

$$(0.7)' \qquad \sup_{\gamma\in\Gamma} \left| \int_\gamma \omega \right| < \infty$$

holds, where Γ is the family of all geodesics on M.

Remark 0.2. In [6] we shall study the Cauchy problem for Schrödinger type equations on a Riemannian manifold without boundary.

Remark 0.3. In section 4 we shall consider the Cauchy problem (0.1) whose $g^{jk}(x)$ do not satisfy the uniform ellipticity (0.3) (Theorem 4.3).

Now, we explain the ideas of the proof of Theorem. We shall prove it by contradiction. We make a change of a variable from t to $\tau = \lambda t$ with a large parameter $\lambda \geq 1$. Then, using (0.5), (0.1) is written in the form

$$(0.1)' \begin{cases} \lambda^2 L_\lambda(u(\tau/\lambda,x)) \\ \equiv \lambda^2[(i\lambda)^{-1}\partial_\tau + H(x,(i\lambda)^{-1}\partial_x) + (i\lambda)^{-1}\ell_\lambda^{(1)}(x,(i\lambda)^{-1}\partial_x) \\ \qquad\qquad\qquad\qquad]u(\tau/\lambda,x) = f(\tau/\lambda,x), \\ u(\tau/\lambda,x)\big|_{\tau=0} = u_0(x). \end{cases}$$

If the Cauchy problem (0.1) is L^2 well posed on $[0,T_0]$, the Cauchy problem for the equation

$$(0.9) \qquad L_\lambda v_\lambda(\tau,x) = f_\lambda(\tau,x)$$

is also L^2 well posed on $[0,\lambda T_0]$. So, we obtain a priori estimate of the Cauchy problem for (0.9). Next, if we assume that (0.7) is not valid, we can take a $t_0 > 0$ and a point $(x^0,p^0) \in R^{2n}$ such that

$$(0.10) \qquad \sum_{j=1}^{n} \int_0^{t_0} \mathrm{Re}\, b^j(X(\theta,x^0,p^0))P_j(\theta,x^0,p^0)d\theta \geq \log \frac{3}{2}C(T_0) .$$

Here, $C(T_0) \geq 1$ is a constant determined from the L^2 well posedness of (0.1) on $[0,T_0]$ (see Lemma 3.1). Then, we can construct asymptotic solutions $v_\lambda(\tau,x)$ of (0.9) on the interval $[0,t_0]$ having L^2-estimates which contradict the a priori estimate of the Cauchy problem for (0.9) derived from the L^2 well posedness on $[0,t_0]$.

We note that the above t_0 may be very large. So, we must

construct on the global interval $[0,t_0]$ asymptotic solutions of (0.9). We would like to remark that such a construction is in contrast with the one for the study of hyperbolic equations (c.f.[9]). As it will be shown in the appendix, the fact we would like to emphasize is that if at least one of $g^{jk}(x)$ $(j,k = 1,2,\cdots,n)$ is not constant, in almost cases the Hamilton-Jacobi equation

$$\partial_\tau \Phi + H(x,\frac{\partial \Phi}{\partial x}) = 0$$

has no smooth solution on the global interval $[0,t_0]$. It seems to us that this fact has obstructed the progress of the study for (0.1) with variable coefficients $g^{jk}(x)$. Of course, if all $g^{jk}(x)$ are constant, we can construct asymptotic solutions of (0.9) on $[0,t_0]$ via the form $e^{i\lambda\Phi(\tau,x)}\{\psi_0(x) + (i\lambda)^{-1}\psi_1(x) + \cdots\}$ by using the solution $\Phi(\tau,x)$ of the Hamilton-Jacobi equation.

To avoid the above obstruction, we shall use the Maslov method originally due to [8]. For the proof of our theorem it is necessary to estimate asymptotic solutions $v_\lambda(\tau,x)$ of (0.9) by the L^2-norm on R^n_x for each $\tau \in [0,t_0]$ and also to estimate on $[0,t_0]$ the remainder terms by the L^2-norm on R^n_x. In [8] the remainder terms are estimated only on the compact sets in $R^{n+1}_{\tau,x}$. Hence, in the present paper the modifications of [8] are necessary mainly in the following two aspects. First, we shall consider not a Lagrangian manifold in $R^{2(n+1)}_{\tau,x,E,p}$ (E denotes the

dual variable of τ) , but a family of Lagrangian manifolds in $R^{2n}_{x,p}$ with a parameter $\tau \in [0,t_0]$. Secondly, we shall estimate the remainder terms not on the compact set in $R^{n+1}_{\tau,x}$, but on $[0,t_0] \times R^n_x$ by the L^2-norm .

The plan of the present paper is as follows. Theorem and Theorem' will be proved in section 3. The main results on the Maslov method will be stated in section 3 (Proposition 3.4) and will be proved in section 4. Sections 1 and 2 are devoted to the preliminaries for sections 3 and 4.

The author wishes to express his sincere gratitude for their advices and encouragements to Professor M.Ikawa, Professor S.Mizohata, Professor K.Shinkai and Professor D.Fujiwara.

1. Pseudo-differential operators with a large parameter λ.

Let $(x_1,\cdots,x_n)$ and $p = (p_1,\cdots,p_n)$ denote the points of R^n and let $\alpha = (\alpha_1,\cdots,\alpha_n)$ be a multi-index whose components α_j are non-negative integers. Then, we use the usual notations :

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \alpha! = \alpha_1! \cdots \alpha_n!,$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j},$$

$$D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}.$$

Let $K = \{k_1,\cdots,k_\ell\}(1 \le k_1 < k_2 < \cdots < k_\ell \le n)$ be a subset of the set $\{1,2,\cdots,n\}$. We permit that K is empty. For the sake of simplicity we denote the complementary set of K by K' in the present paper. Then, as in [8] we denote

$$|K| = \ell, \quad x_K = (x_{k_1},\cdots,x_{k_\ell}), \quad x_K \cdot p_K = \sum_{j=1}^{\ell} x_{k_j} p_{k_j},$$

$$x_K \cdot dp_K = \sum_{j=1}^{\ell} x_{k_j} dp_{k_j}, \quad \langle x_K \rangle = (1 + |x_K|^2)^{1/2}.$$

Also, let K_1 be another subset of $\{1,2,\cdots,n\}$ and let $\varphi(x)$ and $f_j(x)$ $(j = 1,2,\cdots,m)$ be C^∞ functions on R^n. Then, we denote for $f(x) = (f_1(x),\cdots,f_m(x))$

$$\frac{\partial \varphi}{\partial x_K} = \left(\frac{\partial \varphi}{\partial x_{k_1}},\cdots,\frac{\partial \varphi}{\partial x_{k_\ell}}\right), \quad \frac{\partial f}{\partial x_K} = \left(\frac{\partial f_i}{\partial x_{k_j}} \; ; \; \begin{matrix} i \downarrow 1,2,\cdots,m \\ j \rightarrow 1,2,\cdots,\ell \end{matrix}\right),$$

$$\frac{\partial^2 \varphi}{\partial x_{K_1} \partial x_K} = \frac{\partial}{\partial x_{K_1}}\left(\frac{\partial \varphi}{\partial x_K}\right), \quad \frac{\partial^2 \varphi}{\partial x_K^2} = \frac{\partial}{\partial x_K}\left(\frac{\partial \varphi}{\partial x_K}\right),$$

If $|K| = m$, $\frac{D(f)}{D(x_K)}$ denotes the Jacobian determinant.

Let $\mathcal{S} = \mathcal{S}(R^n)$ be the Schwartz space of rapidly decreasing functions on R^n. Following [8], we define the λ-Fourier

transformation $(\mathcal{F}_{\lambda,x_K\to p_K}u)(x_{K'},p_K)$ over a part of variables for $u(x) \in \mathcal{S}$ by

$$(1.1)\quad (\lambda/2\pi)^{|K|/2}\int e^{-i\lambda x_K\cdot p_K}u(x)dx_K$$

and then, the inverse λ-Fourier transformation $(\mathcal{F}^{-1}_{\lambda,p_K\to x_K}v)(x_K,p_{K'})$ for $v(p) \in \mathcal{S}$ is defined by

$$(1.2)\quad (\lambda/2\pi)^{|K|/2}\int e^{i\lambda x_K\cdot p_K}v(p)dp_K.$$

If $\lambda = 1$, $\mathcal{F}_{\lambda,x_K\to p_K}$ and $\mathcal{F}^{-1}_{\lambda,p_K\to x_K}$ denote the usual partial Fourier and the inverse Fourier transformations respectively.

Remark 1.1. We remark that the definitions (1.1) and (1.2) of the λ-Fourier and the inverse λ-Fourier transformations in the present paper are slightly different from those in [8]. If we multiply (1.1) and (1.2) by constants $e^{-i|K|\pi/4}$ and $e^{i|K|\pi/4}$ respectively, we obtain corresponding transformations in [8]. But, we shall use the same symbols as those in [8].

The following lemma is easily shown from the Plancherel theorem for the usual Fourier transformation.

Lemma 1.1. Let K be a subset of $\{1,2,\cdots,n\}$. Then, we get

$$\iint(\mathcal{F}_{\lambda,x_K\to p_K}u_1)(x_{K'},p_K)\overline{(\mathcal{F}_{\lambda,x_K\to p_K}u_2)(x_{K'},p_K)}dx_K,dp_K$$
$$= \int u_1(x)\overline{u_2(x)}dx$$

for $u_j(x) \in \mathcal{S}$ $(j = 1,2)$.

We introduce for a real m the symbol class $T^m(R^n_x)$ of pseudo-differential operators from [8].

Definition 1.1. $T^m(R^n_x)$ denotes the class of all C^∞ functions $h(x,p)$ on $T^*R^n_x = R^{2n}_{x,p}$ satisfying

$$|h^{(\alpha)}_{(\beta)}(x,p)| \le C_{\alpha,\beta}(\langle x\rangle\langle p\rangle)^m$$

for all multi-indices α and β with constants $C_{\alpha,\beta}$, where $h^{(\alpha)}_{(\beta)}(x,p) = \partial_p^\alpha D_x^\beta h(x,p)$.

We define semi-norms $|h|^{(m)}_\ell$ $(\ell = 0,1,2,\cdots)$ of $h(x,p) \in T^m(R^n_x)$ by

$$\sup_{x,p} (\langle x\rangle\langle p\rangle)^{-m} \sum_{|\alpha+\beta|\le\ell} |h^{(\alpha)}_{(\beta)}(x,p)|.$$

The λ-pseudo-differential operator $h(x,\lambda^{-1}D_x)$ with a symbol $h(x,p) \in T^m(R^n_x)$ is defined by

$$(1.3)\quad h(x,\lambda^{-1}D_x)u(x) = \iint e^{i(x-x')\cdot p}\, h(x,\lambda^{-1}p)u(x')dx'\bar{d}p$$

for $u(x) \in \mathcal{S}$, where $\bar{d}p = (2\pi)^{-n}dp$. Next, as in section 2 of chapter 2 in [7], we shall introduce the class of double symbols.

Definition 1.2. We denote by $\tilde{T}^{m,m'}(R^n_x)$ the class of all C^∞ functions $h(x,p,x',p')$ on $T^*R^{2n}_{x,x'} = R^{4n}_{x,p,x',p'}$ satisfying

$$|h^{(\alpha,\alpha')}_{(\beta,\beta')}(x,p,x',p')| \le C_{\alpha,\alpha',\beta,\beta'}(\langle x\rangle\langle p\rangle)^m(\langle x'\rangle\langle p'\rangle)^{m'}$$

for all multi-indices α, α', β and β' with constant

$C_{\alpha,\alpha',\beta,\beta'}$, where $h^{(\alpha,\alpha')}_{(\beta,\beta')}(x,p,x',p') = \partial_p^{\alpha}\partial_p^{\alpha'}D_x^{\beta}D_x^{\beta'}h(x,p,x',p')$.

We define semi-norms $|h|^{(m,m')}_{\ell}$ $(\ell = 0,1,2,\cdots)$ of $h(x,p,x',p') \in \tilde{T}^{m,m'}(R^n_x)$ by

$$\sup_{x,p,x',p'} (\langle x\rangle\langle p\rangle)^{-m} (\langle x'\rangle\langle p'\rangle)^{-m'} \sum_{|\alpha+\alpha'+\beta+\beta'|\le\ell} |h^{(\alpha,\alpha')}_{(\beta,\beta')}(x,p,x',p')|.$$

The λ-pseudo-differential operator $h(x,\lambda^{-1}D_x,x',\lambda^{-1}D_{x'})$ with a double symbol $h(x,p,x',p') \in \tilde{T}^{m,m'}(R^n_x)$ is defined by

$$(1.4)\quad h(x,\lambda^{-1}D_x,x',\lambda^{-1}D_{x'})u(x) = \iiiint e^{i(x-x')\cdot p + i(x'-x'')\cdot p'} h(x,\lambda^{-1}p,x',\lambda^{-1}p') u(x'')dx''\!\!\!^{-}p'dx'\!\!\!^{-}p$$

for $u(x) \in \mathcal{S}$.

Let $h(x,p) \in T^m(R^n_x)$ and K be a subset of $\{1,2,\cdots,n\}$. For the sake of simplicity we shall write $h(x,p)$ as $h(x_{K'},x_K,p_{K'},p_K)$. Then, a λ-pseudo-differential operator $h(x_{K'},-\lambda^{-1}D_{p_K},\lambda^{-1}D_{x_{K'}},p'_K)$, which acts on the functions on $R^n_{x_{K'},p_K}$, with a double symbol can be defined. We can easily see that its double symbol $h(x_{K'},-x_K,p_{K'},p'_K) \in C^{\infty}(R^{4n}_{x,p,x',p'}) = C^{\infty}(T^*R^{2n}_{x_{K'},p_K,x'_{K'},p'_K})$ belongs to the class $\tilde{T}^{m,m}(R^n_{x_{K'},p_K})$.

The following lemmas 1.2 - 1.5 can be proved by the same arguments as those in chapter 2 of [7].

Lemma 1.2. Let $h(x,p) \in T^m(R^n_x)$ and K be a subset of $\{1,2,\cdots,n\}$. Then, it follows for $u(x) \in \mathcal{S}$ that

$$h(x,\lambda^{-1}D_x)u(x)$$
$$= \mathcal{F}^{-1}_{\lambda,p_K\to x_K}\{h(x_{K'},-\lambda^{-1}D_{p_K},\lambda^{-1}D_{x_{K'}},p'_K)\mathcal{F}_{\lambda,x_K\to p_K}u\}(x).$$

Lemma 1.3. Let $h(x,p,x',p') \in \tilde{T}^{m,m'}(R^n_x)$ and N be a non-negative integer. Then, we have for $u(x) \in \mathcal{S}$

$$h(x,\lambda^{-1}D_x,x',\lambda^{-1}D_{x'})u(x)$$
$$= \sum_{|\alpha|<N} \frac{1}{\alpha!}\lambda^{-|\alpha|}h^{(\alpha,0)}_{(0,\alpha)}(x,\lambda^{-1}D_x,x,\lambda^{-1}D_x)u(x) + \lambda^{-N}r_{N,\lambda}(x,\lambda^{-1}D_x)u(x),$$

where $h^{(\alpha,0)}_{(0,\alpha)}(x,\lambda^{-1}D_x,x,\lambda^{-1}D_x)$ denotes the λ-pseudo-differential operator with symbol $h^{(\alpha,0)}_{(0,\alpha)}(x,p,x,p) \in T^{m+m'}(R^n_x)$ and $r_{N,\lambda}(x,p)$ belongs to $T^{m+m'}(R^n_x)$. Setting $s = [(n + |m| + |m'|)/2 + 1]$, we get the estimates for $\ell = 0,1,2,\cdots$

$$|r_{N,\lambda}|^{(m+m')}_{\ell} \le C_{N,\ell}\,|h|^{(m,m')}_{2(N+2s+\ell)}$$

with constants $C_{N,\ell}$ independent of $h(x,p,x',p')$ and $\lambda \ge 1$. For a real m $[m]$ denotes the largest integer not greater than m.

Lemma 1.4. Let $h(x,p) \in T^m(R^n_x)$ and set $s = [(n + |m|)/2 + 1]$. Then, if we denote the L^2-norm of $u(x) \in \mathcal{S}$ by $||u(\cdot)||$, it follows for $u(x) \in \mathcal{S}$ that

$$||h(x,\lambda^{-1}D_x)u(\cdot)|| \le C\,|h|^{(m)}_{2s} \sum_{|\alpha|\le 2s} ||\langle\cdot\rangle^m D^\alpha_x u(\cdot)||,$$

where C is a constant independent of $h(x,p)$, $u(x)$ and $\lambda \ge 1$.

Lemma 1.5. Assume that $h(x,p)$ belongs to $T^m(R^n_x)$ and that $S(x) \in \mathcal{B}^\infty(R^n)$ is a real valued function. Let $\varphi(x) \in \mathcal{S}$. Then,

$e^{-i\lambda S(x)}h(x,\lambda^{-1}D_x)(e^{i\lambda S(x)}\varphi(x))$ has the asymptotic expression

$$(1.5)\qquad \sum_{j=0}^{N-1}(i\lambda)^{-j}\mathcal{D}_j(x,D_x)\varphi(x) + (i\lambda)^{-N}R_N\varphi(x)$$

for $N = 1,2,\cdots$, where the remainder term $R_N\varphi(x)$ satisfies

$$(1.6)\qquad |R_N\varphi(x)| \le C_{N,N'}\langle x\rangle^{-N'}$$

for $N' = 0,1,\cdots$ with constant $C_{N,N'}$ independent of x and $\lambda \ge 1$. $\mathcal{D}_j(x,D_x)$ $(0 \le j \le N-1)$ are linear differential operators of order at most j with C^∞ coefficients and are defined independently of $\varphi(x)$ and $\lambda \ge 1$. In particular, $\mathcal{D}_0(x,D_x)$ and $\mathcal{D}_1(x,D_x)$ have the forms

$$(1.7)\qquad \begin{cases} \mathcal{D}_0(x,D_x)\varphi(x) = h(x,\dfrac{\partial S}{\partial x}(x))\varphi(x), \\ \mathcal{D}_1(x,D_x)\varphi(x) = \displaystyle\sum_{j=1}^{n}\frac{\partial h}{\partial p_j}(x,\frac{\partial S}{\partial x}(x))\frac{\partial \varphi}{\partial x_j}(x) + \\ \qquad\qquad \dfrac{1}{2}\{\mathrm{Tr}\ \dfrac{\partial^2 h}{\partial p^2}(x,\dfrac{\partial S}{\partial x}(x))\dfrac{\partial^2 S}{\partial x^2}(x)\}\varphi(x), \end{cases}$$

where Tr A for a square matrix A denotes the trace of A.

2. The family of Lagrangian manifolds. An immersed submanifold Λ in $T^*R^n_x = R^{2n}_{x,p}$ is called a Lagrangian manifold, if the two form $\sum_{j=1}^{n} dp_j \wedge dx_j$ is identically zero on Λ (see Definition 4.1 in [8]). We shall state the fundamental lemma in the Maslov theory without proof. See Proposition 4.6 in [8] for the proof.

Lemma 2.1. Assume that Λ is an n-dimensional Lagrangian manifold. Then, for any point $(x^0,p^0) \in \Lambda$ there exist an open neighborhood $\mathcal{O}$ of (x^0,p^0) on Λ and a subset K of $\{1,2,\cdots,n\}$ such that $(x_K,,p_K)$ is a local coordinate system on $\mathcal{O}$. K' is the complementary set of K.

Since we assume that the uniform ellipticity (0.3) holds , we can easily see that for each $(x,p) \in T^*R^n_x$ there exist the solutions $(X(\tau,x,p),P(\tau,x,p))$ of (0.6) for all $\tau \in R$. So, for each $\tau \in R$ we can define a mapping h^τ from $R^{2n}_{x,p}$ to $R^{2n}_{x,p}$ by

$$(2.1) \qquad h^\tau(x,p) = (X(\tau,x,p),P(\tau,x,p)).$$

It is easy to see that a family of mappings $\{h^\tau\}_{\tau\in R}$ forms a one-parameter group of diffeomorphism from $R^{2n}_{x,p}$ onto $R^{2n}_{x,p}$. Let Λ^n_0 be an n-dimensional immersed submanifold in $T^*R^n_x$. Then, we set for each $\tau \in R$

$$(2.2) \qquad \Lambda^n_\tau = \{r' = h^\tau(x,p);\ (x,p) \in \Lambda^n_0\}$$

and set

(2.3) $\Lambda^{n+1}(T) = \{r = (\tau, h^{\tau}(x,p));\ 0 \le \tau \le T,\ (x,p) \in \Lambda_0^n\}$.

Then, we know well

Lemma 2.2. If Λ_0^n is a C^{∞} n-dimensional Lagrangian manifold, Λ_{τ}^n defined by (2.2) is also a C^{∞} n-dimensional Lagrangian manifold for each $\tau \in R$.

Let Λ_0^n be a C^{∞} n-dimensional Lagrangian manifold and define the family of Lagrangian manifolds $\Lambda^{n+1}(T)$ by (2.3) for any $T > 0$. Then, we get the following lemma needed in section 3 : We remark that only Lemma 2.2 is necessary in [8].

Lemma 2.3. Let Λ_0^n be a C^{∞} n-dimensional Lagrangian manifold. Then, the two form $\sum_{j=1}^{n} dp_j \wedge dx_j - dH \wedge d\tau$ is identically zero on $\Lambda^{n+1}(T)$.

Proof. Let $\xi^0 = (x^0, p^0) \in \Lambda_0^n$ and $\mathcal{O}_0$ be an open neighborhood of ξ^0 on Λ_0^n with local coordinates $y = (y_1, \cdots, y_n)$. We write

$$\mathcal{O}_0 = \{(x^0(y), p^0(y)); y \in U\}$$

by using C^{∞} functions $x^0(y) = (x_1^0(y), \cdots, x_n^0(y))$ and $p^0(y) = ((p_1^0(y), \cdots, p_n^0(y))$ on U. If we set for each $\tau \in R$

$$\mathcal{O}_{\tau} = \{h^{\tau}\xi;\ \xi \in \mathcal{O}_0\},$$

y are also local coordinates on $\mathcal{O}_{\tau}$. Now, since Λ_{τ}^n is a Lagrangian manifold from Lemma 2.2, we get for any point $\xi = (x^0(y), p^0(y)) \in \mathcal{O}_0$ and any $\tau \in R$

$$
\begin{aligned}
&\Big(\sum_{j=1}^{n} dp_j \wedge dx_j - dH \wedge d\tau\Big)_{(\tau,h^{\tau}\xi)}(\partial_{y_k},\partial_{y_\ell}) \\
&= \Big(\sum_{j} dp_j \wedge dx_j\Big)_{(\tau,h^{\tau}\xi)}(\partial_{y_k},\partial_{y_\ell}) \\
&= 0 \qquad (k,\ \ell = 1,2,\cdots,n).
\end{aligned}
$$

So, we have only to prove for the completeness of the proof that

$$
\Big(\sum_{j=1}^{n} dp_j \wedge dx_j - dH \wedge d\tau\Big)_{(\tau,h^{\tau}\xi)}(\partial_{\tau},\partial_{y_k}) = 0 \quad (k = 1,2,\cdots,n)
$$

are valid for any $\xi = (x^0(y),p^0(y)) \in \mathcal{O}_0$ and any $\tau \in R$. For the sake of simplicity, we write $(X(\tau,x^0(y),p^0(y)),P(\tau,x^0(y),p^0(y)))$ as $(X(\tau,y),P(\tau,y))$. Then, it follows from (0.6) that

$$
\begin{aligned}
&\Big(\sum_{j} dp_j \wedge dx_j - dH \wedge d\tau\Big)_{(\tau,h^{\tau}\xi)}(\partial_{\tau},\partial_{y_k}) \\
&= \sum_{j} \{(\frac{d}{d\tau} P_j(\tau,y))(\frac{\partial}{\partial y_k} X_j(\tau,y)) - (\frac{d}{d\tau} X_j(\tau,y))(\frac{\partial}{\partial y_k} P_j(\tau,y))\} + \\
&\qquad\qquad \frac{\partial}{\partial y_k} H(X(\tau,y),P(\tau,y)) \\
&= -\sum_{j} \{\frac{\partial H}{\partial x_j}(X,P)\frac{\partial}{\partial y_k} X_j + \frac{\partial H}{\partial p_j}(X,P)\frac{\partial}{\partial y_k} P_j\} + \\
&\qquad\qquad \frac{\partial}{\partial y_k} H(X(\tau,y),P(\tau,y)) \\
&= 0,
\end{aligned}
$$

which completes the proof. Q.E.D.

Let $S_0(x)$ be a real valued C^{∞} function on R^n and set

$$
\Lambda_0^n = \{(x,\frac{\partial S_0}{\partial x}(x));\ x \in R^n\}. \tag{2.4}
$$

Then, we know well that Λ_0^n is an n-dimensional Lagrangian manifold. Let Ξ' be the diffeomorphic mapping from R_x^n onto Λ_0^n defined by

$$(2.5)\qquad \Xi' : R^n_x \ni x \longrightarrow (x, \frac{\partial S_0}{\partial x}(x)) \in \Lambda^n_0.$$

Then, if we use the natation $(\Xi'^{-1})^* S_0(\xi) = S_0(\Xi'^{-1}(\xi))$ $(\xi \in \Lambda^n_0)$, we get

Lemma 2.4. It follows that

$$(2.6)\qquad \sum_{j=1}^{n} p_j dx_j = d(\Xi'^{-1})^* S_0 \qquad \text{on} \quad \Lambda^n_0 .$$

Proof. We have at each point $\xi = (x, \frac{\partial}{\partial x} S_0(x))$ on Λ^n_0

$$\sum_{j=1}^{n} p_j dx_j = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} S_0(x) dx_j$$
$$= d(\Xi'^{-1})^* S_0 . \qquad \text{Q.E.D.}$$

Let Λ^n_0 be an n-dimensional Lagrangian manifold defined by (2.4). We define Λ^n_τ and $\Lambda^{n+1}(T)$ by (2.2) and (2.3) for this Λ^n_0, respectively. Then, since we can determine the diffeomorphic mapping $h^\tau \circ \Xi'$ from R^n_x onto Λ^n_τ, we can introduce a volume form dV^n_τ on Λ^n_τ by

$$(2.7)\qquad dV^n_\tau = ((h^\tau \circ \Xi')^{-1})^* dx_1 \wedge \cdots \wedge dx_n,$$

where $dx_1 \wedge \cdots \wedge dx_n$ is an n-form on R^n_x and $((h^\tau \circ \Xi')^{-1})^* dx_1 \wedge \cdots \wedge dx_n$ denotes the pull back by the mapping $(h^\tau \circ \Xi')^{-1}$ of the n-form $dx_1 \wedge \cdots \wedge dx_n$. Let Ω be a sufficiently small open set on $\Lambda^{n+1}(T)$. Then, there exists a subset K of $\{1,2, \cdots, n\}$ from Lemma 2.1 so that $(\tau, I_K) \equiv (\tau, x_{K'}, p_K)$ become local coordinates on Ω. So, there exists a C^∞ positive function $J_K(r) \equiv J_K(r;\Omega)$ on Ω which satisfies

(2.8) $dV^n_\tau = J_K(r)^{-1} dx_K, \wedge dp_K$ or $- J_K(r)^{-1} dx_K, \wedge dp_K$

at $r' \in \Lambda^n_\tau$ such that $r = (\tau, r') \in \Omega$. Here, dx_K, and dp_K denote a $|K'|$-form $dx_{k'_1} \wedge \cdots \wedge dx_{k'_\ell}$ ($K' = \{k'_1, \cdots, k'_\ell\} (k'_1 < k'_2 < \cdots < k'_\ell)$) and a $|K|$-form $dp_{k_1} \wedge \cdots \wedge dp_{k_{n-\ell}}$ ($K = \{k_1, \cdots, k_{n-\ell}\}$ $(k_1 < \cdots < k_{n-\ell})$), respectively. Since dV^n_τ does not vanish at any point of Λ^n_τ, $J_K(r)$ is well defined. Any point $r' \in \Lambda^n_\tau$ is written in the form

$$r' = h^\tau(y, \frac{\partial S_0}{\partial x}(y))$$
$$= (X(\tau, y, \frac{\partial S_0}{\partial x}(y)), P(\tau, y, \frac{\partial S_0}{\partial x}(y)))$$

for a $y \in R^n$. So, we can take $y = (y_1, \cdots, y_n)$ as local coordinates on Λ^n_τ. Then,

(2.7)' $dV^n_\tau = dy_1 \wedge \cdots \wedge dy_n$

is valid from (2.7). So, if we take (τ, y) as local coordinates on Ω, we have from (2.8)

(2.9) $J_K(r;\Omega)$

$$= |\det \frac{\partial}{\partial y}(X_K, (\tau, y, \frac{\partial S_0}{\partial x}(y)), P_K(\tau, y, \frac{\partial S_0}{\partial x}(y)))|$$

for $r = (\tau, h^\tau(y, \frac{\partial S_0}{\partial x}(y))) \in \Omega$.

3. Proofs of Theorem and Theorem'. As was mentioned in the introduction, we shall prove Theorem by contradiction. That is, we assume

(As.1) the Cauchy problem (0.1) is L^2 well posed on $[0,T_0]$ ($T_0 > 0$) or $[T_0,0]$ ($T_0 < 0$) for a $T_0 \neq 0$

and

(As.2) the inequality (0.7) is not valid.

In place of (As.1) we may assume without the loss of generality

(As.1)' the Cauchy problem (0.1) is L^2 well posed on $[0,T_0]$ ($T_0 > 0$).

Let L_λ be the differential operator defined in (0.1)'. Then, we obtain

Lemma 3.1. Assume (As.1)'. Then, there exists a constant $C(T_0) \geq 1$ such that if $v_\lambda(\tau,x) \in \mathcal{E}^0_\tau([0,T];L^2)$ and $L_\lambda v_\lambda(\tau,x) \in \mathcal{E}^0_\tau([0,T];L^2)$, the inequality

$$\max_{0\leq\tau\leq T} ||v_\lambda(\tau,\cdot)|| \leq C(T_0)(||v_\lambda(0,\cdot)|| + \lambda^2 \max_{0\leq\tau\leq T} ||L_\lambda v_\lambda(\tau,\cdot)||)$$

is valid for each $\lambda \geq 1$ and each $T \in (0,\lambda T_0]$.

Theorem 3.2. Assume (As.1)' and (As.2). Then, there exist $p^0 \in R^n$, $t_0 > 0$ and $v(x) \in C_0^\infty(R^n)$ such that we can construct an asymptotic solution $v_\lambda(\tau,x) \in \mathcal{E}^0_\tau([0,t_0];L^2)$ of (0.9) with an initial data $e^{i\lambda x\cdot p^0}v(x)$ at $t = 0$ satisfying

$$(3.1) \qquad \max_{0\le\tau\le t_0} \|L_\lambda v_\lambda(\tau,\cdot)\| = O(\lambda^{-3}),$$

and

$$(3.2) \qquad \|v_\lambda(t_0,\cdot)\| \ge \frac{9}{8} C(T_0)\|v(\cdot)\| + O(\lambda^{-1})$$

for large λ. Here, $C(T_0)$ is the constant in Lemma 3.1.

Theorem is deduced from Lemma 3.1 and Theorem 3.2. Indeed, substitute (3.1) into Lemma 3.1. Then, we have

$$\|v_\lambda(t_0,\cdot)\| \le C(T_0)\|v(\cdot)\| + O(\lambda^{-1}),$$

where λ is large so that $\lambda T_0 \ge t_0$. On the other hand, (3.2) is valid. So, we have

$$\frac{9}{8} C(T_0)\|v(\cdot)\| \le C(T_0)\|v(\cdot)\| + O(\lambda^{-1})$$

for large λ so that $\lambda T_0 \ge t_0$. This inequality shows a contradiction for large λ. Thus, Theorem is proved.

Now, we return to the proofs of Lemma 3.1 and Theorem 3.2. The proof of Theorem 3.2 is the essential part in the present paper.

I. The proof of Lemma 3.1. We shall consider the Cauchy problem (0.1) on the interval $[0,T]$ for a $T \in (0,T_0]$. Recall the definition of the L^2 well posedness of the Cauchy problem (0.1) on $[0,T_0]$. Then, if we assume (As.1)', for any $u_0(x) \in L^2$ and any $f(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ we have one and only one solution $u(t,x)$ of (0.1) in $\mathcal{E}^0_t([0,T];L^2)$. We first get

Lemma 3.1'. Assume (As.1)'. Then, there exists a constant

$C(T_0) \geq 1$ such that if $u(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ is the solution of (0.1) for $f(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ and $u_0(x) \in L^2$,

$$\max_{0\leq t\leq T} ||u(t,\cdot)|| \leq C(T_0)(||u_0(\cdot)|| + \max_{0\leq t\leq T} ||f(t,\cdot)||)$$

is valid for each $T \in (0,T_0]$.

Proof. We first note that $\mathcal{E}^0_t([0,T];L^2)$ is a Banach space with a norm $\max_{0\leq t\leq T} ||g(t,\cdot)||$ and so, $L^2 \times \mathcal{E}^0_t([0,T];L^2)$ is also a Banach space. Now, since the Cauchy problem (0.1) is L^2 well posed on $[0,T_0]$, the mapping : $L^2 \times \mathcal{E}^0_t([0,T_0];L^2) \ni (u_0(x),f(t,x)) \longrightarrow u(t,x) \in \mathcal{E}^0_t([0,T_0];L^2)$ is closed, where $u(t,x)$ is the solution of $Lu(t,x) = f(t,x)$ with $u(0,x) = u_0(x)$. Hence, if we apply the Banach closed graph theorem, the above mapping is continuous. So, there exists a constant $C(T_0) \geq 1$ such that

$$\max_{0\leq t\leq T_0} ||u(t,\cdot)|| \leq C(T_0)||u_0(\cdot)|| + \max_{0\leq t\leq T_0} ||f(t,\cdot)|| \tag{3.3}$$

is valid.

Take a $T \in (0,T_0]$ and let $\tilde{u}(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ be the solution of $L\tilde{u}(t,x) = \tilde{f}(t,x) \in \mathcal{E}^0_t([0,T];L^2)$ with $\tilde{u}(0,x) = u_0(x) \in L^2$. We extend $\tilde{f}(t,x)$ to $f(t,x) \in \mathcal{E}^0_t([0,T_0];L^2)$ by

$$f(t,x) = \begin{cases} \tilde{f}(t,x) & (0 \leq t \leq T) \\ \tilde{f}(T,x) & (T \leq t \leq T_0) \end{cases}$$

and let $u(t,x) \in \mathcal{E}^0_t([0,T_0];L^2)$ be the solution of (0.1) for this $f(t,x)$ and $u_0(x)$. Then, the uniqueness of the solution on the interval $[0,T]$ shows

$$u(t,x) = \tilde{u}(t,x) \quad (0 \le t \le T).$$

So, if we note the choice of the extension from $\tilde{f}(t,x)$ to $f(t,x)$, we get from (3.3)

$$\begin{aligned} &\max_{0\le t\le T} ||\tilde{u}(t,\cdot)|| \\ &\le \max_{0\le t\le T_0} ||u(t,\cdot)|| \\ &\le C(T_0)(||u_0(\cdot)|| + \max_{0\le t\le T_0} ||f(t,\cdot)||) \\ &= C(T_0)(||u_0(\cdot)|| + \max_{0\le t\le T} ||\tilde{f}(t,\cdot)||), \end{aligned}$$

which completes the proof. Q.E.D.

Let $v_\lambda(\tau,x) \in \mathcal{E}^0_\tau([0,T];L^2)$ be the solution of (0.9) for $f_\lambda(\tau,x) \in \mathcal{E}^0_\tau([0,T];L^2)$ with $v_\lambda(0,x) = v^{(0)}(x) \in L^2$ $(0 \le T \le \lambda T_0)$. Then, setting $u(t,x) = v_\lambda(\lambda t,x)$, we can see from (0.1)' that $u(t,x) \in \mathcal{E}^0_t([0,T/\lambda];L^2)$ is the solution of

$$\begin{aligned} Lu(t,x) &= \lambda^2 L_\lambda v_\lambda(\tau,x) \\ &= \lambda^2 f_\lambda(\lambda t,x) \end{aligned}$$

with $u(0,x) = v^{(0)}(x)$. Hence, noting $T/\lambda \le T_0$, we get Lemma 3.1 from Lemma 3.1'.

II. The proof of Theorem 3.2. Assume (As.2). Then, there exist a $t_0 \in R$ and a $(x^0,p^0) \in R^{2n}$ such that

$$\left|\sum_{j=1}^n \int_0^{t_0} \mathrm{Re}\, b^j(X(\theta,x^0,p^0))P_j(\theta,x^0,p^0)d\theta\right| \ge \log\frac{3}{2}\, C(T_0)$$

is valid for the constant $C(T_0)$ in Lemma 3.1. Here, we can assume that t_0 is positive and (0.10) is valid. In fact, a

family of mappings $\{h^\tau\}_{\tau\in R}$ defined by (2.1) forms a one-parameter group and

$$(X(-t,x,p),-P(-t,x,p)) = (X(t,x,-p),P(t,x,-p))$$

is valid from (0.5) and (0.6). So, we get

$$\sum_j \int_0^{-t} b^j(X(\theta,x,-p))P_j(\theta,x,-p)d\theta$$
$$= \sum_j \int_0^{t} b^j(X(\theta,x,p))P_j(\theta,x,p)d\theta,$$

and setting $(x',p') = (X(t,x,p),P(t,x,p))$, we also have

$$-\sum_j \int_0^{t} b^j(X(\theta,x',-p'))P_j(\theta,x',-p')d\theta$$
$$= \sum_j \int_0^{t} b^j(X(-\theta,x',p'))P_j(-\theta,x',p')d\theta$$
$$= \sum_j \int_0^{t} b^j(X(-\theta+t,x,p))P_j(-\theta+t,x,p)d\theta$$
$$= \sum_j \int_0^{t} b^j(X(\theta,x,p))P_j(\theta,x,p)d\theta.$$

Hence, we can take a $t_0 > 0$ and a $(x^0,p^0) \in R^{2n}$ so that (0.10) holds. We fix these t_0, x^0 and p^0 hereafter.

Let $v(x)$ be a C^∞ function with a compact support and set

(3.4) $$S_0(x) = x\cdot p^0.$$

Then, we shall consider the following equation

(3.5) $$\begin{cases} L_\lambda v_\lambda(\tau,x) = 0 & \text{on } [0,t_0]\times R^n_x, \\ v_\lambda(0,x) = e^{i\lambda S_0(x)} v(x). \end{cases}$$

We write L_λ in the form

(3.6) $$L_\lambda v_\lambda(\tau,x)$$

$$= [\lambda^{-1}D_\tau + H(x,\lambda^{-1}D_x) + \sum_{j=1}^{2}(i\lambda)^{-j}H_j(x,\lambda^{-1}D_x)]v_\lambda(\tau,x),$$

where $H(x,p)$ is the function defined by (0.5) and

$$(3.7)\qquad \begin{cases} H_1(x,p) = -\sum_{j=1}^{n} b^j(x)p_j + \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial g^{jk}}{\partial x_j}(x)p_k\,, \\ H_2(x,p) = -c(x). \end{cases}$$

Let U_0 be an open neighborhood of x^0 and set

$$(3.8)\qquad \begin{cases} \Lambda_0^n = \{(x,\frac{\partial S_0}{\partial x}(x)) = (x,p^0);\ x \in R^n\}, \\ \tilde{\Lambda}_0^n = \{(x,\frac{\partial S_0}{\partial x}(x)) = (x,p^0);\ x \in U_0\} \end{cases}$$

which are n-dimensional Lagrangian manifolds. We define Λ_τ^n and $\Lambda^{n+1}(T)$ for this Λ_0^n by (2.2) and (2.3) respectively, that is,

$$(3.9)\qquad \begin{cases} \Lambda_\tau^n = \{h^\tau(x,p^0);\ (x,p^0) \in \Lambda_0^n\}, \\ \Lambda^{n+1}(T) = \{(\tau,h^\tau(x,p^0));\ 0 \le \tau \le T,\ (x,p^0) \in \Lambda_0^n\}, \end{cases}$$

where $h^\tau(x,p^0) = (X(\tau,x,p^0),P(\tau,x,p^0))$ are the solutions of (0.6). In the same way we define

$$(3.9)'\qquad \begin{cases} \tilde{\Lambda}_\tau^n = \{h^\tau(x,p^0);\ (x,p^0) \in \tilde{\Lambda}_0^n\}, \\ \tilde{\Lambda}^{n+1}(T) = \{(\tau,h^\tau(x,p^0));\ 0 \le \tau \le T,\ (x,p^0) \in \tilde{\Lambda}_0^n\}. \end{cases}$$

Notice Lemma 2.1 and take a sufficiently small U_0 so that we can choose a family of connected open sets $\{\Omega_j\}_{j=0}^{s}$ on $\Lambda^{n+1}(t_0)$ satisfying three properties below, where s is a non-negative integer. We fix such an open neighborhood U_0 of x^0. First, for each j there exists a subset K_j of $\{1,2,\cdots,n\}$ such that $(\tau,I_{K_j}) \equiv (\tau,x_{K_j'},p_{K_j})$ become local coordinates on Ω_j. Secondly, we have

(3.10) $\quad \Lambda^{n+1}(t_0) \subset \bigcup_{j=0}^{s} \Omega_j$

and thirdly,

$$\text{(3.11)} \quad \begin{cases} \Omega_j \cap (0,\Lambda_0^n) = \phi & (1 \le j \le s), \\ \Omega_k \cap (t_0,\Lambda_{t_0}^n) = \phi & (0 \le k \le s-1), \\ \Omega_j \cap \Omega_k = \phi & (2 \le |j-k|,\ j,k = 0,1,2,\cdots s) \end{cases}$$

hold. We also fix the above $\{\Omega_j\}_{j=0}^{s}$ and local coordinates (τ,I_{K_j}) $(j = 0,1,\cdots,s)$. The local coordinate system $(\tau,I_{K_j}) = (\tau,x_{K_j'},p_{K_j})$ defines the diffeomorphism from Ω_j into $R^{n+1}_{\tau,x_{K_j'},p_{K_j}}$. We denote its inverse diffeomorphism by $r = r_{K_j}(\tau,I_{K_j}) = r_{K_j}(\tau,x_{K_j'},p_{K_j})$.

Now, hereafter to avoid confusion, we denote the identity mapping from $\Lambda^{n+1}(t_0)$ into $R^1_\tau \times T^*R^n_x$ as $\Lambda^{n+1}(t_0) \ni r \longrightarrow (\tilde{\tau}(r),\tilde{x}(r),\tilde{p}(r)) \in R^1_\tau \times T^*R^n_x$. Let Ξ' be the diffeomorphism from R^n_x onto Λ^n_0 defined by (2.5) with $S_0(x) = x\cdot p^0$ and dV^n_τ the volume form on Λ^n_τ defined by (2.7). Then, we define a C^∞ positive function $J_{K_j}(r) \equiv J_{K_j}(r;\Omega_j)$ on Ω_j by (2.8). Let Ξ be the diffeomorphism from R^n_x onto $(0,\Lambda^n_0)(\subset \Lambda^{n+1}(t_0))$ defined by

(3.12) $\quad \Xi: R^n_x \ni x \longrightarrow (0,x,p^0) \in (0,\Lambda^n_0).$

We shall define a real valued C^∞ function $S(r)$ on $\Lambda^{n+1}(t_0)$ by

(3.13) $\quad S(r) = \int_{r^0}^{r} \tilde{p}\cdot d\tilde{x} - Hd\tilde{\tau} + (\Xi^{-1})^* S_0(r^0),$

where $H(x,p)$ is defined by (0.5) and $r^0 = (0,x^0,p^0) \in (0,\Lambda_0^n)$. Here, the integral in (3.13) is taken along a path from r^0 to r on $\Lambda^{n+1}(t_0)$. Then, we see from Lemma 2.3 that $S(r)$ is independent of the choice of paths. So, we get

$$(3.14) \qquad dS = \tilde{p}\cdot d\tilde{x} - Hd\tilde{\tau} \qquad \text{on } \Lambda^{n+1}(t_0).$$

Also, it follows from Lemma 2.4 that

$$(3.15) \qquad S(r) = (\Xi^{-1})^* S_0(r) \qquad \text{for } r \in (0,\Lambda_0^n),$$

that is,

$$(3.15)' \qquad S(0,x,p^0) = S_0(x) = x\cdot p^0 \qquad \text{for } x \in R^n.$$

Next, let define a real valued C^∞ function $S_{K_j}(r) \equiv S_{K_j}(r;\Omega_j)$ on Ω_j for each j by

$$(3.16) \qquad S_{K_j}(r;\Omega_j) = S(r) - \tilde{x}_{K_j}(r)\cdot\tilde{p}_{K_j}(r).$$

Then, it follows from (3.14) that

$$(3.17) \qquad dS_{K_j} = \tilde{p}_{K_j'}\cdot d\tilde{x}_{K_j'} - \tilde{x}_{K_j}\cdot d\tilde{p}_{K_j} - Hd\tilde{\tau} \qquad \text{on } \Omega_j.$$

Now, we shall define a pre-canonical operator $\mathcal{K}(\Omega_j, I_{K_j})$ acting from $C_0^\infty(\Omega_j)$ to $C^\infty(R^{n+1}_{\tau,x})$ for each j, which corresponds to (6.3) in [8], by

$$(3.18) \qquad \mathcal{K}(\Omega_j, I_{K_j})\varphi(\tau,x)$$
$$= \mathcal{F}^{-1}_{\lambda,p_{K_j}\to x_{K_j}}\left[\frac{e^{i\lambda S_{K_j}(r)}}{J_{K_j}(r)^{1/2}}\ \varphi(r)\Big|_{r=r_{K_j}(\tau,x_{K_j'},p_{K_j})}\right]$$

for $\varphi(r) \in C_0^\infty(\Omega_j)$.

Lemma 3.3. We have

$$||\mathcal{K}(\Omega_j, I_{K_j})\varphi(\tau,\cdot)||^2 = \int_{\Lambda_\tau^n} |\varphi(\tau,r')|^2 dv_\tau^n$$

for any $\varphi(r) \in C_0^\infty(\Omega_j)$.

Proof. It follows from Lemma 1.1 and (2.8) that

$$\begin{aligned} &||\mathcal{K}(\Omega_j, I_{K_j})\varphi(\tau,\cdot)||^2 \\ &= \int_{R^n} (|\varphi(r)|^2 / J_{K_j}(r))\big|_{r=r_{K_j}(\tau,x_{K_j}',p_{K_j})} dx_{K_j}' dp_{K_j} \\ &= \int_{\Lambda_\tau^n} |\varphi(\tau,r')|^2 dv_\tau^n \end{aligned}$$ Q.E.D.

Taking account of (3.10), there exist $e_j(r) \in C_0^\infty(\Omega_j)$ $(j = 0,1,\cdots,s)$ satisfying

$$\sum_{j=0}^{s} e_j(r) = 1 \quad \text{on} \quad \tilde{\Lambda}^{n+1}(t_0). \tag{3.19}$$

We fix these functions $e_j(r)$ $(j = 0,1,\cdots,s)$. Set

$$F(x,p) = \frac{1}{2} \operatorname{Tr} \frac{\partial^2 H}{\partial x \partial p}(x,p) - H_1(x,p). \tag{3.20}$$

We have from (3.7)

$$F(x,p) = \sum_{j=1}^{n} b^j(x) p_j. \tag{3.20}'$$

Then, we shall define $W^{(1)}$ acting from $C^\infty(\Lambda^{n+1}(t_0))$ to $C^\infty(\Lambda^{n+1}(t_0))$ by

$$W^{(1)}\varphi(r) = \{\frac{d}{d\tau} - F(h^\tau(x,p^0))\}\varphi(\tau,h^\tau(x,p^0)) \tag{3.21}$$

at $r = (\tau,h^\tau(x,p^0))$. Then, we obtain the following correspondingly to Theorem 9.3 in [8].

Proposition 3.4. By choosing a real constant $\sigma(\Omega_j)$ $(1 \le j \le$

s) in a suitable way, we can define an operator K acting from $C^\infty(\Lambda^{n+1}(t_0))$ to $C^\infty(R^{n+1}_{\tau,x})$ by

$$(3.22)\qquad K\varphi(\tau,x) = \sum_{j=0}^{s} e^{i\sigma(\Omega_j)} \mathcal{K}(\Omega_j, I_{K_j})(e_j\varphi)(\tau,x)$$

($\sigma(\Omega_0) = 0$) so that $K\varphi(\tau,x)$ for $\varphi(r) \in C_0^\infty(\Lambda^{n+1}(t_0))$ satisfies (i), (ii) and (iii) below.

(i) $K\varphi(0,x) = e^{i\lambda x\cdot p^0}\varphi(0,x,p^0)$.

(ii) $\|K\varphi(t_0,\cdot)\|^2 = \int_{\Lambda^n_{t_0}} |\varphi(t_0,r')|^2 dV^n_{t_0}$.

(iii) $L_\lambda K\varphi(\tau,x)$ has the asymptotic expression

$$(3.23)\qquad L_\lambda K\varphi(\tau,x) = \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} K(W^{(\ell)}\varphi)(\tau,x) + R_N\varphi(\tau,x)$$

for $N = 1,2,\cdots$, where we have for the remainder terms $R_N\varphi(\tau,x)$

$$(3.24)\qquad \max_{0\le\tau\le t_0} \|R_N\varphi(\tau,\cdot)\| = O(\lambda^{-N}).$$

$W^{(\ell)}$ $(2 \le \ell \le N-1)$ are linear differential operators in $\Lambda^{n+1}(t_0)$ of order at most 2ℓ independent of λ.

Remark 3.1. We do not study the index of a curve on $\Lambda^{n+1}(t_0)$ (c.f. [8]). So, we shall determine $\sigma(\Omega_j)$ $(1 \le j \le s)$ directly in the proof of Proposition 3.4.

If we admit Proposiotion 3.4, we can give the proof of Theorem 3.2 as follows. Let $v(x) \in C_0^\infty(U_0)$. We shall construct an asymptotic solution $v_\lambda(\tau,x)$ of (3.5) satisfying

$$(3.25)\qquad \begin{cases} \max_{0\le\tau\le t_0} \|L_\lambda v_\lambda(\tau,\cdot)\| = O(\lambda^{-3}), \end{cases}$$

$$\left\{ \quad v_\lambda(0,x) = e^{i\lambda x\cdot p^0} v(x) \right.$$

in the form

$$(3.26) \qquad v_\lambda(\tau,x) = K[\sum_{j=0}^{1} (i\lambda)^{-j}\varphi](\tau,x).$$

If $\varphi_j(r)$ $(j = 0,1)$ belong to $C_0^\infty(\tilde{\Lambda}^{n+1}(t_0))$ and satisfy the following equations

$$\begin{cases} \{\frac{d}{d\tau} - F(h^\tau(y,p^0))\}\varphi_0(\tau,h^\tau(y,p^0)) = 0, \\ \varphi_0(0,y,p^0) = v(y) \end{cases}$$

and

$$\begin{cases} \{\frac{d}{d\tau} - F(h^\tau(y,p^0))\}\varphi_1(\tau,h^\tau(y,p^0)) + W^{(2)}\varphi_0(\tau,h^\tau(y,p^0)) = 0, \\ \varphi_1(0,y,p^0) = 0, \end{cases}$$

we see that $v_\lambda(\tau,x)$ satisfies (3.25) from (i) and (iii) of Proposition 3.4. Obviously, $\varphi_0(r)$ and $\varphi_1(r)$ are written as

$$(3.27) \begin{cases} \varphi_0(\tau,h^\tau(y,p^0)) = \{\exp\int_0^\tau F(h^\theta(y,p^0))d\theta\}v(y), \\ \varphi_1(\tau,h^\tau(y,p^0)) \\ = -\int_0^\tau \{\exp\int_{\tau'}^\tau F(h^\theta(y,p^0))d\theta\}W^{(2)}\varphi_0(\tau',h^{\tau'}(y,p^0))d\tau'. \end{cases}$$

So, $\varphi_0(r)$ belongs to $C_0^\infty(\tilde{\Lambda}^{n+1}(t_0))$. Also, since $W^{(2)}$ is a linear differential operator in $\Lambda^{n+1}(t_0)$, we can see that $\varphi_1(r)$ belongs to $C_0^\infty(\tilde{\Lambda}^{n+1}(t_0))$. Thus, $\varphi_j(r) \in C_0^\infty(\tilde{\Lambda}^{n+1}(t_0))$ $(j = 0,1)$ are determined.

If we take $v(x)$ such that supp $v(\cdot)$ is sufficiently small around $x = x^0$, we get from (ii) of Proposition 3.4 and (2.7)'

$$(3.28) \qquad \|v_\lambda(t_0,\cdot)\|$$

$$= \|K\varphi_0(t_0,\cdot)\| + O(\lambda^{-1})$$

$$= \left(\int_{\Lambda^n_{t_0}} \left|\left\{\exp \mathrm{Re} \int_0^{t_0} F(h^\theta(y,p^0))d\theta\right\} v(y)\right|^2 dV^n_{t_0}\right)^{1/2} + O(\lambda^{-1})$$

$$\geq \frac{3}{4} \left\{\exp \mathrm{Re} \int_0^{t_0} F(h^\theta(x^0,p^0))d\theta\right\} \left(\int_{R^n} |v(y)|^2 dy\right)^{1/2} + O(\lambda^{-1}).$$

Hence, we obtain by (0.10) and (3.20)'

$$\|v_\lambda(t_0,\cdot)\| \geq \frac{9}{8} C(T_0)\|v(\cdot)\| + O(\lambda^{-1}) \qquad \text{Q.E.D.}$$

Proof of Theorem'. Using the Legendre transformation, for the Hamiltonian function $H(x,p)$ defined by (0.5) we define the Lagrangian function $L(x,\eta)$ on the tangent bundle $TR^n_x = R^{2n}_{x,\eta}$ by

$$(3.29) \qquad L(x,\eta) = p\cdot\eta - H(x,p),$$

where p is expressed in terms of η by the formula

$$(3.30) \qquad \eta = \frac{\partial H}{\partial p}(x,p) = p(g^{ij}(x); \overset{i}{j} \overset{\downarrow}{\rightarrow} 1,2,\cdots,n).$$

Then, it is easy to see that $L(x,\eta)$ is written in the form

$$(3.29)' \qquad L(x,\eta) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(x)\eta_i\eta_j .$$

The following two facts are well known in the theory of analytical dynamics (for example, see section 15 in [1]). If $(X(t,x,p),P(t,x,p))$ are the solutions of (0.6), we have

$$(3.31) \qquad \frac{dX}{dt}(t,x,p) = \frac{\partial H}{\partial p}(X(t,x,p),P(t,x,p))$$

and

$$(3.32)\qquad \frac{d}{dt}\frac{\partial L}{\partial \eta_j}(X,\frac{dX}{dt}) - \frac{\partial L}{\partial x_j}(X,\frac{dX}{dt}) = 0 \qquad (1 \le j \le n).$$

The equation (3.32) is called Lagranges' equations of motion. Conversely, let $X(t,x,p)$ be the solutions of (3.32) with initial data $X(0,x,p) = x$ and $\frac{dX}{dt}(0,x,p) = \frac{\partial H}{\partial p}(x,p)$, and set

$$(3.33)\qquad P(t,x,p) = \frac{\partial L}{\partial \eta}(X,\frac{dX}{dt}).$$

Then, $(X(t,x,p),P(t,x,p))$ are the solutions of (0.6).

Now, we can see that the equations (3.32) are also the equations of geodesics on M

$$(3.32)'\qquad (\frac{d}{dt})^2 x_j(t) + \sum_{\ell,k=1}^{n} \Gamma^j_{\ell k}\,(\frac{d}{dt}\,x_\ell(t))(\frac{d}{dt}\,x_k(t)) = 0 \qquad (1 \le j \le n),$$

where $\Gamma^j_{\ell k}$ are the Christoffel symbols $(1/2)\sum_{i=1}^{n} g^{ij}(\partial_{x_\ell} g_{ki} + \partial_{x_k} g_{\ell i} - \partial_{x_i} g_{\ell k})$ (for example, see p167 in [14]). Hence, it follows from (0.8)' and (3.31) that

$$\begin{aligned}
&\sum_{j=1}^{n} \int_0^t \mathrm{Re}\; b^j(X(\theta,x,p))P_j(\theta,x,p)d\theta \\
&= \sum_{j,k=1}^{n} \int_0^t \mathrm{Re}\; b^j(X(\theta,x,p))g_{jk}(X(\theta,x,p))dX_k(\theta,x,p) \\
&= \int_\gamma \omega\ ,
\end{aligned}$$

where γ denotes a geodesic $\{X(\theta,x,p);\ 0 \le \theta \le t\}$.

Consequently, we can easily complete the proof from Theorem.

Q.E.D.

4. Proof of Proposition 3.4. We shall first prove (i) and (ii) of Proposition 3.4.

Since Ω_0 includes $(0,\Lambda_0^n) = \{(0,x,p^0);\ x \in U_0\}$ from (3.11), K_0 is the empty set. Let $\varphi(r) = \varphi(\tau,r')$ belong to $C_0^\infty(\Lambda^{n+1}(t_0))$. Then, it follows from (3.11) and (3.19) that

$$e_0(0,r') = 1 \quad \text{on} \quad \operatorname{supp} \varphi(0,\cdot).$$

So, we get together with (3.15)'

$$\begin{aligned} & K\varphi(0,x) \\ &= \mathcal{K}(\Omega_0, I_{K_0})(e_0\varphi)(0,x) \\ &= (e^{i\lambda x\cdot p^0}/J_{K_0}(0,x,p^0)^{1/2})\varphi(0,x,p^0). \end{aligned}$$

Since K_0 is the empty set, $J_{K_0}(0,x,p^0) = 1$ for $x \in U_0$ is valid from (2.9) as $S_0(x) = x\cdot p^0$. Hence, we obtain (i).

In the same way to the proof of (i) we have

$$\begin{aligned} & K\varphi(t_0,x) \\ &= e^{i\sigma(\Omega_s)}\mathcal{K}(\Omega_s, I_{K_s})\varphi(t_0,x). \end{aligned}$$

This shows (ii) by Lemma 3.3 independently of the choice of real constants $\sigma(\Omega_j)$.

We shall prove (iii) of Proposition 3.4. Let $\mathcal{K}(\Omega_j, I_{K_j})$ be the pre-canonical operator defined by (3.18). We shall first consider $L_\lambda\{\mathcal{K}(\Omega_j, I_{K_j})\varphi(\tau,x)\}$ for $\varphi(r) \in C_0^\infty(\Omega_j)$. Here, we omit the suffix j. Then, we get from (3.17)

$$(4.1)\quad \begin{cases} \dfrac{\partial}{\partial x_{K,}} S_K(r_K(\tau,x_{K,},p_K)) = \tilde{p}_{K,}(r_K(\tau,x_{K,},p_K)), \\ \dfrac{\partial}{\partial p_{K}} S_K(r_K(\tau,x_{K,},p_K)) = -\tilde{x}_K(r_K(\tau,x_{K,},p_K)), \\ \dfrac{\partial}{\partial \tau} S_K(r_K(\tau,x_{K,},p_K)) + H(x_{K,},-\dfrac{\partial}{\partial p_K} S_K(r_K), \dfrac{\partial}{\partial x_{K,}} S_K(r_K), p_K) = 0. \end{cases}$$

We also have from (2.9) as $S_0(x) = x\cdot p^0$

$$(4.2)\quad J_K(\tau,h^\tau(y,p^0)) = |\det \frac{\partial}{\partial y}(X_{K,}(\tau,y,p^0),P_K(\tau,y,p^0))|$$

at $(\tau,h^\tau(y,p^0)) \in \Omega$. Hence, if we apply Lemmas 1.1 - 1.5 to $L_\lambda\{\mathcal{K}(\Omega,I_K)\varphi(\tau,x)\}$, we obtain in the similar way to the proof of Theorem 8.4 in [8]

Proposition 4.1. Let $\varphi(r) \in C_0^\infty(\Omega_j)$. Then, $L_\lambda\{\mathcal{K}(\Omega_j,I_{K_j})\varphi(\tau,x)\}$ has the following asymptotic expression

$$(4.3)\quad \mathcal{K}(\Omega_j,I_{K_j})[(i\lambda)^{-1}W^{(1)}\varphi(r) + \sum_{\ell=2}^{N-1}(i\lambda)^{-\ell}\mathcal{D}^{(\ell)}(\Omega_j,I_{K_j})\varphi(r)](\tau,x) + R_{1,N}(\Omega_j,I_{K_j})\varphi(\tau,x)$$

for $N = 1,2,\cdots$, where $W^{(1)}$ is an operator defined by (3.21) and $\mathcal{D}^{(\ell)}(\Omega_j,I_{K_j})$ $(2 \le \ell \le N-1)$ are linear differential operators independent of λ and N in $r' \in \Lambda_\tau^n$ of order at most ℓ. We also have for the remainder terms

$$(4.4)\quad \max_{0\le\tau\le t_0} ||R_{1,N}(\Omega_j,I_{K_j})\varphi(\tau,\cdot)|| = O(\lambda^{-N}).$$

Next, let Ω_i and Ω_j be connected open sets in $\{\Omega_j\}_{j=0}^s$ such that $\Omega_i \cap \Omega_j$ is not empty. Then, we shall study the transformation formula from $\mathcal{K}(\Omega_i,I_{K_i})$ to $\mathcal{K}(\Omega_j,I_{K_j})$.

We set up the decomposition into disjoint sets as in the proof of Lemma 6.3 in [8]

$$\{1,2,\cdots,n\} = a \cup b \cup c \cup d$$

with the property

(4.5) $$K_i = a \cup b\ ,\ K_i' = c \cup d,\ K_j = a \cup c,\ K_j' = b \cup d.$$

Then, we have for $\varphi(r) \in C_0^\infty(\Omega_i \cap \Omega_j)$

$$\begin{aligned}(4.6)\quad & \mathcal{F}_{\lambda,x_{K_j}\to p_{K_j}}\{\mathcal{K}(\Omega_i,I_{K_i})\varphi\}(\tau,x_{K_j'},p_{K_j}) \\ &= \mathcal{F}_{\lambda,x_c\to p_c}\circ\mathcal{F}^{-1}_{\lambda,p_b\to x_b}\{(\exp i\lambda S_{K_i}(r))\varphi(r) \\ &\qquad J_{K_i}(r)^{-1/2}\Big|_{r=r_{K_i}(\tau,x_{K_i'},p_{K_i})}\} \\ &= (\lambda/2\pi)^{(|b|+|c|)/2}\iint(\exp i\lambda\Phi(x_c,p_b;\tau,x_{K_j'},p_{K_j})) \\ &\qquad \varphi(r)J_{K_i}(r)^{-1/2}\Big|_{r=r_{K_i}(\tau,x_{K_i'},p_{K_i})}\,dp_b dx_c,\end{aligned}$$

where

$$\begin{aligned}(4.7)\quad \Phi &= \Phi(x_c,p_b) = \Phi(x_c,p_b;\tau,x_{K_j'},p_{K_j}) \\ &\equiv -x_c\cdot p_c + x_b\cdot p_b + S_{K_i}(r_{K_i}(\tau,x_{K_i'},p_{K_i})).\end{aligned}$$

It follows from (4.1) and (4.5) that

$$(4.8)\quad \begin{cases} \dfrac{\partial\Phi}{\partial x_c}(x_c,p_b) = -p_c + \tilde{p}_c(r_{K_i}(\tau,x_{K_i'},p_{K_i})), \\ \dfrac{\partial\Phi}{\partial p_b}(x_c,p_b) = x_b - \tilde{x}_b(r_{K_i}(\tau,x_{K_i'},p_{K_i})). \end{cases}$$

Take an open connected set Ω_{ij} so that

(4.9) $$\text{supp}\,\varphi(\cdot) \subset \Omega_{ij} \subset \overline{\Omega}_{ij} \subset \Omega_i \cap \Omega_j.$$

Then, we may consider $\Phi(x_c,p_b;\tau,x_{K_j'},p_{K_j})$ in (4.6) as a function on the set

$$(4.10)\quad \{(x_c,p_b;\tau,x_{K_j'},p_{K_j});\ r_{K_i}(\tau,x_{K_i'},p_{K_i})\in\overline{\Omega}_{ij},\ x_b\in R^{|b|},\ p_c\in R^{|c|}\}.$$

We shall study the proof of Lemma 6.3 in [8] in more detail mainly in order to obtain the estimate (4.15) below. We see from (4.8) that a stationary point (x_c,p_b) of $\Phi(x_c,p_b;\tau,x_{K_j'},p_{K_j})$ on the set (4.10) is determined by

$$(4.11)\quad x_b=\tilde{x}_b(r_{K_i}(\tau,x_{K_i'},p_{K_i})),\ p_c=\tilde{p}_c(r_{K_i}(\tau,x_{K_i'},p_{K_i})).$$

On the other hand, the equalities

$$(4.12)\quad x_d=\tilde{x}_d(r_{K_i}(\tau,x_{K_i'},p_{K_i})),\ p_a=\tilde{p}_a(r_{K_i}(\tau,x_{K_i'},p_{K_i}))$$

always hold from (4.5). So, (4.11) is equivalent to

$$(4.11)'\quad x_{K_j'}=\tilde{x}_{K_j'}(r_{K_i}(\tau,x_{K_i'},p_{K_i})),\ p_{K_j}=\tilde{p}_{K_j}(r_{K_i}(\tau,x_{K_i'},p_{K_i})).$$

Since $(\tau,x_{K_i'},p_{K_i})$ and $(\tau,x_{K_j'},p_{K_j})$ are local coordinates on $\Omega_i\cap\Omega_j$ respectively, (4.11)' is also equivalent to

$$(4.11)''\quad x_{K_i'}=\tilde{x}_{K_i'}(r_{K_j}(\tau,x_{K_j'},p_{K_j})),\ p_{K_i}=\tilde{p}_{K_i}(r_{K_j}(\tau,x_{K_j'},p_{K_j})).$$

Hence, we can see from (4.11)' that if and only if $\Phi(x_c,p_b;\tau,x_{K_j'},p_{K_j})$ on (4.10) has a stationary point, $(\tau,x_{K_j'},p_{K_j})$ belongs to the set

$$(4.13)\quad \{(\tau,x_{K_j'},p_{K_j})=(\tilde{\tau}(r),\tilde{x}_{K_j'}(r),\tilde{p}_{K_j}(r));\ r\in\overline{\Omega}_{ij}\}.$$

Also, we see from (4.5) and (4.11)'' that the stationary point of $\Phi(x_c,p_b;\tilde{\tau}(r),\tilde{x}_{K_j'}(r),\tilde{p}_{K_j}(r))$ $(r \in \overline{\Omega}_{ij})$ is determined by

$$(4.14) \qquad (x_c,p_b) = (\tilde{x}_c(r),\tilde{p}_b(r)).$$

Moreover, we obtain the fact below from (4.11)' together with (4.8). There exists a constant $C_0 > 0$ such that if $(\tau,x_{K_j'},p_{K_j})$ does not belong to the set (4.13), for any point in the set (4.10) satisfying

$$r_{K_i}(\tau,x_{K_i'},p_{K_i}) \in \operatorname{supp}\varphi(\cdot)$$

we have

$$(4.15) \qquad \left|\frac{\partial\Phi}{\partial x_c}(x_c,p_b;\tau,x_{K_j'},p_{K_j})\right| + \left|\frac{\partial\Phi}{\partial p_b}\right| \geq C_0(1 + |x_b|^2 + |p_c|^2)^{1/2}.$$

Let $(\tau,x_{K_j'},p_{K_j})$ be not contained in the set (4.13). Set $T = \frac{1}{i}(\left|\frac{\partial\Phi}{\partial x_c}\right|^2 + \left|\frac{\partial\Phi}{\partial p_b}\right|^2)^{-1}(\frac{\partial\Phi}{\partial x_c}\cdot\frac{\partial}{\partial x_c} + \frac{\partial\Phi}{\partial p_b}\cdot\frac{\partial}{\partial p_b})$ and let ${}^tT$ be the transposed operator of T. Then, since the supports of $\varphi(r_{K_i}(\tau,x_{K_i'},p_{K_i})) = \varphi(r_{K_i}(\tau,x_c,x_d,p_a,p_b))$ with respect to variables (x_d,p_a) are compact, we get from (4.6) and (4.15) for $N' = 1,2,\cdots$

$$(4.16) \qquad \left|\mathcal{F}_{\lambda,x_{K_j'}\to p_{K_j}}\{\mathcal{K}(\Omega_i,I_{K_i})\varphi\}(\tau,x_{K_j'},p_{K_j})\right| = (\lambda/2\pi)^{(|b|+|c|)/2}\lambda^{-N'}\left|\iint e^{i\lambda\Phi}({}^tT)^{N'}\{\varphi(r) J_{K_i}(r)^{-1/2}\Big|_{r=r_{K_i}(\tau,x_{K_i'},p_{K_i})}\}dp_b dx_c\right|$$

$$\leq C_{N'}\,\lambda^{(|b|+|c|)/2-N'}\{(1+|x_b|^2+|p_c|^2)^{1/2}\langle x_d\rangle\langle p_a\rangle\}^{-N'}$$
$$\leq C'_{N'}\,\lambda^{(|b|+|c|)/2-N'}(1+|x_{K'_j}|^2+|p_{K_j}|^2)^{-N'/2},$$

where $C_{N'}$ and $C'_{N'}$ are constants independent of $(\tau,x_{K'_j},p_{K_j}) \notin \{(\tilde{\tau}(r),\tilde{x}_{K'_j}(r),\tilde{p}_{K_j}(r));\ r \in \bar{\Omega}_{ij}\}$ and $\lambda \geq 1$.

Next, let $r \in \tilde{\Omega}_{ij}$. Then, we can easily obtain the following two results in the similar way to the proof of Lemma 6.3 in [8]. First, we have from (4.7) and the definitions (3.16) of $S_{K_\ell}(r;\Omega_\ell)$ $(\ell = i,j)$

$$\Phi(\tilde{x}_c(r),\tilde{p}_b(r);\tilde{\tau}(r),\tilde{x}_{K'_j}(r),\tilde{p}_{K_j}(r)) = S_{K_j}(r;\Omega_j). \tag{4.17}$$

Secondly, setting

$$A_{ji}(r) = \begin{pmatrix} \dfrac{\partial^2\Phi}{\partial x_c^2} & \dfrac{\partial^2\Phi}{\partial p_b \partial x_c} \\ \dfrac{\partial^2\Phi}{\partial x_c \partial p_b} & \dfrac{\partial^2\Phi}{\partial p_b^2} \end{pmatrix}(\tilde{x}_c(r),\tilde{p}_b(r);\tilde{\tau}(r),\tilde{x}_{K'_j}(r),\tilde{p}_{K_j}(r)), \tag{4.18}$$

we have from (4.8) and the definitions (2.8) of $J_{K_\ell}(r;\Omega_\ell)(\ell = i, j)$

$$|\det A_{ji}(r)| = J_{K_j}(r;\Omega_j)/J_{K_i}(r;\Omega_i). \tag{4.19}$$

Now, we see from (4.19) that we can apply the stationary phase method to (4.6) with $(\tau,x_{K'_j},p_{K_j}) = (\tilde{\tau}(r), \tilde{x}_{K'_j}(r),\tilde{p}_{K_j}(r))$. See Theorem 1.4 in [8], Theorem 2.4 in [2] or Theorem 7.7.6 in [3] for the stationary phase method. Then, it follows from (4.14),

(4.17) and (4.19) for $N = 1,2,\cdots$ that

$$(4.20)\quad \mathcal{F}_{\lambda,x_{K_j}\to p_{K_j}}\{\mathcal{K}(\Omega_i,I_{K_i})\varphi\}(\tilde{\tau}(r),\tilde{x}_{K_j'}(r),\tilde{p}_{K_j}(r))$$

$$= (\lambda/2\pi)^{(|b|+|c|)/2}\{\exp i\lambda\Phi(\tilde{x}_c(r),\tilde{p}_b(r);\tilde{\tau}(r),\tilde{x}_{K_j'}(r),\tilde{p}_{K_j}(r))\}$$

$$\{\exp\tfrac{\pi}{4}i\ \operatorname{sgn}A_{ji}(r)\}|\det A_{ji}(r)/2\pi|^{-1/2}\ \lambda^{-(|b|+|c|)/2}$$

$$\{(1+\sum_{m=1}^{N-1}\lambda^{-m}\mathcal{L}_m)\varphi(r_{K_i}(\tau,x_{K_i'},p_{K_i})$$

$$J_{K_i}(r_{K_i})^{-1/2}\}\Big|_{\tau=\tilde{\tau}(r),x_{K_i'}=\tilde{x}_{K_i'}(r),p_{K_i}=\tilde{p}_{K_i}(r)}+$$

$$R_{2,N}'(I_{K_j},I_{K_i})\varphi(r)$$

$$= \{\exp\tfrac{\pi}{4}i\ \operatorname{sgn}A_{ji}(r)\}(\exp i\lambda S_{K_j}(r))J_{K_j}(r)^{-1/2}\{1+$$

$$\sum_{\ell=1}^{N-1}(i\lambda)^{-\ell}V^{(\ell)}(I_{K_j},I_{K_i})\}\varphi(r)+R_{2,N}'(I_{K_j},I_{K_i})\varphi(r)$$

and

$$(4.21)\qquad \max_{r\in\overline{\Omega}_{ij}}|R_{2,N}'(I_{K_j},I_{K_i})\varphi(r)| = O(\lambda^{-N}),$$

where $\operatorname{sgn}A_{ji}(r)$ denotes the signature of $A_{ji}(r)$, $\mathcal{L}_m$ $(m = 1,2,\cdots,N-1)$ are linear differential operators independent of λ in $x_c \in R^{|c|}$ and $p_b \in R^{|b|}$ of order at most $2m$ and $V^{(\ell)}(I_{K_j},I_{K_i})$ $(\ell = 1,2,\cdots,N-1)$ are linear differential operators independent of λ and N in $r' \in \Lambda_\tau^n$ of order at most 2ℓ.

If we note that $\varphi(r) \in C_0^\infty(\Omega_i \cap \Omega_j)$, we can combine (4.16) and (4.20) with (4.21). We obtain

$$(4.22)\quad \mathcal{F}_{\lambda,x_{K_j}\to p_{K_j}}\{\mathcal{K}(\Omega_i,I_{K_i})\varphi\}(\tau,x_{K_j'},p_{K_j})$$

$$= \{\exp \frac{\pi}{4} i \operatorname{sgn} A_{ji}(r)\}(\exp i\lambda S_{K_j}(r))J_{K_j}(r)^{-1/2}\{1 + \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} V^{(\ell)}(I_{K_j}, I_{K_i})\}\varphi(r)\Big|_{r=r_{K_j}(\tau, x_{K_j}, p_{K_j})} + R^{(2)}_{2,N}(I_{K_j}, I_{K_i})\varphi(\tau, x_{K_j}, p_{K_j})$$

and

$$\max_{0\le\tau\le t_0} \|R^{(2)}_{2,N}(I_{K_j}, I_{K_i})\varphi(\tau,\cdot,\cdot)\| = O(\lambda^{-N}). \tag{4.23}$$

Here, $\operatorname{sgn} A_{ji}(r)$ is constant on $\overline{\Omega}_{ij}$, because $\det A_{ji}(r) \neq 0$ on $\overline{\Omega}_{ij}$ and $\overline{\Omega}_{ij}$ is connected. So, we set

$$\sigma(\Omega_i, \Omega_j) = \frac{\pi}{4} \operatorname{sgn} A_{ji}(r) \quad (r \in \overline{\Omega}_{ij}). \tag{4.24}$$

Operate $\mathcal{F}^{-1}_{\lambda, p_{K_j} \to x_{K_j}}$ on both sides of (4.22) and apply Lemma 1.1 to the remainder term. Then, we have

Proposition 4.2. $\mathcal{K}(\Omega_i, I_{K_i})\varphi(\tau,x)$ for $\varphi(r) \in C_0^\infty(\Omega_i \cap \Omega_j)$ has the another expression for $N = 1,2,\cdots$

$$e^{i\sigma(\Omega_i,\Omega_j)} \mathcal{K}(\Omega_j, I_{K_j})\{\varphi(r) + \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} V^{(\ell)}(I_{K_j}, I_{K_i})\varphi(r)\}(\tau,x) + R_{2,N}(I_{K_j}, I_{K_i})\varphi(\tau,x), \tag{4.25}$$

where the remainder term $R_{2,N}(I_{K_j}, I_{K_i})\varphi(\tau,x)$ satisfies

$$\max_{0\le\tau\le t_0} \|R_{2,N}(I_{K_j}, I_{K_i})\varphi(\tau,\cdot)\| = O(\lambda^{-N}). \tag{4.26}$$

If we use Propositions 4.1 and 4.2, we can complete the proof of Proposition 3.4 as follows.

First, we see from (4.25) that

$$\mathcal{K}(\Omega_i, I_{K_i})\varphi(\tau,x) = e^{i\sigma(\Omega_i,\Omega_j)}\mathcal{K}(\Omega_j, I_{K_j})\varphi + R_{2,1}(I_{K_j}, I_{K_i})\varphi$$

and

$$\mathcal{K}(\Omega_j, I_{K_j})\varphi(\tau,x) = e^{i\sigma(\Omega_j,\Omega_i)}\mathcal{K}(\Omega_i, I_{K_i})\varphi + R_{2,1}(I_{K_i}, I_{K_j})\varphi$$

for any $\varphi(r) \in C_0^\infty(\Omega_i \cap \Omega_j)$. So, we get from (4.26) and Lemma 1.1

$$\begin{aligned} O(\lambda^{-2}) &= \| (1 - e^{i\sigma(\Omega_i,\Omega_j)+i\sigma(\Omega_j,\Omega_i)})\mathcal{K}(\Omega_i, I_{K_i})\varphi(\tau,\cdot)\|^2 \\ &= \iint |(1 - e^{i\sigma(\Omega_i,\Omega_j)+i\sigma(\Omega_j,\Omega_i)})\{\varphi(r) \\ &\qquad J_{K_i}(r)^{-1/2}\Big|_{r=r_{K_i}(\tau,x_{K_i},p_{K_i})}\}|^2 \, dx_{K_i} dp_{K_i} \end{aligned}$$

for any $\varphi(r) \in C_0^\infty(\Omega_i \cap \Omega_j)$. Hence,

(4.27) $$\sigma(\Omega_j,\Omega_i) = -\sigma(\Omega_i,\Omega_j)$$

is valid.

We shall determine real constants $\sigma(\Omega_j)$ $(1 \le j \le s)$ by

(4.28) $$\sigma(\Omega_j) = -\sum_{k=1}^{j} \sigma(\Omega_k,\Omega_{k-1}).$$

We choosed $\{\Omega_j\}_{j=0}^{s}$ in section 3 so that if and only if $\Omega_j \cap \Omega_m$ is not empty, m is equal to j-1, j or j+1. So, if $\Omega_j \cap \Omega_m$ is not empty, we have from (4.27)

(4.29) $$\sigma(\Omega_j) - \sigma(\Omega_m) = \sigma(\Omega_m,\Omega_j).$$

Using these $\sigma(\Omega_j)$, let define K by (3.22).

We shall use Proposition 4.1. For the sake of simplicity we set

$$(4.30)\qquad \mathcal{D}_N(\Omega_j, I_{K_j}) = (i\lambda)^{-1} W^{(1)} + \sum_{\ell=2}^{N-1} (i\lambda)^{-\ell} \mathcal{D}^{(\ell)}(\Omega_j, I_{K_j}),$$

which acts from $C_0^\infty(\Omega_j)$ to $C_0^\infty(\Omega_j)$. Then, we have for $\varphi(r) \in C_0^\infty(\Lambda^{n+1}(t_0))$

$$\begin{aligned}(4.31)\qquad & L_\lambda(K\varphi)(\tau, x) \\ &= \sum_{j=0}^{s} e^{i\sigma(\Omega_j)} L_\lambda(\mathcal{K}(\Omega_j, I_{K_j}) e_j \varphi)(\tau, x) \\ &= \sum_{j=0}^{s} e^{i\sigma(\Omega_j)} \mathcal{K}(\Omega_j, I_{K_j}) \mathcal{D}_N(\Omega_j, I_{K_j}) e_j \varphi + \sum_{j=0}^{s} R_{1,N}(\Omega_j, I_{K_j}) e_j \varphi.\end{aligned}$$

Next, we also set for $V^{(\ell)}(I_{K_j}, I_{K_i})$ in Proposition 4.2

$$(4.32)\qquad V_N(I_{K_j}, I_{K_i}) = I + \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} V^{(\ell)}(I_{K_j}, I_{K_i}),$$

which acts from $C_0^\infty(\Omega_i \cap \Omega_j)$ to $C_0^\infty(\Omega_i \cap \Omega_j)$. Then, since we have from (3.19) for $\varphi(r) \in C_0^\infty(\Omega_j \cap \tilde{\Lambda}^{n+1}(t_0))$

$$\begin{aligned}&\sum_{m=0}^{s} V_N(I_{K_j}, I_{K_m})(e_m \varphi)(r) \\ &= [I + \sum_{m=0}^{s} \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} V^{(\ell)}(I_{K_j}, I_{K_m}) e_m] \varphi(r),\end{aligned}$$

there exists for each j

$$(4.33)\qquad \tilde{V}_N(\Omega_j, I_{K_j}) \equiv I + \sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} \tilde{V}^{(\ell)}(\Omega_j, I_{K_j})$$

such that we have for any $\varphi(r) \in C_0^\infty(\Omega_j \cap \tilde{\Lambda}^{n+1}(t_0))$

$$\begin{aligned}(4.34)\qquad & \Big(\sum_{m=0}^{s} V_N(I_{K_j}, I_{K_m}) e_m \Big) \tilde{V}_N(\Omega_j, I_{K_j}) \varphi(r) \\ &= [I + \sum_{\ell=N}^{2(N-1)} (i\lambda)^{-\ell} \tilde{\tilde{V}}^{(\ell)}(\Omega_j, I_{K_j})] \varphi(r),\end{aligned}$$

where $\tilde{V}^{(\ell)}(\Omega_j, I_{K_j})$ and $\tilde{\tilde{V}}^{(\ell')}(\Omega_j, I_{K_j})$ are linear differential operators in $r' \in \Lambda_\tau^n$ of order at most 2ℓ and $2\ell'$ respectively.

Now, let $\varphi(r) \in C_0^\infty(\tilde{\Lambda}^{n+1}(t_0))$. Then, if we note the function spaces on which operators $\mathcal{K}(\Omega_j, I_{K_j})$, $V_N(I_{K_j}, I_{K_m})$ and the like act, we can apply proposition 4.2 and (4.34) to (4.31) as follows. We obtain

$$
\begin{aligned}
(4.35)\quad & L_\lambda(K\varphi)(\tau,x) \\
&= \sum_{j=0}^{s} e^{i\sigma(\Omega_j)} \mathcal{K}(\Omega_j, I_{K_j}) \Big(\sum_{m=0}^{s} V_N(I_{K_j}, I_{K_m}) e_m \Big) \tilde{V}_N(\Omega_j, I_{K_j}) \\
&\qquad \mathcal{D}_N(\Omega_j, I_{K_j}) e_j \varphi - \sum_{j=0}^{s} \sum_{\ell=N}^{2(N-1)} (i\lambda)^{-\ell} e^{i\sigma(\Omega_j)} \mathcal{K}(\Omega_j, I_{K_j}) \\
&\qquad\qquad \tilde{\tilde{V}}^{(\ell)}(\Omega_j, I_{K_j}) \mathcal{D}_N(\Omega_j, I_{K_j}) e_j \varphi + \sum_{j=0}^{s} R_{1,N}(\Omega_j, I_{K_j}) e_j \varphi \\
&= \sum_{j,m} e^{i\sigma(\Omega_j) - i\sigma(\Omega_m, \Omega_j)} \mathcal{K}(\Omega_m, I_{K_m}) e_m \tilde{V}_N(\Omega_j, I_{K_j}) \\
&\qquad\qquad \mathcal{D}_N(\Omega_j, I_{K_j}) e_j \varphi(\tau,x) + R_N' \varphi(\tau,x).
\end{aligned}
$$

There, it follows from Lemma 3.3, Propositions 4.1 and 4.2 that

$$
(4.36)\qquad \max_{0 \le \tau \le t_0} \| R_N' \varphi(\tau, \cdot) \| = O(\lambda^{-N}).
$$

If we use (3.19), (4.29), (4.30) and (4.33), we get from (4.35)

$$
\begin{aligned}
& L_\lambda(K\varphi)(\tau,x) \\
&= \sum_{m=0}^{s} e^{i\sigma(\Omega_m)} \mathcal{K}(\Omega_m, I_{K_m}) e_m \sum_{j=0}^{s} \tilde{V}_N(\Omega_j, I_{K_j}) \mathcal{D}_N(\Omega_j, I_{K_j}) e_j \varphi(\tau,x) + \\
&\qquad\qquad R_N' \varphi(\tau,x) \\
&= K\Big(\sum_{\ell=1}^{N-1} (i\lambda)^{-\ell} W^{(\ell)} \varphi \Big)(\tau,x) + R_N \varphi(\tau,x),
\end{aligned}
$$

which shows (3.23) and (3.24). Thus, we can complete the proof of Proposition 3.4.

As was mentioned in Remark 0.3, Theorem in the present paper

is generalized as follows. We consider the Cauchy problem

$$(4.37)\quad \begin{cases} L^{(1)}u(t,x) \equiv \frac{1}{i}\partial_t u(t,x) - \frac{1}{2}\sum_{j,k=1}^{n} \partial_{x_j}(g^{jk}(t,x)\partial_{x_k}u) + \\ \qquad\qquad \sum_{j=1}^{n} b^j(t,x)\partial_{x_j}u + c(t,x)u = f(t,x), \\ u(0,x) = u_0(x), \end{cases}$$

where $g^{jk}(t,x)$, $b^j(t,x)$ and $c(t,x)$ are complex vaued C^∞ functions on $R^{n+1}_{t,x}$. We assume that

(4.38) all $g^{jk}(t,x)$ $(j, k = 1,2,\cdots,n)$ are real valued and satisfy $g^{jk}(t,x) = g^{kj}(t,x)$.

Here, we do not suppose even the ellipticity.

As in the proof of Theorem, let make a change of a variable from t to $\tau = \lambda t$ with a large parameter $\lambda \geq 1$. Then, if we use the Taylor expansion, (4.37) is written in the form for $v_\lambda(\tau,x) = u(\tau/\lambda,x)$

$$(4.37)'\quad \begin{cases} \lambda^2 L^{(1)}_\lambda v_\lambda(\tau,x) \\ \equiv \lambda^2[\lambda^{-1}D_\tau + H_0^{(1)}(x,\lambda^{-1}D_x) + \sum_{j=1}^{2}(i\lambda)^{-j}H_j^{(1)}(\tau,x,\lambda^{-1}D_x) + \\ \qquad (i\lambda)^{-3}H_3^{(1)}(\tau,x,\lambda^{-1}D_x;\lambda)]v_\lambda(\tau,x) = f(\tau/\lambda,x), \\ v_\lambda(0,x) = u_0(x), \end{cases}$$

where

$$(4.39)\quad \begin{cases} H_0^{(1)}(x,p) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(0,x)p_j p_k, \\ H_1^{(1)}(\tau,x,p) = -\sum_{j=1}^{n} b^j(0,x)p_j + \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial g^{jk}}{\partial x_j}(0,x)p_k + \end{cases}$$

$$\frac{i}{2}\tau\sum_{j,k=1}^{n}\frac{\partial g^{jk}}{\partial t}(0,x)p_jp_k,$$

$H_2^{(1)}(\tau,x,p)$ belongs to $T^2(R_x^n)$ and we have

$$\sup_{0\le\tau\le T,\,1\le\lambda}\|H_3^{(1)}(\tau,x,\lambda^{-1}D_x;\lambda)\varphi(\cdot)\|<\infty \tag{4.40}$$

for any $\varphi(x)\in\mathcal{S}$ and any $T>0$.

Let $(X(\tau,x,p),P(\tau,x,p))$ be the solutions of (0.6) as $H(x,p)=H_0^{(1)}(x,p)$ and set $\theta(x,p)>0$ by

$$\theta(x,p)=\sup\{\tau;\ |X(\theta,x,p)|+|P(\theta,x,p)|<\infty,\ 0<\theta<\tau\}.$$

Then, we obtain

Theorem 4.3. It is necessary for the Cauchy problem (4.37) to be L^2 well posed on $[0,T_0]$ or $[T_0,0]$ for a $T_0\neq 0$ that the inequality

$$\sup_{\substack{(x,p)\in R^{2n},\\ 0\le\rho<\theta(x,p)}}\Big|\sum_{j=1}^{n}\int_0^{\rho}\mathrm{Re}\,b^j(0,X(\theta,x,p))P_j(\theta,x,p)d\theta\Big|<\infty$$

holds.

Proof. We can prove Theorem 4.3 in the same way to the proof of Theorem. There, we note that $F(x,p)$ defined by (3.20) is replaced by

$$\frac{1}{2}\mathrm{Tr}\,\frac{\partial^2H_0^{(1)}}{\partial x\partial p}(x,p)-H_1^{(1)}(\tau,x,p)$$
$$=\sum_{j=1}^{n}b^j(0,x)p_j-\frac{i}{2}\tau\sum_{j,k=1}^{n}\frac{\partial g^{jk}}{\partial t}(0,x)p_jp_k.$$

Take a $(x^0,p^0)\in R^{2n}$ and a t_0 $(0<t_0<\theta(x^0,p^0))$. Then, if $v(x)$ is a C^∞ function with sufficiently small compact support around $x=x^0$, we can construct the asymptotic solution

$v_\lambda(\tau,x)$ satisfying

$$\begin{cases} \max\limits_{0\le\tau\le t_0} \| [\lambda^{-1}D_\tau + H_0^{(1)}(x,\lambda^{-1}D_x) + \\ \qquad \sum\limits_{j=1}^{2} (i\lambda)^{-j} H_j^{(1)}(\tau,x,\lambda^{-1}D_x)] v_\lambda(\tau,\cdot)] \| = O(\lambda^{-3}), \\ v_\lambda(0,x) = e^{i\lambda x\cdot p^0} v(x) \end{cases}$$

in the corresponding form to (3.26). We can easily see for this $v_\lambda(\tau,x)$ from (4.40) that

$$\max_{0\le\tau\le t_0} \| L_\lambda^{(1)} v_\lambda(\tau,\cdot) \| = O(\lambda^{-3})$$

is valid. Using such $v_\lambda(\tau,x)$, since $g^{jk}(t,x)$ are real valued functions, we can complete the proof of Theorem 4.3 in the same way to the proof of Theorem. Q.E.D.

Appendix. Here, we shall show that if at least one of $g^{jk}(x)$ $(j,k = 1,2,\cdots,n)$ is not constant, in almost cases the Hamilton-Jacobi equation $\partial_t\Phi + H(x,\frac{\partial\Phi}{\partial x}) = 0$ has no smooth solution on the global interval $[0,t_0]$. Let $(X(t,x,p),P(t,x,p))$ be the solutions of (0.6). Then, to show it, we have only to prove that the family of rays $\{X(t,x,p)\}_{x\in R^n}$ for any fixed p very often has a focal point on $[0,t_0]$, if we use the terminology in optics. Here, a point $(t,X(t,x,p)) \in R^{n+1}$ is called a focal point, if $\det \frac{\partial X}{\partial x}(t,x,p) = 0$. Though we consider only the simple examles, the result for them indicates the above.

We consider the Hamiltonian function

$$H(x,p) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(x)p_jp_k, \tag{A.1}$$

where we assume that

all $g^{jk}(x)$ are real valued and satisfy $g^{jk}(x) = g^{kj}(x)$.

We suppose the assumption (*):

(i) We have for any $(x,p) \in R^{2n}$

$$|\sum_{j,k=1}^{n} g^{jk}(x)p_jp_k| \le \delta|p|^2 \tag{A.2}$$

with a positive constant δ independent of $(x,p) \in R^{2n}$.

(ii) There exists a family of real valued C^2 functions $\{y_j(x)\}_{j=1}^{n}$ such that

$$\frac{\partial y}{\partial x}(x)(g^{ij}(x);\overset{i\downarrow}{j\rightarrow}1,2,\cdots,n)\,{}^t\frac{\partial y}{\partial x}(x) = I \quad \text{or} \quad -I \tag{A.3}$$

holds at each point $x \in R^n$, where $y(x) = (y_1(x),\cdots,y_n(x))$,

${}^t\frac{\partial y}{\partial x}(x)$ denotes the transposed matrix of $\frac{\partial y}{\partial x}(x)$ and I the identity matrix.

Remark A.1. Let $n = 1$, and let $g^{11}(x)$ be a real valued C^1 function and be bounded function which does not vanish on R^1. Set $y(x) = \int_0^x \frac{1}{|g^{11}(\theta)|^{1/2}}\, d\theta$. Then, the assmption (*) is automatically satisfied.

Let $(X(t,x,p),P(t,x,p))$ be the solutions of the canonical equations (0.6) for $H(x,p)$ defined by (A.1). Then, we get

Theorem A.1. Assume (*). Then, the following (i) and (ii) are equivalent.

(i) There exists a $t \neq 0$ such that $\det \frac{\partial X}{\partial x}(t,x^0,p)$ does not vanish for any $p \in R^n$.

(ii) $\frac{\partial g^{ij}}{\partial x_k}(x^0) = 0$ holds for all i, j and k.

We first introduce the result on the global homeomorphism from [13] without proof.

Theorem A.2 (Theorem 1.22 in [13]). Let f be a C^1 mapping : $R^n \ni x \longrightarrow f(x) = (f_1(x),\cdots,f_n(x)) \in R^n$. If there exists the inverse matrix $\frac{\partial f}{\partial x}(x)^{-1}$ of $\frac{\partial f}{\partial x}(x)$ for each $x \in R^n$ and we have

$$\sup_{x\in R^n} \left\|\frac{\partial f}{\partial x}(x)^{-1}\right\| < \infty,$$

then, f is a homeomorphism of R^n onto R^n. Here, $\|Q\|$ for a matrix Q denotes the operator norm of Q as the mapping from R^n to R^n.

Proof of Theorem A.1. Since we may replace t by $-t$, in place of (A.3) we can assume

$$(A.3)' \qquad \frac{\partial y}{\partial x}(x)(g^{ij}(x);\overset{i\downarrow}{j\rightarrow}1,2,\cdots,n)\,{}^t\frac{\partial y}{\partial x}(x) = I.$$

It is easily seen from the assymptions (A.2) and (A.3)' that we can apply Theorem A.2 to the mapping : $R^n \ni x \longrightarrow y(x) \in R^n$. That is, the above mapping is a diffeomorpism of R^n onto R^n. There, we denote the inverse mapping by the mapping : $R^n \ni y \longrightarrow x(y) = (x_1(y),\cdots,x_n(y)) \in R^n$. Then, since we have

$$\partial_{y_j} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial y_j}(y)\partial_{x_k},$$

one can define the canonical transformation Φ from $R^{2n}_{x,p}$ onto $R^{2n}_{y,q}$ by

$$(A.4) \qquad \Phi : R^{2n}_{x,p} \ni (x,p) \longrightarrow (y(x), p\frac{\partial x}{\partial y}(y(x))) \in R^{2n}_{y,q}.$$

Then, the inverse canonical transformation Φ^{-1} is given by

$$(A.4)' \qquad \Phi^{-1}: R^{2n}_{y,q} \ni (y,q) \longrightarrow (x(y), q\frac{\partial y}{\partial x}(x(y))) \in R^{2n}_{x,p}.$$

See section 4 in [8] or chapter 9 in [1] for the theory of the canonical transformations. Using this canonical transformation Φ, $H(x,p)$ defined by (A.1) is given from the assumption (A.3)' by the formula

$$(A.5) \qquad H(x,p) = \frac{1}{2}\sum_{j,k=1}^{n} g^{jk}(x)p_j p_k = \frac{1}{2}|q|^2.$$

It is well known in the theory of the analytical dynamics that canonical transformations map the solutions of the canonical equations into the other ones. Consequently, noting (A.5), we get

(A.6) $(X(t,x^0,p^0),P(t,x^0,p^0)) = (x(y^0+tq^0),q^0\ \frac{\partial y}{\partial x}(x(y^0+tq^0)))$,

where $(y^0,q^0) = \Phi(x^0,p^0)$. So, we have

(A.6)' $X(t,x^0,p^0) = x(y(x^0) + tp^0\ \frac{\partial x}{\partial y}(y(x^0)))$.

Then, it is easy to see that $\det \frac{\partial X}{\partial x}(t,x^0,p^0) \neq 0$ is equivalent to

$$0 \neq \det\ [I + t\{\frac{\partial}{\partial x^0}\ p^0\ \frac{\partial x}{\partial y}(y(x^0))\}\frac{\partial x}{\partial y}(y(x^0))]$$
$$\equiv \det\ [I + tM(x^0,p^0)].$$

The (i,j) component of $M(x^0,p^0)$ is given by

(A.7) $$\sum_{k,\ell=1}^{n} p_k^0\ \{\frac{\partial}{\partial x_\ell^0}\ \frac{\partial x_k}{\partial y_i}(y(x^0))\}\frac{\partial x_\ell}{\partial y_j}(y(x^0))$$
$$= \sum_{k,\ell,m} p_k^0\ \frac{\partial^2 x_k}{\partial y_m \partial y_i}(y(x^0))\frac{\partial y_m}{\partial x_\ell}(x^0)\frac{\partial x_\ell}{\partial y_j}(y(x^0))$$
$$= \sum_{k=1}^{n} p_k^0\ \frac{\partial^2 x_k}{\partial y_i \partial y_j}(y(x^0)).$$

So, $M(x^0,p^0)$ is a real symmetric matrix and satisfies $M(x^0,\mu p^0) = \mu M(x^0,p^0)$ for $\mu \in R$. Consequently, (i) in Theorem A.1 is equivalent to $M(x^0,p^0) = 0$ for any $p^0 \in R^n$, which is also equivalent to

(A.8) $$\frac{\partial^2 x_k}{\partial y_i \partial y_j}(y(x^0)) = 0 \qquad (i,\ j,\ k = 1,2,\cdots,n)$$

from (A.7).

We have only to prove that (A.8) is equivalent to (ii) in Theorem A.1. Assume (A.8). Then, since we have from (A.3)'

$$(g^{ij}(x);\ {}^{i\downarrow}_{j\rightarrow}\ 1,2,\cdots,n) = \frac{\partial x}{\partial y}(y(x))\ {}^t\frac{\partial x}{\partial y}(y(x)),$$

it follows that (ii) in Theorem A.1 is valid. Conversely, assume (ii) in Theorem A.1. Then, we have from (A.3)'

$$\text{(A.9)} \qquad \frac{\partial}{\partial y_k}\{\frac{\partial x}{\partial y}(y)\,{}^t\frac{\partial x}{\partial y}(y)\}\Big|_{y=y(x^0)} = 0 \qquad (k = 1,2,\cdots,n).$$

We may assume $y(x^0) = 0$. Set

$$\text{(A.10)} \qquad z(y) = x(y)\,{}^t\frac{\partial x}{\partial y}(0)^{-1}.$$

Then, since we have

$$\frac{\partial z}{\partial y}(y) = \frac{\partial x}{\partial y}(0)^{-1}\,\frac{\partial x}{\partial y}(y),$$

we get from (A.9)

$$\begin{aligned} 0 &= \frac{\partial}{\partial y_k}\{\frac{\partial z}{\partial y}(y)\,{}^t\frac{\partial z}{\partial y}(y)\}\Big|_{y=0} \\ &= \frac{\partial}{\partial y_k}\,\frac{\partial z}{\partial y}(y)\Big|_{y=0} + \frac{\partial}{\partial y_k}\,{}^t\frac{\partial z}{\partial y}(y)\Big|_{y=0} \\ &\equiv (\alpha^i_{jk}; {}^{i\downarrow}_{j\rightarrow}1,2,\cdots,n) + (\alpha^j_{ik}; {}^{i\downarrow}_{j\rightarrow}1,2,\cdots,n) \quad (k = 1,2,\cdots,n), \end{aligned}$$

where $\alpha^i_{jk} = \dfrac{\partial^2}{\partial y_k \partial y_j}\, z_i(y)\Big|_{y=0}$. So,

$$\text{(A.9)}' \qquad \alpha^i_{jk} + \alpha^j_{ki} = 0$$

for all i, j and k are valid. Consequently, for any fixed i, j and k we have

$$\alpha^i_{jk} + \alpha^j_{ik} = 0,\ \alpha^j_{ki} + \alpha^k_{ji} = 0,\ \alpha^k_{ij} + \alpha^i_{kj} = 0.$$

There, if we add the first equality to the third one, we obtain $\alpha^i_{jk} = 0$ by the second one and $\alpha^i_{jk} = \alpha^i_{kj}$. Hence, we can see together with (A.10) that (A.8) is valid.

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