# Multicommodity Flows in Graphs II

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## 1. Introduction

Let $G=(V,E)$ be a graph (finite undirected, possibly with multiple edges but without loops), and let $V=V(G)$, $E=E(G)$ be the sets of vertices and edges of $G$ respectively. In this paper a path has no repeated edges, and we permit paths with one vertex and no edges. For two distinct vertices $x,y$, let $\lambda(x,y)=\lambda_G(x,y)$ be the maximum number of edge-disjoint paths between $x$ and $y$, and let $\lambda(x,x)=\infty$.

We first consider the following problem.

Let $(s_1,t_1),\ldots,(s_k,t_k)$ be pairs (not necessarily distinct) of vertices of $G$. When is the following true ?

(1.1) There exist edge-disjoint paths $P_1,\ldots,P_k$ such that $P_i$ has ends $s_i,t_i$ $(1\leq i\leq k)$.

Seymour [10] and Thomassen [12] characterized such graphs when $k=2$, and Seymour [10] when $s_1,\ldots,s_k,t_1,\ldots,t_k$ take only three distinct values.

Our result is the following

Theorem 1. Suppose that $s_1,s_2,s_3,t_1,t_2,t_3$ are vertices of a graph $G$. If for each $i=1,2,3$,

$$\lambda(s_i,t_i)\geq 3,$$

then there exist edge-disjoint paths $P_1,P_2,P_3$ of $G$, such that $P_i$ has ends $s_i$ and $t_i$ $(i=1,2,3)$.

If $\lambda(s_i, t_i) \leq 2$ for some i, then the conclusion does not always hold. Figure 1 gives a counterexample.

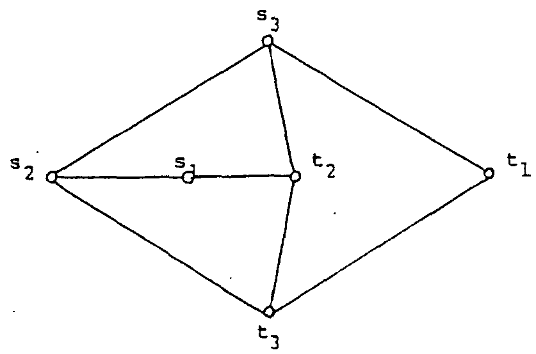


Figure 1.

For a positive integer k, let g(k) be the smallest integer such that for every g(k)-edge-connected graph and for every vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ of the graph, (1.1) holds. Thomassen [12] conjectured the following.

Conjecture. For each odd integer $k \geq 1$, $g(k)=k$, and for each even integer $k \geq 2$, $g(k)=k+1$.

If k is even then $g(k) > k$ (see [12]). It follows easily from Menger's theorem that $g(k) \leq 2k-1$, thus $g(1)=1$, $g(2)=3$; and Cypher [1] proved $g(4) \leq 6$ and $g(5) \leq 7$. As a corollary of Theorem 1 we have the following.

Corollary. $g(3)=3$.

The second problem we consider is the multicommodity flow problem.

Suppose that each edge $e \in E$ has a real-valued capacity $w(e) \geq 0$, and each path has a positive value. We assume that $w \equiv 1$ and each path has value 1 when there is no explanation. For a positive number $\alpha$, paths $\alpha P$, $P$ denote paths of value $\alpha$, 1 respectively. We say that a set of paths $\alpha_1 P_1, \ldots, \alpha_n P_n$ is feasible if for each edge $e \in E$,

$$\sum_{i \in \{i \mid e \in E(P_i)\}} \alpha_i \leq w(e),$$

where $E(P_i)$ is the set of edges of $P_i$.

For two vertices $x, y$ and a real number $q > 0$, a flow $F$ of value $q$ between $x$ and $y$ is a set of paths $\alpha_1 P_1, \ldots, \alpha_n P_n$ between $x$ and $y$ such that $\alpha_1 + \ldots + \alpha_n = q$. When $\alpha_1, \ldots, \alpha_n$ are all integers (half-integers), $F$ is called an integer (half-integer) flow. We say that a set of flows $F_1, \ldots, F_k$ is feasible if the set of paths of $F_1, \ldots, F_k$ is feasible.

Now the multicommodity flow problem is as follows.

Let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of vertices of $G$, as before, and suppose that $q_i \geq 0$ $(1 \leq i \leq k)$ are real-valued demands. When is the following true ?

(1.2) There exist feasible flows $F_1, \ldots, F_k$, such that $F_i$ has ends $s_i$ and $t_i$ and value $q_i$ $(1 \leq i \leq k)$.

Remark. When $k=3$, $w \equiv 1$, and $q_i = 1$ $(1 \leq i \leq 3)$, Theorem 1 implies that (1.2) is true if $\lambda(s_i, t_i) \geq 3$ $(1 \leq i \leq 3)$, and then the flows may be chosen as integer flows.

For a set $X \subseteq V$, let $\partial(X) = \partial_G(X) \subseteq E$ be the set of edges with one end in X and the other in V−X, and let $D(X) = D_G(X) \subseteq \{1, 2, \ldots, k\}$ be

$\{ i \mid 1 \leq i \leq k,\ X \cap \{s_i, t_i\} \neq \emptyset \neq (V-X) \cap \{s_i, t_i\} \}$.

It is clear that if (1.2) is true, then the following holds.

(1.3) For each $X \subseteq V$,

$$\sum_{e \in \partial(X)} w(e) \geq \sum_{i \in D(X)} q_i .$$

Note that $\sum_{e \in \partial(X)} w(e) = |\partial(X)|$ if $w \equiv 1$, and $\sum_{i \in D(X)} q_i = |D(X)|$ if $q_i = 1$ for any i.

Our second result is the following

Theorem 2. Suppose that G is a graph and w is integer-valued, and that k=3, $q_1 = q_2 = q_3 = 1$. Then (1.2) and (1.3) are equivalent.

Moreover if (1.3) holds, then the flows $F_i$ in (1.2) may be chosen as half-integer flows.

(1.4) In general (1.2) and (1.3) are not equivalent, but in the following cases they are equivalent.

(1.4.1) $k=1$ (Ford and Fulkerson [2]).

(1.4.2) $k=2$ (Hu [3] and Seymour [8])

(1.4.3) $k=5$ , $t_i=s_{i+1}$ $(i=1,2,3,4)$ and $t_5=s_1$ (Papernov [7]).

(1.4.4) $k=6$, and $(s_1,t_1),\ldots,(s_6,t_6)$ correspond to the six pairs of a set of four vertices (Papernov [7] and Seymour [9]).

(1.4.5) $s_1=s_2=\ldots=s_j$ and $s_{j+1}=\ldots=s_K$ (obvious extention of (1.4.2)).

(1.4.6) The graph $(V,E\cup\{e_1,\ldots,e_K\})$ is planar, where the edge $e_i$ has ends $s_i$ and $t_i$ $(1\leq i\leq k)$ (Seymour [11]).

(1.4.7) $G$ is planar and can be drawn in the plane so that $s_1,\ldots,s_K,t_1,\ldots,t_K$ are all on the boundary of the infinite face (Okamura and Seymour [5]).

(1.4.8) $G$ is planar and can be drawn in the plane so that $s_1,\ldots,s_j,t_1,\ldots,t_j$ are all on the boundary of a face and $s_{j+1},\ldots,s_K,t_{j+1},\ldots,t_K$ are all on the boundary of the infinite face (Okamura [6]).

(1.4.9) $G$ is planar and can be drawn in the plane so that $s_{j+1},\ldots,s_K,t_1,t_2,\ldots,t_K$ are all on the boundary of the infinite face, and $t_1=\ldots=t_j$ (Okamura [6]).

Moreover if (1.3) and the following (1.5) hold in each case except (1.4.3), or if (1.3) holds and $w$, $q_i$ are even-integer valued in the case (1.4.3), then the flows $F_i$ of (1.2) may be chosen as integer flows.

(1.5) $w$ and $q_i$ are integer-valued, and for each vertex $x \in V$,

$$\sum_{e \in \partial(x)} w(e) - \sum_{i \in D(x)} q_i$$

is even.

(1.4.1),...,(1.4.5) are all the configurations of $(s_i, t_i)$ for which (1.2) and (1.3) are equivalent for all graphs $G$ and all $w, q_i$ (see [9]). When $q_i > 0$ $(1 \leq i \leq 3)$, the case of Theorem 2 is the only case for which (1.2) and (1.3) are equivalent for all graphs $G$ and all $w$ ,$(s_i, t_i)$. Figure 1 gives a counterexample with $q_1=2, q_2=q_3=1$.

Notations and definitions. We call $S \subseteq E$ an n-cut if $|S|=n$ and $S=\partial(X)$ for some $X \subseteq V$ such that $\langle X \rangle$ (which is the subgraph induced by $X$) and $\langle V-X \rangle$ are both connected; and an n-cut $\partial(X)$ is called nontrivial if $|X| \geq 2$ and $|V-X| \geq 2$, trivial otherwise. For two vertices $x,y$ a path $P[x,y]$ or a path $[x,y]$ denotes a path between $x$ and $y$, and let $xy$ be an edge with ends $x,y$, and let $d(x,y)=d_G(x,y)$ be the distance between $x$ and $y$. If vertices $x,y$ belong to a path $P$, then $P(x,y)$ denotes the subpath of $P$ between $x$ and $y$. For a vertex $x$ $\deg(x)=\deg_G(x)$ denotes the degree of $x$, and we let $N(x)=N_G(x)$ be $\{y \in V \mid xy \in E\}$. For a set $X \subseteq V$ and an edge $e$, we denote graphs $\langle V-X \rangle$, $(V, E-e)$ by $G-X$, $G-e$ respectively. For a set $X \subseteq V$ $(S \subseteq E)$ and an element $x \in V$ $(e \in E)$, we denote $X \cup \{x\}$ $(S \cup \{e\})$ by $X \cup x$ $(S \cup e)$.

## 2. Proof of Theorem 1.

In this section disjoint means edge-disjoint. We require the following lemmas.

Lemma 2.1. Suppose that $s_1, s_2, t_1, t_2$ are vertices of a graph G. If $\lambda(s_1, t_1) \geq 3$ and $\lambda(s_2, t_2) \geq 1$, then G contains disjoint paths $[s_1, t_1]$ and $[s_2, t_2]$.

Proof. Since $\lambda(s_1, t_1) \geq 3$, G contains disjoint paths $P_1[s_1, t_1]$, $P_2[s_1, t_1]$ and $P_3[s_1, t_1]$. G contains a path $P_4[s_2, t_2]$. There exist vertices $x, y \in V(P_4)$ such that $P_4(s_2, x)$ and $P_4(t_2, y)$ are disjoint from $P_1, P_2, P_3$. Choose x,y with this property such that $P_4(s_2, x), P_4(t_2, y)$ have the maximum length respectively. If x or $y \notin V(P_1) \cup V(P_2) \cup V(P_3)$, then $x = t_2$ or $y = s_2$, and so the result follows. We may therefore assume that $x \in V(P_2)$ and $y \in V(P_i)$ (i=2 or 3). When i=2 (i=3), let $P_5$ be the path obtained by combining $P_4(s_2, x)$, $P_2(x, y)$ and $P_4(y, t_2)$ ($P_4(s_2, x), P_2(x, s_1), P_3(s_1, y)$ and $P_4(y, t_2)$). Now $P_1$ and $P_5$ are required paths of G.

Lemma 2.2. If G is 3-regular 3-edge-connected graph with no nontrivial 3-cut and with $4 \leq |V| \leq 8$, then G is $K_4, K_{3,3}$, a cube or the graph in Figure 2.

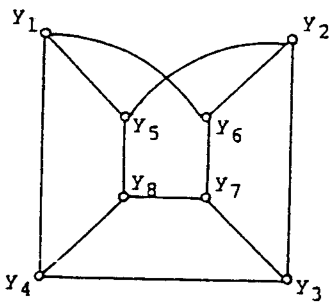


Figure 2.

Proof. Since G is 3-regular 3-edge-connected, G has no multiple edges. Thus if $|V|=4$, then G is $K_4$. If $|V|>4$, then G has no cycle of length three. If $|V|=6$, then let $V=\{x_1,\ldots,x_6\}$. We may let $N(x_1)=\{x_2,x_3,x_4\}$. Since $x_ix_j \notin E$ $(2\leq i<j\leq 4)$, we have $x_ix_j \in E$ $(i=2,3,4;j=5,6)$. Thus G is $K_{3,3}$. If $|V|=8$, then it easily follows that G is a cube or the graph in Figure 2.

Lemma 2.3. Suppose that G is a 3-regular 3-edge-connected graph, and that $a,x_1,x_2,x_3,x_4$ are vertices such that $a\neq x_i$ $(1\leq i\leq 4)$. Then $G-a$ contains disjoint paths $[x_1,x_2]$ and $[x_3,x_4]$.

Proof. We proceed by induction on $|V|$. If $|V|=2$, then G is the graph of triple edges, and the result holds. Therefore we assume $|V|\geq 4$.

First we assume that G contains a nontrivial 3-cut $\{e_1,e_2,e_3\}=\partial(X)$ $(X\subseteq V)$. Let $b_i\in X$, $c_i\in V-X$, $e_i=b_ic_i$ $(i=1,2,3)$, then $b_i \neq b_j$, $c_i \neq c_j$ if $i \neq j$, since G is 3-edge-connected. Let H, K be the graphs obtained from G by contracting V−X, X to one vertex respectively. Let $V(H)=X \cup v$, $V(K)=(V-X) \cup u$. Then H,K are 3-regular 3-edge-connected graphs and $|V(H)| < |V|$, $|V(K)| < |V|$. We may assume $a \in V-X$. It suffices to prove the lemma in the following cases.

Case 1. $\{x_1, x_2, x_3, x_4\} \subseteq V-X$. By induction the result holds in K, and so in G.

Case 2. $x_1 \in X$ and $\{x_2, x_3, x_4\} \subseteq V-X$. By induction the result holds in K (note that $x_1=u$ in K). Thus the result holds in G, since G contains a subgraph $G_1$ homeomorphic to K, such that $x_1$ corresponds to u and each vertex of V−X to itself.

Case 3. $\{x_1, x_2, x_3, x_4\} \subseteq X$. G contains a subgraph $G_2$ homeomorphic to H, such that a corresponds to v and each vertex of X to itself, and so the result holds in G .

Case 4. $\{x_1, x_2\} \subseteq X$ and $\{x_3, x_4\} \subseteq V-X$. Since $K-\{a,u\}$ is connected, this contains a path $[x_3, x_4]$; and H−v contains a path $[x_1, x_2]$.

Case 5. $\{x_1, x_3\} \subseteq X$ and $\{x_2, x_4\} \subseteq V-X$. By induction K−a contains disjoint paths $P_1[u, x_2]$ and $P_2[u, x_4]$. We may let $c_i \in V(P_i)$ (i=1,2), and H−v contains disjoint paths $[x_1, b_1]$ and $[x_3, b_2]$. Thus the result follows.

Case 6. $\{x_1, x_2, x_3\} \subseteq X$ and $x_4 \in V-X$. K−a contains a path $P[u, x_4]$, and we may let $c_1 \in V(P)$. H−v contains disjoint paths $[x_1, x_2]$ and $[x_3, b_1]$. Thus the result follows.

Next we assume that G does not contain a nontrivial 3-cut. If G contains an edge e which is not incident to any of $a, x_1, x_2, x_3, x_4$, then let $\widetilde{G-e}$ be the 3-regular graph homeomorphic to the graph G−e. Then $\widetilde{G-e}$ is 3-edge-connected. By induction the result holds in $\widetilde{G-e}$, and so in G. Thus we assume that any edge is incident to one of $a, x_1, x_2, x_3, x_4$. Then $|E| \leq 15$ and $|V| \leq 10$. We put $T=\{a, x_1, x_2, x_3, x_4\}$. We may assume that $x_1, x_2, x_3$ and $x_4$ are all distinct. For if not, then the result follows, since G−a is 2-edge-connected. Thus $|V| \geq 5$. If $|V|=10$, then $N(x_i) \subseteq V-T$ $(1 \leq i \leq 4)$ and $|V-T|=5$. Thus for some $y \in V-T$, $y \in N(x_1) \cap N(x_2)$. $G-\{a,y\}$ is connected, and so the result follows. If $|V|=6$ or 8, then by Lemma 2.2 G is $K_{3,3}$, a cube, or the graph in Figure 2. We ommit the proofs for them.

Lemma 2.4. Suppose that G is a 3-regular 3-edge-connected graph, and that $a, a_1, a_2, a_3, x_1, x_2, x_3$ are vertices such that $N(a)=\{a_1, a_2, a_3\}$ and $a \neq x_i$ $(1 \leq i \leq 3)$. Then

$$|I_G| \geq 4.$$

Here $I_G = I_G(a, a_1, a_2, a_3, x_1, x_2, x_3)$ is

$$\left\{ (i,j,k) \;\middle|\; \begin{array}{l} \{i,j,k\}=\{1,2,3\}.\ G-a \text{ contains disjoint paths} \\ [x_1, a_i], [x_2, a_j] \text{ and } [x_3, a_k]. \end{array} \right\}.$$

Proof. We proceed by induction on $|V|$. We assume $|V| \geq 4$. First we assume that G contains a nontrivial 3-cut

$\{e_1, e_2, e_3\} = \partial(X)$ $(X \subseteq V)$. We define $b_i, c_i$ $(i=1,2,3)$, $H$, $K$, $v$ and $u$ similarly as in the proof of Lemma 2.3. We may assume $a \in V-X$. Then $|X \cap N(a)| \leq 1$. If $a_i \in X$ for some $i$, then $a_i = u$ in $K$. It suffices to prove the lemma in the following cases.

Case 1. $\{x_1, x_2, x_3\} \subseteq V-X$; $x_1 \in X$ and $\{x_2, x_3\} \subseteq V-X$ ; or $\{x_1, x_2, x_3\} \subseteq X$. Similar as Case 1,2 or 3 in the proof of Lemma 2.3.

Case 2. $\{x_1, x_2\} \subseteq X$ and $x_3 \in V-X$. By induction $|I_K(a, a_1, a_2, a_3, u, u, x_3)| \geq 4$. For each $(i,j,k)$ of $I_K$, $K-a$ contains disjoint paths $P_1[u, a_i]$, $P_2[u, a_j]$ and $P_3[x_3, a_k]$. If $u \notin N_K(a)$, then we may let $c_i \in V(P_i)$ $(i=1,2)$. By Induction $H-v$ contains disjoint paths $[x_1, b_1]$ and $[x_2, b_2]$. Thus $(i,j,k) \in I_G(a, a_1, a_2, a_3, x_1, x_2, x_3)$, and so $|I_G| \geq 4$. If $u \in N_K(a)$, then we may let $a_1 = u, a = c_1$. Now $k \neq 1$ and we may let $i=1$, $j=2$, $k=3$, $c_2 \in V(P_2)$. Since $H-v$ contains disjoint paths $[x_1, b_1]$ and $[x_2, b_2]$, $|I_G| \geq 4$.

Next we assume that $G$ does not contain a nontrivial 3-cut. We may assume that any edge is incident to one of $a$, $x_1, x_2, x_3$ (see the proof of Lemma 2.3). Thus $|E| \leq 12$ and $|V| \leq 8$. By Lemma 2.2 $G$ is $K_4$, $K_{3,3}$, a cube or the graph in Figure 2, but in the last graph any four vertices do not cover all edges of the graph. Thus $G$ is one of the first three graphs. If $G$ is a cube, then in Figure 3 it suffices to check the case $y_1 = a$, $y_3 = x_1$, $y_6 = x_2$, $y_8 = x_3$. We ommit the proofs for $K_4$, $K_{3,3}$.

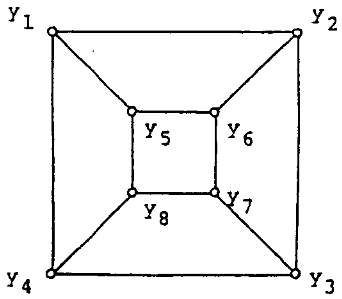


Figure 3.

Lemma 2.5. Suppose that $s_1, s_2, s_3, t_1, t_2, t_3$ are vertices of a graph G. If G is 3-regular 3-edge-connected, then G contains disjoint paths $[s_1, t_1]$, $[s_2, t_2]$ and $[s_3, t_3]$.

Proof. We proceed by induction on $|V|$. We put $T=\{s_1, s_2, s_3, t_1, t_2, t_3\}$. If $s_i=t_i$ for some i, then the result follows by Lemma 2.1, and if $s_1=s_2=s_3$, then the result follows from Menger's theorem. Thus we may assume that these are not the cases.

First we assume that G contains a nontrivial 3-cut $\{e_1, e_2, e_3\}=\partial(X)$ $(X\subseteq V)$. We define $b_i, c_i$ $(i=1,2,3)$, H, K, v and u similarly as in the proof of Lemma 2.3. It suffices to prove the lemma in the following cases.

Case 1. $T\cap X=\emptyset$. By induction the result holds in K, and so in G.

Case 2. $s_1\in X$ and $\{s_2, s_3, t_1, t_2, t_3\}\subseteq V-X$. G contains a subgraph $G_1$ homeomorpic to K, such that $s_1$ corresponds to u and each vertex of $V-X$ to itself.

Case 3. $\{s_1, t_1\} \subseteq X$ and $\{s_2, s_3, t_2, t_3\} \subseteq V-X$. By Lemma 2.3 $K-u$ contains disjoint paths $[s_2, t_2]$ and $[s_3, t_3]$, and $H-v$ contains a path $[s_1, t_1]$.

Case 4. $\{s_1, s_2\} \subseteq X$ and $\{s_3, t_1, t_2, t_3\} \subseteq V-X$. By induction $K$ contains disjoint paths $P_1[u, t_1], P_2[u, t_2]$ and $[s_3, t_3]$. Let $c_i \in V(P_i)$ $(i=1,2)$. By Lemma 2.3 $H-v$ contains disjoint paths $[s_1, b_1]$ and $[s_2, b_2]$. Now the result follows.

Case 5. $\{s_1, s_2, t_1\} \subseteq X$ and $\{s_3, t_2, t_3\} \subseteq V-X$. We can get the result by applying Lemma 2.3 on $H$ and $K$.

Case 6. $\{s_1, s_2, s_3\} \subseteq X$ and $\{t_1, t_2, t_3\} \subseteq V-X$. By Lemma 2.4

$$I_H(v, b_1, b_2, b_3, s_1, s_2, s_3) \cap I_K(u, c_1, c_2, c_3, t_1, t_2, t_3) \neq \emptyset,$$

and so the result follows.

Next we assume that $G$ does not contain a nontrivial 3-cut. We may assume that every edge of $G$ is incident to a vertex of $T$ (see the proof of Lemma 2.3). Thus $|E| \leq 18$ and $|V| \leq 12$. We require the following.

(2.1) We may assume that $d(s_i, t_i) \geq 2$ $(i=1,2,3)$. If $d(s_i, t_i)=2$ for some $i$ and $s_i, t_i$ are adjacent to a common vertex $x$, say for $i=1$, then we may assume that

$$x \in \{s_2, t_2\} \cap \{s_3, t_3\}.$$

Proof. Let $d(s_1, t_1)=1$. If $\{s_i, t_i\} \cap \{s_1, t_1\} = \emptyset$, for $i=2$ or 3, say $i=2$, then $\lambda_{G-s_1t_1}(s_2, t_2)=3$ and by Lemma 2.1

$G-s_1t_1$ contains disjoint paths $[s_2,t_2]$ and $[s_3,t_3]$, and so the result of Lemma 2.5 follows; if not, then we may let $s_2=s_1$, $s_3=t_1$ and $s_1\neq t_i$ $(i=2,3)$. Let $y\in N(s_1)-t_1$. By Lemma 2.3 $G-s_1$ contains disjoint paths $[s_3,t_3]$ and $[t_2,y]$. Thus the result of Lemma 2.5 follows. Hence we may assume that $d(s_i,t_i)\geq 2$ $(i=1,2,3)$. Assume that $s_1$ and $t_1$ are adjacent to a vertex x. Let $y\in N(x)-\{s_1,t_1\}$. If $x\notin T$, then by Lemma 2.3 G-x contains diajoint paths $[s_2,t_2]$ and $[s_3,t_3]$. If $x\in T$ and $x\notin\{s_2,t_2\}\cap\{s_3,t_3\}$, then we may let $x=s_2$ and $s_3\neq x\neq t_3$. By Lemma 2.3 G-x contains disjoint paths $[s_3,t_3]$ and $[t_2,y]$, hence Lemma 2.5 holds. Thus (2.1) is proved.

Now we return to the proof of Lemma 2.5. If $G=K_4$, then $d(s_1,t_1)=1$, and if $G=K_{3,3}$, then $s_1$ and $t_1$ are adjacent to common three vertices, contrary to (2.1). If G is the graph in Figure 2, then we may let $s_1=y_1$ without loss of generality. Then $t_1\neq y_i$ $(i=4,5,6)$ by (2.1). If $t_1=y_i$ (i=2 or 8), say for i=8, then $\{y_4,y_5\}\subseteq\{s_2,t_2\}\cap\{s_3,t_3\}$ by (2.1). So we may let $y_4=s_2=s_3$ and $y_5=t_2=t_3$, contrary to (2.1). If $t_1=y_i$ (i=3 or 7), say for i=3, then we may let $y_4=s_2=s_3$ by (2.1). Now we can not choose $\{t_2,t_3\}$ such that T covers E, a contradiction. When G is a cube, in Figure 3 we may let $s_1=y_1$ and $t_1\neq y_i$ $(i=2,4,5)$. If $t_1=y_i$ (i=3,6 or 8), say for i=3, then we may let $y_2=s_2=s_3$ and $y_4=t_2=t_3$, and the result follows. Thus we may let $t_1=y_7$. Since T covers all edges, we may let $\{s_2,t_2\}=\{y_2,y_8\}$ and $\{s_3,t_3\}=\{y_3,y_5\}$, then the result easily follows.

By Lemma 2.2 we may let $|V|=10$ or 12. Thus $|T|\geq 5$. Note that for each distinct vertices $x,y\in V$, $N(x)\neq N(y)$, because G has no nontrivial 3-cut. We distinguish three cases.

Case 1. $|T|=5$. Let $s_1=s_2$. Now $|V|=10$, and G is a bipartite graph and the partition of V is $(T,V-T)$. The number of vertices which have distance two from $s_1=s_2$ is at least three, and so $d(s_i,t_i)=2$ for i=1 or 2, contrary to (2.1).

Case 2. $|T|=6$ and $|V|=12$. Now G is a bipartite graph and the partition of V is $(T,V-T)$. If the number of vertices which have distance two from $s_1$ is at least five, then one of such vertices is $t_1$, a contradiction; if not, then the number is four, since G does not contain a nontrivial 3-cut. Thus G contains a subgraph as illustrated in Figure 4, where $T=\{s_1,x_1,x_2,x_3,x_4,x_5\}$. By (2.1) $t_1\neq x_i$ $(i=1,2,3,4)$ and $\{s_j,t_j\}$ is not $\{x_1,x_2\},\{x_1,x_4\}$ nor $\{x_2,x_3\}$ $(j=2.3)$, and so we may let $\{x_1,x_3\}=\{s_2,t_2\}$, $\{x_2,x_4\}=\{s_3,t_3\}$ and $x_5=t_1$. Now $\{x_5y_1,x_5y_2,x_5y_3\}\subseteq E$. If $x_1y_i\in E$ (i=1 or 2), say for i=1, then $\{x_3y_2,x_3y_3\}\subseteq E$ and $x_2y_3\in E$. Now the result follows. If $x_1y_3\in E$, then $x_3y_3\notin E$, and so $\{x_3y_1,x_3y_2\}\subseteq E$, contrary to $N(y_1)\neq N(y_2)$.

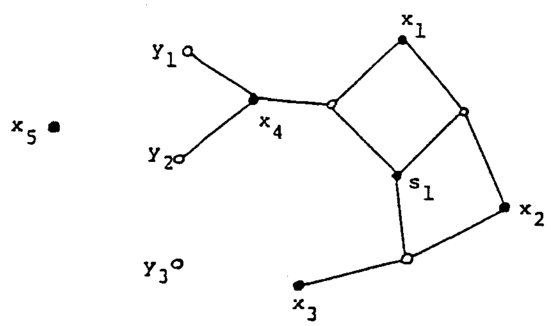


Figure 4.

Case 3. $|T|=6$ and $|V|=10$. Now both ends of just three edges are in T, and by (2.1) $d(s_i,t_i)\geq 3$ ($i=1,2,3$). Thus G contains a subgraph as illustrated in Figures 5a,5b,5c or 5d, where $T=\{x_1,\ldots,x_6\}$ and $V-T=\{y_1,\ldots,y_4\}$.

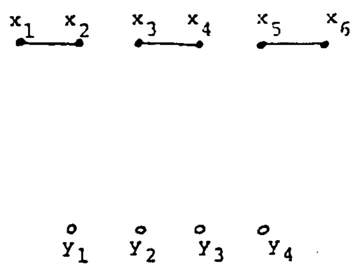


Figure 5a.

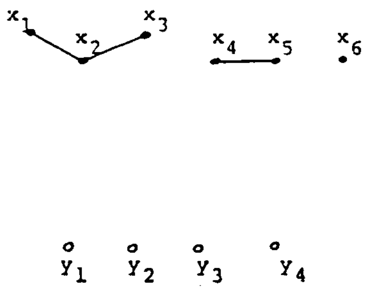


Figure 5b.

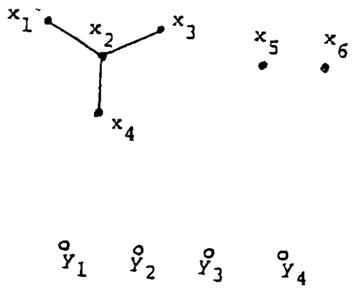


Figure 5c.

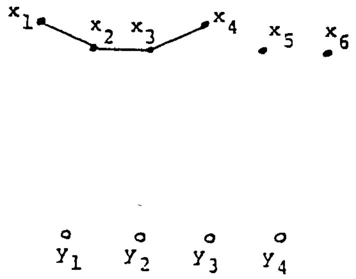


Figure 5d.

In Figure 5a, we may let $\{x_1,x_3\}=\{s_1,t_1\}$, $\{x_2,x_5\}=\{s_2,t_2\}$, $\{x_4,x_6\}=\{s_3,t_3\}$ and $\{x_1y_1, x_1y_2\}\subseteq E$. Then $x_iy_j\in E$ ($i=2,3$; $j=1, 2$). Since $N(y_1)\neq N(y_2)$, one of them contains $\{x_5,x_6\}$ or $\{x_4,x_6\}$, a contradiction. In Figure 5b, we may let $\{x_6y_1, x_6y_2, x_6y_3\}\subseteq E$. If for some $i=1,3,4,5$ $\{x_i,x_6\}=\{s_1,t_1\}$, then $d(s_1,t_1)=2$, a contradiction. Thus we may let $\{x_2,x_6\}=\{s_1,t_1\}$, $\{x_1,x_4\}=\{s_2,t_2\}$ and $\{x_3,x_5\}=\{s_3,t_3\}$. Thus $x_2y_4\in E$. We may let $\{x_1y_1, x_1y_2\}\subseteq E$, and so $\{x_4y_3, x_4y_4, x_5y_1, x_5y_2\}\subseteq E$, contrary to $N(y_1)\neq N(y_2)$. In Figure 5c, for some $i=1,2,3$ $d(s_i,t_i)\leq 2$, a contradiction. In Figure 5d, we may let $\{x_2,x_5\}=\{s_1,t_1\}$, $\{x_3,x_6\}=\{s_2,t_2\}$, $\{x_1,x_4\}=\{s_3,t_3\}$ and $\{x_1y_1, x_1y_2, x_4y_3, x_4y_4\}\subseteq E$.

Now $x_2 y_i \in E$ (i=3 or 4), say for i=3, then $\{x_5 y_1, x_5 y_2, x_5 y_4\} \subseteq E$. $x_3 y_i \in E$ (i=1 or 2), say for i=1, then $\{x_6 y_2, x_6 y_3, x_6 y_4\} \subseteq E$. Now the result easily follows.

Proof of Theorem 1. We proceed by induction on $|V|$. If G is not 2-connected, then we can deduce the result by using induction on blocks. Thus we may assume that G is 2-connected. If G contains a vertex of degree k ($\geq 4$), then we replace this vertex by a k-gon with k vertices of degree 3. (Figure 6 gives an example with k=5.) If this vertex of G is $s_i$ ($t_i$) for some i, then we assign $s_i$ ($t_i$) on any vertex of this k-gon, producing a 3-regular graph $G'$ such that $\lambda_{G'}(s_i, t_i) \geq 3$ for each i. If the result holds in $G'$, then the result clearly holds in G, and so we may assume that G

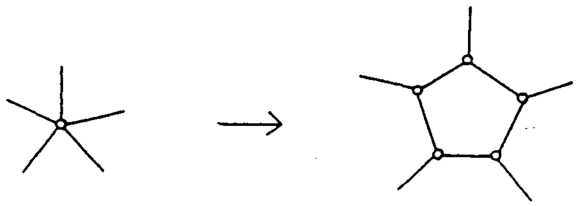

Figure 6.

is 3-regular. By Lemma 2.5 we may assume that G contains a 2-cut $\{e_1, e_2\} = \partial(X)$ ($X \subseteq V$). Let $b_i \in X$, $c_i \in V-X$ and $e_i = b_i c_i$ (i=1,2). We define new graphs H,K as follows.

$$H=(X,\ E(\langle X\rangle)\cup f),$$

$$K=(V-X,\ E(\langle V-X\rangle)\cup g),$$

where $f$, $g$ are new edges with ends $b_1, b_2$ and $c_1, c_2$ respectively. Then $H$ and $K$ are 2-edge-connected. Since $\lambda_G(s_i, t_i) \geq 3$, $\{s_i, t_i\} \subseteq X$ or $\{s_i, t_i\} \subseteq V-X$ for each $i$. Thus it suffices to consider the following cases.

Case 1. $\{s_1, s_2, s_3, t_1, t_2, t_3\} \subseteq X$. By induction the result holds in $H$.

Case 2. $\{s_1, s_2, t_1, t_2\} \subseteq X$ and $\{s_3, t_3\} \subseteq V-X$. By Lemma 2.1 $H$ contains disjoint paths $P_1[s_1, t_1]$ and $P_2[s_2, t_2]$. Let $P_3, P_4$ $P_5$ be disjoint paths of $K$ between $s_3$ and $t_3$, and let $c_1 c_2 \notin E(P_3) \cup E(P_4)$. If $b_1 b_2 \notin E(P_1) \cup E(P_2)$, then $P_1, P_2, P_3$ are required paths of $G$. Thus let $b_1 b_2 \in E(P_1)$. If $c_1 c_2 \notin E(P_5)$, then by Lemma 2.1 $K - c_1 c_2$ contains disjoint paths $[s_3, t_3]$ and $[c_1, c_2]$; and if $c_1 c_2 \in E(P_5)$, then let $P_6[c_1, c_2]$ be the path obtained by combining $P_5 - c_1 c_2$ and $P_4$. In each case we can construct required paths of $G$.

3. Proof of Therem 2.

For an integer $n \geq 3$ and vertices $x_1, x_2, \ldots, x_n$, we denote feasible paths $\frac{1}{2}[x_1, x_2], \frac{1}{2}[x_2, x_3], \ldots, \frac{1}{2}[x_{n-1}, x_n]$, and $\frac{1}{2}[x_n, x_1]$ by $\frac{1}{2}[x_1, \ldots, x_n, x_1]$. For a vertex $x \in V$ and $a, b \in N(x)$, let $G_x^{a,b}$ be the graph $(V, E \cup e_1 - \{e_2, e_3\})$, where $e_1$ is a new edge with ends $a, b$ and $e_2, e_3$ are edges of $E$ with ends $a, x$ and $b, x$ respectively.

Lemma 3.1 (Mader [4]). Suppose that $G$ is a graph and $x$ is a non-separating vertex of $G$ with $\deg x \geq 4$ and with $|N(x)| \geq 2$. Then there exist two vertices $a, b \in N(x)$, such that for each two vertices $y, z \in V - x$,

$$\lambda_{G_x^{a,b}}(y,z) = \lambda_G(y,z).$$

Lemma 3.2. Suppose that $x_1, \ldots, x_5$ are vertices of a graph $G$. If for each $1 \leq i < j \leq 5$,

$$\lambda(x_i, x_j) \geq 4,$$

and each vertex of $G$ has even degree, then $G$ contains edge-disjoint paths $[x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]$, and $[x_5, x_1]$.

Proof. We proceed by induction on $|E|$. We put $T = \{x_1, \ldots, x_5\}$. If $|T| \leq 4$, then the result follows from (1.4.4), and so we may let $|T| = 5$. We may assume that $G$ is 2-connected, and that for each vertex $x$ of $G$ $\deg x \geq 4$. If there exists a vertex $x$ in $V - T$, then by Lemma 3.1 there exist two vertices $a,b \in N(x)$ such that $\lambda_{G_x^{a,b}}(x_i,x_j)=\lambda_G(x_i,x_j)$ $(1 \le i < j \le 5)$. $|E(G_x^{a,b})| < |E|$ and each vertex of $G_x^{a,b}$ has even degree, thus by induction the result holds in $G_x^{a,b}$, and so in G. Let V=T. If $x_5x_1 \in E$, then we can apply (1.4.5) for the graph $G-x_5x_1$, and for pairs $(s_i,t_i)=(x_i,x_{i+1})$ and $q_i=1$ $(1 \le i \le 4)$. Thus we may let $x_5x_1 \notin E$ and $x_ix_{i+1} \notin E$ $(1 \le i \le 4)$. Now G contains a subgraph as illustratrd in Figure 7a or 7b, and the result holds.

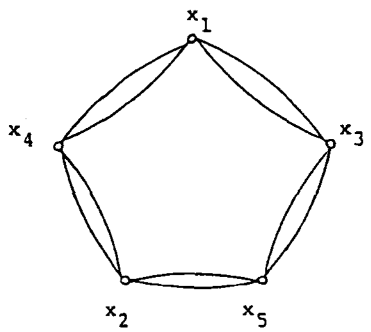


Figure 7a.

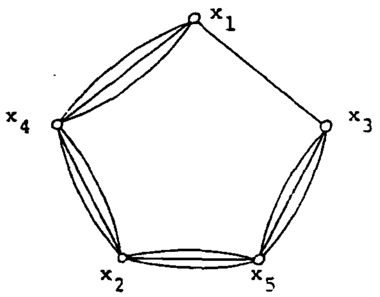


Figure 7b.

Lemma 3.3. Suppose that G is a 2-edge-connected graph and a,b,c,d,x,y are vertices such that deg a=3, N(a)={b,c,d}, deg $b \ge 3$, and a, x, y are all distinct, and that for each

2-cut $\partial(X)$ $(X \subseteq V,\ |X| \leq |V-X|)$,

$X=\{x\}$, $X=\{y\}$ or $X=\{x,y\}$ and $|E(\langle X\rangle)|=1$.

Then G−a contains $\frac{1}{2}[b,c,x,d,y,b]$, if it is not the cases that

deg c=2, c=x, deg $c_1 = 2$ $(N(c)=\{a,c_1\})$ or,

deg c=2, c=y.

Proof. We distinguish four cases.

Case 1. deg $c \geq 3$ and deg $d \geq 3$. Now G−a is 2-edge-connect. Let $G'$ be the graph obtained by replacing each edge of G by double edges. Then $G'-a$ is 4-edge-connected, and so by applying Lemma 3.2 on $G'-a$ we can deduce the result.

Case 2. deg c=2 and deg $d \geq 3$. Let $N(c)=\{a,c_1\}$. By the hypothesis $c \neq y$, and so c=x and deg $c_1 \geq 3$. $G-\{a,c\}$ is 2-edge-connected, and so this contains $\frac{1}{2}[b,c_1,d,y,b]$ by Lemma 3.2.

Case 3. deg $c \geq 3$ and deg d=2. Let d=x and $N(d)=\{a,d_1\}$. If deg $d_1 \geq 3$, then $G-\{a,d\}$ is 2-edge-connected, and so this contains $\frac{1}{2}[b,c,d_1,y,b]$. If deg $d_1=2$, then $d_1=y$. By (1.4.4) $G-\{a,d\}$ contains $\frac{1}{2}[b,c,d_1,b]$, thus G contains $\frac{1}{2}[b,c,x,d,y,b]$ When d=y, the proof is similar.

Case 4. deg c=deg d=2. Now $c \neq d$ and $c \neq y$, thus c=x,d=y, and $G-\{a,c,d\}$ is 2-edge-connected. By (1.4.4) G−a contains $\frac{1}{2}[b,c,d,b]$.

If we prove following Lemma 3.4, Theorem 2 follows.

Lemma 3.4. Suppose that G is a graph with $w \equiv 1$, $(s_1,t_1)$, $(s_2,t_2)$, $(s_3,t_3)$ are pairs of vertices of G, and $q_1=q_2=q_3=1$.

If (1.3) holds, then G contains feasible paths $\frac{1}{2}P_1[s_1,t_1]$, $\frac{1}{2}P_2[s_1,t_1]$, $\frac{1}{2}P_3[s_2,t_2]$, $\frac{1}{2}P_4[s_2,t_2]$, $\frac{1}{2}P_5[s_3,t_3]$, and $\frac{1}{2}P_6[s_3,t_3]$.

Proof. We proceed by induction on $|E|$. We put $T=\{s_1,s_2,s_3,t_1,t_2,t_3\}$. We require the following

(3.1) We may assume the following.

(3.1.1) G is 2-connected, and $|T|=6$.

(3.1.2) For each 2-cut $\partial(X)$ $(X\subseteq V)$, $|X|=|X\cap T|=1$ or $|X\cap T|\geq 2$.

(3.1.3) For each edge $e\in E$, there exists $X\subseteq V$ such that $|\partial(X)|=|D(X)|$ and $e\in\partial(X)$.

(3.1.4) For each $1\leq i\leq 3$, $s_i$ and $t_i$ are not adjacent.

(3.1.5) If for vertices $x_1,x_2,x_3,x_4$ of G $\deg x_2=\deg x_3=2$ and $\{x_1x_2,x_2x_3,x_3x_4\}\subseteq E$, then $\deg x_1\geq 3$ and $\deg x_4\geq 3$.

Proof. (1) If $|T|\leq 5$, then Lemma 3.4 follows from (1.4.5).

(2) Let $\{e_1,e_2\}=\partial(X)$ be a 2-cut , and let $a_i\in X$, $b_i\in V-X$ and $a_ib_i=e_i$ $(i=1,2)$. We define new graphs H,K as follows.

$H=(X,E(\langle X\rangle)\cup f)$,

$K=(V-X,E(\langle V-X\rangle)\cup g)$,

where f, g are new edges with ends $a_1,a_2$ and $b_1,b_2$ respectively. If $X\cap T=\emptyset$, then by induction the result of Lemma 3.4 holds in K, and so in G. If $|X\cap T|=1$ (say $s_1\in X$) and $|X|\geq 2$, then we assign $s_1$ on the midpoint of g in K, producing a new graph $K'$. Now by induction the result of Lemma 3.4 holds in $K'$, and so in $G$.

(3) If there exists $e \in E$ such that for each $X \subseteq V$ with $e \in \partial(X)$ and $|\partial(X)| > |D(X)|$, then the hypothesis of Lemma 3.4 holds in $G-e$, and so we can apply induction on $G-e$.

(4) If $s_3 t_3 \in E$, then we can apply (1.4.2) for the graph $G-s_3 t_3$, and for two pairs $(s_1, t_1), (s_2, t_2)$ and $q_1 = q_2 = 1$.

(5) If $\deg x_1 = 2$, then $x_i \in T$ ($1 \leq i \leq 3$) by (3.1.2), and so we may let $x_1 = s_2$, $x_2 = s_1$ and $x_3 = t_2$ by (3.1.4) and (1.3). Let $x_0 \in N(x_1) - x$ . Let $G'$ be the graph obtained by contracting the edge $x_0 x_1$. By induction $G'$ contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[s_2, t_2]$, $\frac{1}{2}P_4[s_2, t_2]$, $\frac{1}{2}P_5[s_3, t_3]$ and $\frac{1}{2}P_6[s_3, t_3]$. Let $Q_1, \ldots, Q_6$ be the corresponding paths of $G$. We may let $x_1 x_2 \in E(Q_1) \cap E(Q_2)$ or $x_1 x_2 \in E(Q_1) \cap E(Q_3)$. In the former case, let $Q_7$ be the path of $G$ such that $E(Q_7) = \{x_1 x_2, x_2 x_3\}$ and let $Q_8$ be the path of $G$ obtained by combining $x_2 x_3$, $Q_3(t_2, x_0)$ and $Q_2(x_0, t_1)$. Then $\frac{1}{2}Q_1$, $\frac{1}{2}Q_8$, $\frac{1}{2}Q_7$, $\frac{1}{2}Q_4$, $\frac{1}{2}Q_5$, $\frac{1}{2}Q_6$ are required paths of $G$. In the latter case $\frac{1}{2}Q_1, \ldots, \frac{1}{2}Q_6$ are required paths of $G$.

Now we come to the proof of Lemma 3.4. We distinguish three cases.

Case 1. $G$ contains a nontrivial 2-cut $\{e_1, e_2\} = \partial(X)$ ($X \subseteq V$). We define $H$, $K$, $a_i$, $b_i$, $f$ and $g$ similarly as in the proof of (3.1.2). Then $H$ and $K$ are 2-edge-connected. It suffices to consider the following cases by (3.1.2).

Case 1a. $\{s_1, t_1\} \subseteq X$ and $\{s_2, s_3, t_2, t_3\} \subseteq V-X$. Assume that K-g contains feasible paths $\frac{1}{2}P_1[s_2, t_2]$, $\frac{1}{2}P_2[s_2, t_2]$, $\frac{1}{2}P_3[s_3, t_3]$ and $\frac{1}{2}P_4[s_3, t_3]$. Then H-f contains a path $P_5[s_1, t_1]$, and $P_5$, $\frac{1}{2}P_1$, $\frac{1}{2}P_2$, $\frac{1}{2}P_3$, $\frac{1}{2}P_4$ are required paths of G. If this is not the case, then by (1.4.2) for the graph K-g, and for two pairs $(s_2, t_2), (s_3, t_3)$ and $q_2 = q_3 = 1$, (1.3) does not hold. Thus for some $Y \subseteq V-X$ with $b_1 \in Y$,

$$D_{K-g}(Y) = \{2,3\} \text{ and } |\partial_{K-g}(Y)| = 1.$$

For each $Z \subseteq X$ such that $a_1 \in Z$, $f \in \partial_H(Z)$ and $D_H(Z) = \{1\}$,

$$|D_G(Y \cup Z)| = 3,$$

and so $\quad |\partial_G(Y \cup Z)| = |\partial_{H-f}(Z)| + |\partial_{K-g}(Y)| \geq 3,$

thus $\quad |\partial_{H-f}(Z)| \geq 2.$

Hence by (1.4.2) H-f contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[a_1, a_2]$ and $\frac{1}{2}P_4[a_1, a_2]$, and K contains feasible paths $\frac{1}{2}P_5[s_2, t_2]$, $\frac{1}{2}P_6[s_2, t_2]$, $\frac{1}{2}P_7[s_3, t_3]$ and $\frac{1}{2}P_8[s_3, t_3]$. Now we can construct required paths of G.

Case 1b. $\{s_1, s_2\} \subseteq X$ and $\{s_3, t_1, t_2, t_3\} \subseteq V-X$. If H-f is 2-edge-connected, then we assign a new vertex u on the midpoint of g, producing a new graph $K'$. By induction $K'$ contains feasible paths $\frac{1}{2}P_1[u, t_1]$, $\frac{1}{2}P_2[u, t_1]$, $\frac{1}{2}P_3[u, t_2]$, $\frac{1}{2}P_4[u, t_2]$, $\frac{1}{2}P_5[s_3, t_3]$ and $\frac{1}{2}P_6[s_3, t_3]$. We may let $ub_1 \in E(P_1) \cap E(P_2)$ or $ub_1 \in E(P_1) \cap E(P_3)$. In each case we can construct required paths of G, since by (1.4.5) H-f contains feasible paths $\frac{1}{2}P_7[s_1, a_1]$, $\frac{1}{2}P_8[s_1, a_1]$, $\frac{1}{2}P_9[s_2, a_2]$ and $\frac{1}{2}P_{10}[s_2, a_2]$ and contains $\frac{1}{2}[s_1, a_1, s_2, a_2, s_1]$. Thus we may assume that H-f is not 2-edge-connected, and so H contains a 2-cut

$\{f,f'\}$. Then $\{f',e_1\}$ and $\{f',e_2\}$ are 2-cuts of G. By (3.1.2) $H=(\{a_1,a_2\},\{f,f'\})$, and so we may let $a_1=s_1$ and $a_2=s_2$. By (3.1.5) $\deg b_i \geq 3$ $(i=1,2)$. If K contains a 2-cut $\{g,g'\}=\partial_K(Y)$ $(Y\subseteq V-X)$, then $\{g',e_1\}$ and $\{g',e_2\}$ are 2-cuts of G. Since $\deg b_i \geq 3$ $(i=1,2)$, by (3.1.2) $|Y\cap T|=2$ and $|(V-X-Y)\cap T|=2$ By Case 1a we may let $Y\cap T\neq\{s_3,t_3\}$, and so we may let $\langle Y\rangle$ is an edge, contrary to (3.1.5). Thus assume that $K-g$ is 2-edge-connected. By (3.1.3), there exists $X\subseteq V$ such that $|\partial(X)|=|D(X)|$ and $s_1s_2\in\partial(X)$. Thus we may assume that G is the graph as illustrated in Figure 8. Let $Y_1,Y_2$ be the subsets

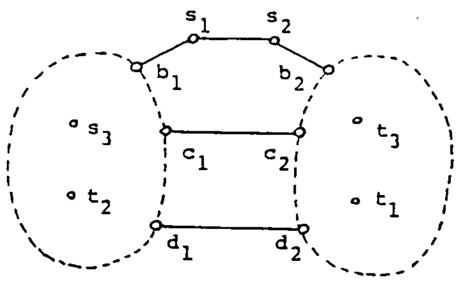


Figure 8.

of V such that $b_i\in Y_i$ and $\partial(Y_i)=\{s_ib_i, c_1c_2, d_1d_2\}$ $(i=1,2)$. We construct new graphs $K_1,K_2$ as follows.

$$K_i=(Y_i\cup v_i,\ E(\langle Y_i\rangle)\cup\{b_iv_i, c_iv_i, d_iv_i\}),\ i=1,2,$$

where $v_1,v_2$ are new vertices. If for i=1 or 2, $K_i$ contains a 2-cut $\partial_{K_i}(Z_i)$ $(Z_i\subseteq Y_i)$ such that $|Z_i|\geq 2$ and $|V(K_i)-Z_i|\geq 2$, say i=1, then $\partial_G(Z_1)$ is a 2-cut of G, and by (3.1.2) $Z_1\cap T=\{s_3,t_2\}$. Thus we may assume that $Z_1=\{s_3,t_2\}$ and $\deg s_3=\deg t_2=2$. This allows that we can apply Lemma 3.3 on

$K_1$ and $K_2$.

Assume that deg $c_1 \geq 3$, deg $d_2 \geq 3$ or deg $c_2 \geq 3$, deg $d_1 \geq 3$, say the former. By Lemma 3.3 $K_1 - v_1$ contains $\frac{1}{2}[b_1, c_1, s_3, d_1, t_2, b_1]$ and $K_2 - v_2$ contains $\frac{1}{2}[b_2, d_2, t_3, c_2, t_1, b_2]$. Now we can construct required paths of G. Assume that for $i=1$ or 2, deg $c_i \geq 3$ and deg $d_i \geq 3$, say for $i=1$. Now we may assume that deg $c_2$=deg $d_2=2$. $c_2 = t_1$, $d_2 = t_3$ or $c_2 = t_3$, $d_2 = t_1$, say the former, then by Lemma 3.3 $K_2 - v_2$ contains $\frac{1}{2}[b_2, d_2, t_3, c_2, t_1, b_2]$ and $K_1 - v_1$ contains $\frac{1}{2}[b_1, c_1, s_3, d_1, t_2, b_1]$. Assume that deg $c_i=2$ ($i=1,2$) or deg $d_j=2$ ($j=1,2$), say the former. Let $y_1 \in N(c_1) - c_2$, and let $y_2 \in N(c_2) - c_1$, then by (3.1.5) deg $y_i \geq 3$ ($i=1,2$). By (3.1.4) we may let $c_1 = t_2$, $c_2 = t_1$ or $c_1 = s_3$, $c_2 = t_1$. If $c_1 = t_2$, then by (1.4.2) $K_1 - \{v_1, c_1\}$ contains feasible paths $\frac{1}{2}P_1[s_3, d_1]$, $\frac{1}{2}P_2[s_3, d_1]$, $\frac{1}{2}P_3[b_1, y_1]$ and $\frac{1}{2}P_4[b_1, y_1]$, and $K_2 - \{v_2, c_2\}$ contains feasible paths $\frac{1}{2}P_5[t_3, d_2]$, $\frac{1}{2}P_6[t_3, d_2]$, $\frac{1}{2}P_7[b_2, y_2]$ and $\frac{1}{2}P_8[b_2, y_2]$, and so the result follows. If $c_1 = s_3$, then by Lemma 3.3 $K_1 - \{v_1, c_1\}$ contains $\frac{1}{2}[b_1, y_1, d_1, t_2, b_1]$ and $K_2 - \{v_2, c_2\}$ contains $\frac{1}{2}[b_2, d_2, t_3, y_2, b_2]$, and so the result follows.

Case 1c. $\{s_1, s_2, t_1\} \subseteq X$ and $\{s_3, t_2, t_3\} \subseteq V-X$. We may assume that neither Case 1a nor Case 1b occurs. If deg $a_1=2$, then $\partial(X-a_1)$ is a 2-cut of G and $|(X-a_1) \cap T|=2$, a contradiction. Thus deg $a_i \geq 3$ and deg $b_i \geq 3$ ($i=1,2$). We assign new vertices $v_2, u_2$ on the midpoints of f,g respectively, producing new graphs $H', K'$. For the graph $H'$, and for two pairs $(s_1, t_1), (s_2, v_2)$ and $q_1=1$, $q_2=2$, if (1.3) does not hold, then there exists $Z \subseteq V(H')$ such that $D_{H'}(Z)=\{1,2\}$ and $|\partial_{H'}(Z)|=2$. Now Case 1b occurs in G, thus (1.3) holds, and so (1.2) holds. Hence $H'-v_2$ contans feasible paths $\frac{1}{2}P_1[s_1,t_1]$, $\frac{1}{2}P_2[s_1,t_1]$, $\frac{1}{2}P_3[s_2,a_1]$ and $\frac{1}{2}P_4[s_2,a_2]$. Similarly $K'-u_2$ contains feasible paths $\frac{1}{2}P_5[s_3,t_3]$, $\frac{1}{2}P_6[s_3,t_3]$, $\frac{1}{2}P_7[t_2,b_1]$ and $\frac{1}{2}P_8[t_2,b_2]$, and so the result follows.

Case 2. Every 2-cut of G is trivial, and G contains a 2-cut. Now we may let $\deg s_1=2$, and let $e_1$, $e_2$ be the edges incident to $s_1$. By (3.1.3), for i=1,2 there exists $X_i \subseteq V$ such that $s_1 \in X_i$, $|\partial(X_i)|=|D(X_i)|$ and $e_i \in \partial(X_i)$. For i=1,2, since $|\partial(X_i)|=3$, let $\partial(X_i)=\{e_i,f_i,g_i\}$. We put $X_3=V-(X_1 \cup X_2)$, then $t_1 \in X_3$. By simple counting we have

$$(3.2)\ |\partial(X_1 \cup X_2)|=|\partial(X_1)|+|\partial(X_2)|-|\partial(X_1 \cap X_2)| \\ -2|\partial(X_1-X_2)\cap\partial(X_2-X_1)|.$$

If $|\partial(X_1 \cap X_2)| \geq 4$, then by (3.2)

$$|\partial(X_3)|=|\partial(X_1 \cup X_2)| \leq 3+3-4=2.$$

Thus $|\partial(X_3)|=2$ and $|\partial(X_1 \cap X_2)|=4$ . Then $|X_3|=1$ and $X_1 \cap X_2=\{s_1,x\}$ for some $x \in V$ with $\deg x=2$. We may let $x=s_2$, then $t_2 \in X_3$, and so $t_1=t_2$, a contradiction. Thus $|\partial(X_1 \cap X_2)|=2$ and $X_1 \cap X_2=\{s_1\}$.

Case 2a. $f_1,f_2,g_1,g_2$ are not all distinct. We may let $f_1=f_2$. Since $f_1 \notin \partial(X_1 \cap X_2)=\{e_1,e_2\}$, $f_1 \in \partial(X_1-X_2)\cap\partial(X_2-X_1)$. By (3.2)

$$|\partial(X_3)|=|\partial(X_1 \cup X_2)| \leq 3+3-2-2=2.$$

Thus $X_3=\{t_1\}$, and we may assume that G is the graph as illustrated in Figure 9.

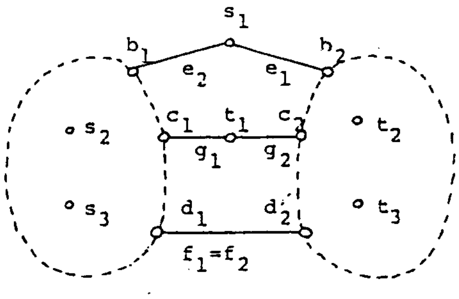


Figure 9.

Since every 2-cut is trivial, deg $b_i \geq 3$ and deg $c_i \geq 3$ ($i=1,2$). By Lemma 3.3 $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, s_3, d_1, s_2, c_1, b_1]$ and $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, t_3, d_2, t_2, c_2, b_2]$, and so the result follows.

Case 2b. $f_1, f_2, g_1, g_2$ are all distinct. Now $\partial(X_1 - X_2) \cap \partial(X_2 - X_1) = \emptyset$. From (3.2) we have

$$|\partial(X_3)| = 3+3-|\partial(X_1 \cap X_2)| = 4.$$

Thus we may assume that G is the graph as illustrated in Figure 10.

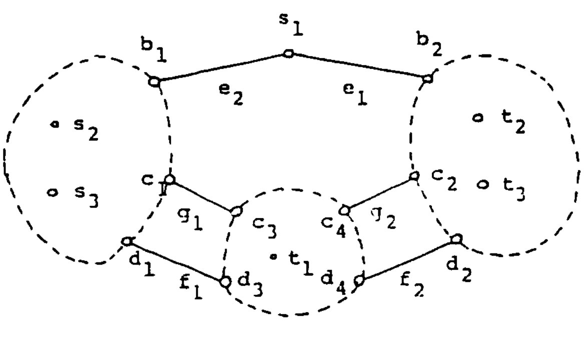


Figure 10.

$\langle X_3 \rangle$ is connected. For if not, then there exist $Y_1, Y_2 \subseteq X_3$ such that $X_3 = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$, and $|\partial(Y_1)| = |\partial(Y_2)| = 2$. Then $|X_3 \cap T| = 2$, a contradiction. deg $c_i \geq 3$ and deg $d_i \geq 3$ ($i=3,4$), for if not, then deg $t_1 = 2$ and one of $f_1, f_2, g_1, g_2$ is incident to

$t_1$, say $g_1$, and Case 2a occurs for $X_1$, $X_2 \cup X_3 - t_1$ instead of $X_1$, $X_2$. If $\langle X_3 \rangle$ contains a 1-cut $\{h\} = \partial_{\langle X_3 \rangle}(Y_1)$ $(Y_1 \subseteq X_3)$, then $Y_1$ contains just two of $c_3, c_4, d_3, d_4$. Put $Y_2 = X_3 - Y_1$. $\{c_3, d_3\} \not\subseteq Y_1$, thus we may let $\{c_3, c_4\} \subseteq Y_1$, $\{d_3, d_4\} \subseteq Y_2$ and $t_1 \in Y_1$. Let $v_1, v_2$ be the vertices such that $v_i \in Y_i$ $(i=1,2)$ and $v_1 v_2 = h$. $\langle Y_1 \rangle$ contains $\frac{1}{2}[c_3, c_4, v_1, t_1, c_3]$, for if $\langle Y_1 \rangle$ is not 2-edge-connected, then $\deg v_1 = 2$ and $v_1 = t_1$. $\langle Y_2 \rangle$ contains $\frac{1}{2}[d_3, d_4, v_2, d_3]$. Thus $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$ and feasible paths $\frac{1}{2}[c_3, c_4]$, $\frac{1}{2}[c_3, t_1]$, $\frac{1}{2}[t_1, c_4]$ and $[d_3, d_4]$. If $\langle X_3 \rangle$ is 2-edge-connected, then $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$ by Lemma 3.2. Assume that $\deg c_1 = 2$. We may let $c_1 = s_2$. $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, s_2, d_1, s_3, b_1]$ and $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$. If $\deg d_2 \geq 3$ or $\deg d_2 = 2$, $d_2 = t_2$, then by Lemma 3.3 $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, t_3, c_2, t_2, d_2, b_2]$. If $\deg d_2 = 2$ and $d_2 = t_3$, then $\langle X_2 \rangle$ contains feasible paths $\frac{1}{2}P_1[b_2, t_3]$, $\frac{1}{2}P_2[b_2, t_3]$, $\frac{1}{2}P_3[c_2, t_2]$ and $\frac{1}{2}P_4[c_2, t_2]$. Now we can dedeuce the result. Thus we may assume that $\deg c_i \geq 3$ $(i=1,2)$. By Lemma 3.3 $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, c_1, s_2, d_1, s_3, b_1]$ and $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, c_2, t_2, d_2, t_3, b_2]$. If $\langle X_3 \rangle$ is 2-edge-connected, then by Lemma 3.2 $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, c_4, d_3, d_4, c_3]$; and if not, then $\langle X_3 \rangle$ contains feasible paths $\frac{1}{2}[c_3, c_4]$, $\frac{1}{2}[c_3, t_1]$, $\frac{1}{2}[t_1, c_4]$ and $[d_3, d_4]$. Now we can deduce the result.

Case 3. G is 3-edge-connected. By Theorem 1 the result follows.

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