

Title	Multicommodity Flows in Graphs II
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Citation	大阪大学, 1984, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/1691
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Multicommodity Flows in Graphs II

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1. Introduction

Let $G=(V,E)$ be a graph (finite undirected, possibly with multiple edges but without loops), and let $V=V(G)$, $E=E(G)$ be the sets of vertices and edges of G respectively. In this paper a path has no repeated edges, and we permit paths with one vertex and no edges. For two distinct vertices x,y , let $\lambda(x,y)=\lambda_G(x,y)$ be the maximum number of edge-disjoint paths between x and y , and let $\lambda(x,x)=\infty$.

We first consider the following problem.

Let $(s_1,t_1),\dots,(s_k,t_k)$ be pairs (not necessarily distinct) of vertices of G . When is the following true ?

(1.1) There exist edge-disjoint paths P_1,\dots,P_k such that P_i has ends s_i,t_i ($1 \leq i \leq k$).

Seymour [10] and Thomassen [12] characterized such graphs when $k=2$, and Seymour [10] when $s_1,\dots,s_k,t_1,\dots,t_k$ take only three distinct values.

Our result is the following

Theorem 1. Suppose that s_1,s_2,s_3,t_1,t_2,t_3 are vertices of a graph G . If for each $i=1,2,3$,

$$\lambda(s_i,t_i) \geq 3,$$

then there exist edge-disjoint paths P_1,P_2,P_3 of G , such that P_i has ends s_i and t_i ($i=1,2,3$).

If $\lambda(s_i, t_i) \leq 2$ for some i , then the conclusion does not always hold. Figure 1 gives a counterexample.

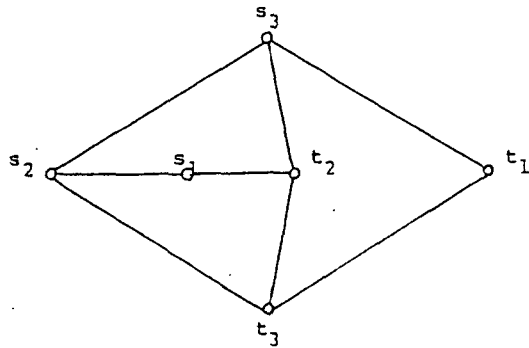


Figure 1.

For a positive integer k , let $g(k)$ be the smallest integer such that for every $g(k)$ -edge-connected graph and for every vertices $s_1, \dots, s_k, t_1, \dots, t_k$ of the graph, (1.1) holds. Thomassen [12] conjectured the following.

Conjecture. For each odd integer $k \geq 1$, $g(k) = k$, and for each even integer $k \geq 2$, $g(k) = k + 1$.

If k is even then $g(k) > k$ (see [12]). It follows easily from Menger's theorem that $g(k) \leq 2k - 1$, thus $g(1) = 1$, $g(2) = 3$; and Cypher [1] proved $g(4) \leq 6$ and $g(5) \leq 7$. As a corollary of Theorem 1 we have the following.

Corollary. $g(3) = 3$.

The second problem we consider is the multicommodity flow problem.

Suppose that each edge $e \in E$ has a real-valued capacity $w(e) \geq 0$, and each path has a positive value. We assume that $w \equiv 1$ and each path has value 1 when there is no explanation. For a positive number α , paths αP , P denote paths of value $\alpha, 1$ respectively. We say that a set of paths $\alpha_1 P_1, \dots, \alpha_n P_n$ is feasible if for each edge $e \in E$,

$$\sum_{i \in \{i | e \in E(P_i)\}} \alpha_i \leq w(e),$$

where $E(P_i)$ is the set of edges of P_i .

For two vertices x, y and a real number $q > 0$, a flow F of value q between x and y is a set of paths $\alpha_1 P_1, \dots, \alpha_n P_n$ between x and y such that $\alpha_1 + \dots + \alpha_n = q$. When $\alpha_1, \dots, \alpha_n$ are all integers (half-integers), F is called an integer (half-integer) flow. We say that a set of flows F_1, \dots, F_k is feasible if the set of paths of F_1, \dots, F_k is feasible.

Now the multicommodity flow problem is as follows.

Let $(s_i, t_i), \dots, (s_k, t_k)$ be pairs of vertices of G , as before, and suppose that $q_i \geq 0$ ($1 \leq i \leq k$) are real-valued demands. When is the following true?

(1.2) There exist feasible flows F_1, \dots, F_k , such that F_i has ends s_i and t_i and value q_i ($1 \leq i \leq k$).

Remark. When $k=3$, $w \equiv 1$, and $q_i = 1$ ($1 \leq i \leq 3$), Theorem 1

implies that (1.2) is true if $\lambda(s_i, t_i) \geq 3$ ($1 \leq i \leq 3$), and then the flows may be chosen as integer flows.

For a set $X \subseteq V$, let $\partial(X) = \partial_G(X) \subseteq E$ be the set of edges with one end in X and the other in $V-X$, and let $D(X) = D_G(X) \subseteq \{1, 2, \dots, k\}$ be

$$\{i \mid 1 \leq i \leq k, X \cap (s_i, t_i) \neq \emptyset \neq (V-X) \cap (s_i, t_i)\}.$$

It is clear that if (1.2) is true, then the following holds.

(1.3) For each $X \subseteq V$,

$$\sum_{e \in \partial(X)} w(e) \geq \sum_{i \in D(X)} q_i.$$

Note that $\sum_{e \in \partial(X)} w(e) = |\partial(X)|$ if $w \equiv 1$, and $\sum_{i \in D(X)} q_i = |D(X)|$

if $q_i = 1$ for any i .

Our second result is the following

Theorem 2. Suppose that G is a graph and w is integer-valued, and that $k=3$, $q_1=q_2=q_3=1$. Then (1.2) and (1.3) are equivalent.

Moreover if (1.3) holds, then the flows F_i in (1.2) may be chosen as half-integer flows.

(1.4) In general (1.2) and (1.3) are not equivalent, but in the following cases they are equivalent.

(1.4.1) $k=1$ (Ford and Fulkerson [2]).

(1.4.2) $k=2$ (Hu [3] and Seymour [8])

(1.4.3) $k=5$, $t_i=s_{i+1}$ ($i=1,2,3,4$) and $t_5=s_1$ (Papernov [7]).

(1.4.4) $k=6$, and $(s_1, t_1), \dots, (s_6, t_6)$ correspond to the six pairs of a set of four vertices (Papernov [7] and Seymour [9]).

(1.4.5) $s_1=s_2=\dots=s_j$ and $s_{j+1}=\dots=s_k$ (obvious extension of (1.4.2)).

(1.4.6) The graph $(V, E \cup \{e_1, \dots, e_k\})$ is planar, where the edge e_i has ends s_i and t_i ($1 \leq i \leq k$) (Seymour [11]).

(1.4.7) G is planar and can be drawn in the plane so that $s_1, \dots, s_k, t_1, \dots, t_k$ are all on the boundary of the infinite face (Okamura and Seymour [5]).

(1.4.8) G is planar and can be drawn in the plane so that $s_1, \dots, s_j, t_1, \dots, t_j$ are all on the boundary of a face and $s_{j+1}, \dots, s_k, t_{j+1}, \dots, t_k$ are all on the boundary of the infinite face (Okamura [6]).

(1.4.9) G is planar and can be drawn in the plane so that $s_{j+1}, \dots, s_k, t_1, t_2, \dots, t_k$ are all on the boundary of the infinite face, and $t_1 = \dots = t_j$ (Okamura [6]).

Moreover if (1.3) and the following (1.5) hold in each case except (1.4.3), or if (1.3) holds and w, q_j are even-integer valued in the case (1.4.3), then the flows F_i of (1.2) may be chosen as integer flows.

(1.5) w and q_i are integer-valued, and for each vertex $x \in V$,

$$\sum_{e \in \partial(x)} w(e) - \sum_{i \in D(x)} q_i$$

is even.

(1.4.1), ..., (1.4.5) are all the configurations of (s_i, t_i) for which (1.2) and (1.3) are equivalent for all graphs G and all w, q_i (see [9]). When $q_i > 0$ ($1 \leq i \leq 3$), the case of Theorem 2 is the only case for which (1.2) and (1.3) are equivalent for all graphs G and all $w, (s_i, t_i)$. Figure 1 gives a counterexample with $q_1=2, q_2=q_3=1$.

Notations and definitions. We call $S \subseteq E$ an n -cut if $|S|=n$ and $S=\partial(X)$ for some $X \subseteq V$ such that $\langle X \rangle$ (which is the subgraph induced by X) and $\langle V-X \rangle$ are both connected; and an n -cut $\partial(X)$ is called nontrivial if $|X| \geq 2$ and $|V-X| \geq 2$, trivial otherwise. For two vertices x, y a path $P[x, y]$ or a path $[x, y]$ denotes a path between x and y , and let xy be an edge with ends x, y , and let $d(x, y) = d_G(x, y)$ be the distance between x and y . If vertices x, y belong to a path P , then $P(x, y)$ denotes the subpath of P between x and y . For a vertex x $\deg(x) = \deg_G(x)$ denotes the degree of x , and we let $N(x) = N_G(x)$ be $\{y \in V \mid xy \in E\}$. For a set $X \subseteq V$ and an edge e , we denote graphs $\langle V-X \rangle$, $(V, E-e)$ by $G-X$, $G-e$ respectively. For a set $X \subseteq V$ ($S \subseteq E$) and an element $x \in V$ ($e \in E$), we denote $X \cup \{x\}$ ($S \cup \{e\}$) by $X \cup x$ ($S \cup e$).

2. Proof of Theorem 1.

In this section disjoint means edge-disjoint. We require the following lemmas.

Lemma 2.1. Suppose that s_1, s_2, t_1, t_2 are vertices of a graph G . If $\lambda(s_1, t_1) \geq 3$ and $\lambda(s_2, t_2) \geq 1$, then G contains disjoint paths $[s_1, t_1]$ and $[s_2, t_2]$.

Proof. Since $\lambda(s_1, t_1) \geq 3$, G contains disjoint paths $P_1[s_1, t_1], P_2[s_1, t_1]$ and $P_3[s_1, t_1]$. G contains a path $P_4[s_2, t_2]$. There exist vertices $x, y \in V(P_4)$ such that $P_4(s_2, x)$ and $P_4(t_2, y)$ are disjoint from P_1, P_2, P_3 . Choose x, y with this property such that $P_4(s_2, x), P_4(t_2, y)$ have the maximum length respectively. If x or $y \notin V(P_1) \cup V(P_2) \cup V(P_3)$, then $x=t_2$ or $y=s_2$, and so the result follows. We may therefore assume that $x \in V(P_2)$ and $y \in V(P_i)$ ($i=2$ or 3). When $i=2$ ($i=3$), let P_5 be the path obtained by combining $P_4(s_2, x), P_2(x, y)$ and $P_4(y, t_2)$ ($P_4(s_2, x), P_2(x, s_1), P_3(s_1, y)$ and $P_4(y, t_2)$). Now P_1 and P_5 are required paths of G .

Lemma 2.2. If G is 3-regular 3-edge-connected graph with no nontrivial 3-cut and with $4 \leq |V| \leq 8$, then G is $K_4, K_{3,3}$, a cube or the graph in Figure 2.

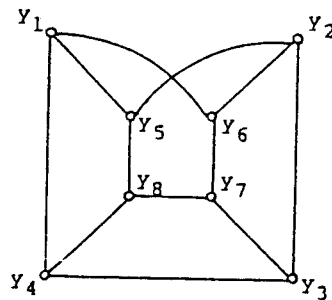


Figure 2.

Proof. Since G is 3-regular 3-edge-connected, G has no multiple edges. Thus if $|V|=4$, then G is K_4 . If $|V|>4$, then G has no cycle of length three. If $|V|=6$, then let $V=\{x_1, \dots, x_6\}$. We may let $N(x_1)=\{x_2, x_3, x_4\}$. Since $x_i x_j \notin E$ ($2 \leq i < j \leq 4$), we have $x_i x_j \in E$ ($i=2,3,4; j=5,6$). Thus G is $K_{3,3}$. If $|V|=8$, then it easily follows that G is a cube or the graph in Figure 2.

Lemma 2.3. Suppose that G is a 3-regular 3-edge-connected graph, and that a, x_1, x_2, x_3, x_4 are vertices such that $a \neq x_i$ ($1 \leq i \leq 4$). Then $G-a$ contains disjoint paths $[x_1, x_2]$ and $[x_3, x_4]$.

Proof. We proceed by induction on $|V|$. If $|V|=2$, then G is the graph of triple edges, and the result holds. Therefore we assume $|V| \geq 4$.

First we assume that G contains a nontrivial 3-cut $\{e_1, e_2, e_3\} = \partial(X)$ ($X \subseteq V$). Let $b_i \in X$, $c_i \in V-X$, $e_i = b_i c_i$ ($i=1,2,3$),

then $b_i \neq b_j, c_i \neq c_j$ if $i \neq j$, since G is 3-edge-connected. Let H, K be the graphs obtained from G by contracting $V-X, X$ to one vertex respectively. Let $V(H)=X \cup v, V(K)=(V-X) \cup u$. Then H, K are 3-regular 3-edge-connected graphs and $|V(H)| < |V|, |V(K)| < |V|$. We may assume $a \in V-X$. It suffices to prove the lemma in the following cases.

Case 1. $\{x_1, x_2, x_3, x_4\} \subseteq V-X$. By induction the result holds in K , and so in G .

Case 2. $x_1 \in X$ and $\{x_2, x_3, x_4\} \subseteq V-X$. By induction the result holds in K (note that $x_1 = u$ in K). Thus the result holds in G , since G contains a subgraph G_1 homeomorphic to K , such that x_1 corresponds to u and each vertex of $V-X$ to itself.

Case 3. $\{x_1, x_2, x_3, x_4\} \subseteq X$. G contains a subgraph G_2 homeomorphic to H , such that a corresponds to v and each vertex of X to itself, and so the result holds in G .

Case 4. $\{x_1, x_2\} \subseteq X$ and $\{x_3, x_4\} \subseteq V-X$. Since $K-(a, u)$ is connected, this contains a path $[x_3, x_4]$; and $H-v$ contains a path $[x_1, x_2]$.

Case 5. $\{x_1, x_3\} \subseteq X$ and $\{x_2, x_4\} \subseteq V-X$. By induction $K-a$ contains disjoint paths $P_1 [u, x_2]$ and $P_2 [u, x_4]$. We may let $c_i \in V(P_i)$ ($i=1,2$), and $H-v$ contains disjoint paths $[x_1, b_1]$ and $[x_3, b_2]$. Thus the result follows.

Case 6. $\{x_1, x_2, x_3\} \subseteq X$ and $x_4 \in V-X$. $K-a$ contains a path $P [u, x_4]$, and we may let $c_1 \in V(P)$. $H-v$ contains disjoint paths $[x_1, x_2]$ and $[x_3, b_1]$. Thus the result follows.

Next we assume that G does not contain a nontrivial 3-cut. If G contains an edge e which is not incident to any of a, x_1, x_2, x_3, x_4 , then let $\widetilde{G-e}$ be the 3-regular graph homeomorphic to the graph $G-e$. Then $\widetilde{G-e}$ is 3-edge-connected. By induction the result holds in $\widetilde{G-e}$, and so in G . Thus we assume that any edge is incident to one of a, x_1, x_2, x_3, x_4 . Then $|E| \leq 15$ and $|V| \leq 10$. We put $T = \{a, x_1, x_2, x_3, x_4\}$. We may assume that x_1, x_2, x_3 and x_4 are all distinct. For if not, then the result follows, since $G-a$ is 2-edge-connected. Thus $|V| \geq 5$. If $|V| = 10$, then $N(x_i) \subseteq V-T$ ($1 \leq i \leq 4$) and $|V-T| = 5$. Thus for some $y \in V-T$, $y \in N(x_1) \cap N(x_2)$. $G-\{a, y\}$ is connected, and so the result follows. If $|V| = 6$ or 8 , then by Lemma 2.2 G is $K_{3,3}$, a cube, or the graph in Figure 2. We omit the proofs for them.

Lemma 2.4. Suppose that G is a 3-regular 3-edge-connected graph, and that $a, a_1, a_2, a_3, x_1, x_2, x_3$ are vertices such that $N(a) = \{a_1, a_2, a_3\}$ and $a \neq x_i$ ($1 \leq i \leq 3$). Then

$$|I_G| \geq 4.$$

Here $I_G = I_G(a, a_1, a_2, a_3, x_1, x_2, x_3)$ is

$$\left\{ \begin{array}{l} (i, j, k) \\ \left. \begin{array}{l} (i, j, k) = (1, 2, 3). \text{ } G-a \text{ contains disjoint paths} \\ [x_1, a_i], [x_2, a_j] \text{ and } [x_3, a_k]. \end{array} \right\} \end{array} \right\}.$$

Proof. We proceed by induction on $|V|$. We assume $|V| \geq 4$. First we assume that G contains a nontrivial 3-cut

$(e_1, e_2, e_3) = \partial(X)$ ($X \subseteq V$). We define b_i, c_i ($i=1,2,3$), H, K, v and u similarly as in the proof of Lemma 2.3. We may assume $a \in V-X$. Then $|X \cap N(a)| \leq 1$. If $a_i \in X$ for some i , then $a_i = u$ in K . It suffices to prove the lemma in the following cases.

Case 1. $(x_1, x_2, x_3) \subseteq V-X$; $x_1 \in X$ and $(x_2, x_3) \subseteq V-X$; or $(x_1, x_2, x_3) \subseteq X$. Similar as Case 1, 2 or 3 in the proof of Lemma 2.3.

Case 2. $(x_1, x_2) \subseteq X$ and $x_3 \in V-X$. By induction $|I_K(a, a_1, a_2, a_3, u, u, x_3)| \geq 4$. For each (i, j, k) of I_K , $K-a$ contains disjoint paths $P_1 [u, a_i]$, $P_2 [u, a_j]$ and $P_3 [x_3, a_k]$. If $u \notin N_K(a)$, then we may let $c_i \in V(P_i)$ ($i=1,2$). By Induction $H-v$ contains disjoint paths $[x_1, b_1]$ and $[x_2, b_2]$. Thus $(i, j, k) \in I_G(a, a_1, a_2, a_3, x_1, x_2, x_3)$, and so $|I_G| \geq 4$. If $u \in N_K(a)$, then we may let $a_1 = u, a_2 = c_1$. Now $k \neq 1$ and we may let $i=1, j=2, k=3, c_2 \in V(P_2)$. Since $H-v$ contains disjoint paths $[x_1, b_1]$ and $[x_2, b_2]$, $|I_G| \geq 4$.

Next we assume that G does not contain a nontrivial 3-cut. We may assume that any edge is incident to one of a, x_1, x_2, x_3 (see the proof of Lemma 2.3). Thus $|E| \leq 12$ and $|V| \leq 8$. By Lemma 2.2 G is $K_4, K_{3,3}$, a cube or the graph in Figure 2, but in the last graph any four vertices do not cover all edges of the graph. Thus G is one of the first three graphs. If G is a cube, then in Figure 3 it suffices to check the case $y_1 = a, y_3 = x_1, y_6 = x_2, y_8 = x_3$. We omit the proofs for $K_4, K_{3,3}$.

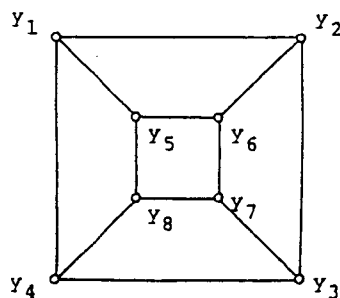


Figure 3.

Lemma 2.5. Suppose that $s_1, s_2, s_3, t_1, t_2, t_3$ are vertices of a graph G . If G is 3-regular 3-edge-connected, then G contains disjoint paths $[s_1, t_1], [s_2, t_2]$ and $[s_3, t_3]$.

Proof. We proceed by induction on $|V|$. We put $T = \{s_1, s_2, s_3, t_1, t_2, t_3\}$. If $s_i = t_i$ for some i , then the result follows by Lemma 2.1, and if $s_1 = s_2 = s_3$, then the result follows from Menger's theorem. Thus we may assume that these are not the cases.

First we assume that G contains a nontrivial 3-cut $(e_1, e_2, e_3) = \partial(X)$ ($X \subseteq V$). We define b_i, c_i ($i=1,2,3$), H, K, v and u similarly as in the proof of Lemma 2.3. It suffices to prove the lemma in the following cases.

Case 1. $T \cap X = \emptyset$. By induction the result holds in K , and so in G .

Case 2. $s_1 \in X$ and $\{s_2, s_3, t_1, t_2, t_3\} \subseteq V - X$. G contains a subgraph G_1 homeomorphic to K , such that s_1 corresponds to u

and each vertex of $V-X$ to itself.

Case 3. $\{s_1, t_1\} \subseteq X$ and $\{s_2, s_3, t_2, t_3\} \subseteq V-X$. By Lemma 2.3 $K-u$ contains disjoint paths $[s_2, t_2]$ and $[s_3, t_3]$, and $H-v$ contains a path $[s_1, t_1]$.

Case 4. $\{s_1, s_2\} \subseteq X$ and $\{s_3, t_1, t_2, t_3\} \subseteq V-X$. By induction K contains disjoint paths $P_1 [u, t_1], P_2 [u, t_2]$ and $[s_3, t_3]$. Let $c_i \in V(P_i)$ ($i=1,2$). By Lemma 2.3 $H-v$ contains disjoint paths $[s_1, b_1]$ and $[s_2, b_2]$. Now the result follows.

Case 5. $\{s_1, s_2, t_1\} \subseteq X$ and $\{s_3, t_2, t_3\} \subseteq V-X$. We can get the result by applying Lemma 2.3 on H and K .

Case 6. $\{s_1, s_2, s_3\} \subseteq X$ and $\{t_1, t_2, t_3\} \subseteq V-X$. By Lemma 2.4 $I_H(v, b_1, b_2, b_3, s_1, s_2, s_3) \cap I_K(u, c_1, c_2, c_3, t_1, t_2, t_3) \neq \emptyset$, and so the result follows.

Next we assume that G does not contain a nontrivial 3-cut. We may assume that every edge of G is incident to a vertex of T (see the proof of Lemma 2.3). Thus $|E| \leq 18$ and $|V| \leq 12$. We require the following.

(2.1) We may assume that $d(s_i, t_i) \geq 2$ ($i=1,2,3$). If $d(s_i, t_i)=2$ for some i and s_i, t_i are adjacent to a common vertex x , say for $i=1$, then we may assume that

$$x \in (s_2, t_2) \cap (s_3, t_3).$$

Proof. Let $d(s_1, t_1)=1$. If $(s_i, t_i) \cap (s_1, t_1) = \emptyset$, for $i=2$ or 3, say $i=2$, then $\lambda_{G-s_1 t_1}(s_2, t_2)=3$ and by Lemma 2.1

$G - s_1, t_1$ contains disjoint paths $[s_2, t_2]$ and $[s_3, t_3]$, and so the result of Lemma 2.5 follows; if not, then we may let $s_2 = s_1, s_3 = t_1$ and $s_i \neq t_i$ ($i=2,3$). Let $y \in N(s_1) - t_1$. By Lemma 2.3 $G - s_1$ contains disjoint paths $[s_3, t_3]$ and $[t_2, y]$. Thus the result of Lemma 2.5 follows. Hence we may assume that $d(s_i, t_i) \geq 2$ ($i=1,2,3$). Assume that s_1 and t_1 are adjacent to a vertex x . Let $y \in N(x) - \{s_1, t_1\}$. If $x \notin T$, then by Lemma 2.3 $G - x$ contains disjoint paths $[s_2, t_2]$ and $[s_3, t_3]$. If $x \in T$ and $x \notin \{s_2, t_2\} \cap \{s_3, t_3\}$, then we may let $x = s_2$ and $s_3 \neq x \neq t_3$. By Lemma 2.3 $G - x$ contains disjoint paths $[s_3, t_3]$ and $[t_2, y]$, hence Lemma 2.5 holds. Thus (2.1) is proved.

Now we return to the proof of Lemma 2.5. If $G = K_4$, then $d(s_1, t_1) = 1$, and if $G = K_{3,3}$, then s_1 and t_1 are adjacent to common three vertices, contrary to (2.1). If G is the graph in Figure 2, then we may let $s_1 = y_1$ without loss of generality. Then $t_i \neq y_i$ ($i=4,5,6$) by (2.1). If $t_i = y_i$ ($i=2$ or 8), say for $i=8$, then $\{y_4, y_5\} \subseteq \{s_2, t_2\} \cap \{s_3, t_3\}$ by (2.1). So we may let $y_4 = s_2 = s_3$ and $y_5 = t_2 = t_3$, contrary to (2.1). If $t_i = y_i$ ($i=3$ or 7), say for $i=3$, then we may let $y_4 = s_2 = s_3$ by (2.1). Now we can not choose $\{t_2, t_3\}$ such that T covers E , a contradiction. When G is a cube, in Figure 3 we may let $s_1 = y_1$ and $t_i \neq y_i$ ($i=2,4,5$). If $t_i = y_i$ ($i=3,6$ or 8), say for $i=3$, then we may let $y_2 = s_1 = s_3$ and $y_4 = t_2 = t_3$, and the result

follows. Thus we may let $t_1 = y_7$. Since T covers all edges, we may let $(s_1, t_2) = (y_2, y_8)$ and $(s_3, t_3) = (y_3, y_5)$, then the result easily follows.

By Lemma 2.2 we may let $|V| = 10$ or 12 . Thus $|T| \geq 5$. Note that for each distinct vertices $x, y \in V$, $N(x) \neq N(y)$, because G has no nontrivial 3-cut. We distinguish three cases.

Case 1. $|T| = 5$. Let $s_1 = s_2$. Now $|V| = 10$, and G is a bipartite graph and the partition of V is $(T, V-T)$. The number of vertices which have distance two from $s_1 = s_2$ is at least three, and so $d(s_i, t_i) = 2$ for $i = 1$ or 2 , contrary to (2.1).

Case 2. $|T| = 6$ and $|V| = 12$. Now G is a bipartite graph and the partition of V is $(T, V-T)$. If the number of vertices which have distance two from s_1 is at least five, then one of such vertices is t_1 , a contradiction; if not, then the number is four, since G does not contain a nontrivial 3-cut. Thus G contains a subgraph as illustrated in Figure 4, where $T = \{s_1, x_1, x_2, x_3, x_4, x_5\}$. By (2.1) $t_1 \neq x_i$ ($i = 1, 2, 3, 4$) and (s_j, t_j) is not (x_1, x_2) , (x_1, x_4) nor (x_2, x_3) ($j = 2, 3$), and so we may let $(x_1, x_3) = (s_2, t_2)$, $(x_2, x_4) = (s_3, t_3)$ and $x_5 = t_1$. Now $\{x_5 y_1, x_5 y_2, x_5 y_3\} \subseteq E$. If $x_i y_i \in E$ ($i = 1$ or 2), say for $i = 1$, then $\{x_3 y_2, x_3 y_3\} \subseteq E$ and $x_2 y_3 \in E$. Now the result follows. If $x_i y_3 \in E$, then $x_3 y_3 \notin E$, and so $\{x_3 y_1, x_3 y_2\} \subseteq E$, contrary to $N(y_1) \neq N(y_2)$.

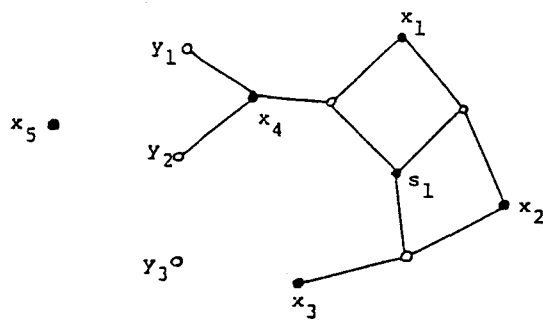


Figure 4.

Case 3. $|T|=6$ and $|V|=10$. Now both ends of just three edges are in T , and by (2.1) $d(s_i, t_i) \geq 3$ ($i=1,2,3$). Thus G contains a subgraph as illustrated in Figures 5a, 5b, 5c or 5d, where $T=(x_1, \dots, x_6)$ and $V-T=(y_1, \dots, y_4)$.

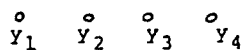
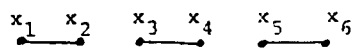


Figure 5a.

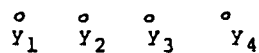
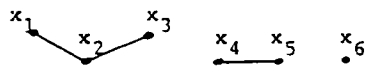


Figure 5b.

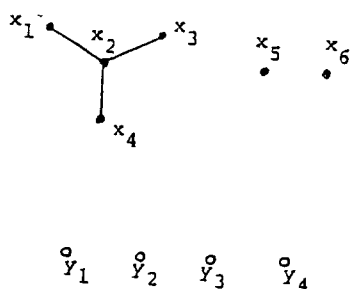


Figure 5c.

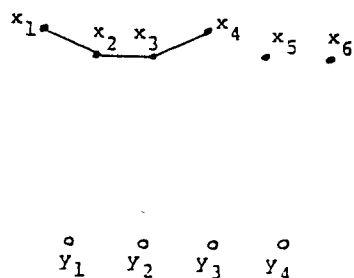


Figure 5d.

In Figure 5a, we may let $\{x_1, x_3\} = \{s_1, t_1\}$, $\{x_2, x_5\} = \{s_2, t_2\}$, $\{x_4, x_6\} = \{s_3, t_3\}$ and $\{x_1, y_1, x_1, y_2\} \subseteq E$. Then $x_i y_j \in E$ ($i=2,3$; $j=1,2$). Since $N(y_1) \neq N(y_2)$, one of them contains $\{x_5, x_6\}$ or $\{x_4, x_6\}$, a contradiction. In Figure 5b, we may let $\{x_6, y_1, x_6, y_2, x_6, y_3\} \subseteq E$. If for some $i=1,3,4,5$ $\{x_i, x_6\} = \{s_1, t_1\}$, then $d(s_1, t_1) = 2$, a contradiction. Thus we may let $\{x_2, x_6\} = \{s_1, t_1\}$, $\{x_1, x_4\} = \{s_2, t_2\}$ and $\{x_3, x_5\} = \{s_3, t_3\}$. Thus $x_2 y_4 \in E$. We may let $\{x_1, y_1, x_1, y_2\} \subseteq E$, and so $\{x_4 y_3, x_4 y_4, x_5 y_1, x_5 y_2\} \subseteq E$, contrary to $N(y_1) \neq N(y_2)$. In Figure 5c, for some $i=1,2,3$ $d(s_i, t_i) \leq 2$, a contradiction. In Figure 5d, we may let $\{x_2, x_5\} = \{s_1, t_1\}$, $\{x_3, x_6\} = \{s_2, t_2\}$, $\{x_1, x_4\} = \{s_3, t_3\}$ and $\{x_1, y_1, x_1, y_2, x_4 y_3, x_4 y_4\} \subseteq E$.

Now $x_2 y_i \in E$ ($i=3$ or 4), say for $i=3$, then $\{x_5 y_1, x_5 y_2, x_5 y_4\} \subseteq E$. $x_3 y_i \in E$ ($i=1$ or 2), say for $i=1$, then $\{x_6 y_2, x_6 y_3, x_6 y_4\} \subseteq E$. Now the result easily follows.

Proof of Theorem 1. We proceed by induction on $|V|$. If G is not 2-connected, then we can deduce the result by using induction on blocks. Thus we may assume that G is 2-connected. If G contains a vertex of degree k (≥ 4), then we replace this vertex by a k -gon with k vertices of degree 3. (Figure 6 gives an example with $k=5$.) If this vertex of G is $s_i(t_i)$ for some i , then we assign $s_i(t_i)$ on any vertex of this k -gon, producing a 3-regular graph G' such that $\lambda_{G'}(s_i, t_i) \geq 3$ for each i . If the result holds in G' , then the result clearly holds in G , and so we may assume that G

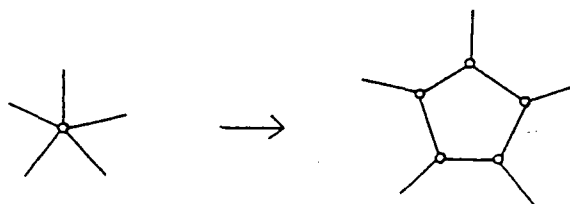


Figure 6.

is 3-regular. By Lemma 2.5 we may assume that G contains a 2-cut $(e_1, e_2) = \partial(X)$ ($X \subseteq V$). Let $b_i \in X$, $c_i \in V-X$ and $e_i = b_i c_i$ ($i=1,2$). We define new graphs H, K as follows.

$$H=(X, E(\langle X \rangle) \cup f),$$

$$K=(V-X, E(\langle V-X \rangle) \cup g),$$

where f, g are new edges with ends b_1, b_2 and c_1, c_2 respectively. Then H and K are 2-edge-connected. Since $\lambda_G(s_i, t_i) \geq 3$, $\{s_i, t_i\} \subseteq X$ or $\{s_i, t_i\} \subseteq V-X$ for each i .

Thus it suffices to consider the following cases.

Case 1. $\{s_1, s_2, s_3, t_1, t_2, t_3\} \subseteq X$. By induction the result holds in H .

Case 2. $\{s_1, s_2, t_1, t_2\} \subseteq X$ and $\{s_3, t_3\} \subseteq V-X$. By Lemma 2.1 H contains disjoint paths $P_1 [s_1, t_1]$ and $P_2 [s_2, t_2]$. Let P_3, P_4, P_5 be disjoint paths of K between s_3 and t_3 , and let $c_1, c_2 \notin E(P_3) \cup E(P_4)$. If $b_1, b_2 \notin E(P_1) \cup E(P_2)$, then P_1, P_2, P_3 are required paths of G . Thus let $b_1, b_2 \in E(P_1)$. If $c_1, c_2 \notin E(P_5)$, then by Lemma 2.1 $K - c_1, c_2$ contains disjoint paths $[s_3, t_3]$ and $[c_1, c_2]$; and if $c_1, c_2 \in E(P_5)$, then let $P_6 [c_1, c_2]$ be the path obtained by combining $P_5 - c_1, c_2$ and P_4 . In each case we can construct required paths of G .

3. Proof of Theorem 2.

For an integer $n \geq 3$ and vertices x_1, x_2, \dots, x_n , we denote feasible paths $\frac{1}{2}[x_1, x_2], \frac{1}{2}[x_2, x_3], \dots, \frac{1}{2}[x_{n-1}, x_n]$, and $\frac{1}{2}[x_n, x_1]$ by $\frac{1}{2}[x_1, \dots, x_n, x_1]$. For a vertex $x \in V$ and $a, b \in N(x)$, let $G_x^{a,b}$ be the graph $(V, E \cup e_1 - \{e_2, e_3\})$, where e_1 is a new edge with ends a, b and e_2, e_3 are edges of E with ends a, x and b, x respectively.

Lemma 3.1 (Mader [4]). Suppose that G is a graph and x is a non-separating vertex of G with $\deg x \geq 4$ and with $|N(x)| \geq 2$. Then there exist two vertices $a, b \in N(x)$, such that for each two vertices $y, z \in V - x$,

$$\lambda_{G_x^{a,b}}(y, z) = \lambda_G(y, z).$$

Lemma 3.2. Suppose that x_1, \dots, x_5 are vertices of a graph G . If for each $1 \leq i < j \leq 5$,

$$\lambda(x_i, x_j) \geq 4,$$

and each vertex of G has even degree, then G contains edge-disjoint paths $[x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]$, and $[x_5, x_1]$.

Proof. We proceed by induction on $|E|$. We put $T = \{x_1, \dots, x_5\}$. If $|T| \leq 4$, then the result follows from (1.4.4), and so we may let $|T| = 5$. We may assume that G is 2-connected, and that for each vertex x of G $\deg x \geq 4$. If there exists a vertex x in $V - T$, then by Lemma 3.1 there

exist two vertices $a, b \in N(x)$ such that $\lambda_{G-x}^{a,b}(x_i, x_j) = \lambda_G(x_i, x_j)$ ($1 \leq i < j \leq 5$). $|E(G_x^{a,b})| < |E|$ and each vertex of $G_x^{a,b}$ has even degree, thus by induction the result holds in $G_x^{a,b}$, and so in G . Let $V=T$. If $x_5 x_i \in E$, then we can apply (1.4.5) for the graph $G - x_5 x_i$, and for pairs $(s_i, t_i) = (x_i, x_{i+1})$ and $q_i = 1$ ($1 \leq i \leq 4$). Thus we may let $x_5 x_i \notin E$ and $x_i x_{i+1} \notin E$ ($1 \leq i \leq 4$). Now G contains a subgraph as illustrated in Figure 7a or 7b, and the result holds.

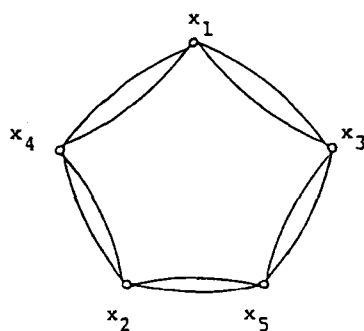


Figure 7a.

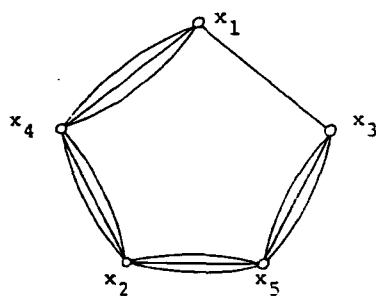


Figure 7b.

Lemma 3.3. Suppose that G is a 2-edge-connected graph and a, b, c, d, x, y are vertices such that $\deg a = 3$, $N(a) = \{b, c, d\}$, $\deg b \geq 3$, and a, x, y are all distinct, and that for each

2-cut $\partial(X)$ ($X \subseteq V$, $|X| \leq |V-X|$),

$X=\{x\}$, $X=\{y\}$ or $X=\{x,y\}$ and $|E(\langle X \rangle)|=1$.

Then $G-a$ contains $\frac{1}{2}[b,c,x,d,y,b]$, if it is not the cases that

$\deg c=2$, $c=x$, $\deg c_1=2$ ($N(c)=\{a,c_1\}$) or,

$\deg c=2$, $c=y$.

Proof. We distinguish four cases.

Case 1. $\deg c \geq 3$ and $\deg d \geq 3$. Now $G-a$ is 2-edge-connect. Let G' be the graph obtained by replacing each edge of G by double edges. Then $G'-a$ is 4-edge-connected, and so by applying Lemma 3.2 on $G'-a$ we can deduce the result.

Case 2. $\deg c=2$ and $\deg d \geq 3$. Let $N(c)=\{a,c_1\}$. By the hypothesis $c \neq y$, and so $c=x$ and $\deg c_1 \geq 3$. $G-\{a,c\}$ is 2-edge-connected, and so this contains $\frac{1}{2}[b,c_1,d,y,b]$ by Lemma 3.2.

Case 3. $\deg c \geq 3$ and $\deg d=2$. Let $d=x$ and $N(d)=\{a,d_1\}$. If $\deg d_1 \geq 3$, then $G-\{a,d\}$ is 2-edge-connected, and so this contains $\frac{1}{2}[b,c,d_1,y,b]$. If $\deg d_1=2$, then $d_1=y$. By (1.4.4) $G-\{a,d\}$ contains $\frac{1}{2}[b,c,d_1,b]$, thus G contains $\frac{1}{2}[b,c,x,d,y,b]$. When $d=y$, the proof is similar.

Case 4. $\deg c=\deg d=2$. Now $c \neq d$ and $c \neq y$, thus $c=x, d=y$, and $G-\{a,c,d\}$ is 2-edge-connected. By (1.4.4) $G-a$ contains $\frac{1}{2}[b,c,d,b]$.

If we prove following Lemma 3.4, Theorem 2 follows.

Lemma 3.4. Suppose that G is a graph with $w \equiv 1$, (s_1, t_1) , (s_2, t_2) , (s_3, t_3) are pairs of vertices of G , and $q_1=q_2=q_3=1$.

If (1.3) holds, then G contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[s_2, t_2]$, $\frac{1}{2}P_4[s_2, t_2]$, $\frac{1}{2}P_5[s_3, t_3]$, and $\frac{1}{2}P_6[s_3, t_3]$.

Proof. We proceed by induction on $|E|$. We put $T = \{s_1, s_2, s_3, t_1, t_2, t_3\}$. We require the following

(3.1) We may assume the following.

(3.1.1) G is 2-connected, and $|T| = 6$.

(3.1.2) For each 2-cut $\partial(X)$ ($X \subseteq V$), $|X| = |X \cap T| = 1$ or $|X \cap T| \geq 2$.

(3.1.3) For each edge $e \in E$, there exists $X \subseteq V$ such that $|\partial(X)| = |D(X)|$ and $e \in \partial(X)$.

(3.1.4) For each $1 \leq i \leq 3$, s_i and t_i are not adjacent.

(3.1.5) If for vertices x_1, x_2, x_3, x_4 of G $\deg x_2 = \deg x_3 = 2$ and $\{x_1 x_2, x_2 x_3, x_3 x_4\} \subseteq E$, then $\deg x_1 \geq 3$ and $\deg x_4 \geq 3$.

Proof. (1) If $|T| \leq 5$, then Lemma 3.4 follows from (1.4.5).

(2) Let $\{e_1, e_2\} = \partial(X)$ be a 2-cut, and let $a_i \in X$, $b_i \in V - X$ and $a_i b_i = e_i$ ($i = 1, 2$). We define new graphs H, K as follows.

$$H = (X, E(\langle X \rangle) \cup f),$$

$$K = (V - X, E(\langle V - X \rangle) \cup g),$$

where f, g are new edges with ends a_1, a_2 and b_1, b_2 respectively. If $X \cap T = \emptyset$, then by induction the result of Lemma 3.4 holds in K , and so in G . If $|X \cap T| = 1$ (say $s_i \in X$) and $|X| \geq 2$, then we assign s_i on the midpoint of g in K ,

producing a new graph K' . Now by induction the result of Lemma 3.4 holds in K' , and so in G .

(3) If there exists $e \in E$ such that for each $X \subseteq V$ with $e \in \partial(X)$ and $|\partial(X)| > |D(X)|$, then the hypothesis of Lemma 3.4 holds in $G-e$, and so we can apply induction on $G-e$.

(4) If $s_3 t_3 \in E$, then we can apply (1.4.2) for the graph $G-s_3 t_3$, and for two pairs $(s_1, t_1), (s_2, t_2)$ and $q_1 = q_2 = 1$.

(5) If $\deg x_i = 2$, then $x_i \in T$ ($1 \leq i \leq 3$) by (3.1.2), and so we may let $x_1 = s_2$, $x_2 = s_1$ and $x_3 = t_2$ by (3.1.4) and (1.3). Let $x_0 \in N(x_1) - x$. Let G' be the graph obtained by contracting the edge $x_0 x_1$. By induction G' contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[s_2, t_2]$, $\frac{1}{2}P_4[s_2, t_2]$, $\frac{1}{2}P_5[s_3, t_3]$ and $\frac{1}{2}P_6[s_3, t_3]$. Let Q_1, \dots, Q_6 be the corresponding paths of G . We may let $x_1 x_2 \in E(Q_1) \cap E(Q_2)$ or $x_1 x_2 \in E(Q_1) \cap E(Q_3)$. In the former case, let Q_7 be the path of G such that $E(Q_7) = (x_1 x_2, x_2 x_3)$ and let Q_8 be the path of G obtained by combining $x_2 x_3$, $Q_3(t_2, x_0)$ and $Q_2(x_0, t_1)$. Then $\frac{1}{2}Q_1, \frac{1}{2}Q_8, \frac{1}{2}Q_7, \frac{1}{2}Q_4, \frac{1}{2}Q_5, \frac{1}{2}Q_6$ are required paths of G . In the latter case $\frac{1}{2}Q_1, \dots, \frac{1}{2}Q_6$ are required paths of G .

Now we come to the proof of Lemma 3.4. We distinguish three cases.

Case 1. G contains a nontrivial 2-cut $(e_1, e_2) = \partial(X)$ ($X \subseteq V$). We define H, K, a_i, b_i, f and g similarly as in the proof of (3.1.2). Then H and K are 2-edge-connected. It suffices to consider the following cases by (3.1.2).

Case 1a. $\{s_1, t_1\} \subseteq X$ and $\{s_2, s_3, t_2, t_3\} \subseteq V-X$. Assume that $K-g$ contains feasible paths $\frac{1}{2}P_1[s_2, t_2]$, $\frac{1}{2}P_2[s_2, t_2]$, $\frac{1}{2}P_3[s_3, t_3]$ and $\frac{1}{2}P_4[s_3, t_3]$. Then $H-f$ contains a path $P_5[s_1, t_1]$, and $P_5, \frac{1}{2}P_1, \frac{1}{2}P_2, \frac{1}{2}P_3, \frac{1}{2}P_4$ are required paths of G . If this is not the case, then by (1.4.2) for the graph $K-g$, and for two pairs $(s_2, t_2), (s_3, t_3)$ and $q_2=q_3=1$, (1.3) does not hold. Thus for some $Y \subseteq V-X$ with $b_1 \in Y$,

$$D_{K-g}(Y) = \{2, 3\} \text{ and } |\partial_{K-g}(Y)| = 1.$$

For each $Z \subseteq X$ such that $a_1 \in Z$, $f \in \partial_H(Z)$ and $D_H(Z) = \{1\}$,

$$|D_G(Y \cup Z)| = 3,$$

and so $|\partial_G(Y \cup Z)| = |\partial_{H-f}(Z)| + |\partial_{K-g}(Y)| \geq 3$,

thus $|\partial_{H-f}(Z)| \geq 2$.

Hence by (1.4.2) $H-f$ contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[a_1, a_2]$ and $\frac{1}{2}P_4[a_1, a_2]$, and K contains feasible paths $\frac{1}{2}P_5[s_2, t_2]$, $\frac{1}{2}P_6[s_2, t_2]$, $\frac{1}{2}P_7[s_3, t_3]$ and $\frac{1}{2}P_8[s_3, t_3]$. Now we can construct required paths of G .

Case 1b. $\{s_1, s_2\} \subseteq X$ and $\{s_3, t_1, t_2, t_3\} \subseteq V-X$. If $H-f$ is 2-edge-connected, then we assign a new vertex u on the midpoint of g , producing a new graph K' . By induction K' contains feasible paths $\frac{1}{2}P_1[u, t_1]$, $\frac{1}{2}P_2[u, t_1]$, $\frac{1}{2}P_3[u, t_2]$, $\frac{1}{2}P_4[u, t_2]$, $\frac{1}{2}P_5[s_3, t_3]$ and $\frac{1}{2}P_6[s_3, t_3]$. We may let $ub_1 \in E(P_1) \cap E(P_2)$ or $ub_1 \in E(P_1) \cap E(P_3)$. In each case we can construct required paths of G , since by (1.4.5) $H-f$ contains feasible paths $\frac{1}{2}P_7[s_1, a_1]$, $\frac{1}{2}P_8[s_1, a_1]$, $\frac{1}{2}P_9[s_2, a_2]$ and $\frac{1}{2}P_{10}[s_2, a_2]$ and contains $\frac{1}{2}[s_1, a_1, s_2, a_2, s_1]$. Thus we may assume that $H-f$ is not 2-edge-connected, and so H contains a 2-cut

$\{f, f'\}$. Then $\{f', e_1\}$ and $\{f', e_2\}$ are 2-cuts of G . By (3.1.2) $H = (\{a_1, a_2\}, \{f, f'\})$, and so we may let $a_1 = s_1$ and $a_2 = s_2$. By (3.1.5) $\deg b_i \geq 3$ ($i=1,2$). If K contains a 2-cut $\{g, g'\} = \partial_K(Y)$ ($Y \subseteq V-X$), then $\{g', e_1\}$ and $\{g', e_2\}$ are 2-cuts of G . Since $\deg b_i \geq 3$ ($i=1,2$), by (3.1.2) $|Y \cap T| = 2$ and $|(V-X-Y) \cap T| = 2$. By Case 1a we may let $Y \cap T \neq \{s_3, t_3\}$, and so we may let $\langle Y \rangle$ is an edge, contrary to (3.1.5). Thus assume that $K-g$ is 2-edge-connected. By (3.1.3), there exists $X \subseteq V$ such that $|\partial(X)| = |D(X)|$ and $s_1, s_2 \in \partial(X)$. Thus we may assume that G is the graph as illustrated in Figure 8. Let Y_1, Y_2 be the subsets

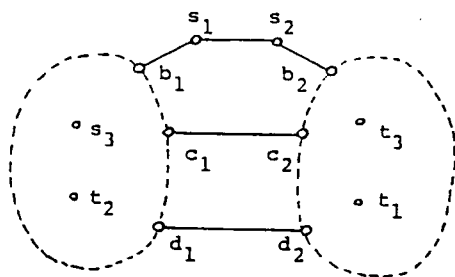


Figure 8.

of V such that $b_i \in Y_i$ and $\partial(Y_i) = \{s_i, b_i, c_1, c_2, d_1, d_2\}$ ($i=1,2$). We construct new graphs K_1, K_2 as follows.

$$K_i = (Y_i \cup v_i, E(\langle Y_i \rangle) \cup \{b_i, v_i, c_i, v_i, d_i, v_i\}), \quad i=1,2,$$

where v_1, v_2 are new vertices. If for $i=1$ or 2 , K_i contains a 2-cut $\partial_{K_i}(Z_i)$ ($Z_i \subseteq Y_i$) such that $|Z_i| \geq 2$ and $|V(K_i) - Z_i| \geq 2$, say $i=1$, then $\partial_G(Z_1)$ is a 2-cut of G , and by (3.1.2) $Z_1 \cap T = \{s_3, t_2\}$. Thus we may assume that $Z_1 = \{s_3, t_2\}$ and $\deg s_3 = \deg t_2 = 2$. This allows that we can apply Lemma 3.3 on

K_1 and K_2 .

Assume that $\deg c_1 \geq 3$, $\deg d_2 \geq 3$ or $\deg c_2 \geq 3$, $\deg d_1 \geq 3$, say the former. By Lemma 3.3 $K_1 - v_1$ contains $\frac{1}{2}[b_1, c_1, s_3, d_1, t_2, b_1]$ and $K_2 - v_2$ contains $\frac{1}{2}[b_2, d_2, t_3, c_2, t_1, b_2]$. Now we can construct required paths of G . Assume that for $i=1$ or 2 , $\deg c_i \geq 3$ and $\deg d_j \geq 3$, say for $i=1$. Now we may assume that $\deg c_2 = \deg d_2 = 2$. $c_2 = t_1$, $d_2 = t_3$ or $c_2 = t_3$, $d_2 = t_1$, say the former, then by Lemma 3.3 $K_2 - v_2$ contains $\frac{1}{2}[b_2, d_2, t_3, c_2, t_1, b_2]$ and $K_1 - v_1$ contains $\frac{1}{2}[b_1, c_1, s_3, d_1, t_2, b_1]$. Assume that $\deg c_i = 2$ ($i=1,2$) or $\deg d_j = 2$ ($j=1,2$), say the former. Let $y_1 \in N(c_1) - c_2$, and let $y_2 \in N(c_2) - c_1$, then by (3.1.5) $\deg y_i \geq 3$ ($i=1,2$). By (3.1.4) we may let $c_1 = t_2$, $c_2 = t_1$ or $c_1 = s_3$, $c_2 = t_1$. If $c_1 = t_2$, then by (1.4.2) $K_1 - (v_1, c_1)$ contains feasible paths $\frac{1}{2}P_1[s_3, d_1]$, $\frac{1}{2}P_2[s_3, d_1]$, $\frac{1}{2}P_3[b_1, y_1]$ and $\frac{1}{2}P_4[b_1, y_1]$, and $K_2 - (v_2, c_2)$ contains feasible paths $\frac{1}{2}P_5[t_3, d_2]$, $\frac{1}{2}P_6[t_3, d_2]$, $\frac{1}{2}P_7[b_2, y_2]$ and $\frac{1}{2}P_8[b_2, y_2]$, and so the result follows. If $c_1 = s_3$, then by Lemma 3.3 $K_1 - (v_1, c_1)$ contains $\frac{1}{2}[b_1, y_1, d_1, t_2, b_1]$ and $K_2 - (v_2, c_2)$ contains $\frac{1}{2}[b_2, d_2, t_3, y_2, b_2]$, and so the result follows.

Case 1c. $\{s_1, s_2, t_1\} \subseteq X$ and $\{s_3, t_2, t_3\} \subseteq V - X$. We may assume that neither Case 1a nor Case 1b occurs. If $\deg a_1 = 2$, then $\partial(X - a_1)$ is a 2-cut of G and $|(X - a_1) \cap T| = 2$, a contradiction. Thus $\deg a_i \geq 3$ and $\deg b_i \geq 3$ ($i=1,2$). We assign new vertices v_1, u_2 on the midpoints of f, g respectively, producing new graphs H', K' . For the graph H' , and for two pairs $(s_1, t_1), (s_2, v_2)$ and $q_1 = 1$, $q_2 = 2$, if (1.3) does not hold, then

there exists $Z \subseteq V(H')$ such that $D_{H'}(Z) = \{1, 2\}$ and $|\partial_{H'}(Z)| = 2$. Now Case 1b occurs in G , thus (1.3) holds, and so (1.2) holds. Hence $H' - v_2$ contains feasible paths $\frac{1}{2}P_1[s_1, t_1]$, $\frac{1}{2}P_2[s_1, t_1]$, $\frac{1}{2}P_3[s_2, a_1]$ and $\frac{1}{2}P_4[s_2, a_2]$. Similarly $K' - u_2$ contains feasible paths $\frac{1}{2}P_5[s_3, t_3]$, $\frac{1}{2}P_6[s_3, t_3]$, $\frac{1}{2}P_7[t_2, b_1]$ and $\frac{1}{2}P_8[t_2, b_2]$, and so the result follows.

Case 2. Every 2-cut of G is trivial, and G contains a 2-cut. Now we may let $\deg s_1 = 2$, and let e_1, e_2 be the edges incident to s_1 . By (3.1.3), for $i=1, 2$ there exists $X_i \subseteq V$ such that $s_1 \in X_i$, $|\partial(X_i)| = |D(X_i)|$ and $e_i \in \partial(X_i)$. For $i=1, 2$, since $|\partial(X_i)| = 3$, let $\partial(X_i) = \{e_i, f_i, g_i\}$. We put $X_3 = V - (X_1 \cup X_2)$, then $t_1 \in X_3$. By simple counting we have

$$(3.2) \quad |\partial(X_1 \cup X_2)| = |\partial(X_1)| + |\partial(X_2)| - |\partial(X_1 \cap X_2)| \\ - 2|\partial(X_1 - X_2) \cap \partial(X_2 - X_1)|.$$

If $|\partial(X_1 \cap X_2)| \geq 4$, then by (3.2)

$$|\partial(X_3)| = |\partial(X_1 \cup X_2)| \leq 3 + 3 - 4 = 2.$$

Thus $|\partial(X_3)| = 2$ and $|\partial(X_1 \cap X_2)| = 4$. Then $|X_3| = 1$ and $X_1 \cap X_2 = \{s_1, x\}$ for some $x \in V$ with $\deg x = 2$. We may let $x = s_2$, then $t_2 \in X_3$, and so $t_1 = t_2$, a contradiction. Thus $|\partial(X_1 \cap X_2)| = 2$ and $X_1 \cap X_2 = \{s_1\}$.

Case 2a. f_1, f_2, g_1, g_2 are not all distinct. We may let $f_1 = f_2$. Since $f_1 \notin \partial(X_1 \cap X_2) = \{e_1, e_2\}$, $f_1 \in \partial(X_1 - X_2) \cap \partial(X_2 - X_1)$. By (3.2)

$$|\partial(X_3)| = |\partial(X_1 \cup X_2)| \leq 3 + 3 - 2 - 2 = 2.$$

Thus $X_3 = \{t_1\}$, and we may assume that G is the graph as illustrated in Figure 9.

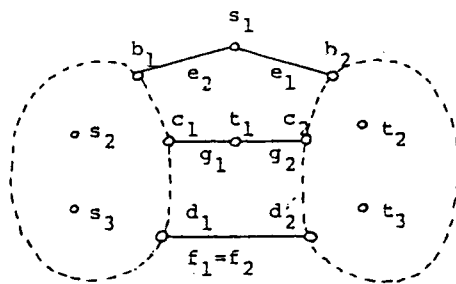


Figure 9.

Since every 2-cut is trivial, $\deg b_i \geq 3$ and $\deg c_i \geq 3$ ($i=1,2$). By Lemma 3.3 $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, s_3, d_1, s_2, c_1, b_1]$ and $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, t_3, d_2, t_2, c_2, b_2]$, and so the result follows.

Case 2b. f_1, f_2, g_1, g_2 are all distinct. Now $\partial(X_1 - X_2) \cap \partial(X_2 - X_1) = \emptyset$. From (3.2) we have

$$|\partial(X_3)| = 3 + 3 - |\partial(X_1 \cap X_2)| = 4.$$

Thus we may assume that G is the graph as illustrated in Figure 10.

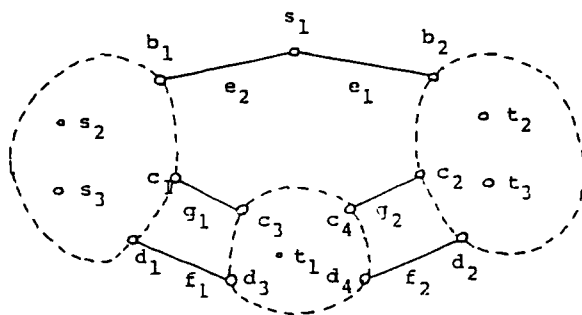


Figure 10.

$\langle X_3 \rangle$ is connected. For if not, then there exist $Y_1, Y_2 \subseteq X_3$ such that $X_3 = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$, and $|\partial(Y_1)| = |\partial(Y_2)| = 2$. Then $|\partial(X_3 \cap T)| = 2$, a contradiction. $\deg c_i \geq 3$ and $\deg d_i \geq 3$ ($i=3,4$), for if not, then $\deg t_1 = 2$ and one of f_1, f_2, g_1, g_2 is incident to

t_1 , say g_1 , and Case 2a occurs for $X_1, X_2 \cup X_3 - t_1$ instead of X_1, X_2 . If $\langle X_3 \rangle$ contains a 1-cut $(h) = \partial_{\langle X_3 \rangle}(Y_1)$ ($Y_1 \subseteq X_3$), then Y_1 contains just two of c_3, c_4, d_3, d_4 . Put $Y_2 = X_3 - Y_1$. $(c_3, d_3) \not\subseteq Y_1$, thus we may let $(c_3, c_4) \subseteq Y_1, (d_3, d_4) \subseteq Y_2$ and $t_1 \in Y_1$. Let v_1, v_2 be the vertices such that $v_i \in Y_i$ ($i=1,2$) and $v_1 v_2 = h$. $\langle Y_1 \rangle$ contains $\frac{1}{2}[c_3, c_4, v_1, t_1, c_3]$, for if $\langle Y_1 \rangle$ is not 2-edge-connected, then $\deg v_1 = 2$ and $v_1 = t_1$. $\langle Y_2 \rangle$ contains $\frac{1}{2}[d_3, d_4, v_2, d_3]$. Thus $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$ and feasible paths $\frac{1}{2}[c_3, c_4], \frac{1}{2}[c_3, t_1], \frac{1}{2}[t_1, c_4]$ and $[d_3, d_4]$. If $\langle X_3 \rangle$ is 2-edge-connected, then $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$ by Lemma 3.2. Assume that $\deg c_1 = 2$. We may let $c_1 = s_2$. $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, s_2, d_1, s_3, b_1]$ and $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, d_4, d_3, c_4, c_3]$. If $\deg d_2 \geq 3$ or $\deg d_2 = 2, d_2 = t_2$, then by Lemma 3.3 $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, t_3, c_2, t_2, d_2, b_2]$. If $\deg d_2 = 2$ and $d_2 = t_3$, then $\langle X_2 \rangle$ contains feasible paths $\frac{1}{2}P_1[b_2, t_3], \frac{1}{2}P_2[b_2, t_3], \frac{1}{2}P_3[c_2, t_2]$ and $\frac{1}{2}P_4[c_2, t_2]$. Now we can deduce the result. Thus we may assume that $\deg c_i \geq 3$ ($i=1,2$). By Lemma 3.3 $\langle X_1 \rangle$ contains $\frac{1}{2}[b_1, c_1, s_2, d_1, s_3, b_1]$ and $\langle X_2 \rangle$ contains $\frac{1}{2}[b_2, c_2, t_2, d_2, t_3, b_2]$. If $\langle X_3 \rangle$ is 2-edge-connected, then by Lemma 3.2 $\langle X_3 \rangle$ contains $\frac{1}{2}[c_3, t_1, c_4, d_3, d_4, c_3]$; and if not, then $\langle X_3 \rangle$ contains feasible paths $\frac{1}{2}[c_3, c_4], \frac{1}{2}[c_3, t_1], \frac{1}{2}[t_1, c_4]$ and $[d_3, d_4]$. Now we can deduce the result.

Case 3. G is 3-edge-connected. By Theorem 1 the result follows.

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