Title	SOME APPLICATIONS OF STOCHASTIC PROGRAMMING TO NETWORK PROBLEMS
Author(s)	塩出, 省吾
Citation	大阪大学, 1985, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/1715
rights	
Note	

SOME APPLICATIONS
OF
STOCHASTIC PROGRAMMING
TO
NETWORK PROBLEMS

（ネットワーク問題への確率計画法の応用）

SHŌGO SHIODE

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CHAPTER 1

INTRODUCTION

In this thesis we investigate stochastic versions of network problems, such as transportation problem, minimal spanning tree problem, and facility location problem. Researches on stochastic programming have made a remarkable development in recent years, and applicable areas thereof have been extended to a variety of fields, e.g., agriculture, finance, marketing, warehousing, etc., (See [8, and 14].) However, research works on stochastic programming have been mainly devoted to theoretical aspects, so far.

Stochastic programming deals with the methods for incorporating stochastic fluctuations in the framework of mathematical programming and for making optimal decisions with respect to certain criterions. ([7,8,9, and 15]) Such stochastic fluctuations may occur in the objective function and/or the constraints. Various approaches have been proposed to deal with problems of mathematical programming in such fluctuating situations, since the initiating papers by Dantzig and Beale in 1955. There are two main approaches, i.e., the "wait-and-see" approach and the "here-and-now" approach named by Madansky [11]. In the former approach we wait until an observation is made on random elements and then solve the deterministic problem. On the other hand, in the latter approach a decision is made before we observe the stochastic elements. The former approach caused the so-called distribution problems. Concerning the latter approach the so-called two-stage problems have been studied. (Walkup and Wets [16] generalized this latter approach to stochastic programming with recourse.)

Two-stage formulation was considered independently by Beale [1] and by Dantzig [4]. For example, we consider the following linear programming problem.

$$\begin{aligned} &\text{Minimize } \sum_{j=1}^{n} c_j x_j \\ &\text{subject to } \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i=1,2,\ldots,m, \\ &\qquad x_j \geqq 0,\ j=1,2,\ldots,n. \end{aligned} \tag{1.1}$$

If b_i, $i=1,2,\ldots,m$, are random variables, then two-stage formulation is as follows:

$$\begin{aligned} &\text{Minimize } \sum_{j=1}^{n} c_j x_j + E[\sum_{i=1}^{m} (p_i y_i^{+} + q_i y_i^{-})] \\ &\text{subject to } \sum_{j=1}^{n} a_{ij} x_j + y_i^{+} - y_i^{-} = b_i,\ i=1,2,\ldots,m, \\ &\qquad x_j \geqq 0,\ j=1,2,\ldots,n, \\ &\qquad y_i^{+},\ y_i^{-} \geqq 0,\ i=1,2,\ldots,m, \end{aligned} \tag{1.2}$$

where E denotes the expectation, and p_i, q_i are penalties for the positive and negative discrepancies between the right and left side values of the i-th constraint (1.1).

Charnes and Cooper [3] have also initiated another probabilistic approach, i.e., introduced chance constraints to mathematical programming problem. In their approach, the constraints do not hold necessarily, but they have only to hold with the probability greater than a given level. For instance we consider the following linear constraints;

$$\sum_{j=1}^{n} a_{ij} x_j \geqq b_i,\ i=1,2,\ldots,m. \tag{1.3}$$

If a_{ij} and b_i, $i=1,2,\ldots,m$; $j=1,2,\ldots,n$, are random variables, the solution set of the above inequalities may be empty. Therefore we consider the following chance constraints.

$$\Pr\{\sum_{j=1}^{n} a_{ij}x_j \geqq b_i\} \geqq \alpha_i, \ i=1,2,\ldots,m, \tag{1.4}$$

where α_i, $i=1,2,\ldots,m$, are given probability levels, and Pr{A} means the probability of A.

Bracken and Soland [2] introduced "the value of information" to stochastic mathematical programming problems, though this concept had been originally considered in [13]. Generally speaking, additional information may reduce the uncertainty on stochastic situation. If we can get "perfect information", then the problem under uncertainty becomes a problem under certainty. The value of information is the difference between "here-and-now" approach and "wait-and-see" approach for mathematical programming problem. Two types of values of information, i.e., the expected value of perfect information (EVPI) and the expected value of sample information (EVSI) are considered. The EVPI is the upper bound to what extent one would be willing to pay for perfect information. Usually, perfect information may not be available, and we have to take sample if we want the more information. Since sampling incurs some cost, the EVSI would be helpful in such a situation where we have to decide whether or not to take sample. Since EVSI is not greater than EVPI and approaches EVPI as the sample size increases, EVPI is useful as an upper bound for the EVSI.

In chapter 2 we consider a partially chance-constrained E-model with constraint of a random linear inequality and provide algorithms for solving it. There are few solution algorithms to solve such a chance-constrained programming problem. To solve this problem, we first transform the problem into the equivalent deterministic problem. We introduce subsidiary problem and derive useful properties for solving the deterministic problem. The algorithm is provided for solving the subsidiary problem with finite number of iterations. Moreover another type of subsidiary problem is introduced and the properties of the problem are derived. Then we provide the main algorithm for solving the main problem by utilizing the above algorithm and properties. Moreover we prove the validity and finiteness of the algorithm.

In chapter 3 we consider a stochastic transportation problem with simple recourse. For stochastic programs with recourse, there are few exact algorithms for obtaining the optimal solution.

The transportation problem was introduced by F.L. Hitchcock, discussed in details by T.C. Koopmans, and solved efficiently by G.B. Dantzig, L.R. Ford, Jr. and D.R. Fulkerson. (See the reference [5].)

We consider the two-stage formulation of this problem and derive some useful properties concerning the optimal solution and the optimal value. Moreover we investigate the behavior of the objective function and provide an algorithm for obtaining the optimal solution.

In chapter 4 we investigate minimal spanning tree problems in which edge costs are considered to be random variables. Several methods, e.g., Kruskal's algorithm [10] and Prim's algorithm [12], etc., are available for finding a minimal spanning tree in polynomial time order.

We consider two types of problems. One problem is to find an optimal spanning tree and optimal budget under the chance constraint that the probability with which total cost does not exceed budget is larger than a certain level. Another problem is to find an optimal spanning tree and optimal satisficing probability level under the same chance constraint. For the first type problem we propose a parametric type algorithm which finds an optimal spanning tree in $O(m^2n^2)$, where m and n are the number of edges and the number of vertices in a given graph respectively. And for the second type problem we propose another parametric type algorithm. Though this problem is more complicated than the first one, this algorithm also finds an optimal solution in $O(m^2n^2)$ computational time.

In chapter 5 we deal with single facility location problems where the weights and/or the locations of demand points are randomly distributed. The deterministic single facility location problems have been investigated so far by many researchers. ([6]) Concerning these deterministic problems, we consider the following problems specifically and the corresponding stochastic problems in this thesis. Suppose there are n demand points distributed on a plane, whose coordinates are (a_i, b_i), $i=1,2,\ldots,n$. Now, let (x,y) be the location of the facility point and we consider the distance $d_i(x,y)$ between the facility point and the i-th demand point. Now we consider two types of problems. One is the weighted minisum problem, i.e., the problem to minimize $\sum_{i=1}^{n} w_i d_i(x,y)$ with respect to x,y, where w_i is the weight by which the distance $d_i(x,y)$ is converted in terms of the cost required. Another is the minimax problem, i.e., the problem to minimize $\max_i d_i(x,y)$ with respect to x,y. This problem may be applicable to the location analysis of emergency service facility, for example. We consider several kinds of distances, e.g., the shortest Euclidean distance, socalled rectangular distance, i.e., the distance the admissible rectangular routes, for measuring the distances between the facility and the demand points, etc.. Euclidean distance is used to some network problems, e.g., electrical wiring problems, pipeline design problems. And rectangular distance is appropriate in urban location analysis where we travel along an orthogonal set of streets.

The first two problems in this chapter are concerned with the value of information in facility location. The value of information was first introduced by Wesolowsky [17]. He treated the EVPI in one-dimensional facility location model in which the weights have a multivariate normal distribution. One problem is to obtain the expected value of sample information. We discuss the model in which we consider the rectangular distances between the facility and demand points. And each weight is independently distributed normal random variable with unknown mean and known variance. We investigate the behavior the EVSI as the function of changing sample size. In addition, the expected net gain of our sampling and then the optimal sample size are found. Another problem is to find the expected value of perfect information in facility location model in which the distances are rectangular and both the weights and the locations of demand points are known only probabilistically. We give an explicit representation of EVPI.

The second problem is the model in which the weights of demand points are random variables and the distances are ℓp distances, i. e., $(|x-a_i|+|y-b_i|)$, $i=1,2,\ldots,n$. Our objective is to find a solution which maximizes the probability of satisfying the cost restriction. Especially the problem whose distances are rectangular is investigated in detail and an $O(n^3)$ time algorithm is given for solving the problem.

Finally we deal with a minimax facility location problem under locational uncertainty. In our model the number of demand points is assumed to be a random variable with a Poisson distribution and the location of each demand point is also random variable with uniform distribution on rectangular area or piecewise uniform distribution on the separated two areas. Our objective is to locate an emergency service facility, minimizing the reachable distance under constraint so that the probability of reaching all the locations of accidents (demand points) is larger than a certain predetermined value. First we analyze the problem parametrically, and provide the sensitivity analysis with respect to the aspired probability level. Finally an explicit optimal solution of the problem is parametrically derived.

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CHAPTER 2

AN ALGORITHM FOR A PARTIALLY CHANCE-CONSTRAINED E-MODEL

2.1 Introduction

Many types of chance-constrained programming problems have been considered [1-5, 7, 9 and 10] since Charnes and Cooper [1] introduced chance constraints into mathematical programming problems. This chapter considers an E-model having a random linear inequality constraint and provides an algorithm to solve it. There are few solution algorithms for solving the problem with stochastic constraints.

In Section 2.2 Problem P_0 and its deterministic equivalent Problem P are formulated. In Section 2.3 we introduce subsidiary problem $P(\mu)$ parametrized with μ and derives useful relations between P and $P(\mu)$. In Section 2.4 we give Algorithm 2.1 for solving $P(\mu)$ based on the parametric procedure ([5]) and prove validity and finiteness of the algorithm. In Section 2.5 we introduce another type of the subsidiary problem P^R and provide the main Algorithm 2.2 for solving P utilizing Algorithm 2.1 and properties of P^R. The validity and finiteness of Algorithm 2.2 are also proved. And in Section 2.6 we give an illustrative example.

2.2 Problem Formulation

In this chapter we consider the following problem P_0.

P_0: Maximize $E(\mathbf{c}^T\mathbf{x})$

subject to $\Pr\{\mathbf{a}^T\mathbf{x} \leq b\} \geq \alpha,$ (2.1)

$$A_1 \mathbf{x} \leqq B_1, \quad \mathbf{x} \geqq 0,$$

where T and E mean transpose and expectation respectively; $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ is an n-dimensional random vector and distributed according to multivariate normal distribution with mean vector $E(\mathbf{a}) = (E(a_1), E(a_2), \ldots, E(a_n))^T$ and variance-covariance matrix W; b is distributed according to a normal distribution with mean E(b) and variance σ_0^2; a_i and b are mutually independent $i=1,2,\ldots,n$; $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$ is an n-dimensional random vector with mean $E(\mathbf{c}) = (E(c_1), E(c_2), \ldots, E(c_n))^T$; A_1 is an m by n matrix; B_1 is an m-dimensional vector; $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is an n-dimensional decision variable vector; $\alpha(>1/2)$ is a probability level at least with which constraint $\mathbf{a}^T \mathbf{x} \leqq b$ must hold.

The chance constraint in (2.1) can be transformed into the following form by simple calculations.

$$\Pr\{\mathbf{a}^T \mathbf{x} \leqq b\} = \Pr\{\frac{\mathbf{a}^T \mathbf{x} - b - E(\mathbf{a})^T \mathbf{x} + E(b)}{\sqrt{\sigma_0^2 + \mathbf{x}^T W \mathbf{x}}} \leqq \frac{E(b) - E(\mathbf{a})^T \mathbf{x}}{\sqrt{\sigma_0^2 + \mathbf{x}^T W \mathbf{x}}}\} \geqq \alpha. \tag{2.2}$$

Since $\mathbf{a}$ and b are distributed according to $N(E(\mathbf{a}), W)$ and $N(E(b), \sigma_0^2)$ respectively,

$$\frac{\mathbf{a}^T \mathbf{x} - b - E(\mathbf{a})^T \mathbf{x} + E(b)}{\sqrt{\sigma_0^2 + \mathbf{x}^T W \mathbf{x}}}$$

is distributed according to the standard normal distribution N(0,1). Therefore (2.2) becomes as follows.

$$\frac{E(b) - E(\mathbf{a})^T \mathbf{x}}{\sqrt{\sigma_0^2 + \mathbf{x}^T W \mathbf{x}}} \geqq \Phi^{-1}(\alpha), \tag{2.3}$$

where Φ is the cumulative distribution function of standard normal distribution. The inequality (2.3) is further transformed into

$$E(\mathbf{a})^T\mathbf{x} + K_\alpha\sqrt{\sigma_0^2 + \mathbf{x}^T W \mathbf{x}} \leq E(b),$$

where $K_\alpha \overset{\Delta}{=} \Phi^{-1}(\alpha)$. $E(\mathbf{c}^T\mathbf{x})$ is equivalent to $E(\mathbf{c})^T\mathbf{x}$ by the linearity of expectation. Then the problem P_0 is equivalent to the following deterministic problem P.

$$P: \quad \text{Maximize } E(\mathbf{c})^T\mathbf{x}$$

$$\text{subject to } E(\mathbf{a})^T\mathbf{x} + K_\alpha(\sigma_0^2 + \mathbf{x}^T W \mathbf{x})^{\frac{1}{2}} \leq E(b)$$

$$A_1\mathbf{x} \leq B_1, \quad \mathbf{x} \geq \mathbf{0}.$$

Moreover we assume that the feasible set of P,

$$S \overset{\Delta}{=} \{\mathbf{x} \mid E(\mathbf{a})^T\mathbf{x} + K_\alpha(\sigma_0^2 + \mathbf{x}^T W \mathbf{x})^{\frac{1}{2}} \leq E(b),\ A_1\mathbf{x} \leq B_1,\ \mathbf{x} \geq \mathbf{0}\}$$

is not empty and bounded. As is easily shown, S is a convex set and therefore P is a convex programming problem.

2.3 Subsidiary Problem of P

Let $\mathbf{x}^*$ and μ^* denote an optimal solution and the optimal value of Problem P respectively. To solve P we introduce the following subsidiary problem $P(\mu)$.

$$P(\mu): \text{ Minimize } E(\mathbf{a})^T\mathbf{x} + K_\alpha(\sigma_0^2 + \mathbf{x}^T W \mathbf{x})^{\frac{1}{2}}$$

$$\text{subject to } E(\mathbf{c})^T\mathbf{x} \geq \mu,\ A_1\mathbf{x} \leq B_1,\ \mathbf{x} \geq \mathbf{0}.$$

Denoting the optimal solution and the optimal value of $P(\mu)$ by $\mathbf{x}(\mu)$ and $z(\mu)$ respectively. As is easily proved, $P(\mu)$ is a strictly convex programming problem, and so $\mathbf{x}(\mu)$ is unique. Then the following relation between P(μ) and P holds.

Theorem 2.1 If $\mathbf{x}(\mu)$ satisfies

$$E(\mathbf{a})^T\mathbf{x}(\mu)+K_\alpha(\sigma_0^2+\mathbf{x}(\mu)^T W\mathbf{x}(\mu))^{\frac{1}{2}}=E(b)$$

and $$E(\mathbf{c})^T\mathbf{x}(\mu)=\mu,$$

then $\mathbf{x}(\mu)$ is also an optimal solution of P.

Proof: Kuhn-Tucker condition of Problem P (KTP) is as follows ([8]).

$$\text{KTP:}\quad \mathbf{v}-pE(\mathbf{a})-K_\alpha p\frac{W\mathbf{x}}{(\sigma_0^2+\mathbf{x}^T W\mathbf{x})^{\frac{1}{2}}}-A_1^T\mathbf{q}=-E(\mathbf{c}),$$

$$E(\mathbf{a})^T\mathbf{x}+K_\alpha(\sigma_0^2+\mathbf{x}^T W\mathbf{x})^{\frac{1}{2}}+s_0=E(b),$$

$$A_1\mathbf{x}+\mathbf{s}=B_1,\quad \mathbf{v}^T\mathbf{x}+\mathbf{s}^T\mathbf{q}+s_0p=0,$$

$$\mathbf{v},\ \mathbf{x},\ \mathbf{s},\ \mathbf{q}\geq\mathbf{0},\ s_0,p\geq 0,$$

where $\mathbf{v}$ is an n-dimensional vector; $\mathbf{s}$, $\mathbf{q}$ are m-dimensional vectors ; s_0, p are scolors. On the other hand, Kuhn-Tucker condition of Problem P(μ) (KTP(μ)) becomes as follows.

$$\text{KTP}(\mu):\quad \bar{\mathbf{v}}-E(\mathbf{a})-K_\alpha\frac{W\mathbf{x}}{(\sigma_0^2+\mathbf{x}^T W\mathbf{x})^{\frac{1}{2}}}-A_1^T\bar{\mathbf{q}}=-\bar{r}E(\mathbf{c}),$$

$$E(\mathbf{c})^T\mathbf{x}-\bar{s}_0=\mu,\quad A_1^T\mathbf{x}+\bar{\mathbf{s}}=B_1,$$

$$\bar{\mathbf{v}}^T\mathbf{x}+\bar{s}_0\bar{r}+\bar{\mathbf{q}}^T\bar{\mathbf{s}}=0,$$

$$\bar{\mathbf{v}},\ \mathbf{x},\ \bar{\mathbf{s}},\ \bar{\mathbf{q}}\geq\mathbf{0},\ \bar{s}_0,\ \bar{r}\geq 0,$$

where $\bar{\mathbf{v}}$ is an n-dimensional vector; $\bar{\mathbf{s}}$, $\bar{\mathbf{q}}$ are m-dimensional vectors; $\bar{s}_0$, $\bar{r}$ are scalars. Let

$$X(\mu)=(\mathbf{x}(\mu)^T,\bar{\mathbf{v}}(\mu)^T,\bar{\mathbf{q}}(\mu)^T,\bar{r}(\mu),\bar{s}_0(\mu),\bar{\mathbf{s}}(\mu)^T)^T$$

denote the solution of KTP(μ). Since $E(\mathbf{c})^T\mathbf{x}(\mu)=\mu$ means $\bar{s}_0(\mu)=0$, $\bar{r}(\mu)$ must be positive. By the positivity of $\bar{r}(\mu)$ and the condition

$$E(\mathbf{a})^T\mathbf{x}(\mu)+K_\alpha(\sigma_0^2+\mathbf{x}(\mu)^TW\mathbf{x}(\mu))^{\frac{1}{2}}=E(b),$$

the solution of KTP is constructed from X(μ) as follows:

$$\mathbf{v}=\bar{\mathbf{v}}(\mu)/\bar{r}(\mu),\ p=1/\bar{r}(\mu),\ \mathbf{q}=\bar{\mathbf{q}}(\mu)/\bar{r}(\mu),\ s_0=0,$$

$$\mathbf{x}=\mathbf{x}(\mu),\ \mathbf{s}=\bar{\mathbf{s}}(\mu).$$

(Indeed this solution satisfies KTP.) Since P and P(μ) are strictly concave programming problem and strictly convex programming problem respectively, feasible solutions of KTP and KTP(μ) are optimal solutions of P and P(μ) respectively. Therefore $\mathbf{x}(\mu)$ satisfying conditions of this theorem is the optimal solution of P. ☐

Moreover the following properties of P(μ) can be derived.

__Property 2.1__ $z(\mu)$ is a convex function of μ.

__Proof:__ For $\mu_1<\mu_2$, $0<\lambda<1$ and $\bar{\lambda}=1-\lambda$,

$$\begin{aligned}
&\lambda z(\mu_1)+\bar{\lambda}z(\mu_2)-z(\lambda\mu_1+\bar{\lambda}\mu_2)\\
&=\lambda E(\mathbf{a})^T\mathbf{x}(\mu_1)+\bar{\lambda}E(\mathbf{a})^T\mathbf{x}(\mu_2)-E(\mathbf{a})^T\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2)\\
&\quad+\lambda K_\alpha(\sigma_0^2+\mathbf{x}(\mu_1)^TW\mathbf{x}(\mu_1))^{\frac{1}{2}}+\bar{\lambda}K_\alpha(\sigma_0^2+\mathbf{x}(\mu_2)^TW\mathbf{x}(\mu_2))^{\frac{1}{2}}\\
&\quad-K_\alpha(\sigma_0^2+\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2)^TW\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2))^{\frac{1}{2}}\\
&\geq E(\mathbf{a})^T(\lambda\mathbf{x}(\mu_1)+\bar{\lambda}\mathbf{x}(\mu_1))-E(\mathbf{a})^T\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2)\\
&\quad+K_\alpha\{\sigma_0^2+(\lambda\mathbf{x}(\mu_1)+\bar{\lambda}\mathbf{x}(\mu_2))^TW(\lambda\mathbf{x}(\mu_1)+\bar{\lambda}\mathbf{x}(\mu_2))\}^{\frac{1}{2}}\\
&\quad-K_\alpha\{\sigma_0^2+\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2)^TW\mathbf{x}(\lambda\mu_1+\bar{\lambda}\mu_2)\}^{\frac{1}{2}}
\end{aligned}$$

(since $(\sigma_0^2 + \mathbf{x}^T W \mathbf{x})^{\frac{1}{2}}$ is convex in $\mathbf{x}$)

≥ 0 (by the feasibility of $\lambda\mathbf{x}(\mu_1) + \bar{\lambda}\mathbf{x}(\mu_2)$ and optimality of $\mathbf{x}(\lambda\mu_1 + \bar{\lambda}\mu_2)$ for $P(\lambda\mu_1 + \bar{\lambda}\mu_2)$). ☐

Property 2.2 $z(\mu)$ is a nondecreasing function of μ.

Proof: It is clear from the fact that the feasible region of $P(\mu)$ becomes smaller as μ increases. ☐

Theorem 2.2 Without any loss of generality, we can always assume $\bar{s}_0(\mu) = 0$.

Proof: Assume that there exists a $\hat{\mu}$ such that $\bar{s}_0(\hat{\mu}) > 0$. Then $z(\mu) = z(\hat{\mu})$ and $\mathbf{x}(\mu) = \mathbf{x}(\hat{\mu})$ for any $\hat{\mu} + \bar{s}_0(\hat{\mu}) \geq \mu \geq \hat{\mu}$ since $\bar{s}_0(\hat{\mu}) > 0$ implies

$$E(\mathbf{c})^T \mathbf{x}(\hat{\mu}) \geq \hat{\mu} + \bar{s}_0(\hat{\mu}) , \qquad (2.4)$$

and (2.4) means that $\mathbf{x}(\hat{\mu})$ is optimal for any μ among $\hat{\mu} + \bar{s}_0(\hat{\mu}) \geq \mu \geq \hat{\mu}$ from Property 2.2. Convexity of $z(\mu)$ shows that this occurs only the first portion of $z(\mu)$. Since $S \neq \phi$ implies $z(\hat{\mu}) \leq E(b)$, this portion can be excluded from further consideration by Theorem 2.1. That is, we can assume $\bar{s}_0(\mu) = 0$ without any loss of generality. ☐

From Theorem 2.2 we can assume that $E(\mathbf{c})^T \mathbf{x} = \mu$ in Theorem 2.1. In addition we have $z(\mu) > z(\mu')$ for $\mu > \mu'$ as a byproduct. Therefore Property 2.2 is strengthened as follows.

Property 2.2' There exists $\ddot{\mu}$ such that $z(\mu)$ is monotonically increasing function of μ for any $\mu \geq \ddot{\mu}$.

Now we must check whether μ such that $z(\mu) = E(b)$ exists or not. For this purpose let

$$\tilde{\mu} \overset{\Delta}{=} \max\{E(\mathbf{c})^T \mathbf{x} \mid A_1 \mathbf{x} \leq B_1, \ \mathbf{x} \geq \mathbf{0}\}.$$

Note that $\tilde{\mu}$ may not exist. If $\tilde{\mu}$ exists, then $\mathbf{x}(\mu)$ for $\mu > \tilde{\mu}$ does not exist. Moreover if $E(b) > z(\tilde{\mu})$ holds, μ such that $z(\mu) = E(b)$ is not defined. But in this case, $\mathbf{x}(\tilde{\mu})$ becomes an optimal solution of P as is easily shown.

Property 2.3 μ such that $z(\mu) = E(b)$ (that is, the optimal value of P) is unique if it exists.

Proof: This is clear from $z(\ddot{\mu}) \leq E(b)$ and Property 2.2'. Note that $z(\ddot{\mu}) \leq E(b)$ is derived from $S \neq \phi$. □

2.4 Algorithm for Solving P(μ)

In order to solve $P(\mu)$, we introduce in this section an auxiliary parametrized problem $P^R(\mu)$.

$$P^R(\mu):\quad \text{Minimize } RE(\mathbf{a})^T\mathbf{x} + \frac{1}{2}K_\alpha(\sigma_0^2 + \mathbf{x}^TW\mathbf{x})$$

$$\text{subject to } E(\mathbf{c})^T\mathbf{x} \geq \mu,\ A_1\mathbf{x} \leq B_1,\ \mathbf{x} \geq \mathbf{0}.$$

Note that the feasible region of $P^R(\mu)$ coincides with that of $P(\mu)$. Let $\mathbf{x}^R(\mu)$ and $z^R(\mu)$ denote the optimal solution and optimal value of $P^R(\mu)$. The objective function of Problem $P^R(\mu)$ is a strictly convex function, and so $\mathbf{x}^R(\mu)$ is unique.

Theorem 2.3 If $\mathbf{x}^R(\mu)$ satisfies $R^2 = \sigma_0^2 + \mathbf{x}^R(\mu)^TW\mathbf{x}^R(\mu)$, then it is the optimal solution $\mathbf{x}(\mu)$ of $P(\mu)$.

Proof: Each $P^R(\mu)$ is a convex programming problem and corresponding Kuhn-Tucker condition $KTP^R(\mu)$ becomes as follows.

$$KTP^R(\mu):\quad \hat{\mathbf{v}} - K_\alpha W\mathbf{x} - A_1^T\hat{\mathbf{q}} = RE(\mathbf{a}),$$

$$A_1\mathbf{x} + \hat{\mathbf{s}} = B_1,\ E(\mathbf{c})^T\mathbf{x} - \hat{s}_0 = \mu,$$

$$\hat{\mathbf{v}}^T\mathbf{x} + \hat{\mathbf{s}}^T\hat{\mathbf{q}} + \hat{r}\hat{s}_0 = 0,\ \hat{\mathbf{v}},\ \mathbf{x},\ \hat{\mathbf{s}},\ \hat{\mathbf{q}} \geq \mathbf{0},\ \hat{r},\ \hat{s}_0 \geq 0.$$

If $\mathbf{x}^R(\mu)$ satisfies $R^2 = \sigma_0^2 + \mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu)$, then $X(\mu)$ can be constructed from a solution $X^R(\mu) \triangleq (\mathbf{x}^R(\mu)^T, \hat{\mathbf{v}}^R(\mu)^T, \hat{\mathbf{q}}^R(\mu), \hat{r}^R(\mu), \hat{s}_0^R(\mu), \hat{\mathbf{s}}^R(\mu)^T)^T$ of $\mathrm{KTP}^R(\mu)$ as follows.

$$X(\mu):\quad \mathbf{x}(\mu) = \mathbf{x}^R(\mu),\ \bar{\mathbf{v}}(\mu) = \hat{\mathbf{v}}^R(\mu)/R,\ \bar{\mathbf{q}}(\mu) = \hat{\mathbf{q}}^R(\mu)/R,$$

$$\bar{r}(\mu) = \hat{r}^R(\mu)/R,\ \bar{s}_0(\mu) = \hat{s}_0^R(\mu),\ \bar{\mathbf{s}}(\mu) = \hat{\mathbf{s}}^R(\mu).$$

Indeed the solution constructed as above satisfies $\mathrm{KTP}(\mu)$ as is easily checked. Therefore $\mathbf{x}^R(\mu)$ becomes an optimal solution of $P(\mu)$. □

Property 2.4 $z^R(\mu)$ is a monotonically increasing function of μ.

Proof: We can show it similarly to Property 2.2. □

Property 2.5 $E(\mathbf{a})^T \mathbf{x}^R(\mu)$ is a nonincreasing function of R.

Proof: For $R' < R$ the following inequalities hold;

$$RE(\mathbf{a})^T \mathbf{x}^R(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + \mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu))$$
$$\leq RE(\mathbf{a})^T \mathbf{x}^{R'}(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + \mathbf{x}^{R'}(\mu)^T W \mathbf{x}^{R'}(\mu))$$

$$R'E(\mathbf{a})^T \mathbf{x}^{R'}(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + \mathbf{x}^{R'}(\mu)^T W \mathbf{x}^{R'}(\mu))$$
$$\leq R'E(\mathbf{a})^T \mathbf{x}^R(\mu) + \frac{1}{2}K_\alpha(\sigma_0^2 + \mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu)),$$

since $\mathbf{x}^R(\mu)$ and $\mathbf{x}^{R'}(\mu)$ are optimal solutions of $P^R(\mu)$ and $P^{R'}(\mu)$ respectively. These imply

$$R\{E(\mathbf{a})^T \mathbf{x}^R(\mu) - E(\mathbf{a})^T \mathbf{x}^{R'}(\mu)\}$$
$$+\frac{1}{2}K_\alpha\{\mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu) - \mathbf{x}^{R'}(\mu)^T W \mathbf{x}^{R'}(\mu)\} \leq 0, \quad (2.5)$$

$$R'\{E(\mathbf{a})^T \mathbf{x}^R(\mu) - E(\mathbf{a})^T \mathbf{x}^{R'}(\mu)\}$$
$$+\frac{1}{2}K_\alpha\{\mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu) - \mathbf{x}^{R'}(\mu)^T W \mathbf{x}^{R'}(\mu)\} \geq 0. \quad (2.6)$$

From (2.5) and (2.6) we have

$$(R-R')\{E(\mathbf{a})^T\mathbf{x}^R(\mu)-E(\mathbf{a})^T\mathbf{x}^{R'}(\mu)\}\leq 0.$$

Therefore from $R'<R$

$$E(\mathbf{a})^T\mathbf{x}^R(\mu)\leq E(\mathbf{a})^T\mathbf{x}^{R'}(\mu). \quad \square$$

Property 2.6 $\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$ is a nondecreasing function of R.

Proof: From the optimality of $\mathbf{x}^R(\mu)$ for $P^R(\mu)$ we have

$$R'E(\mathbf{a})^T\mathbf{x}^{R'}(\mu)+\frac{1}{2}K_\alpha\{\sigma_0^2+\mathbf{x}^{R'}(\mu)^T W\mathbf{x}^{R'}(\mu)\}$$

$$\leq R'E(\mathbf{a})^T\mathbf{x}^R(\mu)+\frac{1}{2}\{\sigma_0^2+\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)\}.$$

From this inequality and Property 2.5 we have

$$\mathbf{x}^{R'}(\mu)^T W\mathbf{x}^{R'}(\mu)\leq \mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu). \quad \square$$

We define $R(\mu)\overset{\Delta}{=}\{\sigma_0^2+\mathbf{x}(\mu)^T W\mathbf{x}(\mu)\}^{\frac{1}{2}}$. The following theorem provides some useful informations about $R(\mu)$ even if $R\neq R(\mu)$.

Theorem 2.4

(i) $R>R(\mu) \longleftrightarrow R^2>\sigma_0^2+\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$

(ii) $R<R(\mu) \longleftrightarrow R^2<\sigma_0^2+\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$

(iii) $R=R(\mu) \longleftrightarrow R^2=\sigma_0^2+\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$.

Proof: For each $\mathbf{x}^R(\mu)$, $\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)<\infty$ holds since $P^R(\mu)$ has the same feasible region as $P(\mu)$ and boundedness of S implies $E(\mathbf{a})^T\mathbf{x}^R(\mu)>-\infty$. Therefore from Property 2.6 there exists a sufficiently large $\bar{R}$ such that $\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$ is constant for $R>\bar{R}$. The continuity of $\mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$ with respect to R can be derived from the continuity of $\mathbf{x}^R(\mu)$ with respect to R. Therefore Mean-Value Theorem, Theorem 2.3 and the uniqueness of $\mathbf{x}(\mu)$ together prove

Theorem 2.4. $\square$

Now we are ready to solve $P(\mu)$ by utilizing $P^R(\mu)$. Generally $X^R(\mu)$ depends upon μ and R, which determines a basic matrix B. Being based on B, there exist constant vectors $\mathbf{d}'_B$, $\mathbf{e}'_B$, $\mathbf{g}'_B$ and a certain interval $L_B(\mu) \leq R \leq U_B(\mu)$ determined by the basic matrix B and μ, and so $X^R(\mu)$ can be written as follows.

$$X^R(\mu) = R\mathbf{d}'_B + \mu\mathbf{e}'_B + \mathbf{g}'_B \qquad (L_B(\mu) \leq R \leq U_B(\mu)).$$

Moreover taking $\mathbf{x}$ part of $X^R(\mu)$, we can write down as

$$\mathbf{x}^R(\mu) = R\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B,$$

using $\mathbf{d}_B$, $\mathbf{e}_B$ and $\mathbf{g}_B$ ($\mathbf{x}$ part of $\mathbf{d}'_B$, $\mathbf{e}'_B$ and $\mathbf{g}'_B$ respectively). By the above discussion, the condition

$$R^2 = \sigma_0^2 + \mathbf{x}^R(\mu)^T W \mathbf{x}^R(\mu)$$

is equivalent to the condition that one of roots of the equation

$$(\mathbf{d}_B^T W \mathbf{d}_B - 1)R^2 + 2(\mu\mathbf{e}_B + \mathbf{g}_B)^T W \mathbf{d}_B R + (\mu\mathbf{e} + \mathbf{g}_B)^T W(\mu\mathbf{e}_B + \mathbf{g}_B) + \sigma_0^2 = 0$$

exists on the interval $[L_B(\mu), U_B(\mu)]$. Hereafter let us refer this equation to Q-equation. The roots of Q-equation are as follows:

(Case a) $\mathbf{d}_B^T W \mathbf{d}_B = 1$,

$$R = \frac{-(\mu\mathbf{e}_B + \mathbf{g}_B)^T W(\mu\mathbf{e}_B + \mathbf{g}_B) - \sigma_0^2}{2(\mu\mathbf{e}_B + \mathbf{g}_B)^T W \mathbf{d}_B},$$

(Case b) $\mathbf{d}_B^T W \mathbf{d}_B \neq 1$,

$$R = \frac{-(\mu\mathbf{e}_B + \mathbf{g}_B)^T W \mathbf{d}_B \pm \sqrt{D}}{\mathbf{d}_B^T W \mathbf{d}_B - 1},$$

where $D \overset{\Delta}{=} \{(\mu\mathbf{e}_B + \mathbf{g}_B)^T W\mathbf{d}_B\}^2 - (\mathbf{d}_B^T W\mathbf{d}_B - 1)\{(\mu\mathbf{e}_B + \mathbf{g}_B)^T W(\mu\mathbf{e}_B + \mathbf{g}_B) + \sigma_0^2\}$.

Remark 2.1 $R \geq \sigma_0$ only must be checked for $R^2 = \sigma_0^2 + \mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu)$ since W is positive definite.

Let $K^\mu(R) \overset{\Delta}{=} \sigma_0^2 + \mathbf{x}^R(\mu)^T W\mathbf{x}^R(\mu) - R^2$. Then if $K^\mu(L_B(\mu)) \geq 0$ and $K^\mu(U_B(\mu)) \leq 0$, one root of Q-equation exists in the interval $[L_B(\mu), U_B(\mu)]$.

Algorithm 2.1 for solving $P(\mu)$

Step 1: Set $R_\ell \longleftarrow \sigma_0$, $R_u \longleftarrow M$ (M is a sufficiently large positive number) and $R \longleftarrow R_0$ (R_0 is an arbitrary number such that $R_0 \geq \sigma_0$). Solve $P^R(\mu)$ and find B, $\mathbf{d}_B$, $\mathbf{e}_B$, $\mathbf{g}_B$, $L_B(\mu)$ and $U_B(\mu)$. Go to Step 2.

Step 2: If $K^\mu(L_B(\mu)) < 0$, then set $R_u \longleftarrow L_B(\mu)$ and $R \longleftarrow (R_u + R_\ell)/2$, and go to Step 4. If $K^\mu(L_B(\mu)) = 0$, then set $\mathbf{x}(\mu) = L_B(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$ and terminate. If $K^\mu(L_B(\mu)) > 0$, then go to Step 3.

Step 3: If $K^\mu(U_B(\mu)) < 0$, then solve Q-equation, find roots β_1, β_2 and go to Step 5. If $K^\mu(U_B(\mu)) = 0$, then set $\mathbf{x}(\mu) = U_B(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$ and terminate. If $K^\mu(U_B(\mu)) > 0$, then set $R \longleftarrow U_B(\mu)$ and $R \longleftarrow (R_\ell + R_u)/2$, and go to Step 4.

Step 4: Solve $P^R(\mu)$ and find B, $\mathbf{d}_B$, $\mathbf{e}_B$, $\mathbf{g}_B$, $L_B(\mu)$ and $U_B(\mu)$. Return to Step 2.

Step 5: If β_1 (or β_2) belongs to $[L_B(\mu), U_B(\mu)]$, then set $\mathbf{x}(\mu) = \beta_1\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$ (or $\mathbf{x}(\mu) = \beta_2\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$) and terminate.

Remark 2.2 (i) In Step 5, if $\mathbf{d}_B^T W\mathbf{d}_B = 1$, then we consider

$$\beta_1 = \beta_2 = \frac{(\mu\mathbf{e}_B + \mathbf{g}_B)^T W(\mu\mathbf{e}_B + \mathbf{g}_B) + \sigma_0^2}{2(\mu\mathbf{e}_B + \mathbf{g}_B)^T W\mathbf{d}_B} .$$

(ii) If $K^\mu(L_B(\mu)) < 0$ holds, $K^\mu(U_B(\mu)) < 0$ necessarily holds by Theorem 2.4. Thus the test for $K^\mu(U_B(\mu))$ is to be omitted. On

the other hand if $K^{\mu}(U_B(\mu)) > 0$ holds, then $K^{\mu}(L_B(\mu)) > 0$ holds and the test for $K^{\mu}(L_B(\mu))$ is also omitted.
(iii) $[L_B(\mu),U_B(\mu)] \subseteq [R_\ell,R_u]$ and $U_B(\mu) - L_B(\mu) \leq \frac{1}{2}(R_u - R_\ell)$ hold except the first $[L_B(\mu),U_B(\mu)]$.

Theorem 2.5 Algorithm 2.1 terminates after finite iterations, and upon termination it finds $\mathbf{x}(\mu)$.

Proof: (Finiteness) After each calculation of Step 4, five cases (a)-(e) as illustrated in Figure 2.1a-2.1e are possible. In case (d) (or (e)) it is clear that

$$\mathbf{x}(\mu) = L_B(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B \quad (\text{or } \mathbf{x}(\mu) = U_B(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B)$$

holds. In case (c) either β_1 or β_2 (but not both) must belong to the interval $[L_B(\mu),U_B(\mu)]$ according to the continuity and Mean-Value Theorem with respect to $K^{\mu}(R)$. Thus in cases (c)-(e), Algorithm 2.1 terminates. In cases (a) and (b), neither β_1 nor β_2 belongs to the interval $[L_B(\mu),U_B(\mu)]$ by Theorem2.4. First note that

$$L_B(\mu) \leq (R_\ell + R_u)/2 \leq U_B(\mu) \qquad (2.7)$$

holds as is easily known from the updating procedure of R in Step 2 or Step 3.
Case (a): R_u is set to $L_B(\mu)$ since $K^{\mu}(L_B(\mu)) < 0$.
Case (b): R_ℓ is set to $U_B(\mu)$ since $K^{\mu}(U_B(\mu)) > 0$.
In any case it follows from (2.7) that the difference $R_u - R_\ell$ is at least halved except the first excecution of Step 2 and Step 3. Therefore after finite iterations, case (c), (d) or (e) occurs since $R(\mu)$ belongs to a certain interval $[L_B(\mu),U_B(\mu)]$ with $U_B(\mu)-L_B(\mu)>0$.
(Validity) Termination condition itself assures validity of Algorithm 2.1. ☐

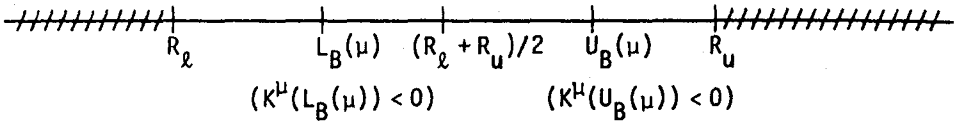

Figure 2.1a. Case (a)

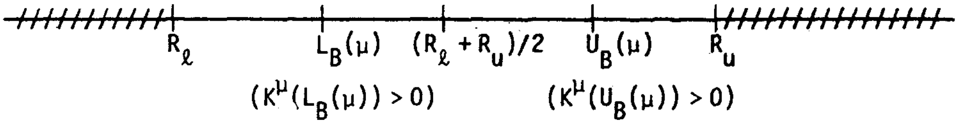

Figure 2.1b. Case (b)

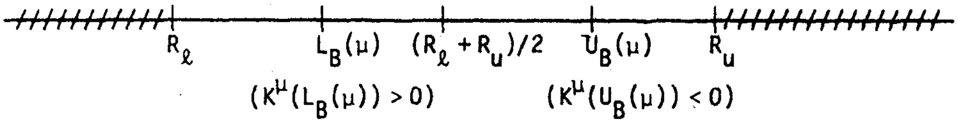

Figure 2.1c. Case (c)

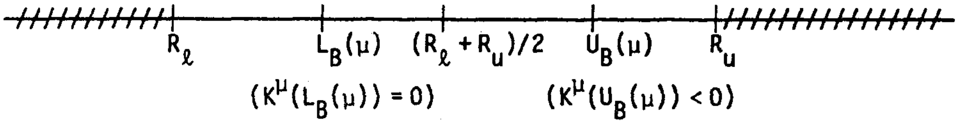

Figure 2.1d. Case (d)

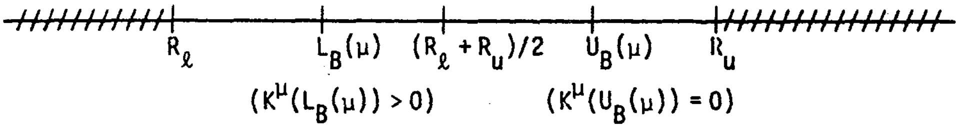

Figure 2.1e. Case (e)

2.5 Main Algorithm for Solving P

Let B denote the optimal basic matrix of KTP(μ), that is, let $\mathbf{x}(\mu) = \beta(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$. (Of course, $\beta(\mu) = R(\mu)$, but for convenience, we denote $R(\mu)$ with $\beta(\mu)$.) Now we define

$$I(B) \overset{\Delta}{=} \{\mu \mid L_B(\mu) \leq \beta(\mu) \leq U_B(\mu) \text{ and } \mathbf{x}(\mu) \geq \mathbf{0}\}.$$

Then I(B) is the set of μ where B becomes the optimal basic matrix of KTP(μ) and for μ on I(B) we can write down $\mathbf{x}(\mu) = \beta(\mu)\mathbf{d}_B + \mu\mathbf{e}_B + \mathbf{g}_B$. In other words, the shape of $z(\mu)$ with respect to μ on I(B) is determined. If $z(\mu)$ crosses $z(\mu) = E(b)$ on I(B), then the optimal solution will be found. For this purpose, let

$$\bar{\mu}_B \overset{\Delta}{=} \sup\{\mu \mid \mu \in I(B)\}$$

and
$$\mu'_B \overset{\Delta}{=} \sup\{\mu \mid \mu \in I(B),\ z(\mu) \leq E(b)\}.$$

When $\mu'_B = \mu^*$,

$$\mathbf{x}^* = \beta(\mu'_B)\mathbf{d}_B + \mu'_B\mathbf{e}_B + \mathbf{g}_B$$

holds. But in case that $\mu'_B < \mu^*$, we have to continue the search for μ^*. Now define another type subsidiary problem P^R with a parameter $R \geq \sigma_0$.

$$P^R: \quad \text{Maximize } E(\mathbf{c})^T\mathbf{x}$$

$$\text{subject to } E(\mathbf{a})^T\mathbf{x} \leq E(b) - K_\alpha R,$$

$$A_1\mathbf{x} \leq B_1, \quad \mathbf{x} \geq \mathbf{0}.$$

Let $\mathbf{x}^R$ and μ^R denote an optimal solution and the optimal value of P^R respectively.

Proposition 2.1 If an optimal solution $\mathbf{x}^{\sigma_0}$ of P^{σ_0} satisfies

$$E(\mathbf{a})^T\mathbf{x}^{\sigma_0}+K_\alpha\{\sigma_0^2+(\mathbf{x}^{\sigma_0 T}W\mathbf{x}^{\sigma_0})^{\frac{1}{2}}\}\leq E(b),$$

then $\mathbf{x}^{\sigma_0}$ becomes an optimal solution of P.

Proof: Since any $\mathbf{x}\in S$ satisfies

$$K_\alpha(\sigma_0^2+\mathbf{x}^TW\mathbf{x})^{\frac{1}{2}}\geq K_\alpha\sigma_0,$$

P^{σ_0} is a relaxation problem of P. Therefore by the assumption $\mathbf{x}^{\sigma_0}\in S$, it is clear that $\mathbf{x}^{\sigma_0}$ is also an optimal solution of P. ☐

Proposition2.2 If $\mathbf{x}^R$ satisfies

$$E(\mathbf{a})^T\mathbf{x}^R<E(b)-K_\alpha R,$$

that is, there exists a gap between $E(b)-K_\alpha R$ and $E(\mathbf{a})^T\mathbf{x}^R$, then $\mu^R=E(\mathbf{c})^T\mathbf{x}^R\geq\mu^*$ holds.

Proof: Assume $\mu^R<\mu^*$, then

$$E(\mathbf{a})^T\mathbf{x}^*>E(b)-K_\alpha R$$

holds, for otherwise $\mathbf{x}^*$ is feasible for P^R and $\mu^R\geq\mu^*=E(\mathbf{c})^T\mathbf{x}^*$ holds. Now consider $\bar{\mathbf{x}}^\lambda\triangleq\lambda\mathbf{x}^R+\bar{\lambda}\mathbf{x}^*$. Then

$$\begin{aligned}E(b)-E(\mathbf{a})^T\bar{\mathbf{x}}^\lambda-K_\alpha R&=\lambda(E(b)-E(\mathbf{a})^T\mathbf{x}^R-K_\alpha R)+\bar{\lambda}(E(b)-E(\mathbf{a})^T\mathbf{x}^*-K_\alpha R)\\&=\lambda S^R+\bar{\lambda}S^*=\lambda(S^R-S^*)+S^*\end{aligned}$$

holds, where $S^R\triangleq E(b)-E(\mathbf{a})^T\mathbf{x}^R-K_\alpha R<0$ and $S^*\triangleq E(b)-E(\mathbf{a})^T\mathbf{x}^*-K_\alpha R>0$. If λ is taken to be $1>\lambda>-S^*/(S^R-S^*)>0$, then $E(b)-E(\mathbf{a})^T\mathbf{x}^\lambda-K_\alpha R>0$ and $A_1\mathbf{x}^\lambda\geq B_1$, $\mathbf{x}^\lambda\geq\mathbf{0}$, and therefore $\mathbf{x}^\lambda$ is feasible for P^R. Besides,

$$E(\mathbf{c})^T\mathbf{x}^\lambda=\lambda E(\mathbf{c})^T\mathbf{x}^R+\bar{\lambda}E(\mathbf{c})^T\mathbf{x}^*=\lambda\mu^R+\bar{\lambda}\mu^*>\mu^R$$

and it contradicts the optimality of $\mathbf{x}^R$. Therefore $\mu^R \geq \mu^*$ results. □

Property 2.7 μ^R is a nonincreasing function of R.

Proof: As R increases, the feasible region of P^R reduces. Therefore μ^R is a nonincreasing function of R. □

Property 2.8 μ^R is a concave function of R.

Proof: For $R_1 > R_2$ and $1 \geq \lambda \geq 0$, let $\bar{R}_\lambda \triangleq \lambda R_1 + \bar{\lambda} R_2$. Then

$$E(\mathbf{a})^T(\lambda \mathbf{x}^{R_1} + \bar{\lambda}\mathbf{x}^{R_2}) \leq \lambda(E(b) - K_\alpha R_1) + \bar{\lambda}(E(b) - K_\alpha R_2)$$

$$= E(b) - K_\alpha \bar{R}_\lambda,$$

and $A_1(\lambda \mathbf{x}^{R_1} + \bar{\lambda}\mathbf{x}^{R_2}) \leq \lambda B_1 + \bar{\lambda} B_1 = B_1$, $\lambda \mathbf{x}^{R_1} + \bar{\lambda}\mathbf{x}^{R_2} \geq 0$

hold, i.e., $\lambda \mathbf{x}^{R_1} + \bar{\lambda}\mathbf{x}^{R_2}$ is feasible for $P^{\bar{R}_\lambda}$. Since

$$\lambda\mu^{R_1} + \bar{\lambda}\mu^{R_2} = E(\mathbf{c})^T(\lambda \mathbf{x}^{R_1} + \bar{\lambda}\mathbf{x}^{R_2})$$

and $\lambda\mu^{R_1} + \bar{\lambda}\mu^{R_2} \leq \mu^{\bar{R}_\lambda} = E(\mathbf{c})^T \mathbf{x}^{\bar{R}_\lambda}$

hold from the optimality of $\mathbf{x}^{\bar{R}_\lambda}$ for $P^{\bar{R}_\lambda}$. Therefore μ^R is a concave function of R. □

Now let $R^* \triangleq (\mathbf{x}^{*T} W \mathbf{x}^* + \sigma_0^2)^{\frac{1}{2}}$, then $\mathbf{x}^*$ is feasible for P^{R^*} and so $\mu^* \leq \mu^{R^*}$ follows. By Property 2.8, Property 2.7 is strengthened as follows.

Property 2.7' Except a first portion μ^R is a monotonically decreasing function of R.

Figure 2.2 and Figure 2.3 show the shapes of $z(\mu)$ and μ^R respectively. Note that the optimal value of $P^{R(\mu)}$ is not less than μ since $\mathbf{x}(\mu)$ is a feasible solution of $P^{R(\mu)}$. Now we are ready to describe our main algorithm for solving P. In the algorithm, the

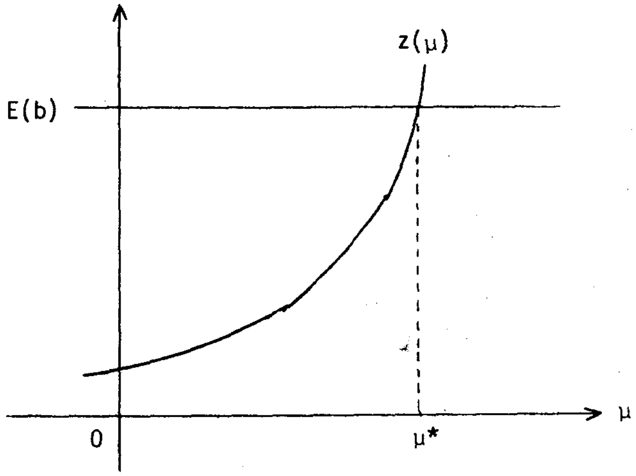

Figure 2.2. $z(\mu)$ v.s. μ

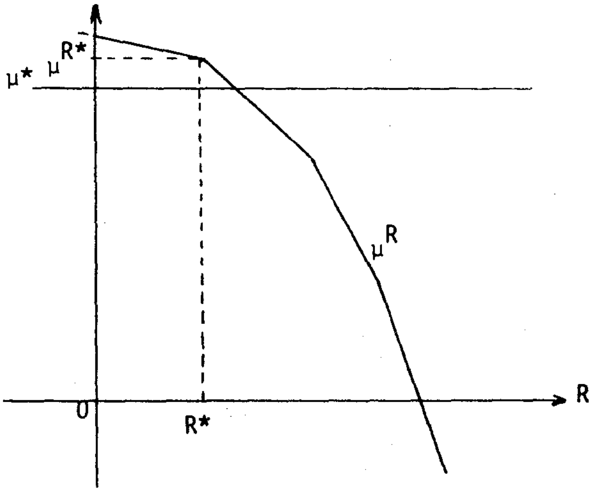

Figure 2.3. μ^R v.s. R

following notations are used.

μ_c; current μ, $\bar{\mu}$; an upper bound of μ^*, $R(\mathbf{x}) \triangleq (\sigma_0^2 + \mathbf{x}^T W \mathbf{x})^{\frac{1}{2}}$,

B_c; basic matrix corresponding to the current optimal solution,

β_c; current solution of Q-equation, $\mathbf{x}' \triangleq \mathbf{x}(\mu'_{B_c})$.

Algorithm 2.2

Step 0: Calculate $\tilde{\mu}$, solve $P(\tilde{\mu})$ and find $\mathbf{x}(\tilde{\mu})$ and $z(\tilde{\mu})$ by using Algorithm 2.1. If $z(\tilde{\mu}) \leq E(b)$, then set $\mathbf{x}^* \leftarrow \mathbf{x}(\tilde{\mu})$ and terminate. Otherwise set $\bar{\mu} \leftarrow \tilde{\mu}$, $\underline{R} \leftarrow \sigma_0$ and $\mu_c \leftarrow (-M)$ (M is a sufficiently large number). Go to Step 1.

Step 1: Solve $P(\mu_c)$ and find $\mathbf{x}(\mu_c)$, optimal basis B_c and $I(B_c)$. If $\mu^* \in I(B_c)$, then $\mathbf{x}^* \leftarrow \beta_c(\mu^*)\mathbf{d}_{B_c} + \mu^* \mathbf{e}_{B_c} + \mathbf{g}_{B_c}$ and terminate. If $\mu^* \notin I(B_c)$ and $\bar{\mu}_{B_c} > \mu^*$, then go to Step 2. If $\mu^* \notin I(B_c)$ and $\bar{\mu}_{B_c} < \mu^*$ (in this case $\bar{\mu}_{B_c} = \mu'_{B_c}$), then go to Step 3.

Step 2: If $\bar{\mu} > \bar{\mu}_{B_c}$, then set $\bar{\mu} \leftarrow \bar{\mu}_{B_c}$ and

$$\mu_c \longleftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\mu'_{B_c}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})}$$

and return to Step 1. If $\bar{\mu} \leq \bar{\mu}_{B_c}$ and $R(\mathbf{x}') > \underline{R}$, then set $\underline{R} \leftarrow R(\mathbf{x}')$, solve $P^{R(\mathbf{x}')}$ and calculate $\mu^{R(\mathbf{x}')}$. Go to Step 3. If $\bar{\mu} \leq \bar{\mu}_{B_c}$ and $\underline{R} \geq R(\mathbf{x}')$, then set

$$\mu_c \longleftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu} z(\bar{\mu}) + \mu'_{B_c} z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})}$$

and return to Step 1.

Step 3: Solve $P(\mu^{R(\mathbf{x}')})$ and calculate $z(\mu^{R(\mathbf{x}')})$. If $E(b) > z(\mu^{R(\mathbf{x}')})$ and

$$\frac{(\bar{\mu}-\mu'_{B_c})E(b)-\bar{\mu}z(\bar{\mu})+\mu'_{B_c}z(\bar{\mu})}{z(\bar{\mu})-z(\mu'_{B_c})} < \mu^{R(\mathbf{x}')},$$

then set $\mu_c \leftarrow \mu^{R(\mathbf{x}')}$ and return to Step 1. If $E(b)=z(\mu^{R(\mathbf{x}')})$, then set $\mathbf{x}^* \leftarrow \mathbf{x}(\mu^{R(\mathbf{x}')})$ and terminate. Otherwise set

$$\mu_c \leftarrow \frac{(\bar{\mu}-\mu'_{B_c})E(b)-\bar{\mu}z(\mu'_{B_c})+\mu'_{B_c}z(\bar{\mu})}{z(\bar{\mu})-z(\mu'_{B_c})}$$

and return to Step 1.

Theorem 2.6 Algorithm 2.2 finds $\mathbf{x}^*$ at finite iterations.

Proof: (Finiteness) Each $P^R(\mu)$ has the same constraint condition $KTP^R(\mu)$ except parametrized right hand side with respect to R and μ. The number of basic matrices satisfying nonnegativity and complementary condition is finite, and by the theory of parametric quadratic programming, $R(\mu)$ corresponds to an optimal basis $B=B(\mu)$. That is, μ is divided into $I(B)$'s determined by basic matrix B. Algorithm 2.2 searches for μ^* among those regions $I(B)$ at most once for each B. Therefore finiteness of Algorithm 2.2 follows from finiteness of the number of $I(B)$.
(Validity) Theorem 2.2 assures the condition $\bar{s}_0(\bar{\mu})=0$ in Theorem 2.1. Termination condition that $z(\mu)=E(b)$ assures validity by Theorem 2.1. □

2.6 An Example

We consider the following example P_0.

P_0: Maximize $E(c_1x_1+c_2x_2)$

subject to $\Pr\{a_1x_1+a_2x_2 \leq b\} \geq 0.7$,

$$3x_1 + 2x_2 \leq 18, \quad x_1 + 2x_2 \leq 10, \quad x_1, x_2 \geq 0,$$

where $E(\mathbf{c}) = (8,6)^T$, $E(b) = 32$, $\sigma_0 = 4$, $E(\mathbf{a}) = (5,6)^T$ and $W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. P_0 is transformed into the following deterministic equivalent problem P.

P: Maximize $8x_1 + 6x_2$

subject to $5x_1 + 6x_2 + 0.5(16 + x_1^2 + x_2^2)^{\frac{1}{2}} \leq 32$,

$$3x_1 + 2x_2 \leq 18, \quad x_1 + 2x_2 \leq 10, \quad x_1, \ x_2 \geq 0.$$

<u>Step 0:</u> $\tilde{\mu} \overset{\Delta}{=} \max\{8x_1 + 6x_2 | 3x_1 + 2x_2 \leq 18, \ x_1 + 2x_2 \leq 10, \ x_1, \ x_2 \geq 0\} = 48$. Solve $P(\tilde{\mu})$ and find $x(\tilde{\mu})$ and $z(\tilde{\mu})$.

$P(\tilde{\mu})$: Minimize $5x_1 + 6x_2 + 0.5(16 + x_1^2 + x_2^2)^{\frac{1}{2}}$

subject to $8x_1 + 6x_2 \geq 48 \ (= \tilde{\mu})$, $3x_1 + 2x_2 \leq 18$,

$$x_1 + 2x_2 \leq 10, \quad x_1, \ x_2 \geq 0.$$

<u>Algorithm 1</u>

<u>Step 1:</u> Set $R_\ell \leftarrow 4$, $R_u \leftarrow M$ and $R \leftarrow 5$.

$P^R(\tilde{\mu})$: Minimize $R(5x_1 + 6x_2) + 0.25(16 + x_1^2 + x_2^2)$

subject to $8x_1 + 6x_2 \geq 48$, $3x_1 + 2x_2 \leq 18$,

$$x_1 + 2x_2 \leq 10, \quad x_1, \ x_2 \geq 0.$$

$KTP^R(\tilde{\mu})$: $\hat{v}_1 - 0.5x_1 + 8\hat{r} - 3\hat{q}_1 - \hat{q}_2 = 5R$, $\hat{v}_2 - 0.5x_2 + 6\hat{r} - 2\hat{q}_1 - 2\hat{q}_2 = 6R$,

$$3x_1 + 2x_2 + \hat{s}_1 = 18, \quad x_1 + 2x_2 + \hat{s}_2 = 10, \quad 8x_1 + 6x_2 - \hat{s}_0 = \tilde{\mu} (= 48)$$

$$x_1\hat{v}_1 + x_2\hat{v}_2 + \hat{s}_1\hat{q}_1 + \hat{s}_2\hat{q}_2 + \hat{r}\hat{s}_0 = 0,$$

$$x_1,\ x_2,\ \hat{v}_1,\ \hat{v}_2,\ \hat{s}_1,\ \hat{s}_2,\ \hat{q}_1,\ \hat{q}_2,\ \hat{r},\ \hat{s}_0 \geq 0.$$

$X^R(\mu)$ is given as follows: $x_1 = \tilde{\mu}/8$, $x_2 = 0$, $\hat{v}_1 = 0$, $\hat{v}_2 = 9R/4 - 3\tilde{\mu}/64$, $\hat{s}_1 = 18 - 3\tilde{\mu}/8$, $\hat{s}_2 = 10 - \tilde{\mu}/8$, $\hat{s}_0 = 0$, $\hat{q}_1 = \hat{q}_2 = 0$, $\hat{r} = 5R/8 + \tilde{\mu}/128$.

$$B = \begin{array}{c} \begin{array}{ccccc} x_1 & \hat{v}_2 & \hat{s}_1 & \hat{s}_2 & \hat{r} \end{array} \\ \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},\quad \mathbf{d}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\quad \mathbf{e}_B = \begin{pmatrix} 1/8 \\ 0 \end{pmatrix},\quad \mathbf{g}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$R \geq \tilde{\mu}/48$ $(=L_B(\tilde{\mu}))$.

<u>Step 2:</u> $K^{\mu}(L_B(\tilde{\mu})) = 16 + \frac{\tilde{\mu}^2}{64} - \frac{\tilde{\mu}^2}{(48)^2} > 0.$

<u>Step 3:</u> $K^{\mu}(U_B(\tilde{\mu})) < 0$. Therefore $R(\tilde{\mu})$ exists on $[\frac{\tilde{\mu}}{48}, \infty)$ and given as follows.

$$R(\tilde{\mu}) = (16 + \tilde{\mu}^2/64)^{\frac{1}{2}}.$$

<u>Step 5:</u> $\mathbf{x}(\tilde{\mu}) = \begin{pmatrix} \tilde{\mu}/8 \\ 0 \end{pmatrix}$. Return to Main Algorithm.

Since $z(\tilde{\mu}) = 5\tilde{\mu}/8 + 0.5\sqrt{16 + \tilde{\mu}^2/64} = 30 + \sqrt{13} > 32 = E(b)$, set $\bar{\mu} \longleftarrow 48 (= \tilde{\mu})$ and $\underline{R} \longleftarrow 4\ (= \sigma_0)$. Go to Step 1.

$$P(\mu_c):\quad \text{Minimize } 5x_1 + 6x_2 + 0.5\sqrt{16 + x_1^2 + x_2^2}$$

$$\text{subject to } 8x_1 + 6x_2 \geq \mu_c\ (= -M),\ 3x_1 + 2x_2 \leq 18,$$

$$x_1 + 2x_2 \leq 10,\ x_1,\ x_2 \geq 0.$$

Using Algorithm 2.1, we obtain $\mathbf{d}_B = \mathbf{e}_B = \mathbf{g}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $I(B_c) = (-\infty, 0)$. Therefore $\bar{\mu}_{B_c} = \mu'_{B_c} = 0$ and $z(\mu) = 2$ on $I(B_c)$. $\bar{\mu}_{B_c} < \mu^*$, i.e., $\mu^* \notin I(B_c)$. Go to Step 3.

Step 3: $\mathbf{x}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $R(\mathbf{x}') = \sqrt{16} = 4$, $\mu^{R(\mathbf{x}')} = \mu^{4} = \mu^{\sigma_0} = 48$. Since $z(48) > E(b) = 32$,

$$\mu_c \longleftarrow \frac{(\bar{\mu} - \mu'_{B_c})E(b) - \bar{\mu}z(\mu'_{B_c}) + \mu'_{B_c}\, z(\bar{\mu})}{z(\bar{\mu}) - z(\mu'_{B_c})} \cong 45.5616.$$

Return to Step 1.

Step 1: Solving $P(\mu_c)$, we obtain $X(\mu_c)$ given as follows.

$X(\mu_c)$: $x_1 = \mu_c/8$, $x_2 = 0$, $\hat{v}_1 = 0$, $\hat{v}_2 = 9R/4 - 3\mu_c/64$, $\hat{s}_1 = 18 - 3\mu_c/8$, $\hat{s}_2 = 10 - \mu_c/8$, $\hat{s}_0 = 0$, $\hat{q}_1 = \hat{q}_2 = 0$, $\hat{r} = 5R/8 + \mu_c/128$, $(R \geq \mu_c/48 = L_B(\mu_c))$.

$R(\mu_c) = \sqrt{16 + \mu_c^2/64}$, $z(\mu_c) = 5\mu_c/8 + 0.5\sqrt{16 + \mu_c^2/64}$, $\mathbf{d}_{B_c} = \mathbf{g}_{B_c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_{B_c} = \begin{pmatrix} 1/8 \\ 0 \end{pmatrix}$, $I(B_c) = \{\mu \,|\, 0 \leq \mu \leq 48\}$. Obviously $\mu^* \in I(B_c)$. Thus we solve

$$z(\mu) = 5\mu/8 + 0.5\sqrt{16 + \mu_c^2/64} = 32$$

and obtain $\mu^* \cong 45.62$ and $\mathbf{x}^* = \begin{pmatrix} \mu^*/8 \\ 0 \end{pmatrix} = \begin{pmatrix} 5.70 \\ 0 \end{pmatrix}$. $(E(\mathbf{c})^T \mathbf{x}^* = \mu^*.)$

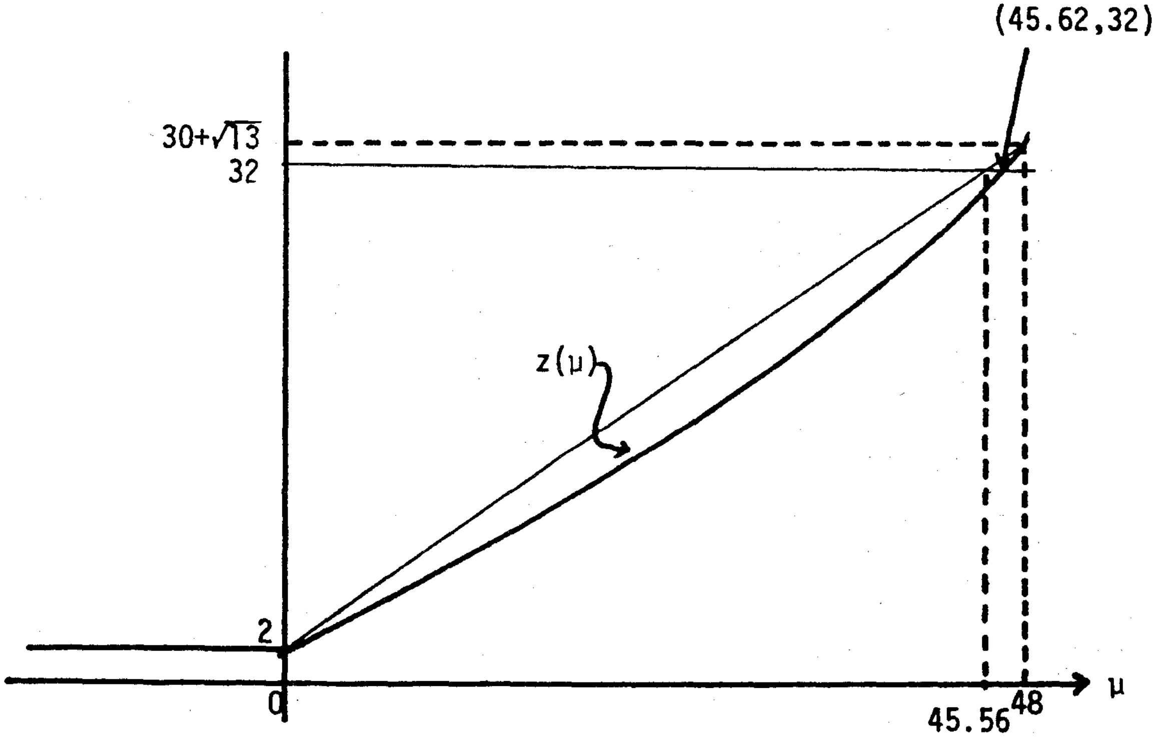

Figure 2.4. z(μ) v.s. μ

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CHAPTER 3

A STOCHASTIC TRANSPORTATION PROBLEM

3.1 Introduction

In this chapter we consider a stochastic transportation problem with simple recourse. Stochastic programs with recourse have been investigated by many authors ([4, 5, and 9]), but there are few papers giving the exact algorithms to obtain the optimal solution for them.

The main purpose of this chapter is to derive a new algorithm for obtaining the optimal solution of stochastic transportation problem. In the following section, two-stage formulation of this problem and some properties of its optimal solution and optimal value are described. In Section 3.3 we investigate the behavior of the objective function. In Section 3.4 we give an algorithm for obtaining the optimal solution. And in Section 3.5 an illustrative example is given.

3.2 Two-Stage Formulation and Some Properties

Suppose that there are m-sources and n-destinations, then the following transportation problem TP_0 is considered.

$$TP_0: \text{ Minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i=1,2,\ldots,m, \tag{3.1}$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j=1,2,\ldots,n, \tag{3.2}$$

$$x_{ij} \geq 0, \ i=1,2,\ldots,m; \ j=1,2,\ldots,n,$$

where

x_{ij} is the quantity of items shipping from source i to destination j.
c_{ij} is the shipping cost per unit from source i to destination j,
a_i is the quantity of items supplied at source i,
b_j is the positive random demand at destination j, whose marginal distribution function is $F_j(\cdot)$.

Assume that a_i and c_{ij} are positive. Since each b_j is a random variable, the constraints (3.2) may not be satisfied. For the discrepancy of each j-th constraint we impose the penalty p_j per unit. And in addition we consider the penalty for oversupplying to the j-th destination, which is denoted by q_j. Then the following two stage problem TP_1 is considered.

$$TP_1: \quad \text{Minimize} \ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + E[\min \sum_{j=1}^{n} (p_j y_j^+ + q_j y_j^-)]$$

$$\text{subject to} \ \sum_{j=1}^{n} x_{ij} \leq a_i, \ i=1,2,\ldots,m,$$

$$\sum_{i=1}^{m} x_{ij} + y_j^+ - y_j^- = b_j, \ j=1,2,\ldots,n,$$

$$x_{ij}, \ y_j^+, \ y_j^- \geq 0, \ i=1,2,\ldots,m; \ j=1,2,\ldots,n,$$

where y_j^+ and y_j^- are the undersupply and the oversupply to the j-th destination respectively, which are represented as follows; for $j=1,2,\ldots,n$,

$$y_j^+ = b_j - \sum_{i=1}^{m} x_{ij} \ \text{and} \ y_j^- = 0 \ \text{if} \ b_j > \sum_{i=1}^{m} x_{ij},$$

$$y_j^+ = 0 \ \text{and} \ y_j^- = \sum_{i=1}^{m} x_{ij} - b_j \ \text{if otherwise.}$$

Let X* denote the optimal solution of Problem TP_1. According to the well-known results of two-stage programming under uncertainty

([6, and 8]), Problem TP_1 can be rewritten as the following problem TP_2:

$$TP_2: \quad \text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} \{p_j \int_{u_j}^{\infty} (b_j - u_j)dF_j(b_j) + q_j \int_{-\infty}^{u_j}(u_j - b_j)dF_j(b_j)\}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i=1,2,\ldots,m, \tag{3.3}$$

$$\sum_{i=1}^{m} x_{ij} \geq u_j, \quad j=1,2,\ldots,n, \tag{3.4}$$

$$x_{ij} \geq 0, \quad i=1,2,\ldots,m; \; j=1,2,\ldots,n.$$

In the inequalities (3.4) the equalities hold at the optimal solution. In order to solve Problem TP_2, we consider the following transportation type problem TP_3.

$$TP_3: \quad \text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i=1,2,\ldots,m,$$

$$\sum_{i=1}^{m} x_{ij} \geq u_j, \quad j=1,2,\ldots,n,$$

$$x_{ij} \geq 0, \quad i=1,2,\ldots,m; \; j=1,2,\ldots,n.$$

Let $\mathbf{x}^*(\mathbf{u}) = (x^*_{ij}(\mathbf{u}),\ i=1,2,\ldots,m;\ j=1,2,\ldots,n)$ denote the optimal solution of Problem TP_3. Then the following property holds.

Property 3.1 For arbitrarily fixed u_j $(j=1,2,\ldots,n)$, Problem TP_2 has the same solution as Problem TP_3.

Proof: For arbitrarily fixed u_j $(j=1,2,\ldots,n)$, the penalty term of Problem TP_2 is a constant. Therefore Problem TP_2 is equivalent to Problem TP_3. ☐

Furthermore we consider the following dual problem TP_4 of Problem TP_3.

$$TP_4: \quad \text{Maximize } -\sum_{i=1}^{m} a_i\alpha_i + \sum_{j=1}^{n} u_j\beta_j$$

$$\text{subject to } -\alpha_i + \beta_j \leq c_{ij}, \ i=1,2,\dots,m; \ j=1,2,\dots,n,$$

$$\alpha_i, \ \beta_j \geq 0, \ i=1,2,\dots,m; \ j=1,2,\dots,n.$$

Let $\alpha_i^*(\mathbf{u})$, $\beta_j^*(\mathbf{u})$ $(i=1,2,\dots,m; \ j=1,2,\dots,n)$ denote the optimal solution of Problem TP_4. Then from the duality theorem of Linear Programming ([1]) in case that both the problems TP_3 and TP_4 have finite solutions,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}^*(\mathbf{u}) = -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}) + \sum_{j=1}^{n} u_j\beta_j^*(\mathbf{u}) \tag{3.5}$$

holds. Moreover the following Property 3.2, Property 3.3 and Corollary 3.1 hold.

Property 3.2 The optimal value of Problem TP_4 is piecewise linear and nondecreasing convex function of u_j $(j=1,2,\dots,n)$.

Proof: For any $\mathbf{u}'$ and $\mathbf{u}''$, we define

$$\mathbf{u}^\lambda \overset{\Delta}{=} \lambda\mathbf{u}' + (1-\lambda)\mathbf{u}'', \text{ for } 0 \leq \lambda \leq 1.$$

From the optimality of $(\boldsymbol{\alpha}^*(\mathbf{u}), \boldsymbol{\beta}^*(\mathbf{u})) \overset{\Delta}{=} (\alpha_1^*(\mathbf{u}), \dots, \alpha_m^*(\mathbf{u}), \beta_1^*(\mathbf{u}), \dots, \beta_n^*(\mathbf{u}))$

$$-\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}') + \sum_{j=1}^{n} u_j'\beta_j^*(\mathbf{u}') \geq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}^\lambda) + \sum_{j=1}^{n} u_j'\beta_j^*(\mathbf{u}^\lambda) \tag{3.6}$$

$$-\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}'') + \sum_{j=1}^{n} u_j''\beta_j^*(\mathbf{u}'') \geq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}^\lambda) + \sum_{j=1}^{n} u_j''\beta_j^*(\mathbf{u}^\lambda) \tag{3.7}$$

hold. From (3.6) and (3.7)

$$\lambda\{-\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}') + \sum_{j=1}^{n} u_j'\beta_j^*(\mathbf{u}')\} + (1-\lambda)\{-\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}'') + \sum_{j=1}^{n} u_j''\beta_j^*(\mathbf{u}'')\}$$

$$\geq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}^\lambda) + \sum_{j=1}^{n} (\lambda u_j' + (1-\lambda)u_j'')\beta_j^*(\mathbf{u}^\lambda)$$

holds. This proves the covexity of the optimal value of Problem TP_4.

Now let $u_k^1 < u_k^2$ for some k, then

$$-\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}_k^2) + \sum_{\substack{j=1\\ \neq k}}^{n} u_j\beta_j(\mathbf{u}_k^2) + u_k^2\beta_k(\mathbf{u}_k^2)$$

$$\geq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}_k^1) + \sum_{\substack{j=1\\ \neq k}}^{n} u_j\beta_j(\mathbf{u}_k^1) + u_k^2\beta_k(\mathbf{u}_k^1)$$

$$\geq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}_k^1) + \sum_{\substack{j=1\\ \neq k}}^{n} u_j\beta_j(\mathbf{u}_k^1) + u_k^1\beta_k(\mathbf{u}_k^1)$$

from the nonnegativity of β_k, where $\mathbf{u}_k^1 = (u_1,\ldots,u_k^1,\ldots,u_n)$ and $\mathbf{u}_k^2 = (u_1,\ldots,u_k^2,\ldots,u_n)$. Therefore the optimal value of Problem TP_4 is nondecreasing function of u_j. According to the theory of sensitivity analysis of linear programming the optimal solution $(\boldsymbol{\alpha}^*(\mathbf{u}),\boldsymbol{\beta}^*(\mathbf{u}))$ does not change in some region of $\mathbf{u}$. Hence the optimal value of Problem TP_4 is a piecewise linear function of $\mathbf{u}$. □

Property 3.3 The objective function of Problem TP_2 is a convex function of $\mathbf{u}$. If the distribution of b_j is continuous, it is a strictly convex function of $\mathbf{u}$.

Proof: See the reference [8]. □

Consequently instead of the objective function in Problem TP_2, it is sufficiently to consider the following objective function hereafter.

$$G(\mathbf{u}) = -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}) + \sum_{j=1}^{n} u_j\beta_j^*(\mathbf{u}) + \sum_{j=1}^{n} \{p_j\int_{u_j}^{\infty} (b_j-u_j)dF_j(b_j)$$

$$+q_j\int_{-\infty}^{u_j}(u_j-b_j)dF_j(b_j)\}. \qquad (3.8)$$

Let $\mathbf{u}^*$ denote the minimum solution of $G(\mathbf{u})$. From Property 3.2 and Property 3.3, the following Corollary 3.1 holds.

Corollary 3.1 $G(\mathbf{u})$ is a convex function of $\mathbf{u}$. If the distribution of b_j is continuous, it is a strictly convex function of $\mathbf{u}$.

We can solve Problem TP_2 because $\mathbf{u}$ giving the minimum of $G(\mathbf{u})$ is identical with one giving the minimum of Problem TP_2. Actual optimal solution may be found by solving Problem TP_3 with this $\mathbf{u}$, and so we may consider that this $\mathbf{u}$ is also the optimal solution hereafter.

3.3 Behavior of $G(\mathbf{u})$

In this section we investigate the behavior of $G(\mathbf{u})$ in detail. The domain of $\mathbf{u}$ can be divided into several regions so that the optimal solution of Problem TP_4 does not change in each of them. Though $G(\mathbf{u})$ is continuous, it is not necessarily differentiable on the boundary of each region. Let $A(\mathbf{u})$ denote the region in which $\alpha_i^*(\mathbf{u})$ and $\beta_j^*(\mathbf{u})$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$ do not change. The finiteness of the number of these regions is assured ([1]). Each boundary may be contained in several regions. If we choose an arbitrary $\mathbf{u}^0$, the function $G(\mathbf{u})$ is differentiable in the region $A(\mathbf{u}^0)$. (If $\mathbf{u}^0$ is on the boundary, $A(\mathbf{u}^0)$ is one of the regions containing the boundary.) Moreover the following function

$$G^0(\mathbf{u};\mathbf{u}^0) \triangleq -\sum_{i=1}^{m} a_i\alpha_i^*(\mathbf{u}^0) + \sum_{j=1}^{n} u_j\beta_j^*(\mathbf{u}^0) + \sum_{j=1}^{m}\{p_j\int_{u_j}^{\infty}(b_j-u_j)dF_j(b_j)$$

$$+q_j\int_{-\infty}^{u_j}(u_j-b_j)dF_j(b_j)\} \qquad (3.9)$$

is differentiable over the whole domain of $\mathbf{u}$. Then $G^0(\mathbf{u};\mathbf{u}^0)$ coincides with $G(\mathbf{u})$ in the region $A(\mathbf{u}^0)$. The partial derivative of $G^0(\mathbf{u};\mathbf{u}^0)$ with respect to u_j is given as follows:

$$\frac{\partial G^0(\mathbf{u};\mathbf{u}^0)}{\partial u_j} = \beta_j^*(\mathbf{u}^0) + (p_j + q_j)F_j(u_j) - p_j. \tag{3.10}$$

Note that (3.10) is the equation of u_j only. Therefore the n-dimensional vector $\mathbf{u}^*(\mathbf{u}^0)$ giving the minimum of $G^0(\mathbf{u};\mathbf{u}^0)$ becomes as follows:

$$u_j^*(\mathbf{u}^0) = \begin{cases} \max\{u_j \mid u_j \in D_j\}, \text{ if } \beta_j^*(\mathbf{u}^0) < -q_j, \\ \min\{u_j \mid F_j(u_j) \geq \dfrac{p_j - \beta_j^*(\mathbf{u}^0)}{p_j + q_j}\}, \text{ if } -q_j \leq \beta_j^*(\mathbf{u}^0) < p_j, \\ \min\{u_j \mid u_j \in D_j\}, \text{ otherwise,} \end{cases} \tag{3.11}$$

where $D_j \overset{\Delta}{=} \{u_j \mid \mathbf{x}^*(\mathbf{u}) \text{ is feasible}\}$. Then the following theorem holds.

Theorem 3.1 If $\mathbf{u}^*(\mathbf{u}^0)$ belongs to $A(\mathbf{u}^0)$, then it also gives the minimum of $G(\mathbf{u})$.

Proof: From the minimality of $\mathbf{u}^*(\mathbf{u}^0)$ and the optimality of $(\boldsymbol{\alpha}^*(\mathbf{u}), \boldsymbol{\beta}^*(\mathbf{u}))$, the following inequalities hold.

$$G^0(\mathbf{u}^*(\mathbf{u}^0);\mathbf{u}^0) \leq G^0(\mathbf{u};\mathbf{u}^0) \leq G(\mathbf{u}),$$

where the second equality holds at $\mathbf{u} \in A(\mathbf{u}^0)$. Since $\mathbf{u}^*(\mathbf{u}^0) \in A(\mathbf{u}^0)$, $G^0(\mathbf{u}^*(\mathbf{u}^0);\mathbf{u}^0) = G(\mathbf{u}^*(\mathbf{u}^0))$ holds. Therefore

$$G^0(\mathbf{u}^*(\mathbf{u}^0);\mathbf{u}^0) = \min G^0(\mathbf{u};\mathbf{u}^0) \leq \min G(\mathbf{u})$$

since $G^0(\mathbf{u};\mathbf{u}^0)$ is a lower approximation function of $G(\mathbf{u})$. □

By testing whether $(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^0)), \boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^0))) = (\boldsymbol{\alpha}^*(\mathbf{u}^0), \boldsymbol{\beta}^*(\mathbf{u}^0))$ or not, we can know whether $\mathbf{u}^*(\mathbf{u}^0)$ exists in $A(\mathbf{u}^0)$ or not. Even if $(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^0)), \boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^0))) \neq (\boldsymbol{\alpha}^*(\mathbf{u}^0), \boldsymbol{\beta}^*(\mathbf{u}^0))$ for any $\mathbf{u}^0 \in D$, the following informations about the optimal solution of $G(\mathbf{u})$ can be obtained from the theory of convex programming ([2]).

Theorem 3.2 If $\mathbf{u}^*(\mathbf{u}^0)$ does not belong to $A(\mathbf{u}^0)$ for any $\mathbf{u}^0 \in D$, then $\mathbf{u}$ giving the minimum of $G(\mathbf{u})$ is attained on one of the boundaries.

Proof: Since $G(\mathbf{u})$ is convex and D is closed, there exists an optimal solution. Suppose that this optimal solution $\mathbf{u}^*$ does not exist on any boundary. Then $\mathbf{u}^*$ belongs to the interior of some $A(\mathbf{u}^0)$. Therefore $\mathbf{u}^*$ is the minimal solution of $G^0(\mathbf{u};\mathbf{u}^0)$, i.e., $\mathbf{u}^*(\mathbf{u}^0)$. This contradicts to the above assumption. ☐

3.4 Algorithm

In this section we give an algorithm utilizing the results in previous sections.

From Property 3.2 we obtain an upperbound for the optimal solution as the following. We define $Q_j(u_j)$ as follows:

$$Q_j(u_j) \overset{\Delta}{=} p_j \int_{u_j}^{\infty} (b_j-u_j)dF_j(b_j) + q_j \int_{-\infty}^{u_j} (u_j-b_j)dF_j(b_j). \qquad (3.12)$$

We define the minimum solution of $Q_j(u_j)$ by u_j^s. Then u_j^s can be given as follows:

$$u_j^s = \min\{u_j \mid F_j(u_j) \geq \frac{p_j}{p_j+q_j}\}.$$

From (3.8) and (3.12) we have

$$G(\mathbf{u}) = -\sum_{i=1}^{m} a_i \alpha_i^*(\mathbf{u}) + \sum_{j=1}^{n} u_j \beta_j^*(\mathbf{u}) + \sum_{j=1}^{n} Q_j(u_j). \qquad (3.13)$$

Then the following property holds.

Property 3.4 For any j, $u_j^* \leq u_j^s$.

Proof: This property is easily proved from Property 3.2 and the equation (3.13). ☐

Now we propose an algorithm based on the above properties. First we define subgradient $\partial G_j (j=1,2,\dots,n)$ as follows:

$$\partial G_j(\mathbf{u}) \triangleq \{h \mid \lim_{\varepsilon \to -0} \frac{\partial G^0(\mathbf{u};\mathbf{u}+\boldsymbol{\epsilon}_j)}{\partial u_j} \leq h \leq \lim_{\varepsilon \to +0} \frac{\partial G^0(\mathbf{u};\mathbf{u}+\boldsymbol{\epsilon}_j)}{\partial u_j}\},$$

where $\boldsymbol{\epsilon}_j \triangleq (0,\dots,0,\overset{j}{\overset{\vee}{\varepsilon}},0,\dots,0)$.

Algorithm

Step 0: Set $\mathbf{u}^c \leftarrow \mathbf{u}^\dagger$ ($\mathbf{u}^\dagger$ is an arbitrary feasible solution so that $\mathbf{0} \leq \mathbf{u} \leq \mathbf{u}^s$). Solve Problem TP_4 with $\mathbf{u}=\mathbf{u}^c$.

Step 1: If every $\partial G_j(\mathbf{u}^c)$ contain 0, then $\mathbf{u}^* \leftarrow \mathbf{u}^c$ and go to Step 4. Otherwise solve Problem TP_4 with $\mathbf{u}=\mathbf{u}^*(\mathbf{u}^c)$ and go to Step 2.

Step 2: If $(\boldsymbol{\alpha}^*(\mathbf{u}^c),\boldsymbol{\beta}^*(\mathbf{u}^c)) = (\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^c)),\boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^c)))$, then $\mathbf{u}^c \leftarrow \mathbf{u}^*(\mathbf{u}^c)$ and go to Step 4. Otherwise go to Step 3.

Step 3: For j with the greatest value of $\min\{|h| \mid h \in \partial G_j(\mathbf{u}^c)\}$, find u_j' so that $\partial G_j(u_1^c,\dots,u_{j-1}^c,u_j',u_{j+1}^c,\dots,u_n^c)$ contains 0. Set $\mathbf{u}^c \leftarrow (u_1^c,\dots,u_{j-1}^c,u_j',u_{j+1}^c,\dots,u_n^c)$ and go to Step 1.

Step 4: Solve Problem TP_3 with $\mathbf{u}=\mathbf{u}^c$ (for example, by using an algorithm due to [3]) and terminate.

Theorem 3.3 This algorithm finds X* at finite iterations.

Proof: From the theory of sensitivity analysis the number of regions $A(\mathbf{u}^0)$ is finite. And the dimension of $\mathbf{u}$ is finite. Therefore the algorithm finds X* at finite iterations. □

3.5 An Example

We consider the following problem with 3-sources and 2-destinations (see Figure 3.1):

Minimize $c_{11}x_{11}+c_{12}x_{12}+c_{21}x_{21}+c_{22}x_{22}+c_{31}x_{31}+c_{32}x_{32}$

subject to $x_{11}+x_{12} \leq 6$,

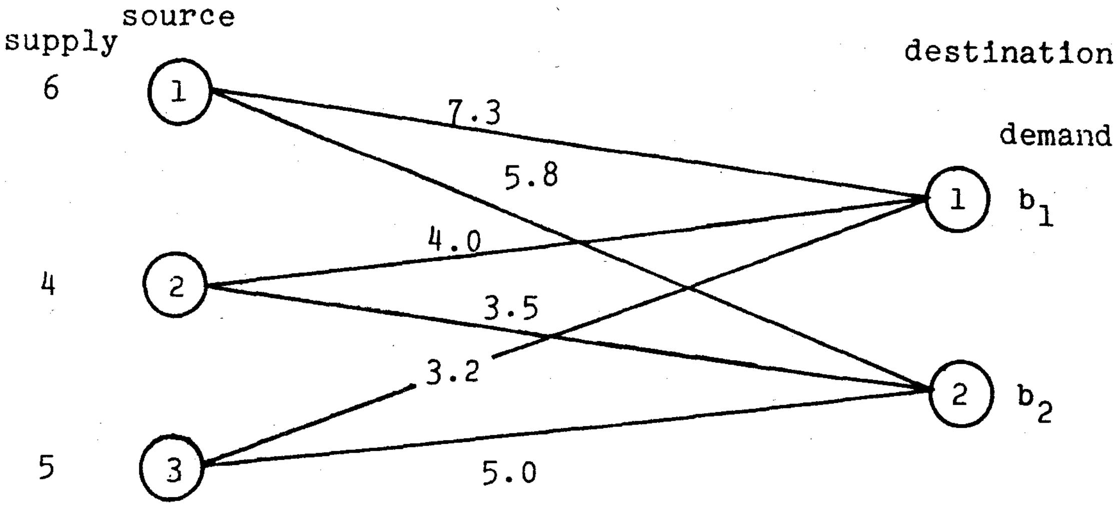

Figure 3.1. Example

$$x_{21}+x_{22}\leq 4,$$

$$x_{31}+x_{32}\leq 5,$$

$$x_{11}+x_{21}+x_{31}\geq b_1,$$

$$x_{12}+x_{22}+x_{32}\geq b_2,$$

$$x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{31},\ x_{32}\geq 0,$$

where
$$(c_{ij}) = \begin{pmatrix} 7.3 & 5.8 \\ 4.0 & 3.5 \\ 3.2 & 5.0 \end{pmatrix},$$

and each b_j has identical uniform distribution between 0 and 16. We assume $p_1=p_2=10$ and $q_1=q_2=6$. Then $(u_1^s,u_2^s)=(10,10)$.

Step 0: $\mathbf{u}^c=(6,6)$.

$(\boldsymbol{\alpha}^*(\mathbf{u}^c),\boldsymbol{\beta}^*(\mathbf{u}^c))=(0,2.3,3.1,6.3,5.8)$.

Step 1: $\partial G_1(\mathbf{u}^c)=\{2.3\}$ and $\partial G_2(\mathbf{u}^c)=\{1.8\}$.
$\mathbf{u}^*(\mathbf{u}^c)=(3.7,4.2)$ and $(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^c)),\boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^c)))=(0,1.5,0,3.2,5.0)$.

Step 2: $(\boldsymbol{\alpha}^*(\mathbf{u}^c),\ \boldsymbol{\beta}^*(\mathbf{u}^c))\neq(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^c)),\boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^c)))$.

Step 3: $u_1'=5$ and hence $\mathbf{u}^c=(5,6)$.
$(\boldsymbol{\alpha}^*(\mathbf{u}^c),\ \boldsymbol{\beta}^*(\mathbf{u}^c))=(0,2.3,3.1,6.3,5.8)$.

Step 1: $\partial G_1(\mathbf{u}^c)=\{h\,|\,-1.0\leq h\leq 1.3\}$ and $\partial G_2(\mathbf{u}^c)=\{1.8\}$. $\mathbf{u}^*(\mathbf{u}^c)=(5,4.2)$ and $(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^c)),\boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^c)))=(0,2.3,3.1,6.3,5.8)$.

Step 2: $(\boldsymbol{\alpha}^*(\mathbf{u}^c),\boldsymbol{\beta}^*(\mathbf{u}^c))=(\boldsymbol{\alpha}^*(\mathbf{u}^*(\mathbf{u}^c)),\boldsymbol{\beta}^*(\mathbf{u}^*(\mathbf{u}^c)))$, therefore $\mathbf{u}^c=(5,4.2)$.

Step 4:
$$X^*=\begin{pmatrix} 0 & 0.2 \\ 0 & 4 \\ 5 & 0 \end{pmatrix}.$$

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CHAPTER 4

STOCHASTIC SPANNING TREE PROBLEMS

4.1 Introduction

Until today the minimal spanning tree problem has been well studied and many efficient algorithms such as [3, 6, and 7] are known. This chapter generalizes it and proposes stochastic versions of minimal spanning tree problems where edge costs are not constant, but random variables.

Consider the construction of a communication network which connects some cities each other directly or indirectly. If each construction cost of line between one city and other city is deterministic, the problem becomes the minimal spanning tree problem as is well known. In reality, however, those costs vary with time, and so they can be considered as random variables. In this chapter we consider two types of problems. One problem is to find an optimal spanning tree and optimal budget under the condition that the probability with which total cost exceeds budget is below a certain level. Another problem is to find an optimal spanning tree and optimal satisficing probability level under the same chance constraint.

In Section 4.2 we consider the first type problem and propose a parametric type algorithm which finds an optimal spanning tree in $O(m^2n^2)$, where m and n are the number of edges and the number of vertices in a given graph G respectively. In Section 4.3 we consider the second type problem and propose another parametric type algorithm. Though the problem is complicated, the algorithm also finds an optimal solution in $O(m^2n^2)$ computational time.

4.2 Problem Formulation

Let $G=(N,E)$ denote undirected graph consisting of vertex set $N=\{v_1,v_2,\ldots,v_n\}$ and edge set $E=\{e_1,e_2,\ldots,e_m\}\subseteq N\times N$. Moreover cost c_j is attached to each edge e_j. Spanning tree $T=(N,S)$ of G is a partial graph satisfying the following conditions. (See [2] for details.)

(a) T has the same vertex set as G.

(b) $S\subseteq E$ and $|S|=n-1$, where $|S|$ denotes the cardinality of set S.

(c) T is connected.

Then T can be denoted with 0-1 variables x_1, $x_2,\ldots,$ x_m as follows:

$$T:\quad x_i=1 \text{ if } e_i\in S,$$

$$x_i=0 \text{ if } e_i\notin S.$$

Conversely, if $\{e_i|x_i=1\}$ becomes a spanning tree of G with vertex set N, $\mathbf{x}=(x_1,x_2,\ldots,x_m)$ is also called spanning tree hereafter in this paper.

Ordinary minimal spanning tree problem is to seek a spanning tree $\mathbf{x}$ minimizing $\sum_{j=1}^{m}c_jx_j$. In many real situations, however, c_j's are not constant but random variables. So we consider the following two types of stochastic minimal spanning tree problem.

Type (I): Specified Probability Level Model

Minimize f

subject to $\Pr\{\sum_{j=1}^{m}c_jx_j\leq f\}\geq\alpha$,

$$x_j = 0 \text{ or } 1, \ \mathbf{x}: \text{ spanning tree},$$

Type (II): Variable Probability Level Model

$$\text{Minimize } f - \lambda\alpha$$

$$\text{subject to } \Pr\{\sum_{j=1}^{m} c_j x_j \leq f\} \geq \alpha,$$

$$x_j = 0 \text{ or } 1, \ \mathbf{x}: \text{ spanning tree},$$

where each c_j is assumed to be distributed according to the normal distribution with mean μ_j and variance σ_j^2 and they are mutually independent. We assume that $1/2 < \alpha \leq 1$.

The chance constraint which is common to both problems is transformed as follows, if $\sum_{j=1}^{m} \mu_j^2 x_j^2 \neq 0$. ([1, 7, 10, and 11]).

$$\Pr\{\frac{\sum_{j=i}^{m}(c_j - \mu_j)x_j}{\sqrt{\sum_{j=1}^{m} \sigma_j^2 x_j^2}} \leq \frac{f - \sum_{j=1}^{m} \mu_j x_j}{\sqrt{\sum_{j=1}^{m} \sigma_j^2 x_j^2}}\} \geq \alpha, \tag{4.1}$$

Since $\sum_{j=1}^{m}(c_j - \mu_j)x_j/(\sum_{j=1}^{m}\sigma_j^2 x_j^2)^{\frac{1}{2}}$ is a random variable according to the standard normal dictribution, (4.1) is further transformed into the following deterministic inequality.

$$\frac{f - \sum_{j=1}^{m} \mu_j x_j}{\sqrt{\sum_{j=1}^{m} \sigma_j^2 x_j^2}} \geq \Phi^{-1}(\alpha), \tag{4.2}$$

where $\Phi(\cdot)$ is the distribution function of standard normal distribution. Therefore (4.2) is rewritten as follows.

$$f \geq \sum_{j=1}^{m} \mu_j x_j + K_\alpha \sqrt{\sum_{j=1}^{m} \sigma_j^2 x_j^2}\ , \tag{4.3}$$

where $K_\alpha \overset{\Delta}{=} \Phi^{-1}(\alpha) > 0$ (since $1/2 < \alpha \leq 1$). By using (4.3) the problems of Type (I) and Type(II) is transformed into the following deterministic equivalent problems respectively.

Type(I)

$$\text{SP:}\quad \text{Minimize} \quad \sum_{j=1}^{m} \mu_j x_j + K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j \Big)^{\frac{1}{2}}$$

subject to $x_j = 0$ or 1, $\mathbf{x}$: spanning tree.

Type(II)

$$\text{VP:}\quad \text{Minimize} \quad \sum_{j=1}^{m} \mu_j x_j + K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j \Big)^{\frac{1}{2}} - \lambda \Phi(K_\alpha)$$

subject to $x_j = 0$ or 1, $\mathbf{x}$: spanning tree.

Note that the fact $x_j^2 = x_j$ is used in the above transformation.

4.3 Chance Constrained Spanning Tree Problem with Specified Probability Level

In this section we treat tne problem of Type (I), i.e., Problem SP in detail. In order to solve SP, the following auxiliary problem SP(R) with a positive parameter R is introduced.

$$\text{SP(R):}\quad \text{Minimize } R \sum_{j=1}^{m} \mu_j x_j + K_\alpha \sum_{j=1}^{m} \sigma_j^2 x_j$$

subject to $x_j = 0$ or 1, $\mathbf{x}$: spanning tree.

Problem SP(R) is an ordinary minimal spanning tree problem with each edge cost $R\mu_j + K_\alpha \sigma_j^2$. Let $\mathbf{x}^R$ denote an optimal solution of SP(R) and

$$z(R) \overset{\Delta}{=} \sum_{j=1}^{m} \mu_j x_j^R + K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^R \Big)^{\frac{1}{2}}.$$

Then the following properties hold.

Property 4.1 $\sum_{j=1}^{m}\mu_j x_j^R$ is a monotonically nonincreasing function of $R > 0$.

Proof: For $\bar{R} > R > 0$, from the optimality of $\mathbf{x}^R$ and $\mathbf{x}^{\bar{R}}$ for SP(R) and SP($\bar{R}$),

$$R \sum_{j=1}^{m} \mu_j x_j^R + K_\alpha \sum_{j=1}^{m} \sigma_j^2 x_j^R \leq R \sum_{j=1}^{m} \mu_j x_j^{\bar{R}} + K_\alpha \sum_{j=1}^{m} \sigma_j^2 x_j^{\bar{R}}, \qquad (4.4)$$

$$\bar{R} \sum_{j=1}^{m} \mu_j x_j^{\bar{R}} + K_\alpha \sum_{j=1}^{m} \sigma_j^2 x_j^{\bar{R}} \leq \bar{R} \sum_{j=1}^{m} \mu_j x_j^R + K_\alpha \sum_{j=1}^{m} \sigma_j^2 x_j^R, \qquad (4.5)$$

hold respectively. Therefore from the above two inequalities we have

$$(R - \bar{R}) \sum_{j=1}^{m} \mu_j x_j^R \leq (R - \bar{R}) \sum_{j=1}^{m} \mu_j x_j^{\bar{R}} .$$

Since $\bar{R} > R$, we have

$$\sum_{j=1}^{m} \mu_j x_j^R \geq \sum_{j=1}^{m} \mu_j x_j^{\bar{R}} . \quad \square \qquad (4.6)$$

Property 4.2 $\sum_{j=1}^{m}\sigma_j^2 x_j^R$ is a monotonically nondecreasing function of R.

Proof: Let $\bar{R} > R > 0$. From (4.4) and (4.6)

$$\sum_{j=1}^{m} \sigma_j^2 x_j^R \leq \sum_{j=1}^{m} \sigma_j^2 x_j^{\bar{R}}$$

since $K_\alpha > 0$. $\square$

Now let $D(\mathbf{x}) \overset{\Delta}{=} (\sum_{j=1}^{m} \sigma_j^2 x_j)^{\frac{1}{2}}$ for each spanning tree $\mathbf{x}$ and let $\mathbf{x}^*$ denote an optimal solution of Problem SP. Moreover, for convenience, $D(\mathbf{x}^*)$ is denoted D^* simply. Then the following lemmas hold.

Lemma 4.1 For $R \leq 2D^*$ and any spanning tree $\bar{\mathbf{x}}$ such that $D(\bar{\mathbf{x}}) > D^*$,

$$R\sum_{j=1}^{m}\mu_j\bar{x}_j + K_\alpha\sum_{j=1}^{m}\sigma_j^2\bar{x}_j > R\sum_{j=1}^{m}\mu_j x_j^* + K_\alpha\sum_{j=1}^{m}\sigma_j^2 x_j^*$$

holds.

Proof: From the optimality of $\mathbf{x}^*$ for Problem SP,

$$\sum_{j=1}^{m}\mu_j x_j^* + K_\alpha\Big(\sum_{j=1}^{m}\sigma_j^2 x_j^*\Big)^{\frac{1}{2}} \leq \sum_{j=1}^{m}\mu_j\bar{x}_j + K_\alpha\Big(\sum_{j=1}^{m}\sigma_j^2\bar{x}_j\Big)^{\frac{1}{2}} \tag{4.7}$$

holds. Multiplying both hands of (4.7) by R such that $2D^* \geq R > 0$ and rearranging appropriately, we have

$$R\sum_{j=1}^{m}\mu_j x_j^* + K_\alpha\sum_{j=1}^{m}\sigma_j^2 x_j{}^* \leq R\sum_{j=1}^{m}\mu_j\bar{x}_j + K_\alpha\sum_{j=1}^{m}\sigma_j^2\bar{x}_j + K_\alpha\varepsilon,$$

where

$$\varepsilon \overset{\Delta}{=} \sum_{j=1}^{m}\sigma_j^2 x_j^* - \sum_{j=1}^{m}\sigma_j^2\bar{x}_j + R\Big\{\Big(\sum_{j=1}^{m}\sigma_j^2\bar{x}_j\Big)^{\frac{1}{2}} - \Big(\sum_{j=1}^{m}\sigma_j^2 x_j^*\Big)^{\frac{1}{2}}\Big\}.$$

Then it is sufficient to prove $\varepsilon < 0$. Using D^* and $D(\bar{\mathbf{x}})$, ε is rewritten as follows.

$$\varepsilon = D^{*2} - D(\bar{\mathbf{x}})^2 + R(D(\mathbf{x}) - D^*) = (D^* - D(\mathbf{x}))(D^* + D(\bar{\mathbf{x}}) - R).$$

Since $D^* < D(\bar{\mathbf{x}})$ from the assumption of this lemma and $D^* + D(\bar{\mathbf{x}}) - R >$ $2D^* - R \geq 0$, $\varepsilon < 0$ is deduced. □

Lemma 4.2 For $R \geq 2D^*$ and any spanning tree $\hat{\mathbf{x}}$ such that $D(\hat{\mathbf{x}}) <$ D^*,

$$R\sum_{j=1}^{m}\mu_j\hat{x}_j + K_\alpha\sum_{j=1}^{m}\sigma_j^2\hat{x}_j > R\sum_{j=1}^{m}\mu_j x_j^* + K_\alpha\sum_{j=1}^{m}\sigma_j^2 x_j^*$$

holds.

Proof: We assume

$$R\sum_{j=1}^{m}\mu_j\hat{x}_j + K_\alpha\sum_{j=1}^{m}\sigma_j^2\hat{x}_j \leq R\sum_{j=1}^{m}\mu_j x_j^* + K_\alpha\sum_{j=1}^{m}\sigma_j^2 x_j^*. \tag{4.8}$$

From the optimality of $\mathbf{x}^*$, the inequality

$$\sum_{j=1}^{m} \mu_j x_j^* + K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^* \Big)^{\frac{1}{2}} \leq \sum_{j=1}^{m} \mu_j \hat{x}_j + K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \Big)^{\frac{1}{2}} \tag{4.9}$$

holds. Then from the assumption $D(\hat{\mathbf{x}}) < D^*$ and (4.9) we have

$$\sum_{j=1}^{m} \mu_j x_j^* < \sum_{j=1}^{m} \mu_j \hat{x}_j . \tag{4.10}$$

Therefore (4.8) can be rewritten as follows.

$$R \leq \frac{K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^* - \sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \Big)}{\sum_{j=1}^{m} \mu_j \hat{x}_j - \sum_{j=1}^{m} \mu_j x_j^*} . \tag{4.11}$$

Since

$$\sum_{j=1}^{m} \mu_j \hat{x}_j - \sum_{j=1}^{m} \mu_j x_j^* \geq K_\alpha \Big\{ \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^* \Big)^{\frac{1}{2}} - \Big(\sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \Big)^{\frac{1}{2}} \Big\} \tag{4.12}$$

holds from (4.9), (4.11) and (4.12) together imply

$$R \leq \frac{K_\alpha \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^* - \sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \Big)}{K_\alpha \Big\{ \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^* \Big)^{\frac{1}{2}} - \Big(\sum_{j=1}^{m} \sigma_j^2 \hat{x}_j \Big)^{\frac{1}{2}} \Big\}}$$

$$= \frac{D^{*2} - D(\hat{\mathbf{x}})^2}{D^* - D(\hat{\mathbf{x}})} = D^* + D(\hat{\mathbf{x}}) < 2D^* .$$

This contradicts the assumption $R \geq 2D^*$. Thus this lemma holds. □

From Lemma 4.1 and Lemma 4.2 the following theorem holds.

Theorem 4.1 An optimal solution of SP(2D*), i.e., $\mathbf{x}^{2D^*}$, is also optimal for Problem SP.

Therefore we must find 2D*. The following theorems give some informations to find $R^* = 2D^*$.

Theorem 4.2 If $0 < R' < 2D(\mathbf{x}^{R'})$, then either

$$R^* \geq 2D(\mathbf{x}^{R'}) \text{ or } R^* < R'$$

holds.

Proof: From Property 4.2, for R such that $R' \leq R < 2D(\mathbf{x}^{R'})$,

$$D(\mathbf{x}^{R'}) \leq D(\mathbf{x}^{R})$$

holds. Therefore

$$R < 2D(\mathbf{x}^{R'}) \leq 2D(\mathbf{x}^{R})$$

holds. This means R* does not exist on the interval $[R', 2D(\mathbf{x}^{R'}))$ by Theorem 4.1. Thus either $R^* \geq 2D(\mathbf{x}^{R'})$ or $R^* < R'$. ☐

Theorem 4.3 If $R' > 2D(\mathbf{x}^{R'})$, then either

$$R^* > R' \text{ or } R^* \leq 2D(\mathbf{x}^{R'})$$

holds.

Proof: For R such that $R' \geq R > 2D(\mathbf{x}^{R'})$,

$$D(\mathbf{x}^{R}) \leq D(\mathbf{x}^{R'})$$

holds from Property 4.2. This implies

$$2D(\mathbf{x}^{R}) \leq 2D(\mathbf{x}^{R'}) < R.$$

Thus R* does not exist on the interval $(2D(\mathbf{x}^{R'}), R']$, i.e., either $R^* > R'$ or $R^* \leq 2D(\mathbf{x}^{R'})$. ☐

Now define R_{ij} for e_i, e_j $(i < j)$ as follows.

$$R_{ij} \overset{\Delta}{=} K_\alpha \frac{\sigma_i^2 - \sigma_j^2}{\mu_j - \mu_i} \quad . \quad (i,j=1,2,\ldots,m,\ i<j) \tag{4.13}$$

Moreover let $\mathbf{x}^L$ ($\mathbf{x}^U$) denote a minimal spanning tree (maximal spanning tree) of G with each edge cost σ_j^2, and m_D (M_D) denote its value respectively. Rearranging R_{ij} such that $2\sqrt{m_D} \le R_{ij} \le 2\sqrt{M_D}$ in ascending order of magnitude, let

$$R_1 < R_2 < \cdots < R_k,$$

where k is the number of different R_{ij}'s belonging to the interval $[2\sqrt{m_D}, 2\sqrt{M_D}]$.

<u>**Theorem 4.4**</u> An optimal solution $\mathbf{x}^{\bar{R}}$ of SP($\bar{R}$) for $\bar{R} \in [R_i, R_{i+1}]$ is also optimal for all SP(R), $R \in [R_i, R_{i+1}]$.

<u>**Proof:**</u> Let T($\bar{R}$) be a corresponding spanning tree of $\mathbf{x}^{\bar{R}}$, i.e., T($\bar{R}$) consists of N and edge set $E(\bar{R}) \overset{\Delta}{=} \{e_i \,|\, x_i^{\bar{R}} = 1\}$. Then from the optimality of $\mathbf{x}^{\bar{R}}$

$$\bar{R}\mu_t + K_\alpha \sigma_t^2 \le \bar{R}\mu_r + K_\alpha \sigma_r^2 \tag{4.14}$$

must hold for any $e_t \in E(\bar{R})$ and $e_r \in £(e_t, T(\bar{R}))$, where $£(e_t, T_{\bar{R}}) = \{e_r \,|$ edge e_t is contained in the loop in $\{e_r\} \cup T_{\bar{R}}\}$. By the definition of R_i, i=1,2,...,k, the order of edge length does not change among the interval $[R_i, R_{i+1}]$. Thus once (4.14) holds for a certain $\bar{R}$ such that $\bar{R} \in [R_i, R_{i+1}]$, then for any $R \in [R_i, R_{i+1}]$ (4.14) also holds, i.e., $\mathbf{x}^{\bar{R}}$ is optimal for SP(R). □

Let

$$Z(R) \overset{\Delta}{=} \sum_{j=1}^{m} \mu_j x_j^R + K_\alpha \left(\sum_{j=1}^{m} \sigma_j^2 x_j^R\right)^{\frac{1}{2}}$$

and $$Z^L \overset{\Delta}{=} \sum_{j=1}^{m} \mu_j x_j^L + K_\alpha \left(\sum_{j=1}^{m} \sigma_j^2 x_j^L\right)^{\frac{1}{2}}.$$

Now we are ready to construct our algorithm.

Algorithm 4.1

Step 1: Calculate R_1, R_2,...,R_k and set $i \leftarrow 1$, $\mathbf{x}^c \leftarrow \mathbf{x}^L$, and $c \leftarrow Z^L$. Go to Step 2.

Step 2: If $i = k$, then go to Step 4. Otherwise, set $R \leftarrow (R_i + R_{i+1})/2$ and solve SP(R). If $Z(R) < c$, then set $\mathbf{x}^c \leftarrow \mathbf{x}^R$ and $c \leftarrow Z(R)$. Go to Step 3.

Step 3: Set $i \leftarrow \max[\min\{q - 1 | R_q \geq 2D(\mathbf{x}^R)\}, i+1]$ and return to Step 2.

Step 4: Set $R \leftarrow 2\sqrt{M_D}$ and solve SP(R). If $Z(R) < c$, then set $\mathbf{x}^* \leftarrow \mathbf{x}^R$ and terminate. Otherwise, set $\mathbf{x}^* \leftarrow \mathbf{x}^c$ and terminate.

Theorem 4.5 The above algorithm finds an optimal solution $\mathbf{x}^*$ in at most $O(m^2n^2)$ iterations.

Proof: First note that the calculation of R_1, R_2,..., R_k can be done in at most $O(m^2 \log m)$. For each R, $\mathbf{x}^R$ can be found in at most $O(n^2)$ if using Prim's algorithm [9] or Kruskal's one [8]. Clearly the number of $\mathbf{x}^R$ checked in the above algorithm is at most $m(m-1)/2 + 2$ in order to find $\mathbf{x}^*$. Thus in at most $O(m^2n^2)$ iterations, the algorithm finds $\mathbf{x}^*$. □

Example

Consider the following graph G given in Figure 4.1.

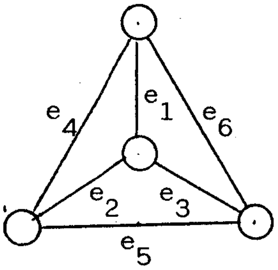

Figure 4.1. Graph in Example

For G each cost is given in Table 4.1.

Table 4.1 Edge costs of G ($N(\mu_j,\sigma_j^2)$ denotes a normal distribution with mean μ_j and variance σ_j^2)

edge	edge cost distribution
e_1	N(16,0.6)
e_2	N(49/3,0.1)
e_3	N(14,1)
e_4	N(44/3,0.7)
e_5	N(15,0.2)
e_6	N(43/3,0.2)

Then the problem of Type (I) is as follows.

Minimize f

$$\text{subject to } \Pr\{\sum_{j=1}^{6} c_j x_j \leq f\} \geq 0.8413$$

$$x_j = 0 \text{ or } 1,\ j=1,2,\ldots,6,\ \mathbf{x}\text{:spanning tree.}$$

Since $F^{-1}(0.8413) = 1.0$, SP and SP(R) become as follows.

$$\text{SP: Minimize } \sum_{j=1}^{6} \mu_j x_j + 1.0(\sum_{j=1}^{6} \sigma_j^2 x_j)^{\frac{1}{2}}$$

$$\text{subject to } x_j = 0 \text{ or } 1,\ j=1,2,\ldots,6,\ \mathbf{x}\text{:spanning tree.}$$

$$\text{SP(R): Minimize } R\sum_{j=1}^{6} \mu_j x_j + 1.0 \sum_{j=1}^{6} \sigma_j^2 x_j$$

$$\text{subject to } x_j = 0 \text{ or } 1,\ j=1,2,\ldots,6,\ \mathbf{x}\text{:spanning tree.}$$

Then $\mathbf{x}^L$ and $\mathbf{x}^U$ are shown in Figure 4.2. Therefore $m_D = 0.5$ and $M_D = 2.3$.

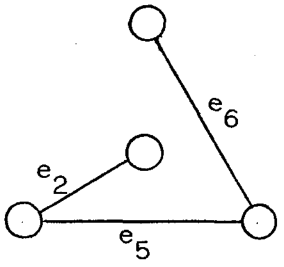

Figure 4.2(a). $\mathbf{x}^L$

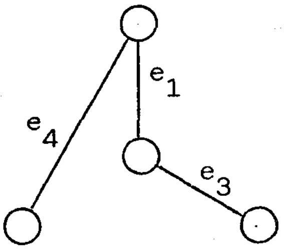

Figure 4.2(b). $\mathbf{x}^U$

Step 1: Since $2\sqrt{m_D}=1.414$ and $2\sqrt{M_D}=3.033$, R_{ij} is shown in Table 4.2. Based on these quantities, $R_1(=R_{12}=R_{45})=1.5$, $R_2(=R_{36})=2.4$ and $k=2$. $i \leftarrow 1$, $\mathbf{x}^c \leftarrow (0,1,0,0,1,1)$ and $c \leftarrow 46.3(=Z^L)$.

Table 4.2 R_{ij}, $1 \le i \le j \le 6$ (Encircled figures constitute R_ℓ)

	2	3	4	5	6
1	1.5	0.2	0.3/4	-0.4	1.2/5
2		2.7/7	1.8/5	0.3/4	0.05
3			0.9/2	0.8	2.4
4				1.5	-1.5/2
5					0

Step 2: Since $i=1\neq 2=k$, set $R \leftarrow (R_1+R_2)/2=1.95$ and solve P(1.95). Then $\mathbf{x}^R=(0,0,1,1,0,1)$. (See Figure 4.3.) Since $Z(R)=44.378 < c\ (=46.3)$, set $c \leftarrow 44.378$. Go to Step 3.

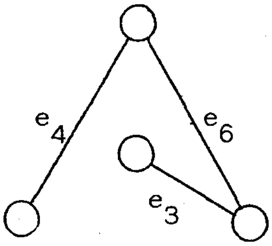

Figure 4.3. Spanning tree $\mathbf{x}^{1.95}$

Step 3: Since $k = 2$, $i \leftarrow 2$. Return to Step 2.

Step 2: Since $i = k = 2$, then go to Step 4.

Step 4: Set $R \leftarrow 3.033$ ($= 2\sqrt{M_D}$) and solve SP(R). Again $\mathbf{x}^R = (0,0,1,1,0,1)$ and so $\mathbf{x}^* \leftarrow (0,0,1,1,0,1)$ ($= \mathbf{x}^c$). Terminate.

4.4 Chance Constrained Spanning Tree Problem with Variable Probability Level

In this section we consider the problem of Type (II),

$$\text{VP:} \quad \text{Minimize } g(\mathbf{x},q) \overset{\Delta}{=} \sum_{j=1}^{m} \mu_j x_j + q\Big(\sum_{j=1}^{m} \sigma_j^2 x_j\Big)^{\frac{1}{2}} - \lambda\Phi(q)$$

$$\text{subject to } x_j = 0 \text{ or } 1,\ j=1,2,\ldots,m,\ \mathbf{x}\text{:spanning tree,}$$

where $q \overset{\Delta}{=} K_\alpha = \Phi^{-1}(\alpha)$ (> 0). First we introduce the following subproblem VP^q in order to solve VP.

$$\mathrm{VP}^q\text{:} \quad \text{Minimize } \sum_{j=1}^{m} \mu_j x_j + q\Big(\sum_{j=1}^{m} \sigma_j^2 x_j\Big)^{\frac{1}{2}}$$

$$\text{subject to } x_j = 0 \text{ or } 1,\ \mathbf{x}\text{:spanning tree.}$$

Let $\mathbf{x}^q$ denote an optimal solution of VP^q, $X(q)$ set of all $\mathbf{x}^q$ and $(\mathbf{x}^*, q^*)$ an optimal solution of Problem VP. Further we define

$$E(\mathbf{x}) \overset{\Delta}{=} \sum_{j=1}^{m} \mu_j x_j \quad \text{and} \quad D(q) \overset{\Delta}{=} \{D(\mathbf{x}^q) \mid \mathbf{x}^q \in X(q)\}\ .$$

Then the following property holds.

Property 4.3 $D(\mathbf{x}^q)$ is a monotonically nonincreasing function of q.

Proof: From the optimality of $\mathbf{x}^{q_1}$ and $\mathbf{x}^{q_2}$ $(q_1 < q_2)$, the following inequalities hold.

$$\sum_{j=1}^{m} \mu_j x_j^{q_1} + q_1\Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_1}\Big)^{\frac{1}{2}} \leq \sum_{j=1}^{m} \mu_j x_j^{q_2} + q_1\Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_2}\Big)^{\frac{1}{2}}, \qquad (4.15)$$

$$\sum_{j=1}^{m} \mu_j x_j^{q_2} + q_2 \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_2} \Big)^{\frac{1}{2}} \leq \sum_{j=1}^{m} \mu_j x_j^{q_1} + q_2 \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_1} \Big)^{\frac{1}{2}}. \tag{4.16}$$

From (4.15) and (4.16) we have

$$(q_1 - q_2) \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_1} \Big)^{\frac{1}{2}} \leq (q_1 - q_2) \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_2} \Big)^{\frac{1}{2}}.$$

Therefore we have

$$D(\mathbf{x}^{q_1}) = \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_1} \Big)^{\frac{1}{2}} \geq \Big(\sum_{j=1}^{m} \sigma_j^2 x_j^{q_2} \Big)^{\frac{1}{2}} = D(\mathbf{x}^{q_2})$$

because $q_1 - q_2 < 0$. □

In order to solve VP^q, we consider an auxiliary problem $VP^q(R)$ with positive parameter R as follows.

$VP^q(R)$: Minimize $R \sum_{j=1}^{m} \mu_j x_j + q \sum_{j=1}^{m} \sigma_j^2 x_j$

subject to $x_j = 0$ or 1, $\mathbf{x}$:spanning tree.

Let $\mathbf{x}^q(R)$ denote an optimal solution of $VP^q(R)$. This problem is the same as Problem SP(R) because $q = K_\alpha$. Therefore Property 4.2, Lemma 4.1 and Lemma 4.2 hold by rewriting $\mathbf{x}(R)$ and $\mathbf{x}^*$ as $\mathbf{x}^q(R)$ and $\mathbf{x}^q$ respectively.

Remark 4.1 All optimal solutions of $VP^q(2D(\mathbf{x}^q))$ have the same value with respect to $D(\cdot)$ and $E(\cdot)$. Thus they have the same value with respect to $g(\cdot, q)$.

From Lemma 4.1 and Lemma 4.2 the following theorem holds.

Theorem 4.6 An optimal solution of Problem $VP^q(2D(\mathbf{x}^q))$, i.e., $\mathbf{x}^q(2D(\mathbf{x}^q))$, is also optimal for Problem VP^q.

Now let define R_{ij}^q as follows.

$$R_{ij}^q \triangleq q(\sigma_j^2 - \sigma_i^2)/(\mu_i - \mu_j) \quad (i,j=1,2,\ldots,m,\ i < j).$$

Rearranging R^q_{ij} such that $0 < R^q_{ij} < \infty$ in ascending order of magnitude, let

$$R^q_1 < R^q_2 < \cdots < R^q_k \text{ and } R^q_0 \overset{\Delta}{=} 0,$$

where k is the number of different R^q_{ij}'s belonging to the interval $(0,\infty)$. Note that the order of R^q_i, $i=0,1,\ldots,k$, and the number k are independent of value q.

Theorem 4.7 For $\overline{R} \in [R^q_i, R^q_{i+1}]$, $\mathbf{x}^q(\overline{R})$ is also an optimal solution of all $VP^q(R)$ for $R \in [R^q_i, R^q_{i+1}]$ so long as the latter interval includes $\overline{R}$.

Proof: Let $T^q(\overline{R})$ be a corresponding spanning tree of $\mathbf{x}^q(\overline{R})$, i.e., $T^q(\overline{R})$ consists of $\underline{N}$ and edge set $E^q(\overline{R}) = \{e_i \,|\, x^q_i(\overline{R}) = 1\}$. Then from the optimality of $\mathbf{x}^q(\overline{R})$,

$$\overline{R}\mu_r + \overline{q}\sigma_r^2 \geq \overline{R}\mu_t + \overline{q}\sigma_t^2 \tag{4.17}$$

must hold for any $e_t \in E^q(\overline{R})$ and $e_r \in \pounds(e_t, T^q(\overline{R}))$. By the definition of R^q_ℓ, $\ell=1,2,\ldots,k$, order of edge cost does not change among the interval $[R^q_i, R^q_{i+1}]$. Thus once (4.17) holds for a certain $\overline{R}$ such that $\overline{R} \in [R^q_i, R^q_{i+1}]$, for any R on $[R^q_i, R^q_{i+1}]$ including $\overline{R}$, (4.17) holds, i.e., $\mathbf{x}^q(\overline{R})$ is optimal for $P^q(R)$. □

Theorem 4.8 $g(\mathbf{x},q)$ is a convex function with respect to $q > 0$.

Proof: For $q > 0$

$$\frac{\partial^2 g(\mathbf{x},q)}{\partial q^2} = \frac{q}{\sqrt{2\pi}} e^{-\frac{1}{2}q^2} > 0.$$

This inequality shows the convexity of $g(\mathbf{x},q)$ with respect to $q > 0$. □

By Theorem 4.8, the optimal $q = q(\mathbf{x})$ for each spanning tree $\mathbf{x}$ becomes as follows.

$$q(\mathbf{x}) = \begin{cases} \sqrt{\log(\frac{\lambda^2}{2\pi \sum_{j=1}^{m} \sigma_j x_j^2})} & \lambda \geq \sqrt{2\pi} D(\mathbf{x}) \\ 0 & \lambda < \sqrt{2\pi} D(\mathbf{x}). \end{cases}$$

Based on $q(\mathbf{x})$, transformation $T(q)$ with respect to $q > 0$ is defined as follows.

$$T(q) = \begin{cases} \sqrt{\log(\frac{\lambda^2}{2\pi (D(\mathbf{x}^q))^2})} & \lambda \geq \sqrt{2\pi} D(\mathbf{x}^q) \\ 0 & \lambda < \sqrt{2\pi} D(\mathbf{x}^q). \end{cases}$$

Note that $T(q)$ is not necessarily unique, but the followings hold.

Property 4.4 $T(q)$ is a nondecreasing function of q.

Proof: By Property 3,

$$\sqrt{\log(\frac{\lambda^2}{2\pi(D(\mathbf{x}^q))^2})}$$

is nondecreasing function of q. Therefore this property holds. ☐

Theorem 4.9 $(\mathbf{x}^*, q^*)$, an optimal solution of Q, satisfies $q^* = T(q^*)$, $\mathbf{x}^{q^*} = \mathbf{x}^*$. (That is, q^* is a fixed point with respect to $T(\cdot)$.)

Proof: If $q^* \neq T(q^*)$, then $q^* \neq q(\mathbf{x}^{q^*})$. Therefore from the definition of $q(\mathbf{x})$ we have

$$g(\mathbf{x}^{q^*}, q(\mathbf{x}^{q^*})) < g(\mathbf{x}^{q^*}, q^*).$$

This contradicts the optimality of q^*. ☐

Theorem 4.10 For q_1 and $q_2 = T(q_1)$

$$q_1 > q_2 \longrightarrow q^* \notin (q_2, q_1]$$

and $$q_1 < q_2 \longrightarrow q^* \notin [q_1, q_2)$$

hold.

Proof: If $q_1 > q_2$, for any $\hat{q} \in [q_2, q_1]$

$$T(\hat{q}) - \hat{q} < T(\hat{q}) - q_2 \leq T(q_1) - q_2 = 0$$

holds from Property 4.4. Therefore $\hat{q}$ does not satisfy the necessary condition of q^*. In case of $q_1 < q_2$, the proof can be done similarly. □

Now we are ready to construct our algorithm. In the algorithm, we use the following simplified notations.

$$q^L \overset{\Delta}{=} q(\mathbf{x}^L) \text{ and } q^U \overset{\Delta}{=} q(\mathbf{x}^U).$$

Algorithm 4.2

Step 1: Set $q \leftarrow 1$ and calculate $R_0^q, \ldots, R_k^q$. Then set $c \leftarrow g(\mathbf{x}^L, q^L)$, $\bar{\mathbf{x}} \leftarrow \mathbf{x}^L$, $\bar{q} \leftarrow q^L$ and $i \leftarrow 0$. Go to Step 2.

Step 2: Set $R \leftarrow \frac{1}{2}(R_i^q + R_{i+1}^q)$, find $\mathbf{x}^q(R)$ and calculate $g(\mathbf{x}^q(R), q(\mathbf{x}^q(R))$. If $c > g(\mathbf{x}^q(R), q(\mathbf{x}^q(R)))$, set $c \leftarrow g(\mathbf{x}^q(R), q(\mathbf{x}^q(R)))$, $\bar{\mathbf{x}} \leftarrow \mathbf{x}^q(R)$ and $\bar{q} \leftarrow q(\mathbf{x}^q(R))$, and go to Step 3. Otherwise, go to Step 3 directly.

Step 3: Set $i \leftarrow i+1$. If $i = k$, go to Step 4. Otherwise return to Step 2.

Step 4: If $g(\mathbf{x}^U, q^U) < c$, set $\mathbf{x}^* \leftarrow \mathbf{x}^U$ and $q^* \leftarrow q^u$, and terminate. Otherwise, set $\mathbf{x}^* \leftarrow \bar{\mathbf{x}}$ and $q^* \leftarrow \bar{q}$ and terminate.

Theorem 4.11 The above algorithm finds an optimal solution $(\mathbf{x}^*, q^*)$ in at most $O(m^2n^2)$ iterations.

Proof: (Validity) By Theorem 4.9, $\mathbf{x}^* \in S^{q^*}$ holds where S^{q^*}

is the set of all optimal solutions of P^{q^*}. Moreover by Theorem 4.6, $S^{q^*} \subset S^{q^*}(2D^{q^*})$ holds, where $S^{q^*}(2D^{q^*})$ is the set of all optimal solutions of $P^{q^*}(2D^{q^*})$. Above discussion and Theorem 4.7 together show that $\mathbf{x}^*$ is included among $\mathbf{x}^q(R)$'s for (q,R) such that $R \in [R_i^q, R_{i+1}^q]$, $i=1,2,\dots,k$, $R < R_1^q$ and $R > R_k^q$, because of Remark 4.1. Further the order of R_i^q and k are independent of q. The algorithm tests all these candidates and finds a minimal solution of them.
(Complexity), First note that the calculation of $R_1^q, \dots, R_k^q$ can be done in at most $O(m^2 \log m)$. For each (q,R), $\mathbf{x}^q(R)$ can be found in at most $O(n^2)$ if using Prim's algorithm [9] or Kruskal's one [8]. Clearly, the number of $\mathbf{x}^q(R)$ checked by the algorithm is at most $m(m-1)/2+2$ in order to find $(\mathbf{x}^*, q^*)$. Thus in at most $O(m^2n^2)$ computational time, the algorithm finds $(\mathbf{x}^*, q^*)$. ☐

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CHAPTER 5

STOCHASTIC FACILITY LOCATION PROBLEMS

5.1 Introduction

Up to now, there are stochastic facility location problems are investigated in many papers [1,2,5-7,10,11,15,17, and 20]. In this chapter we deal with four types of single facility location problems in which the weights and/or the locations of demand points are random. The first two problems refer to the value of information in facility location. The value of information was discussed in the reference [16] and introduced to stochastic linear programming by Bracken and Soland [2] and to facility location model by Wesolowsky [16]. Generally speaking, additional information may reduce the uncertainty on stochastic situation. If we can have "perfect information", that is, we can know the realization of random elements in advance, then the stochastic problem becomes deterministic one. Therefore perfect information has some value, which is so called "the expected value of perfect information (EVPI)". The EVPI is the upper bound on what one would be willing to pay for perfect information about the random variables. Usually, perfect information is seldom available, and so we must take a sample if we want to obtain more information. Since sampling incurs some cost, it would be helpful for deciding whether or not to take a sample to know the worth of sample information, i.e., the expected value of sample information (EVSI). EVSI is not greater than EVPI and approaches EVPI as the sample size increases. Therefore EVPI is usable as an upper bound for the EVSI, though perfect information can not be received. In the other problem, we are interested in finding the optimal location.

In Section 5.2 we investigate the EVSI in facility location. We assume each weight is normally distributed independent random variable with unknown mean and known variance, and a distance between the facility and each demand point is rectangular. The EVPI of facility location problem with random weights is investigated in [11] and [20]. They deal with the case where the weights have a multivariate normal distribution with known means and a known covariance matrix. We evaluate the EVSI by utilizing the computational method developed in [20]. Moreover we investigate the behavior of the EVSI as the sample size changes and provide the optimal additional sample size maximizing the expected net gain of sampling.

In Section 5.3 we evaluate the EVPI in facility location model under locational and weighted uncertainties. In the references [6, and 15], the locations of demand points are assumed to be identical, independent distributed (i.i.d.) random variables. But there are few papers in which the locations and the weights of the demand points are random variables. We assume that the locations of demand points are i.i.d. random variables and the weights of demand points are also i.i.d. normal random variables, and the distances between a facility and demand points are rectangular. We evaluate the EVPI and give its explicit representation.

In Section 5.4 we find an optimal facility location which maximizes the probability satisfying the cost restriction. We assume the weights of demand points are mutually independent normal random variables and distances are lp ones. Especially in the rectangular distance case, we construct an algorithm which finds an optimal solution in $O(n^3)$ time. In stochastic programming problems, some polynomial time algorithms are developed (e.g., [12, 13, and 14]).

However there are few as for stochastic facility location problems.

In Section 5.5 we consider a minimax facility location problem under locational uncertainty. When a demand call appears, e.g., an accident happens in a certain point, we rush a relief squad from emergency service facility to the scene. In this case we cannot know certainly a priori when and where an accident happens. If we cannot restore the scene to the original state in a restricted time, it may not be relieved. Moreover the reachable distance in a restricted time depends on the ability of facility. Therefore we minimize the reachable distance so that the probability to relieve all of the accidents is larger than a certain value. In this section the locations and number of demand points are assumed to be random, that is, the former are random variables with uniform distribution on rectangular area or piecewise uniform distribution on separated two areas, which are independent and identical, and the latter is a random variables with a Poisson distribution. We derive some useful properties to solve the problem and give an explicit optimal solution of the problem by parametrically.

5.2 Expected Value of Sample Information in Facility Location under Probabilistic Weights

Let (a_i, b_i), i=1,2,...,n, denote the locations of n demand points on a plane, and W_i the weight which converts the distance between the i-th demand point (a_i, b_i) and the facility into cost. We assume the distances are rectangular and W_i (i=1,2,...,n) have independent normal distributions with unknown means M_i and known variances $1/r_i$ (parameter r_i is called the precision of W_i). And we assume that the prior distribution of M_i is a normal distribution with a positive mean μ_i and a positive variance $1/\tau_i$. The parameter τ_i is the precision of M_i.

If the minimum expected cost is used as a criterion of optimality, the problem is as follows.

$$\underset{x,y}{\text{Minimize}}\ E[\sum_{i=1}^{n} W_i(|x-a_i|+|y-b_i|)],$$

where (x,y) is the location of the facility. We define $(\hat{x},\hat{y})$ to be a solution which is optimal under the prior distribution, i.e.,

$$\sum_{i=1}^{n} \mu_i(|\hat{x}-a_i|+|\hat{y}-b_i|)=\min_{x,y}\sum_{i=1}^{n} \mu_i(|x-a_i|+|y-b_i|). \tag{5.1}$$

Now suppose that $W_i^{(1)},\ldots,W_i^{(k_i)}$ are random samples of W_i, where k_i is the number of samples. Then the posterior distribution of M_i when $W_i^{(j)}=w_{ij}$ $(j=1,2,\ldots,k_i)$ is a normal distribution with mean μ_i' and precision $\tau_i+k_ir_i$ (see [8]), where

$$\mu_i' \overset{\Delta}{=} \frac{\tau_i\mu_i+k_ir_i\bar{w}_i}{\tau_i+k_ir_i},\quad (\bar{w}_i\text{: sample mean}). \tag{5.2}$$

Under the posterior distribution determined by the sample mean $\bar{w}_i$, the problem reduces as follows.

$$\underset{x,y}{\text{Minimize}}\ \sum_{i=1}^{n} \mu_i'(|x-a_i|+|y-b_i|).$$

Then the conditional value of the sample information (CVSI) is defined as follows. (See [16].)

$$\begin{aligned}\text{CVSI}(\bar{w}_1,\ldots,\bar{w}_n) \overset{\Delta}{=} &\sum_{i=1}^{n} \mu_i'(|\hat{x}-a_i|+|\hat{y}-b_i|)\\ &-\min_{x,y}\sum_{i=1}^{n} \mu_i'(|x-a_i|+|y-b_i|).\end{aligned} \tag{5.3}$$

The CVSI cannot be evaluated until $\bar{w}_i$'s are known, but we can compute the expected value of sample information (EVSI) before $\bar{w}_i$'s are known:

$$\text{EVSI}=E[\sum_{i=1}^{n} Z_i(|\hat{x}-a_i|+|\hat{y}-b_i|)-\min_{x,y}\sum_{i=1}^{n} Z_i(|x-a_i|+|y-b_i|)], \tag{5.4}$$

where

$$Z_i \overset{\Delta}{=} \frac{\tau_i \mu_i + k_i r_i \overline{W}_i}{\tau_i + k_i r_i} . \tag{5.5}$$

Here each Z_i has an independent normal distribution with mean μ_i and variance $1/\tau_i - 1/(\tau_i + k_i r_i)$.

It will be useful to separate the EVSI as follows:

$$\mathrm{EVSI} = \mathrm{EVSI}_x + \mathrm{EVSI}_y , \tag{5.6}$$

where $$\mathrm{EVSI}_x \overset{\Delta}{=} E\,[\sum_{i=1}^{n} Z_i |\hat{x}-a_i| - \min_x \sum_{i=1}^{n} Z_i |x-a_i|], \tag{5.7}$$

and $$\mathrm{EVSI}_y \overset{\Delta}{=} E\,[\sum_{i=1}^{n} Z_i |\hat{y}-b_i| - \min_y \sum_{i=1}^{n} Z_i |y-b_i|]. \tag{5.8}$$

Because it is easy to treat one dimensional case at a time, we shall deal only with finding EVSI_x hereafter. EVSI_y can be calculated similarly.

The equation (5.7) can be reduced to

$$\mathrm{EVSI}_x = \sum_{i=1}^{n} \mu_i |\hat{x}-a_i| - E[\min_x \sum_{i=1}^{n} Z_i |x-a_i|]. \tag{5.9}$$

To evaluate the second term in the right hand side of (5.9), we define $x^*(\mathbf{Z})$ $(\mathbf{Z} \overset{\Delta}{=} (Z_1, Z_2, \ldots, Z_n))$ to be the optimal solution of the following problem $P(\mathbf{Z})$.

$$P(\mathbf{Z}): \quad \underset{x}{\text{Minimize}} \sum_{i=1}^{n} Z_i |x - a_i| .$$

Then we obtain

$$\begin{aligned} &E[\min_x \sum_{i=1}^{n} Z_i |x-a_i|] \\ &= \sum_{j=1}^{n} [\sum_{i=1}^{n} \int_{-\infty}^{\infty} z_i |a_j - a_i| \Pr\{Z_i = z_i | x^*(\mathbf{Z}) = a_j\}\, dz_i] \Pr\{x^*(\mathbf{Z}) = a_j\} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{-\infty}^{\infty} z_i |a_j - a_i| \Pr\{x^*(\mathbf{Z}) = a_j | Z_i = z_i\} g_i(z_i)\, dz_i , \end{aligned} \tag{5.10}$$

where $g_i(\cdot)$ is the p.d.f. of Z_i. The first equality holds because $\Pr\{x^*(\mathbf{Z}) \neq a_j$ for $j=1,2,\ldots,n\} = 0$, and the second one is derived from Bayes' theorem ([8]).

Now we renumber a_i, $i=1,2,\ldots,n$, according to the nondecreasing order of magnitude and assume $a_1 \leq a_2 \leq \cdots \leq a_n$. For practical purposes, we will place restrictions, e.g., $\mu_i \geq 3/\tau_i$, $i=1,2,\ldots,n$, on μ_i and τ_i so that we can neglect the probability of getting a negative value of each Z_i. Then the probability that $x^*(\mathbf{Z}) = a_j$ becomes as follows([20]).

$$\Pr\{x^*(\mathbf{Z}) = a_j\} = \Pr\{\{\sum_{i=1}^{j-1} Z_i \leq \sum_{i=j}^{n} Z_i\} \cap \{\sum_{i=1}^{j} Z_i > \sum_{i=j+1}^{n} Z_i\}\}$$
$$= \Phi(\frac{-u_j + \mu_j}{\sqrt{v}}) - \Phi(\frac{-u_j - \mu_j}{\sqrt{v}}), \tag{5.11}$$

where $\Phi(\cdot)$ is the standard normal distribution ,

$$u_j \overset{\Delta}{=} \sum_{i=1}^{j-1} \mu_i - \sum_{i=j+1}^{n} \mu_i \tag{5.12}$$

and
$$v \overset{\Delta}{=} \sum_{i=1}^{n} (\frac{1}{\tau_i} - \frac{1}{\tau_i + k_i r_i}). \tag{5.13}$$

Similarly, if $i < j$,

$$\Pr\{x^*(\mathbf{Z})=a_j | Z_i=z_i\} = \Phi(\frac{-u_j+\mu_j-z_i+\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j-z_i+\mu_i}{\sqrt{v_i}}), \tag{5.14}$$

and if $i > j$,

$$\Pr\{x^*(\mathbf{Z})=a_j | Z_i=z_i\} = \Phi(\frac{-u_j+\mu_j+z_i-\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j+z_i-\mu_i}{\sqrt{v_i}}), \tag{5.15}$$

where

$$v_i \overset{\Delta}{=} \sum_{\substack{j=1 \\ \neq i}}^{n} (\frac{1}{\tau_i} - \frac{1}{\tau_i + k_i r_i}). \tag{5.16}$$

Therefore substituting (5.14) and (5.15) into (5.10), we obtain

$$
\begin{aligned}
&E[\min_{x} \sum_{i=1}^{n} Z_i |x-a_i|] \\
&= \sum_{i=1}^{n} \int_{-\infty}^{\infty} z_i [\sum_{j=1}^{i-1} (a_i-a_j)\{\Phi(\frac{-u_j+\mu_j+z_i-\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j+z_i-\mu_i}{\sqrt{v_i}})\} \\
&+ \sum_{j=i+1}^{n} (a_j-a_i)\{\Phi(\frac{-u_j+\mu_j-z_i+\mu_i}{\sqrt{v_i}}) - \Phi(\frac{-u_j-\mu_j-z_i+\mu_i}{\sqrt{v_i}})\}]g_i(z_i)dz_i
\end{aligned}
\tag{5.17}
$$

The integrals in the right-hand side of (5.17) can be calculated by numerical integration such as Simpson's rule. Evaluating EVSI_y similarly, we can find the EVSI by (5.6).

Considering the function EVSI($\mathbf{k}$) of sample size vector $\mathbf{k}=(k_1, k_2,\ldots,k_n)$, we shall derive some properties, treating each k_i as if it were a continuous variable.

$$
\mathrm{EVSI}_x(\mathbf{k}) = \sum_{i=1}^{n} \mu_i |\hat{x}-a_i| - \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{\min_{x} \sum_{i=1}^{n} Z_i |x-a_i|\} \prod_{i=1}^{n} \{g_i(z_i)dz_i\}.
\tag{5.18}
$$

Now we have the following property.

Property 5.1 EVSI($\mathbf{k}$) is a nondecreasing function of each k_i.

To prove this property, we first show the following lemma.

Lemma 5.1 Suppose that $\Psi(x)$ is a concave function of x and the random variable X has a normal distribution with mean μ and variance σ^2. Then the expected value of $\Psi(X)$, i.e., $E[\Psi(X)]$ is non-increasing with respect to σ.

Proof: Considering $E[\Psi(X)]$ as a function of σ,

$$
\begin{aligned}
L(\sigma) \triangleq E[\Psi(X)] &= \int_{-\infty}^{\infty} \Psi(x) \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \{\Psi(\sigma x+\mu)+\Psi(-\sigma x+\mu)\}\exp(-\frac{x^2}{2})dx.
\end{aligned}
$$

For $0 < \sigma_1 < \sigma_2$ and $x > 0$, we have

$$L(\sigma_1) - L(\sigma_2)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty [\{\Psi(\sigma_1 x+\mu)+\Psi(-\sigma_1 x+\mu)\}-\{\Psi(\sigma_2 x+\mu)+\Psi(-\sigma_2 x+\mu)\}]\exp(-\frac{x^2}{2})dx$$

$$= \frac{\sigma_2-\sigma_1}{\sqrt{2\pi}} \int_0^\infty x\{\frac{\Psi(-\sigma_1 x+\mu)-\Psi(-\sigma_2 x+\mu)}{(-\sigma_1 x+\mu)-(-\sigma_2 x+\mu)} - \frac{\Psi(\sigma_2 x+\mu)-\Psi(\sigma_1 x+\mu)}{(\sigma_2 x+\mu)-(\sigma_1 x+\mu)}\}\exp(-\frac{x^2}{2})dx$$

≥ 0 (by the concavity of $\Psi(x)$). ☐

Now we are ready to prove Property 5.1.

Proof of Property 5.1 Let $\mathbf{k}=(k_1,k_2,\ldots,k_i,\ldots,k_n)$ and $\mathbf{k}^i=(k_1, k_2,\ldots,\tilde{k}_i,\ldots,k_n)$, i.e., $\mathbf{k}^i$ has the same components as $\mathbf{k}$ except i-th one. Moreover we assume $\tilde{k}_i > k_i$. Then by (4.1) we have

$$\mathrm{EVSI}_x(\mathbf{k}^i) - \mathrm{EVSI}_x(\mathbf{k}) = \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty H_i(\mathbf{z}^{(i)}) \prod_{\substack{j=1\\ \neq i}}^n (g_j(z_j)dz_j),$$

where

$$H_i(\mathbf{z}^{(i)}) \triangleq \int_{-\infty}^\infty \min_x \sum_{j=1}^n z_j|x-a_j|g_i(z_i)dz_i - \int_{-\infty}^\infty \min_x \sum_{j=1}^n z_j|x-a_j|\tilde{g}_i(z_i)dz_i,$$

$$z^{(i)} \triangleq (z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_n),$$

$g_i(\cdot)$ and $\tilde{g}_i(\cdot)$ are the density functions of normal distribution with the same mean μ_i but different variances $1/\tau_i-1/(\tau_i+k_i r_i)$ and $1/\tau_i-1/(\tau_i+\tilde{k}_i r_i)$ respectively. Let

$$f(q) \triangleq \min_x \{\sum_{\substack{j=1\\ \neq i}}^n (z_j|x-a_j|+q|x-a_i|)\}.$$

Then it is easily shown $f(\cdot)$ is concave. By Lemma 5.1,

$$\int_{-\infty}^\infty f(z_i)g_i(z_i)dz_i$$

is nondecreasing with respect to the standard deviation (or variance) of Z_i. Since

$$\left(\frac{1}{\tau_i} - \frac{1}{\tau_i + \tilde{k}_i r_i}\right) - \left(\frac{1}{\tau_i} - \frac{1}{\tau_i + k_i r_i}\right) > 0,$$

$H_i(\mathbf{z}^{(i)}) \geq 0$ results. Therefore

$$\mathrm{EVSI}_x(\mathbf{k}^i) - \mathrm{EVSI}_x(\mathbf{k}) \geq 0.$$

As to EVSI_y, we can also obtain similar results. □

We define the expected value of perfect information (EVPI):

$$\mathrm{EVPI} = \mathrm{EVPI}_x + \mathrm{EVPI}_y, \tag{5.19}$$

where

$$\mathrm{EVPI}_x = \sum_{i=1}^{n} \mu_i |\hat{x}-a_i| - E[\min_x \sum_{i=1}^{n} \bar{Z}_i |x-a_i|], \tag{5.20}$$

$$\mathrm{EVPI}_y = \sum_{i=1}^{n} \mu_i |\hat{y}-b_i| - E[\min_y \sum_{i=1}^{n} \bar{Z}_i |y-b_i|], \tag{5.21}$$

and assume each $\bar{Z}_i$ has independent normal distribution with mean μ_i and variance $1/\tau_i$. Then the following corollary holds.

Corollary 5.1 $0 \leq \mathrm{EVSI} \leq \mathrm{EVPI}$.

Proof: By (5.4) and the definition of $(\hat{x},\hat{y})$, the first inequality holds. Furthermore, as the sample size k_i increases, the variance of random variable Z_i approaches to the variance $\bar{Z}_i$, i.e., $1/\tau_i$. Thus Theorem 5.1 implies the second inequality. □

This corollary shows that the EVPI gives an upper bound of EVSI.

The EVSI is the value of sample information without considering sampling cost. On the other hand, if the sample information involves some costs, this sampling cost, $CS(\mathbf{k})=\sum_{j=1}^{n} c_j k_j + b$, should be subtracted from the EVSI, where each c_j is a unit cost taking one sample about the i-th location and b is a fixed charge taking the sample. Then the net result called the expected net gain of sampling (ENGS) becomes as follows:

$$ENGS(\mathbf{k}) = EVSI(\mathbf{k}) - CS(\mathbf{k}). \tag{5.22}$$

The optimal vector sample size is defined as the vector size $\mathbf{k}$ which maximizes ENGS($\mathbf{k}$).

In the following example, we consider the optimal sampling in which all k_1, $k_2, \ldots,$ k_n are restricted to the same value k. If we define $c=\sum_{j=1}^{n} c_j$, then the sampling cost reduces to the following scalar function:

$$CS(k) = ck + b. \tag{5.23}$$

Now the ENGS is determined by the value of k, and so the function of k. Therefore the optimal sample size k* is defined as the size k which maximizes ENGS(k). In the following we give an example and find its optimal sample size.

Example 5.1 We want to locate a wholesale store in the town where there are 5 retail stores. Let (a_i, b_i) and W_i denote the location of i-th retail store and the random amount sold in a week there respectively. When the location of wholesale store is (x,y), the distance between the wholesale store and the i-th retail store is $|x-a_i| + |y-b_i|$ kilometers. We assume that W_i is mutually independent normal random variable with unknown mean M_i and known variance $1/r_i$. And we assume that the prior distribution of M_i is a normal distribution with mean μ_i and variance $1/\tau_i$. ((a_i, b_i), r_i, μ_i and τ_i are given in Table 5.1.) We assume the transportation cost per kilometer and ton is 1000 yen. Then the optimal location under the prior distribution becomes (8,6). If we assume that all sample sizes k_1, $k_2, \ldots,$ k_n have the same value k, the EVSI and the ENGS as a function of sample size k are shown in Figure 5.1. In Figure 5.1 we assume the sampling cost (×1000 yen) is as follows

$$CS(k) = 0.02k + 5.$$

Then the optimal sample size is about 60 and ENGS(60) is 2240 yen.

Table 5.1. Data for Example 5.1.

i	1	2	3	4	5
(a_i,b_i)	(3,2)	(4,9)	(8,12)	(12,1)	(14,6)
r_i	0.01	0.01	0.01	0.01	0.01
μ_i	50	38	30	35	25
τ_i	0.1	0.1	0.1	0.1	0.1

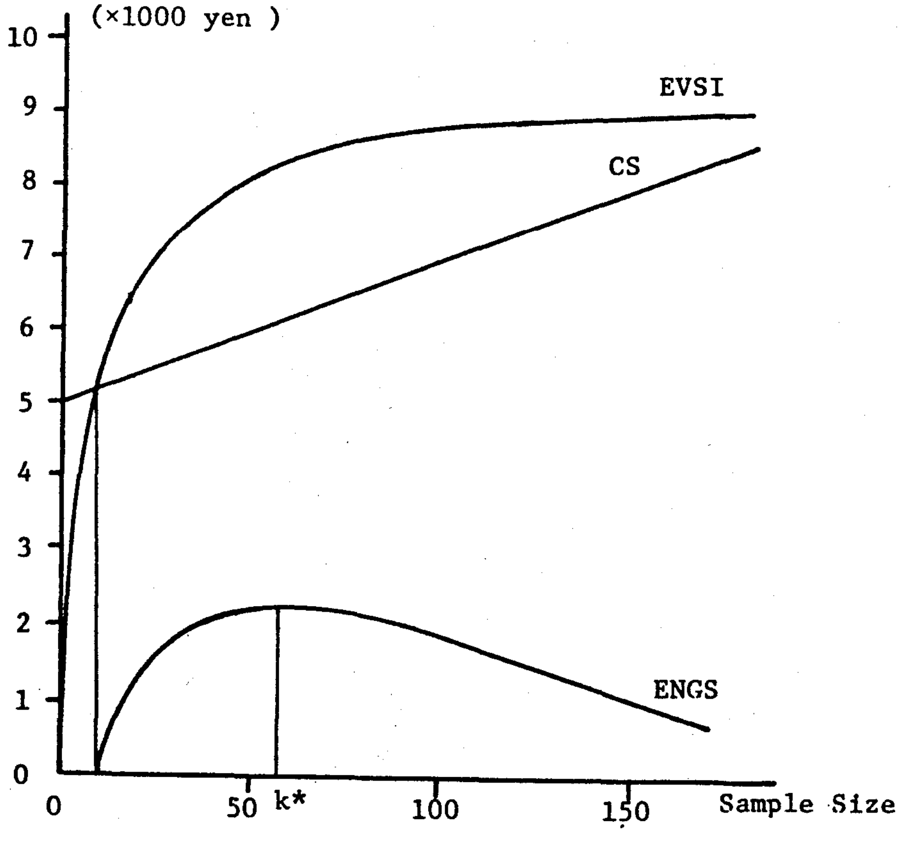


Figure 5.1. EVSI and ENGS of Example 5.1.

5.3 Expected Value of Perfect Information in Facility Location under Locational and Weighted Uncertainties

There are n demand points in the plane and both their locations and weights are known probabilistically. Let (X_i, Y_i) and W_i denote the location and the weight of i-th demand point respectively. We assume that X_i, Y_i $(i=1,2,...,n)$ are identical, pairwise independent random variables with density function $f(\cdot,\cdot)$ and cumulative distribution function $F(\cdot,\cdot)$, and W_i $(i=1,2,...,n)$ are identical, independent , normally distributed random variables with mean μ and variance σ^2. We assume that we can neglect the occurrence probability of negative W_i (e.g., $\mu \geq 3\sigma$). Further we assume the distances between a facility (x,y) and the demand points (X_i, Y_i), $i=1, 2,...,n$, are rectangular. Then the cost function is as follows:

$$C(x,y;\mathbf{X},\mathbf{Y},\mathbf{W}) = \sum_{i=1}^{n} W_i(|x-X_i| + |y-Y_i|), \tag{5.24}$$

where $\mathbf{X} = (X_1, X_2, ..., X_n)$, $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ and $\mathbf{W} = (W_1, W_2, ..., W_n)$.

Now we consider the expected value of perfect information (EVPI). We define the EVPI as follows (see [20]):

$$\text{EVPI} = \min_{x,y} E[C(x,y;\mathbf{X},\mathbf{Y},\mathbf{W})] - E[\min_{x,y} C(x,y;\mathbf{X},\mathbf{Y},\mathbf{W})], \tag{5.25}$$

where E stands for the mathematical expectation. We can transform (5.24) into the separate form with respsct to X-coordinate and Y-coordinate as follows:

$$C(x,y;\mathbf{X},\mathbf{Y},\mathbf{W}) = C_x(x;\mathbf{X},\mathbf{W}) + C_y(y;\mathbf{Y},\mathbf{W}), \tag{5.26}$$

where $$C_x(x;\mathbf{X},\mathbf{W}) = \sum_{i=1}^{n} W_i|x-X_i|$$

and $$C_y(y;\mathbf{Y},\mathbf{W}) = \sum_{i=1}^{n} W_i|y-Y_i|.$$

Therefore it can easily be shown that equation(5.25) becomes as follows:

$$\mathrm{EVPI} = \mathrm{EVPI}_x + \mathrm{EVPI}_y, \tag{5.27}$$

where $\mathrm{EVPI}_x = \min_x E[C_x(x;\mathbf{X},\mathbf{W})] - E[\min_x C_x(x;\mathbf{X},\mathbf{W})]$,

and $\mathrm{EVPI}_y = \min_y E[C_y(y;\mathbf{Y},\mathbf{W})] - E[\min_y C_y(y;\mathbf{Y},\mathbf{W})]$.

Because it is easy to consider one dimension at a time, in what follows we have only to consider either EVPI_x or EVPI_y. Without any loss of generality, we shall find EVPI_x hereafter. The EVPI_y can be found similarly.

Now we consider two types of problems. One is as follows:

EP: $\underset{x}{\text{Minimize}}\ E[C_x(x;\mathbf{X},\mathbf{W})]$,

and the other is as follows:

$P(\mathbf{x},\mathbf{w})$: $\underset{x}{\text{Minimize}}\ C_x(x;\mathbf{x},\mathbf{w})$,

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{w}=(w_1,w_2,\ldots,w_n)$. Let x^o and $x^*(\mathbf{x},\mathbf{w})$ denote the optimal solutions of problems EP and $P(\mathbf{x},\mathbf{w})$ respectively.

Define T(x) as follows:

$$T(x) \overset{\Delta}{=} E[C_x(x;\mathbf{X},\mathbf{W})]. \tag{5.28}$$

Then T(x) can be rewritten as follows:

$$T(x) = n\{\int_{-\infty}^{x}\mu(x-t)f(t)dt + \int_{x}^{\infty}\mu(t-x)f(t)dt\},$$

where $f(t) \overset{\Delta}{=} \int_{-\infty}^{\infty} f(t,u)du$. Differentiating T(x), we have

$$\frac{dT(x)}{dx} = n\{2F(x) - 1\}\mu,$$

where $F(x) \overset{\Delta}{=} \int_{-\infty}^{x} f(t)dt$. Therefore the optimal solution x^o of Problem

EP is obtained from the convexity of T(x), i.e.,

$$x^{o}=F^{-1}(1/2). \tag{5.29}$$

Now we consider the second type of problem, i.e., Problem $P(\mathbf{x},\mathbf{w})$. Since X_i, $i=1,2,...,n$, are i.i.d. random variables, the expected optimal value of Problem $P(\mathbf{x},\mathbf{w})$ is as follows:

$$E[\min_{x} C_x(x;\mathbf{X},\mathbf{W})] = n!E[\min_{x} \sum_{i=1}^{n} W_i |x-X_i| \,|X_1\leq\cdots\leq X_n]\Pr\{X_1\leq\cdots\leq X_n\}.$$

This equation can be rewritten by using $x^*(\mathbf{X},\mathbf{W})$ as follows:

$$E[\min_{x} C_x(x;\mathbf{X},\mathbf{W})] = n! \sum_{i=1}^{n} \int_{-\infty}^{\infty} w\{ \sum_{j=1}^{i-1} EP_{ij}(w) - \sum_{j=i+1}^{n} EP_{ij}(w)\} dw, \tag{5.30}$$

where

$$EP_{ij}(w) = E[X_i-X_j|X_1\leq\cdots\leq X_n,\ x^*(\mathbf{X},\mathbf{W})=X_j \text{ and } W_i=w]$$

$$\times\Pr\{W_i=w,\ x^*(\mathbf{X},\mathbf{W})=X_j \text{ and } X_1\leq\cdots\leq X_n\}. \tag{5.31}$$

Rewriting the equation (5.31), we have

$$EP_{ij}(w) = \int_{-\infty}^{\infty} xP_{jii}(w,x)dx - \int_{-\infty}^{\infty} xP_{jij}(w,x)dx, \tag{5.32}$$

where

$$P_{ijk}(w,x) = \Pr\{X_1\leq\cdots\leq X_n,\ x^*(\mathbf{X},\mathbf{W})=X_i,\ W_i=w \text{ and } X_k=x\}.$$

Now we define

$$A_i(x) \overset{\Delta}{=} \Pr\{X_1 \leq \cdots \leq X_n | X_i=x\}, \tag{5.33}$$

$$B_{ij}(w) \overset{\Delta}{=} \Pr\{ \sum_{k=1}^{j-1} W_k < \sum_{k=j}^{n} W_k \text{ and } \sum_{k=1}^{j} W_k > \sum_{k=j+1}^{n} W_k | W_i=w\}, \tag{5.34}$$

and

$$g(w) \overset{\Delta}{=} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(w-\mu)^2}{2\sigma^2}}$$

Since X_i, $i=1,2,...,n$, are i.i.d. random variables, the equation

(5.33) becomes as follows:

$$A_i(x) = \frac{1}{(i-1)!(n-i)!} \{F(x)\}^{i-1} \{1-F(x)\}^{n-i}.$$

If we can neglect the probability of negative W_j, the equation (5.34) becomes as follows:

$$B_{ij}(w) = \Pr\{ \sum_{k=1}^{j-1} W_k < \sum_{k=j}^{n} W_k | W_i = w\} - \Pr\{ \sum_{k=1}^{j} W_k \leq \sum_{k=j+1}^{n} W_k | W_i = w\}.$$

By standardizing them, we have

$$B_{ij}(w) = \Phi(\frac{w+(n-2j+1)\mu}{\sqrt{n-1}\,\sigma}) - \Phi(\frac{w+(n-2j-1)\mu}{\sqrt{n-1}\,\sigma}) \text{ if } i > j, \qquad (5.35)$$

$$B_{ij}(w) = \Phi(\frac{-w+(n-2j+3)\mu}{\sqrt{n-1}\,\sigma}) - \Phi(\frac{-w+(n-2j+1)\mu}{\sqrt{n-1}\,\sigma}) \text{ if } i < j, \qquad (5.36)$$

where $\Phi(\cdot)$ stands for the cumulative distribution function of a standard normal variate. We simply denote the functions (5.35) and (5.36) as $V_j(w)$ and $V_j'(w)$ respectively. Then $P_{jii}(w,x)$ and $P_{jij}(w, x)$ can be rewritten as follows:

$$P_{jii}(w,x) = A_i(x)f(x)B_{ij}(w)g(w),$$

$$P_{jij}(w,x) = A_j(x)f(x)B_{ij}(w)g(w).$$

Substituting these two equations into the equation (5.32), we have

$$EP_{ij}(w) = \int_{-\infty}^{\infty} x\{A_i(x) - A_j(x)\}B_{ij}(w)f(x)g(w)dx.$$

Therefore the equation (5.30) becomes as follows:

$$E[\min_x C_x(x;\mathbf{X},\mathbf{W})] = n!\int_{-\infty}^{\infty} w \int_{-\infty}^{\infty} x\{H_1(x,w)-H_2(x,w)\}f(x)g(w)dxdw, \qquad (5.37)$$

where $$H_1(x,w) \triangleq \sum_{i=1}^{n} \sum_{j=1}^{i-1} \{A_i(x) - A_j(x)\}V_j(w),$$

$$H_2(x,w) \overset{\Delta}{=} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \{A_i(x) - A_j(x)\} V_j'(w).$$

Since $\Phi(-x) = 1 - \Phi(x)$, we have

$$V_j'(w) = \Phi(\frac{w+\{n-2(n-j+1)+1\}\mu}{\sqrt{n-1}\,\sigma}) - \Phi(\frac{w+\{n-2(n-j+1)-1\}\mu}{\sqrt{n-1}\,\sigma})$$
$$= V_{n-j+1}(w).$$

Hence by simply rewriting summation, we have

$$H_2(x,w) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} \{A_{n-i+1}(x) - A_{n-j+1}(x)\} V_j(w).$$

Therefore the equation (5.37) becomes as follows:

$$E[\min_x C_x(x;\mathbf{X},\mathbf{W})] = n! \sum_{i=1}^{n} \sum_{j=1}^{i-1} (L_i - L_j) M_j,$$
$$= n! \sum_{i=1}^{n} L_i \{ \sum_{j=1}^{i-1} M_j - (n-i) M_i \}, \qquad (5.38)$$

where $L_i \overset{\Delta}{=} \int_{-\infty}^{\infty} x\{A_i(x) - A_{n-i+1}(x)\} f(x)dx,$ (5.39)

and $M_j \overset{\Delta}{=} \int_{-\infty}^{\infty} wV_j(w)g(w)dw.$ (5.40)

Rewriting the equation (5.40) by using the density function of standard normal variate $\phi(\cdot)$, we obtain

$$M_j = \sigma\int_{-\infty}^{\infty} wV_j(w\sigma+\mu)\phi(w)dw + \mu\int_{-\infty}^{\infty} V_j(w\sigma+\mu)\phi(w)dw. \qquad (5.41)$$

Now we define T_1 and T_2 as follows:

$$T_1 = \int_{-\infty}^{\infty} wV_j(w\sigma+\mu)\,\phi(w)dw,$$

and $T_2 = \int_{-\infty}^{\infty} V_j(w\sigma+\mu)\phi(w)dw.$

Then by the integration by parts, T_1 can be rewritten as follows:

$$T_1 = \frac{1}{\sqrt{n-1}} \int_{-\infty}^{\infty} \phi(w)\{\phi(\frac{w\sigma+(n-2j+2)\mu}{\sqrt{n-1}\,\sigma}) - \phi(\frac{w\sigma+(n-2j)\mu}{\sqrt{n-1}\,\sigma})\}dw.$$

By the simple integral calculation, we have

$$T_1 = \frac{1}{\sqrt{2\pi n}}(e^{-\frac{u_j^2}{2}} - e^{-\frac{u_{j+1}^2}{2}}), \tag{5.42}$$

where $u_j = \dfrac{(n-2j+2)\mu}{\sqrt{n}\,\sigma}$.

And T_2 is rewritten as follows:

$$T_2 = \int_{-\infty}^{\infty}\int_{t_{j+1}}^{t_j} \phi(t)\phi(w)dtdw,$$

where $t_j = \dfrac{w+\sqrt{n}u_j}{\sqrt{n-1}}$.

Transforming (t,w) into (t',w') by

$$w' = \frac{1}{\sqrt{n}} w - \frac{\sqrt{n-1}}{\sqrt{n}} t,$$

$$t' = \frac{\sqrt{n-1}}{\sqrt{n}} w + \frac{1}{\sqrt{n}} t,$$

we have $T_2 = \Phi(-u_{j+1}) - \Phi(-u_j).$ (5.43)

Therefore from the equations (5,41), (5.42) and (5.43), we obtain

$$\sum_{j=1}^{i-1} M_j = \frac{\sigma}{\sqrt{2\pi n}}(e^{-u_1^2/2} - e^{-u_i^2/2}) + \mu\{\Phi(-u_i) - \Phi(-u_1)\}.$$

Hence substituting this equation into (5.38), we have

$$\begin{aligned} E[\min_x C_x(x;\mathbf{X},\mathbf{W})] = n! \sum_{i=1}^{n} \int_{-\infty}^{\infty} x\{A_i(x) - A_{n-i+1}(x)\} f(x)dx \\ \times[\frac{\sigma}{\sqrt{2\pi n}}\{e^{-u_1^2/2} - (n-i+1)e^{-u_i^2/2} + (n-i)e^{-u_{i+1}^2/2}\} \\ +\mu\{(n-i+1)\Phi(-u_i) - \Phi(-u_1) - (n-i)\Phi(-u_{i+1})\}]. \end{aligned}$$

(5.44)

The integrals in right hand side of the equation (5.44) can be calculated by numerical integration.

Special Case

If the distribution of X_i is symmetric, then we can set the center of symmetry to 0 by a suitable translation. Then we have

$$\int_{-\infty}^{\infty} x A_{n-i+1}(x) f(x) dx = -\int_{-\infty}^{\infty} x A_i(x) f(x) dx. \tag{5.45}$$

Therefore the equation (5.44) becomes a slightly simple form. In the following example we show that the EVPI can be represented by elementary functions and normal distribution functions if each X_i has a uniform (therefore symmetric) distribution.

Example 5.2 We assume the distribution of X_i is uniform on (-h,h). Then the optimal solution x^o of Problem EP is 0. Therefore the optimal value of Problem EP is

$$T(x^o) = n\{\int_{-\infty}^{0} \mu(-t) f(t) dt + \int_{0}^{\infty} \mu t f(t) dt\}$$

$$= \frac{n\mu h}{2}.$$

And from the equations (5.39) and (5.45), we have

$$L_i = \frac{2h}{(i-1)!(n-i)!} \sum_{j=1}^{n-i} {}_{n-i}C_j (-1)^j \frac{i+j-1}{(i+j)(i+j+1)}.$$

Therefore

$$\begin{aligned} E[\min_x C_x(x;\mathbf{X},\mathbf{W})] = 2h \sum_{i=1}^{n} i \cdot {}_nC_i \sum_{j=1}^{n-i} {}_{n-i}C_j (-1)^j \frac{i+j-1}{(i+j)(i+j+1)} \\ \times[\frac{\sigma}{\sqrt{2\pi n}}\{e^{-u_1^2/2} - (n-i+1)e^{-u_i^2/2} + (n-i)e^{-u_{i+1}^2/2}\} \\ + \mu\{(n-i+1)\Phi(-u_i) - \Phi(-u_1) - (n-i)\Phi(-u_{i+1})\}\ . \end{aligned}$$

We calculate the $EVPI_x$ for various r ($\triangleq \mu/\sigma$) and n. (See Table 5.2.)

Table 5.2. Calculation of the EVPI

n	r	EVPI (× μh)
5	3.0	0.52070
	4.0	0.50547
	5.0	0.50132
	6.0	0.50028
10	3.0	0.53362
	4.0	0.51073
	5.0	0.49877
	6.0	0.49107

5.4 Stochastic Facility Location Problem under Aspiration Level Criterion

There are n demand points (a_i, b_i), i=1,2,...,n, on a plane. Let (x,y) be the location of the facility on the plane and let $d_i(x,y)$ be the lp distance between the facility and the i-th demand point . We assume each weight W_i, which converts the distance $d_i(x,y)$ into cost, is the independent normal random variable with positive mean μ_i and positive variance σ_i^2.

If we use the aspiration-level criterion, then the problem is as follows:

$$\text{AP}_0\text{:} \quad \text{Maximize } \Pr\{ \sum_{i=1}^{n} W_i d_i(x,y) \leq c \},$$

where $d_i(x,y) = (|x-a_i|^p + |y-b_i|^p)^{1/p}$, $p \geq 1$.

Let (x^*, y^*) denote an optimal solution of Problem AP_0. In this paper we assume that

$$c > \sum_{i=1}^{n} \mu_i d_i(x,y) \text{ for some } (x,y). \tag{5.46}$$

This assumption is not so restrictive because maximum probability should be larger than 1/2 usually. Then Problem AP_0 is equivalent to the following deterministic problem AP. (See [19].)

$$\text{AP:} \quad \text{Maximize } V(x,y) \triangleq \frac{c - \sum_{i=1}^{n} \mu_i d_i(x,y)}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 (d_i(x,y))^2}} .$$

The optimal value of'Problem AP is denoted by λ^*. Then the assumption (5.46)implies $\lambda^* > 0$. In order to solve the fractional problem AP, we consider the following parametric subproblem AP(λ).

$$\text{AP}(\lambda):\quad \text{Minimize } F(x,y;\lambda) = \sum_{i=1}^{n} \mu_i d_i(x,y) + \lambda \sqrt{\sum_{i=1}^{n} \sigma_i^2 (d_i(x,y))^2}$$

Let (x_λ, y_λ) be the optimal solution of Problem AP(λ). Then the following properties hold.

Property 5.2 $F(x,y;\lambda)$ is a convex function of (x,y) for $\lambda \geq 0$.

Proof: The first term in the objective function of Problem AP(λ) is convex because it is positively weighted sum of convex functions $d_i(x,y)$. Now define

$$D(x,y) \overset{\Delta}{=} \sqrt{\sum_{i=1}^{n} \sigma_i^2 (d_i(x,y))^2}, \tag{5.47}$$

then for $0 \leq \alpha \leq 1$ we can show

$$\begin{aligned}
&\{\alpha D(x_1,y_1)+\bar{\alpha} D(x_2,y_2)\}^2 - \{D(\alpha x_1+\bar{\alpha} x_2, \alpha y_1+\bar{\alpha} y_2)\}^2 \\
&= \sum_{i=1}^{n} \sigma_i^2 [\{\alpha d_i(x_1,y_1)+\bar{\alpha} d_i(x_2,y_2)\}^2 - \{d_i(\alpha x_1+\bar{\alpha} x_2, \alpha y_1+\bar{\alpha} y_2)\}^2 \\
&+2\alpha\bar{\alpha}[\sqrt{\sum_{i=1}^{n} \sigma_i^2 (d_i(x_1,y_1))^2}\sqrt{\sum_{i=1}^{n} \sigma_i^2 (d_i(x_2,y_2))^2} \\
&\qquad - \sum_{i=1}^{n} \sigma_i^2 d_i(x_1,y_1) d_i(x_2,y_2)] \geq 0 \qquad (\bar{\alpha} \overset{\Delta}{=} 1-\alpha)
\end{aligned}$$

by using the convexity of $d_i(x,y)$ and Cauchy's inequality. Since $D(x,y) \geq 0$ and $\lambda > 0$, this property holds. ☐

Property 5.3 Let $F_\lambda \overset{\Delta}{=} F(x_\lambda, y_\lambda; \lambda)$, then F_λ is continuous and strictly monotone increasing function of λ, and

$$F_\lambda < c \longleftrightarrow \lambda < \lambda^*,$$

$$F_\lambda = c \longleftrightarrow \lambda = \lambda^*,$$

$$F_\lambda > c \longleftrightarrow \lambda > \lambda^*.$$

Proof: See [9]. ☐

From Property 5.3, we can derive the followings.

i) There exists an unique λ such that $F_\lambda=c$.

ii) If $F_\lambda=c$, then (x_λ,y_λ) is the optimal solution of Problem AP, and if $F_\lambda\neq c$, then we replace current value of λ by a better one, e.g., $\lambda=V(x_\lambda,y_\lambda)$.

Further we consider the following auxiliary problem AP^R of subproblem $AP(\lambda)$.

$$AP^R:\quad \text{Minimize } G(x,y,;R)=R\sum_{i=1}^{n}\mu_i d_i(x,y)+\sum_{i=1}^{n}\sigma_i^2(d_i(x,y))^2.$$

Let (x^R,y^R) denote the optimal solution of this problem. Then the following properties hold.

Property 5.4 For $R\geq 0$, $G(x,y;R)$ is a convex function of (x,y).

Proof: The first term is convex for $R\geq 0$ as well as Property 5.2. The convexity of $(d_i(x,y))^2$ is ensured as follows:

For $0\leq\alpha\leq 1$, $(\bar{\alpha}=1-\alpha)$

$$\alpha(d_i(x_1,y_1))^2+\bar{\alpha}(d_i(x_2,y_2))^2-(d_i(\alpha x_1+\bar{\alpha}x_2,\alpha y_1+\bar{\alpha}y_2))^2$$

$$\geq\alpha(d_i(x_1,y_1))^2+\bar{\alpha}(d_i(x_2,y_2))^2-(\alpha d_i(x_1,y_1)+\bar{\alpha}d_i(x_2,y_2))^2$$

(by the convexity of $d_i(x,y)$)

$$=\alpha\bar{\alpha}(d_i(x_1,y_1)-d_i(x_2,y_2))^2\geq 0.$$

Thus $G(x,y;R)$ is convex because it is the positive sum of convex functions. □

Property 5.5 Suppose $x^R\neq a_i$ and $y^R\neq b_j$, $i=1,2,\ldots,n$; $j=1,2,\ldots,n$. Then (x^R,y^R) is an optimal solution of Problem $AP(\lambda)$ if and only if $\lambda R=2D(x^R,y^R)$.

Proof: For $x\neq a_i$, $y\neq b_j$, $i=1,2,\ldots,n$; $j=1,2,\ldots,n$, both $F(x,y;\lambda)$ and $G(x,y;R)$ are differentiable, then

$$\frac{\partial F(x,y;\lambda)}{\partial x}\bigg|_{x=x^R,\ y=y^R}=\frac{\partial G(x,y;R)}{\partial x}\bigg|_{x=x^R,\ y=y^R}=0,$$

$$\frac{\partial F(x,y;\lambda)}{\partial y}\bigg|_{x=x^R,\ y=y^R}=\frac{\partial G(x,y;R)}{\partial y}\bigg|_{x=x^R,\ y=y^R}=0,$$

if and only if $\lambda R=2D(x^R,y^R)$. Therefore (x^R,y^R) is an optimal solution of $P(\lambda)$ by Property 5.2 and Property 5.4. □

If $x^R=a_m$ or $y^R=b_m$ for some m, then the above property cannot be used to solve the problem. So we consider the following restricted problems.

$$AP_{x,m}(\lambda):\quad \text{Minimize}\ \sum_{i=1}^{n}\mu_i d_i(x,b_m)+\lambda\sqrt{\sum_{i=1}^{n}\sigma_i^2(d_i(x,b_m))^2}.$$

$$AP^R_{x,m}:\quad \text{Minimize}\ G(x,b_m;R)=R\sum_{i=1}^{n}\mu_i d_i(x,b_m)+\sum_{i=1}^{n}\sigma_i^2(d_i(x,b_m))^2.$$

$$AP_{y,m}(\lambda):\quad \text{Minimize}\ \sum_{i=1}^{n}\mu_i d_i(a_m,y)+\lambda\sqrt{\sum_{i=1}^{n}\sigma_i^2(d_i(a_m,y))^2}.$$

$$AP^R_{y,m}:\quad \text{Minimize}\ G(a_m,y;R)=R\sum_{i=1}^{n}\mu_i d_i(a_m,y)+\sum_{i=1}^{n}\sigma_i^2(d_i(a_m.y))^2.$$

Then the following corollary of Property 5.5 holds.

Corollary 5.2 Suppose $x^R\neq a_i$, $i=1,2,...,n$, then x^R is an optimal solution for Problem $AP_{x,m}(\lambda)$ if $\lambda R=2D(x^R,b_m)$. And suppose $y^R\neq b_j$, $j=1,2,...,n$, then y^R is an optimal solution for Problem $AP_{y,m}(\lambda)$ if $\lambda R=2D(a_m,y^R)$.

Proof: This corollary can be easily proved by the similar manner to Property 5.5. □

Now we rearrange a_i's and b_i's according to the ascending order such as $\bar{a}_1<\bar{a}_2<\cdots<\bar{a}_j$ and $\bar{b}_1<\bar{b}_2<\cdots<\bar{b}_k$ respectively, where j and k are numbers of different a_i's and b_i's. And we divide the area $A=\{(x,y)\,|\,\bar{a}_1\leq x\leq\bar{a}_j,\ \bar{b}_1\leq y\leq\bar{b}_k\}$ into three types of areas $S_{h,m}$, $T^x_{h,m}$ and $T^y_{h,m}$, and point sets $U_{h,m}$ which are disjoint each other.

$$S_{h,m} \overset{\Delta}{=} \{(x,y) \mid \bar{a}_h < x < \bar{a}_{h+1},\ \bar{b}_m < y < \bar{b}_{m+1}\},\quad h=1,2,\dots,j-1$$
$$m=1,2,\dots,k-1,$$

$$T^x_{h,m} \overset{\Delta}{=} \{(x,y) \mid \bar{a}_h < x < \bar{a}_{h+1},\ y=\bar{b}_m\},\ h=1,2,\dots,j-1;\ m=1,2,\dots,k,$$

$$T^y_{h,m} \overset{\Delta}{=} \{(x,y) \mid x=\bar{a}_h,\ \bar{b}_m < y < \bar{b}_{m+1}\},\ h=1,2,\dots,j;\ m=1,2,\dots,k-1,$$

$$U_{h,m} \overset{\Delta}{=} \{(\bar{a}_h,\bar{b}_m)\},\ h=1,2,\dots,j;\ m=1,2,\dots,k.$$

The optimal solution of Problem AP^R, i.e., (x^R,y^R), is contained in only one of the above sets. Therefore we concentrate the search on one set at a time.

[Area $S_{h,m}$]

We consider (x,y) contained in the area $S_{h,m}$. Define the following disjoint index sets:

$$I_1(h,m) \overset{\Delta}{=} \{\, i \mid (a_i,b_i) \in (-\infty,\bar{a}_h] \times (-\infty,\bar{b}_m]\},$$

$$I_2(h,m) \overset{\Delta}{=} \{\, i \mid (a_i,b_i) \in [\bar{a}_{h+1},\infty) \times (-\infty,\bar{b}_m]\},$$

$$I_3(h,m) \overset{\Delta}{=} \{\, i \mid (a_i,b_i) \in (-\infty,\bar{a}_h] \times [\bar{b}_{m+1},\infty)\},$$

$$I_4(h,m) \overset{\Delta}{=} \{\, i \mid (a_i,b_i) \in [\bar{a}_{h+1},\infty) \times [\bar{b}_{m+1},\infty)\},$$

where × denotes the Cartesian product. Then we can easily find

$$I_1(h,m) \cup I_2(h,m) \cup I_3(h,m) \cup I_4(h,m) = \{1,2,\dots,n\}.$$

For $(x,y) \in S_{h,m}$, we can differentiate $G(x,y;R)$ as follows:

$$g_x(x,y;R) = \frac{\partial G(x,y;R)}{\partial x} = R(M_1-M_2)+2(S_1+S_2)x+2(S_1-S_2)y-2(H_1+H_2), \tag{5.48}$$

$$g_y(x,y;R) = \frac{\partial G(x,y;R)}{\partial y} = R(M_1+M_2)+2(S_1-S_2)x+2(S_1+S_2)y-2(H_1-H_2), \tag{5.49}$$

where $M_1 = \sum_{I_1} \mu_i - \sum_{I_4} \mu_i$,

$M_2 = \sum_{I_2} \mu_i - \sum_{I_3} \mu_i$,

$S_1 = \sum_{I_1} \sigma_i^2 + \sum_{I_4} \sigma_i^2$,

$S_2 = \sum_{I_2} \sigma_i^2 + \sum_{I_3} \sigma_i^2$,

$H_1 = \sum_{I_1} \sigma_i^2(a_i+b_i) + \sum_{I_4} \sigma_i^2(a_i+b_i)$,

$H_2 = \sum_{I_2} \sigma_i^2(a_i-b_i) + \sum_{I_3} \sigma_i^2(a_i-b_i)$,

and I_i stands for $I_i(h,m)$, $i=1,2,\ldots,4$. Let $(\bar{x}_{h,m}(R),\bar{y}_{h,m}(R))$ denote the solution of the following equations:

$$g_x(x,y;R) = 0, \tag{5.50}$$

$$g_y(x,y;R) = 0. \tag{5.51}$$

Then $\bar{x}_{h,m}(R)$ and $\bar{y}_{h,m}(R)$ are as follows:

$$\begin{aligned} \bar{x}_{h,m}(R) &= \frac{-(M_1S_2-M_2S_1)R+2(H_1S_2+H_2S_1)}{4S_1S_2}, \\ \bar{y}_{h,m}(R) &= \frac{-(M_1S_2+M_2S_1)R+2(H_1S_2-H_2S_1)}{4S_1S_2}, \end{aligned} \tag{5.52}$$

where $S_1S_2 > 0$ since σ_i^2, $i=1,2,\ldots,n$, are positive. If for $R > 0$ the above solution satisfies the following inequalities;

$$\bar{a}_h < \bar{x}_{h,m}(R) < \bar{a}_{h+1} \text{ and } \bar{b}_m < \bar{y}_{h,m}(R) < \bar{b}_{m+1},$$

then it is the optimal solution of AP^R, i.e., (x^R,y^R); otherwise, (x^R,y^R) does not exist in $S_{h,m}$. Now we define the area of R,

$$\mathbb{R}_{h,m} = \{R > 0 \mid \bar{a}_h < \bar{x}_{h,m} < \bar{a}_{h+1} \text{ and } \bar{b}_m < \bar{y}_{h,m}(R) < \bar{b}_{m+1}\}.$$

From Property 5.5, when $x^R \neq a_i$, $y^R \neq b_j$, $i=1,2,\ldots,n$; $j=1,2,\ldots,n$,

$$\lambda R = 2 \sum_{i=1}^{n} \sigma_i^2 (|x^R - a_i| + |y^R - b_i|)^2, \tag{5.53}$$

if and only if (x^R,y^R) is an optimal solution of Problem AP(λ). In addition from Property 5.3,

$$F(x^R,y^R;\lambda) = c, \tag{5.54}$$

if and only if (x^R,y^R) is an optimal solution of Problem AP. Therefore if there exist the parameters λ and R satisfying the above two equations (5.53) and (5.54), then (x^R,y^R) is an optimal solution of Problem AP. Multiplying (5.54) by R and substituting it into (5.53), we have

$$R \sum_{i=1}^{n} \mu_i (|x^R - a_i| + |y - b_i|) + 2 \sum_{i=1}^{n} \sigma_i^2 (|x^R - a_i| + |y^R - b_i|)^2 = cR.$$

This equation is rewritten as follows:

$$R\{(M_1 - M_2)x^R + (M_1 + M_2)y^R - A_1\} + 2(S_1 + S_2)\{(x^R)^2 + (y^R)^2\}$$
$$+4(S_1 - S_2)x^R y^R - 4H_1(x^R + y^R) - 4H_2(x^R - y^R) + 2A_2 = cR, \tag{5.55}$$

where A_1 and A_2 are defined as follows:

$$A_1 = \sum_{I_1} \mu_i (a_i + b_i) - \sum_{I_2} \mu_i (a_i - b_i) + \sum_{I_3} \mu_i (a_i - b_i) - \sum_{I_4} \mu_i (a_i + b_i),$$

$$A_2=\sum_{I_1}\sigma_i^2(a_i+b_i)^2+\sum_{I_2}\sigma_i^2(a_i-b_i)^2+\sum_{I_3}\sigma_i^2(a_i-b_i)^2+\sum_{I_4}\sigma_i^2(a_i+b_i)^2.$$

While equations (5.50) and (5.51) are rewritten as follows:

$$R(M_1-M_2)+2(S_1+S_2)x^R+2(S_1-S_2)y^R-2(H_1+H_2)=0, \tag{5.56}$$

$$R(M_1+M_2)+2(S_1-S_2)x^R+2(S_1+S_2)y^R-2(H_1-H_2)=0. \tag{5.57}$$

From (5.55), (5.56) and (5.57), we have

$$2(H_1+H_2)x^R+2(H_1-H_2)y^R+(A_1+c)R-2A_2=0. \tag{5.58}$$

If $\mathbb{R}_{h,m} \neq \phi$, for $R \in \mathbb{R}_{h,m}$ x^R and y^R are given by (5.52). Therefore the equation (5.58) is linear in R. Then if we define R satisfying the equation (5.58) as R', R' is given as follows:

$$R'=\frac{H_1^2S_2+H_2^2S_1-2A_2S_1S_2}{H_1M_1S_2-H_2M_2S_1-S_1S_2(A_1+c)}. \tag{5.59}$$

If $R' \in \mathbb{R}_{h,m}$, then $(x^{R'},y^{R'})$ is an optimal solution of Problem AP.

[Area $T^x_{h,m}$]

If the optimal solution does not exist in any area $S_{h,m}$, then it exists on $T^x_{h,m}$, $T^y_{h,m}$ or $U_{h,m}$. At first we consider the case that it exists on $T^x_{h,m}$. Then the following property holds.

Property 5.6 For some R (> 0), if

$$\bar{y}_{h,m}(R) \leq \bar{b}_m,\ \bar{y}_{h,m-1} \geq \bar{b}_m, \tag{5.60}$$

$$g_x(\bar{a}_h,\bar{b}_m;R)< 0 \text{ and } g_x(\bar{a}_{h+1},\bar{b}_m;R)> 0, \tag{5.61}$$

then the optimal solution of Problem AP^R exists on $T^x_{h,m}$.

Proof: From (5.61) the minimum solution of $G(x,\bar{b}_m;R)$ exists in $(\bar{a}_h,\bar{a}_{h+1})$ and it is denoted by $x^o_{h,m}$. Now consider sufficiently small neighborhood of $x^o_{h,m}$ denoted by $N(x^o_{h,m})$. For any $(x,y) \in N(x^o_{h,m}) \cap S_{h,m-1}$, let $(x',\bar{b}_m)$ denote the intersection of $T^x_{h,m}$ and line segment between $(\bar{x}_{h,m-1}(R),\bar{y}_{h,m-1}(R))$ and (x,y). Then

$$G(x,y;R) > G(x',\bar{b}_m;R) \qquad \text{(by the convexity of } G(x,y;R))$$

$$> G(x^o_{h,m},\bar{b}_m;R) \qquad \text{(by the definition of } x^o_{h,m}).$$

Similarly for any $(x,y) \in N(x^o_{h,m}) \cap S_{h,m}$,

$$G(x,y;R) > G(x^o_{h,m},\bar{b}_m;R).$$

Therefore $(x^o_{h,m},\bar{b}_m)$ is a local optimal solution of Problem AP^R. Since $G(x,y;R)$ is convex, $(x^o_{h,m},\bar{b}_m)$ is a global optimal solution. ☐

If (5.60) holds, we define $\hat{x}_{h,m}(R)$ as follows:

$$\hat{x}_{h,m}(R) = \frac{-R(M_1-M_2)-2(S_1-S_2)\bar{b}_m+2(H_1+H_2)}{2(S_1+S_2)}. \tag{5.62}$$

Then R satisfying the condition (5.61) is

$$\mathbb{R}^x_{h,m} = \{R > 0 \mid \bar{a}_h < \hat{x}_{h,m}(R) < \bar{a}_{h+1}\}.$$

If $\mathbb{R}^x_{h,m} = \phi$, then the optimal solution of Problem AP^R does not exist in $(\bar{a}_h,\bar{a}_{h+1})$; otherwise, $\hat{x}_{h,m}(R)$ is the optimal solution of Problem AP^R, i.e., $x^R=\hat{x}_{h,m}(R)$. From Corollary 5.2, if

$$\lambda R = 2\sqrt{\sum_{i=1}^{n} \sigma_i^2(|x^R-a_i| + |\bar{b}_m-b_i|)^2},$$

then $(x^R,\bar{b}_m)$ is the optimal solution of $AP_{x,m}(\lambda)$. In addition,

$$F(x^R,\bar{b}_m;\lambda) = c$$

if and only if $(x^R,\bar{b}_m)$ is an optimal solution of Problem AP. From the above two equations, we have

$$R\sum_{i=1}^{n}\mu_i(|x^R-a_i|+|\bar{b}_m-b_i|)+2\sum_{i=1}^{n}\sigma_i^2(|x^R-a_i|+|\bar{b}_m-b_i|)^2=cR.$$

This equation is rewritten as follows:

$$R\{(M_1-M_2)x^R+(M_1+M_2)\bar{b}_m-A_1\}+2(S_1+S_2)\{(x^R)^2+\bar{b}_m^2\}+4(S_1-S_2)\bar{b}_m x^R$$
$$-4H_1(x^R+\bar{b}_m)-4H_2(x^R-\bar{b}_m)+2A_2=cR. \tag{5.63}$$

Since $(x^R,\bar{b}_m)$ is the solution of (5.50), we have

$$R(M_1-M_2)+2(S_1+S_2)x^R+2(S_1-S_2)\bar{b}_m-2(H_1+H_2)=0. \tag{5.64}$$

From (5.63) and (5.64) we have

$$2\{H_1+H_2-(S_1-S_2)\bar{b}_m\}x^R+\{A_1-(M_1+M_2)\bar{b}_m+c\}R$$
$$-2\{(S_1+S_2)\bar{b}_m^2-2(H_1-H_2)\bar{b}_m+A_2\}=0. \tag{5.65}$$

If $\mathbb{R}^x_{h,m}\neq\phi$, for $R\in\mathbb{R}^x_{h,m}$ (5.65) is a linear equation of R. Therefore substituting $\hat{x}_{h,m}(R)$ into x^R and solving the linear equation, we have the following solution R''.

$$R''=\frac{2\{(H_1+H_2)^2+4(H_1S_2-H_2S_1)\bar{b}_m-4S_1S_2\bar{b}_m^2-A_2(S_1+S_2)\}}{(H_1+H_2)(M_1-M_2)-(A_1+c)(S_1+S_2)+2(S_1M_2+S_2M_1)\bar{b}_m}. \tag{5.66}$$

If R'' is contained in $\mathbb{R}^x_{h,m}$, then $(x^R,\bar{b}_m)$ is an optimal solution of Problem AP. In case of $T^y_{h,m}$, we have the following corollary of Property 5.6.

Corollary 5.3 For some R (> 0), if

$$\bar{x}_{h,m}(R)<\bar{a}_h,\ \bar{x}_{h,m-1}>\bar{a}_h, \tag{5.67}$$

$$g_y(\bar{a}_h,\bar{b}_m;R)<0 \text{ and } g_y(\bar{a}_h,\bar{b}_{m+1};R)>0, \tag{5.68}$$

then the optimal solution of Problem AP^R exists on $T^y_{h,m}$.

Proof: We can prove it similarly to Property 5.6. □

In addition $\hat{y}_{h,m}(R)$ and R'' are given as follows.

$$\hat{y}_{h,m}(R)=\frac{-R(M_1+M_2)-2(S_1-S_2)\bar{a}_h+2(H_1-H_2)}{2(S_1+S_2)}, \tag{5.69}$$

$$R''=\frac{2\{(H_1-H_2)^2+4(H_1S_2+H_2S_1)\bar{a}_h-4S_1S_2\bar{a}_h^2-A_2(S_1+S_2)\}}{(H_1-H_2)(M_1+M_2)-(A_1+c)(S_1+S_2)-2(S_1M_2-S_2M_1)\bar{a}_h}. \tag{5.70}$$

[Point Set $U_{h,m}$]

If the optimal solution does not exist in the above areas, then it exists in $U_{h,m}$. The following property holds.

Property 5.7 If for some $R>0$

$$\hat{x}_{h,m}(R)\leq\bar{a}_h \text{ and } \hat{x}_{h-1,m}(R)\geq\bar{a}_h$$

or

$$\hat{y}_{h,m}(R)\leq\bar{b}_m \text{ and } \hat{y}_{h,m-1}(R)\geq\bar{b}_m, \tag{5.71}$$

then $(\bar{a}_h,\bar{b}_m)$ is the optimal solution of Problem AP^R.

Proof: $\hat{x}_{h,m}(R)$ is defined only when $\bar{y}_{h,m}(R)<\bar{b}_m$ and $\bar{y}_{h,m-1}(R)>\bar{b}_m$ hold. Therefore $(\bar{a}_h,\bar{b}_m)$ is the optimal solution of Problem AP^R by the convexity of $G(x,y;R)$. □

Let $\hat{\mathbb{R}}_{h,m}\overset{\Delta}{=}\{R>0\,|\,R \text{ satisfies } (5.71)\}$.

Now we obtain the algorithm by using the above properties.

Algorithm

Step 1: Rearranging a_i and $b_i (i=1,2,...,n)$ in ascending order of magnitude respectively. Set $h\longleftarrow[j/2]$, $m\longleftarrow[k/2]$ and go to Step 2.

Step 2: Consider Problem AP^R in the area $S_{h,m}$. If $\mathbb{R}_{h,m}\neq\phi$, then go to Step 3; otherwise go to Step 4.

Step 3: If $R' \in \mathbb{R}_{h,m}$, then $x^* \longleftarrow \bar{x}_{h,m}(R')$, $y^* \longleftarrow \bar{y}_{h,m}(R')$ and stop; otherwise go to Step 4.

Step 4: If $\hat{x}_{h,m}(R)$ (or $\hat{y}_{h,m}(R)$) is not defined, then go to Step 6; otherwise go to Step 5.

Step 5: If there exists R satisfying (5.60) (or (5.67)), then go to Step 7; otherwise go to Step 6.

Step 6: If $\bar{x}_{h,m}(R') \leq \bar{a}_h$ (or $\bar{x}_{h,m}(R') \geq \bar{a}_{h+1}$), then $h' \longleftarrow h$, $h \longleftarrow h-1$ (or $h \longleftarrow h+1$, $h' \longleftarrow h$) and return to Step 2. If $\bar{y}_{h,m}(R') \leq \bar{b}_m$ (or $\bar{y}_{h,m}(R') \geq \bar{b}_{m+1}$), then $m' \longleftarrow m$, $m \longleftarrow m-1$ (or $m \longleftarrow m+1$, $m' \longleftarrow m$) and return to Step 2.

Step 7: Consider Problem $AP^R_{x,m'}$ (or $AP^R_{y,h'}$) in the area $T^x_{h',m'}$ (or $T^y_{h',m'}$). If $\mathbb{R}^x_{h',m'}=\phi$ (or $\mathbb{R}^y_{h',m'}=\phi$), then go to Step 9; otherwise go to Step 8.

Step 8: If $R'' \in \mathbb{R}^x_{h,m}$ (or $R'' \in \mathbb{R}^y_{h,m}$), $y^* \longleftarrow \bar{b}_m$ (or $x^* \longleftarrow \bar{a}_h$) and $x^* \longleftarrow \hat{x}_{h,m}(R'')$ (or $y^* \longleftarrow \hat{y}_{h,m}(R'')$) and stop; otherwise go to Step 9.

Step 9: If $\hat{x}_{h,m}(R'') \leq \bar{a}_h$ (or $\hat{y}_{h,m}(R'') \leq \bar{b}_m$), $h' \longleftarrow h$, $h \longleftarrow h-1$ (or $m' \longleftarrow m$, $m \longleftarrow m-1$). If $\hat{x}_{h,m}(R'') \geq \bar{a}_{h+1}$ (or $\hat{y}_{h,m}(R'') \geq \bar{b}_{m+1}$), $h \longleftarrow h+1$, $h' \longleftarrow h$ (or $m \longleftarrow m+1$, $m' \longleftarrow m$). If the condition (5.71) is satisfied, then go to Step 10. Otherwise, return to Step 2.

Step 10: Set $x^* \longleftarrow \bar{a}_{h'}$, $y^* \longleftarrow \bar{b}_{m'}$ and stop.

Theorem 5.1 The above algorithm finds an optimal solution in at most $O(n^3)$ time.

Proof: The number of areas is at most $(n-1)^2+2n(n-1)+n^2$ and the computational time necessary to search each area is $O(n)$. Therefore this algorithm finds an optimal solution in at most $O(n^3)$ time. □

In the following, we give some toy examples illustrating the above algorithm.

Example 5.3 The location (a_i, b_i) and the mean μ_i and variance σ_i^2 of weight W_i are given in Table 5.3. And the available cost c is 1000.

Step 1: $\bar{a}_1=0$, $\bar{a}_2=1$, $\bar{a}_3=3$, $\bar{a}_4=4$, $\bar{a}_5=7$, $\bar{b}_1=-3$, $\bar{b}_2=-1$, $\bar{b}_3=2$, $\bar{b}_4=3$, $\bar{b}_5=4$, $h=2$ and $m=2$.

Step 2: $S_{22}=\{(x,y) |\ 1 < x < 3,\ -1 < y < 2\}$. $\mathbb{R}_{22}=\phi$.

Step 4: Neither $\hat{x}_{h,m}(R)$ nor $\hat{y}_{h,m}(R)$ are not defined.

Step 6: $\bar{x}_{22} \geq 3$, and therefore $h=3$ and $h'=3$.

Step 2: $S_{32}=\{(x,y) |\ 3 < x < 4,\ -1 < y < 2\}$ and $\mathbb{R}_{32} \neq \phi$.

Step 3: $R'=13.82$ and $\mathbb{R}_{h,m}=\{R |\ 9.04 < R < 18.09\}$, therefore $R' \in \mathbb{R}_{h,m}$. $x^*=3.53$ and $y^*=1.32$.

Example 5.4 (a_i, b_i) and c are the same as in Example 5.3. μ_i and σ_i^2 are given in Table 5.4.

Step 1, Step 2, Step 4 and Step 6 are the same as in Example 5.3.

Step 2: $S_{32}=\{(x,y) |\ 3 < x < 4,\ -1 < y < 2\}$ and $\mathbb{R}_{32}=\phi$.

Step 4: $\hat{y}_{h,m}(R)$ is defined.

Step 5: The condition (5.67) is satisfied for some R (e.g., R=5).

Step 7: $T_{32}^{y}=\{(3,y) |\ -1 < y < 2\}$ and $\mathbb{R}_{32}^{y} \neq \phi$.

Step 8: $R''=28.13$ and $\mathbb{R}_{32}^{y}=\{R |\ 0 < R < 48\}$. Since $R'' \in \mathbb{R}_{32}^{y}$, $x^*=3$ and $y^*=1.32$.

Example 5.5 (a_i, b_i) and c are the same as in Example 5.3. μ_i and σ_i^2 are given in Table 5.5.

The first seven steps are advanced in the same way as in Example 5.4.

Step 7: $T_{32}^{y}=\{(3,y) |\ -1 < y < 2\}$ and $\mathbb{R}_{32}^{y}=\phi$.

Step 9: Since $\hat{y}_{32}(R) > 2$, $m=3$ and $m'=3$. The condition (5.71) is not satisfied.

Step 2: $S_{33}=\{(x,y) |\ 3 < x < 4,\ 2 < y < 3\}$ and $\mathbb{R}_{33}=\phi$.

Step 4: $\hat{x}_{33}(R)$ is defined.

Step 5: The condition (5.60) is satisfied for some R.

Step 7: $T^{x}_{33}=\{(x,2) \mid 3 < x < 4\}$ and $\mathbb{R}^{x}_{33}=\phi$.

Step 9: Since $\hat{x}_{33}(R) < 3$, $h'=3$ and $h=2$. The condition (5.71) is not satisfied.

Step 2: $S_{23}=\{(x.y) \mid 1 < x < 3,\ 2 < y < 3\}$ and $\mathbb{R}_{23}=\phi$.

Step 4: $\hat{y}_{33}(R)$ is defined.

Step 5: The condition (5.67) is satisfied for some R.

Step 7: $T^{y}_{33}=\{(3,y) \mid 2 < y < 3\}$ and $\mathbb{R}^{y}_{33}=\phi$.

Step 9: Since $\hat{y}_{33}(R) < 2$, $m'=3$ and $m=2$. The condition (5.71) is satisfied.

Step 10: $x^{*}=3$ and $y^{*}=2$.

The above examples show the three cases whose optimal solutions are in $S_{h,m}$, $T^{y}_{h,m}$ and $U_{h,m}$ respectively.

Table 5.3. Data for Example 5.3.

i	(a_i, b_i)	μ_i	σ_i^2
1	(0,2)	22	18
2	(1,4)	28	11
3	(3,-3)	24	21
4	(4,3)	35	12
5	(7,-1)	55	17

Table 5.4. Data for Example 5.4.

	1	2	3	4	5
μ_i	22	32	52	50	47
σ_i^2	12	11	21	12	17

Table 5.5. Data for Example 5.5.

	1	2	3	4	5
μ_i	25	32	46	35	30
σ_i^2	10	8	9	12	5

5.5 Chance Constrained Minimax Facility Location Problem

There appears some demand points in the plane and their locations and the number are known only probabilistically. Let (X_i,Y_i) and N denote the location of i-th demand point and the number of demand points respectively. We assume that (X_i,Y_i) are independent, identically distributed (i.i.d.) variates and that N has a Poisson distribution with positive parameter β.

Our objective is to decide the reachable distance r in a restricted time and the location (x,y) of facility to solve the following problem.

$$MP_0: \quad \text{Minimize } r$$

$$\text{subject to } \Pr\{\max_i D_i(x,y) \leq r\} \geq \alpha,$$

$$\text{where} \quad D_i(x,y) \overset{\Delta}{=} |X_i - x| + |Y_i - y| \tag{5.72}$$

and $0 \leq \alpha \leq 1$. Now let $\{(x^*,y^*),r^*\}$ denote an optimal solution of this problem.

To solve the problem, we introduce the following subproblem MP(r) with nonnegative parameter r.

$$MP(r): \quad \text{Maximize } \Pr\{\max_i D_i(x,y) \leq r\}.$$

The optimal solution and the optimal value of this problem P(r) are denoted by (x_r,y_r) and $f(r)$ respectively. Then the following property holds.

Property 5.8 Let $r' \overset{\Delta}{=} \min\{r \mid f(r) \geq \alpha,\ r \geq 0\}$, then $\{(x_{r'},y_{r'}),$ $r'\}$ is an optimal solution of MP_0.

Proof: The chance constraint of Problem MP_0 is satisfied by $\{(x_{r'},y_{r'}),r'\}$. And f(r) is a nondecreasing function. Therefore if $r^* < r'$, then by the definition of r' we have $f(r^*) < \alpha$. It contradicts that r* is feasible for Problem MP_0. Hence r*=r'. ☐

Since (X_i, Y_i) are i.i.d. variates, $D_i(x,y)$ are also i.i.d. variates. Thus we have

$$\Pr\{\max_i D_i(x,y) \le r \mid N=n\} = (\Pr\{D(x,y) \le r\})^n \text{ for } n \ge 1, \quad (5.73)$$

where we drop the subscript of $D_i(x,y)$. If there are not any demand points, i.e., n=0, then the maximum distance between the facility and demand points is zero, hence

$$\Pr\{\max_i D_i(x,y) \le r \mid N=0\} = 1. \quad (5.74)$$

Therefore we have

$$\Pr\{\max_i D_i(x,y) \le r\} = \sum_{n=0}^{\infty} \Pr\{\max_i D_i(x,y) \le r \mid N=n\} \Pr\{N=n\}$$

$$= \sum_{n=0}^{\infty} (\Pr\{D(x,y) \le r\})^n \beta^n e^{-\beta}/n!$$

(by (5.73) and (5.74))

$$= \exp[\beta \Pr\{D(x,y) \le r\} - \beta]. \quad (5.75)$$

Now we consider the following problem MP'(r).

$$MP'(r): \quad \text{Maximize } \Pr\{D(x,y) \le r\}.$$

Then Problem MP(r) is equivalent to the above Problem MP'(r) from (5.75). Thus if we define the optimal value of MP'(r) by g(r), we have

$$f(r) = \exp\{\beta g(r) - \beta\}. \quad (5.76)$$

If $\alpha \le e^{-\beta}$, then from (5.75) the chance constraint of Problem MP_0 holds for any $\{(x,y),r\}$, $r > 0$. Thus we consider the nontrivial case $\alpha > e^{-\beta}$ hereinafter.

Now we consider the case that the demand points are uniformly distributed in a rectangular area $U \overset{\Delta}{=} \{(x,y) \mid 0 \leq x \leq a,\ 0 \leq y \leq b\}$. We assume $a \geq b$ without any loss of generality. Then the following property holds.

Property 5.9 For any x, y and r, the following inequalities hold.

$$\Pr\{D(x,b/2) \leq r\} \geq \Pr\{D(x,y) \leq r\},$$

$$\Pr\{D(a/2,y) \leq r\} \geq \Pr\{D(x,y) \leq r\}.$$

Proof: The probability $\Pr\{D(x,y) \leq r\}$ is in proportion to the area of intersection of the rectangular U and the rotated square area V(x,y) (see Figure 5.2), where $V(x,y) \overset{\Delta}{=} \{(x',y') \mid |x'-x|+|y'-y| \leq r\}$. Therefore the proof of this property is easily done. ☐

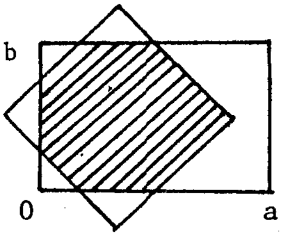


Figure 5.2. Intersection of U and V.(Shaded area.)

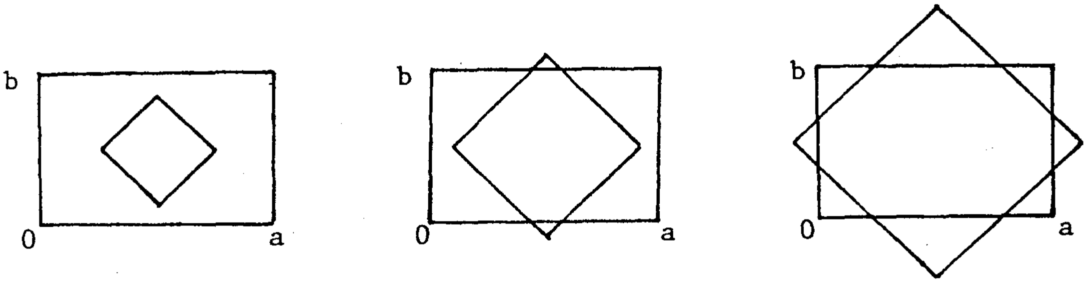
From this property, it follows that (a/2,b/2) is an optimal location of the problem MP'(r). Therefore we have only to consider three cases. (See Figure 5.3.) If $r > (a+b)/2$, then we can cover the area U with the area V entirely. Therefore we have

$$g(r) = 1, \tag{5.77}$$

and by (5.76)

$$f(r) = 1, \tag{5.78}$$

that is, we can reach any point of the rectangular area U with prob-



(i) $0 \leqq r \leqq \frac{b}{2}$ (ii) $\frac{b}{2} < r \leqq \frac{a}{2}$ (iii) $\frac{a}{2} < r \leqq \frac{a+b}{2}$

Figure 5.3 Three cases.

ability 1. Hence from Property 5.8 we do not consider the case r > (a+b)/2 anymore.

The optimal value of Problem P'(r) in three cases of Figure 5.3 are

$$g(r) = \frac{2r^2}{ab}, \tag{5.79}$$

$$g(r) = \frac{4r-b}{2a}, \tag{5.80}$$

$$g(r) = 1-\frac{(a+b-2r)^2}{2ab} \tag{5.81}$$

respectively. Therefore from Property 5.8, (5.79), (5.80) and (5.81) we have

$$r^* = \frac{ab(1+\ln \alpha/\beta\ )}{2}, \tag{5.82}$$

$$r^* = \frac{2a(1+\ln \alpha/\beta\ )+b}{4}, \tag{5.83}$$

$$r^* = \frac{a+b-\sqrt{-2ab\ln\ \alpha/\beta}}{2} \tag{5.84}$$

respectively for the cases (i), (ii) and (iii) in Figure 5.3. Transforming the intervals of r into corresponding ones of α, we obtain the parametric solutions. (See Table 5.6.)

Table 5.6. Parametric solution in the uniform case.

Interval of α	(x^*,y^*)	r^*
$0 \leqq \alpha \leqq e^{-\beta}$	$(\frac{a}{2},\frac{b}{2})$	0
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{b}{2a})}$	$(\frac{a}{2},\frac{b}{2})$	(5.82)
$e^{-\beta(1-\frac{b}{2a})} < \alpha \leqq e^{-\frac{\beta b}{2a}}$	$(\frac{a}{2},\frac{b}{2})$	(5.83)
$e^{-\frac{\beta b}{2a}} < \alpha \leqq 1$	$(\frac{a}{2},\frac{b}{2})$	(5.84)

Next we partition a rectangular region U into two subregions $U_1 \triangleq \{(x,y) \mid 0 \leqq x \leqq a_1,\ 0 \leqq y \leqq b\}$ and $U_2 \triangleq \{(x,y) \mid a_1 \leqq x \leqq a,\ 0 \leqq y \leqq b\}$ in which the demand points have uniform distributions with densities q_1 (>0) and q_2 (>0) respectively. Then the following equation holds.

$$a_1 b q_1 + (a-a_1) b q_2 = 1. \tag{5.85}$$

In this section we do not assume $a \geqq b$. But we assume $q_1 \geqq q_2 > 0$ without any loss of generality. Then we obtain the following property.

Property 5.10 The following inequalities hold.

(i) $\Pr\{D(x,b/2) \leqq r\} \geqq \Pr\{D(x,y) \leqq r\}$,

(ii) $\Pr\{D(a_1/2,b/2) \leqq r\} \geqq \Pr\{D(x,b/2) \leqq r\}$ for $r \leqq a_1/2$,

(iii) $\Pr\{D(r,b/2) \leq r\} \geq \Pr\{D(x,b/2) \leq r\}$ for $r > a_1/2$ and $x \geq r$

(iv) $\Pr\{D(a_1-r,b/2) \leq r\} \geq \Pr\{D(x,b/2) \leq r\}$ for $r > a_1/2$ and $x \leq a_1-r$.

Proof: Let $A(\cdot)$ denote a function which represents the size of areas on R^2. Then we have

$$\Pr\{D(x,y) \leq r\} = q_1 A(V(x,y) \cap U_1) + q_2 A(V(x,y) \cap U_2), \qquad (5.86)$$

where $V(s,t) = \{(x,y) \mid |x-s|+|y-s| \leq t\}$.

The equation (5.86) can be rewritten as follows:

$$\Pr\{D(x,y) \leq r\} = (q_1-q_2) A(V(x,y) \cap U_1) + q_2 A(V(x,y) \cap U). \qquad (5.87)$$

(i) Both of the areas U_1 and U_2 are considered separately. Therefore we can prove this inequality similarly as Property 5.9.

(ii) If $r \leq a_1/2$, then $(x,y)=(a_1/2,b/2)$ maximizes $A(V(x,y) \cap U_1)$ and $A(V(x,y) \cap U)$. Therefore this inequality is proved from (5.87).

(iii) For $r > a_1/2$ and $x \geq r$, both $A(V(x,b/2) \cap U_1)$ and $A(V(x,b/2) \cap U)$ are nonincreasing in x. (See Figure 5.4.)

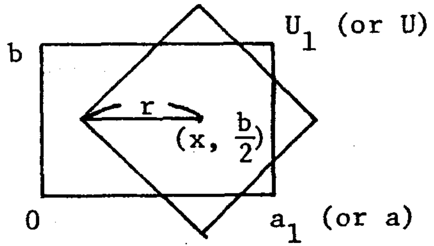


Figure 5.4. $r > \frac{a_1}{2}$, $x \geq r$.

(iv) For $r > a_1/2$ and $x \leq a_1-r$, both $A(V(x,b/2) \cap U_1)$ and $A(V(x,b/2) \cap U)$ are nondecreasing in x. (See Figure 5.5.) □

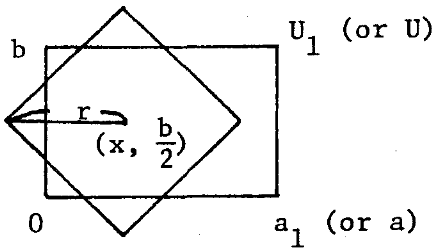


Figure 5.5. $r > \frac{a_1}{2}$, $x \leqq a_1 - r$.

From this property it follows that we have only to consider the twelve cases. (See Figure 5.6.) The optimal values and the optimal solutions of Problem MP'(r) in the above twelve cases are as follows:

Case (i) $g(r)=2q_1r^2$

$(x_r, y_r)=(a_1/2, b/2)$

Case (ii) $g(r)=2q_1r^2-\frac{q_1(q_1-q_2)}{2q_1-q_2}(2r-a_1)^2$

$(x_r, y_r)=(\frac{(q_1-q_2)a_1+q_2r}{2q_1-q_2}, b/2)$

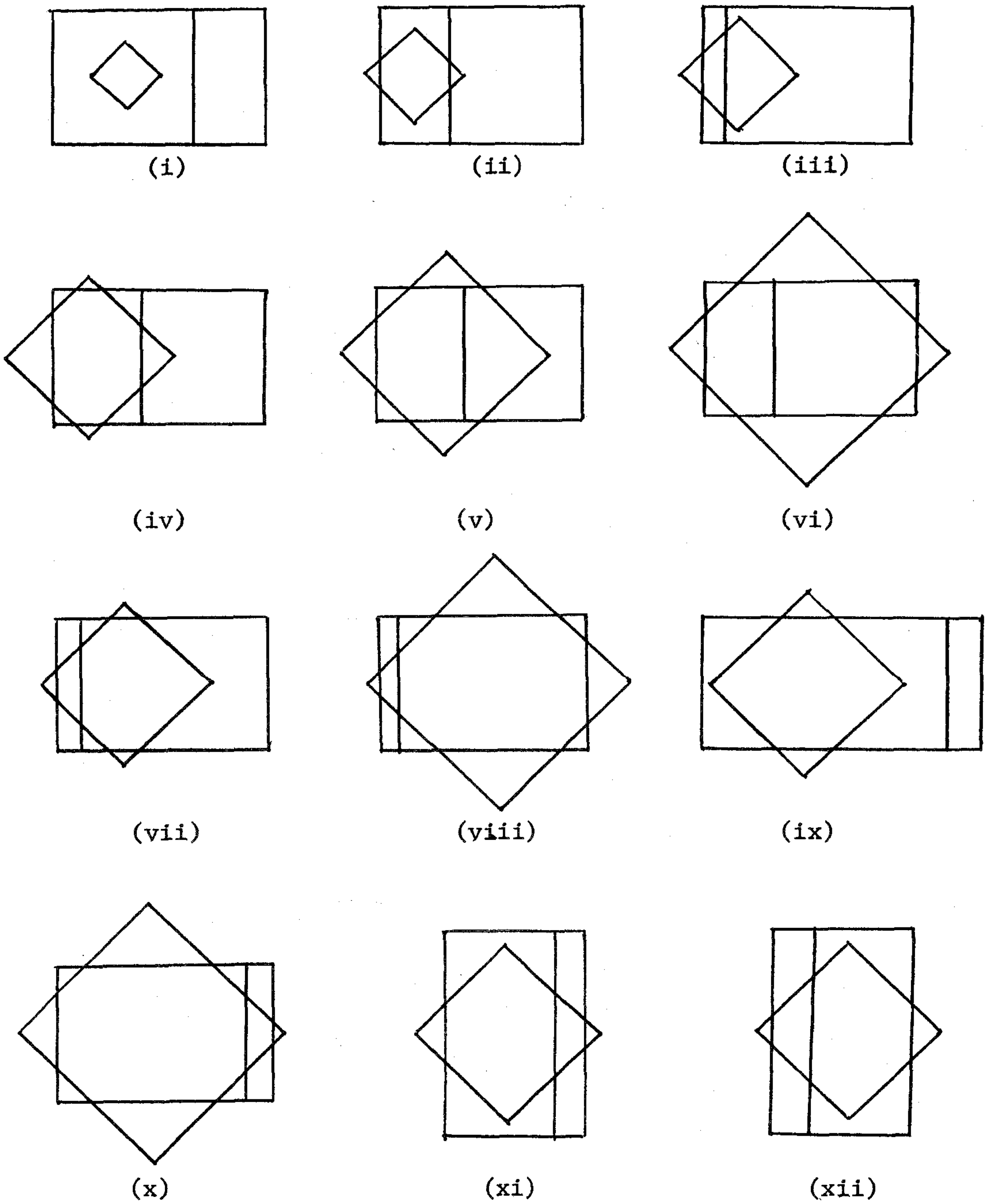
Case (iii) $g(r)=2q_2r^2+\frac{q_1(q_1-q_2)}{q_2}a_1^2$

$(x_r, y_r)=(-\frac{q_1-q_2}{q_2}a_1+r, b/2)$

Case (iv) $g(r)=-\frac{q_1(q_1-q_2)}{2q_1-q_2}(2r-a_1)^2+\frac{q_1b(4r-b)}{2}$

$(x_r, y_r)=(\frac{(q_1-q_2)a_1+q_2r}{2q_1-q_2}, b/2)$

Case (v) $g(r)=2q_2br+(q_1-q_2)a_1b-\frac{q_2(3q_1-q_2)}{4q_1}b^2$



(i) (ii) (iii)

(iv) (v) (vi)

(vii) (viii) (ix)

(x) (xi) (xii)

Figure 5.6 Twelve cases

$(x_r, y_r)=(r-\frac{q_1-q_2}{2q_1}b, b/2)$

Case (vi) $g(r)=-\frac{q_1q_2}{q_1+q_2}(a+b-2r)^2+1$

$(x_r, y_r)=(\frac{(q_1-q_2)(2r-b)+2q_2a}{2(q_1+q_2)}, b/2)$

Case (vii) $g(r)=\frac{q_1(q_1-q_2)}{q_2}a_1^2+q_2b(2r-\frac{b}{2})$

$(x_r, y_r)=(-\frac{q_1-q_2}{q_2}a_1+r, b/2)$

Case (viii) $g(r)=-2q_2r^2+2(\frac{1}{b}+q_2b)r-\frac{1}{2q_2b^2}+\frac{q_1(q_1-q_2)}{q_2}a_1^2-\frac{q_2}{2}b^2$

$(x_r, y_r)=(\frac{1}{2q_2b}, b/2)$

Case (ix) $g(r)=q_1(2r^2-\frac{(2r-b)^2}{2})$

$(x_r, y_r)=(a_1/2, b/2)$

Case (x) $g(r)=-2q_1r^2+2(\frac{1}{b}+q_1b)r-\frac{1}{2q_1b^2}-\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2-\frac{q_1}{2}b^2$

$(x_r, y_r)=(\frac{1}{2q_1b}, b/2)$

Case (xi) $g(r)=\frac{2r}{b}-\frac{1}{2q_1b^2}-\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2$

$(x_r, y_r)=(\frac{1}{2q_1b}, b/2)$

Case (xii) $g(r)=\frac{2r}{b}-\frac{1}{2q_2b^2}+\frac{q_1(q_1-q_2)}{q_2}a_1^2$

$(x_r, y_r)=(\frac{1}{2q_2b}, b/2)$

From Property 5.8 and (5.76) we have the following optimal solutions for Problem MP_0 in cases (i)-(xii) respectively.

(S-1)
$$r^*=\sqrt{\frac{1+\ln\alpha/\beta}{2q_1}}, \quad (x^*, y^*)=(a_1/2, b/2)$$

(S-2)
$$r^*=\frac{-2q_1(q_1-q_2)a_1+\sqrt{2q_1(2q_1-q_2)\{q_1(q_1-q_2)a_1^2+q_2(1+\ln\alpha/\beta)\}}}{2q_1q_2}$$
$$(x^*, y^*)=(\frac{(q_1-q_2)a_1+q_2r^*}{2q_1-q_2}, b/2)$$

(S-3)
$$r^*=\sqrt{\frac{-q_1(q_1-q_2)a_1^2+q_2(1+\ln\alpha/\beta)}{2q_2}}$$
$$(x^*, y^*)=(-\frac{q_1-q_2}{q_2}a_1+r^*, b/2)$$

(S-4)
$$r^*=\begin{cases} \frac{b}{4}+\frac{1+\ln\alpha/\beta}{2q_1b} & \text{if } q_1=q_2, \\ \frac{2q_1(q_1-q_2)a_1+q_1(2q_1-q_2)b-\sqrt{D}}{4q_1(q_1-q_2)} & \text{if otherwise,} \end{cases}$$

where $D=q_1(2q_1-q_2)\{q_1q_2b^2+4q_1(q_1-q_2)a_1b-4(q_1-q_2)(1+\ln\alpha/\beta)\}$
$$(x^*, y^*)=(\frac{(q_1-q_2)a_1+q_2r^*}{2q_1-q_2}, b/2)$$

(S-5)
$$r^*=-\frac{q_1-q_2}{2q_2}a_1+\frac{3q_1-q_2}{8q_1}b+\frac{1+\ln\alpha/\beta}{2q_2b}$$
$$(x^*, y^*)=(r^*-\frac{q_1-q_2}{2q_1}b, b/2)$$

(S-6)
$$r^*=\frac{a+b}{2}-\frac{1}{2}\sqrt{\frac{-(q_1+q_2)\ln\alpha/\beta}{q_1q_2}}$$
$$(x^*, y^*)=(\frac{2q_2a+(q_1-q_2)(2r^*-b)}{2(q_1+q_2)}, b/2)$$

(S-7)

$$r^* = -\frac{q_1(q_1-q_2)}{2q_2^2 b} a_1^2 + \frac{b}{4} + \frac{1+\ln\alpha/\beta}{2q_2 b}$$

$$(x^*,\ y^*) = \left(-\frac{q_1-q_2}{q_2} a_1 + r^*,\ b/2\right)$$

(S-8)

$$r^* = \frac{q_2^2(a-a_1)+q_2^2 b+q_1 q_2 a_1 - \sqrt{2q_1 q_2^2(q_1-q_2) a_1^2 - 2q_2^3 \ln\alpha/\beta}}{2q_2^2}$$

$$(x^*,\ y^*) = \left(\frac{a}{2} - \frac{q_1-q_2}{2q_2} a_1,\ b/2\right)$$

(S-9)

$$r^* = \frac{b}{4} + \frac{1+\ln\alpha/\beta}{2q_1 b}$$

$$(x^*,\ y^*) = (a_1/2,\ b/2)$$

(S-10)

$$r^* = \frac{1/b + q_1 b - \sqrt{2\{-q_2(q_1-q_2)(a-a_1)^2 - q_1 \ln\alpha/\beta\}}}{2q_1}$$

$$(x^*,\ y^*) = \left(\frac{1}{2q_1 b},\ b/2\right)$$

(S-11)

$$r^* = \frac{1+2b^2\{q_2(q_1-q_2)(a-a_1)^2 + q_1(1+\ln\alpha/\beta)\}}{4q_1 b}$$

$$(x^*,\ y^*) = \left(\frac{1}{2q_1 b},\ b/2\right)$$

(S-12)

$$r^* = \frac{1-2b^2\{q_1(q_1-q_2)a_1^2 - q_2(1+\ln\alpha/\beta)\}}{4q_2 b}$$

$$(x^*,\ y^*) = \left(\frac{a}{2} - \frac{q_1-q_2}{2q_2} a_1,\ b/2\right)$$

Now all the cases are shown in Table 5.7. Table 5.8 shows the parametric solutions.

Table 5.7. All the cases of b and a_1

b	a_1	policies
$0 < b \leq 2a/3$	$0 \leq a_1 \leq q_2b/2q_1$	Table 5.8 (i)
	$q_2b/2q_1 < a_1 \leq b$	Table 5.8 (ii)
	$b < a_1 \leq a-b/2$	Table 5.8 (iii)
	$a-b/2 < a_1 \leq a$	Table 5.8 (iv)
$2a/3 < b \leq a$	$0 \leq a_1 \leq q_2b/2q_1$	Table 5.8 (i)
	$q_2b/2q_1 < a_1 \leq a-b/2$	Table 5.8 (ii)
	$a-b/2 < a_1 \leq b$	Table 5.8 (v)
	$b < a_1 \leq a$	Table 5.8 (iv)
$a<b\leq 2q_1a/(q_1+q_2)$ †	$0 \leq a_1 \leq q_2(b-a)/(q_1-q_2)$	Table 5.8 (vi)
	$q_2(b-a)/(q_1-q_2) < a_1 \leq q_2b/2q_1$	Table 5.8 (i)
	$q_2b/2q_1 < a_1 \leq a-b/2$	Table 5.8 (ii)
	$a-b/2 < a_1 \leq a$	Table 5.8 (v)
$2q_1a/(q_1+q_2) \leq b$	$0 \leq a_1 \leq q_2a/(q_1+q_2)$	Table 5.8 (vi)
	$q_2a/(q_1+q_2) < a_1 \leq a$	Table 5.8 (vii)

† In this q_1 is not equal to q_2.

Table 5.8. Parametric solutions

(i)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 a_1^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 a_1^2}{2})} < \alpha \leqq e^{-\beta(1-\frac{q_1(3q_1-q_2)}{q_2}a_1^2)}$	(S-2)
$e^{-\beta(1-\frac{q_1(3q_1-q_2)}{q_2}a_1^2)} < \alpha \leqq e^{-\beta(1-\frac{q_1(q_1-q_2)}{q_2}a_1^2-\frac{q_2}{2}b^2)}$	(S-3)
$e^{-\beta(1-\frac{q_1(q_1-q_2)}{q_2}a_1^2-\frac{q_2}{2}b^2)} < \alpha \leqq e^{-\beta(-\frac{q_1(q_1-q_2)}{q_2}a_1^2+\frac{q_2}{2}b^2)}$	(S-7)
$e^{-\beta(-\frac{q_1(q_1-q_2)}{q_2}a_1^2+\frac{q_2}{2}b^2)} < \alpha \leqq e^{-\beta\frac{q_1(q_1-q_2)}{q_2}a_1^2}$	(S-8)
$e^{-\beta\frac{q_1(q_1-q_2)}{q_2}a_1^2} < \alpha \leqq 1$	(S-6)

(ii)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 a_1^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 a_1^2}{2})} < \alpha \leqq e^{-\beta(1+\frac{q_1(q_1-q_2)}{2(2q_1-q_2)}(b-a_1)^2-\frac{q_1}{2}b^2)}$	(S-2)
$e^{-\beta(1+\frac{q_1(q_1-q_2)}{2(2q_1-q_2)}(b-a_1)^2-\frac{q_1}{2}b^2)} < \alpha \leqq e^{-\beta q_2 b(a-a_1-\frac{q_1-q_2}{4q_1}b)}$	(S-4)
$e^{-\beta q_2 b(a-a_1-\frac{q_1-q_2}{4q_1}b)} < \alpha \leqq e^{-\beta\frac{q_2(q_1+q_2)}{4q_1}b^2}$	(S-5)
$e^{-\beta\frac{q_2(q_1+q_2)}{4q_1}b^2} < \alpha \leqq 1$	(S-6)

(iii)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1b^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1b^2}{2})} < \alpha \leqq e^{-\beta(1-q_1a_1b+\frac{q_1b^2}{2})}$	(S-9)
$e^{-\beta(1-q_1a_1b+\frac{q_1b^2}{2})} < \alpha \leqq e^{-\beta q_2b(a-a_1-\frac{q_1-q_2}{4q_1}b)}$	(S-4)
$e^{-\beta q_2b(a-a_1-\frac{q_1-q_2}{4q_1}b)} < \alpha \leqq e^{-\beta\frac{q_2(q_1+q_2)}{4q_1}b^2}$	(S-5)
$e^{-\beta\frac{q_2(q_1+q_2)}{4q_1}b^2} < \alpha \leqq 1$	(S-6)

(iv)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 b^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 b^2}{2})} < \alpha \leqq e^{-\beta(1-q_1 a_1 b+\frac{q_1 b^2}{2})}$	(S-9)
$e^{-\beta(1-q_1 a_1 b+\frac{q_1 b^2}{2})} < \alpha \leqq e^{-\beta(\frac{(q_1-q_2)^2}{q_1}(a-a_1)^2+(q_1-q_2)(a-a_1-b)^2-\frac{(q_1-2q_2)}{2}b^2)}$	(S-4)
$e^{-\beta(\frac{(q_1-q_2)^2}{q_1}(a-a_1)^2+(q_1-q_2)(a-a_1-b)^2-\frac{(q_1-2q_2)}{2}b^2)} < \alpha \leqq e^{-\beta\frac{q_2(q_1+q_2)}{q_1}(a-a_1)^2}$	(S-10)
$e^{-\beta\frac{q_2(q_1+q_2)}{q_1}(a-a_1)^2} < \alpha \leqq 1$	(S-6)

(v)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 a_1^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 a_1}{2})} < \alpha \leqq e^{-\beta(1+\frac{q_1(q_1-q_2)}{2(2q_1-q_2)}(b-a_1)^2-\frac{q_1}{2}b^2)}$	(S-2)
$e^{-\beta(1+\frac{q_1(q_1-q_2)}{2(2q_1-q_2)}(b-a_1)^2-\frac{q_1}{2}b^2)} < \alpha \leqq e^{-\beta(\frac{(q_1-q_2)}{q_1}(a-a_1)^2+(q_1-q_2)(a-a_1-b)^2-\frac{q_1-2q_2}{2}b^2)}$	(S-4)
$e^{-\beta(\frac{(q_1-q_2)}{q_1}(a-a_1)^2+(q_1-q_2)(a-a_1-b)^2-\frac{q_1-2q_2}{2}b^2)} < \alpha \leqq e^{-\beta\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2}$	(S-10)
$e^{-\beta\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2} < \alpha \leqq 1$	(S-6)

(vi)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 a_1^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 a_1^2}{2})} < \alpha \leqq e^{-\beta(1-\frac{q_1(3q_1-q_2)}{q_2} a_1^2)}$	(S-2)
$e^{-\beta(1-\frac{q_1(3q_1-q_2)}{q_2} a_1^2)} < \alpha \leqq e^{-\beta(1-\frac{1}{2q_2 b^2}-\frac{q_1(q_1-q_2)}{q_2} a_1^2)}$	(S-3)
$e^{-\beta(1-\frac{1}{2q_2 b^2}-\frac{q_1(q_1-q_2)}{q_2} a_1^2)} < \alpha \leqq e^{-\beta(\frac{1}{2q_2 b^2}-\frac{q_1(q_1-q_2)}{q_2} a_1^2)}$	(S-12)
$e^{-\beta(\frac{1}{2q_2 b^2}-\frac{q_1(q_1-q_2)}{q_2} a_1^2)} < \alpha \leqq e^{-\beta\frac{q_1(q_1+q_2)}{q_2} a_1^2}$	(S-8)
$e^{-\beta\frac{q_1(q_1+q_2)}{q_2} a_1^2} < \alpha \leqq 1$	(S-6)

(vii)

Interval of α	solution
$0 \leqq \alpha \leqq e^{-\beta}$	(S-0)
$e^{-\beta} < \alpha \leqq e^{-\beta(1-\frac{q_1 a_1^2}{2})}$	(S-1)
$e^{-\beta(1-\frac{q_1 a_1^2}{2})} < \alpha \leqq e^{-\beta(1-\frac{2q_1-q_2}{2q_1q_2b^2}+\frac{q_1(q_1-q_2)}{q_2}a_1^2)}$	(S-2)
$e^{-\beta(1-\frac{2q_1-q_2}{2q_1q_2b^2}+\frac{q_1(q_1-q_2)}{q_2}a_1^2)} < \alpha \leqq e^{-\beta(\frac{1}{2q_1b^2}+\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2)}$	(S-11)
$e^{-\beta(\frac{1}{2q_1b^2}+\frac{q_2(q_1-q_2)}{q_1}(a-a_1)^2)} < \alpha \leqq e^{-\beta\frac{q_2(q_1+q_2)}{q_1}(a-a_1)^2}$	(S-10)
$e^{-\beta\frac{q_2(q_1+q_2)}{q_1}(a-a_1)^2} < \alpha \leqq 1$	(S-6)

In the above tables, the solution (S-0) is as follows:

(S-0) $r^*=0$, $(x^*,y^*)=(a_1/2,b/2)$.

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ACKNOWLEDGEMENT

First of all, the author wishes to express his sincere appreciation to Professor T. Nishida for supervising this thesis. His continuous encouragement and invaluable comments have helped to accomplish the thesis.

The author would also like to express his sincere appreciation to Professor H. Sugiyama, Professor M. Yamamoto, and Professor Y. Tezuka for their valuable suggestions and helpful comments for improving the thesis.

The author is also indebted to Associate Professor H. Ishii who guided the author to the present study and has been giving the continuous encouragement and invaluable comments to complete the thesis.

Furthermore, the author would like to express his gratitude to Professor T. Sunaga of Kyushu University and Associate Professor Y. Tabata for their advices and continuous encouragement.

Finally, the author wishes to thank Dr. F. Ohi for his helpful and stimulating discussions.