Title	SCATTERING PROBLEM FOR A SYSTEM OF NONLINEAR KLEIN-GORDON EQUATIONS
Author(s)	馬, 具巣
Citation	大阪大学, 2009, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/1720
rights	
Note	

SCATTERING PROBLEM FOR A SYSTEM OF NONLINEAR KLEIN-GORDON EQUATIONS
非線形Klein-Gordon方程式系の散乱問題

RATNO BAGUS EDY WIBOWO

馬具巣

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy (Ph.D.)

OSAKA UNIVERSITY

2009

Program Authorized to Offer Degree: MATHEMATICS

OSAKA UNIVERSITY
GRADUATE SCHOOL OF SCIENCE DEPARTMENT OF MATHEMATICS

This is to certify that I have examined this copy of a doctoral dissertation by

RATNO BAGUS EDY WIBOWO
馬具巣

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Chair of the Supervisory Committee:

____
Professor Nakao Hayashi

Reading Committee:

____
Professor Tatsuo Nishitani

____
Professor Shin-ichi Doi

____
Professor Hideaki Sunagawa

Date: ______________________

TABLE OF CONTENTS

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to his dissertation advisor, Professor Nakao Hayashi and families, for his invaluable guidance, continuous encouragement, support and faith in him throughout his graduate studies at Department of Mathematics, Graduate School of Science, Toyonaka campuss, Osaka University. Without his efforts, this dissertation could not have been completed. Thanks are also expressed to other members of the author's dissertation committee: Professor Tatsuo Nishitani, Professor Shin-ichi Doi, and Professor Hideaki Sunagawa. The author is grateful to Professor Pavel I. Naumkin for joint work, many useful sugesstions and comments. The author would like also to thank Professor Hideo Kubo and Professor Soichiro Katayama, Professor Akitaka Matsumura, Professor Hideki Takuwa for many important advices in many mathematics research meetings. The author thanks to Dr. Yuichiro Kawahara and Dr. Hashimoto Itsuko for kindly attention and help on how to read mathematics books in japanese language when the author arrived in Japan. A note of appreciation from the author is also extended to the faculty members and staff at the Department of Mathematics. The Department provides the good equipment and office environment to the author.

DEDICATION

To my beloved daugther in the heaven *Diva*

To my dear wife, daughter and son; *Rossy, Bilqis and Akihiko.*

CHAPTER 1

INTRODUCTION

We investigate asymptotic behavior in time of small solutions to some kinds of nonlinear wave equations, which are appeared in geometric optics, fluid mechanics and quantum mechanics to describe the phenomena of nature. In particular, we concentrate on the study of nonlinear Klein Gordon equations and nonlinear systems of Klein-Gordon equations including Dirac-Klein-Gordon equations.

Global existence in time of small solutions to nonlinear Klein-Gordon equations with quadratic nonlinearities were shown by Klainerman [11] and Shatah [14], independently. Their methods are different each other and now very important tools in this field.

The method by Klainerman [11] is called the method of vector field and the method of Shatah [14] is called the method of normal form. The method of vector field is based on the time decay estimates obtained by using operators which commute with the linear Klein-Gordon operator. Compactness conditions on data were used to consider the problems by the use of hyperbolic coordinates. On the other hand, the method of normal form is based on a suitable nonlinear transformation which transforms the original equation to another one with cubic nonlinearities.

The method of vector field was improved by Bachelot[2], Georgiev [5, 6] and Hörmander [1] and compactness conditions on initial data were removed. However the higher order Sobolev spaces were needed. Then, Ozawa, Tsutaya and Tsutsumi [13] succeeded in obtaining global existence of small solutions and the existence of the scattering operator to quadratic semi-linear Klein-Gordon equations in two space dimensions by making use of time decay estimate by Georgiev and the method of normal form. It is known that cubic nonlinearities in one space dimension are critical ones from the balance of time decay of linear part and that of nonlinear terms. Katayama [10] found that nonlinearities (if they are non resonance terms) are divided into two groups, one of them are eliminated by choosing a suitable transformation and another group have better time decay through vector field. As a product, he obtained the global existence of small solutions to cubic nonlinear Klein-Gordon equations in one space dimension. I note here that his method is much simpler than previous works by Shatah [14] or Ozawa, Tsutaya and Tsutsumi [13]. However the previous works are not sufficient in view of the scattering theory since improper regularity and decaying assumptions in space are imposed on the data.

The main purpose of this thesis is to refine the time decay estimates by the method of vector field to apply the scattering problem. After that we apply them to a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations. Indeed, we prove the existence of a scattering operator for a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations in three space dimensions [19].

The plan in this thesis as follows: In Chapter 1, we summerize the previous related results as a motivations of our research.

Chapter 2 is devoted to study a scattering problems for a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations in three space dimensions [19]. I note here that this is the first results for the asymptotic completness of scattering operator for a system of nonlinear Klein-Gordon equations with a derevative of unknown functions in the nonlinearities.

In Chapter 3, we consider the initial value problems for a system of nonlinear Klein-Gordon equations with more general quadratic nonlinearities than ones considered in Chapter 2. We prove the existence of scattering states, namely, the asymptotic stability of small solutions in the neigborhood of free solutions for small initial data in the lower weighthed Sobolev spaces comparing with the previous works [2], [4], [5], [16], [18], [17]. If nonliearities satisfy the strong null condition [5], then the same result is true in two dimensions for small data in $\mathbf{H}^{5,3} \times \mathbf{H}^{4,3}$. A system of massive Dirac-massless Klein-Gordon equations in three spaces dimensions is also considered by our method [9].

CHAPTER 2

SCATTERING PROBLEM FOR A SYSTEM OF NONLINEAR KLEIN-GORDON EQUATIONS RELATED TO DIRAC-KLEIN-GORDON EQUATIONS

We consider the scattering problem for the system of Klein-Gordon equations in $\mathbf{R}^3$

$$\begin{cases} \left(\partial_t^2 - \Delta + M^2\right)\psi = -g^2\phi^2\psi - g\sum_{\mu=0}^{3}\gamma^\mu\gamma^4\psi\partial_\mu\phi, \\ \left(\partial_t^2 - \Delta + m^2\right)\phi = g\psi^*\gamma^0\psi, \end{cases} \tag{2.0.1}$$

where $m > 0$, $M > 0$, $g \in \mathbf{R}$, $\{\gamma^\mu\}$ are the 4×4 Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

the 4×4 matrix γ^4 is defined by $\gamma^4 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, A^* is transposed conjugate matrix of A, and the 4×1 vector $\psi = \left(\psi_j\right)_{1\le j\le 4} \in \mathbf{C}^4$ is a spinor field and $\phi \in \mathbf{R}$ is a scalar field. Equation (2.0.1) has an important physical meaning since it is derived from the massive Dirac-Klein-Gordon equation

$$\begin{cases} \left(-i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\psi = ig\phi\gamma^4\psi, \\ \left(\partial_t^2 - \Delta + m^2\right)\phi = g\psi^*\gamma^0\psi, \end{cases} \tag{2.0.2}$$

where $t = x_0, x = (x_1, x_2, x_3)$ and $\partial_\mu = \partial/\partial x_\mu$. Note that $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$ for $\nu \neq \mu$ and $\gamma^0\gamma^0 = \mathbf{E}$, $\gamma^\mu\gamma^\mu = -\mathbf{E}$ for $\mu = 1, 2, 3$, where $\mathbf{E} = [\delta_{jk}]_{1\le j,k\le 4}$, $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$.

Therefore by a direct computation we have

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu}=\mathbf{E}\left(\partial_t^2-\Delta\right).$$

Multiplying both sides of (2.0.2) by $i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu}+M$ and using the identity $\gamma^4\gamma^{\mu}=-\gamma^{\mu}\gamma^4$ for $\mu=0,1,2,3$ and $\gamma^4\gamma^4=\mathbf{E}$, we obtain

$$\begin{aligned}\left(\partial_t^2-\Delta+M^2\right)\psi &= \left(i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu}+M\right)\left(-i\sum_{\nu=0}^{3}\gamma^{\nu}\partial_{\nu}+M\right)\psi\\ &= ig\left(i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu}+M\right)\phi\gamma^4\psi\\ &= -g\sum_{\mu=0}^{3}\gamma^{\mu}\gamma^4\psi\partial_{\mu}\phi+ig\phi\left(i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu}+M\right)\gamma^4\psi\\ &= -g\sum_{\mu=0}^{3}\gamma^{\mu}\gamma^4\psi\partial_{\mu}\phi+ig\phi\gamma^4\left(-i\sum_{\mu=0}^{3}\gamma^{\mu}\partial_{\mu}+M\right)\psi\\ &= -g\sum_{\mu=0}^{3}\gamma^{\mu}\gamma^4\psi\partial_{\mu}\phi-g^2\phi^2\psi\end{aligned}$$

and we also have

$$\left(\partial_t^2-\Delta+m^2\right)\phi=g\left(|\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2\right).$$

Therefore we have the system of Klein-Gordon equations (2.0.1).

Denote the usual Lebesgue space by $\mathbf{L}^p=\{\phi\in S':\|\phi\|_{\mathbf{L}^p}<\infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p}=\left(\int_{\mathbf{R}^3}|\phi(x)|^p\,dx\right)^{\frac{1}{p}}$ if $1\le p<\infty$ and $\|\phi\|_{\mathbf{L}^\infty}=\sup_{x\in\mathbf{R}^3}|\phi(x)|$ if $p=\infty$. Weighted Sobolev space

$$\mathbf{H}_p^{m,k}=\left\{\phi:\|\phi\|_{\mathbf{H}_p^{m,k}}\equiv\left\|\langle x\rangle^k\langle i\nabla\rangle^m\phi\right\|_{\mathbf{L}^p}<\infty\right\},$$

where $m,k\in\mathbf{R}$, $1\le p\le\infty$, $\langle x\rangle=\sqrt{1+|x|^2}$, $\langle i\nabla\rangle=\sqrt{1-\Delta}$. We also write for simplicity $\mathbf{H}^{m,k}=\mathbf{H}_2^{m,k}$, $\mathbf{H}^m=\mathbf{H}_2^{m,0}$, $\mathbf{H}_p^m=\mathbf{H}_p^{m,0}$, so we usually omit the index 0 if it does not cause a confusion. We use the same notations for the vector functions. The direct Fourier transform $\hat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi=\hat{\phi}=(2\pi)^{-\frac{3}{2}}\int_{\mathbf{R}^3}e^{-i(x\cdot\xi)}\phi(x)\,dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = (2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} e^{i(x\cdot\xi)}\phi(\xi)\, d\xi.$$

Existence of global small solutions of the Cauchy problem for nonlinear Klein-Gordon equations with quadratic nonlinearity was shown by Klainerman [11]. His method depends on the new time decay estimates through the operators $x_j\partial_t + t\partial_j$ and can be applied the Cauchy problem (2.0.1). Compactness conditions on the data were used to consider the problem by the use of hyperbolic coordinates. His method was improved by [2], [5], [6], [1] and compactness conditions on the data were removed. However the higher order Sobolev spaces were needed for the initial data. Recently, Hayashi, Naumkin and the author [9] showed a global existence of small solutions to a system of nonlinear Klein-Gordon equation including (2.0.1) under the initial conditions such that

$$\|\psi(0)\|_{\mathbf{H}^{4,3}} + \|\partial_t\psi(0)\|_{\mathbf{H}^{3,3}} + \|\phi(0)\|_{\mathbf{H}^{4,3}} + \|\partial_t\phi(0)\|_{\mathbf{H}^{3,3}}$$

are small. Furthermore in [9], the inverse wave operator from the neighborhood at the origin of $\left(\mathbf{H}^{4,3}\times\mathbf{H}^{3,3}\right)^4 \times \left(\mathbf{H}^{4,3}\times\mathbf{H}^{3,3}\right)$ to the neighborhood at the origin of $\left(\mathbf{H}^{4,1}\times\mathbf{H}^{3,1}\right)^4 \times \left(\mathbf{H}^{4,1}\times\mathbf{H}^{3,1}\right)$ was constructed. However the scattering operator for the problem (2.0.1) is not obtained. Our purpose in the present paper is to prove the existence of scattering operator for (2.0.1). Scattering operator is obtained by showing the range of wave operator includes the domain of inverse wave operator. We also show the existence of the inverse scattering operator. Our main point is to show a global existence of small solutions to (2.0.1) under the initial conditions such that

$$\|\psi(0)\|_{\mathbf{H}^{5/2,1}} + \|\partial_t\psi(0)\|_{\mathbf{H}^{3/2,1}} + \|\phi(0)\|_{\mathbf{H}^{3,1}} + \|\partial_t\phi(0)\|_{\mathbf{H}^{2,1}}$$

are small which enables us to construct the scattering operator from the neighborhood at the origin of $\left(\mathbf{H}^{5/2,1}\times\mathbf{H}^{3/2,1}\right)^4 \times \left(\mathbf{H}^{3,1}\times\mathbf{H}^{2,1}\right)$ to the neighborhood at the origin of $\left(\mathbf{H}^{5/2,1}\times\mathbf{H}^{3/2,1}\right)^4 \times \left(\mathbf{H}^{3,1}\times\mathbf{H}^{2,1}\right)$.

In order to state our result precisely, we introduce the operator $\mathcal{L}_m = E\partial_t + iA\left\langle i\nabla\right\rangle_m,$ where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The free evolution group associated with $\mathcal{L}_m$ is defined by

$$\mathcal{U}_m(t) = \begin{pmatrix} e^{-it\langle i\nabla\rangle_m} & 0 \\ 0 & e^{it\langle i\nabla\rangle_m} \end{pmatrix}$$

with $\langle i\nabla\rangle_m = \sqrt{m^2-\Delta}$. We put

$$w_j = \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \psi_j + i\langle i\nabla\rangle_M^{-1} \partial_t \psi_j \\ \psi_j - i\langle i\nabla\rangle_M^{-1} \partial_t \psi_j \end{pmatrix},$$

and

$$w_5 = \begin{pmatrix} w_5^{(1)} \\ w_5^{(2)} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \phi + i\langle i\nabla\rangle_m^{-1} \partial_t \phi \\ \phi - i\langle i\nabla\rangle_m^{-1} \partial_t \phi \end{pmatrix}.$$

Then the system (2.0.1) can be rewritten as

$$\begin{cases} \mathcal{L}_M w_j = \begin{pmatrix} i\langle i\nabla\rangle_M^{-1} F_j(w, \partial_\mu w_5) \\ -i\langle i\nabla\rangle_M^{-1} F_j(w, \partial_\mu w_5) \end{pmatrix}, \\ \mathcal{L}_m w_5 = \begin{pmatrix} i\langle i\nabla\rangle_m^{-1} F_5(\widetilde{w}) \\ -i\langle i\nabla\rangle_m^{-1} F_5(\widetilde{w}) \end{pmatrix}, \end{cases} \tag{2.0.3}$$

where $j = 1,2,3,4$, $\widetilde{w} = (w_j)_{1\le j\le 4}$,

$$\begin{aligned} & F_j(w, \partial_\mu w_5) \\ = \; & \frac{1}{2}\left(-g^2\left(w_5^{(1)}+w_5^{(2)}\right)^2\left(w_j^{(1)}+w_j^{(2)}\right) - g\sum_{\mu=0}^{3}\left(\gamma^\mu\gamma^4\left(w_j^{(1)}+w_j^{(2)}\right)\right)\partial_\mu\left(w_5^{(1)}+w_5^{(2)}\right)\right) \end{aligned}$$

for $1 \le j \le 4$ and

$$F_5(\widetilde{w}) = \frac{1}{2}\left(g\left(\sum_{j=1}^{2}\left|w_j^{(1)}+w_j^{(2)}\right|^2 - \sum_{j=3}^{4}\left|w_j^{(1)}+w_j^{(2)}\right|^2\right)\right).$$

We consider the problem (2.0.3) with the initial condition

$$\begin{aligned} \overset{\circ}{w}_j &= \begin{pmatrix} \overset{\circ}{w}{}_j^{(1)} \\ \overset{\circ}{w}{}_j^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \overset{\circ}{\psi}{}_j^{(1)} + i\langle i\nabla\rangle_M^{-1} \overset{\circ}{\psi}{}_j^{(2)} \\ \overset{\circ}{\psi}{}_j^{(1)} - i\langle i\nabla\rangle_M^{-1} \overset{\circ}{\psi}{}_j^{(2)} \end{pmatrix}, \\ \overset{\circ}{w}_5 &= \begin{pmatrix} \overset{\circ}{w}{}_5^{(1)} \\ \overset{\circ}{w}{}_5^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \overset{\circ}{\phi}{}^{(1)} + i\langle i\nabla\rangle_m^{-1} \overset{\circ}{\phi}{}^{(2)} \\ \overset{\circ}{\phi}{}^{(1)} - i\langle i\nabla\rangle_m^{-1} \overset{\circ}{\phi}{}^{(2)} \end{pmatrix} \end{aligned}$$

and the final condition

$$\begin{aligned} w_j^+ &= \begin{pmatrix} w_j^{(1)+} \\ w_j^{(2)+} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \psi_j^{(1)+} + i \langle i\nabla \rangle_M^{-1} \psi_j^{(2)+} \\ \psi_j^{(1)+} - i \langle i\nabla \rangle_M^{-1} \psi_j^{(2)+} \end{pmatrix}, \\ w_5^+ &= \begin{pmatrix} w_5^{(1)+} \\ w_5^{(2)+} \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} \phi^{(1)+} + i \langle i\nabla \rangle_m^{-1} \phi^{(2)+} \\ \phi^{(1)+} - i \langle i\nabla \rangle_m^{-1} \phi^{(2)+} \end{pmatrix}. \end{aligned}$$

We note that the second term of $F_j(w, \partial_\mu w_5)$ contains the full derivative $\partial_\mu \left(w_5^{(1)} + w_5^{(2)} \right)$, and so has a derivative loss. This is the reason why we consider $\phi = w_5^{(1)} + w_5^{(2)}$ in higher regularity class than the class in which $\psi_j = w_j^{(1)} + w_j^{(2)}$ is considered.

We let a closed ball $\mathbf{H}_q^{m,k}(\varepsilon)$ with a radius ε and a center at the origin in the function space

$$\mathbf{H}_q^{m,k} = \left\{ v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} : \|v\|_{\mathbf{H}_q^{m,k}} = \left\| v^{(1)} \right\|_{\mathbf{H}_q^{m,k}} + \left\| v^{(2)} \right\|_{\mathbf{H}_q^{m,k}} < \infty \right\}.$$

We introduce the function space

$$\mathbf{Z}_I = \left\{ v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix} \in \mathbf{C}\left(I; \left(\mathbf{L}^2\right)^5 \times \left(\mathbf{L}^2\right)^5 \right);\ \|\phi\|_{\mathbf{Z}_I} < \infty \right\},$$

with the norm

$$\begin{aligned} \|v\|_{\mathbf{Z}_I} &= \sum_{j=1}^{4} \sum_{|\beta| \le 1} \left(\left\| \mathcal{P}_M^\beta v_j \right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{5/2-|\beta|}\right)} + \left\| \partial_t \mathcal{P}_M^\beta v_j \right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3/2-|\beta|}\right)} \right. \\ &+ \left. \left\| \mathcal{P}_M^\beta v_j \right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{2-|\beta|}\right)} + \left\| \partial_t \mathcal{P}_M^\beta v_j \right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{1-|\beta|}\right)} \right) \\ &+ \sum_{|\beta| \le 1} \left(\left\| \mathcal{P}_m^\beta v_5 \right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3-|\beta|}\right)} + \left\| \partial_t \mathcal{P}_m^\beta v_5 \right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{2-|\beta|}\right)} \right. \\ &+ \left. \left\| \mathcal{P}_m^\beta v_5 \right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{5/2-|\beta|}\right)} + \left\| \partial_t \mathcal{P}_m^\beta v_5 \right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2-|\beta|}\right)} \right), \frac{1}{q} = \frac{2}{5}, \frac{1}{r} = \frac{3}{20}, \end{aligned}$$

where $\mathcal{P}_M = (\mathcal{Z}, \mathcal{J}_M)$. We are now in a position to present our main results

Theorem 2.0.1. *We let* $\overset{\circ}{w} = \left(\overset{\circ}{w}_j, \overset{\circ}{w}_5 \right)_{1 \le j \le 4} \in \left(\mathbf{H}^{5/2,1}\right)^4 \times \mathbf{H}^{3,1}$. *Then there exists an* $\varepsilon > 0$ *and a positive constant* C *such that (2.0.3) has a unique global solution*

$$w \in \mathbf{C}\left((-\infty, 0]; \left(\mathbf{L}^2\right)^5 \right)$$

such that $\|w\|_{\mathbf{Z}_{(-\infty,0]}} \le C\varepsilon$ *for any* $\overset{\circ}{w} \in \left(\mathbf{H}^{5/2,1}(\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(\varepsilon)$. *Furthermore there exists a unique* $w^- = \left(w_j^-, w_5^-\right)_{1\le j\le 4} \in \left(\mathbf{H}^{5/2,1}(C\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(C\varepsilon)$ *such that*

$$\left\|\mathcal{U}_M(-t)\,w_j(t) - w_j^-\right\|_{\mathbf{H}^{5/2,1}} + \left\|\mathcal{U}_m(-t)\,w_5(t) - w_5^-\right\|_{\mathbf{H}^{3,1}} \to 0$$

as $t \to -\infty$.

Theorem 2.0.2. *We let* $w^+ = \left(w_j^+, w_5^+\right)_{1\le j\le 4} \in \left(\mathbf{H}^{5/2,1}\right)^4 \times \mathbf{H}^{3,1}$. *Then there exists an* $\varepsilon > 0$ *and a positive constant* C *such that (2.0.3) has a unique global solution* w *satisfying*

$$\left\|\mathcal{U}_M(-t)\,w_j(t) - w_j^+\right\|_{\mathbf{H}^{5/2,1}} + \left\|\mathcal{U}_m(-t)\,w_5(t) - w_5^+\right\|_{\mathbf{H}^{3,1}} \to 0$$

as $t \to \infty$ *and* $\|w\|_{\mathbf{Z}_{[0,\infty)}} \le C\varepsilon$ *for any* $w^+ \in \left(\mathbf{H}^{5/2,1}(\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(\varepsilon)$.

By Theorem 2.0.1, we can define the inverse wave operator W_-^{-1} which maps $\left(\mathbf{H}^{5/2,1}(\widetilde{\varepsilon})\right)^4 \times \mathbf{H}^{3,1}(\widetilde{\varepsilon})$ to $\left(\mathbf{H}^{5/2,1}(C\widetilde{\varepsilon})\right)^4 \times \mathbf{H}^{3,1}(C\widetilde{\varepsilon})$, and by Theorem 2.0.2, we can define the wave operator W_+ which maps $\left(\mathbf{H}^{5/2,1}(\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(\varepsilon)$ to $\left(\mathbf{H}^{5/2,1}(C_1\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(C_1\varepsilon)$. Therefore the scattering operator which maps $\left(\mathbf{H}^{5/2,1}(\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(\varepsilon)$ to $\left(\mathbf{H}^{5/2,1}(CC_1\varepsilon)\right)^4 \times \mathbf{H}^{3,1}(CC_1\varepsilon)$ can be constructed if we let $C_1\varepsilon \le \widetilde{\varepsilon}$. It is easy to see that the inverse scattering operator is also defined. We have the relations

$$\begin{aligned}
\psi_j &= w_j^{(1)} + w_j^{(2)}, \partial_t\psi_j = -i\,\langle i\nabla\rangle_M\left(w_j^{(1)} - w_j^{(2)}\right), \\
\phi &= w_5^{(1)} + w_5^{(2)}, \partial_t\phi = -i\,\langle i\nabla\rangle_m\left(w_5^{(1)} - w_5^{(2)}\right).
\end{aligned}$$

Therefore for the solutions to (2.0.1) we find that

$$\begin{aligned}
&\left\|U_M(-t)\begin{pmatrix}\psi_j(t)\\ \langle i\nabla\rangle_M^{-1}\partial_t\psi_j(t)\end{pmatrix} - \begin{pmatrix}\psi_j^{(1)+}\\ \langle i\nabla\rangle_M^{-1}\psi_j^{(2)+}\end{pmatrix}\right\|_{\mathbf{H}^{5/2,1}} \\
&+\left\|U_m(-t)\begin{pmatrix}\phi(t)\\ \langle i\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} - \begin{pmatrix}\phi^{(1)+}\\ \langle i\nabla\rangle_m^{-1}\phi^{(2)+}\end{pmatrix}\right\|_{\mathbf{H}^{3,1}} \to 0, \qquad (2.0.4)
\end{aligned}$$

where

$$U_m(t) = \begin{pmatrix}\cos\left(\langle i\nabla\rangle_m t\right) & \sin\left(\langle i\nabla\rangle_m t\right)\\ -\sin\left(\langle i\nabla\rangle_m t\right) & \cos\left(\langle i\nabla\rangle_m t\right)\end{pmatrix}$$

is the usual free Klein-Gordon evolution group. From (2.0.4) and the fact that the operator $U_m(t)$ is a unitary operator in $\mathbf{H}^{m,0}$ it follows that

$$\left\|\psi_j(t)-\left(\cos\left(\langle i\nabla\rangle_M t\right)\psi_j^{(1)+}+\langle i\nabla\rangle_M^{-1}\sin\left(\langle i\nabla\rangle_M t\right)\psi_j^{(2)+}\right)\right\|_{\mathbf{H}^{5/2,0}}$$
$$+\left\|\phi(t)-\left(\cos\left(\langle i\nabla\rangle_m t\right)\phi^{(1)+}+\langle i\nabla\rangle_m^{-1}\sin\left(\langle i\nabla\rangle_m t\right)\phi^{(2)+}\right)\right\|_{\mathbf{H}^{3,0}}\to 0.$$

One of our main tools is the operator

$$\mathcal{J}_m=\langle i\nabla\rangle_m\mathcal{U}_m(t)\,x\mathcal{U}_m(-t)=\langle i\nabla\rangle_m\left(xE+itA\langle i\nabla\rangle_m^{-1}\nabla\right)$$

which was used to obtain the time decay estimates for smooth and decaying functions and applied to asymptotic problem of nonlinear Klein-Gordon equations with super critical nonlinearities in [7], [8]. By a direct calculation we see that the commutation relation

$$[\mathcal{L}_m,\mathcal{J}_m]=0$$

is true. However it is difficult to calculate the action of $\mathcal{J}_m$ on the nonlinearity since $\mathcal{J}_m$ can not be considered as a first order differential operator on the power nonlinearities. Therefore we use the first order differential operator

$$\mathcal{Z}=E\left(t\nabla+x\partial_t\right)=\left(\mathcal{Z}^{(1)},\mathcal{Z}^{(2)},\mathcal{Z}^{(3)}\right)$$

which is closely related to $\mathcal{J}_m$ through the identity

$$\mathcal{Z}^{(j)}=\mathcal{L}_m x_j-iA\mathcal{J}_m^{(j)}$$

and we find that it almost commutes with $\mathcal{L}_m$ by

$$[\mathcal{L}_m,\mathcal{Z}]=E\nabla-iA\left[x,\langle i\nabla\rangle_m\right]\partial_t=-iA\langle i\nabla\rangle_m^{-1}\nabla\mathcal{L}_m. \tag{2.0.5}$$

The operator $\mathcal{Z}$ was introduced by Klainerman [11] to obtain time decay estimates of solutions to linear Klein-Gordon equations and the estimates were improved by [2], [5], [6], [1], [7], [8]. We often use the following commutation relations

$$[x,\mathcal{L}_m]=iA\left[x,\langle i\nabla\rangle_m\right]=iA\langle i\nabla\rangle_m^{-1}\nabla,\ \left[x_j,\mathcal{Z}^{(k)}\right]=Et\left[x_j,\partial_{x_k}\right]=-Et\delta_{jk}. \tag{2.0.6}$$

2.1 LEMMAS

We first state time decay estimates through the operator $\mathcal{J}_m$ for any smooth and decaying functions which was shown in [9].

Lemma 2.1.1. *Let $m > 0$. Then the estimate*

$$\|\phi\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-n(1-2/p)/2} \|\phi\|_{\mathbf{H}^\nu}^{1-n/2+n/p} \|\mathcal{J}_m \phi\|_{\mathbf{H}^{\nu-1}}^{n/2-n/p} + C \langle t \rangle^{-n(1-2/p)/2} \|\phi\|_{\mathbf{H}^\nu}$$

is valid for all $t \geq 0$, where $\nu = (n/2+1)(1-2/p)$, $2 \leq p < 2n/(n-2)$ and $n \geq 2$, provided that the right-hand side is finite, where n denotes the space dimension.

We now state the Strichartz type estimates used in this paper. Denote the space-time norm

$$\|\phi\|_{\mathbf{L}_t^r(\mathbf{I};\mathbf{L}_x^q)} = \left\| \|\phi(t)\|_{\mathbf{L}_x^q} \right\|_{\mathbf{L}_t^r(\mathbf{I})},$$

where $\mathbf{I}$ is a bounded or unbounded time interval. By the duality argument of [20] along with the $\mathbf{L}^p - \mathbf{L}^q$ time decay estimates of [12] we obtain the Strichartz estimate. Define

$$\Psi[g](t) = \int_T^t e^{-i(t-\tau)\langle i\nabla\rangle_m} \langle i\nabla \rangle^{-1} g(\tau)\, d\tau, m > 0$$

Lemma 2.1.2. *Let $2 \leq q < 2n/(n-2)$ and $2/r = n(1-2/q)/2$. Then for any time interval $\mathbf{I}$ and for any $T \in \overline{\mathbf{I}}$ the following estimates are true*

$$\|\Psi[g]\|_{\mathbf{L}_t^r(\mathbf{I};\mathbf{L}_x^q)} \leq C \|g\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^{2\mu-1}\right)},$$

$$\|\Psi[g]\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{L}_x^2)} \leq C \|g\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^{\mu-1}\right)}$$

and

$$\left\| e^{-it\langle i\nabla\rangle_m} \phi \right\|_{\mathbf{L}_t^r(\mathbf{I};\mathbf{L}_x^q)} \leq C \|\phi\|_{\mathbf{H}^\mu},$$

where $r' = r/(r-1)$, $q' = q/(q-1)$ and $\mu = (1+n/2)(1-2/q)/2$.

Proof. For convenience of the reader, we give a proof. Denote

$$\Phi_\nu[f](t) = \int_{\mathbf{R}} K(t,\tau) e^{-i(t-\tau)\langle i\nabla\rangle_m} \langle i\nabla \rangle^{-\nu} f(\tau)\, d\tau,$$

where $K(t,\tau) \in \mathbf{L}^{\infty}\left(\mathbf{R}^2\right)$ is a piecewise continuous complex-valued function. In particular, to get the integral $\Psi[f]$ of the lemma we choose $K(t,\tau)=1$ for $T \le \tau \le t$ and $K(t,\tau)=0$ otherwise. By the $\mathbf{L}^p - \mathbf{L}^q$ time decay estimates of [12] we have for $n(1-2/q)/2 < 1$

$$\|\Phi_{2\mu}[f](t)\|_{\mathbf{L}^q_x} \le \int_T^t |t-\tau|^{-\frac{n}{2}\left(1-\frac{2}{q}\right)} \left\| \langle i\nabla\rangle^{\left(1+\frac{n}{2}\right)\left(1-\frac{2}{q}\right)-2\mu} f(\tau)\right\|_{\mathbf{L}^{q'}_x} d\tau$$

from which by the Sobolev inequality it follows that

$$\|\Phi_{2\mu}[f]\|_{\mathbf{L}^r_t\left(\mathbf{I};\mathbf{L}^q_x\right)} \le C \left\| \langle i\nabla\rangle^{\left(1+\frac{n}{2}\right)\left(1-\frac{2}{q}\right)-2\mu} f\right\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{L}^{q'}_x\right)} = C\|f\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{L}^{q'}_x\right)},$$

where $2/r = n(1-2/q)/2$ and $\mu = (1+n/2)(1-2/q)/2$. Then substituting $f = \langle i\nabla\rangle^{2\mu-1} g$ gives us the first estimate of the lemma. We have

$$\begin{aligned}
&\|\Phi_\mu[f](t)\|^2_{\mathbf{L}^2_x} \\
=\ &\left(\int_T^t e^{-i(t-\tau)\langle i\nabla\rangle_m}\langle i\nabla\rangle^{-\mu} f(\tau)\,d\tau, \int_T^t e^{-i(t-\tau')\langle i\nabla\rangle_m} U\left(t-\tau'\right)\langle i\nabla\rangle^{-\mu} f\left(\tau'\right)d\tau'\right)_{\mathbf{L}^2_x} \\
\le\ &\int_T^t \left(\int_T^t \left\| e^{-i(\tau'-\tau)\langle i\nabla\rangle_m}\langle i\nabla\rangle^{-2\mu} f(\tau)\right\|_{\mathbf{L}^q_x} d\tau\right)\left\|f\left(\tau'\right)\right\|_{\mathbf{L}^{q'}_x} d\tau' \\
\le\ &C\|f\|^2_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{L}^{q'}_x\right)}
\end{aligned}$$

which implies the second estimate of the lemma if we take $f = \langle i\nabla\rangle^{\mu-1} g$. The last estimate is dual to the second one

$$\begin{aligned}
&\left|\int_{\mathbf{I}}\left(e^{-it\langle i\nabla\rangle_m}\phi, g(t)\right)_{\mathbf{L}^2_x} dt\right| = \left|\left(\langle i\nabla\rangle^{\mu}\phi, \int_{\mathbf{I}} e^{it\langle i\nabla\rangle_m}\langle i\nabla\rangle^{-\mu} g(t)\,dt\right)_{\mathbf{L}^2_x}\right| \\
\le\ &C\|\langle i\nabla\rangle^{\mu}\phi\|_{\mathbf{L}^2_x}\|\Phi_\mu[g](0)\|_{\mathbf{L}^2_x} \le C\|\phi\|_{\mathbf{H}^\mu}\|g\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{L}^{q'}_x\right)}.
\end{aligned}$$

Lemma is proved. ∎

We next consider time decay estimates involving the operators $\mathcal{Z}$ and $\mathcal{L}_m$.

Lemma 2.1.3. *Let $m>0$. Then the estimate is valid*

$$\|\phi\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\left(\|\phi\|_{\mathbf{H}^\nu} + \|\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-1}}\right) + C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1,1}}$$

for all $t \ge 0$, where $\nu = (n/2+1)(1-2/p)$, $2 \le p < 2n/(n-2)$ and $n \ge 2$, provided that the right-hand side is finite.

Proof. By (2.0.6), we have the identity

$$\mathcal{J}_m = iA\mathcal{Z} - E\left\langle i\nabla\right\rangle_m^{-1}\nabla - iAx\mathcal{L}_m. \tag{2.1.1}$$

Therefore we get

$$\left\|\mathcal{J}_m\phi\right\|_{\mathbf{H}^{\nu-1}} \leq C\left\|\mathcal{Z}\phi\right\|_{\mathbf{H}^{\nu-1}} + \left\|\phi\right\|_{\mathbf{H}^{\nu-1}} + C\left\|x\mathcal{L}_m\phi\right\|_{\mathbf{H}^{\nu-1}}. \tag{2.1.2}$$

In view of Lemma 2.1.1 and (2.1.2) the estimate of the lemma follows. ∎

2.2 *PROOF OF THEOREM 2.0.1 AND 2.0.2*

Denote by $\mathbf{Z}_I(\varepsilon)$ a closed ball of a radius ε with a center in the origin in the space $\mathbf{Z}_I$. Let us consider the linearized version of the Cauchy problem

$$\begin{cases} \mathcal{L}_M w_j = \begin{pmatrix} i\left\langle i\nabla\right\rangle_M^{-1} F_j\left(v, \partial_\mu v_5\right) \\ -i\left\langle i\nabla\right\rangle_M^{-1} F_j\left(v, \partial_\mu v_5\right) \end{pmatrix}, \\ \mathcal{L}_m w_5 = \begin{pmatrix} i\left\langle i\nabla\right\rangle_m^{-1} F_5\left(\widetilde{v}\right) \\ -i\left\langle i\nabla\right\rangle_m^{-1} F_5\left(\widetilde{v}\right) \end{pmatrix}, \\ w_k(0,x) = \overset{\circ}{w}_k(x), \end{cases} \tag{2.2.1}$$

with $j = 1, ..., 4, k = 1, ..., 5$ and a given function

$$v = (\widetilde{v}, v_5) \in \mathbf{Z}_{\mathbf{I}}(\rho), \widetilde{v} = (v_1, ..., v_4), \mathbf{I} = (-\infty, 0]$$

where

$$\rho = C\left(\sum_{j=1}^{4}\left\|\overset{\circ}{w}_j\right\|_{\mathbf{H}^{5/2,1}} + \left\|\overset{\circ}{w}_5\right\|_{\mathbf{H}^{3,1}}\right) = C\varepsilon.$$

The integration of (2.2.1) with respect to time yields

$$w_j(t) = \mathcal{U}_M(t)\overset{\circ}{w}_j + \int_0^t \mathcal{U}_M(t-\tau)\begin{pmatrix} i\left\langle i\nabla\right\rangle_M^{-1} F_j\left(v, \partial_\mu v_5\right) \\ -i\left\langle i\nabla\right\rangle_M^{-1} F_j\left(v, \partial_\mu v_5\right) \end{pmatrix} d\tau. \tag{2.2.2}$$

Taking the $\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{5/2}\right)$ - norm of (2.2.2) and applying the second estimate of Lemma 2.1.2 we find with $1/p+1/p'=1/2$

$$\begin{aligned}
&\|w_j\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{5/2}\right)} \\
\leq\ & C\left\|\overset{\circ}{w}_j\right\|_{\mathbf{H}^{5/2}} + C\left\|v\partial_\mu v_5\right\|_{\mathbf{L}_t^1\left(\mathbf{I};\mathbf{H}^{3/2}\right)} \\
\leq\ & C\left\|\overset{\circ}{w}_j\right\|_{\mathbf{H}^{5/2}} + C\left\|v\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^{10})}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)} \\
&+C\left\|v\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_{10}^{1/2}\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^1\right)} \\
&+C\left\|v\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_{p'}^1\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_p^{1/2}\right)} \\
&+C\left\|v\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^{10})}.
\end{aligned} \tag{2.2.3}$$

By Sobolev's inequality with $s\geq\frac{3}{p}-\frac{3}{10}, 2<p<6$ and Lemma 2.1.1

$$\begin{aligned}
&\|v\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^{10})} \leq C\left\|\langle i\nabla\rangle^s v\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^p)} \\
\leq\ & C\left\|\langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)}\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I})}\Big(\left\|\mathcal{J}_M\langle i\nabla\rangle^s\widetilde{v}\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu-1})} \\
&+\left\|\mathcal{J}_m\langle i\nabla\rangle^s v_5\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu-1})}+\|v\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu+s})}\Big) \\
\leq\ & C\left(\left\|\mathcal{J}_M\widetilde{v}\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu+s-1})}+\left\|\mathcal{J}_m v_5\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu+s-1})}+\|v\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^{\nu+s})}\right) \\
\leq\ & C\rho,
\end{aligned}$$

where $\nu=\frac{5}{2}\left(1-\frac{2}{p}\right)$ and we have assumed $6>p>\frac{60}{13}$. We also have

$$\begin{aligned}
&\|v\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_{10}^{1/2}\right)} \\
\leq\ & C\left(\left\|\mathcal{J}_M\widetilde{v}\right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3/2}\right)}+\left\|\mathcal{J}_m v_5\right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3/2}\right)}+\|v\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{5/2}\right)}\right)\leq C\rho
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_p^{1/2}\right)} \\
\leq\ & C\left(\left\|\mathcal{J}_m\partial_\mu v_5\right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{\nu-1/2}\right)}+\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{\nu+1/2}\right)}\right)\leq C\rho.
\end{aligned}$$

By Sobolev's inequality

$$\|v\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_{p'}^1\right)}\leq C\|v\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^2\right)}\leq C\rho.$$

Therefore by (2.2.3)

$$\|w_j\|_{\mathbf{L}^\infty_t(\mathbf{I};\mathbf{H}^{5/2})} \le C\varepsilon + C\rho^2.$$

Taking the $\mathbf{L}^r_t\left(\mathbf{I};\mathbf{H}^2_q\right)$ - norm of (2.2.2) with $1/q = 2/5$ and applying the third and the first estimates of Lemma 2.1.2 we obtain with $2\mu = 1/2$

$$\begin{aligned} \|w_j\|_{\mathbf{L}^r_t(\mathbf{I};\mathbf{H}^2_q)} &\le \left\|\mathcal{U}_M(t)\, \mathring{w}_j\right\|_{\mathbf{L}^r_t(\mathbf{I};\mathbf{H}^2_q)} \\ &+ C\left\|\int_0^t \mathcal{U}_M(t-\tau)\begin{pmatrix} i\left\langle i\nabla\right\rangle_M^{-1} F_j(v,\partial_\mu v_5) \\ -i\left\langle i\nabla\right\rangle_M^{-1} F_j(v,\partial_\mu v_5)\end{pmatrix} d\tau\right\|_{\mathbf{L}^r_t(\mathbf{I};\mathbf{H}^2_q)} \\ \le\ & C\left\|\mathring{w}_j\right\|_{\mathbf{H}^{5/2}} + C\left\|v\partial_\mu v_5\right\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{H}^{3/2}_{q'}\right)}. \end{aligned} \tag{2.2.4}$$

By the Hölder inequality with $1/q' = 3/5, 1/q = 2/5, 1/r' = 17/20, 1/r = 3/20$ we get

$$\begin{aligned} &\left\|v\partial_\mu v_5\right\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{H}^{3/2}_{q'}\right)} \\ \le\ & C\|v\|_{\mathbf{L}^{10/7}_t(\mathbf{I};\mathbf{L}^5)}\|v_5\|_{\mathbf{L}^r_t\left(\mathbf{I};\mathbf{H}^{5/2}_q\right)} + C\|v\|_{\mathbf{L}^r_t\left(\mathbf{I};\mathbf{H}^{3/2}_q\right)}\|v_5\|_{\mathbf{L}^{10/7}_t\left(\mathbf{I};\mathbf{H}^1_5\right)}. \end{aligned}$$

We use Lemma 2.1.1 to find

$$\|v(t)\|_{\mathbf{L}^5} \le C\langle t\rangle^{-\frac{9}{10}}\left(\|v\|_{\mathbf{H}^{5/3}} + \|\mathcal{J}_M v\|_{\mathbf{H}^{2/3}}\right) \le C\rho\langle t\rangle^{-\frac{9}{10}}. \tag{2.2.5}$$

Hence

$$\begin{aligned} &\left\|v\partial_\mu v_5\right\|_{\mathbf{L}^{r'}_t\left(\mathbf{I};\mathbf{H}^{3/2}_{q'}\right)} \\ \le\ & C\rho\left\|\langle t\rangle^{-\frac{9}{10}}\right\|_{\mathbf{L}^{10/7}_t(\mathbf{I})}\left(\|v\|_{\mathbf{L}^\infty_t(\mathbf{I};\mathbf{H}^{5/3})} + \|\mathcal{J}_M\widetilde{v}\|_{\mathbf{L}^\infty_t(\mathbf{I};\mathbf{H}^{2/3})} + \|\mathcal{J}_m v_5\|_{\mathbf{L}^\infty_t(\mathbf{I};\mathbf{H}^{5/3})}\right) \\ \le\ & C\rho^2. \end{aligned}$$

Thus by (2.2.3) we obtain the estimate

$$\|w_j\|_{\mathbf{L}^\infty_t(\mathbf{I};\mathbf{H}^{5/2})} + \|w_j\|_{\mathbf{L}^r_t(\mathbf{I};\mathbf{H}^2_q)} \le C\varepsilon + C\rho^2. \tag{2.2.6}$$

For the estimate of w_5, we get

$$
\begin{aligned}
&\|w_5\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{5/2}\right)} \leq \left\|\mathcal{U}_m(t)\,\mathring{w}_5\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{5/2}\right)} \\
&+C\left\|\int_0^t \mathcal{U}_m(t-\tau)\begin{pmatrix} i\langle i\nabla\rangle_m^{-1} F_5(\widetilde{v}) \\ -i\langle i\nabla\rangle_m^{-1} F_5(\widetilde{v})\end{pmatrix} d\tau\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{5/2}\right)} \\
\leq\ & C\left\|\mathring{w}_5\right\|_{\mathbf{H}^3} + C\left\|\widetilde{v}^2\right\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^2\right)} \\
\leq\ & C\left\|\mathring{w}_5\right\|_{\mathbf{H}^3} + C\left\|\widetilde{v}\right\|_{\mathbf{L}_t^{10/7}(\mathbf{I};\mathbf{L}^5)}\left\|\widetilde{v}\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^2\right)} \leq C\varepsilon + C\rho^2 \qquad (2.2.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\|w_5\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^3)} &\leq C\left\|\mathring{w}_5\right\|_{\mathbf{H}^3} + C\left\|\widetilde{v}^2\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{H}^2)} \\
&\leq C\left\|\mathring{w}_5\right\|_{\mathbf{H}^3} + C\left\|\widetilde{v}\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^{10})}\left\|\widetilde{v}\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^2\right)} \\
&\leq C\varepsilon + C\rho^2. \qquad (2.2.8)
\end{aligned}
$$

Since $[\mathcal{L}_M, \mathcal{Z}] = -iA\langle i\nabla\rangle_M^{-1}\nabla\mathcal{L}_M$, the application of the operator $\mathcal{Z}$ to equation (2.2.1) yields

$$
\mathcal{L}_M\mathcal{Z}w_j = \left(\mathcal{Z} - iA\langle i\nabla\rangle_M^{-1}\nabla\right)\begin{pmatrix} i\langle i\nabla\rangle_M^{-1} F_j(v,\partial_\mu v_5) \\ -i\langle i\nabla\rangle_M^{-1} F_j(v,\partial_\mu v_5)\end{pmatrix}.
$$

Then by integrating with respect to time

$$
\mathcal{Z}w_j = \mathcal{U}_M(t)\,\mathcal{Z}w_j + \int_0^t \mathcal{U}_M(t-\tau)\left(\mathcal{Z} - iA\langle i\nabla\rangle_M^{-1}\nabla\right)\begin{pmatrix} i\langle i\nabla\rangle_M^{-1} F_j(v,\partial_\mu v_5) \\ -i\langle i\nabla\rangle_M^{-1} F_j(v,\partial_\mu v_5)\end{pmatrix} d\tau. \qquad (2.2.9)
$$

As above, taking the $\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3/2}\right)$ and $\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{1/2}\right)$ - norms of (2.2.9) and applying Lemma 2.1.2 we obtain

$$
\begin{aligned}
&\|\mathcal{Z}w_j\|_{\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^{3/2}\right)} + \|\mathcal{Z}w_j\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^1\right)} \\
\leq\ & C\|\mathring{w}\|_{\mathbf{H}^{5/2,1}} \\
&+C\sum_{|\beta|\leq 1}\left(\left\|\mathcal{Z}^\beta F_j(v,\partial_\mu v_5)\right\|_{\mathbf{L}_t^1\left(\mathbf{I};\mathbf{H}^{1/2}\right)} + \left\|\mathcal{Z}^\beta F_j(v,\partial_\mu v_5)\right\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^{1/2}\right)}\right). \qquad (2.2.10)
\end{aligned}
$$

We have by Hölder's inequality

$$\begin{aligned}
&\left\|\mathcal{Z}F_j\left(v,\partial_\mu v_5\right)\right\|_{\mathbf{L}_t^1\left(\mathbf{I};\mathbf{H}^{1/2}\right)}\\
\le\ & C\left\|v\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_{10}^{1/2}\right)}\left\|\mathcal{Z}\partial_\mu v_5\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{H}_q^{1/2}\right)}+C\left\|\mathcal{Z}v\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{L}^{p'}\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_p^{1/2}\right)}\\
&+C\left\|\mathcal{Z}v\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{H}_q^{1/2}\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}^{20/17}\left(\mathbf{I};\mathbf{L}^{10}\right)}\\
\le\ & C\varepsilon+C\rho^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\mathcal{Z}F_j\left(v,\partial_\mu v_5\right)\right\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^{1/2}\right)}\\
\le\ & C\left\|v\right\|_{\mathbf{L}_t^{10/7}\left(\mathbf{I};\mathbf{L}^{5}\right)}\left\|\mathcal{Z}\partial_\mu v_5\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{H}_q^{1/2}\right)}+C\left\|\mathcal{Z}v\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{H}_q^{1/2}\right)}\left\|\partial_\mu v_5\right\|_{\mathbf{L}_t^{10/7}\left(\mathbf{I};\mathbf{H}_5^{1}\right)}\\
\le\ & C\varepsilon+C\rho^2.
\end{aligned}$$

In the same manner we find the estimate

$$\left\|\partial_t w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)}+\left\|\partial_t w_j\right\|_{\mathbf{L}_t^{r}\left(\mathbf{I};\mathbf{H}_q^{1}\right)}\le C\varepsilon+C\rho^2. \tag{2.2.11}$$

By the relation $\mathcal{J}_M=iA\mathcal{Z}-iA\mathcal{L}_Mx$ we get

$$\left\|\mathcal{J}_M w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}\le\left\|\mathcal{Z}w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}+\left\|\mathcal{L}_M x w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}. \tag{2.2.12}$$

Multiplying both sides of (2.2.1) by x, we find

$$\mathcal{L}_M x w_j=-iA\left[x,\langle i\nabla\rangle_M\right]w_j+x\begin{pmatrix} i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\\ -i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\end{pmatrix}. \tag{2.2.13}$$

Since $Ex=\langle i\nabla\rangle_M^{-1}\mathcal{J}_M-iAt\nabla\langle i\nabla\rangle_M^{-1}$ by the Sobolev inequality we have

$$\left\|x\begin{pmatrix} i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\\ -i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\end{pmatrix}\right\|_{\mathbf{H}^{3/2}}\le C\varepsilon+C\rho^2. \tag{2.2.14}$$

Therefore by (2.2.10), (2.2.12) - (2.2.14)

$$\begin{aligned}
&\left\|\mathcal{J}_M w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}\\
\le\ & \left\|\mathcal{Z}w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}+\left\|w_j\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}\\
&+C\left\|x\begin{pmatrix} i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\\ -i\langle i\nabla\rangle_M^{-1}F_j\left(v,\partial_\mu v_5\right)\end{pmatrix}\right\|_{\mathbf{L}_t^{\infty}\left(\mathbf{I};\mathbf{H}^{3/2}\right)}\\
\le\ & C\varepsilon+C\rho^2,
\end{aligned}$$

similarly,

$$\left\|\mathcal{J}_M w_j\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^1\right)} \le C\varepsilon + C\rho^2.$$

We next consider the estimate of w_5. Taking the $\mathbf{L}_t^\infty\left(\mathbf{I};\mathbf{H}^2\right)$ and $\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)$ - norms to the integral equation

$$\mathcal{Z}w_5 = \mathcal{U}_m(t)\,\mathcal{Z}w_5 + \int_0^t \mathcal{U}_m(t-\tau)\left(\mathcal{Z} - iA\left\langle i\nabla\right\rangle_m^{-1}\nabla\right)\begin{pmatrix} i\left\langle i\nabla\right\rangle_m^{-1} F_5(\widetilde{v}) \\ -i\left\langle i\nabla\right\rangle_m^{-1} F_5(\widetilde{v}) \end{pmatrix} d\tau$$

and applying Lemma 2.1.2 we get

$$\begin{aligned}
& \left\|\mathcal{Z}w_5\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^2)} + \left\|\mathcal{Z}w_5\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)} \\
\le\ & C\left\|\mathring{w}\right\|_{\mathbf{H}^{3,1}} + C\sum_{|\beta|\le 1}\left(\left\|\mathcal{Z}^\beta F_5(\widetilde{v})\right\|_{\mathbf{L}_t^1(\mathbf{I};\mathbf{H}^1)} + \left\|\mathcal{Z}^\beta F_5(\widetilde{v})\right\|_{\mathbf{L}_t^{r'}\left(\mathbf{I};\mathbf{H}_{q'}^1\right)}\right) \\
\le\ & C\left\|\mathring{w}\right\|_{\mathbf{H}^{3,1}} + C\left\|\widetilde{v}\right\|_{\mathbf{L}_t^{20/17}(\mathbf{I};\mathbf{L}^{10})}\left\|\mathcal{Z}\widetilde{v}\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^1\right)} \\
& + C\left\|\widetilde{v}\right\|_{\mathbf{L}_t^{10/7}(\mathbf{I};\mathbf{L}^5)}\left\|\mathcal{Z}\widetilde{v}\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^1\right)} + C\left\|\widetilde{v}\right\|_{\mathbf{L}_t^{20/17}\left(\mathbf{I};\mathbf{H}_p^1\right)}\left\|\mathcal{Z}\widetilde{v}\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{L}^{p'}\right)} \\
\le\ & C\varepsilon + C\rho^2. \qquad (2.2.15)
\end{aligned}$$

In the same way

$$\left\|\mathcal{J}_m w_5\right\|_{\mathbf{L}_t^\infty(\mathbf{I};\mathbf{H}^2)} + \left\|\mathcal{J}_m w_5\right\|_{\mathbf{L}_t^r\left(\mathbf{I};\mathbf{H}_q^{3/2}\right)} \le C\varepsilon + C\rho^2 \qquad (2.2.16)$$

which in view of (2.2.6), (2.2.10) and (2.2.11) implies

$$\|w\|_{\mathbf{Z_I}} \le \rho + C\rho^2. \qquad (2.2.17)$$

Therefore the mapping $\mathcal{M} : w = \mathcal{M}(v)$ defined by the problem (2.2.1), transforms a ball $\mathbf{Z_I}(2\rho)$ into itself. Denote $\widetilde{w} = \mathcal{M}(\widetilde{v})$, then in the same way as in the proof of (2.2.17) we have

$$\|w - \widetilde{w}\|_{\mathbf{X}} \le C\rho\,\|v - \widetilde{v}\|_{\mathbf{X}}.$$

Thus we find that there exists an ρ such that $\mathcal{M}$ is a contraction mapping in $\mathbf{Z_I}(2\rho)$ and so there exists a unique solution $w = \mathcal{M}(w)$. To prove the asymptotics of solutions, we replace v by w in (2.2.9), then we have for $t \ge t'$

$$\begin{aligned}
& \mathcal{U}_M(-t)\,\mathcal{Z}w_j(t) - \mathcal{U}_M(-t')\,\mathcal{Z}w_j(t') \\
=\ & \int_{t'}^t \mathcal{U}_M(-\tau)\left(\mathcal{Z} - iA\left\langle i\nabla\right\rangle_M^{-1}\nabla\right)\begin{pmatrix} i\left\langle i\nabla\right\rangle_M^{-1} F_j(w,\partial_\mu w_5) \\ -i\left\langle i\nabla\right\rangle_M^{-1} F_j(w,\partial_\mu w_5) \end{pmatrix} d\tau.
\end{aligned}$$

Hence by lemma with $\widetilde{\mathbf{I}} = (-\infty, 0]$ and estimate (2.2.5)

$$\begin{aligned} & \left\| \mathcal{U}_M(-t) \mathcal{Z} w_j(t) - \mathcal{U}_M(-t') \mathcal{Z} w_j(t') \right\|_{\mathbf{H}^{3/2}} \\ \leq & \left\| \int_{t'}^{t} \mathcal{U}_M(-\tau) \left(\mathcal{Z} - iA \langle i\nabla \rangle_M^{-1} \nabla \right) \begin{pmatrix} i \langle i\nabla \rangle_M^{-1} F_j(w, \partial_\mu w_5) \\ -i \langle i\nabla \rangle_M^{-1} F_j(w, \partial_\mu w_5) \end{pmatrix} d\tau \right\|_{\mathbf{H}^{3/2}} \\ \leq & \; C\rho^2 \langle t \rangle^{-\gamma} \end{aligned}$$

since

$$\left| \int_{t'}^{t} \langle \tau \rangle^{-\frac{9}{7}} d\tau \right|^{\frac{7}{10}} \leq C \langle t \rangle^{-\frac{1}{5}}, \left| \int_{t'}^{t} \langle \tau \rangle^{-\frac{3}{2}\left(1-\frac{2}{p}\right)\frac{20}{17}} d\tau \right|^{\frac{17}{20}} \leq C \langle t \rangle^{-\gamma},$$

where $\gamma = -3(1-2/p)/2 + 17/20, 60/13 < p < 6$. Similarly,

$$\left\| \mathcal{U}_M(-t) w_j(t) - \mathcal{U}_M(-t') w_j(t') \right\|_{\mathbf{H}^{5/2}} \leq C\rho^2 \langle t \rangle^{-\gamma}$$

for all $0 > t > t'$. As in derivation of estimate (2.2.12) we obtain

$$\left\| \mathcal{U}_M(-t) \mathcal{J}_M w_j(t) - \mathcal{U}_M(-t') \mathcal{J}_M w_j(t') \right\|_{\mathbf{H}^{3/2}} \leq C\rho^2 \langle t \rangle^{-\gamma}.$$

Hence

$$\left\| \mathcal{U}_M(-t) w_j(t) - \mathcal{U}_M(-t') w_j(t') \right\|_{\mathbf{H}^{5/2,1}} \leq C\rho \langle t \rangle^{-\gamma}$$

for all $0 > t > t'$. Thus we see that there exists a unique final state $w_j^- \in \left(\mathbf{H}^{5/2,1}\right)^2$ such that

$$\left\| \mathcal{U}_M(-t) w_j(t) - w_j^- \right\|_{\mathbf{H}^{5/2,1}} \leq C\rho^2 \langle t \rangle^{-\gamma}.$$

In the same way as in the proofs of (2.2.7), (2.2.8), (2.2.15), (2.2.16) we find that there exists a unique final state $w_5^- \in \left(\mathbf{H}^{3,1}\right)^2$

$$\left\| \mathcal{U}_m(-t) w_5(t) - w_5^- \right\|_{\mathbf{H}^{3,1}} \leq C\rho^2 \langle t \rangle^{-\gamma}.$$

This completes the proof of Theorem 2.0.1. In order to prove Theorem 2.0.2, we consider the problem

$$\begin{cases} \mathcal{L}_M \left(w_j - \mathcal{U}_M(t) w_j^+ \right) = \begin{pmatrix} i \langle i\nabla \rangle_M^{-1} F_j(w, \partial_\mu w_5) \\ -i \langle i\nabla \rangle_M^{-1} F_j(w, \partial_\mu w_5) \end{pmatrix}, \\ \mathcal{L}_m \left(w_5 - \mathcal{U}_m(t) w_5^+ \right) = \begin{pmatrix} i \langle i\nabla \rangle_m^{-1} F_5(\widetilde{w}) \\ -i \langle i\nabla \rangle_m^{-1} F_5(\widetilde{w}) \end{pmatrix}, \end{cases} \tag{2.2.18}$$

under the condition that

$$\left\|\mathcal{U}_M(-t)\,w_j(t)-w_j^+\right\|_{\mathbf{H}^{5/2,1}}+\left\|\mathcal{U}_m(-t)\,w_5(t)-w_5^+\right\|_{\mathbf{H}^{3,1}}\to 0$$

as $t\to\infty$. In the same way as in the proof of Theorem 2.0.1, we find that there exists a unique solution w and an ε such that $\|w\|_{\mathbf{Z}_{[0,\infty)}}\le C\varepsilon$ for any $w^+\in\left(\mathbf{H}^{5/2,1}(\varepsilon)\right)^4\times\mathbf{H}^{3,1}(\varepsilon)$. We also have the time decay estimates

$$\left\|\mathcal{U}_M(-t)\,w_j(t)-w_j^+\right\|_{\mathbf{H}^{5/2,1}}+\left\|\mathcal{U}_m(-t)\,w_5(t)-w_5^+\right\|_{\mathbf{H}^{3,1}}\le C\rho^2\langle t\rangle^{-\gamma}.$$

This completes the proof of Theorem 2.0.2.

CHAPTER 3

SCATTERING PROBLEM FOR A SYSTEM OF NONLINEAR KLEIN-GORDON EQUATIONS

We consider a system of semi-linear Klein-Gordon equations

$$\begin{cases} \left(\Box + m_j^2\right) u_j = \mathcal{N}_j\left(u, \partial u\right), \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n, \\ u_j\left(0, x\right) = \mathring{u}_j^{(1)}\left(x\right), \partial_t u_j\left(0, x\right) = \mathring{u}_j^{(2)}\left(x\right), x \in \mathbf{R}^n, \end{cases} \tag{3.0.1}$$

where $\Box = \partial_t^2 - \Delta, m_j > 0, u = (u_1, \ldots, u_l)$, the partial derevative $\partial = (\partial_t, \partial_x) = (\partial_0, \partial_1, \ldots, \partial_n)$, the spatial dimension $n = 2, 3$. We assume that the nonlinearities $\mathcal{N}_j\left(y\right) \in C^{p_0}\left(\mathbf{C}^{(2+n)l}; \mathbf{C}\right)$ satisfy the estimates

$$|\mathcal{N}_j\left(y\right)| \leq C|y|^2, |\partial_y \mathcal{N}_j\left(y\right)| \leq C|y|, |\partial_y^\alpha \mathcal{N}_j\left(y\right)| \leq C, 2 \leq |\alpha| \leq p_0 \tag{3.0.2}$$

fol all $|y| \leq 1$. Thus the nonlinearities $\mathcal{N}_j$ include quadratic terms, when we restrict our attention to small solutions. Our purpose in the present paper is to prove asymptotic stability of small solutions to (3.0.1) in the neighborhood of free solutions in the lower order Sobolev spaces comparing with the previous works [2], [4], [5], [16], [17], [18]. The main ingredient which will be used here is the method of “Klainerman vector fields”. We will rely on the expression of the Klainerman vector fields in terms of the operator itself and of a convenient pseudodifferential operator of order one. It allows us to treat the system involving Klein-Gordon equation with positive masses and the wave equations (i.e. zero masses Klein-Gordon equations). As an example of such a system will be considered the system of massive Dirac-massless Klein-Gordon equations in three space dimensions.

For $n = 2, 3$ we denote the usual Lebesgue space by $\mathbf{L}^p = \{\phi \in S'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbf{R}^n} |\phi\left(x\right)|^p dx\right)^{\frac{1}{p}}$, if $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x \in \mathbf{R}^n} |\phi\left(x\right)|$, if $p = \infty$. Weighted Sobolev space

$$\mathbf{H}_p^{m,k} = \left\{\phi; \|\phi\|_{\mathbf{H}_p^{m,k}} \equiv \|\left\langle x \right\rangle^k \left(1 - \Delta\right)^{\frac{m}{2}} \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $m,k \in \mathbf{R}, 1 \leq p \leq \infty, \langle x\rangle = \sqrt{1+|x|^2}, \Delta = \sum_{i=1}^{n}\partial_i^2$. Homogeneous weighted Sobolev space is defined by

$$\dot{\mathbf{H}}_p^{m,k} = \left\{\phi; \|\phi\|_{\dot{\mathbf{H}}_p^{m,k}} \equiv \| \langle x\rangle^k (1-\Delta)^{\frac{m-1}{2}} |\nabla|\phi\|_{\mathbf{L}^p} < \infty\right\}.$$

In the sequal we use widely the fact that the spaces $\mathbf{H}_p^{m,k}$ for $1 < p < \infty$ are stable under Fourier multipliers of order 0, in particular by the operator $\langle i\nabla\rangle^{-1} i\nabla$ (see [15]). We also write for simplicity $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}, \mathbf{H}^m = \mathbf{H}_2^{m,0}$, $\mathbf{H}_p^m = \mathbf{H}_p^{m,0}$, so we usually omit the index 0 if it does not cause a confusion. We use the same notations for vector valued functions. The direct Fourier transform $\widehat{\phi}(\xi)$ of the function $\phi(x)$ is defined by

$$\mathcal{F}\phi = \widehat{\phi} = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i(x.\xi)}\phi(x)dx,$$

then the inverse Fourier transformation is given by

$$\mathcal{F}^{-1}\phi = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i(x.\xi)}\phi(\xi)d\xi.$$

We denote the free Klein-Gordon evolution group by

$$U_m(t) = \begin{pmatrix} \cos\left(\langle i\nabla\rangle_m t\right) & \sin\left(\langle i\nabla\rangle_m t\right) \\ -\sin\left(\langle i\nabla\rangle_m t\right) & \cos\left(\langle i\nabla\rangle_m t\right) \end{pmatrix},$$

where $\langle x\rangle_m = \sqrt{m^2+|x|^2}$, so that $\langle i\nabla\rangle_m = \sqrt{m^2-\Delta}$. For the case of the massive Klein-Gordon equations we introduce a closed ball $\mathbf{H}_q^{m,k}(\varepsilon)$ with radius $\varepsilon > 0$ and a center at the origin in the function space

$$\mathbf{H}_q^{m,k} = \left\{ v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}; \|v^{(1)}\|_{\mathbf{H}_q^{m,k}} + \|v^{(2)}\|_{\mathbf{H}_q^{m,k}} < \infty \right\}.$$

Different positive constants we denote by the same letter C.

Theorem 3.0.1. *Let $n = 3$ and condition* (3.0.2) *be fulfilled with $p_0 = 4$. Then there exists $\varepsilon > 0$ such that for any initial data $\begin{pmatrix} \mathring{u}_j^{(1)} \\ \mathring{u}_j^{(2)} \end{pmatrix} \in \boldsymbol{H}^{4,3}(\varepsilon_0), 1 \leq j \leq l$, with $\varepsilon_0 \in \left(0, \varepsilon^{\frac{4}{3}}\right]$ the Cauchy problem* (3.0.1) *has a unique global solution*

$$u_j \in C\left([0,\infty); \boldsymbol{H}^{4,3}\right) \cap C^1\left([0,\infty); \boldsymbol{H}^{3,3}\right)$$

and

$$\|u_j(t)\|_{L^\infty} \le C \langle t\rangle^{-\frac{3}{2}\left(1-\frac{2}{q}\right)}$$

for all $t \ge 0, 1 \le j \le l$, *where* $2 \le q < \infty$. *Furthermore, there exists a unique final states* $\begin{pmatrix} u_j^{+(1)} \\ u_j^{+(2)} \end{pmatrix} \in \boldsymbol{H}^{4,1}(\varepsilon), 1 \le j \le l$, *satisfying*

$$\left\| U_{m_j}(-t) \begin{pmatrix} u_j(t) \\ \langle i\nabla\rangle_{m_j}^{-1} \partial_t u_j(t) \end{pmatrix} - \begin{pmatrix} u_j^{+(1)} \\ \langle i\nabla\rangle_{m_j}^{-1} u_j^{+(2)} \end{pmatrix} \right\|_{\boldsymbol{H}^{4,1}} \le C\varepsilon^2 \langle t\rangle^{-\delta}$$

for all $t \ge 0, 1 \le j \le l$, *where* $0 < \delta < \frac{1}{2}$.

Theorem 3.0.1 shows the existence of final states $\begin{pmatrix} u_j^{+(1)} \\ u_j^{+(2)} \end{pmatrix} \in \mathbf{H}^{4,1}(\varepsilon)$. In other words the small solution of (3.0.1) are asymptotically stable in the neighborhood of the free solutions. Therefore, the inverse wave operator is well defined from $(\mathbf{H}^{4,3}(\varepsilon_0))^l$ to $(\mathbf{H}^{4,1}(\varepsilon))^l$.

Remark 3.0.1. *Since the evolution operator* $U_{m_j}(t)$ *is the unitary one in* $\boldsymbol{L}^2 \times \boldsymbol{L}^2$, *we find*

$$\left\| \begin{pmatrix} u_j(t) \\ \langle i\nabla\rangle_{m_j}^{-1} \partial_t u_j(t) \end{pmatrix} - U_{m_j}(t) \begin{pmatrix} u_j^{+(1)} \\ \langle i\nabla\rangle_{m_j}^{-1} u_j^{+(2)} \end{pmatrix} \right\|_{\boldsymbol{H}^4} \le C\varepsilon^2 t^{-\delta}$$

for all $t \ge 0, 1 \le j \le l$, *where* $0 < \delta < \frac{1}{2}$. *Hence,*

$$\|u_j(t) - \left(\cos(\langle i\nabla\rangle_{m_j} t) u_j^{+(1)} + \sin(\langle i\nabla\rangle_{m_j} t) \langle i\nabla\rangle_{m_j}^{-1} u_j^{+(2)}\right) \|_{\boldsymbol{H}^4} \le C\varepsilon^2 \langle t\rangle^{-\delta}.$$

We next consider the two dimensional case $n = 2$. Assume that nonlinear terms have a special complex-conjugate structure,

$$\mathcal{N}(u, \partial u) = \sum_{0\le r,s\le 2, 1\le m,k\le l} q^{mkrs} \partial_r u_m \overline{\partial_s u_k}, \tag{3.0.3}$$

with complex vector coefficients q^{mkrs}. Following [5], we introduce the strong null condition.

Definition 3.0.1. *We say that nonlinearities written by* (3.0.3) *satisfy the strong null condition if*

$$\sum_{1\le r,s\le 2} q^{mkrs} \eta_r \eta_s = 0$$

for any $\eta = (\eta_0, \eta_1, \eta_2) \in \left(\boldsymbol{R}^3\right), 1 \le m, k \le l$.

For example, the nonlinearity

$$\mathcal{N}(u,\partial u) = q^{1212}\left(\partial_1 u_1 \overline{\partial_2 u_2} - \partial_2 u_1 \overline{\partial_1 u_2}\right)$$

satisfies the strong null condition. Denote the operators $\mathcal{Z}^{(1)} = t\partial_1 + x_1\partial_t$ and $\mathcal{Z}^{(2)} = t\partial_2 + x_2\partial_t$. Then by the identity

$$\mathcal{N}(u,\partial u) = q^{1212}t^{-2}\Big((\mathcal{Z}^{(1)}u_1)\overline{\mathcal{Z}^{(2)}u_2} - (\mathcal{Z}^{(2)}u_1)\overline{\mathcal{Z}^{(1)}u_2}$$
$$-(\mathcal{Z}^{(1)}u_1)\overline{x_2\partial_t u_2} + (x_2\partial_t u_1)\overline{\mathcal{Z}^{(1)}u_2} - (x_1\partial_t u_1)\overline{\mathcal{Z}^{(2)}u_2} + (\mathcal{Z}^{(2)}u_1)\overline{x_1\partial_t u_2}\Big)$$

it is clear that the strong null condition helps us gain more time decay properties of the nonlinearity through the operator $t\partial_j + x_j\partial_t$, so that the problem with the qudratic nonlinearity in two space dimensions behaves like the asymptotically free one. Indeed, we have the following result.

Theorem 3.0.2. *Let condition* (3.0.2) *be fulfilled with* $p_0 = 5$*. Suppose that the quadratic nonlinear terms of the nonlinearity satisfy the strong null condition. Then there exists* $\varepsilon > 0$ *such that for any initial data* $\begin{pmatrix} \mathring{u}_j^{(1)} \\ \mathring{u}_j^{(2)} \end{pmatrix} \in \boldsymbol{H}^{5,4}(\varepsilon_0), 1 \le j \le l$, *with* $\varepsilon_0 \in \left(0, \varepsilon^{\frac{5}{4}}\right]$, *Cauchy problem* (3.0.1) *has a unique global solution,*

$$u_j \in C\left([0,\infty);\boldsymbol{H}^{5,4}\right) \cap C^1\left([0,\infty);\boldsymbol{H}^{4,4}\right)$$

and

$$\|u_j(t)\|_{L^\infty} \le C\langle t\rangle^{-1}$$

for all $t \ge 0, 1 \le j \le l$*. Furthermore there exists a unique final states* $\begin{pmatrix} u_j^{+(1)} \\ u_j^{+(2)} \end{pmatrix} \in \boldsymbol{H}^{4,1}(\varepsilon), 1 \le j \le l$, *satisfying*

$$\left\| U_{m_j}(-t)\begin{pmatrix} u_j(t) \\ \langle i\nabla\rangle_{m_j}^{-1}\partial_t u_j(t)\end{pmatrix} - \begin{pmatrix} u_j^{+(1)} \\ \langle i\nabla\rangle_{m_j}^{-1} u_j^{+(2)}\end{pmatrix}\right\|_{\boldsymbol{H}^{4,1}} \le C\varepsilon^2\langle t\rangle^{-\delta}$$

for all $t \ge 0, 1 \le j \le l$*, where* $0 < \delta < \frac{1}{2}$*.*

Theorem 3.0.2 shows the existence of final scattering states $\begin{pmatrix} u_j^{+(1)} \\ u_j^{+(2)} \end{pmatrix} \in \mathbf{H}^{4,1}(\varepsilon)$. Namely, the inverse wave operator is well defined from $(\mathbf{H}^{5,4}(\varepsilon_0))^l$ to $(\mathbf{H}^{4,1}(\varepsilon))^l$, and the small solutions of (3.0.1) are asymptotically stable in the neighborhood of the free solutions.

Remark 3.0.2. *By the proof of Theorem* 3.0.2 *we find that*

$$\|u_j(t) - \left(\cos(\langle i\nabla\rangle_{m_j} t)u_j^{+(1)} + \sin(\langle i\nabla\rangle_{m_j} t)\,\langle i\nabla\rangle_{m_j}^{-1} u_j^{+(2)}\right)\|_{\boldsymbol{H}^4} \leq C\varepsilon^2 t^{-1}.$$

We now apply our method to the Dirac-Klein-Gordon equations

$$\begin{cases} \left(-i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\psi = ig\phi\gamma^4\psi, \\ \left(\Box + m^2\right)\phi = g\psi^*\gamma^0\psi, \end{cases} \tag{3.0.4}$$

where $\Box = \partial_t^2 - \Delta, m \geq 0, M > 0, g \in \mathbf{R}$,

$$\psi = \psi(t,x) = (\psi_1(t,x), \psi_2(t,x), \psi_3(t,x), \psi_4(t,x)) \in \mathbf{C}^4$$

is a spinor field, and $\phi = (t,x) \in \mathbf{R}$ is a scalar field. We use the coordinates $t = x_0$, $x = (x_1,x_2,x_3)$ on $\mathbf{R}^{1+3}$, and $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where $\{\gamma^\mu\}$ are the 4×4 Dirac matrices, given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Here ψ^* denotes the adjoint. We have

$$\psi^*\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2.$$

The 4×4 matrix γ^4 is defined by

$$\gamma^4 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Note that $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \mathbf{0}$ for $\nu \neq \mu$ and $\gamma^0\gamma^0 = \mathbf{1}$ and $\gamma^\mu\gamma^\mu = \mathbf{-1}$ for $\mu = 1, 2, 3$, where $\mathbf{1} = [\delta_{jk}]_{1\leq j,k\leq 4}, \delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$. Then by a direct computation we get

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \mathbf{1}\Box$$

and so by (3.0.4)

$$\begin{aligned}
\left(\Box + M^2\right)\psi &= \left(i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\left(-i\sum_{\nu=0}^{3}\gamma^\nu\partial_\nu + M\right)\psi \\
&= ig\left(i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\phi\gamma^4\psi \\
&= -g\sum_{\mu=0}^{3}\gamma^\mu\gamma^4\psi\partial_\mu\phi + ig\phi\left(i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\gamma^4\psi \\
&= -g\sum_{\mu=0}^{3}\gamma^\mu\gamma^4\psi\partial_\mu\phi + ig\phi\gamma^4\left(-i\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu + M\right)\psi \\
&= -g\sum_{\mu=0}^{3}\gamma^\mu\gamma^4\psi\partial_\mu\phi - g^2\phi^2\psi
\end{aligned}$$

and

$$\left(\Box + m^2\right)\phi = g\left(|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2\right),$$

where we have used the identities $\gamma^4\gamma^\mu = -\gamma^\mu\gamma^4$ for $\mu = 0, 1, 2, 3$ and $\gamma^4\gamma^4 = \mathbf{1}$. Therefore, we consider the system of Klein-Gordon equations instead of (3.0.4)

$$\begin{cases} \left(\Box + M^2\right)\psi = -g^2\phi^2\psi - g\sum_{\mu=0}^{3}\gamma^\mu\gamma^4\psi\partial_\mu\phi, \\ \left(\Box + m^2\right)\phi = g\left(|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2\right), \\ \psi_j(0,x) = \mathring{\psi}_j^{(1)}, \partial_t\psi_j(0,x) = \mathring{\psi}_j^{(2)}, j = 1,2,3,4, \\ \phi(0,x) = \mathring{\phi}^{(1)}, \partial_t\phi(0,x) = \mathring{\phi}^{(2)}, x \in \mathbf{R}^3. \end{cases} \tag{3.0.5}$$

In the case of $M > 0, m > 0$, the asymptotic stability of small solutions was obtained in Theorem 3.0.1 .

We now consider the case $M > 0, m = 0$. We introduce a closed ball $\dot{\mathbf{H}}_q^{m,k}(\varepsilon)$ with a radius ε and a center in the origin in the vector function space $\dot{\mathbf{H}}_q^{m,k}$ of initial data for massless Klein-Gordon equations,

$$\dot{\mathbf{H}}_q^{m,k} = \left\{ v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \end{pmatrix}; \|v^{(1)}\|_{\dot{\mathbf{H}}_q^{m,k}} + \|v^{(2)}\|_{\mathbf{H}_q^{m-1,k}} < \infty \right\}.$$

Theorem 3.0.3. *Let $M > 0$ and $m = 0$. Then there exists $\varepsilon > 0$ such that for any initial data* $\begin{pmatrix} \mathring{\psi}_j^{(1)} \\ \mathring{\psi}_j^{(2)} \end{pmatrix} \in \boldsymbol{H}^{4,3}(\varepsilon_0)$, $\begin{pmatrix} \mathring{\phi}^{(1)} \\ \mathring{\phi}^{(2)} \end{pmatrix} \in \dot{\boldsymbol{H}}^{4,3}(\varepsilon_0)$ *with* $\varepsilon_0 \in \left(0, \varepsilon^{\frac{4}{3}}\right]$, *Cauchy problem* (3.0.5) *has a unique global solution,*

$$\psi_j \in C\left([0,\infty); \boldsymbol{H}^{4,3}\right) \cap C^1\left([0,\infty); \boldsymbol{H}^{3,3}\right),$$
$$\phi \in C\left([0,\infty); \dot{\boldsymbol{H}}^{4,3}\right) \cap C^1\left([0,\infty); \boldsymbol{H}^{3,3}\right),$$

and

$$\|\psi_j(t)\|_{L^\infty} \le Ct^{-\frac{3}{2}\left(1-\frac{2}{q}\right)}, \|i\nabla\phi(t)\|_{L^\infty} \le C\langle t\rangle^{-1},$$

where $2 \le p < \infty$, for all $t \ge 0, 1 \le j \le 4$. Furthermore, there exists a unique final states $\begin{pmatrix} \psi_j^{+(1)} \\ \psi_j^{+(2)} \end{pmatrix} \in \boldsymbol{H}^{4,1}(\varepsilon)$ *satisfying*

$$\left\| U_M(-t)\begin{pmatrix} \psi_j(t) \\ \langle i\nabla\rangle_M^{-1}\partial_t\psi_j(t) \end{pmatrix} - \begin{pmatrix} \psi_j^{+(1)} \\ \langle i\nabla\rangle_M^{-1}\psi_j^{+(2)} \end{pmatrix} \right\|_{\boldsymbol{H}^{4,1}} \le C\varepsilon^2 t^{-\delta}$$

for all $t \geq 0, 1 \leq j \leq 4$, *where* $0 < \delta < \frac{1}{2}$ *and there exists a unique* $\begin{pmatrix} \phi^{+(1)} \\ \phi^{+(2)} \end{pmatrix} \in \dot{\boldsymbol{H}}^{4,1}(\varepsilon)$ *satisfying*

$$\left\| U_0(-t) \begin{pmatrix} \phi(t) \\ |i\nabla|^{-1}\partial_t \phi(t) \end{pmatrix} - \begin{pmatrix} \phi^{+(1)} \\ |i\nabla|^{-1}\phi^{+(2)} \end{pmatrix} \right\|_{\dot{\boldsymbol{H}}^{4,1}} \leq C\varepsilon^2 t^{-\delta}$$

for all $t \geq 0$.

Theorem 3.0.3 shows the existence of final states $\begin{pmatrix} \psi_j^{+(1)} \\ \psi_j^{+(2)} \end{pmatrix} \in \mathbf{H}^{4,1}(\varepsilon)$ and $\begin{pmatrix} \phi^{+(1)} \\ \phi^{+(2)} \end{pmatrix} \in \dot{\mathbf{H}}^{4,1}(\varepsilon)$. Namely, the inverse wave operator is well-defined from $(\mathbf{H}^{4,3}(\varepsilon_0))^4 \times \dot{\mathbf{H}}^{4,3}(\varepsilon_0)$ to $(\mathbf{H}^{4,1}(\varepsilon))^4 \times \dot{\mathbf{H}}^{4,1}(\varepsilon)$ and the small solutions of (3.0.5) are asymptotically stable in the neighborhood of the free solutions.

Remark 3.0.3. *By virtue of Theorem* 3.0.3*, we have*

$$\|\psi_j(t) - \left(\cos(\langle i\nabla\rangle_M t)\psi_j^{+(1)} + \sin(\langle i\nabla\rangle_M t)\langle i\nabla\rangle_M^{-1}\psi_j^{+(2)}\right)\|_{\boldsymbol{H}^4} \leq C\varepsilon^2 t^{-\delta}$$

and

$$\|\phi(t) - \left(\cos(|i\nabla|t)\phi^{+(1)} + \sin(|i\nabla|t)|i\nabla|^{-1}\phi^{+(2)}\right)\|_{\dot{\boldsymbol{H}}^4} \leq C\varepsilon^2 t^{-\delta}.$$

We can see the previous work on (4) *in [2], where the higher order Sobolev spaces* $\boldsymbol{H}^{26}$ *used for the initial data.*

Our result is obtained through a new time decay estimates of solutions to the inhomogeneous equations

$$\begin{cases} \mathcal{L}_{m_j} w = \begin{pmatrix} i\langle i\nabla\rangle_m^{-1} f \\ -i\langle i\nabla\rangle_m^{-1} f \end{pmatrix}, \\ w(0,x) = \mathring{w}, \end{cases} \tag{3.0.6}$$

where $\mathcal{L}_m = E\partial_t + A\langle i\nabla\rangle_m$, $\langle i\nabla\rangle_m = \sqrt{m^2-\Delta}, m>0$,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$w = \begin{pmatrix} w^{(1)} \\ w^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u + i\langle i\nabla\rangle_m^{-1}\partial_t u \\ u - i\langle i\nabla\rangle_m^{-1}\partial_t u \end{pmatrix},$$

$$\mathring{w} = \begin{pmatrix} \mathring{w}^{(1)} \\ \mathring{w}^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathring{u}^{(1)} + i\langle i\nabla\rangle_m^{-1}\mathring{u}^{(2)} \\ \mathring{u}^{(1)} - i\langle i\nabla\rangle_m^{-1}\mathring{u}^{(2)} \end{pmatrix}.$$

Then the solution u of the usual inhomogeneous Klein-Gordon equation

$$\begin{cases} \left(\partial_t^2 - \Delta + m^2\right)u = f, \\ u(0) = \mathring{u}^{(1)}, \partial_t u(0) = \mathring{u}^{(2)}, \end{cases} \tag{3.0.7}$$

can be representated by $u = w^{(1)} + w^{(2)}$. We introduce the free evolution group of (3.0.6)

$$\mathcal{U}_m(t) = \begin{pmatrix} e^{-it\langle i\nabla\rangle_m} & 0 \\ 0 & e^{it\langle i\nabla\rangle_m} \end{pmatrix}.$$

The operator

$$\begin{aligned} \mathcal{J}_m &= \langle i\nabla\rangle_m \mathcal{U}_m(t) x \mathcal{U}_m(-t) = \langle i\nabla\rangle_m \left(xE + itA\langle i\nabla\rangle_m^{-1}\nabla\right) \\ &= xE\langle i\nabla\rangle_m - E\langle i\nabla\rangle_m^{-1}\nabla + itA\nabla \end{aligned} \tag{3.0.8}$$

is useful for obtaining the time decay estimates of solutions (see Lemma 3.1.1 below), where we have used the commutation relation

$$\left[x, \langle i\nabla\rangle_m^{\lambda}\right] = \mathcal{F}^{-1}\left[i\nabla, \langle\xi\rangle_m^{\lambda}\right]\mathcal{F} = \lambda\langle i\nabla\rangle_m^{\lambda-2}\nabla \tag{3.0.9}$$

and the notation

$$\begin{aligned} \mathcal{J}_m &= \left(\mathcal{J}_m^{(1)}, \ldots, \mathcal{J}_m^{(n)}\right) \\ &= \left(\langle i\nabla\rangle_m \mathcal{U}_m(t) x_1 \mathcal{U}_m(-t), \ldots, \langle i\nabla\rangle_m \mathcal{U}_m(t) x_n \mathcal{U}_m(-t)\right). \end{aligned}$$

By a direct calculation we see that the commutation relation

$$[\mathcal{L}_m, \mathcal{J}_m] = 0$$

is true. However, it is difficult to calculate the action of $\mathcal{J}_m$ on the nonlinearity since $\mathcal{J}_m$ can not be considered as a first order differential operator on the power nonlinearities. Therefore, we use the first order differential operator

$$\mathcal{Z} = E(t\nabla + x\partial_t) = \left(\mathcal{Z}^{(1)}, \ldots, \mathcal{Z}^{(n)}\right)$$

which is closed related to $\mathcal{J}_m$ by

$$\mathcal{Z}^{(j)} = \mathcal{L}_m x_j - iA\mathcal{J}_m^{(j)} \tag{3.0.10}$$

and it almost commutes with $\mathcal{L}_m$ since by a direct calculation,

$$[\mathcal{L}_m, \mathcal{Z}] = E\nabla - iA\left[x, \langle i\nabla\rangle_m\right]\partial_t = -iA\langle i\nabla\rangle_m^{-1}\nabla\mathcal{L}_m. \tag{3.0.11}$$

The operator $\mathcal{Z}$ was introduced by Klainerman [11] to obtain decay estimates of solutions to linear Klein-Gordon equations and the estimates were improved by [2], [5], [6], [1]. We also use the following commutation relations

$$\left[\mathcal{Z}^{(j)}, \langle i\nabla\rangle_m^{-1}\partial_{x_k}\right] = \left(\langle i\nabla\rangle_m^{-3}\partial_{x_k}\partial_{x_j} - \langle i\nabla\rangle_m^{-1}\delta_{jk}\right)\left(\mathcal{L}_m - iA\langle i\nabla\rangle_m\right) \tag{3.0.12}$$

$$[x, \mathcal{L}_m] = iA\left[x, \langle i\nabla\rangle_m\right] = iA\langle i\nabla\rangle_m^{-1}\nabla, \left[x_j, \mathcal{Z}^{(k)}\right] = Et\left[x_j, \partial_{x_k}\right] = -iEt\delta_{jk}, \tag{3.0.13}$$

$$\begin{aligned}\left[\mathcal{Z}^{(j)}, \mathcal{J}_m^{(k)}\right] &= \left[E\left(t\partial_{x_j} + x_j\partial_t\right), \langle i\nabla\rangle_m x_k + iAt\partial_{x_k}\right] \\ &= Et\langle i\nabla\rangle_m\delta_{jk} + E\langle i\nabla\rangle_m^{-1}\partial_{x_j}x_k\partial_t + iAx_j\partial_{x_k} - iA\delta_{jk}t\partial_t \\ &= -i\delta_{jk}t\mathcal{L}_m + \langle i\nabla\rangle_m^{-1}\left(E\partial_{x_j}\mathcal{Z}^{(k)} + iA\mathcal{J}_m^{(j)}\partial_{x_k}\right).\end{aligned}$$

3.1 LEMMAS

We first prove time decay estimates through the operator $\mathcal{J}_m$ for any smooth and decaying functions. We note here that $\mathcal{J}_m$ was used to study nonlinear Klein-Gordon equations with a super critical nonlinearity in two space dimensions (see, [7], [8]).

Lemma 3.1.1. *Let $m > 0$. Then the estimate*

$$\|\phi\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\phi\|_{\mathbf{H}^\nu}^{1-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\nu-1}}^{\frac{n}{2}\left(1-\frac{2}{p}\right)} + C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\phi\|_{\mathbf{H}^\nu}$$

is valid for all $t \geq 0$, *where* $\nu = \left(\frac{n}{2}+1\right)\left(1-\frac{2}{p}\right), 2 \leq p < \frac{2n}{n-2}$ *and* $n \geq 2$, *provided the right-hand side is finite. Furthermore, the estimates are true,*

$$\|\phi\|_{\boldsymbol{L}^p} \leq C\left\langle t\right\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\left\|\mathcal{U}_m\left(-t\right)\mathcal{J}_m\phi\right\|_{\boldsymbol{H}^{\nu-1,\frac{1}{2}}}^{\frac{n}{3}\left(1-\frac{2}{p}\right)}\|\phi\|_{\boldsymbol{H}^{\nu}}^{1-\frac{n}{3}\left(1-\frac{2}{p}\right)} + C\left\langle t\right\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\phi\|_{\boldsymbol{H}^{\nu}}$$

and

$$\|\phi\|_{\boldsymbol{L}^p} \leq C\left\langle t\right\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\sum_{|\alpha|\leq 2}\|\mathcal{J}_m^{\alpha}\phi\|_{\boldsymbol{H}^{\nu-2}}^{\frac{n}{3}\left(1-\frac{2}{p}\right)}\|\phi\|_{\boldsymbol{H}^{\nu}}^{1-\frac{n}{3}\left(1-\frac{2}{p}\right)} + C\left\langle t\right\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)}\|\phi\|_{\boldsymbol{H}^{\nu}}$$

for all $t \geq 0$, *where* $\nu = \left(\frac{n}{2}+1\right)\left(1-\frac{2}{p}\right), 2 \leq p < \frac{2n}{n-3}$ *and* $n \geq 3$, *provided the right-hand sides are finite.*

Proof. By the Sobolev inequality we find

$$\|\phi\|_{\mathbf{L}^p} \leq C\|\phi\|_{\mathbf{H}^{\frac{n}{2}(1-\frac{2}{p})}}$$

for $2 \leq p < \infty, n \geq 2$; hence the estimates of lemma follow for all $|t| \leq 1$. We consider now $|t| \geq 1$. We have the $\mathbf{L}^{\infty} - \mathbf{L}^1$ time decay estimate for the free evolution group $\mathcal{U}_m(t)$ (see Lemma 1 in [12]).

$$\|\phi\|_{\mathbf{L}^{\infty}} = \|\mathcal{U}_m(t)\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^{\infty}} \leq C|t|^{-\frac{n}{2}}\|\left\langle i\nabla\right\rangle^{\frac{n+2}{2}}\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^1}. \tag{3.1.1}$$

Since $\|\phi\|_{\mathbf{L}^2} = \|\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}$, then by interpolation we get

$$\|\phi\|_{\mathbf{L}^p} \leq C|t|^{-\frac{n}{2}(1-\frac{2}{p})}\|\left\langle i\nabla\right\rangle_m^{\nu}\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^{p\prime}}, \tag{3.1.2}$$

where $p\prime = \frac{p}{p-1}, \nu = (\frac{n+2}{2})(1-\frac{2}{p})$. Taking $\rho = \left\||x|^{\sigma}\phi\right\|_{\mathbf{L}^2}^{\frac{1}{\sigma}}\|\phi\|_{\mathbf{L}^2}^{-\frac{1}{\sigma}}$,and applying the Hölder inequality, we obtain

$$\begin{aligned}\|\phi\|_{\mathbf{L}^{p\prime}} &\leq \|(\rho+|x|)^{-\sigma}\|_{\mathbf{L}^{\frac{2p}{p-2}}}\|(\rho+|x|)^{\sigma}\phi\|_{\mathbf{L}^2} \\ &\leq C\rho^{\frac{n}{2}(1-\frac{2}{p})-\sigma}\left\||x|^{\sigma}\phi\right\|_{\mathbf{L}^2} + C\rho^{\frac{n}{2}(1-\frac{2}{p})}\|\phi\|_{\mathbf{L}^2} \\ &\leq C\left\||x|^{\sigma}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\sigma}(1-\frac{2}{p})}\|\phi\|_{\mathbf{L}^2}^{1-\frac{n}{2\sigma}(1-\frac{2}{p})}\end{aligned} \tag{3.1.3}$$

since $2 \le p < \frac{2n}{n-2\sigma}, \sigma > 0$. Substitution of (3.1.3) into (3.1.2) yields

$$\begin{aligned} \|\phi(t)\|_{\mathbf{L}^p} \le & C|t|^{-\frac{n}{2}(1-\frac{2}{p})} \| \langle i\nabla\rangle_m^{\nu} \mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}^{1-\frac{n}{2\sigma}(1-\frac{2}{p})} \\ & \times \left\| |x|^{\sigma} \langle i\nabla\rangle_m^{\nu} \mathcal{U}_m(-t)\phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\sigma}(1-\frac{2}{p})}. \end{aligned} \tag{3.1.4}$$

By (3.0.9) we get identity

$$\mathcal{U}_m(t) x \langle i\nabla\rangle_m^{\lambda} \mathcal{U}_m(-t) = \langle i\nabla\rangle_m^{\lambda-1} \mathcal{J}_m + \lambda \langle i\nabla\rangle_m^{\lambda-2} i\nabla. \tag{3.1.5}$$

Hence,

$$\|x \langle i\nabla\rangle_m^{\lambda} \mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{H}^{\lambda-1}} + C\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\lambda-1}}. \tag{3.1.6}$$

We apply (3.1.6) with $\lambda = \nu$ to (3.1.4) with $\sigma = 1$ to obtain the first estimate of Lemma 3.1.1.

We next consider the second estimate. Taking $\sigma = \frac{3}{2}$ in (3.1.3) we find that

$$\|\phi\|_{\mathbf{L}^{p\prime}} \le C \left\| |x|^{\frac{3}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{3}(1-\frac{2}{p})} \|\phi\|_{\mathbf{L}^2}^{1-\frac{n}{3}(1-\frac{2}{p})}$$

for any $2 \le p < \frac{2n}{n-3}$. Therefore, we get by (3.1.2) and (3.1.5)

$$\begin{aligned} \|\phi\|_{\mathbf{L}^p} \le & C|t|^{-\frac{n}{2}(1-\frac{2}{p})} \sum_{j=1}^{3} \left\| |x|^{\frac{1}{2}} x_j \langle i\nabla\rangle_m^{\nu} \mathcal{U}_m(-t)\phi \right\|_{\mathbf{L}^2}^{\frac{n}{3}(1-\frac{2}{p})} \\ & \times \| \langle i\nabla\rangle_m^{\nu} \mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}^{1-\frac{n}{3}(1-\frac{2}{p})} \\ & \le C|\, t|^{-\frac{n}{2}(1-\frac{2}{p})} \left\| |x|^{\frac{1}{2}} \mathcal{U}_m(-t) \langle i\nabla\rangle_m^{\nu-2} i\nabla\phi \right\|_{\mathbf{L}^2}^{\frac{n}{3}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu}}^{1-\frac{n}{3}(1-\frac{2}{p})} \\ & + C|\, t|^{-\frac{n}{2}(1-\frac{2}{p})} \left\| |x|^{\frac{1}{2}} \langle i\nabla\rangle_m^{\nu-1} \mathcal{U}_m(-t)\mathcal{J}_m\phi \right\|_{\mathbf{L}^2}^{\frac{n}{3}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu}}^{1-\frac{n}{3}(1-\frac{2}{p})} \end{aligned} \tag{3.1.7}$$

which implies the second estimate of lemma if we apply the inequality

$$\begin{aligned} \left\| |x|^{\frac{1}{2}} \mathcal{U}_m(-t)\phi \right\|_{\mathbf{L}^2}^2 &= \left(|x|^{\frac{1}{2}} \mathcal{U}_m(-t)\phi, |x|^{\frac{1}{2}} \mathcal{U}_m(-t)\phi \right) \\ &= \left(\phi, \mathcal{U}_m(t)|x|\mathcal{U}_m(-t)\phi \right) \le \|\phi\|_{\mathbf{L}^2} \|x\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}. \end{aligned} \tag{3.1.8}$$

By (3.1.8), (3.1.7) and (3.1.5) we obtain

$$\|\phi\|_{\mathbf{L}^p} \le C|t|^{-\frac{n}{2}(1-\frac{2}{p})} (\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\nu-2}} + \|\phi\|_{\mathbf{H}^{\nu-2}})^{\frac{n}{6}(1-\frac{2}{p})}$$
$$\times \|\phi\|_{\mathbf{H}^{\nu-1}}^{\frac{n}{6}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu}}^{1-\frac{n}{3}(1-\frac{2}{p})}$$
$$+ C|t|^{-\frac{n}{2}(1-\frac{2}{p})} \left\| |x|^{\frac{1}{2}} \langle i\nabla\rangle_m^{\nu-1} \mathcal{U}_m(-t)\mathcal{J}_m\phi \right\|_{\mathbf{L}^2}^{\frac{n}{3}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu}}^{1-\frac{n}{3}(1-\frac{2}{p})}$$
$$\le C|t|^{-\frac{n}{2}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu}}^{1-\frac{n}{3}(1-\frac{2}{p})} \Big(\sum_{|\beta|\le 1} \left(\|\mathcal{J}_m^{\beta}\phi\|_{\mathbf{H}^{\nu-2}} \|\phi\|_{\mathbf{H}^{\nu}} \right)^{\frac{n}{6}(1-\frac{2}{p})}$$
$$+ \sum_{|\beta|\le 2} \|\mathcal{J}_m^{\beta}\phi\|_{\mathbf{H}^{\nu-2}}^{\frac{n}{6}(1-\frac{2}{p})} \|\phi\|_{\mathbf{H}^{\nu-2}}^{\frac{n}{6}(1-\frac{2}{p})} \Big). \tag{3.1.9}$$

By (3.1.9) in view of the inequality $ab \le \frac{1}{2}(a^2 + b^2)$ we get the last estimate of the lemma. This completes the proof of Lemma 3.1.1. ■

We next consider time decay estimates involving the operators $\mathcal{Z}$ and $\mathcal{L}_m$.

Lemma 3.1.2. *Let $m > 0$. Then the following estimate is valid:*

$$\|\phi\|_{\boldsymbol{L}^p} \le C \langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \left(\|\phi\|_{\boldsymbol{H}^{\nu}} + \|\mathcal{Z}\phi\|_{\boldsymbol{H}^{\nu-1}} \right)$$
$$+ C \langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \|\mathcal{L}_m\phi\|_{\boldsymbol{H}^{\nu-1,1}}$$

for all $t \ge 0$, where $\nu = \left(\frac{n}{2}+1\right)\left(1-\frac{2}{p}\right)$, $2 \le p < \frac{2n}{n-2}$ and $n \ge 2$, provided that the right-hand side is finite. Furthermore, the estimate is true:

$$\|\phi\|_{\boldsymbol{L}^p} \le C \langle t\rangle^{-\frac{n}{2}\left(1-\frac{2}{p}\right)} \left(\sum_{|\beta|\le 2} \|\mathcal{Z}^{\beta}\phi\|_{\boldsymbol{H}^{\nu-|\beta|}} + \sum_{|\beta|\le 1} \|\mathcal{L}_m\mathcal{Z}\phi\|_{\boldsymbol{H}^{\nu-2,1}} \right.$$
$$\left. + (\|\mathcal{L}_m\phi\|_{\boldsymbol{H}^{\nu-1,2}} + \langle t\rangle \|\mathcal{L}_m\phi\|_{\boldsymbol{H}^{\nu-1,1}})^{\frac{1}{2}} \|\mathcal{L}_m\phi\|_{\boldsymbol{H}^{\nu-1,1}}^{\frac{1}{2}} \right),$$

for all $t \ge 0$, where $\nu = \left(\frac{n+2}{2}\right)\left(1-\frac{2}{p}\right)$, $2 \le p < \frac{2n}{n-3}$ and $n \ge 3$, provided that the right-hand side is finite.

Proof. By (3.0.10) and (3.0.13) we have the identity

$$\mathcal{J}_m = iA\mathcal{Z} - E\langle i\nabla\rangle_m^{-1} i\nabla - iAx\mathcal{L}_m. \tag{3.1.10}$$

Therefore, we get

$$\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\nu-1}} \le C\|\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-1}} + \|\phi\|_{\mathbf{H}^{\nu-1}} + C\|x\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1}}. \tag{3.1.11}$$

In view of the first estimate of Lemma 3.1.1 and (3.1.11) the first estimate of the lemma follows.

By the second estimate of Lemma 3.1.1 we have

$$\begin{aligned}\|\phi\|_{\mathbf{L}^p} \leq & C\langle t\rangle^{-\frac{n}{2}(1-\frac{2}{p})}\Big(\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\nu-1}}+\|\phi\|_{\mathbf{H}^\nu}\\ &+\Big\||x|^{\frac{1}{2}}\langle i\nabla\rangle^{\nu-1}\mathcal{U}_m(-t)\mathcal{J}_m\phi\Big\|_{\mathbf{L}^2}\Big).\end{aligned} \tag{3.1.12}$$

We again apply (3.1.10) to get

$$\begin{aligned}&\Big\||x|^{\frac{1}{2}}\langle i\nabla\rangle_m^{\nu-1}\mathcal{U}_m(-t)\mathcal{J}_m\phi\Big\|_{\mathbf{L}^2}\leq C\Big\||x|^{\frac{1}{2}}\mathcal{U}_m(-t)\langle i\nabla\rangle_m^{\nu-1}\mathcal{Z}\phi\Big\|_{\mathbf{L}^2}\\ &+C\Big\||x|^{\frac{1}{2}}\mathcal{U}_m(-t)x\langle i\nabla\rangle_m^{\nu-1}\mathcal{L}_m\phi\Big\|_{\mathbf{L}^2}+C\Big\||x|^{\frac{1}{2}}\mathcal{U}_m(-t)\langle i\nabla\rangle_m^{\nu-2}\nabla\phi\Big\|_{\mathbf{L}^2}.\end{aligned}$$

Then via (3.1.8) we find

$$\begin{aligned}&\Big\||x|^{\frac{1}{2}}\langle i\nabla\rangle_m^{\nu-1}\mathcal{U}_m(-t)\mathcal{J}_m\phi\Big\|_{\mathbf{L}^2}\leq C\|\phi\|_{\mathbf{H}^{\nu-1}}+C\|\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-1}}\\ &+C\|x\langle i\nabla\rangle_m^{\nu-1}\mathcal{U}_m(-t)\langle i\nabla\rangle_m^{-2}\nabla\phi\|_{\mathbf{L}^2}+C\|x\langle i\nabla\rangle_m^{\nu-1}\mathcal{Z}\phi\|_{\mathbf{L}^2}\\ &+C\|x\langle i\nabla\rangle_m^{\nu-1}\mathcal{U}_m(-t)x\mathcal{L}_m\phi\|_{\mathbf{L}^2}^{\frac{1}{2}}\|x\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1}}^{\frac{1}{2}}\end{aligned}$$

and by (3.1.6) with $\lambda=\nu-1$ we obtain

$$\begin{aligned}&\Big\||x|^{\frac{1}{2}}\langle i\nabla\rangle_m^{\nu-1}\mathcal{U}_m(-t)\mathcal{J}_m\phi\Big\|_{\mathbf{L}^2}\leq C\|\phi\|_{\mathbf{H}^{\nu-1}}+C\|\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-1}}\\ &+C\|\mathcal{J}_m\langle i\nabla\rangle_m^{-2}\nabla\phi\|_{\mathbf{H}^{\nu-2}}+C\|\mathcal{J}_m\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-2}}\\ &+C\left(\|x\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-2}}^{\frac{1}{2}}+\|\mathcal{J}_m x\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-2}}^{\frac{1}{2}}\right)\|x\mathcal{L}_m\phi\|_{\mathbf{L}^2}^{\frac{1}{2}}.\end{aligned}$$

By (3.0.8),(3.0.9) and (3.1.10) we find

$$\begin{aligned}&\|\mathcal{J}_m\langle i\nabla\rangle_m^{-2}\nabla\phi\|_{\mathbf{H}^{\nu-2}}\leq C\|\mathcal{J}_m\phi\|_{\mathbf{H}^{\nu-2}}+C\|\phi\|_{\mathbf{H}^{\nu-2}}\\ &\leq C\sum_{|\beta|\leq 1}\|\mathcal{Z}^\beta\phi\|_{\mathbf{H}^{\nu-2}}+C\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-2,1}}.\end{aligned}$$

We also have by (3.1.10)

$$\|\mathcal{J}_m\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-2}}\leq C\sum_{|\beta|\leq 2}\|\mathcal{Z}^\beta\phi\|_{\mathbf{H}^{\nu-2}}+C\|\mathcal{L}_m\mathcal{Z}\phi\|_{\mathbf{H}^{\nu-2,1}}$$

and by a direct calculation we get

$$\|\mathcal{J}_m x\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-2}} \le C\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1,2}} + Ct\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1,1}}.$$

Therefore, we have

$$\begin{aligned}
&\left\| |x|^{\frac{1}{2}} \langle i\nabla\rangle_m^{\nu-1} \mathcal{U}_m(-t)\mathcal{J}_m\phi \right\|_{\mathbf{L}^2} \\
&\quad \le C\sum_{|\beta|\le 2} \|\mathcal{Z}^\beta\phi\|_{\mathbf{H}^{\nu-|\beta|}} + C\sum_{|\beta|\le 1} \|\mathcal{L}_m\mathcal{Z}^\beta\phi\|_{\mathbf{H}^{\nu-2,1}} \\
&\quad + C\left(\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1,2}} + t\|\mathcal{L}_m\phi\|_{\mathbf{H}^{\nu-1,1}}\right)^{\frac{1}{2}} \|\mathcal{L}_m\phi\|^{\frac{1}{2}}_{\mathbf{H}^{\nu-1,1}}.
\end{aligned} \tag{3.1.13}$$

By virtue of (3.1.12) and (3.1.13) the last estimate of the lemma follows. This completes the proof of Lemma 3.1.2. ■

In the case $n = 2$ we have the following estimate.

Lemma 3.1.3. *Let $m > 0$. Then the estimate*

$$\|\phi\|_{\boldsymbol{H}^1_\infty} \le C\langle t\rangle^{-1}\|\phi\|_{\boldsymbol{H}^3} + C\langle t\rangle^{-1}\|\mathcal{U}_m(-t)\mathcal{J}_m\phi\|_{\boldsymbol{H}^{2,\delta}}$$

is valid, for all $t \ge 0$, where $\delta > 0$.

Proof. We have by (3.1.1) and (3.0.9)

$$\begin{aligned}
\|\phi\|_{\mathbf{H}^1_\infty} &\le C|t|^{-1}\|\langle i\nabla\rangle^3\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^1} \\
&\le C|t|^{-1}\left(\|\phi\|_{\mathbf{H}^3} + \|\langle x\rangle^\delta\langle i\nabla\rangle_m^3\mathcal{U}_m(-t)\phi\|_{\mathbf{L}^2}\right).
\end{aligned} \tag{3.1.14}$$

Then by identity (3.0.8) we have $\langle i\nabla\rangle_m^3 x\mathcal{U}_m(-t) = \langle i\nabla\rangle_m^2\mathcal{U}_m(-t)\mathcal{J}_m$ which implies the desired result. Lemma 3.1.3 is proved. ■

In the next lemma we state the time decay estimates through the operators $\mathcal{Z}$ and $\mathcal{L}_m$ in the two dimensional case.

Lemma 3.1.4. *Let $m > 0$. Then the estimate*

$$\begin{aligned}
\|\phi\|_{\boldsymbol{H}^1_\infty} \le C\langle t\rangle^{-1}\Bigg(&\|\phi\|_{\boldsymbol{H}^3} \\
&+ \left(\sum_{|\alpha|\le 2}\|\mathcal{Z}^\alpha\phi\|^\delta_{\boldsymbol{H}^1} + \sum_{|\alpha|\le 1}\|x\mathcal{L}_m\mathcal{Z}^\alpha\phi\|^\delta_{\boldsymbol{H}^1}\right)\sum_{|\alpha|\le 1}\|\mathcal{Z}^\alpha\phi\|^{1-\delta}_{\boldsymbol{H}^2} \\
&+ \|\mathcal{L}_m\phi\|^\delta_{\boldsymbol{H}^{2,2}}\|\mathcal{L}_m\phi\|^{1-\delta}_{\boldsymbol{H}^{2,1}} + \langle t\rangle^\delta\|\mathcal{L}_m\phi\|_{\boldsymbol{H}^{2,1}}\Bigg)
\end{aligned}$$

is valid, for all $t \ge 0$, where $\delta \in (0, 1)$.

Proof. We have by Lemma 3.1.3 and identity (3.1.10)

$$\|\phi\|_{\mathbf{H}^1_\infty} \le C\langle t\rangle^{-1}\Big(\|\phi\|_{\mathbf{H}^3} + \|\mathcal{U}_m(-t)\phi\|_{\mathbf{H}^{2,\delta}} + \|\mathcal{U}_m(-t)\mathcal{Z}\phi\|_{\mathbf{H}^{2,\delta}} + \|\mathcal{U}_m(-t)x\mathcal{L}_m\phi\|_{\mathbf{H}^{2,\delta}}\Big).$$

Applying the Hölder inequality and the definition of $\mathcal{J}_m$ by (3.0.8), we obtain

$$\begin{aligned}\|\mathcal{U}_m(-t)\phi\|_{\mathbf{H}^{2,\delta}} &\le C\|\phi\|_{\mathbf{H}^2} + C\Big\||x|^\delta\langle i\nabla\rangle_m\mathcal{U}_m(-t)\phi\Big\|_{\mathbf{H}^1}\\ &\le C\|\phi\|_{\mathbf{H}^2} + C\|x\langle i\nabla\rangle_m\mathcal{U}_m(-t)\phi\|^\delta_{\mathbf{H}^1}\|\phi\|^{1-\delta}_{\mathbf{H}^2}\\ &= C\|\phi\|_{\mathbf{H}^2} + C\|\mathcal{J}_m\phi\|^\delta_{\mathbf{H}^1}\|\phi\|^{1-\delta}_{\mathbf{H}^2}.\end{aligned}$$

Then by (3.1.10) we find the estimates

$$\|\mathcal{U}_m(-t)\phi\|_{\mathbf{H}^{2,\delta}} \le C\|\phi\|_{\mathbf{H}^2} + C\left(\|\mathcal{Z}\phi\|^\delta_{\mathbf{H}^1} + \|x\mathcal{L}_m\phi\|^\delta_{\mathbf{H}^1}\right)\|\phi\|^{1-\delta}_{\mathbf{H}^2}$$

and

$$\|\mathcal{U}_m(-t)\mathcal{Z}\phi\|_{\mathbf{H}^{2,\delta}} \le C\|\mathcal{Z}\phi\|_{\mathbf{H}^2} + C\left(\|\mathcal{Z}^2\phi\|^\delta_{\mathbf{H}^1} + \|x\mathcal{L}_m\mathcal{Z}\phi\|^\delta_{\mathbf{H}^1}\right)\|\mathcal{Z}\phi\|^{1-\delta}_{\mathbf{H}^2}.$$

In the same manner via (3.0.8) we obtain

$$\begin{aligned}&\|\mathcal{U}_m(-t)x\mathcal{L}_m\phi\|_{\mathbf{H}^{2,\delta}}\\ &\le C\|x\mathcal{L}_m\phi\|_{\mathbf{H}^2} + C\|\mathcal{J}_m x\mathcal{L}_m\phi\|^\delta_{\mathbf{H}^1}\|x\mathcal{L}_m\phi\|^{1-\delta}_{\mathbf{H}^2}\\ &\le C\|\mathcal{L}_m\phi\|^\delta_{\mathbf{H}^{2,2}}\|\mathcal{L}_m\phi\|^{1-\delta}_{\mathbf{H}^{2,1}} + C\langle t\rangle^\delta\|\mathcal{L}_m\phi\|_{\mathbf{H}^{2,1}},\end{aligned}$$

from which the estimate of the lemma follows. Lemma 3.1.4 is proved. ■

We next consider the time decay estimates for the case $m = 0$, which is needed to prove Teorem 3.0.3.

Lemma 3.1.5. *The estimates are valid:*

$$\|\phi\|_{\boldsymbol{L}^p} \le C|t|^{-\frac{n-1}{2}\left(1-\frac{2}{p}\right)}\sum_{|\alpha|\le 1}\Big\||i\nabla|^{\nu-1}\mathcal{J}_0^\alpha\phi\Big\|^{\frac{n}{2}\left(1-\frac{2}{p}\right)}_{\boldsymbol{L}^2}\Big\||i\nabla|^\nu\phi\Big\|^{1-\frac{n}{2}\left(1-\frac{2}{p}\right)}_{\boldsymbol{L}^2}$$

and

$$\|\phi\|_{L^q} \leq Ct^{-1}\Big(\|\nabla\mathcal{J}_0\phi\|_{\boldsymbol{L}^{\tilde{q}}} + \|\nabla\phi\|_{\boldsymbol{L}^{\tilde{q}}}\Big) + Ct^{-\frac{n-1}{2}(1-\frac{2}{q})}\|\nabla\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\boldsymbol{H}^{1+[\frac{n-1}{2}]}}\|\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\boldsymbol{H}^{1+[\frac{n-1}{2}]}}\|\phi\|^{\frac{2}{q}}_{\boldsymbol{L}^2}$$

where $\nu = \frac{n+1}{2}\left(1-\frac{2}{p}\right)$ *and* $2 \leq p < \frac{2n}{n-2}, n \geq 2, \frac{1}{q} = \frac{1}{\tilde{q}} - \frac{1}{n}, \tilde{q} \geq 2$, *provided that the right-hand sides are finite.*

Proof. By $\mathbf{L}^p - \mathbf{L}^q$ time decay estimate of solutions to the linear wave equations obtained by Brenner [3] and (3.1.3) we have

$$\|\phi\|_{\mathbf{L}^p} = \|\mathcal{U}_0(t)\mathcal{U}_0(-t)\phi\|_{\mathbf{L}^p} \leq C|t|^{-\frac{n-1}{2}(1-\frac{2}{p})}\Big\||i\nabla|^{\nu}\mathcal{U}_0(-t)\phi\Big\|_{\mathbf{L}^{p\prime}}$$
$$\leq C|t|^{-\frac{n-1}{2}(1-\frac{2}{p})}\Big\||x||i\nabla|^{\nu}\mathcal{U}_0(-t)\phi\Big\|^{\frac{n}{2}(1-\frac{2}{p})}_{\mathbf{L}^2}\Big\||i\nabla|^{\nu}\mathcal{U}_0(-t)\phi\Big\|^{1-\frac{n}{2}(1-\frac{2}{p})}_{\mathbf{L}^2},$$

where $p\prime = \frac{p}{p-1}, \nu = \frac{n+1}{2}(1-\frac{2}{p})$ and $2 \leq p < \frac{2n}{2-n}$. In the same way as in the proofs of (3.1.5) and (3.1.6) we have

$$\Big\|x|i\nabla|^{\nu}\mathcal{U}_0(-t)\phi\Big\|_{\mathbf{L}^2} \leq C\Big\||i\nabla|^{\nu-1}\phi\Big\|_{\mathbf{L}^2} + C\Big\||i\nabla|^{\nu-1}\mathcal{J}_0\phi\Big\|_{\mathbf{L}^2}.$$

Therefore we have the first estimate of lemma.

The second estimate for all $|t| < 1$ is a consequence of the Sobolev inequality. Consider now $|t| \geq 1$. We have the identities with $\mathcal{J}_0 = xE|i\nabla| + itA\nabla - E|i\nabla|^{-1}\nabla$

$$\Big(|x|^{-1}x\cdot\mathcal{J}_0\Big)i\nabla|i\nabla|^{-1} + E\Big(|x|^{-1}x\cdot\nabla|i\nabla|^{-2}\Big)i\nabla = At\Big(|x|^{-1}x\cdot i\nabla\Big)|i\nabla|^{-1}i\nabla + |x|Ei\nabla$$

and

$$A\Big(\mathcal{J}_0\cdot i\nabla|i\nabla|^{-1}\Big)i\nabla|i\nabla|^{-1} - iAi\nabla|i\nabla|^{-1} = A|x|\Big(|x|^{-1}x\cdot i\nabla\Big)|i\nabla|^{-1}i\nabla + tEi\nabla.$$

Therefore,

$$\begin{aligned}&(t\pm|x|)\left(E\pm A\left(|x|^{-1}x\cdot i\nabla|i\nabla|^{-1}\right)\right)i\nabla\\&= -iAi\nabla|i\nabla|^{-1} + A\left(\mathcal{J}_0\cdot i\nabla|i\nabla|^{-1}\right)i\nabla|i\nabla|^{-1}\\&\pm\left(|x|^{-1}x\cdot\mathcal{J}_0\right)i\nabla|i\nabla|^{-1} \pm E\left(|x|^{-1}x\cdot\nabla|i\nabla|^{-1}\right)i\nabla|i\nabla|^{-1}.\end{aligned} \tag{3.1.15}$$

From (3.1.15) we have

$$\begin{aligned}
&\left(E + A\left(|x|^{-1}x \cdot i\nabla|i\nabla|^{-1}\right)\right) i\nabla \\
&= \frac{1}{t+|x|}\Big(-iAi\nabla|i\nabla|^{-1} + A\left(\mathcal{J}_0 \cdot i\nabla|i\nabla|^{-1}\right) i\nabla|i\nabla|^{-1} \\
&+ \left(|x|^{-1}x \cdot \mathcal{J}_0\right) i\nabla|i\nabla|^{-1} + E\left(|x|^{-1}x \cdot \nabla|i\nabla|^{-1}\right) i\nabla|i\nabla|^{-1}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\left(E - A\left(|x|^{-1}x \cdot i\nabla|i\nabla|^{-1}\right)\right) i\nabla \\
&= \frac{1}{t-|x|}\Big(-iAi\nabla|i\nabla|^{-1} + A\left(\mathcal{J}_0 \cdot i\nabla|i\nabla|^{-1}\right) i\nabla|i\nabla|^{-1} \\
&- \left(|x|^{-1}x \cdot \mathcal{J}_0\right) i\nabla|i\nabla|^{-1} - E\left(|x|^{-1}x \cdot \nabla|i\nabla|^{-1}\right) i\nabla|i\nabla|^{-1}\Big).
\end{aligned}$$

Summing up the both identities, and taking the absolut value of the resulting identity, we obtain

$$\begin{aligned}
&|\nabla\phi(x)| \le C\left(\frac{1}{(t+|x|)} + \frac{1}{(t-|x|)}\right) \\
&\times\left(\left|i\nabla|i\nabla|^{-1}\phi(x)\right| + \left|(\mathcal{J}_0 \cdot i\nabla) i\nabla|i\nabla|^{-2}\phi(x)\right|\right. \\
&\left.+ \left|\mathcal{J}_0 i\nabla|i\nabla|^{-1}\phi(x)\right| + \left|\nabla^2|i\nabla|^{-2}\phi(x)\right|\right).
\end{aligned}$$

Taking $\mathbf{L}^q\left(|x| < \frac{t}{2}\right)$, we find that

$$\begin{aligned}
&\|\nabla\phi\|_{\mathbf{L}^q\left(|x|<\frac{t}{2}\right)} \le Ct^{-1} \\
&\times\left(\left\|i\nabla|i\nabla|^{-1}\phi\right\|_{\mathbf{L}^q} + \left\|(\mathcal{J}_0 \cdot i\nabla) i\nabla|i\nabla|^{-2}\phi\right\|_{\mathbf{L}^q}\right. \\
&\left.+ \left\|\mathcal{J}_0 i\nabla|i\nabla|^{-1}\phi\right\|_{\mathbf{L}^q} + \left\|\nabla^2|i\nabla|^{-2}\phi\right\|_{\mathbf{L}^q}\right).
\end{aligned}$$

By the commutation relation $\left[\mathcal{J}_0^{(j)}, |i\nabla|^{-1}\partial_k\right] = -E\delta_{jk} - iE|i\nabla|^{-2}\partial_j\partial_k$ we find

$$\begin{aligned}
\|\nabla\phi\|_{\mathbf{L}^q\left(|x|<\frac{t}{2}\right)} &\le Ct^{-1}\Big(\|\mathcal{J}_0\phi\|_{\mathbf{L}^q} + \|\phi\|_{\mathbf{L}^q}\Big) \\
&\le Ct^{-1}\Big(\|\nabla\mathcal{J}_0\phi\|_{\mathbf{L}^{\tilde{q}}} + \|\nabla\phi\|_{\mathbf{L}^{\tilde{q}}}\Big) \qquad (3.1.16)
\end{aligned}$$

for any smooth vector-function ϕ and $\frac{1}{q} = \frac{1}{\tilde{q}} - \frac{1}{n} \geq 0$. Using the Sobolev inequality on a sphere we also have for all $|x| > \frac{t}{2}$

$$|\phi(x)|^2 |x|^{n-1} = -\int_{|x|}^{\infty} \partial_r |\phi(r,\theta)|^2 r^{n-1} dr \leq C \|\nabla\phi\|_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}}$$

for any $\phi \in \mathbf{C}_0^{\infty}(\mathbf{R}^n)$, which implies

$$\sup_{|x|>\frac{t}{2}} |\phi(x)| \leq C t^{-\frac{n-1}{2}} \|\nabla\phi\|^{\frac{1}{2}}_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|^{\frac{1}{2}}_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}}. \tag{3.1.17}$$

Therefore by (3.1.17)

$$\begin{aligned} \|\phi\|_{\mathbf{L}^q(|x|>\frac{t}{2})} &\leq C \|\phi\|^{1-\frac{2}{q}}_{\mathbf{L}^{\infty}(|x|>\frac{t}{2})} \|\phi\|^{\frac{2}{q}}_{\mathbf{L}^2(|x|>\frac{t}{2})} \\ &\leq C t^{-\frac{n-1}{2}(1-\frac{2}{q})} \|\nabla\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\mathbf{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|^{\frac{2}{q}}_{\mathbf{L}^2}. \end{aligned} \tag{3.1.18}$$

From (3.1.16) and (3.1.18) the second estimate of the lemma follows. Lemma 3.1.5 is proved.

■

Lemma 3.1.6. *The following estimates are true:*

$$\begin{aligned} \|\phi\|_{\boldsymbol{L}^p} \leq C|t|^{-\frac{n-1}{2}\left(1-\frac{2}{p}\right)} \Bigg(\left\| |i\nabla|^{\nu-1} \mathcal{Z}\phi \right\|_{\boldsymbol{L}^2} \\ + \left\| |i\nabla|^{\nu-1}\phi \right\|_{\boldsymbol{L}^2} + \left\| |i\nabla|^{\nu-1} x\mathcal{L}_0\phi \right\|_{\boldsymbol{L}^2} \Bigg)^{\frac{n}{2}\left(1-\frac{2}{p}\right)} \left\| |i\nabla|^{\nu}\phi \right\|^{1-\frac{n}{2}\left(1-\frac{2}{p}\right)}_{\boldsymbol{L}^2} \end{aligned}$$

and

$$\begin{aligned} \|\phi\|_{\boldsymbol{L}^q} \leq C t^{-1} \Big(\left\|\nabla\mathcal{Z}\phi\right\|_{\boldsymbol{L}^{\tilde{q}}} + \left\|\nabla x\mathcal{L}_0\phi\right\|_{\boldsymbol{L}^{\tilde{q}}} + \left\|\nabla\phi\right\|_{\boldsymbol{L}^{\tilde{q}}} \Big) \\ + C t^{-\frac{n-1}{2}(1-\frac{2}{q})} \|\nabla\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\boldsymbol{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|^{\frac{1}{2}(1-\frac{2}{q})}_{\boldsymbol{H}^{1+\left[\frac{n-1}{2}\right]}} \|\phi\|^{\frac{2}{q}}_{\boldsymbol{L}^2} \end{aligned}$$

where $\nu = \frac{n+1}{2}\left(1 - \frac{2}{p}\right)$ *and* $2 \leq p < \frac{2n}{n-2}$, $\quad n \geq 2, \frac{1}{q} = \frac{1}{\tilde{q}} - \frac{1}{n} \geq 0, \tilde{q} \geq 2$, *provided that the right-hand sides are finite.*

Proof. By the identity (3.1.10) we obtain

$$\begin{aligned} \left\| |i\nabla|^{\nu-1} \mathcal{J}_0\phi \right\|_{\mathbf{L}^2} \leq & C \left\| |i\nabla|^{\nu-1} \mathcal{Z}\phi \right\|_{\mathbf{L}^2} + C \left\| |i\nabla|^{\nu-1} x\mathcal{L}_0\phi \right\|_{\mathbf{L}^2} \\ & + C \left\| |i\nabla|^{\nu-1}\phi \right\|_{\mathbf{L}^2} \end{aligned}$$

and

$$\|\mathcal{J}_0\phi\|_{\mathbf{L}^{\tilde{q}}} \leq C \|\nabla\mathcal{Z}\phi\|_{\mathbf{L}^{\tilde{q}}} + C \|\nabla x\mathcal{L}_0\phi\|_{\mathbf{L}^{\tilde{q}}} + C \|\nabla\phi\|_{\mathbf{L}^{\tilde{q}}}.$$

Therefore, by Lemma 3.1.5 we have the desired result. ■

3.2 PROOF OF THEOREM 3.0.1

We put

$$w_j(t,x) = \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u_j + i\langle i\nabla\rangle_{m_j}^{-1}\partial_t u_j \\ u_j - i\langle i\nabla\rangle_{m_j}^{-1}\partial_t u_j \end{pmatrix},$$

$$w_j(0,x) = \mathring{w}_j(x) = \begin{pmatrix} \mathring{w}_j^{(1)} \\ \mathring{w}_j^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathring{u}_j^{(1)} + i\langle i\nabla\rangle_{m_j}^{-1}\mathring{u}_j^{(2)} \\ \mathring{u}_j^{(1)} - i\langle i\nabla\rangle_{m_j}^{-1}\mathring{u}_j^{(2)} \end{pmatrix}.$$

Then the nonlinear Klein-Gordon equation (3.0.1) can be rewritten as a system of equations,

$$\begin{cases} \mathcal{L}_{m_j} w_j = \langle i\nabla\rangle_{m_j}^{-1} F_j(w), (t,x) \in \mathbf{R}\times\mathbf{R}^3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1,\ldots,l, \\ w_j(0,x) = \mathring{w}_j(x), x \in \mathbf{R}^3, \end{cases} \tag{3.2.1}$$

where $\mathcal{L}_{m_j} = E\partial_t + iA\langle i\nabla\rangle_{m_j}, \langle i\nabla\rangle_{m_j} = \sqrt{m_j^2 - \Delta}, \quad m_j > 0$, and

$$F_j(w) = \begin{pmatrix} F_j^{(1)}(w) \\ F_j^{(2)}(w) \end{pmatrix}$$
$$= i\mathbf{b}\mathcal{N}_j\left(w^{(1)} + w^{(2)}, \langle i\nabla\rangle_{m_j}\left(w^{(1)} - w^{(2)}\right), \nabla\left(w^{(1)} + w^{(2)}\right)\right)$$

with $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We introduce function space

$$\mathbf{X}_T = \left\{\phi \in C\Big([0,T]; (\mathbf{L}^2)^l \times (\mathbf{L}^2)^l\Big); \|\phi\|_{\mathbf{X}_T} < \infty\right\},$$

with the norm

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]} \sum_{j=1}^{l} \sum_{|\beta|\le 3} \left(\|\mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{4-|\beta|}} + \|\partial_t \mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{3-|\beta|}} \right)$$
$$= \sup_{t\in[0,T]} \sum_{j=1}^{l} \Big(\langle t\rangle^{-1} \|\phi_j\|_{\mathbf{H}^{3,1}} + \langle t\rangle^{-2} \|\phi_j\|_{\mathbf{H}^{2,2}} \Big).$$

Denote by $\mathbf{X}_{T,\varepsilon}$ a closed ball of a radius $\varepsilon > 0$ with a center at the origin in the space $\mathbf{X}_T$. By the contraction mapping principle we can easily obtain the local existence of solutions to (3.2.1).

Proposition 3.2.1. *Let the initial functions* $\mathring{w}_j \in \left(\boldsymbol{H}^{4,3}\right)^2, 1 \le j \le l,$ *and the norm* $\|\mathring{w}_j\|_{\boldsymbol{H}^{4,3}} = \varepsilon_0$. *Then there exists a time* $T \ge O(\varepsilon_0^{-1})$ *and a unique solution* $w \in \boldsymbol{X}_{T,2\varepsilon}$ *of* (3.2.1).

It is sufficient to prove that there exists $\varepsilon > 0$ such that *a priori* estimates $\|w\|_{\mathbf{X}_T} < \varepsilon(\ge \varepsilon_0^{\frac{3}{4}})$ hold for any $T > 0$ to get global solutions to (3.2.1). In order to prove *a priori* estimates we devide the proof into several lemmas.

We note here that in what follows constants C appearing in the proof below do not depend on T. The integral equation associated with (3.2.1) is written as

$$w_j(t) = \mathcal{U}_{m_j}(t)\mathring{w}_j + \int_0^t \mathcal{U}_{m_j}(t-\tau) \langle i\nabla\rangle_{m_j}^{-1} F_j(w)(\tau)d\tau. \tag{3.2.2}$$

Taking the $\mathbf{H}^4$-norm of (3.2.2) we obtain

$$\begin{aligned} \|w_j(t)\|_{\mathbf{H}^4} &\le \|\mathring{w}_j\|_{\mathbf{H}^4} + C\int_0^t \|F_j(w)(\tau)\|_{\mathbf{H}^3} d\tau \\ &\le \|\mathring{w}_j\|_{\mathbf{H}^4} + C\int_0^t \Big(\varepsilon\|w(\tau)\|_{\mathbf{H}^1_\infty} \\ &+\|w(\tau)\|_{\mathbf{H}^2_6}\|w(\tau)\|_{\mathbf{H}^3_3} + \|w(\tau)\|^3_{\mathbf{H}^2_6}\Big) d\tau. \end{aligned} \tag{3.2.3}$$

Since $w \in \mathbf{X}_{T,\varepsilon}$ we also have for $(k = 1, 2 \le p \le \infty)$, and $(k = 3, p = 2)$

$$\begin{aligned} \|\partial_t w_j(t)\|_{\mathbf{H}^k_p} &\le C\|w_j(t)\|_{\mathbf{H}^{k+1}_p} + C\|F_j(w)(t)\|_{\mathbf{H}^{k-1}_p} \le C\|w_j(t)\|_{\mathbf{H}^{k+1}_p} \\ &\le 2\|w(t)\|_{\mathbf{H}^{k+1}_p}. \end{aligned}$$

Therefore, *a priori* estimates of time derevative of solutions are obtained through the estimates of space derevative solutions. We consider second term of (3.2.3). By the Sobolev inequality and the first estimate of Lemma 3.1.2 we get

$$\begin{aligned} &\langle t\rangle^{\frac{1}{2}} \|w\|_{\mathbf{H}^3_3} \le C\|w\|_{\mathbf{H}^4} + C\|\mathcal{Z}w\|_{\mathbf{H}^3} \\ &+C\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}} \le C\varepsilon + C\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}} \end{aligned} \tag{3.2.4}$$

and for $\frac{6}{1+2\delta} < q < 6, \delta \in (0, \frac{1}{2})$

$$\begin{aligned}\langle t \rangle^{\frac{3}{2}(1-\frac{2}{q})} \|w\|_{\mathbf{H}_6^2} &\leq C \langle t \rangle^{\frac{3}{2}(1-\frac{2}{q})} \| \langle i\nabla \rangle^{2+\delta} w\|_{\mathbf{L}^q} \\ &\leq C\|w\|_{\mathbf{H}^4} + C\|\mathcal{Z}w\|_{\mathbf{H}^3} + C\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}} \\ &\leq C\varepsilon + C\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}}. \end{aligned} \tag{3.2.5}$$

Also by the Sobolev inequality, identity (3.0.11), and the second estimate of Lemma 3.1.2 we obtain for $\frac{3}{\delta} < p < \infty, \delta \in (0, \frac{1}{2})$

$$\begin{aligned} &\|w\|_{\mathbf{H}_\infty^1} \leq C\| \langle i\nabla \rangle^{1+\delta} w\|_{\mathbf{L}^p} \\ &\leq C \langle t \rangle^{-\frac{3}{2}(1-\frac{2}{p})} \sum_{j=1}^{l} \Bigg(\sum_{|\beta|\leq 2} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}} + \sum_{|\beta|\leq 1} \|\mathcal{Z}^\beta \langle i\nabla \rangle_{m_j}^{-1} F_j(w)\|_{\mathbf{H}^{2,1}} \\ &+ (\|F_j(w)\|_{\mathbf{H}^{2,2}} + \langle t \rangle \|F_j(w)\|_{\mathbf{H}^{2,1}})^{\frac{1}{2}} \|F_j(w)\|_{\mathbf{H}^{2,1}}^{\frac{1}{2}} \Bigg). \end{aligned} \tag{3.2.6}$$

In the next lemma we estimate the nonlinearity $F_j(w)$ in the norms $\mathbf{H}^{2,1}$ and $\mathbf{H}^{2,2}$.

Lemma 3.2.2. *Let* $w \in \boldsymbol{X}_{T,\varepsilon}$ *be a local solution of* (3.2.1) *such that the inequality*

$$\langle t \rangle^{\frac{1}{2}} \|w\|_{\boldsymbol{H}_3^3} + \langle t \rangle^{\frac{3}{2}(1-\frac{2}{q})} \|w\|_{\boldsymbol{H}_6^2} \leq C\varepsilon + C\sum_{j=1}^{l} \|F_j(w)\|_{\boldsymbol{H}^{2,1}} \tag{3.2.7}$$

is true for $3 \leq q < 6$*. Then the following estimate are valid:*

$$\langle t \rangle^{\frac{1}{2}} \|w\|_{\boldsymbol{H}_3^3} + \langle t \rangle^{1-\frac{3}{2q}} \|w\|_{\boldsymbol{H}_4^2} + \langle t \rangle^{\frac{3}{2}(1-\frac{2}{q})} \|w\|_{\boldsymbol{H}_6^2} \leq C\varepsilon$$

and

$$\sum_{j=1}^{l} \Bigg(\langle t \rangle \|F_j(w)\|_{\boldsymbol{H}^{2,1}} + \|F_j(w)\|_{\boldsymbol{H}^{2,2}} \Bigg) \leq C\varepsilon^2 \langle t \rangle^{\frac{3}{q}}$$

for all $t \in [0, T]$*, where* $3 \leq q < 6$*.*

Proof. By applying the identities(3.0.8) and (3.1.10) we find

$$xE = iA \langle i\nabla \rangle_{m_j}^{-1} \mathcal{Z} - E\langle i\nabla \rangle_{m_j}^{-2} i\nabla - iAt \langle i\nabla \rangle_{m_j}^{-1} i\nabla - iA \langle i\nabla \rangle_{m_j}^{-1} x\mathcal{L}_{m_j}.$$

Therefore, by the Sobolev inequality

$$\|\phi\|_{\mathbf{L}^3} \leq C\|\phi\|_{\mathbf{H}^1}$$

and estimate (3.2.7) we get

$$\begin{aligned}
\|w\|_{\mathbf{H}^{3,1}_3} &\leq \|w\|_{\mathbf{H}^3_3} + \|xw\|_{\mathbf{H}^3_3} \\
&\leq C \sum_{j=1}^{l} \Big(\|\mathcal{Z} w_j\|_{\mathbf{H}^3} + \|\mathcal{L}_{m_j} w_j\|_{\mathbf{H}^{3,1}} + \langle t \rangle \|w_j\|_{\mathbf{H}^3_3} \Big) \\
&\leq C \langle t \rangle^{\frac{1}{2}} \sum_{j=1}^{l} (\varepsilon + \|F_j(w)\|_{\mathbf{H}^{2,1}}) \,. \qquad (3.2.8)
\end{aligned}$$

We use the Hölder inequality and estimate (3.2.7) to obtain

$$\begin{aligned}
\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}} &\leq C \|w\|_{\mathbf{H}^2_6} \|w\|_{\mathbf{H}^{3,1}_3} \\
&\leq C \langle t \rangle^{\frac{3}{q}-1} \sum_{j=1}^{l} (\varepsilon + \|F_j(w)\|_{\mathbf{H}^{2,1}})^2 \,. \qquad (3.2.9)
\end{aligned}$$

Since $w \in \mathbf{X}_{T,\varepsilon}$ we see that $\|F_j(w)\|_{\mathbf{H}^{2,1}} \leq C\|w\|^2_{\mathbf{X}_T} \leq CT\varepsilon^2$. Thus by (3.2.9) we find

$$\sum_{j=1}^{l} \|F_j(w)\|_{\mathbf{H}^{2,1}} \leq C\varepsilon^2 \langle t \rangle^{\frac{3}{q}-1} \,. \qquad (3.2.10)$$

for all $t \in [0,1]$. Then by a standard continuation argument we obtain (3.2.10) for all $t \in [0,T]$. By (3.2.7) and (3.2.10) we have for $3 \leq q < 6$

$$\langle t \rangle^{\frac{1}{2}} \|w\|_{\mathbf{H}^3_3} + \langle t \rangle^{\frac{3}{2}(1-\frac{2}{q})} \|w\|_{\mathbf{H}^2_6} \leq C\varepsilon + C\varepsilon^2. \qquad (3.2.11)$$

Applying the Hölder inequality and estimates (3.2.11) we can write

$$\|w\|_{\mathbf{H}^2_4} \leq \|w\|^{\frac{1}{2}}_{\mathbf{H}^2_3} \|w\|^{\frac{1}{2}}_{\mathbf{H}^2_6} \leq C \langle t \rangle^{\frac{3}{2q}-1} (\varepsilon + C\varepsilon^2).$$

Thus, the first estimates of the lemma is true. Substituting (3.2.10) in to (3.2.8) yields

$$\|w\|_{\mathbf{H}^{3,1}_3} \leq C\varepsilon \langle t \rangle^{\frac{1}{2}} \,. \qquad (3.2.12)$$

As in the proof of (3.2.8) we estimate via (3.2.10) and (3.2.11)

$$\begin{aligned}\|w\|_{\mathbf{H}_6^{1,1}} &\le \|w\|_{\mathbf{H}_6^1} + \|xw\|_{\mathbf{H}_6^1} \\ &\le C\sum_{j=1}^{l}\Big(\|\mathcal{Z}w_j\|_{\mathbf{H}^1} + \|\mathcal{L}_{m_j}w_j\|_{\mathbf{H}^{1,1}} + \langle t\rangle \|w_j\|_{\mathbf{H}_6^1}\Big) \\ &\le C\varepsilon \langle t\rangle^{\frac{3}{q}-\frac{1}{2}}\end{aligned}$$

for $3 \le q < 6$. Then as above by virtue of (3.2.10) and (3.2.11) we get

$$\begin{aligned}\|w\|_{\mathbf{H}_4^{2,1}} &\le \|w\|_{\mathbf{H}_4^2} + \|xw\|_{\mathbf{H}_4^2} \\ &\le C\sum_{j=1}^{l}\Big(\|\mathcal{Z}w_j\|_{\mathbf{H}^2} + \|\mathcal{L}_{m_j}w_j\|_{\mathbf{H}^{2,1}} + \langle t\rangle \|w_j\|_{\mathbf{H}_4^2}\Big) \\ &\le C\varepsilon \langle t\rangle^{\frac{3}{2q}}\end{aligned}$$

for $3 \le q < 6$. Hence, we obtain

$$\begin{aligned}&\sum_{j=1}^{l}\|F_j(w)\|_{\mathbf{H}^{2,2}} \le C\|w\|_{\mathbf{H}_3^{3,1}}\|w\|_{\mathbf{H}_6^{1,1}} + \|w\|^2_{\mathbf{H}_4^{2,1}} \\ &\le C\varepsilon^2 \langle t\rangle^{\frac{3}{q}}.\end{aligned}$$

Thus, the second estimate of the lemma is fulfilled. Lemma (3.2.2) is proved.

We continue to prove Theorem 3.0.1. By equation (3.2.1) we have $\partial_t w_j = -iA\langle i\nabla\rangle_{m_j} w_j + \langle i\nabla\rangle_{m_j}^{-1} F_j(w)$. Hence $\|\partial_t w_j\|_{\mathbf{H}^{0,1}} = C\|w_j\|_{\mathbf{H}^{1,1}} + \|F_j(w)\|_{\mathbf{H}^{0,1}} \le C\varepsilon\langle t\rangle$. Therefore, by (3.0.9) we find

$$\begin{aligned}&\sum_{|\beta|\le 1}\|\mathcal{Z}^\beta \langle i\nabla\rangle_{m_j}^{-1} F_j(w)\|_{\mathbf{H}^{2,1}} \\ &\le C\|\partial_t F_j(w)\|_{\mathbf{H}^{0,1}} + \sum_{|\beta|\le 1}\|\mathcal{Z}^\beta F_j(w)\|_{\mathbf{H}^{1,1}} \\ &\le C\varepsilon\langle t\rangle \|w\|_{\mathbf{H}^1_\infty} + \sum_{|\beta|\le 1}\|\mathcal{Z}^\beta F_j(w)\|_{\mathbf{H}^{1,1}}. \qquad (3.2.13)\end{aligned}$$

We now apply (3.1.10), (3.0.8) and the first estimate of Lemma 3.2.2 (we note that the condition (3.2.7) of the lemma follows from (3.2.4) and (3.2.5)). We have for $3 \le q < 6$

$$\sum_{|\beta|\le 1} \|\mathcal{Z}^\beta F_j(w)\|_{\mathbf{H}^{1,1}} \le C \sum_{|\beta|\le 1} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{2,1}_3} \|w\|_{\mathbf{H}^2_6}$$
$$\le C\varepsilon \langle t\rangle^{-\frac{3}{2}(1-\frac{2}{q})} \sum_{k=1}^{l} \sum_{|\beta|\le 1} \Big(\|\mathcal{J}_{m_k} \mathcal{Z}^\beta w_k\|_{\mathbf{H}^1_3} + \langle t\rangle \|\mathcal{Z}^\beta w_k\|_{\mathbf{H}^2_3} \Big).$$

Using (3.0.10), (3.0.11) and (3.0.13) we find

$$\mathcal{J}_{m_k} \mathcal{Z}^\beta = -x \left(iA\mathcal{Z}^\beta + |\beta| \langle i\nabla\rangle_{m_k}^{-1} i\nabla \right) \mathcal{L}_{m_k} + \left(iA\mathcal{Z} - \langle i\nabla\rangle_{m_k}^{-1} i\nabla \right) \mathcal{Z}^\beta.$$

Therefore, by the Sobolev inequality $\|\phi\|_{\mathbf{H}^1_3} \le C\|\phi\|_{\mathbf{H}^2}$ we get

$$\sum_{|\beta|\le 1} \|\mathcal{Z}^\beta F_j(w)\|_{\mathbf{H}^{1,1}} \le C\varepsilon \langle t\rangle^{\frac{3}{q}-\frac{1}{2}} \sum_{|\beta|\le 2} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}}$$
$$+ C\varepsilon \langle t\rangle^{-\frac{3}{2}(1-\frac{2}{q})} \sum_{k=1}^{l} \sum_{|\beta|\le 1} \|x\mathcal{Z}^\beta \langle i\nabla\rangle_{m_k}^{-1} F_k(w)\|_{\mathbf{H}^2}.$$

Substitution of the last estimate into (3.2.13) yields

$$\sum_{|\beta|\le 1} \|\mathcal{Z}^\beta \langle i\nabla\rangle_{m_j}^{-1} F_j(w)\|_{\mathbf{H}^{2,1}} \le C\varepsilon \langle t\rangle \|w\|_{\mathbf{H}^1_\infty} + C\varepsilon \langle t\rangle^{\frac{3}{q}-\frac{1}{2}} \|w\|_{\mathbf{X}_{T,\varepsilon}}$$
$$+ C\varepsilon \langle t\rangle^{-\frac{3}{2}(1-\frac{2}{q})} \sum_{k=1}^{l} \sum_{|\beta|\le 1} \|x\mathcal{Z}^\beta \langle i\nabla\rangle_{m_k}^{-1} F_k(w)\|_{\mathbf{H}^2}.$$

Hence for $w \in \mathbf{X}_{T,\varepsilon}$ we obtain

$$(1 - C\varepsilon) \sum_{j=1}^{l} \sum_{|\beta|\le 1} \|\mathcal{Z}^\beta \langle i\nabla\rangle_{m_j}^{-1} F_j(w)\|_{\mathbf{H}^{2,1}} \le C\varepsilon \langle t\rangle \|w\|_{\mathbf{H}^1_\infty} + C\varepsilon^2 \langle t\rangle^{\frac{3}{q}-\frac{1}{2}}. \tag{3.2.14}$$

We substitute (3.2.14) and the second estimate of Lemma 3.2.2 (we recall that Lemma 3.2.2 applies because of (3.2.4) and (3.2.5)) into the right-hand side of (3.2.6) to get for $5 \le q < 6$ and $9 \le p < \infty$

$$\|w\|_{\mathbf{H}^1_\infty} \le C\varepsilon \langle t\rangle^{-\frac{3}{2}(1-\frac{2}{p})} \left(1 + \langle t\rangle \|w\|_{\mathbf{H}^1_\infty} + \varepsilon \langle t\rangle^{\frac{3}{q}-\frac{1}{2}} \right).$$

Hence,

$$\|w\|_{\mathbf{H}^1_\infty} \le C\varepsilon \langle t\rangle^{\frac{3}{p}+\frac{3}{q}-2} \le C\varepsilon \langle t\rangle^{-1-\delta} \tag{3.2.15}$$

with some $\delta > 0$. Collecting (3.2.3),(3.2.15) and the estimate of Lemma 3.2.2 we find

$$\|w(t)\|_{\mathbf{H}^4} \leq \|\mathring{w}\|_{\mathbf{H}^4} + C\varepsilon^2 \int_0^t \langle\tau\rangle^{-1-\delta} d\tau \leq \varepsilon_0 + C\varepsilon^2$$

with some $\delta > 0$. Thus, we obtain

$$\sup_{t\in[0,T]} \|w(t)\|_{\mathbf{H}^4} \leq \varepsilon_0 + C\varepsilon^2. \tag{3.2.16}$$

Appliying the operator $\mathcal{Z}^\alpha$ to the equation (3.2.1) using commutator relation (3.0.11), taking the $H^{4-|\alpha|}$-norm of the result, we find after integrating with respect to time

$$\begin{aligned}
&\|\mathcal{Z}^\alpha w_j(t)\|_{\mathbf{H}^{4-|\alpha|}} \leq \|(\mathcal{Z}^\alpha w_j)(0)\|_{\mathbf{H}^{4-|\alpha|}} \\
&+ C\int_0^t \|w\|_{\mathbf{H}^1_\infty} \sum_{|\alpha|\leq 3} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1} d\tau \\
&+ C\int_0^t \sum_{|\alpha|\leq 1} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_6} \sum_{|\alpha|\leq 2} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_3} d\tau
\end{aligned}$$

for $|\alpha| \leq 3$. Since $w \in \mathbf{X}_{T,\varepsilon}$ by (3.2.15) we have the estimate with $5 \leq q < 6$ and $9 \leq p < \infty$

$$\|w\|_{\mathbf{H}^1_\infty} \sum_{|\alpha|=3} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1} \leq C\varepsilon^2 \langle t\rangle^{\frac{3}{p}+\frac{3}{q}-2}.$$

In the same way as in the proofs of (3.2.4) and (3.2.5) we get

$$\sum_{|\alpha|\leq 2} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_3} \leq C\varepsilon \langle t\rangle^{-\frac{1}{2}}$$

and

$$\sum_{|\alpha|\leq 1} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_6} \leq C\varepsilon \langle t\rangle^{-\frac{3}{2}(1-\frac{2}{q})}$$

for $5 \leq q < 6$. Hence,

$$\sum_{|\alpha|\leq 1} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_6} \sum_{|\alpha|\leq 2} \|\mathcal{Z}^\alpha w\|_{\mathbf{H}^1_3} \leq C\varepsilon^2 \langle t\rangle^{\frac{3}{q}-2}.$$

Therefore, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \sum_{|\beta|\leq 3} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}} \\
&\leq \sum_{|\beta|\leq 3} \sup_{t\in[0,T]} \left\| \left(x^\beta \partial_t^{|\beta|} w\right)(0) \right\|_{\mathbf{H}^{4-|\beta|}} + C\varepsilon^2 \leq C\varepsilon_0 + C\varepsilon^2.
\end{aligned}$$

By using this estimate and identities $[\partial_t, \mathcal{Z}] = E\nabla, [\partial_{x_k}, \mathcal{Z}^{(j)}] = E[\partial_{x_k}, x_j]\partial_t = E\delta_{jk}\partial_t, E\partial_t = \mathcal{L}_m - iA \langle i\nabla\rangle_m, xE = \langle i\nabla\rangle_m^{-1} \mathcal{J}_m - iAt \langle i\nabla\rangle_m^{-1} i\nabla$,(3.1.10) we have for the local solution (3.2.1)

$$\sup_{t\in[0,T]} \sum_{|\beta|\le 3} \|\partial_t \mathcal{Z}^\beta w\|_{\mathbf{H}^{3-|\beta|}}$$
$$\le C \sup_{t\in[0,T]} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}} + C \sup_{t\in[0,T]} \sum_{j=1}^{l} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta F_j\|_{\mathbf{H}^{2-|\beta|}}$$
$$\le C\varepsilon_0 + C\|w\|_{\mathbf{X}_T}^2 \le C\varepsilon_0 + C\varepsilon^2,$$

$$\sup_{t\in[0,T]} \langle t\rangle^{-1} \|w\|_{\mathbf{H}^{3,1}}$$
$$\le C \sup_{t\in[0,T]} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}} + C \sup_{t\in[0,T]} \sum_{j=1}^{l} \langle t\rangle^{-1} \|F_j\|_{\mathbf{H}^{1,1}}$$
$$\le C\varepsilon_0 + C\|w\|_{\mathbf{X}_T}^2 \le C\varepsilon_0 + C\varepsilon^2,$$

and

$$\sup_{t\in[0,T]} \langle t\rangle^{-2} \|w\|_{\mathbf{H}^{2,2}} \le C \sup_{t\in[0,T]} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w\|_{\mathbf{H}^{4-|\beta|}}$$
$$+ C \sup_{t\in[0,T]} \sum_{j=1}^{l} \sum_{|\beta|\le 1} \langle t\rangle^{-1} \|\mathcal{Z}^\beta F_j\|_{\mathbf{H}^{1,1}} + C \sup_{t\in[0,T]} \sum_{j=1}^{l} \langle t\rangle^{-2} \|F_j\|_{\mathbf{H}^{1,2}}$$
$$\le C\varepsilon_0 + C\|w\|_{\mathbf{X}_T}^2 \le C\varepsilon_0 + C\varepsilon^2.$$

Thus, we obtain the estimate

$$\|w\|_{\mathbf{X}_T} < \varepsilon$$

for any $T > 0$. Therefore we have a global in time of solutions.

We next consider the asymptotic behavior of solutions. By (3.2.2) in view of (3.2.15) and (3.2.16)

$$\|\mathcal{U}_{m_j}(-t)\, w_j(t) - \mathcal{U}_{m_j}(-s)\, w_j(s)\|_{\mathbf{H}^4} \le C\varepsilon^2 \int_s^t \langle \tau\rangle^{-1-\delta}\, d\tau \le C\varepsilon^2 s^{-\delta} \tag{3.2.17}$$

for all $t > s \geq 1$ with some $\delta > 0$. In the same way

$$\|\mathcal{U}_{m_j}(-t)\,\mathcal{Z}w_j(t) - \mathcal{U}_{m_j}(-s)\,\mathcal{Z}w_j(s)\|_{\mathbf{H}^3} \leq Cs^{-\delta}. \tag{3.2.18}$$

We let $t \to \infty$, then there exist unique final states $w_j^+ \in \mathbf{H}^4$ such that

$$\|w_j^+ - \mathcal{U}_{m_j}(-s)\,w_j(s)\|_{\mathbf{H}^4} \leq C\varepsilon^2 s^{-\delta}.$$

Via Lemma 3.2.2 $\|F_j(w)\|_{\mathbf{H}^{2,1}} \leq C\varepsilon^2 \langle t \rangle^{\frac{3}{q}-1}$ with $3 < q < 6$. Therefore, by identity (3.1.10) and estimate (3.2.18) we get

$$\begin{aligned}
&\|\mathcal{U}_{m_j}(-t)\mathcal{J}_{m_j}w_j(t) - \mathcal{U}_{m_j}(-s)\,\mathcal{J}_{m_j}w_j(s)\|_{\mathbf{H}^3} \\
&\leq \|\mathcal{U}_{m_j}(-t)\left(iA\mathcal{Z} - E\langle i\nabla\rangle_{m_j}^{-1} i\nabla\right) w_j(t) \\
&\quad - \mathcal{U}_{m_j}(-s)\left(iA\mathcal{Z} - E\langle i\nabla\rangle_{m_j}^{-1} i\nabla\right) w_j(s)\|_{\mathbf{H}^3} \\
&\quad + C\|x\mathcal{L}_{m_j}w_j(t)\|_{\mathbf{H}^3} + C\|x\mathcal{L}_{m_j}w_j(s)\|_{\mathbf{H}^3} \\
&\leq C\varepsilon^2 s^{-\delta} + C\|xF_j(w(s))\|_{\mathbf{H}^2} \leq C\varepsilon^2 s^{-\delta}.
\end{aligned}$$

with some $\delta > 0$, from which via (3.2.17) by the relation $\mathcal{U}_m(-t)\mathcal{J}_m = \langle i\nabla\rangle_m x\mathcal{U}_m(-t) = x\langle i\nabla\rangle_m \mathcal{U}_m(-t) - \langle i\nabla\rangle_m^{-1} i\nabla\mathcal{U}_m(-t)$ we get

$$\|x\langle i\nabla\rangle_{m_j}\left(\mathcal{U}_{m_j}(-t)\,w_j(t) - \mathcal{U}_{m_j}(-s)\,w_j(s)\right)\|_{\mathbf{H}^3} \leq C\varepsilon^2 s^{-\delta}$$

for all $t > s \geq 1$. We let $t \to \infty$, then we see that there exists unique final states $w_j^+ \in \mathbf{H}^{4,1}$ such that

$$\|w_j^+ - \mathcal{U}_{m_j}(-s)\,w_j(s)\|_{\mathbf{H}^{4,1}} \leq C\varepsilon^2 s^{-\delta}.$$

The asymptotic behavior stated in the theorem follows from the relations $u_j = w_j^{(1)} + w_j^{(2)}$, $\langle i\nabla\rangle_{m_j}^{-1} \partial_t u_j = w_j^{(1)} - w_j^{(2)}$. Theorem 3.0.1 is proved. ■

3.3 PROOF OF THEOREM 3.0.2

In the same way as in the proofs of the previous theorem we prove *a priori* estimates of the local solution of (3.2.1). We let $\varepsilon \geq \varepsilon_0^{\frac{3}{4}} = \|\mathring{w}\|_{\mathbf{H}^{5,4}}^{\frac{3}{4}}$. We introduce the function space

$$\mathbf{Y}_T = \left\{\phi \in C\left([0,T]; \left(\mathbf{L}^2\right)^l \times \left(\mathbf{L}^2\right)^l\right); \|\phi\|_{\mathbf{Y}_T} < \infty\right\},$$

with the norm

$$\|\phi\|_{\mathbf{Y}_T} = \sup_{t\in[0,T]} \sum_{j=1}^{l} \Big(\sum_{|\beta|\le 4} \langle t\rangle^{-\gamma} \|\mathcal{Z}^{\beta}\phi_j\|_{\mathbf{H}^{5-|\beta|}} + \sum_{|\beta|\le 3} \|\mathcal{Z}^{\beta}\phi_j\|_{\mathbf{H}^{4-|\beta|}} \Big)$$
$$+ \sup_{t\in[0,T]} \sum_{j=1}^{l} \Big(\sum_{|\beta|\le 3} \|\partial_t \mathcal{Z}^{\beta}\phi_j\|_{\mathbf{H}^{3-|\beta|}} + \langle t\rangle^{-1} \|\phi_j\|_{\mathbf{H}^{3,1}} + \langle t\rangle^{-2} \|\phi\|_{\mathbf{H}^{2,2}} \Big),$$

where γ is small. Denote by $\mathbf{Y}_{T,\varepsilon}$ a closed ball of a radius $\varepsilon > 0$ with a center in the origin in the space $\mathbf{Y}_T$.

Taking the $\mathbf{H}^5$-norm of the integral equation (3.2.2) we obtain

$$\|w_j(t)\|_{\mathbf{H}^5} \le \|\mathring{w}\|_{\mathbf{H}^5} + C\int_0^t \|F_j(w)(\tau)\|_{\mathbf{H}^4} d\tau$$
$$\le \|\mathring{w}\|_{\mathbf{H}^5} + C\int_0^t \|w(\tau)\|_{\mathbf{H}^1_\infty} \|w(\tau)\|_{\mathbf{H}^5} d\tau. \tag{3.3.1}$$

In view of $(3.0.9), (3.0.11), (3.1.10)$ and the identity $xE = \langle i\nabla\rangle_{m_j}^{-1} \mathcal{J}_{m_j} - iAt\,\langle i\nabla\rangle_{m_j}^{-1} i\nabla$ we have

$$\|x\mathcal{L}_{m_j}\mathcal{Z}w_j\|_{\mathbf{H}^1} \le C\sum_{|\alpha|\le 1} \|\mathcal{Z}^{\alpha}F_j(w)\|_{\mathbf{H}^{0,1}} \le C\|w\|_{\mathbf{H}^1_\infty} \sum_{|\alpha|\le 1} \|\mathcal{Z}^{\alpha}w\|_{\mathbf{H}^{1,1}}$$
$$\le C\|w\|_{\mathbf{H}^1_\infty} \Big(\langle t\rangle \sum_{j=1}^{l} \sum_{|\alpha|\le 2} \|\mathcal{Z}^{\alpha}w_j\|_{\mathbf{H}^1} + \sum_{|\alpha|\le 1} \|x\mathcal{L}_{m_j}\mathcal{Z}^{\alpha}w_j\|_{\mathbf{L}^2} \Big).$$

Hence for $w \in \mathbf{Y}_{T,\varepsilon}$

$$(1 - C\varepsilon)\|x\mathcal{L}_{m_j}\mathcal{Z}w_j\|_{H^1} \le C\varepsilon\,\langle t\rangle\,\|w\|_{\mathbf{H}^1_\infty}. \tag{3.3.2}$$

By Lemma 3.1.4 we have

$$\langle t\rangle\,\|w\|_{\mathbf{H}^1_\infty} \le C\|w\|_{\mathbf{H}^3}$$
$$+ C\sum_{|\alpha|\le 1} \|\mathcal{Z}^{\alpha}w\|_{\mathbf{H}^2}^{1-\gamma} \Big(\sum_{|\alpha|\le 2} \|\mathcal{Z}^{\alpha}w\|_{\mathbf{H}^1}^{\gamma} + \|x\mathcal{L}_m\mathcal{Z}w\|_{\mathbf{H}^1}^{\gamma} \Big)$$
$$+ C\Big(\|F(w)\|_{\mathbf{H}^{1,2}}^{\gamma} + \langle t\rangle^{\gamma} \|F(w)\|_{\mathbf{H}^{1,1}}^{\gamma} \Big) \|F(w)\|_{\mathbf{H}^{1,1}}^{1-\gamma}$$

with some small $\gamma > 0$. Then for $w \in \mathbf{Y}_{T,\varepsilon}$ we get

$$\langle t\rangle\,\|w\|_{\mathbf{H}^1_\infty} \le C\varepsilon + C\Big(\|F(w)\|_{\mathbf{H}^{1,2}}^{\gamma} + \langle t\rangle^{\gamma} \|F(w)\|_{\mathbf{H}^{1,1}}^{\gamma} \Big) \|F(w)\|_{\mathbf{H}^{1,1}}^{1-\gamma}. \tag{3.3.3}$$

From identities $E\partial_r = t^{-1}\mathcal{Z}^{(r)} - t^{-1}Ex_r\partial_t$ and the strong null condition we can write

$$F_j(w) = i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}\mathcal{Z}^{(r)}\left(w_m^{(1)} + w_m^{(2)}\right)\overline{\mathcal{Z}^{(s)}(w_k^{(1)} + w_k^{(1)})}$$
$$- i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}\mathcal{Z}^{(r)}\left(w_m^{(1)} + w_m^{(2)}\right)\overline{x_s\partial_t(w_k^{(1)} + w_k^{(1)})}$$
$$- i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}x_r\partial_t\left(w_m^{(1)} + w_m^{(2)}\right)\overline{\mathcal{Z}^{(s)}(w_k^{(1)} + w_k^{(1)})}$$

with $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then we have the identity

$$x_px_qF_j(w) = i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}x_p\mathcal{Z}^{(r)}\left(w_m^{(1)} + w_m^{(2)}\right)\overline{x_q\mathcal{Z}^{(s)}(w_k^{(1)} + w_k^{(1)})}$$
$$- i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}x_p\mathcal{Z}^{(r)}\left(w_m^{(1)} + w_m^{(2)}\right)\overline{x_qx_s\partial_t(w_k^{(1)} + w_k^{(1)})}$$
$$- i\mathbf{b}t^{-2} \sum_{1\le m,k\le l} \sum_{0\le r,s\le 2} q_j^{mkrs}x_px_r\partial_t\left(w_m^{(1)} + w_m^{(2)}\right)\overline{x_q\mathcal{Z}^{(s)}(w_k^{(1)} + w_k^{(1)})}.$$

So by identity $xE = \langle i\nabla\rangle_{m_j}^{-1}\mathcal{J}_{m_j} - iAt\langle i\nabla\rangle_{m_j}^{-1}\nabla$, the relation $\partial_t\big(w_j^{(1)}+w_j^{(2)}\big) = -i\langle i\nabla\rangle_{m_j}\big(w_j^{(1)} - w_j^{(2)}\big)$ and by the Sobolev imbedding theorem $\|w\|_{\mathbf{L}^4} \le C\|w\|_{\mathbf{H}^1}$ we get for $w \in \mathbf{Y}_{T,\varepsilon}$

$$\|F_j(w)\|_{\mathbf{H}^{1,2}} \le Ct^{-2}\sum_{k=1}^{l}\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{L}^4}^2 + C\|\mathcal{Z}w\|_{\mathbf{H}_4^1}^2$$
$$+ C\left(t^{-2}\sum_{k=1}^{l}\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{L}^4} + t^{-1}\|\mathcal{Z}w\|\|_{\mathbf{H}_4^1}\right)\|w\|_{\mathbf{H}_4^{1,2}} \le C\varepsilon^2\langle t\rangle, \qquad (3.3.4)$$

where we have used the fact that $\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{L}^4}$ can be estimated through the identity (3.1.10) and (3.3.2) as follows

$$\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{L}^4} \le C\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{H}^1}$$
$$\le C\sum_{|\alpha|\le 2}\|\mathcal{Z}^\alpha w_k\|_{\mathbf{H}^1} + C\|x\mathcal{L}_{m_k}\mathcal{Z}w_k\|_{\mathbf{H}^1} \le C\varepsilon\langle t\rangle.$$

Similarly,

$$x_pF_j(w) = i\mathbf{b}t^{-2}\sum_{1\le m,k\le l}\sum_{0\le r,s\le 2} q_j^{mkrs}x_p\mathcal{Z}^{(r)}\left(w_m^{(1)}+w_m^{(2)}\right)\overline{\mathcal{Z}^{(s)}(w_k^{(1)}+w_k^{(1)})}$$
$$-i\mathbf{b}t^{-2}\sum_{1\le m,k\le l}\sum_{0\le r,s\le 2} q_j^{mkrs}x_p\mathcal{Z}^{(r)}\left(w_m^{(1)}+w_m^{(2)}\right)\overline{x_s\partial_t(w_k^{(1)}+w_k^{(1)})}$$
$$-i\mathbf{b}t^{-2}\sum_{1\le m,k\le l}\sum_{0\le r,s\le 2} q_j^{mkrs}x_r\partial_t\left(w_m^{(1)}+w_m^{(2)}\right)\overline{x_p\mathcal{Z}^{(s)}(w_k^{(1)}+w_k^{(1)})}$$

which by identity $xE = \langle i\nabla\rangle_{m_j}^{-1}\mathcal{J}_{m_j} - iAt\,\langle i\nabla\rangle_{m_j}^{-1}\nabla$, the relation $\partial_t\big(w_j^{(1)}+w_j^{(2)}\big) = -i\,\langle i\nabla\rangle_{m_j}\,\big(w_j^{(1)}-w_j^{(2)}\big)$ and by the Sobolev imbedding theorem $\|w\|_{\mathbf{L}^4}\le C\|w\|_{\mathbf{H}^1}$ we get for $w\in\mathbf{Y}_{T,\varepsilon}$

$$\begin{aligned}
&\|F_j(w)\|_{\mathbf{H}^{1,1}} \le C\bigg(t^{-2}\sum_{k=1}^{l}\|\mathcal{J}_{m_k}\mathcal{Z}w_k\|_{\mathbf{H}^1} + t^{-1}\|\mathcal{Z}w\|_{\mathbf{H}^1_4}\bigg)\\
&\times\left(\sum_{k=1}^{l}\|\mathcal{J}_{m_k}w_k\|_{\mathbf{H}^1} + t\|w(t)\|_{\mathbf{H}^1_\infty}^{\frac12}\|w\|_{\mathbf{H}^1}^{\frac12}\right)\\
&\le C\varepsilon\,\langle t\rangle^{-\frac12}\left(\varepsilon + \langle t\rangle^{\frac12}\|w(t)\|_{\mathbf{H}^1_\infty}^{\frac12}\right).
\end{aligned}\tag{3.3.5}$$

We substitute (3.3.4) and (3.3.5) into (3.3.3) to get

$$\langle t\rangle\,\|w(t)\|_{\mathbf{H}^1_\infty} \le C\varepsilon + C\varepsilon^2. \tag{3.3.6}$$

Substitution of (3.3.6) into (3.3.1) yields

$$\|w(t)\|_{\mathbf{H}^5} \le \varepsilon + C\varepsilon\int_0^t \langle\tau\rangle^{-1}\|w(\tau)\|_{\mathbf{H}^5}d\tau.$$

from which it follows that

$$\langle t\rangle^{-\gamma}\|w(t)\|_{\mathbf{H}^5} \le C\varepsilon + C\varepsilon^2. \tag{3.3.7}$$

In the same way as in the proof of (3.3.7), we estimate the other terms in the norm of $\mathbf{Y}_T$ to find $\|w(t)\|_{\mathbf{Y}_T} \le C\varepsilon_0 + C\varepsilon^2$. The asymptotics stated in the theorem follows from the same arguments as in the proof of Theorem 3.0.1. Theorem 3.0.2 is proved. ∎

3.4 PROOF OF THEOREM 3.0.3

We put $\mathcal{L}_m = E\partial_t + iA\left\langle i\nabla\right\rangle_m,$

$$w_j = \begin{pmatrix} w_j^{(1)} \\ w_j^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \psi_j + i\left\langle i\nabla\right\rangle_M^{-1}\partial_t\psi_j \\ \psi_j - i\left\langle i\nabla\right\rangle_M^{-1}\partial_t\psi_j \end{pmatrix},$$

$$\mathring{w}_j = \begin{pmatrix} \mathring{w}_j^{(1)} \\ \mathring{w}_j^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \mathring{\psi}_j^{(1)} + i\left\langle i\nabla\right\rangle_M^{-1}\mathring{\psi}_j^{(2)} \\ \mathring{\psi}_j^{(1)} - i\left\langle i\nabla\right\rangle_M^{-1}\mathring{\psi}_j^{(2)} \end{pmatrix},$$

and

$$w_5 = \begin{pmatrix} w_5^{(1)} \\ w_5^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \phi + i|i\nabla|^{-1}\partial_t\phi \\ \phi - i|i\nabla|^{-1}\partial_t\phi \end{pmatrix},$$

$$\mathring{w}_5 = \begin{pmatrix} \mathring{w}_5^{(1)} \\ \mathring{w}_5^{(2)} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} \mathring{\phi}^{(1)} + i|i\nabla|^{-1}\mathring{\phi}^{(1)} \\ \mathring{\phi}^{(1)} - i|i\nabla|^{-1}\mathring{\phi}^{(2)} \end{pmatrix}.$$

Then the system (3.0.5) can be rewritten as

$$\begin{cases} \mathcal{L}_M w_j = \begin{pmatrix} i\left\langle i\nabla\right\rangle_M^{-1} F_j(w) \\ -i\left\langle i\nabla\right\rangle_M^{-1} F_j(w) \end{pmatrix}, j = 1,2,3,4 \\ \mathcal{L}_0 w_5 = \begin{pmatrix} i|i\nabla|^{-1}F_5(w) \\ -i|i\nabla|^{-1}F_5(w) \end{pmatrix}, \end{cases} \tag{3.4.1}$$

where

$$F_j(w) = \frac{1}{2}\left(-g^2\left(w_5^{(1)} + w_5^{(2)}\right)^2\left(w_j^{(1)} + w_j^{(2)}\right) - g\sum_{\mu=0}^{3}\left(\gamma^\mu\gamma^4\left(w_j^{(1)} + w_j^{(2)}\right)\right)\partial_\mu\left(w_5^{(1)} + w_5^{(2)}\right)\right)$$

for $1 \le j \le 4$ and

$$F_5(\widetilde{w}) = \frac{1}{2}\left(g\left(\sum_{j=1}^{2}\left|w_j^{(1)} + w_j^{(2)}\right|^2 - \sum_{j=3}^{4}\left|w_j^{(1)} + w_j^{(2)}\right|^2\right)\right).$$

We note that the first term in the nonlinearity $F_j(w)$ is cubic, the second term contains the full derevative $\partial_\mu \left(w_5^{(1)} + w_5^{(2)} \right)$, and so has a good time decay property (see the second estimate of Lemma 3.1.5). Therefore, the nonlinearity $F_j(w)$ is asymptotically free.

We introduce function space

$$\mathbf{Z}_T = \left\{ \phi \in C\Big([0,T] \, ; (\mathbf{L}^2)^5 \times (\mathbf{L}^2)^5 \Big) ; \|\phi\|_{\mathbf{Z}_T} < \infty \right\},$$

with the norm

$$\begin{aligned}
\|\phi\|_{\mathbf{Z}_T} &= \sup_{t\in[0,T]} \sum_{j=1}^{4} \sum_{|\beta|\le 3} \langle t \rangle^{-\gamma} \left(\|\mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{4-|\beta|}} + \|\partial_t \mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{3-|\beta|}} \right) \\
&+ \sup_{t\in[0,T]} \sum_{j=1}^{4} \sum_{|\beta|\le 2} \left(\|\mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{3-|\beta|}} + \|\partial_t \mathcal{Z}^\beta \phi_j\|_{\mathbf{H}^{2-|\beta|}} \right) \\
&+ \sup_{t\in[0,T]} \sum_{|\beta|\le 3} \left(\|\mathcal{Z}^\beta \phi_5\|_{\dot{\mathbf{H}}^{4-|\beta|}} + \|\partial_t \mathcal{Z}^\beta \phi_5\|_{\mathbf{H}^{3-|\beta|}} \right) \\
&+ \sup_{t\in[0,T]} \sum_{j=1}^{4} \left(\langle t \rangle^{-1} \|\phi_j\|_{\mathbf{H}^{3,1}} + \langle t \rangle^{-2} \|\phi_j\|_{\mathbf{H}^{2,2}} \right)
\end{aligned}$$

for small $\gamma > 0$. Denote by $\mathbf{Z}_{T,\varepsilon}$ a closed ball of a radius $\varepsilon > 0$ with a center in the origin in the space $\mathbf{Z}_T$. As in the proof of Theorem 3.0.1 we used Lemma 3.1.2 to estimate w_j, then we get for $|\beta| = 3$

$$\begin{aligned}
\|\mathcal{Z}^\beta w_5\|_{\dot{\mathbf{H}}^1} &\le \varepsilon_0 + \int_0^t \|\mathcal{Z}^\beta F_5(w)\|_{\mathbf{L}^2} d\tau \\
&\le \varepsilon_0 + \int_0^t \Bigg(\sum_{j=1}^{4} \|w_j\|_{\mathbf{L}^\infty} \sum_{j=1}^{4} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^2} \\
&+ \sum_{j=1}^{4} \sum_{|\beta|=1} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^6} \sum_{j=1}^{4} \sum_{|\beta|\le 2} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^3} \Bigg) d\tau \\
&\le \varepsilon_0 + C\|w\|_{\mathbf{Z}_T}^2 \int_0^t \langle \tau \rangle^{-\frac{3}{2}+\gamma} \, d\tau,
\end{aligned}$$

and

$$\|\mathcal{Z}^\beta w_5\|_{\mathbf{L}^2} \le \varepsilon_0 + \int_0^t \|\mathcal{Z}^\beta F_5(w)\|_{\mathbf{L}^2} + \|\mathcal{Z}^\beta F_5(w)\|_{\mathbf{L}^p} d\tau,$$

where $p < \frac{6}{5}$. In the same way as above

$$\int_0^t \|\mathcal{Z}^\beta F_5(w)\|_{\mathbf{L}^p} d\tau$$

is estimated from above by

$$\int_0^t \sum_{j=1}^4 \sum_{|\beta|\le 1} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^q} \sum_{j=1}^4 \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^2} d\tau$$
$$\le C\|w\|^2_{\mathbf{Z}_T} \int_0^t \langle\tau\rangle^{-\frac{1}{2}+\gamma} d\tau \le C\varepsilon^2 \langle t\rangle^{\frac{1}{2}+\gamma}$$

with $q < 3$. Hence we get

$$\|\mathcal{Z}^\beta w_5\|_{\mathbf{L}^2} \le \varepsilon_0 + C\varepsilon^2 \langle t\rangle^{\frac{1}{2}+\gamma}.$$

We have for $|\beta| = 3, \quad 1 \le j \le 4$

$$\|\mathcal{Z}^\beta w_j\|_{\mathbf{H}^1} \le \varepsilon_0 + \sum_{j=1}^4 \int_0^t \|\mathcal{Z}^\beta F_j(w)\|_{\mathbf{L}^2} d\tau$$
$$\le \varepsilon_0 + C\int_0^t \Bigg((\sum_{|\beta|\le 1} \|\mathcal{Z}^\beta w_5\|_{\mathbf{L}^\infty})^2 \sum_{j=1}^4 \sum_{|\beta|=3} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^2}$$
$$+ \sum_{j=1}^4 \sum_{|\beta|\le 1} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^6} \sum_{|\beta|\le 1} \|\mathcal{Z}^\beta w_5\|_{\mathbf{L}^\infty} \sum_{|\beta|=3} \|\mathcal{Z}^\beta w_5\|_{\mathbf{L}^3}$$
$$+ \sum_{j=1}^4 \left(\|w_j\|_{\mathbf{L}^\infty} \sum_{|\beta|=3} \|\mathcal{Z}^\beta \nabla w_5\|_{\mathbf{L}^2} + \|\nabla w_5\|_{\mathbf{L}^\infty} \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^2} \right)$$
$$+ \sum_{|\alpha|\le 2} \|\mathcal{Z}^\alpha \nabla w_5\|_{\mathbf{L}^3} \sum_{j=1}^4 \sum_{|\beta|\le 2} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^6} \Bigg) d\tau$$
$$\le \varepsilon_0 + C\varepsilon \int_0^t \langle\tau\rangle^{-1} \sum_{j=1}^4 \sum_{|\beta|\le 3} \|\mathcal{Z}^\beta w_j\|_{\mathbf{L}^2} d\tau + C\varepsilon^2 \int_0^t \langle\tau\rangle^{-\frac{5}{4}+\gamma} d\tau,$$

and an analogous estimate for the norm $\|\mathcal{Z}^\beta w_5\|_{\dot{\mathbf{H}}^1}$ holds, where we have used Lemma 3.1.6 to estimate w_5 and Lemma 3.1.2, from which with the Gronwall inequality it follows

$$\sum_{|\beta|=3} \left(\sum_{j=1}^4 \langle t\rangle^{-\gamma} \|\mathcal{Z}^\beta w_j\|_{\mathbf{H}^1} + \|\mathcal{Z}^\beta w_5\|_{\dot{\mathbf{H}}^1} \right) \le C\varepsilon_0 + C\varepsilon^2$$

for small $\gamma > 0$. In this way we have for the solutions w_5 of the massless Klein-Gordon equation

$$\sup_{t\in[0,T]} \sum_{|\beta|\le 3} \|\mathcal{Z}^{\beta} w_5\|_{\dot{\mathbf{H}}^{4-|\beta|}} \le C\varepsilon_0 + C\varepsilon^2 \int_0^t \langle\tau\rangle^{-\frac{3}{2}+\gamma}\, d\tau \le C\varepsilon_0 + C\varepsilon^2.$$

Then for the solutions $w_j, 1 \le j \le 4$ of the massive Klein-Gordon equation we obtain

$$\sup_{t\in[0,T]} \sum_{j=1}^{4} \sum_{|\beta|\le 2} \|\mathcal{Z}^{\beta} w_j\|_{\mathbf{H}^{3-|\beta|}} \le C\varepsilon_0 + C\varepsilon^2 \int_0^t \langle\tau\rangle^{-\frac{5}{4}+\gamma}\, d\tau$$

and

$$\sup_{t\in[0,T]} \sum_{j=1}^{4} \sum_{|\beta|\le 3} \langle t\rangle^{-\gamma} \|\mathcal{Z}^{\beta} w_j\|_{\mathbf{H}^{4-|\beta|}} \le C\varepsilon_0 + C\varepsilon^2 \langle t\rangle^{-\gamma} \int_0^t \langle\tau\rangle^{-1+\gamma}\, d\tau.$$

In the same manner as in the proof of Theorem 3.0.1 we estimate the other terms in the norm $\mathbf{Z}_T$ to find the desired a priori estimate of solutions $\|w\|_{\mathbf{Z}_T} \le C\varepsilon_0 + C\varepsilon^2$. The asymptotics is proved in the same way as that Theorem 3.0.1. Theorem 3.0.3 is proved. ■

BIBLIOGRAPHY

[1] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, Berlin-New York-Heidelberg, 1997.

[2] A. Bachelot. Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon. *Ann. Inst. Henri Poincaré*, 48:387–422, 1988.

[3] P. Brenner. On $\mathbf{L}^p - \mathbf{L}^q$ estimate for the wave equation. *Math. Z.*, 145:251–254, 1975.

[4] J. M. Delort, D. Fang, and R. Xue. Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two dimensions. *J. Funct. Anal.*, 211:288–323, 2004.

[5] V. Georgiev. Global solution of the system of wave and Klein-Gordon equations. *Math. Z.*, 203:683–698, 1990.

[6] V. Georgiev. Decay estimates for the Klein-Gordon equation. *Commun. P.D.E.*, 17:1111–1139, 1992.

[7] N. Hayashi and P.I. Naumkin. Scattering operator for the nonlinear Klein-Gordon equations. *Commun. Contempory. Math. In press.*

[8] N. Hayashi and P.I. Naumkin. Scattering operator for the nonlinear Klein-Gordon equations in higher space dimensions. *Differential Equs.*, 244:188–199, 2008.

[9] N. Hayashi, P.I. Naumkin, and Ratno Bagus Edy Wibowo. Nonlinear scattering for a system of nonlinear Klein-Gordon equations. *J. Math. Phys.*, 49(103501):1–24, 2008.

[10] S. Katayama. A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension. *J. Math. Kyoto Univ.*, 39:203–213., 1999.

[11] S. Klainerman. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Commun. Pure Appl. Math.*, 38:631–641, 1985.

[12] B. Marshall, W. Strauss, and S. Wainger. $\mathbf{L}^p - \mathbf{L}^q$ estimates for the Klein-Gordon equation. *Commun. Pure Appl. Math.*, 59:417–440, 1980.

[13] T. Ozawa, K. Tsutaya, and Y. Tsutsumi. Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions. *Math. Z.*, 222:341–362., 1996.

[14] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Commun. Pure Appl. Math.*, 38:685–696, 1985.

[15] E. M. Stein. *Singular Integral and Differentiabilty Properties of Functions*. Princeton Univ. Press, Princeton, NJ, 1970.

[16] H. Sunagawa. On the global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with mass terms in one space dimmension. *J. Differential Equations*, 192:308–325, 2003.

[17] Y. Tsutsumi. Global solutions to Dirac-Proca equations with small initial data in $3+1$ time dimensions. *J. Math. Anal. Appl.*, 278:485–499, 2003.

[18] Y. Tsutsumi. Stability of constant equlibrium for the Maxwell-Higgs equations. *Funkcialaj Ekvacioj*, 46:41–46, 2003.

[19] Ratno Bagus Edy Wibowo. Scattering problem for a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations. *J. Nonlinear. Anal.*, 71:881–890, 2009.

[20] K. Yajima. Existence of solutions for Schrödinger evolution equations. *Commun. Math. Phys.*, 110:415–426, 1987.

LIST OF AUTHOR'S PAPERS CITED IN THIS THESIS

[1] Ratno Bagus Edy Wibowo, *Scattering problem for a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations*. J. Nonlinear Anal.,**71**(881-890) (2009). Published online 18 December 2008.

[2] N. Hayashi, P. I. Naumkin and Ratno Bagus Edy Wibowo, *Nonlinear scattering for a system of nonlinear Klein-Gordon equations,* J. Math. Phys., **49**, 103501(1-24) (2008). Published online 1 October 2008.

VITA

Mr Ratno Bagus Edy Wibowo was born in September 1975 in Blitar, Indonesia. Bagus received a PhD in Mathematics from Osaka University, where he did graduate work in nonlinear partial differential equations. His research interests include scattering problem for a system of nonlinear Klein-Gordon equations. He has been studying and doing a joint research on the asymptotic behavior for systems of some kind of nonlinear wave equations in three space dimensions with Professor Nakao Hayashi and Professor Pavel I. Naumkin (Universidad Nacional Autnoma de Mexico) from 2006 until now. If you have any feedback on this thesis or wish to discuss anything related to nonlinear partial differential equations, contact him via his email at `wibowo@cr.math.sci.osaka-u.ac.jp`.