INTERACTIVE DECISION MAKING
FOR
MULTIOBJECTIVE LINEAR AND NONLINEAR PROGRAMMING PROBLEMS
WITH FUZZY PARAMETERS

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ABSTRACT

In this thesis, considering the imprecise nature of the human judgements in the real-world decision situations, two types of fuzziness of human judgements are incorporated in multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters. One is the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process, and the other is the fuzzy goals of the decision maker for each of the objective functions. The fuzzy parameters, which reflect the experts' ambiguous understanding in the problem-formulation, are characterized by fuzzy numbers. The concept of M-α-Pareto optimality is introduced on the basis of the α-level sets of the fuzzy numbers and the fuzzy goals of the decision maker for each of the objective functions are quantified by eliciting the corresponding membership functions. In our interactive methods, the satisficing solution of the decision maker can be derived efficiently from among an M-α-Pareto optimal solution set by updating his/her reference membership values and/or the degree α together with the trade-off information. Based on the proposed methods, interactive computer programs are written to implement man-machine interactive procedures. To demonstrate the feasibility and efficiency of the proposed methods, several illustrative numerical examples are shown along with the corresponding computer outputs.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction and Historical Remarks

Recently, it is increasingly recognized that most of the real-world decision making problems usually involve multiple, noncommensurate, and often conflicting objectives. For such multiobjective programming problems, multiple objectives are usually noncommensurable and cannot be combined into a single objective. Moreover, the objectives usually conflict with each other in that any improvement of one objective can be achieved only at the expense of another. Consequently, the aim in solving multiobjective programming problems is to find a compromise or satisficing solution of a decision maker (DM) which is also Pareto optimal based on his/her subjective value-judgement (Chankong and Haimes, 1983a,1983b; Cohon, 1978; Grauer, Lewandowski and Wierzbicki, 1982; Grauer and Wierzbicki, 1984; Haimes, Hall and Freedman, 1975; Hwang and Masud, 1979; Steuer, 1986; Zeleny, 1982).

Two types of approaches for the determination of a compromise or satisficing solution of a DM in multiobjective programming problems have been developed. They are:

(1) goal programming approaches (e.g. Charnes and Cooper, 1961,1977; Ignizio, 1976,1983; Lee, 1972)

(2) interactive approaches (e.g. Geoffrion, Dyer and Feinberg, 1972; Musselman and Talavage, 1980; Sakawa, 1981,1982a; Sakawa and Mori, 1983,1984; Sakawa and Seo, 1980,1982,1983; Sakawa and Yano, 1984b; Steuer and Choo, 1983; Wierzbicki, 1979a,1979b,1980; Zionts and Wallenius, 1976)

The goal programming approaches, which assume that the DM can specify his/her goals of the objective functions, first appeared in a 1961 text by Charnes and Cooper (1961) in order to deal with multiobjective linear programming (MOLP) problems. Subsequent works on goal programming approaches have been numerous including Charnes and Cooper (1977), Ignizio (1976,1983) and Lee (1972).

The interactive approaches, which assume that the DM is able to give some preference information on a local level to a particular solution, were first initiated by Geoffrion et al.(1972) and further developed by many researchers such as Sakawa (1981, 1982a), Sakawa and Mori (1983, 1984), Sakawa and Seo (1980,1982,1983), Sakawa and Yano (1984b), Steuer and Choo (1983), Wierzbicki (1979,1980) and Zionts and Wallenius (1976).

The interactive goal programming method proposed by Dyer (1972) is a first attempt to provide a link between goal programming and interactive approaches. Since then several goal programming based interactive methods which combine the attractive features from both goal programming and interactive approaches have been proposed (Masud and Hwang, 1981; Monarchi, Kisiel and Duckstein, 1973; Weistroffer, 1982,1983,1984).

However, considering the imprecise nature of the DM's judgements in multiobjective programming problems, fuzzy programming approaches (e.g.

Kickert, 1978; Zimmermann, 1983; Zimmermann, Gaines and Zadeh, 1984) seem to be very applicable and promising for solving multiobjective programming problems.

An application of the theory of fuzzy set (Zadeh, 1965) to multiobjective linear programming problems was first presented by Zimmermann (1978) and further studied by Leberling (1981) and Hannan (1981). Following the fuzzy decision or the minimum-operator proposed by Bellman and Zadeh (1970) together with linear, hyperbolic or piecewise linear membership functions respectively, they proved that there exist equivalent linear programming problems.

However, suppose that the interaction with the DM establishes that the first membership function should be linear, the second hyperbolic, the third piecewise linear and so forth. In such a situation, the resulting problem becomes a nonlinear programming problem and cannot be solved by a linear programming technique.

In order to overcome such difficulties, Sakawa (1983a, 1983b) has proposed a new method by the combined use of the bisection method and the linear programming method together with five types of membership functions; linear, exponential, hyperbolic, hyperbolic inverse and piecewise linear functions. This method was further extended for solving multiobjective linear fractional (Sakawa and Yumine, 1983) and nonlinear programming problems (Sakawa, 1984a).

In these fuzzy approaches, however, it has been implicitly assumed that the fuzzy decision or the minimum-operator of Bellman and Zadeh (1970) is the proper representation of the DM's fuzzy preferences. Therefore, these approaches are preferable only when the DM feels that the fuzzy decision or the minimum-operator is appropriate when combining the fuzzy goals and/or constraints. However such situations seem to rarely occur, and consequently it becomes evident that an interaction with the DM is necessary.

Under these circumstances, assuming that the DM has a fuzzy goal for each of the objective functions in multiobjective programming problems, several interactive fuzzy decision making methods have been proposed by incorporating the desirable features of both the goal programming methods and the interactive approaches into the fuzzy approaches (Sakawa, Yumine and Yano, 1984a,1984b; Sakawa and Yano, 1984a, 1985b,1985d).

However, when formulating the multiobjective programming problem which closely describes and represents the real decision situation, various factors of the real system should be reflected in the description of the objective functions and the constraints. Naturally these objective functions and the constraints involve many parameters whose possible values may be assigned by the experts. In the previous approaches, such parameters are fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters.

In most practical situations, however, it is natural to consider that the possible values of these parameters are often only ambiguously known to the experts. In this case, it may be more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy subsets of the real line known as fuzzy numbers (Dubois and Prade, 1978,1980). The resulting multiobjective programming problem involving fuzzy parameters would be viewed as the more realistic version of the conventional one.

Recently, Tanaka and Asai (1981,1984) formulated the multiobjective linear programming problems with fuzzy parameters. Following the fuzzy decision or the minimum operator proposed by Bellman and Zadeh (1970) together with triangular membership functions for fuzzy parameters, they considered two types of fuzzy multiobjective linear programming problems; one is to decide the nonfuzzy solution and the other is to decide the fuzzy solution.

More recently, Orlovski (1983,1984) formulated general multiobjective nonlinear programming problems with fuzzy parameters. He presented two approaches to the formulated problems by making systematic use of the extension principle of Zadeh (1975) and demonstrated that there exist in some sense equivalent nonfuzzy formulations.

Very recently, in order to deal with the multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters characterized by fuzzy numbers, Sakawa and Yano (1985c,1985e, 1986b,1986d,1986f,1986g,1986i,1986j) introduced the concept of α-multiobjective programming and (M-)α-Pareto optimality based on the α-level sets of the fuzzy numbers. Then they presented several interactive decision making methods not only in objective spaces but also in membership spaces to derive the satisficing solution of the DM efficiently from among an (M-)α-Pareto optimal solution set for multiobjective linear, linear fractional and nonlinear programming problems as a generalization of their previous results (Sakawa,1983a, 1983b,1984a; Sakawa and Yano, 1985f,1986h; Sakawa, Yano and Yumine, 1986; Sakawa and Yumine, 1983; Sakawa, Yumine and Yano, 1984a,1984b).

Finally, it is appropriate to mention some application areas of the multiobjective approach. Although most of the early practical applications have been accomplished in the areas of water resources planning (see, for example, the texts by Haimes 1977, Cohon 1978), regional planning (e.g. Rietveld 1980) and environmental planning (e.g. Nijikamp 1979). Many other real-world problems are inherently multiobjective. As we look at recent engineering and industrial applications of the multiobjective approach, we can see continuing advances. They can be found, for example, in the areas of optimal design of shallow arches (e.g. Stadler 1983a,b), electronic circuit design (e.g. Lightner 1979), operation of a packaging system in automated warehouses (e.g. Sakawa 1983b), management of the erection of a cablestayed bridge (Ishido, Nakayama, Furukawa, Inoue and Tanikawa 1986) and pass scheduling for hot tandem mills (Sakawa, Narazaki, Konishi, Nose and Morita 1986).

1.2 Outline of the Thesis

In multiobjective programming problems, multiple objectives are often noncommensurable and conflict with each other, and consequently the aim is to find a compromise or satisficing solution of a decision maker (DM) which is also Pareto optimal based on his/her subjective value-judgement.

However, considering the imprecise or fuzzy nature of the human judgements, a fuzzy approach seems to be very applicable and promising for multiobjective programming problems under fuzziness. Two types of fuzziness of human judgements should be incorporated in multiobjective programming problems. One is the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process, and the other is the fuzzy goals of the DM for each of the objective functions.

In order to cope with both types of fuzziness, in this thesis, we formulate multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters and present several interactive decision making methods for obtaining the satisficing solution of the decision maker based on his/her subjective imprecise value-judgements. The fuzzy parameters in the description of the objective functions and the constraints, which reflect the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process, are characterized by fuzzy numbers. The concept of α-multiobjective linear, linear fractional and nonlinear programming together with M-α-Pareto optimality is introduced based on the α-level sets of the fuzzy numbers. The fuzzy goals of the DM for each of the objective functions are quantified by eliciting the corresponding membership functions. Then interactive decision making methods for multiobjective linear, linear fractional and nonlinear programming problems are presented to derive the satisficing solution of the decision maker efficiently from among M-α-Pareto optimal solution sets based on his/her subjective judgement. On the basis of the proposed methods, time-sharing computer programs for all the proposed methods are written in FORTRAN to implement man-machine interactive procedures. Illustrative numerical examples for multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters are demonstrated along with the corresponding computer outputs.

Organization of each Chapter is briefly summarized as follows:

Chapter 2 is concerned with a new linear programming based on an interactive decision making method for the multiobjective linear programming problems with fuzzy parameters in order to derive the satisficing solution of the decision maker from among an M-α-Pareto optimal solution set by updating his/her reference membership values and/or the degree α on the basis of the current M-α-Pareto optimal solution as well as the trade-off rates.

In Chapter 3, a new interactive decision making method for multiobjective linear fractional programming problems with fuzzy parameters is proposed on the basis of the linear programming method. In the proposed interactive method, the satisficing solution of the DM can be derived from among an M-α-Pareto optimal solution set by updating his/her reference membership values and/or the degree α together with trade-off information.

Chapter 4 is devoted to developing a new interactive decision making method for multiobjective nonlinear programming problems with fuzzy parameters. In the proposed interactive decision making method, in order to generate a candidate for the (local) satisficing solution which is also (local) M-α-Pareto optimal, if the DM specifies the degree α of the α-level sets and the reference membership values, the DM is supplied with the corresponding (local) M-α-Pareto optimal solution together with the trade-off rates. Then by considering the current values of the objective or membership functions and α as well as the trade-off rates, the DM acts on this solution by updating his/her reference membership values and/or degree α.

Chapter 5 develops new interactive computer programs on the basis of the methods proposed in Chapters 2,3 and 4 to facilitate the interaction processes. Moreover, in order to demonstrate the feasibility and efficiency of both the proposed algorithms and the corresponding computer programs, interaction processes for several numerical examples for multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters under the hypothetical decision maker are shown together with the corresponding computer outputs.

The Appendix presents generalized scalarizing methods for multiobjective programming problems, called the hyperplane methods, by putting the special emphasis not only on generating Pareto optimal solutions but also on obtaining trade-off information. The results presented in the Appendix are the theoretical basis for the trade-off information used in Chapters 2,3 and 4.

CHAPTER 2

INTERACTIVE DECISION MAKING FOR MULTIOBJECTIVE LINEAR PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

2.1 Introduction

In multiobjective programming problems, multiple objectives are usually noncommensurable and cannot be combined into a single objective. Moreover, the objectives often conflict with each other in that any improvement of one objective can be achieved only at the expense of another. For most such multiobjective programming problems, in addition to the decision analyst's role, value-judgement-based analysis of a decision maker plays an essential role. To be more explicit, it is particularly important how to combine the roles of a decision maker and a decision analyst in order to find a compromise or satisficing solution of a decision maker which is also Pareto optimal. However, when formulating the multiobjective programming problem which closely describes and represents the real-world decision situation, various factors of the real-world system should be reflected in the description of the objective functions and the constraints. Naturally these objective functions and the constraints involve many parameters whose possible values may be assigned by the experts. In the conventional approaches, such parameters are fixed at some values in an experimental and/or subjective manner through the experts' understanding of the nature of the parameters.

In most real-world situations, however, it may be reasonable to assume that the possible values of these parameters are often only imprecisely or ambiguously known to the experts. In this case, it would certainly be more appropriate to interpret the experts' understanding of the parameters as fuzzy numerical data which can be represented by means of fuzzy subsets of the real line known as fuzzy numbers (Dubois and Prade 1978,1980). The resulting multiobjective programming problem involving fuzzy parameters would be viewed as the more realistic version of the conventional one.

Recently, Tanaka and Asai (1981,1984) formulated the multiobjective linear programming problems with fuzzy parameters. Following the fuzzy decision or the minimum operator proposed by Bellman and Zadeh (1970) together with triangular membership functions for fuzzy parameters, they considered two types of fuzzy multiobjective linear programming problems; one is to decide the nonfuzzy solution and the other is to decide the fuzzy solution.

However, it should be emphasized here that their approaches are preferable only when the decision maker feels that the minimum-operator is appropriate. In other words, in general decision situations, human decision makers do not always use the minimum-operator when they combine the fuzzy goals and/or constraints. Probably the most crucial problem is the identification of an appropriate aggregation function which well represents the human decision makers' fuzzy preferences. If the appropriate aggregation function can be explicitly identified, then the problem reduces to a standard mathematical programming problems.

However, this rarely happens and as an alternative, it becomes evident that an interaction with the decision maker is necessary.

In this chapter, we focus on the multiobjective linear programming problems with fuzzy parameters characterized by fuzzy numbers. To cope with the fuzzy parameters of the experts' together with the fuzzy goals of the decision maker, the concept of M-α-Pareto optimality is introduced by extending the ordinary Pareto optimality concept. Then a new interactive decision making method to derive the satisficing solution of the decision maker efficiently from among an M-α-Pareto optimal solution set is presented on the basis of the linear programming method as a generalization of the results for multiobjective linear programming problems without fuzzy parameters by Sakawa (1983a,1983b).

2.2 Problem Statement and Solution Concept

Consider multiobjective linear programming (MOLP) problems of the following form:

$$\begin{aligned} &\min \quad (c_1 x , c_2 x , \dots , c_k x) \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.1) \\ &x \in X = \{ x \in E^n \mid a_j x \leq b_j ,\ j=1,\dots,m ;\ x \geq 0 \}, \end{aligned}$$

where x is an n-dimensional column vector of decision variables, c_1,

$c_2,\ldots,c_k$ are n-dimensional cost factor row vectors, a_1, $a_2,\ldots,$ a_m are n-dimensional constraint row vectors and b_1, $b_2,\ldots,$ b_m are constants.

Fundamental to the MOLP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MOLP is one where any improvement of one objective function can be achieved only at the expense of another.

Definition 2.1 (Pareto optimal solution)

$x^* \in X$ is said to be a Pareto optimal solution to the MOLP, if and only if there does not exist another $x \in X$ such that $c_i x \leq c_i x^*$, $i=1,\ldots,k$ with strict inequality holding for at least one i.

In practice, however, it would certainly be more appropriate to consider that the possible values of the parameters in the description of the objective functions and the constraints usually involve the ambiguity of the experts' understanding of the real system. For this reason, in this chapter, we consider the following multiobjective linear programming problem involving fuzzy parameters (MOLP-FP) :

$$\begin{aligned}
&\min \quad (\tilde{c}_1 x\ ,\ \tilde{c}_2 x\ ,\ldots,\ \tilde{c}_k x\ ) \\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.2) \\
&x \in X(\tilde{a},\tilde{b}) \triangleq \{\ x \in E^n \mid \tilde{a}_j x \leq \tilde{b}_j\ , j=1,\ldots,m\ ;\ x \geq 0\ \}.
\end{aligned}$$

Here $\tilde{c}_i = (\tilde{c}_{i1}, \ldots, \tilde{c}_{in})$, and $\tilde{a}_j = (\tilde{a}_{j1}, \ldots, \tilde{a}_{jn})$, $\tilde{b}_j$ represent respectively fuzzy parameters involved in the objective function $\tilde{c}_i x$ and the constraint $\tilde{a}_j x \leq \tilde{b}_j$.

These fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced by Dubois and Prade (1978,1980). It is appropriate to review here that a real fuzzy number $\tilde{p}$ is a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{p}}(p)$ is defined as:

(1) A continuous mapping from E^n to the closed interval $[0,1]$,

(2) $\mu_{\tilde{p}}(p) = 0$ for all $p \in (-\infty, p_1]$,

(3) Strictly increasing and continuously differentiable on (p_1, p_2),

(4) $\mu_{\tilde{p}}(p) = 1$ for all $p \in [p_2, p_3]$,

(5) Strictly decreasing and continuously differentiable on (p_3, p_4),

(6) $\mu_{\tilde{p}}(p) = 0$ for all $p \in [p_4, +\infty)$.

Fig. 2.1 illustrates the graph of the possible shape of the fuzzy number $\tilde{p}$.

We now assume that $\tilde{c}_{i1}, \ldots, \tilde{c}_{in}$, $\tilde{a}_{j1}, \ldots, \tilde{a}_{jn}$ and $\tilde{b}_j$ in the MOLP-FP are fuzzy numbers whose membership functions are $\mu_{\tilde{c}_{i1}}(c_{i1}), \ldots, \mu_{\tilde{c}_{in}}(c_{in})$, $\mu_{\tilde{a}_{j1}}(a_{j1}), \ldots, \mu_{\tilde{a}_{jn}}(a_{jn})$ and $\mu_{\tilde{b}_j}(b_j)$ respectively. For simplicity in the notation, define the following vectors:

$c = (c_1, \ldots, c_k)$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_k)$, $a = (a_1, \ldots, a_m)$,

$\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_m)$, $b = (b_1, \ldots, b_m)$, $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_m)$.

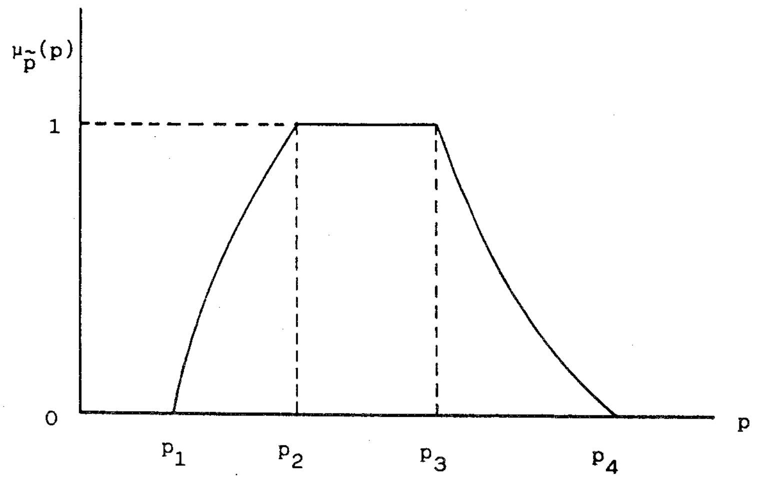

Fig. 2.1 Fuzzy number

Then we can introduce the following α-level set or α-cut (Dubois and Prade 1980) of the fuzzy numbers $\tilde{a}_{jr}$, $\tilde{b}_j$ and $\tilde{c}_{ir}$.

Definition 2.2 (α-level set)

The α-level set of the fuzzy numbers $\tilde{a}_{jr}, \tilde{b}_j$ and $\tilde{c}_{ir}$ is defined as the ordinary set $L_\alpha(\tilde{a}, \tilde{b}, \tilde{c})$ for which the degree of their membership functions exceeds the level α:

$$L_\alpha(\tilde{a}, \tilde{b}, \tilde{c}) = \{(a,b,c) \mid \mu_{\tilde{a}_{jr}}(a_{jr}) \geq \alpha,\ \mu_{\tilde{b}_j}(b_j) \geq \alpha,\ \mu_{\tilde{c}_{ir}}(c_{ir}) \geq \alpha,$$
$$i=1,\ldots,k,\ j=1,\ldots,m,\ r=1,\ldots,n \} . \tag{2.3}$$

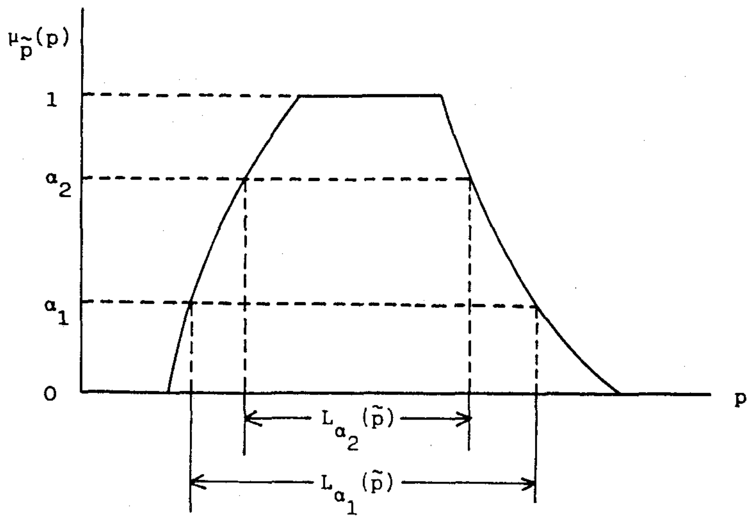

Fig. 2.2 α-level set

The concept of the α-level set is illustrated in Fig. 2.2. As can be seen from Fig. 2.2, it is clear that the level sets have the following property:

$$\alpha_1 \leq \alpha_2 \quad \text{if and only if} \quad L_{\alpha_1}(\tilde{a},\tilde{b},\tilde{c}) \supset L_{\alpha_2}(\tilde{a},\tilde{b},\tilde{c}) \ . \tag{2.4}$$

For a certain degree α, the MOLP-FP (2.2) can be understood as the following nonfuzzy α-multiobjective linear programming (α-MOLP) problem.

$$\min \quad (\ c_1x,\ c_2x\ ,\dots,\ c_kx\ )$$

subject to

$$x \in X(a,b) \triangleq \{\ x \in E^n \mid a_jx \leq b_j\ ,j=1,\dots,m\ ;\ x \geq 0\ \}\ , \tag{2.5}$$

$$(a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c})\ .$$

It should be emphasized here that in the α-MOLP the parameters (a,b,c) are treated as decision variables rather than constants.

On the basis of the α-level sets of the fuzzy numbers, we introduce the concept of α-Pareto optimal solutions to the α-MOLP.

Definition 2.3 (α-Pareto optimal solution)

$x^* \in X(a^*,b^*)$ is said to be an α-Pareto optimal solution to the α-MOLP (2.5), if and only if there does not exist another $x \in X(a,b)$, $(a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c})$ such that $c_i x \leq c_i^* x^*$, $i=1,\dots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters (a^*,b^*,c^*) are called α-level optimal parameters.

2.3 Interactive Decision Making under Fuzziness

2.3.1 Fuzzy Goals

As can be immediately understood from Definition 2.3, in general, α-Pareto optimal solutions to the α-MOLP (2.5) consist of an infinite number of points and some kinds of subjective judgement should be added to the quantitative analyses by the decision maker (DM). Namely, the DM must select his/her compromise or satisficing solution from among α-Pareto optimal solutions based on his/her subjective judgement.

However, considering the imprecise nature of the DM's judgement, it is natural to assume that the DM may have imprecise or fuzzy goals for each of the objective functions in the α-MOLP (2.5). In a minimization problem, a goal stated by the DM may be to achieve "substantially less" than A. This type of statement can be quantified by eliciting a corresponding membership function.

In order to elicit a membership function $\mu_i(c_i x)$ from the DM for each of the objective functions $c_i x$, $i=1,\ldots,k$, in the α-MOLP (2.5), we first calculate the individual minimum and maximum of each objective function under the given constraints for $\alpha = 0$ and $\alpha = 1$. By taking account of the calculated individual minimum and maximum of each objective function for $\alpha = 0$ and $\alpha = 1$ together with the rate of increase of membership of satisfaction, the DM may be able to determine his/her membership function $\mu_i(c_i x)$ in a subjective manner which is a strictly monotone decreasing function with respect to $c_i x$.

So far we have restricted ourselves to a minimization problem and consequently assumed that the DM has a fuzzy goal such as " $c_i x$ should be substantially less than A_i". In the fuzzy approaches, we can further treat a more general case where the DM has two types of fuzzy goals, namely fuzzy goals expressed in words such as " $c_i x$ should be in the vicinity of C_i" (called fuzzy equal) as well as " $c_i x$ should be substantially less than A_i or greater than B_i" (called fuzzy min or fuzzy max). Such a generalized α-MOLP (Gα-MOLP) problem may now be expressed as:

$$\text{fuzzy min} \quad c_i x \quad (i \in I_1)$$

$$\text{fuzzy max} \quad c_i x \quad (i \in I_2)$$

$$
\begin{array}{lll}
\text{fuzzy equal} & c_i x & (i \in I_3) \\
\text{subject to} & x \in X(a,b) & \\
 & (a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c}) &
\end{array}
\tag{2.6}
$$

where $I_1 \cup I_2 \cup I_3 = \{ 1,2,\ldots,k \}$.

In order to elicit a membership function $\mu_i(c_i x)$ from the DM for a fuzzy goal like " $c_i x$ should be in the vicinity of C_i", it is obvious that we can use different functions to the left and right sides of C_i.

Concerning the membership functions of the Gα-MOLP, it is reasonable to assume that $\mu_i(c_i x)$, $i \in I_1$ and the right side functions of $\mu_i(c_i x)$, $i \in I_3$ are strictly monotone decreasing and continuous functions with respect to $c_i x$, and $\mu_i(c_i x)$, $i \in I_2$ and the left side functions of $\mu_i(c_i x)$, $i \in I_3$ are strictly monotone increasing and continuous functions with respect to $c_i x$. To be more explicit, each membership function $\mu_i(c_i x)$ of the Gα-MOLP for $i \in I_1$, $i \in I_2$ or $i \in I_3$ is defined and its possible shape is depicted as follows:

(1) $i \in I_1$:

$$
\mu_i(c_i x) = \begin{cases} 1 \text{ or } \to 1 & \text{if } (c_i x)_R^1 \geq c_i x , \\ D_{iR}(c_i x) & \text{if } (c_i x)_R^1 \leq c_i x \leq (c_i x)_R^0 , \\ 0 \text{ or } \to 0 & \text{if } c_i x \geq (c_i x)_R^0 , \end{cases} \tag{2.7}
$$

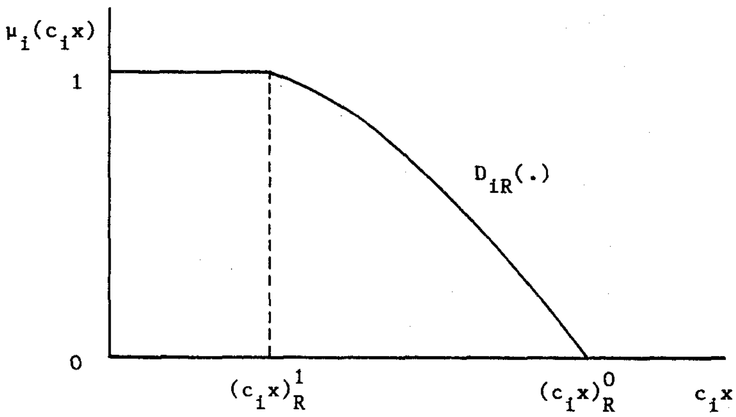

Fig. 2.3 Fuzzy min membership function

(2) $i \in I_2$:

$$\mu_i(c_ix) = \begin{cases} 0 \quad \text{or} \quad \to 0 & \text{if} \quad (c_ix)_L^0 \geq c_ix \ , \\ D_{iL}(\ c_ix\ ) & \text{if} \quad (c_ix)_L^0 \leq c_ix \leq (c_ix)_L^1 \ , \\ 1 \quad \text{or} \quad \to 1 & \text{if} \quad c_ix \ \geq (c_ix)_L^1 \ , \end{cases} \tag{2.8}$$

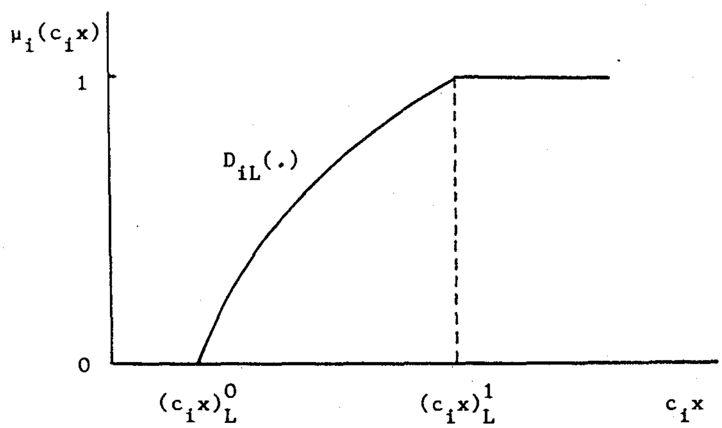

Fig. 2.4 Fuzzy max membership function

(3) $i \in I_3$:

$$\mu_i(c_i x) = \begin{cases} 0 \ \text{ or } \rightarrow 0 & \text{if} \quad c_i x \leq (c_i x)_L^0 , \\ D_{iL}(c_i x) & \text{if} \quad (c_i x)_L^0 \leq c_i x \leq (c_i x)_L^1 , \\ 1 & \text{if} \quad (c_i x)_L^1 \leq c_i x \leq (c_i x)_R^1 , \\ D_{iR}(c_i x) & \text{if} \quad (c_i x)_R^1 \leq c_i x \leq (c_i x)_R^0 , \\ 0 \ \text{ or } \rightarrow 0 & \text{if} \quad (c_i x)_R^0 \leq c_i x . \end{cases} \tag{2.9}$$

Here it is assumed that $D_{iR}(c_i x)$ or $D_{iL}(c_i x)$ is respectively a strictly monotone decreasing or increasing and continuous function with respect to $c_i x$ and may be linear or nonlinear, and $(c_i x)_L^0$ and $(c_i x)_R^0$ are unacceptable levels for $c_i x$ and $(c_i x)_L^1$ and $(c_i x)_R^1$ are totally desirable levels for $c_i x$.

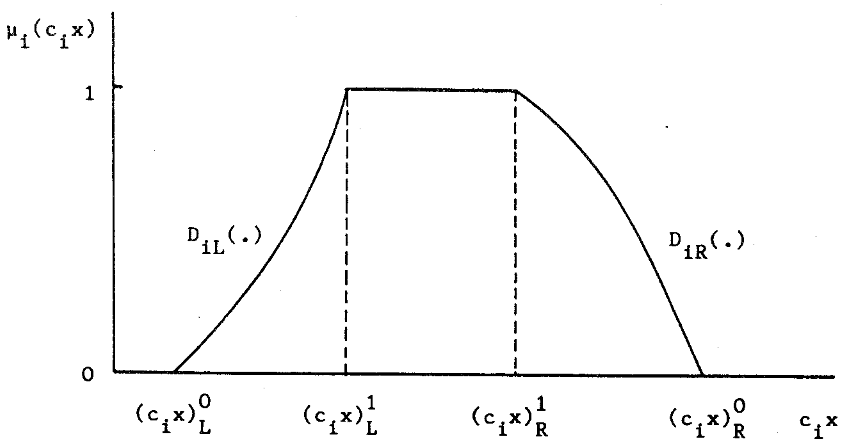

Fig. 2.5 Fuzzy equal membership function

When fuzzy equal is included in the fuzzy goals of the DM, it is desirable that $c_i x$ should be as close to C_i as possible. Consequently, the notion of α-Pareto optimal solutions defined in terms of objective functions cannot be applied. For this reason, we introduce the concept of M-α-Pareto optimal solutions which is defined in terms of membership functions instead of objective functions, where M refers to membership.

Definition 2.4 (M-α-Pareto optimal solution)

$x^* \in X(a^*,b^*)$ is said to be an M-α-Pareto optimal solution to the Gα-MOLP, if and only if there does not exist another $x \in X(a,b)$, $(a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c})$ such that $\mu_i(c_i x) \geq \mu_i(c_i^* x^*), i=1,\ldots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters (a^*,b^*,c^*) are called α-level optimal parameters.

Observe that the concept of M-α-Pareto optimal solutions defined in terms of membership functions is a natural extension of that of α-Pareto optimal solutions defined in terms of objective functions, when fuzzy equal is included in the fuzzy goals of the DM.

Having elicited the membership functions $\mu_i(c_i x), i=1,\ldots,k$ from the DM for each of the objective functions $c_i x$, $i=1,\ldots,k$, if we introduce a general aggregation function

$$\mu_D(\ \mu_1(c_1 x),\ \mu_2(c_2 x),\ldots,\ \mu_k(c_k x),\ \alpha\ ), \tag{2.10}$$

a general fuzzy α-multiobjective decision problem (Fα-MODP) can be defined by:

$$\max \quad \mu_D(\mu_1(c_1x), \mu_2(c_2x), \ldots, \mu_k(c_kx), \alpha), \tag{2.11}$$

subject to

$$(x,a,b,c) \in P(\alpha), \quad \alpha \in [\ 0,\ 1\ ]. \tag{2.12}$$

where $P(\alpha)$ is the set of M-α-Pareto optimal solutions and corresponding α-level optimal parameters to the Gα-MOLP.

Probably the most crucial problem in the Fα-MODP is the identification of an appropriate aggregation function which well represents the human decision makers' fuzzy preferences. If $\mu_D(.)$ can be explicitly identified, then the Fα-MODP reduces to a standard mathematical programming problem. However, this rarely happens and as an alternative, it becomes evident that an interaction with the DM is necessary.

Throughout this section we make the following assumptions.

Assumption 2.1 The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions through the interaction with the DM.

Assumption 2.2 $\mu_D(.)$ exists and is known only implicitly to the DM, which means the DM cannot specify the entire form of $\mu_D(.)$, but he/her can provide local information concerning his/her preference. Moreover, it is strictly increasing and continuous with respect to $\mu_i(.)$ and α.

2.3.2 Minimax Problems

For generating a candidate for the satisficing solution which is also M-α-Pareto optimal in our decision making method, the DM is asked to specify the degree α of the α-level set and the reference levels of achievement of the membership functions, called reference membership values. Observe that the idea of the reference membership values first appeared in Sakawa, Yumine and Yano (1984a,b) can be viewed as an obvious extension of the idea of the reference point of Wierzbicki (1979a). Once the DM's degree α and reference membership values $\bar{\mu}_i$, $i=1,\ldots,k$, are specified, the corresponding M-α-Pareto optimal solution, which is in a sense close to his/her requirement or better than that if the reference levels are attainable, is obtained by solving the following minimax problem.

$$\min_{\substack{x \in X(a,b) \\ (a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c})}} \max_{1 \le i \le k} \left(\bar{\mu}_i - \mu_i(c_i x) \right), \tag{2.13}$$

or equivalently

$$\begin{aligned} &\min \quad v \\ &\text{subject to} \quad \bar{\mu}_i - \mu_i(c_i x) \le v, \quad i=1,\ldots,k, \\ &\qquad a_j x \le b_j, \quad j=1,\ldots,m, \quad x \ge 0, \\ &\qquad (a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c}). \end{aligned} \tag{2.14}$$

Fig. 2.6 illustrates a graphical description of the minimax problem.

However, with the strictly monotone decreasing or increasing membership function given by (2.7)-(2.9), which may be nonlinear, the resulting problem becomes a nonlinear programming problem. In order to solve the formulated problem on the basis of the linear programming method, we first convert each constraint $\bar{\mu}_i - \mu_i(c_i x) \leq v$, $i=1,\ldots,k$, of the minimax problem (2.14) into the following form using the strictly monotone property of $D_{iL}(.)$ and $D_{iR}(.)$.

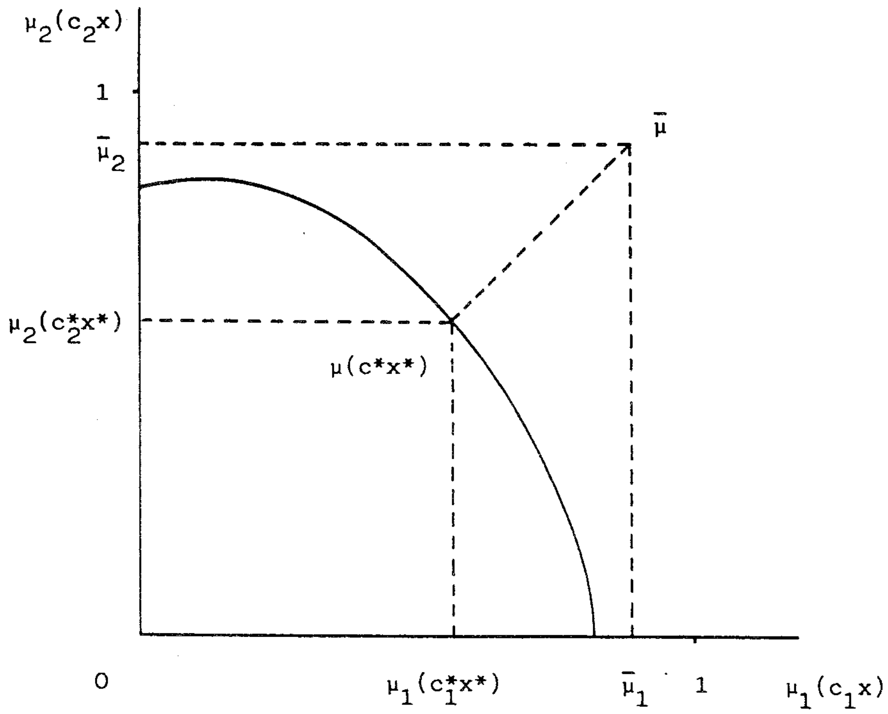

Fig. 2.6 Minimax problem

$$c_i x \leq D_{iR}^{-1}(\bar{\mu}_i - v), \quad i \in I_1 \cup I_3 \tag{2.15}$$

$$c_i x \geq D_{iL}^{-1}(\bar{\mu}_i - v), \quad i \in I_2 \cup I_3 \tag{2.16}$$

Now we introduce the following set-valued functions $S_{iR}(.)$, $S_{iL}(.)$ and $T_j(.,.)$.

$$S_{iR}(c_i) = \{ (x,v) \mid c_i x \leq D_{iR}^{-1}(\bar{\mu}_i - v) \} , \quad i \in I_1 \cup I_3 \qquad (2.17)$$

$$S_{iL}(c_i) = \{ (x,v) \mid c_i x \geq D_{iL}^{-1}(\bar{\mu}_i - v) \} , \quad i \in I_2 \cup I_3 \qquad (2.18)$$

$$T_j(a_j,b_j) = \{ x \mid a_j x \leq b_j \} , \quad j=1,\ldots,k. \qquad (2.19)$$

Then, it can be verified that the following relations hold for $S_{iR}(.)$, $S_{iL}(.)$ and $T_j(.,.)$, when $x \geq 0$.

Proposition 2.1

(1) If $c_i^1 \leq c_i^2$, then $S_{iR}(c_i^1) \supset S_{iR}(c_i^2)$ and $S_{iL}(c^1) \subset S_{iL}(c_i^2)$.

(2) If $a_j^1 \leq a_j^2$, then $T_j(a_j^1,b_j) \supset T_j(a_j^2,b_j)$.

(3) If $b_j^1 \leq b_j^2$, then $T_j(a_j,b_j^1) \subset T_j(a_j,b_j^2)$.

Now from the properties of the α-level sets for the vectors of the fuzzy numbers $\tilde{c}_i$, $\tilde{a}_j$ and the fuzzy numbers $\tilde{b}_j$, it should be noted here that the feasible regions for c_i, a_j and b_j can be denoted respectively by the intervals $[c_{i\alpha}^L, c_{i\alpha}^R]$, $[a_{j\alpha}^L, a_{j\alpha}^R]$ and $[b_{j\alpha}^L, b_{j\alpha}^R]$ as shown in Fig. 2.7.

Consequently, by making use of the results in Proposition 2.1, we can obtain an optimal solution to (2.14) by solving the following problem.

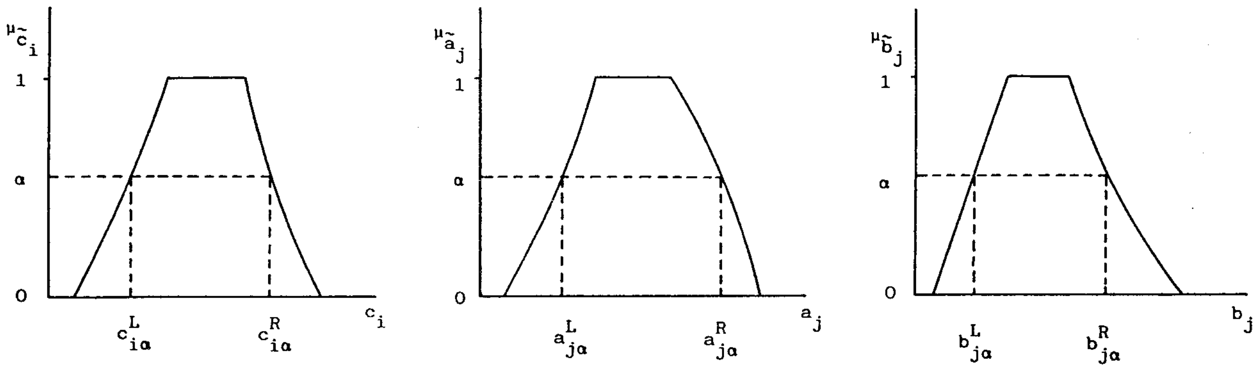

Fig. 2.7 Feasible region for c_i, a_j and b_j

$$
\begin{aligned}
\min \quad & v \\
\text{subject to} \quad & c^L_{i\alpha}x \leq D^{-1}_{iR}(\bar{\mu}_i - v), \quad i \in I_1 \cup I_3, \\
& c^R_{i\alpha}x \geq D^{-1}_{iL}(\bar{\mu}_i - v), \quad i \in I_2 \cup I_3, \\
& a^L_{j\alpha}x \leq b^R_{j\alpha}, \quad j=1,\ldots,m, \quad x \geq 0
\end{aligned}
\tag{2.20}
$$

It is important to note here that in this formulation, if the value of v is fixed, it can be reduced to a set of linear inequalities. Obtaining the optimal solution v* to the above problem is equivalent to determining the minimum value of v so that there exists an admissible set satisfying the constraints of (2.20). Since v satisfies $\bar{\mu}_{max} - 1 \leq v \leq \bar{\mu}_{max}$, where $\bar{\mu}_{max}$ denotes the maximum value of $\bar{\mu}_i$, i=1,...,k, we have the following method for solving this problem by combined use of the bisection method and the simplex method of linear programming.

Step 1. Set $v = \bar{\mu}_{max}$ and test whether an admissible set satisfying the constraints of (2.20) exists or not by making use of phase one of the simplex method. If an admissible set exists, proceed. Otherwise, the DM must reassess his/her membership function.

Step 2. Set $v = \bar{\mu}_{max} - 1$ and test whether an admissible set satisfying the constraints of (2.20) exists or not using phase one of the simplex method. If an admissible set exists, set $v^* = \bar{\mu}_{max} - 1$. Otherwise go to the next step, since the minimum v which satisfies the constraints of (2.20) exists between $\bar{\mu}_{max} - 1$ and $\bar{\mu}_{max}$.

Step 3. For the initial value of $v = \bar{\mu}_{max} - 0.5$, update the value of v using the bisection method as follows :

$v_{n+1} = v_n - 1/2^{n+1}$ if admissible set exists for v_n ,

$v_{n+1} = v_n + 1/2^{n+1}$ if no admissible set exists for v_n .

Namely, for each v_n (n=1,2,...), test whether an admissible set of (2.20) exists or not using the sensitivity analysis technique for the changes in right hand side of the simplex method and determine the minimum value of v satisfying the constraints of (2.20).

In this way, we can determine the optimal solution v^*. Then the DM selects an appropriate standing objective from among the objectives $c_i x$, $i=1,\dots,k$. For notational convenience, in the following, without loss of generality, let it be $c_1 x$ and $1 \in I_1$. Then the following linear programming problem is solved for $v = v^*$.

$$
\begin{aligned}
&\min \quad c^L_{1\alpha}x \\
&\text{subject to} \quad c^L_{i\alpha}x \leq D^{-1}_{iR}(\bar{\mu}_i - v^*), \quad i \in I_1 \cup I_3, \\
&\qquad\qquad\quad c^R_{i\alpha}x \geq D^{-1}_{iL}(\bar{\mu}_i - v^*), \quad i \in I_2 \cup I_3, \\
&\qquad\qquad\quad a^L_{j\alpha}x \leq b^R_{j\alpha}, \qquad j=1,\dots,m, \quad x \geq 0 .
\end{aligned}
\tag{2.21}
$$

For convenience in our subsequent discussion, we assume that the optimal solution x^* to (2.21) satisfies the following conditions:

$$c^L_{i\alpha}x^* = D^{-1}_{iR}(\bar{\mu}_i - v^*), \quad i \in I_1 \cup I_{3R},$$

$$c^R_{i\alpha}x^* = D^{-1}_{iL}(\bar{\mu}_i - v^*), \quad i \in I_2 \cup I_{3L}.$$

where $I_3 = I_{3L} \cup I_{3R}$ and $I_{3L} \cap I_{3R} = \Phi$.

Then it is interesting to note that $c^L_{i\alpha}(i \in I_1 \cup I_{3R})$, $c^R_{i\alpha}(i \in I_2 \cup I_{3L})$ $a^L_{j\alpha}$ and $b^R_{j\alpha}$ $(j=1,\dots,m)$ are α-level optimal parameters for any M-α-Pareto optimal solution.

The relationships between the optimal solutions to (2.20) and the M-α-Pareto optimal concept of the Gα-MOLP can be characterized by the following theorems.

Theorem 2.1

If x* is a unique optimal solution to (2.20), then x* is an M-α-Pareto optimal solution to the Gα-MOLP.

(Proof)

Assume that x* is not an M-α-Pareto optimal solution to the Gα-MOLP. Then, since $c^L_{i\alpha}(i \in I_1 \cup I_{3R})$, $c^R_{i\alpha}(i \in I_2 \cup I_{3L})$ and $a^L_{j\alpha}$, $b^R_{j\alpha}$ $(j=1,\ldots,m)$ are α-level optimal parameters to the Gα-MOLP, there exist $x \in X(a,b)$ and $(a,b,c) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c})$ such that $\mu_i(c_i x) \geq \mu_i(c^L_{i\alpha}x*)$, $(i \in I_1 \cup I_{3R})$, $\mu_i(c_i x) \geq \mu_i(c^R_{i\alpha}x*)$, $(i \in I_2 \cup I_{3L})$, strict inequality holding for at least one i. Then it holds that

$$\max_{i \in I_1 \cup I_{3R}} (\bar{\mu}_i - \mu_i(c^L_{i\alpha}x*)) \geq \max_{i \in I_1 \cup I_{3R}} (\bar{\mu}_i - \mu_i(c_i x))$$

$$\geq \max_{i \in I_1 \cup I_{3R}} (\bar{\mu}_i - \mu_i(c^L_{i\alpha}x)),$$

$$\max_{i \in I_2 \cup I_{3L}} (\bar{\mu}_i - \mu_i(c^R_{i\alpha}x*)) \geq \max_{i \in I_2 \cup I_{3L}} (\bar{\mu}_i - \mu_i(c_i x))$$

$$\geq \max_{i \in I_2 \cup I_{3L}} (\bar{\mu}_i - \mu_i(c^R_{i\alpha}x)),$$

which contradicts the fact that x* is a unique optimal solution to (2.20). Q.E.D.

Theorem 2.2

If x* is an M-α-Pareto optimal solution to the Gα-MOLP, then x* is an optimal solution to (2.20) for some $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)$.

(Proof)

Assume that (x^*, v^*) is not an optimal solution to (2.20) for any $\bar{\mu}$ satisfying

$$\bar{\mu}_i - \mu_i(c^L_{i\alpha}x^*) = v^*, \quad i \in I_1 \cup I_{3R}$$

$$\bar{\mu}_i - \mu_i(c^R_{i\alpha}x^*) = v^*, \quad i \in I_2 \cup I_{3L}$$

Then there exists $x \in X(a,b)$ such that

$$\bar{\mu}_i - \mu_i(c^L_{i\alpha}x) < \bar{\mu}_i - \mu_i(c^L_{i\alpha}x^*), \quad i \in I_1 \cup I_{3R}$$

$$\bar{\mu}_i - \mu_i(c^R_{i\alpha}x) < \bar{\mu}_i - \mu_i(c^R_{i\alpha}x^*), \quad i \in I_2 \cup I_{3L}$$

This implies that $\mu_i(c^L_{i\alpha}x) > \mu_i(c^L_{i\alpha}x^*)$, $(i \in I_1 \cup I_{3R})$, $\mu_i(c^R_{i\alpha}x) > \mu_i(c^R_{i\alpha}x^*)$, $(i \in I_2 \cup I_{3L})$, which contradicts the fact that x^* is an M-α-Pareto optimal solution to the Gα-MOLP. Q.E.D.

It must be observed here that for generating M-α-Pareto optimal solutions using Theorem 2.1, uniqueness of solution must be verified. In the ad hoc numerical approach, however, in order to test the M-α-Pareto optimality of a current optimal solution x^*, we formulate and solve the following linear programming problem:

$$\max \sum_{i=1}^{k} \epsilon_i$$

subject to

$$c^L_{i\alpha}x + \epsilon_i = c^L_{i\alpha}x^*, \quad \epsilon_i \geq 0, \quad i \in I_1 \cup I_{3R} \qquad (2.22)$$

$$c^R_{i\alpha}x - \epsilon_i = c^R_{i\alpha}x^* , \quad \epsilon_i \geq 0 , \quad i \in I_2 \cup I_{3L}$$

$$a^L_{j\alpha}x \leq b^R_{j\alpha} , \quad j = 1,\ldots,m , \quad x \geq 0 .$$

Let $\bar{x}$ and $\bar{\epsilon}$ be an optimal solution to (2.22). If all $\bar{\epsilon}_i = 0$, then x^* is an M-α-Pareto optimal solution. If at least one $\bar{\epsilon}_i > 0$, it can be easily shown that $\bar{x}$ is an M-α-Pareto optimal solution.

2.3.3 Interactive Algorithm

Now given the M-α-Pareto optimal solution for the degree α and the reference membership values specified by the DM by solving the corresponding minimax problem, the DM must either be satisfied with the current M-α-Pareto optimal solution and α, or update his/her reference membership values and/or the degree α . In order to help the DM express his/her degree of preference, trade-off information between a standing membership function and each of the other membership functions as well as between the degree α and the membership functions is very useful. Such a trade-off information is easily obtainable since it is closely related to the simplex multipliers of the problem (2.21).

To derive the trade-off information, define the following Lagrangian function L corresponding to the problem (2.21).

$$L = c^L_{1\alpha}x + \sum_{i \in I_1 \cup I_{3R}} \pi_{iR}\{c^L_{i\alpha}x - D^{-1}_{iR}(\bar{\mu}_i - v^*)\}$$

$$+ \sum_{i \in I_2 \cup I_{3L}} \pi_{iL}\{D_{iL}^{-1}(\bar{\mu}_i - v^*) - c_{i\alpha}^R x\} + \sum_{j=1}^{m} \lambda_j (a_{j\alpha}^L x - b_{j\alpha}^R) \tag{2.23}$$

where π_{iL}, π_{iR} and λ_j are simplex multipliers corresponding to the constraints of (2.21).

Here, we assume that the problem (2.21) has a unique and nondegenerate optimal solution satisfying the following conditions

(1) $\pi_{iR} > 0$, $i \in I_1 \cup I_{3R}$, $i \neq 1$

(2) $\pi_{iL} > 0$, $i \in I_2 \cup I_{3L}$.

Then, by using the results in Haimes and Chankong (1979), the following expression holds .

$$-\frac{\partial(c_{1\alpha}^L x)}{\partial(c_{i\alpha}^L x)} = \pi_{iR}, \quad i \in I_1 \cup I_{3R}, \ i \neq 1 \tag{2.24}$$

$$-\frac{\partial(c_{1\alpha}^L x)}{\partial(c_{i\alpha}^R x)} = -\pi_{iL}, \quad i \in I_2 \cup I_{3L} \tag{2.25}$$

Furthermore, using the strictly monotone decreasing or increasing property of $D_{iR}(.)$ or $D_{iL}(.)$ together with the chain rule, if $D_{iR}(.)$ and $D_{iL}(.)$ are differentiable at the optimal solution to (2.21), it holds that

$$-\frac{\partial\mu_1(c_{1\alpha}^L x)}{\partial\mu_i(c_{i\alpha}^L x)} = \frac{D'_{1R}(c_{1\alpha}^L x)}{D'_{iR}(c_{i\alpha}^L x)}\pi_{iR}, \quad i \in I_1 \cup I_{3R}, \ i \neq 1 \tag{2.26}$$

$$-\frac{\partial\mu_1(c_{1\alpha}^L x)}{\partial\mu_i(c_{i\alpha}^R x)} = -\frac{D'_{1R}(c_{1\alpha}^L x)}{D'_{iL}(c_{i\alpha}^R x)}\pi_{iL}, \quad i \in I_2 \cup I_{3L} \tag{2.27}$$

where $D'_{iR}(.)$ and $D'_{iL}(.)$ denote the differential coefficients of $D_{iR}(.)$ and $D_{iL}(.)$ respectively.

Regarding a trade-off rate between $\mu_1(c^L_{1\alpha}x)$ and α, the following relation holds based on the sensitivity theorem (for details, see, e.g., Luenburger 1973 or Fiacco, 1983).

$$\frac{\partial \mu_1(c^L_{1\alpha}x)}{\partial \alpha} = D'_{1R}(c^L_{1\alpha}x)\left\{\frac{\partial(c^L_{1\alpha})}{\partial\alpha}x + \sum_{i\in I_1 \cup I_{3R}} \pi_{iR}\frac{\partial(c^L_{i\alpha})}{\partial\alpha}x \right.$$
$$\left. - \sum_{i\in I_2 \cup I_{3L}} \pi_{iL}\frac{\partial(c^R_{i\alpha})}{\partial\alpha}x + \sum_{j=1}^{m}\lambda_j\left\{\frac{\partial(a^L_{j\alpha})}{\partial\alpha}x - \frac{\partial(b^R_{j\alpha})}{\partial\alpha}\right\}\right\} \tag{2.28}$$

It should be noted here that in order to obtain the trade-off rate information from (2.26)-(2.27), all the constraints of the problem (2.21) must be active for the current optimal solution. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_i$ for inactive constraints by $D_{iR}(c^L_{i\alpha}x^*) + v^*$ or $D_{iL}(c^R_{i\alpha}x^*) + v^*$ and solve the corresponding problem (2.21) for obtaining the simplex multipliers.

Now, following the above discussions, we can present the interactive algorithm in order to derive the satisficing solution for the DM from among the M-α-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

<u>Step 0.</u> Calculate the individual minimum and maximum of each objective function under given constraints for $\alpha = 0$ and $\alpha = 1$.

Step 1*. Elicit a membership function $\mu_i(c_i x)$ from the DM for each of the objective functions.

Step 2*. Ask the DM to select the initial value of α $(0 \leq \alpha \leq 1)$ and set the initial reference membership values $\bar{\mu}_i = 1$, $i=1,\ldots,k$.

Step 3. For the degree α and the reference membership values specified by the DM, solve the minimax problem and perform the M-α-Pareto optimality test.

Step 4*. The DM is supplied with the corresponding M-α-Pareto optimal solution and the trade-off rates between the membership functions and the degree α. If the DM is satisfied with the current membership function values of the M-α-Pareto optimal solution and α, stop. Otherwise, the DM must update his/her reference membership values and/or the degree α by considering the current values of the membership functions and α together with the trade-off rates between the membership functions and the degree α, and return to step 3.

Here it should be stressed for the DM that (1) any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions for some fixed degree α, and (2) the greater value of the degree α gives worse values of the membership functions for some fixed reference membership values.

2.3.4 Numerical Example

To clarify the concept of M-α-Pareto optimality as well as the proposed method, consider the following three objective linear programming problem with fuzzy parameters.

$$
\begin{array}{lll}
\text{fuzzy min} & z_1(x,\tilde{c}_1) \triangleq 2x_1 + \tilde{c}_{12}x_2 & \\
\text{fuzzy max} & z_2(x,\tilde{c}_2) \triangleq 3x_1 - \tilde{c}_{22}x_2 & \\
\text{fuzzy equal} & z_3(x,\tilde{c}_3) \triangleq \tilde{c}_{31}x_1 - x_2 & \qquad (2.29) \\
\text{subject to} & x \in X \triangleq \{ (x_1,x_2) \mid 3x_1 + x_2 - 12 \leq 0, & \\
 & \quad x_1 + 2x_2 - 12 \leq 0,\ x_i \geq 0,\ i=1,2 \} &
\end{array}
$$

where $\tilde{c}_{12}$, $\tilde{c}_{22}$, and $\tilde{c}_{31}$ are fuzzy numbers whose membership functions are given below:

$$
\begin{aligned}
\mu_{\tilde{c}_{12}}(c_{12}) &= \max(1 - 0.5\,|c_{12} - 4|,\ 0), \\
\mu_{\tilde{c}_{22}}(c_{22}) &= \max(1 - 2\,|c_{22} + 0.75|,\ 0), \qquad (2.30) \\
\mu_{\tilde{c}_{31}}(c_{31}) &= \max(1 - |c_{31} - 2.5|,\ 0).
\end{aligned}
$$

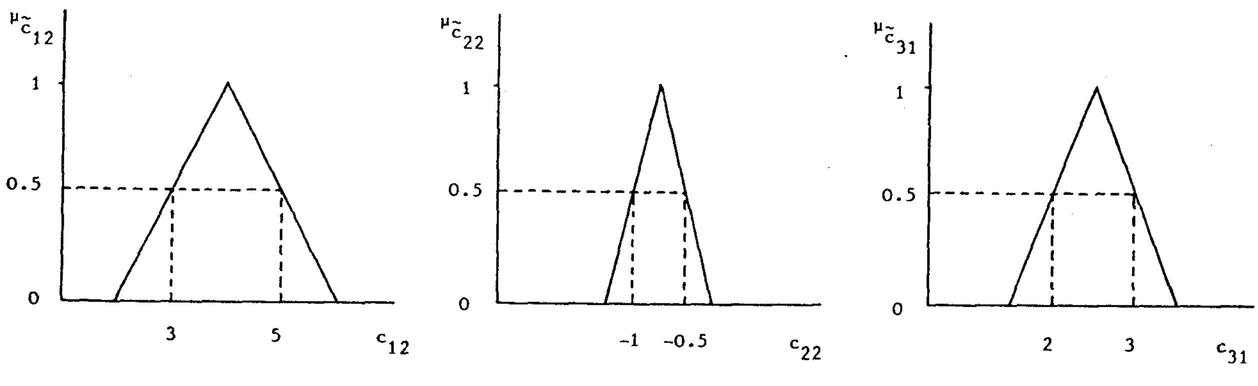

Fig. 2.8 Fuzzy numbers $\tilde{c}_{12}$, $\tilde{c}_{22}$, $\tilde{c}_{31}$

Now, for illustrative purposes, suppose that the interaction with the hypothetical DM establishes the following simple linear membership functions for the three objective functions.

$$\mu_1(z_1) = \begin{cases} 1 & z_1 \leq 5, \\ D_{1R}(z_1) = (20 - z_1)/15, & 5 \leq z_1 \leq 20, \\ 0 & 20 \leq z_1, \end{cases}$$

$$\mu_2(z_2) = \begin{cases} 0 & z_2 \leq 3, \\ D_{2L}(z_2) = (z_2 - 3)/9, & 3 \leq z_2 \leq 12, \\ 1 & 12 \leq z_2, \end{cases}$$

$$\mu_3(z_3) = \begin{cases} 0 & z_3 \leq -3, \\ D_{3L}(z_3) = (z_3 + 3)/3, & -3 \leq z_3 \leq 0, \\ 1 & z_3 = 0, \\ D_{3R}(z_3) = (6 - z_3)/6, & 0 \leq z_3 \leq 6, \\ 0 & 6 \leq z_3. \end{cases}$$

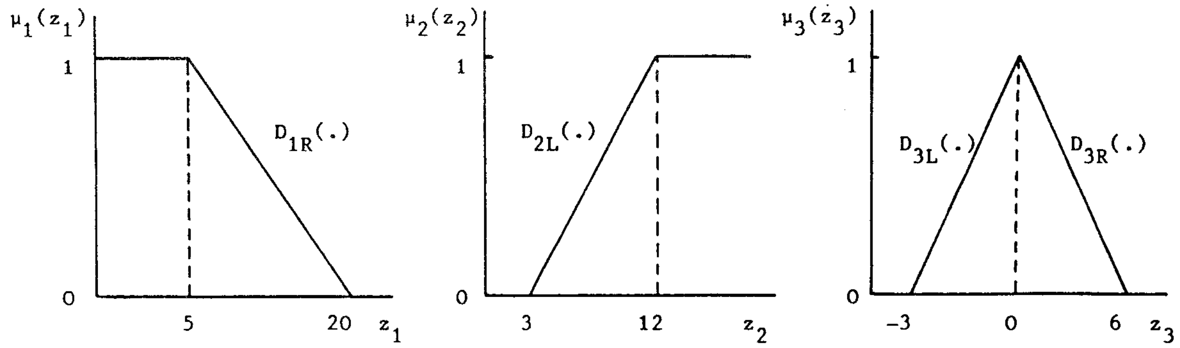

Fig. 2.9 Linear membership functions representing fuzzy goals

Fig. 2.9 illustrates a graphical description of the hypothetical DM's linear membership functions representing his/her fuzzy goals for each of the objective functions of (2.29).

Also assume that the hypothetical DM selects the initial value of the degree α to be 0.5, and the initial reference membership values $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$ to be (19/30, 5/6, 1). Then the corresponding M-α-Pareto optimal solution can be obtained by solving the following problem.

$$\min_{x \in X} \quad v$$

subject to

$$2x_1 + 3x_2 \leq D_{1R}^{-1}(\bar{\mu}_1 - v)$$

$$3x_1 + x_2 \geq D_{2L}^{-1}(\bar{\mu}_2 - v)$$

$$2x_1 - x_2 \leq D_{3R}^{-1}(\bar{\mu}_3 - v)$$

$$3x_1 - x_2 \geq D_{3L}^{-1}(\bar{\mu}_3 - v)$$

Solving this problem by combined use of the bisection method and the simplex method of linear programming, we obtain the optimal solution $v^* = 1/6$. In order to obtain the corresponding optimal values of the decision variable x^*, we solve the following linear programming problem for $v^* = 1/6$.

$$\min_{x \in X} \quad 2x_1 + 3x_2$$

subject to

$$3x_1 + x_2 \geq D_{2L}^{-1}(\bar{\mu}_2 - v^*)$$

$$2x_1 - x_2 \leq D_{3R}^{-1}(\bar{\mu}_3 - v^*)$$

$$3x_1 - x_2 \geq D_{3L}^{-1}(\bar{\mu}_3 - v^*)$$

As a result, we get the following optimal values for v^*, x^*, $z_1^* \triangleq z_1(x^*, c_{1\alpha}^L)$, $z_2^* \triangleq z_2(x^*, c_{2\alpha}^R)$, $z_3^* \triangleq z_3(x^*, c_{3\alpha}^L)$, $\mu_i^* \triangleq \mu_i(z_i^*)$, $i=1,\ldots,k$, and the simplex multipliers $(\pi_{2L}^*, \pi_{3L}^*, \pi_{3R}^*)$.

$$(x_1^*, x_2^*) = (2, 3), \ (z_1^*, z_2^*, z_3^*) = (13, 9, 1),$$

$$(\mu_1^*, \mu_2^*, \mu_3^*) = (7/15, 2/3, 5/6), \ (\pi_{2L}^*, \pi_{3L}^*, \pi_{3R}^*) = (8/5, 0, 7/5).$$

From (2.26) and (2.27), the trade-off rates among the membership functions become as follows:

$$-\frac{\partial\mu_1(z_1^*)}{\partial\mu_2(z_2^*)} = -\frac{D_{1R}'(z_1^*)}{D_{2L}'(z_2^*)}\pi_{2L}^* = -\frac{(-1/15)}{(1/9)}\frac{8}{5} = \frac{72}{75},$$

$$-\frac{\partial\mu_1(z_1^*)}{\partial\mu_3(z_3^*)} = -\frac{D_{1R}'(z_1^*)}{D_{3L}'(z_3^*)}\pi_{3L}^* = -\frac{(-1/15)}{(-1/6)}\frac{7}{5} = \frac{42}{75},$$

Concerning the trade-off rate between $\mu_1(z_1)$ and α, from (2.28) we have:

$$\begin{aligned}\frac{\partial\mu_1(z_1^*)}{\partial\alpha} &= D_{1R}'(z_1^*)\left\{\frac{\partial c_{1\alpha}^L}{\partial\alpha}x_2^* + \pi_{3R}^*\frac{\partial c_{3\alpha}^L}{\partial\alpha}x_1^* - \pi_{2L}^*\frac{\partial c_{2\alpha}^R}{\partial\alpha}x_2^*\right\}\\ &= (-1/15)\{(1/2)\times 3 + (7/5)\times 1\times 2 - (8/5)\times(-2)\times 3\}\\ &= -139/150.\end{aligned}$$

Observe that the DM can obtain his/her satisficing solution from among an M-α-Pareto optimal solution set by updating his/her reference membership values and/or the degree α on the basis of the current values of the membership functions and α together with the trade-off rates among the values of the membership functions and the degree α.

2.4 Conclusion

In this chapter, we have proposed a new interactive decision making method for multiobjective linear programming problems with fuzzy parameters to cope with the imprecise nature of human judgements. As the conclusions of this chapter, the desirable features of our proposed method will be summarized as follows.

(1) The experts' ambiguous understanding of the nature of the parameters in the problem-formulation process can be incorporated.

(2) The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions, which may be nonlinear.

(3) For the degree α and the reference membership values specified by the DM, the corresponding M-α-Pareto optimal solution can be easily obtained by solving the minimax problems based mainly on the well known linear programming method.

(4) M-α-Pareto optimality of the generated solution in each iteration is guaranteed by performing the M-α-Pareto optimality test.

(5) The trade-off information between the membership functions and the degree α is easily obtainable, since it is closely related to the simplex multipliers of the minimax problems.

(6) The satisficing solution of the DM can be derived efficiently from among M-α-Pareto optimal solutions by updating his/her reference membership values and/or the degree α based on the current values of the M-α-Pareto optimal solution together with the trade-off information between the membership functions and the degree α.

In the next chapter, we further proceed to the multiobjective linear fractional programming problems with fuzzy parameters as a slightly generalized version of this chapter.

CHAPTER 3

INTERACTIVE DECISION MAKING FOR MULTIOBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

3.1 Introduction

As indicated in Kornbluth and Steuer (1981b) linear fractional objectives (i.e., ratio objectives that have linear numerator and denominator) are useful in production planning, financial and corporate planning, health care and hospital planning and so forth. Examples of fractional objectives in production planning include inventory/sales, output/employee, etc. However, for single objective linear fractional programming, the Charnes and Cooper (1962) transformation can be used to transform the problem into a linear programming problem.

Concerning multiobjective linear fractional programming (MOLFP) few approaches have appeared in the literature (Choo and Atkins 1980; Kornbluth and Steuer 1981a,1981b; Luhandjula 1984; Sakawa and Yumine 1983). Kornbluth and Steuer (1981a,1981b) present two different approaches to MOLFP; one is the simplex-based approach and the other is the goal programming approach. Choo and Atkins (1980) proposed an interactive approach to MOLFP based on the weighted Tchebycheff norm. Luhandjula (1984) presents a linguistic approach to MOLFP by introducing linguistic variables to represent linguistic aspirations of the decision maker(DM). Sakawa and Yumine (1983) presented a fuzzy approach for solving MOLFP by combined use of the bisection method and the linear programming method together with five types of membership functions ; linear, exponential, hyperbolic, hyperbolic inverse and piecewise linear functions. Recently, Sakawa, Yano and Yumine (1986) have presented a new interactive fuzzy satisficing method by combined use of the bisection method and the linear programming method to derive the satisficing solution for the DM efficiently from among a Pareto optimal solution set by updating his/her reference values for each of the membership functions, called the reference membership values, as a generalization of the result in Sakawa and Yumine (1983).

In this chapter, we further focus on multiobjective linear fractional programming problems with fuzzy parameters, which reflect the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process. Then by considering the imprecise nature of the DM, we present a new interactive decision making method for obtaining the satisficing solution of the DM on the basis of the linear programming method as a generalization of the method of Sakawa, Yano and Yumine (1986) for MOLFP.

3.2 Problem Statement and Solution Concept

Consider multiobjective linear fractional programming (MOLFP) problems of the following form:

$$\min \quad z_1(x) = p_1(x)/q_1(x)$$
$$\min \quad z_2(x) = p_2(x)/q_2(x)$$
$$\cdots\cdots\cdots\cdots \tag{3.1}$$
$$\min \quad z_k(x) = p_k(x)/q_k(x)$$

subject to $x \in X = \{ x \in E^n \mid a_j x \leq b_j,\ j=1,\ldots,k \ ;\ x \geq 0 \}$

where x is an n-dimensional column vector of decision variables, a_j is an n-dimensional constraint row vector, b_j is a constant, $z_1(x),\ldots,z_k(x)$ are k distinct linear fractional objective functions and

$$p_i(x) = c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n + c_{i,n+1}, \tag{3.2}$$
$$q_i(x) = d_{i1}x_1 + d_{i2}x_2 + \ldots + d_{in}x_n + d_{i,n+1}.$$

Here it is customary to assume that the $q_i(x) > 0$ for all $x \in X$.

Fundamental to the MOLFP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MOLFP is one where any improvement of one objective function can be achieved only at the expense of another.

Definition 3.1 (Pareto optimal solution)

$x^* \in X$ is said to be a Pareto optimal solution to the MOLFP, if and only if there does not exist another $x \in X$ such that $z_i(x) \leq z_i(x^*)$, $i=1,\ldots,k$ with strict inequality holding for at least one i.

In practice, however, it would certainly be appropriate to consider that the possible values of the parameters in the description of the objective functions and the constraints usually involve the ambiguity of the experts' understanding of the real system. For this reason, in this chapter, we consider the following multiobjective linear fractional programming problem involving fuzzy parameters (MOLFP-FP):

$$\min \quad z(x,\tilde{c},\tilde{d}) \triangleq (z_1(x,\tilde{c}_1,\tilde{d}_1),\ z_2(x,\tilde{c}_2,\tilde{d}_2),\ldots,\ z_k(x,\tilde{c}_k,\tilde{d}_k)) \tag{3.3}$$

$$\text{subject to} \quad x \in X(\tilde{a},\tilde{b}) \triangleq \{ x \in E^n \mid \tilde{a}_j x \leq \tilde{b}_j,\ j=1,\ldots,m\ ;\ x \geq 0 \},$$

$$\text{where} \qquad z_i(x,\tilde{c}_i,\tilde{d}_i) = p_i(x,\tilde{c}_i)/q_i(x,\tilde{d}_i), \tag{3.4}$$

$$\text{and} \qquad p_i(x,\tilde{c}_i) = \tilde{c}_{i1}x_1 + \tilde{c}_{i2}x_2 + \ldots + \tilde{c}_{in}x_n + \tilde{c}_{i,n+1},$$

$$q_i(x,\tilde{d}_i) = \tilde{d}_{i1}x_1 + \tilde{d}_{i2}x_2 + \ldots + \tilde{d}_{in}x_n + \tilde{d}_{i,n+1}. \tag{3.5}$$

Here $\tilde{c}_i = (\tilde{c}_{i1},\ldots,\tilde{c}_{in},\tilde{c}_{i,n+1})$, $\tilde{d}_i = (\tilde{d}_{i1},\ldots,\tilde{d}_{in},\tilde{d}_{i,n+1})$ and $\tilde{a}_j = (\tilde{a}_{j1},\ldots,\tilde{a}_{jn})$, $\tilde{b}_j$ represent respectively fuzzy parameters involved in the objective function $z_i(x,\tilde{c}_i,\tilde{d}_i)$ and the constraint $\tilde{a}_j x \leq \tilde{b}_j$.

These fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced by Dubois and Prade (1978,1980).

We now assume that $\tilde{c}_i$, $\tilde{d}_i$ and $\tilde{a}_j$, $\tilde{b}_j$ in the MOLFP-FP are fuzzy numbers whose membership functions are $\mu_{\tilde{c}_i}(c_i)$, $\mu_{\tilde{d}_i}(d_i)$ and $\mu_{\tilde{a}_j}(a_j)$, $\mu_{\tilde{b}_j}(b_j)$ respectively. For simplicity in the notation, define the following vectors:

$$c = (c_1,\ldots,c_k),\quad \tilde{c} = (\tilde{c}_1,\ldots,\tilde{c}_k)\ ,\quad d = (d_1,\ldots,d_k),\quad \tilde{d} = (\tilde{d}_1,\ldots,\tilde{d}_k)\ ,$$

$a = (a_1,\ldots,a_m)$, $\tilde{a} = (\tilde{a}_1,\ldots,\tilde{a}_m)$, $b = (b_1,\ldots,b_m)$, $\tilde{b} = (\tilde{b}_1,\ldots,\tilde{b}_m)$,

$\mu_{\tilde{c}}(c) = (\mu_{\tilde{c}_1}(c_1),\ldots,\mu_{\tilde{c}_k}(c_k))$, $\mu_{\tilde{d}}(d) = (\mu_{\tilde{d}_1}(d_1),\ldots,\mu_{\tilde{d}_k}(d_k))$,

$\mu_{\tilde{a}}(a) = (\mu_{\tilde{a}_1}(a_1),\ldots,\mu_{\tilde{a}_m}(a_m))$, $\mu_{\tilde{b}}(b) = (\mu_{\tilde{b}_1}(b_1),\ldots,\mu_{\tilde{b}_m}(b_m))$,

Then we can introduce the following α-level set or α-cut (Dobois and Prade 1980) of the fuzzy numbers $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$.

Definition 3.2 (α-level set)

The α-level set of the fuzzy numbers $\tilde{a},\tilde{b},\tilde{c}$ and $\tilde{d}$ is defined as the ordinary set $L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ for which the degree of their membership functions exceeds the level α:

$$L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = \{ (a,b,c,d) \mid \mu_{\tilde{a}_{jr}}(a_{jr}) \geq \alpha,\ \mu_{\tilde{b}_j}(b_j) \geq \alpha,\ \mu_{\tilde{c}_{ir}}(c_{ir}) \geq \alpha,$$
$$\mu_{\tilde{d}_{ir}}(d_{ir}) \geq \alpha,\ i=1,\ldots,k,\ j=1,\ldots,m,\ r=1,\ldots,n \}. \qquad (3.6)$$

It is clear that the level sets have the following property:

$$\alpha_1 \leq \alpha_2 \text{ if and only if } L_{\alpha_1}(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) \supset L_{\alpha_2}(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) . \qquad (3.7)$$

For a certain degree α, the MOLFP-FP (3.3) can be understood as the following nonfuzzy α-multiobjective linear fractional programming (α-MOLFP) problem.

$$\min\ z(x,c,d) \triangleq (z_1(x,c_1,d_1),z_2(x,c_2,d_2),\ldots,z_k(x,c_k,d_k))$$

subject to

$$x \in X(a,b) \triangleq \{ x \in E^n \mid a_j x \leq b_j, j=1,\ldots,m \; ; \; x \geq 0 \}, \tag{3.8}$$

$$(a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d}),$$

where $$z_i(x,c_i,d_i) = p_i(x,c_i)/q_i(x,d_i), \tag{3.9}$$

and $$p_i(x,c_i) = c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n + c_{i,n+1}, \tag{3.10}$$

$$q_i(x,d_i) = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n + d_{i,n+1}.$$

It should be emphasized here that in the α-MOLFP the parameters (a,b,c,d) are treated as decision variables rather than constants, and it is customary to assume that the $q_i(x,d_i) > 0$ for all $x \in X(a,b)$. In this chapter, for simplicity, we further assume that the $p_i(x,c_i) > 0$ for all $x \in X(a,b)$.

On the basis of the α-level sets of the fuzzy numbers, we introduce the concept of α-Pareto optimal solutions to the α-MOLFP.

Definition 3.3 (α-Pareto optimal solution)

$x^* \in X(a^*,b^*)$ is said to be an α-Pareto optimal solution to the α-MOLFP (3.8), if and only if there does not exist another $x \in X(a,b)$, $(a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ such that $z_i(x,c_i,d_i) \leq z_i(x^*,c_i^*,d_i^*)$, $i=1,\ldots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters (a^*,b^*,c^*,d^*) are called α-level optimal parameters.

3.3 Interactive Decision Making under Fuzziness

3.3.1 Fuzzy Goals

As can be seen from Definition 3.3, usually, α-Pareto optimal solutions consist of an infinite number of points, and the DM must select his/her compromise or satisficing solution from among α-Pareto optimal solutions based on his/her subjective value-judgement.

However, considering the imprecise nature of the DM's judgement, it is reasonable to assume that the DM may have fuzzy goals for each of the objective functions in the α-MOLFP (3.8). For example, a goal stated by the DM may be to achieve "substantially less" than A. This type of statement can be quantified by eliciting a corresponding membership function.

In order to elicit a membership function $\mu_i(z_i(x,c_i,d_i))$ from the DM for each of the objective functions $z_i(x,c_i,d_i)$, $i=1,\ldots,k$, in the α-MOLFP (3.8), we first calculate the individual minimum and maximum of each objective function under the given constraints for $\alpha = 0$ and $\alpha = 1$. By taking account of the calculated individual minimum and maximum of each objective function for $\alpha = 0$ and $\alpha = 1$ together with the rate of increase of membership of satisfaction, the DM must determine his/her membership function $\mu_i(z_i(x,c_i,d_i))$ in a subjective manner.

It is significant to note here that, in the fuzzy approaches, we can treat two types of fuzzy goals; namely, fuzzy goals expressed in words such as " $z_i(x,c_i,d_i)$ should be in the vicinity of C_i" (called fuzzy equal) as well as " $z_i(x,c_i,d_i)$ should be substantially less than

A_i or greater than B_i" (called fuzzy min or fuzzy max). Such a generalized α-MOLP (Gα-MOLP) problem can be expressed as:

$$\begin{array}{lll} \text{fuzzy min} & z_i(x,c_i,d_i) & (i \in I_1) \\ \text{fuzzy max} & z_i(x,c_i,d_i) & (i \in I_2) \\ \text{fuzzy equal} & z_i(x,c_i,d_i) & (i \in I_3) \\ \text{subject to} & x \in X(a,b) & \\ & (a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) & \end{array} \tag{3.11}$$

where $I_1 \cup I_2 \cup I_3 = \{ 1,2,\ldots,k \}$.

In order to elicit a membership function $\mu_i(z_i(x,c_i,d_i))$ from the DM for a fuzzy goal like " $z_i(x,c_i,d_i)$ should be in the vicinity of C_i", it is obvious that we can use different functions to the left and right sides of C_i.

For the membership functions of the Gα-MOLFP, it is reasonable to assume that $\mu_i(z_i(x,c_i,d_i))$, $i \in I_1$ and the right side functions of $\mu_i(z_i(x,c_i,d_i))$, $i \in I_3$ are strictly monotone decreasing and continuous functions with respect to $z_i(x,c_i,d_i)$, and $\mu_i(z_i(x,c_i,d_i))$, $i \in I_2$ and the left side of $\mu_i(z_i(x,c_i,d_i))$, $i \in I_3$ are strictly monotone increasing and continuous functions with respect to $z_i(x,c_i,d_i)$.

To be more specific, each membership function $\mu_i(z_i(x,c_i,d_i))$ of the Gα-MOLFP for $i \in I_1$, $i \in I_2$ or $i \in I_3$ is defined as follows :

(1) $i \in I_1$:

$$\mu_i(z_i(x,c_i,d_i)) = \begin{cases} 1 \quad \text{or} \to 1 \quad \text{if} \quad z_i(x,c_i,d_i) \leq z_{iR}^1, \\ D_{iR}(z_i(x,c_i,d_i)) \quad \text{if} \quad z_{iR}^1 \leq z_i(x,c_i,d_i) \leq z_{iR}^0, \\ 0 \quad \text{or} \to 0 \quad \text{if} \quad z_{iR}^0 \leq z_i(x,c_i,d_i). \end{cases} \qquad (3.12)$$

(2) $i \in I_2$:

$$\mu_i(z_i(x,c_i,d_i)) = \begin{cases} 0 \quad \text{or} \to 0 \quad \text{if} \quad z_i(x,c_i,d_i) \leq z_{iL}^0, \\ D_{iL}(z_i(x,c_i,d_i)) \quad \text{if} \quad z_{iL}^0 \leq z_i(x,c_i,d_i) \leq z_{iL}^1, \\ 1 \quad \text{or} \to 1 \quad \text{if} \quad z_{iL}^1 \leq z_i(x,c_i,d_i) \end{cases} \qquad (3.13)$$

(3) $i \in I_3$:

$$\mu_i(z_i(x,c_i,d_i)) = \begin{cases} 0 \quad \text{or} \to 0 \quad \text{if} \quad z_i(x,c_i,d_i) \leq z_{iL}^0, \\ D_{iL}(z_i(x,c_i,d_i)), \quad \text{if} \quad z_{iL}^0 \leq z_i(x,c_i,d_i) \leq z_{iL}^1, \\ 1 \quad \text{if} \quad z_{iL}^1 \leq z_i(x,c_i,d_i) \leq z_{iR}^1, \\ D_{iR}(z_i(x,c_i,d_i)) \quad \text{if} \quad z_{iR}^1 \leq z_i(x,c_i,d_i) \leq z_{iR}^0, \\ 0 \quad \text{or} \to 0 \quad \text{if} \quad z_{iR}^0 \leq z_i(x,c_i,d_i). \end{cases} \qquad (3.14)$$

where $D_{iR}(z_i(x,c_i,d_i))$ or $D_{iL}(z_i(x,c_i,d_i))$ is respectively a strictly monotone decreasing or increasing continuous function with respect to $z_i(x,c_i,d_i)$ and may be linear or nonlinear, and z_{iL}^0 and z_{iR}^0 are unacceptable levels for $z_i(x,c_i,d_i)$ and z_{iL}^1 and z_{iR}^1 are totally desirable levels for $z_i(x,c_i,d_i)$.

When fuzzy equal is included in the fuzzy goals of the DM, it is desirable that $z_i(x, c_i, d_i)$ should be as close to C_i as possible. Consequently, the notion of α-Pareto optimal solutions defined in terms of objective functions cannot be applied. For this reason, we introduce the concept of M-α-Pareto optimal solutions which is defined in terms of membership functions instead of objective functions, where M refers to membership.

Definition 3.4 (M-α-Pareto optimal solution)

$x^* \in X(a^*, b^*)$ is said to be a M-α-Pareto optimal solution to the Gα-MOLP, if and only if there does not exist another $x \in X(a,b), (a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ such that $\mu_i(z_i(x,c_i,d_i)) \geq \mu_i(z_i(x^*, c_i^*, d_i^*)), i=1,\ldots,k,$ with strict inequality holding for at least one i, where the corresponding values of parameters (a^*, b^*, c^*, d^*) are called α-level optimal parameters.

After eliciting the membership functions $\mu_i(z_i(x,c_i,d_i))$, $i=1,\ldots,k$ from the DM for each of the objective functions $z_i(x,c_i,d_i)$, $i=1,\ldots,k$, if we introduce a general aggregation function

$$\mu_D(\ \mu_1(z_1(x,c_1,d_1)),\ldots,\ \mu_k(z_k(x,c_k,d_k)),\ \alpha\ ), \tag{3.15}$$

a general fuzzy α-multiobjective decision problem (Fα-MODP) can be defined by:

$$\max\ \mu_D(\ \mu_1(z_1(x,c_1,d_1)),\ldots,\ \mu_k(z_k(x,c_k,d_k)),\ \alpha\ ) \tag{3.16}$$

subject to

$$(x,a,b,c,d) \in P(\alpha), \quad \alpha \in [\,0,\,1\,] \tag{3.17}$$

where $P(\alpha)$ is the set of M-α-Pareto optimal solutions and corresponding α-level optimal parameters to the Gα-MOLFP.

Probably the most crucial problem in the Fα-MODP is the identification of an appropriate aggregation function which well represents the human decision makers' fuzzy preferences. If $u_D(.)$ can be explicitly identified, then the Fα-MODP reduces to a standard mathematical programming problem. However, this rarely happens and as an alternative, it becomes evident that an interaction with the DM is necessary.

Throughout this section we make the following assumptions.

Assumption 3.1 The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions through the interaction with the DM.

Assumption 3.2 $u_D(.)$ exists and is known only implicitly to the DM, which means the DM cannot specify the entire form of $u_D(.)$, but he/she can provide local information concerning his/her preference. Moreover, it is strictly increasing and continuous with respect to $u_i(.)$ and α.

3.3.2 Minimax Problems

Having determined the membership functions for each of the objective functions, in order to generate a candidate for the satisficing solution which is also M-α-Pareto optimal, the DM is asked to specify the degree α of the α-level set and the reference levels of achievement of the membership functions, called reference membership values (Sakawa and Yano 1985f). For the DM's degree α and reference membership values $\bar{\mu}_i$, $i=1,\ldots,k$, the corresponding M-α-Pareto optimal solution, which is in a sense close to his/her requirement or better than that if the reference levels are attainable, is obtained by solving the following minimax problem.

$$\min_{\substack{x \in X(a,b) \\ (a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d})}} \max_{1 \leq i \leq k} \left(\bar{\mu}_i - z_i(x,c_i,d_i) \right), \tag{3.18}$$

or equivalently

$$\begin{aligned} &\min \quad v \\ &\text{subject to} \quad \bar{\mu}_i - \mu_i(z_i(x,c_i,d_i)) \leq v, \quad i=1,\ldots,k \;, \\ &\qquad\qquad\quad a_j x \leq b_j \;, \; j=1,\ldots,m, \quad x \geq 0, \\ &\qquad\qquad\quad (a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d}) \;. \end{aligned} \tag{3.19}$$

A graphical description of the minimax problem is depicted in Fig. 3.1.

In order to solve the formulated problem on the basis of the linear programming method, we first convert each constraint $\bar{\mu}_i - \mu_i(z_i(x,c_i,d_i)) \leq v$, $i=1,\ldots,k$, of the minimax problem (3.19) into the following form using the strictly monotone decreasing or increasing property of $D_{iR}(.)$ and $D_{iL}(.)$.

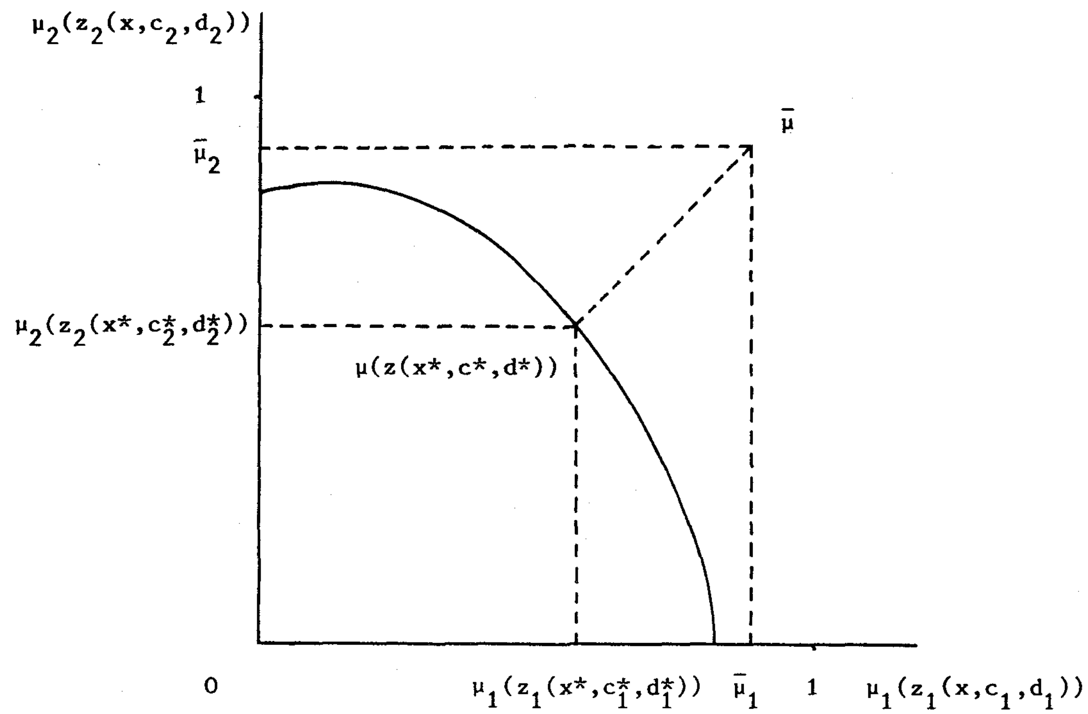

Fig. 3.1 Minimax problem

$$z_i(x,c_i,d_i) \leq D_{iR}^{-1}(\bar{\mu}_i - v), \quad i \in I_1 \cup I_3 \tag{3.20}$$

$$z_i(x,c_i,d_i) \geq D_{iL}^{-1}(\bar{\mu}_i - v), \quad i \in I_2 \cup I_3 \tag{3.21}$$

Since $q_i(x,d_i) \geq 0$ for all $x \in X(a,b)$ (by assumption), each constraint (3.20) and (3.21) can be converted as follows:

$$p_i(x,c_i) \leq D_{iR}^{-1}(\bar{\mu}_i - v)\, q_i(x,d_i), \quad i \in I_1 \cup I_3 \tag{3.22}$$

$$p_i(x,c_i) \geq D_{iL}^{-1}(\bar{\mu}_i - v)\, q_i(x,d_i), \quad i \in I_2 \cup I_3 \tag{3.23}$$

Then, we introduce the following set-valued functions $R_{iL}(.,.)$, $R_{iR}(.,.)$ and $T_j(.,.)$.

$$R_{iL}(c_i,d_i) = \{(x,v) \mid p_i(x,c_i) \geq D_{iL}^{-1}(\bar{\mu}_i - v)q_i(x,d_i),\ i \in I_2 \cup I_3\} \tag{3.24}$$

$$R_{iR}(c_i,d_i) = \{(x,v) \mid p_i(x,c_i) \leq D_{iR}^{-1}(\bar{\mu}_i - v)q_i(x,d_i),\ i \in I_1 \cup I_3\} \tag{3.25}$$

$$T_j(a_j,b_j) = \{x \mid a_j x \leq b_j\} \tag{3.26}$$

Then it can be verified that the following relations hold for $R_{iL}(\cdot,\cdot)$, $R_{iR}(\cdot,\cdot)$ and $T_j(\cdot,\cdot)$, when $x \geq 0$.

Proposition 3.1

(1) If $c_i^1 \leq c_i^2$, then $R_{iL}(c_i^1,d_i) \subset R_{iL}(c_i^2,d_i)$

and $R_{iR}(c_i^1,d_i) \supset R_{iR}(c_i^2,d_i)$.

(2) If $d_i^1 \leq d_i^2$, then $R_{iL}(c_i,d_i^1) \supset R_{iL}(c_i,d_i^2)$

and $R_{iR}(c_i,d_i^1) \subset R_{iR}(c_i,d_i^2)$.

(3) If $a_j^1 \leq a_j^2$, then $T_j(a_j^1,b_j) \supset T_j(a_j^2,b_j)$.

(4) If $b_j^1 \leq b_j^2$, then $T_j(a_j,b_j^1) \subset T_j(a_j,b_j^2)$.

It should be noted here that the feasible regions for c_i, d_i, a_j and b_j can be denoted respectively by the intervals $[c^L_{i\alpha}, c^R_{i\alpha}]$, $[d^L_{i\alpha}, d^R_{i\alpha}]$, $[a^L_{j\alpha}, a^R_{j\alpha}]$ and $[b^L_{j\alpha}, b^R_{j\alpha}]$ shown in Fig. 3.2.

Therefore, by making use of the results in Proposition 3.1, we can obtain an optimal solution to (3.19) by solving the following problem.

min v

subject to

$$p_i(x,c^L_{i\alpha}) \leq D^{-1}_{iR}(\bar{u}_i - v)\, q_i(x,d^R_{i\alpha}), \quad i \in I_1 \cup I_3 , \tag{3.27}$$

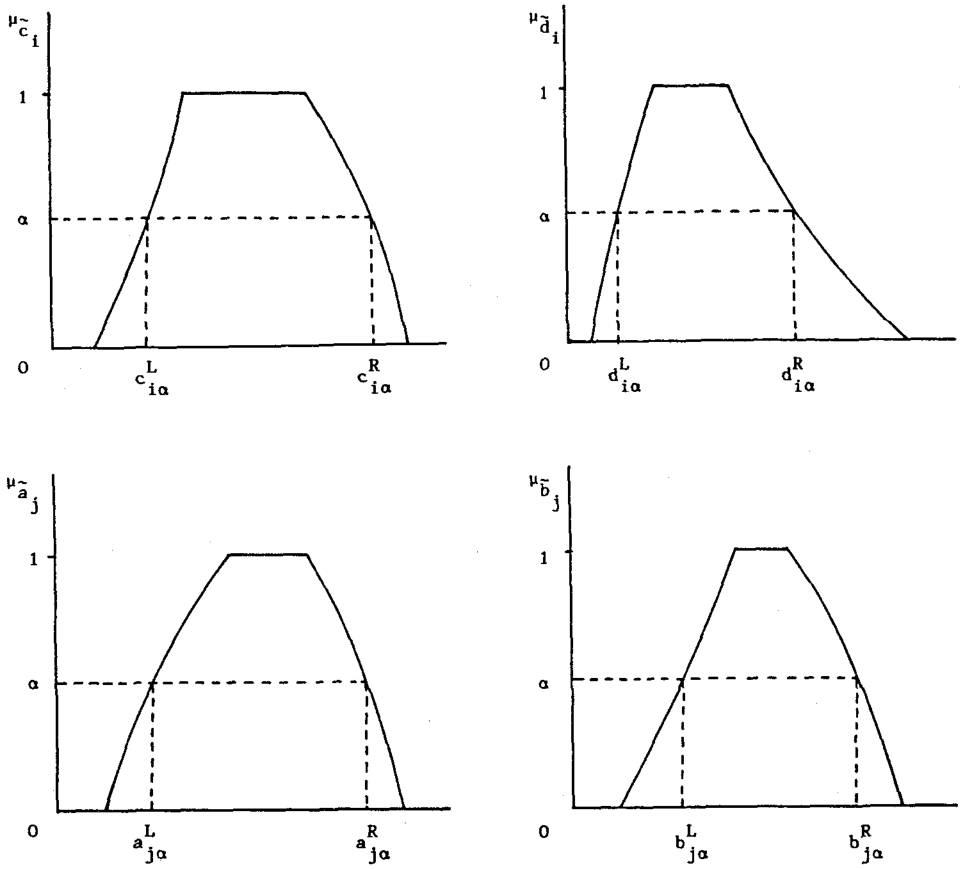

Fig. 3.2 Feasible regions for c_i, d_i a_j and b_j

$$p_i(x, c^R_{i\alpha}) \geq D^{-1}_{iL}(\bar{u}_i - v)\ q_i(x, d^L_{i\alpha}), \quad i \in I_2 \cup I_3 ,$$

$$a^L_{j\alpha} x \leq b^R_{j\alpha} , \quad j=1,\ldots,m ; \quad x \geq 0 .$$

It is important to note here that in this formulation, if the value of v is fixed, it can be reduced to a set of linear inequalities. Obtaining the optimal solution v^* to the above problem is equivalent to determining the minimum value of v so that there exists an admissible set satisfying the constraints of (3.27). Since v satisfies $\bar{u}_{max} - 1 \leq v \leq \bar{u}_{max}$, where $\bar{u}_{max}$ denotes the maximum value of $\bar{u}_i$, $i=1,\ldots,k$, we have the following method for solving this problem by combined use of the bisection method and the simplex method of linear programming.

Step 1. Set $v = \bar{\mu}_{max}$ and test whether an admissible set satisfying the constraints of (3.27) exists or not by making use of phase one of the simplex method. If an admissible set exists, proceed. Otherwise, the DM must reassess his/her membership function.

Step 2. Set $v = \bar{\mu}_{max} - 1$ and test whether an admissible set satisfying the constraints of (3.27) exists or not using phase one of the simplex method. If an admissible set exists, set $v^* = \bar{\mu}_{max} - 1$. Otherwise go to the next step, since the minimum v which satisfies the constraints of (3.27) exists between $\bar{\mu}_{max} - 1$ and $\bar{\mu}_{max}$.

Step 3. For the initial value of $v_1 = \bar{\mu}_{max} - 0.5$, update the value of v using the bisection method as follows :

$v_{n+1} = v_n - 1/2^{n+1}$ if admissible set exists for v_n,

$v_{n+1} = v_n + 1/2^{n+1}$ if no admissible set exists for v_n.

In this way, we can determine the optimal solution v^*. Then the DM selects an appropriate standing objective from among the objectives $z_i(x, c_i, d_i), i=1,\ldots,k$. For notational convenience, in the following, let it be $z_1(x, c_1, d_1)$ and $1 \in I_1$. Then the following linear fractional programming problem is solved for $v = v^*$.

$$\min \quad z_1(x, c^L_{1\alpha}, d^R_{1\alpha})$$

subject to $p_i(x, c^L_{i\alpha}) \leq D^{-1}_{iR}(\bar{u}_i - v*)\, q_i(x, d^R_{i\alpha}),\ i \in I_1 \cup I_3$ (3.28)

$p_i(x, c^R_{i\alpha}) \geq D^{-1}_{iL}(\bar{u}_i - v*)\, q_i(x, d^L_{i\alpha}),\ i \in I_2 \cup I_3$

$a^L_{j\alpha} x \leq b^R_{j\alpha},\quad j=1,\ldots,m,\quad x \geq 0$.

In order to solve this linear fractional programming problems, we can use Charnes-Cooper's (1962) variable transformation :

$$t = 1/q_1(x, d^R_{1\alpha}), \quad y = (x, 1)^T t, \tag{3.29}$$

and formulate the following standard linear programming problems:

$$\min \quad c^L_{1\alpha}\, y$$

$$\begin{aligned} \text{subject to} \quad & c^L_{i\alpha}\, y \leq D^{-1}_{iR}(\bar{u}_i - v*)\, d^R_{i\alpha}\, y, \quad i \in I_1 \cup I_3 \\ & c^R_{i\alpha}\, y \geq D^{-1}_{iL}(\bar{u}_i - v*)\, d^L_{i\alpha}\, y, \quad i \in I_2 \cup I_3 \\ & d^R_{1\alpha}\, y = 1 \\ & (a^L_{j\alpha}, -b^R_{j\alpha})\, y \leq 0, \quad j=1,\ldots,m, \quad y \geq 0 . \end{aligned} \tag{3.30}$$

For convenience in our subsequent discussion, assume that the optimal solution $y*$ to (3.30) satisfies the following conditions:

$$c^L_{i\alpha}\, y* = D^{-1}_{iR}(\bar{u}_i - v*)\, d^R_{i\alpha}\, y*, \quad i \in I_1 \cup I_{3R}$$

$$c^R_{i\alpha}\, y* = D^{-1}_{iL}(\bar{u}_i - v*)\, d^L_{i\alpha}\, y*, \quad i \in I_2 \cup I_{3L}$$

where $I_3 = I_{3L} \cup I_{3R}$ and $I_{3L} \cap I_{3R} = \Phi$.

Then it must be observed here that $c^L_{i\alpha}$, $d^R_{i\alpha}$ $(i \in I_1 \cup I_{3R})$, $c^R_{i\alpha}$, $d^L_{i\alpha}$ $(i \in I_2 \cup I_{3L})$ and $a^L_{j\alpha}$, $b^R_{j\alpha}$ $(j=1,\ldots,m)$ are α-level optimal parameters for any M-α-Pareto optimal solution.

The relationships between the optimal solutions to (3.30) and the M-α-Pareto optimal concept of the Gα-MOLFP can be characterized by the following theorems.

Theorem 3.1

If $x*$ is a unique optimal solution to (3.27), then $x*$ is a M-α-Pareto optimal solution to the Gα-MOLFP.

(Proof)

Assume that $x*$ is not an M-α-Pareto optimal solution to the Gα-MOLFP. Then, since $c^L_{i\alpha}$, $d^R_{i\alpha}$ $(i \in I_1 \cup I_{3R})$, $c^R_{i\alpha}$, $d^L_{i\alpha}$ $(i \in I_2 \cup I_{3L})$ and $a^L_{j\alpha}$, $b^R_{j\alpha}$ $(j=1,\dots,m)$ are α-level optimal parameters to the Gα-MOLFP, there exist $x \in X(a,b)$ and $(a,b,c,d) \in L_\alpha(\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ such that

$$\mu_i(z_i(x,c_i,d_i)) \geq \mu_i(z_i(x*,c^L_{i\alpha},\ d^R_{i\alpha})), \quad i \in I_1 \cup I_{3R},$$

$$\mu_i(z_i(x,c_i,d_i)) \geq \mu_i(z_i(x*,c^R_{i\alpha},\ d^L_{i\alpha})), \quad i \in I_2 \cup I_{3L},$$

with strict inequality holding for at least one i. Then it holds that

$$\max_{i \in I_1 \cup I_{3R}} (\ \bar{\mu}_i - \mu_i(z_i(x*,c^L_{i\alpha},\ d^R_{i\alpha}))\ )$$

$$\geq \max_{i \in I_1 \cup I_{3R}} (\ \bar{\mu}_i - \mu_i(z_i(x\ ,c_i,\ d_i))\ )$$

$$\geq \max_{i \in I_1 \cup I_{3R}} (\ \bar{\mu}_i - \mu_i(z_i(x\ ,c^L_{i\alpha},\ d^R_{i\alpha}))\ )$$

$$\max_{i \in I_2 \cup I_{3L}} (\ \bar{\mu}_i - \mu_i(z_i(x*,c^R_{i\alpha},\ d^L_{i\alpha}))\ )$$

$$\geq \max_{i \in I_2 \cup I_{3L}} (\bar{\mu}_i - \mu_i(z_i(x, c_i, d_i)))$$

$$\geq \max_{i \in I_2 \cup I_{3L}} (\bar{\mu}_i - \mu_i(z_i(x, c^R_{i\alpha}, d^L_{i\alpha})))$$

which contradicts the fact that x* is a unique optimal solution to (3.27). Q.E.D.

Theorem 3.2

If x* is an M-α-Pareto optimal solution to the Gα-MOLFP, then x* is an optimal solution to (3.27) for some $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)$.

(Proof)

Assume that (x*,v*) is not an optimal solution to (3.27) for any $\bar{\mu}$ satisfying

$$\bar{\mu}_i - \mu_i(z_i(x^*, c^L_{i\alpha}, d^R_{i\alpha})) = v^*, \quad i \in I_1 \cup I_{3R}$$

$$\bar{\mu}_i - \mu_i(z_i(x^*, c^R_{i\alpha}, d^L_{i\alpha})) = v^*, \quad i \in I_2 \cup I_{3L}$$

Then there exists $x \in X(a,b)$ such that

$$\bar{\mu}_i - \mu_i(z_i(x, c^L_{i\alpha}, d^R_{i\alpha})) < \bar{\mu}_i - \mu_i(z_i(x^*, c^L_{i\alpha}, d^R_{i\alpha})), \; i \in I_1 \cup I_{3R},$$

$$\bar{\mu}_i - \mu_i(z_i(x, c^R_{i\alpha}, d^L_{i\alpha})) < \bar{\mu}_i - \mu_i(z_i(x^*, c^R_{i\alpha}, d^L_{i\alpha})), \; i \in I_2 \cup I_{3L}.$$

This implies that $\mu_i(z_i(x, c^L_{i\alpha}, d^R_{i\alpha})) > \mu_i(z_i(x^*, c^L_{i\alpha}, d^R_{i\alpha}))$, $i \in I_1 \cup I_{3R}$, $\mu_i(z_i(x, c^R_{i\alpha}, d^L_{i\alpha})) > \mu_i(z_i(x^*, c^R_{i\alpha}, d^L_{i\alpha}))$, $i \in I_2 \cup I_{3L}$, which contradicts the fact that x* is an M-α-Pareto optimal solution to the Gα-MOLP. Q.E.D.

It should be noted here that for generating M-α-Pareto optimal solutions using Theorem 3.1, uniqueness of solution must be verified. In general, however, it is not easy to check numerically whether an optimal solution to (3.30) is unique or not.

Consequently, in order to test the M-α-Pareto optimality of a current optimal solution x*, we formulate and solve the following linear programming problem :

$$\bar{w} = \max \sum_{i=1}^{k} \epsilon_i$$

subject to

$$p_i(x, c_{i\alpha}^L) + \epsilon_i = z_i(x^*, c_{i\alpha}^L, d_{i\alpha}^R) q_i(x, d_{i\alpha}^R), \quad i \in I_1 \cup I_{3R}, \qquad (3.31)$$

$$p_i(x, c_{i\alpha}^R) - \epsilon_i = z_i(x^*, c_{i\alpha}^R, d_{i\alpha}^L) q_i(x, d_{i\alpha}^L), \quad i \in I_2 \cup I_{3L},$$

$$a_{j\alpha}^L x \leqq b_{j\alpha}^R, \; j = 1, \ldots, m, \quad x \geqq 0, \quad \epsilon_i \geqq 0, \; i = 1, \ldots, k.$$

Let $\bar{x}$ and $\bar{\epsilon}$ be an optimal solution to (3.31). If $\bar{w} = 0$, then x* is an M-α-Pareto optimal solution. In case of $\bar{w} > 0$ and consequently at least one $\bar{\epsilon}_\ell > 0$, we perform the following operations.

<u>Step 1.</u> Solve the following problem for any ℓ such that $\bar{\epsilon}_\ell > 0$.

$$\min \quad \bar{z}_\ell(x, c_{\ell\alpha}, d_{\ell\alpha})$$

subject to

$$z_i(x, c_{i\alpha}^L, d_{i\alpha}^R) = z_i(\bar{x}, c_{i\alpha}^L, d_{i\alpha}^R) \quad \{i \mid \bar{\epsilon}_i = 0\} \cap \{I_1 \cup I_{3R}\},$$

$$z_i(x, c^R_{i\alpha}, d^L_{i\alpha}) = z_i(\bar{x}, c^R_{i\alpha}, d^L_{i\alpha}) \quad \{i \mid \bar{\epsilon}_i = 0\} \cap \{I_2 \cup I_{3L}\}, \tag{3.32}$$

$$z_i(x, c^L_{i\alpha}, d^R_{i\alpha}) \leq z_i(\bar{x}, c^L_{i\alpha}, d^R_{i\alpha}) \quad \{i \mid \bar{\epsilon}_i > 0\} \cap \{I_1 \cup I_{3R}\},$$

$$z_i(x, c^R_{i\alpha}, d^L_{i\alpha}) \geq z_i(\bar{x}, c^R_{i\alpha}, d^L_{i\alpha}) \quad \{i \mid \bar{\epsilon}_i > 0\} \cap \{I_2 \cup I_{3L}\},$$

$$a^L_{j\alpha} x \leq b^R_{j\alpha} \quad , \ j = 1, \ldots, m \ , \quad x \geq 0 \ .$$

where $\bar{z}_\ell(x, c_{\ell\alpha}, d_{\ell\alpha})$ is defined as :

$$\bar{z}_\ell(x, c_{\ell\alpha}, d_{\ell\alpha}) \triangleq \begin{cases} z_\ell(x, c^L_{\ell\alpha}, d^R_{\ell\alpha}), & \ell \in I_1 \cup I_{3R}, \\ \\ -\, z_\ell(x, c^R_{\ell\alpha}, d^L_{\ell\alpha}), & \ell \in I_2 \cup I_{3L}. \end{cases}$$

<u>Step 2</u>. Test the M-α-Pareto optimality for the solution to (3.32).

<u>Step 3.</u> If $\bar{w} = 0$, stop. Otherwise, return to step 1.

Repeating this process at least k-1 iterations, an M-α-Pareto optimal solution can be obtained.

3.3.3 Interactive Algorithm

Now given the M-α-Pareto optimal solution for the degree α and the reference membership values specified by the DM by solving the corresponding minimax problem, the DM must either be satisfied with the current M-α-Pareto optimal solution and α, or update his/her reference membership values and/or the degree α. In order to help the DM express his/her degree of preference, trade-off information between a standing membership function and each of the other membership functions as well as between the degree α and the membership functions is very useful. Such a trade-off information is easily obtainable since it is closely related to the simplex multipliers of the problem (3.30).

To derive the trade-off information, we define the following Lagrangian function L corresponding to the problem (3.30).

$$
\begin{aligned}
L = \; & c_{1\alpha}^{L} y + \eta \, (d_{1\alpha}^{R} y - 1) \\
& + \sum_{i \in I_1 \cup I_{3R}} \pi_{iR} \{ c_{i\alpha}^{L} y - D_{iR}^{-1} (\bar{\mu}_i - v^*) d_{i\alpha}^{R} y \} \\
& + \sum_{i \in I_2 \cup I_{3L}} \pi_{iL} \{ D_{iL}^{-1} (\bar{\mu}_i - v^*) d_{i\alpha}^{L} y - c_{i\alpha}^{R} y \} \\
& + \sum_{j=1}^{m} \lambda_j \, (a_{j\alpha}^{L}, - b_{j\alpha}^{R}) \, y \, . \qquad (3.33)
\end{aligned}
$$

where η, π_{iL}, π_{iR} and λ_j are simplex multipliers corresponding to the constraints in the problem (3.30).

Here, we assume that the problem (3.30) has a unique and nondegenerate optimal solution satisfying the following conditions.

(i) $\pi_{iR} > 0 \; , \quad i \in I_1 \cup I_{3R}, \quad i \neq 1,$

(ii) $\pi_{iL} > 0 \; , \quad i \in I_2 \cup I_{3L} \; .$

Then by using the results in Haimes and Chankong (1979), the following expression holds .

$$
- \frac{\partial (z_1 (x, c_{1\alpha}^{L}, d_{1\alpha}^{R}))}{\partial (z_i (x, c_{i\alpha}^{L}, d_{i\alpha}^{R}))} = \pi_{iR} d_{i\alpha}^{R} y \, , \quad i \in I_1 \cup I_{3R}, \; i \neq 1, \qquad (3.34)
$$

$$-\frac{\partial(z_1(x,c^L_{1\alpha},d^R_{1\alpha}))}{\partial(z_i(x,c^R_{i\alpha},d^L_{i\alpha}))} = -\pi_{iL}d^L_{i\alpha}y, \quad i\in I_2\cup I_{3L} \tag{3.35}$$

Furthermore, using the strictly monotone decreasing or increasing property of $D_{iR}(.)$ and $D_{iL}(.)$ together with the chain rule, if $D_{iR}(.)$ and $D_{iL}(.)$ are differentiable at the optimal solution to (3.30), it holds that

$$-\frac{\partial\mu_1(z_1(x,c^L_{1\alpha},d^R_{1\alpha}))}{\partial\mu_i(z_i(x,c^L_{i\alpha},d^R_{i\alpha}))} = \pi_{iR}d^R_{i\alpha}y\,\frac{D'_{1R}(z_1(x,c^L_{1\alpha},d^R_{i\alpha}))}{D'_{iR}(z_i(x,c^L_{i\alpha},d^R_{i\alpha}))},\ i\in I_1\cup I_{3R},\ i\neq 1, \tag{3.36}$$

$$-\frac{\partial\mu_1(z_1(x,c^L_{1\alpha},d^R_{1\alpha}))}{\partial\mu_i(z_i(x,c^R_{i\alpha},d^L_{i\alpha}))} = -\pi_{iL}d^L_{i\alpha}y\,\frac{D'_{1R}(z_1(x,c^L_{1\alpha},d^R_{i\alpha}))}{D'_{iL}(z_i(x,c^R_{i\alpha},d^L_{i\alpha}))},\ i\in I_2\cup I_{3L} \tag{3.37}$$

Regarding a trade-off rate between $\mu_1(.)$ and α, the following relation holds based on the sensitivity theorem (Fiacco 1983).

$$\begin{aligned}\frac{\partial\mu_1(z_1(x,c^L_{1\alpha},d^R_{1\alpha}))}{\partial\alpha} = D'_{1R}(z_1(x,c^L_{1\alpha},d^R_{1\alpha}))\Bigg\{ &\frac{\partial c^L_{1\alpha}}{\partial\alpha}y + n\frac{\partial d^R_{1\alpha}}{\partial\alpha}y \\ &+ \sum_{i\in I_1\cup I_{3R}} \pi_{iR}\left(\frac{\partial c^L_{i\alpha}}{\partial\alpha}y - D^{-1}_{iR}(\bar{u}_i - v*)\frac{\partial d^R_{i\alpha}}{\partial\alpha}y\right) \\ &+ \sum_{i\in I_2\cup I_{3L}} \pi_{iL}\left(D^{-1}_{iL}(\bar{u}_i - v*)\frac{\partial d^L_{i\alpha}}{\partial\alpha}y - \frac{\partial c^R_{i\alpha}}{\partial\alpha}y\right) \\ &+ \sum_{j=1}^{m}\lambda_j\left(\frac{\partial a^L_{j\alpha}}{\partial\alpha}, -\frac{\partial b^R_{j\alpha}}{\partial\alpha}\right)y\Bigg\}\end{aligned} \tag{3.38}$$

It should be noted here that in order to obtain the trade-off rate information from (3.36)-(3.37), all the constraints of the problem (3.30) must be active for the current optimal solution y^*, and y^* must satisfy the M-α-Pareto optimality test. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_i$ for inactive constraints by $\mu_i(z_i(x^*, c^L_{i\alpha}, d^R_{i\alpha})) + v^*$ or $\mu_i(z_i(x^*, c^R_{i\alpha}, d^L_{i\alpha}) + v^*$, and solve the corresponding problem (3.30) for obtaining the simplex multipliers.

Following the above discussions, we can now construct the interactive algorithm in order to derive the satisficing solution for the DM from among the M-α-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

<u>Step 0.</u> Calculate the individual minimum and maximum of each objective function under given constraints for $\alpha = 0$ and $\alpha = 1$.

<u>Step 1*.</u> Elicit a membership function $\mu_i(z_i(x, c_i, d_i))$ from the DM for each of the objective functions.

<u>Step 2*.</u> Ask the DM to select the initial value of α $(0 \leqq \alpha \leqq 1)$ and set the initial reference membership values $\bar{\mu}_i = 1$, $i=1,\ldots,k$.

<u>Step 3.</u> For the degree α and the reference membership values specified by the DM, solve the minimax problem and perform the M-α-Pareto optimality test.

<u>Step 4*.</u> The DM is supplied with the corresponding M-α-Pareto optimal solution and the trade-off rates between the membership functions and the degree α. If the DM is satisfied with the current membership function values of the M-α-Pareto optimal solution and α, stop. Otherwise, the DM must update the reference membership values and/or the degree α by considering

the current values of the membership functions and together with the trade-off rates between the membership functions and the degree α, and return to step 3.

Here it should be stressed for the DM that (1) any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions for some fixed degree α, and (2) the greater value of the degree α gives worse values of the membership functions for some fixed reference membership values.

3.3.4 Numerical Example

To illustrate the proposed method, consider the following three objective linear fractional programming problems with fuzzy parameters.

$$\text{fuzzy min} \quad z_1(x,\tilde{c}_1) \triangleq (-x_1 + \tilde{c}_{13}) / (-x_2 + 9)$$

$$\text{fuzzy max} \quad z_2(x,\tilde{c}_2) \triangleq (x_1 + \tilde{c}_{23}) / (-x_2 + 7)$$

$$\text{fuzzy equal} \quad z_3(x,\tilde{d}_3) \triangleq (x_2 + 2) / (-x_1 + \tilde{d}_{33})$$

subject to

$$x \in X \triangleq \{ (x_1, x_2) \mid 2x_1 + x_2 - 14 \leq 0,$$

$$2x_1 + 5x_2 - 30 \leq 0,\ x_i \geq 0,\ i=1,2 \}$$

where $\tilde{c}_{13}$, $\tilde{c}_{23}$ and $\tilde{d}_{33}$ are fuzzy numbers whose membership functions are:

$$\mu_{\tilde{c}_{13}}(c_{13}) = \max (1 - | c_{13} - 11.5 |, 0),$$

$$\mu_{\tilde{c}_{23}}(c_{23}) = \max (1 - 0.5 | c_{23} - 8 |, 0),$$

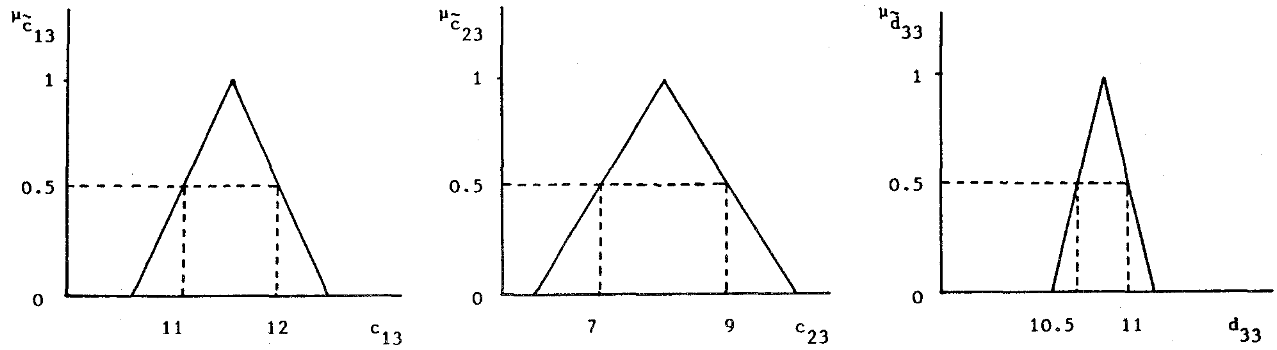

Fig. 3.3 Fuzzy numbers $\tilde{c}_{13}$, $\tilde{c}_{23}$, $\tilde{d}_{33}$

$$\mu_{\tilde{d}_{33}}(d_{33}) = \max\left(1 - 2 \mid d_{33} - 10.75 \mid, 0 \right).$$

Now, suppose that the interaction with the hypothetical DM establishes the following simple linear membership functions for the three objective functions.

$$\mu_1(z_1) = \begin{cases} 1 & z_1 \leqq 0.5 \\ D_{1R}(z_1) = (-z_1 + 3)/2.5 & 0.5 \leqq z_1 \leqq 3 \\ 0 & 3 \leqq z_1 \end{cases}$$

$$\mu_2(z_2) = \begin{cases} 0 & z_2 \leqq 1 \\ D_{2L}(z_2) = (z_2 - 1)/4 & 1 \leqq z_2 \leqq 5 \\ 1 & 5 \leqq z_2 \end{cases}$$

$$\mu_3(z_3) = \begin{cases} 0 & z_3 \leq 0 \\ D_{3L}(z_3) = (10z_3)/3 & 0 \leq z_3 \leq 0.3 \\ 1 & z_3 = 0.3 \\ D_{3R}(z_3) = (1.2 - z_3)/0.9 & 0.3 \leq z_3 \leq 1.2 \\ 0 & 1.2 \leq z_3 \end{cases}$$

Also assume that the hypothetical DM selects the initial value of the degree α to be 0.5, and the initial reference membership values $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$ to be (14/15, 61/80, 233/315). Then, the corresponding M-α-Pareto optimal solution can be obtained by solving the following problem.

$$\min_{x \in X} \quad v$$

subject to

$$(-x_1 + 11) \leq D_{1R}^{-1}(\bar{\mu}_1 - v)(-x_2 + 9)$$

$$(x_1 + 9) \geq D_{2L}^{-1}(\bar{\mu}_2 - v)(-x_2 + 7)$$

$$(x_2 + 2) \leq D_{3R}^{-1}(\bar{\mu}_3 - v)(-x_1 + 11)$$

$$(x_2 + 2) \geq D_{3L}^{-1}(\bar{\mu}_3 - v)(-x_1 + 10.5)$$

Solving this problem by combined use of the bisection method and the simplex method of linear programming, we obtain the optimal solution $v^* = 1/5$. For obtaining the corresponding optimal values of the decision variable x^*, we solve the following linear programming problem for $v^* = 1/5$.

$$\min_{y \in Y} \quad -y_1 + 11y_3$$

subject to

$$y_1 + 9y_3 \geq D_{2L}^{-1}(\bar{\mu}_2 - v^*)(-y_2 + 7y_3)$$

$$y_2 + 2y_3 \leq D_{3R}^{-1}(\bar{\mu}_3 - v^*)(-y_1 + 11y_3)$$

$$y_2 + 2y_3 \geq D_{3L}^{-1}(\bar{\mu}_3 - v^*)(-y_1 + 10.5y_3)$$

$$-y_2 + 9y_3 = 1$$

where $Y \triangleq \{ (y_1, y_2, y_3) \mid 2y_1 + y_2 - 14y_3 \leq 0,$

$2y_1 + 5y_2 - 30y_3 \leq 0,\ y_i \geq 0,\ i=1,2,3 \},$

and

$$y = (x, 1)^T t,\ t = 1/(-x_2+9)$$

As a result, we obtain the optimal solution $y^* = (y_1^*, y_2^*, y_3^*) = (2/3, 1/2, 1/6)$. The corresponding optimal values of the objective functions x^*, $z_1^* \triangleq z_1(x^*, c_{1\alpha}^L)$, $z_2^* \triangleq z_2(x^*, c_{2\alpha}^R)$, $z_3^* \triangleq z_3(x^*, d_{3\alpha}^R)$, and the membership functions $\mu_i^* \triangleq \mu_i(z_i^*)$ $i=1,2,3$, and the simplex multipliers $(\pi_{2L}^*, \pi_{3L}^*, \pi_{3R}^*)$ can be obtained as follows:

$$x^* = (x_1^*, x_2^*) = (4, 3), \quad (z_1^*, z_2^*, z_3^*) = (7/6, 13/4, 5/7),$$

$$(\mu_1^*, \mu_2^*, \mu_3^*) = (11/15, 9/16, 34/63),$$

$$(\pi_{2L}^*, \pi_{3L}^*, \pi_{3R}^*) = (154/111, 0, 371/111).$$

From (3.36) and (3.37), the trade-off rates among the membership functions become as follows:

$$-\frac{\partial \mu_1(z_1^*)}{\partial \mu_2(z_2^*)} = -\pi_{2L}^* d_{2\alpha}^L y^* \frac{D'_{1R}(z_1^*)}{D'_{2L}(z_2^*)}$$

$$= - \frac{154}{111} (0, -1, 7)(2/3, 1/2, 1/6)^T \frac{-1/2.5}{1/4}$$

$$= 2464/1665 ,$$

$$- \frac{\partial \mu_1(z_1^*)}{\partial \mu_3(z_3^*)} = \pi_{3R}^* d_{3\alpha}^R y^* \frac{D'_{1R}(z_1^*)}{D'_{3R}(z_3^*)}$$

$$= - \frac{371}{111} (-1, 0, 11)(2/3, 1/2, 1/6)^T \frac{-1/2.5}{-1/0.9}$$

$$= 7791/5550 .$$

Concerning the trade-off rate between $\mu_1(z_1)$ and α, from (3.38), we have

$$\frac{\partial \mu_1(z_1^*)}{\partial \alpha} = D'_{1R}(z_1^*) \{ c_{1\alpha}^L y_3^* + \pi_{3R}^* (- D_{3R}^{-1}(\bar{\mu}_3 - v^*) \frac{\partial d_{3\alpha}^R}{\partial \alpha} y_3^*)$$

$$+ \pi_{2L}^* (- \frac{\partial c_{2\alpha}^R}{\partial \alpha} y_3^*) \}$$

$$= -718/1665 .$$

Observe that the DM can obtain his/her satisficing solution from among an M-α-Pareto optimal solution set by updating his/her reference membership values and/or the degree α on the basis of the current values of the membership functions and the degree α together with the trade-off rates among the values of the membership functions and the degree α.

3.4 Conclusion

In this chapter, we have proposed an interactive decision making method for multiobjective linear fractional programming problems with fuzzy parameters by considering the imprecise nature of human judgement.

Similar to the previous chapter, the following desirable features of our proposed method will be summarized for the linear fractional objectives extension in this chapter.

(1) The experts' ambiguous understanding of the nature of the parameters in the problem-formulation process can be incorporated.

(2) The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions, which may be nonlinear.

(3) For the degree α and the reference membership values specified by the DM, the corresponding M-α-Pareto optimal solution can be easily obtained by solving the minimax problems based mainly on the well known linear programming method.

(4) M-α-Pareto optimality of the generated solution in each iteration is guaranteed by performing the M-α-Pareto optimality test.

(5) The trade-off information between the membership functions and the degree α is easily obtainable, since it is closely related to the simplex multipliers of the minimax problems.

(6) The satisficing solution of the DM can be derived efficiently from among M-α-Pareto optimal solutions by updating his/her reference membership values and/or the degree α based on the current values of the M-α-Pareto optimal solution together with the trade-off information between the membership functions and the degree α.

In the next chapter, multiobjective nonlinear programming problems with fuzzy parameters are considered as a nonlinear generalization of chapter 2 and 3.

CHAPTER 4

INTERACTIVE DECISION MAKING FOR MULTIOBJECTIVE NONLINEAR PROGRAMMING PROBLEMS WITH FUZZY PARAMETERS

4.1 Introduction

In this chapter, attention is now focused on multiobjective nonlinear programming problems with fuzzy parameters, which reflect the experts' ambiguous or fuzzy understanding of the nature of the parameters in the problem-formulation process. Such general multiobjective nonlinear programming problems with fuzzy parameters were first formulated by Orlovski (1983,1984). He presented two approaches to the formulated problems by making systematic use of the extension principle of Zadeh (1975) and demonstrated that there exist in some sense equivalent nonfuzzy formulations. Unfortunately, however, no interactive decision making methods have been proposed.

In this chapter, in order to deal with the multiobjective nonlinear programming problems with fuzzy parameters characterized by fuzzy numbers, the concept of M-α-Pareto optimality is introduced on the basis of the α-level sets of the fuzzy numbers. Then by considering the fuzzy goals of the decision maker (DM), a new interactive decision making method using augmented minimax problems is presented in order to derive the satisficing solution of the DM efficiently from among an M-α-Pareto optimal solution set as a nonlinear generalization of chapters 2 and 3.

4.2 Problem Statement and Solution Concept

In general, the multiobjective nonlinear programming (MONLP) problem is represented as the following vector-minimization problem:

$$\min f(x) \triangleq (f_1(x), f_2(x), \ldots, f_k(x)) \tag{4.1}$$

$$\text{subject to} \quad x \in X = \{ x \in E^n \mid g_j(x) \leq 0,\ j=1,\ldots,m \}$$

where x is an n-dimensional vector of decision variables, $f_1(x),\ldots,f_k(x)$ are k distinct objective functions of the decision vector x, $g_1(x),\ldots,$ $g_m(x)$ are m inequality constraints, and X is the feasible set of constrained decisions.

Fundamental to the MONLP is the Pareto optimal concept, also known as a noninferior solution. Qualitatively, a Pareto optimal solution of the MONLP is one where any improvement of one objective function can be achieved only at the expense of another. Mathematically, a formal definition of a Pareto optimal solution to the MONLP is given below:

Definition 4.1 (Pareto optimal solution)

$x^* \in X$ is said to be a Pareto optimal solution to the MONLP, if and only if there does not exist another $x \in X$ such that $f_i(x) \leq f_i(x^*)$,

$i=1,\ldots,k$, with strict inequality holding for at least one i.

In practice, however, it would certainly be more appropriate to consider that the possible values of the parameters in the description of the objective functions and the constraints usually involve the ambiguity of the experts' understanding of the real system. For this reason, in this chapter, we consider the following multiobjective nonlinear programming problem with fuzzy parameters (MONLP-FP) :

$$\min f(x,\tilde{a}) \triangleq (f_1(x,\tilde{a}_1),f_2(x,\tilde{a}_2),\ldots,f_k(x,\tilde{a}_k)) \tag{4.2}$$

$$\text{subject to } \ x \in X(\tilde{b}) \triangleq \{ x \in E^n \mid g_j(x,\tilde{b}_j) \leq 0,\ j=1,\ldots m \}$$

where $\tilde{a}_i = (\tilde{a}_{i1},\ldots, \tilde{a}_{ip_i})$, $\tilde{b}_j = (\tilde{b}_{j1},\ldots,\tilde{b}_{jq_j})$ represent respectively a vector of fuzzy parameters involved in the objective function $f_i(x,\tilde{a}_i)$ and the constraint function $g_j(x,\tilde{b}_j)$. These fuzzy parameters are assumed to be characterized as the fuzzy numbers introduced by Dubois and Prade (1978,1980). We now assume that $\tilde{a}_{ir}$ and $\tilde{b}_{js}$ in the MONLP-FP are fuzzy numbers whose membership functions are $\mu_{\tilde{a}_{ir}}(a_{ir})$ and $\mu_{\tilde{b}_{js}}(b_{js})$ respectively. For simplicity in the notation, define the following vectors:

$$a_i = (a_{i1},\ldots,a_{ip_i}),\quad b_j = (b_{j1},\ldots,b_{jq_j})$$

$$a = (a_1,\ldots,a_k),\quad \tilde{a} = (\tilde{a}_1,\ldots,\tilde{a}_k),\quad b = (b_1,\ldots,b_m),\quad \tilde{b} = (\tilde{b}_1,\ldots,\tilde{b}_m).$$

Then we can introduce the following α-level set or α-cut (Dubois and Prade 1980) to the fuzzy numbers $\tilde{a}_{ir}$ and $\tilde{b}_{js}$.

Definition 4.2 (α-level set)

The α-level set of the fuzzy numbers $\tilde{a}_{ir}(i=1,\ldots,k,\ r=1,\ldots,p_i)$ and $\tilde{b}_{js}(j=1,\ldots,m,\ \ s=1,\ldots,q_j)$ is defined as the ordinary set $L_\alpha(\tilde{a},\tilde{b})$ for which the degree of their membership functions exceeds the level α:

$$L_\alpha(\tilde{a},\tilde{b}) = \{ (a,b) \mid \mu_{\tilde{a}_{ir}}(a_{ir}) \geq \alpha,\ i=1,\ldots,k,\ r=1,\ldots,p_i\ ;$$
$$\mu_{\tilde{b}_{js}}(b_{js}) \geq \alpha,\ j=1,\ldots,m,\ s=1,\ldots,q_j\ \} \tag{4.3}$$

For a certain degree α, the MONLP-FP (4.2) can be understood as the following nonfuzzy α-multiobjective nonlinear programming (α-MONLP) problem.

$$\begin{aligned} &\min\ \ f(x,a) \triangleq (f_1(x,a_1),f_2(x,a_2),\ldots,f_k(x,a_k)) \\ &\text{subject to}\quad x \in X(b) \triangleq \{\ x \in E^n \mid g_j(x,b_j) \leq 0,\ j=1,\ldots,m\ \} \\ &\qquad\qquad\quad (a,b) \in L_\alpha(\tilde{a},\tilde{b}) \end{aligned} \tag{4.4}$$

It should be emphasized here that in the α-MONLP the parameters (a,b) are treated as decision variables rather than constants.

On the basis of the α-level sets of the fuzzy numbers, we introduce the concept of α-Pareto optimal solutions to the α-MONLP.

Definition 4.3 (α-Pareto optimal solution)

$x^* \in X(b^*)$ is said to be an α-Pareto optimal solution to the α-MONLP (4.4), if and only if there does not exist another $x \in X(b)$ and $(a,b) \in L_\alpha(\tilde{a},\tilde{b})$ such that $f_i(x,a_i) \leq f_i(x^*,a_i^*)$, $i=1,\dots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters a^* and b^* are called α-level optimal parameters.

For practical purposes, however, since only local solutions are guaranteed in solving a scalar optimization problem by any standard optimization technique, unless the problem is convex, we deal with local α-Pareto optimal solutions instead of global α-Pareto optimal solutions.

Definition 4.4 (local α-Pareto optimal solution)

$x^* \in X(b^*)$ is said to be a local α-Pareto optimal solution to the α-MONLP (4.4), if and only if there does not exist another $x \in X(b) \cap N(x^*;r)$ and $(a,b) \in L_\alpha(\tilde{a},\tilde{b}) \cap N(a^*,b^*;r')$ such that $f_i(x,a_i) \leq f_i(x^*,a_i^*)$, $i=1,\dots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters a^* and b^* are called α-level local optimal parameters and $N(x^*;r)$ denotes the set $\{x \mid x \in E^n, \|x - x^*\| < r\}$.

4.3 Interactive Decision Making under Fuzziness

4.3.1 Fuzzy Goals

As can be immediately seen from Definition 4.4, usually, (local) α-Pareto optimal solutions consist of an infinite number of points, and the DM must select his/her (local) satisficing or compromise solution from among (local) α-Pareto optimal solutions based on his/her subjective judgement.

However, considering the imprecise nature of the DM's judgement, it is natural to assume that the DM may have imprecise or fuzzy goals for each of the objective functions in the α-MONLP (4.4). In a minimization problem, a fuzzy goal stated by the DM may be to achieve "substantially less " than A . This type of statement can be quantified by eliciting a corresponding membership function.

In order to elicit a membership function $\mu_i(f_i(x,a_i))$ from the DM for each of the objective functions $f_i(x,a_i)$ in the α-MONLP (4.4), we first calculate the individual (local) minimum f_i^{min} and maximum f_i^{max} of each objective function $f_i(x,a_i)$ under the given constraints for $\alpha = 0$ and $\alpha = 1$. By taking account of the calculated individual (local) minimum and maximum of each objective function, the DM must determine his/her subjective membership function $\mu_i(f_i(x,a_i))$ which is a strictly monotone decreasing function with respect to $f_i(x,a_i)$. Fig. 4.1 illustrates the graph of the possible shape of the membership function representing the fuzzy goal to achieve substantially less than A_i.

It is now appropriate to point out that, in the fuzzy approaches, we can treat two types of fuzzy goals; namely, fuzzy goals expressed in words such as " $f_i(x,a_i)$ should be in the vicinity of C_i" (called

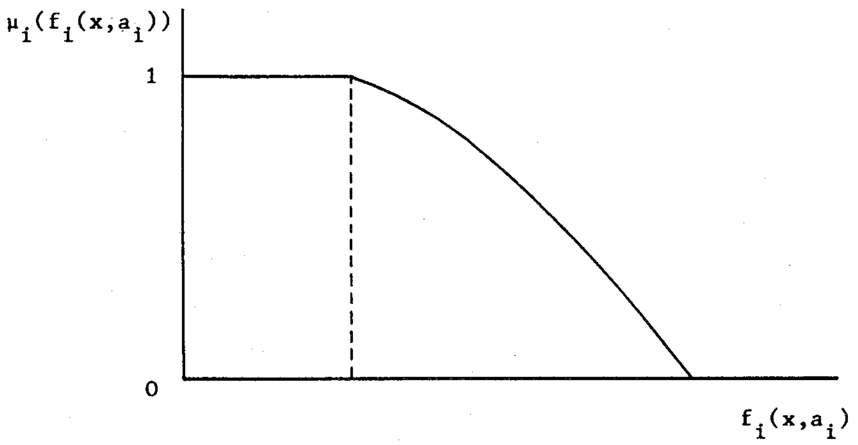

Fig. 4.1 Monotone decreasing membership function

fuzzy equal) as well as " $f_i(x,a_i)$ should be substantially less than A_i or greater than B_i" (called fuzzy min or fuzzy max). Such a generalized α-MONLP (Gα-MONLP) problem may now be expressed as:

$$
\begin{array}{llll}
\text{fuzzy min} & f_i(x,a_i) & (i \in I_1) & \\
\text{fuzzy max} & f_i(x,a_i) & (i \in I_2) & \\
\text{fuzzy equal} & f_i(x,a_i) & (i \in I_3) & (4.5) \\
\text{subject to} & x \in X(b), & (a,b) \in L_\alpha(\tilde{a},\tilde{b}) &
\end{array}
$$

where $I_1 \cup I_2 \cup I_3 = \{ 1,2,\ldots,k \}$.

In order to elicit a membership function from the DM for a fuzzy goal like "$f_i(x,a_i)$ should be in the vicinity of C_i", it is obvious that we can use different functions to the left and right sides of C_i.

As an example, Fig. 4.2 illustrates the graph of the possible shape of the fuzzy equal membership function representing the fuzzy goal to be in the vicinity of C_i.

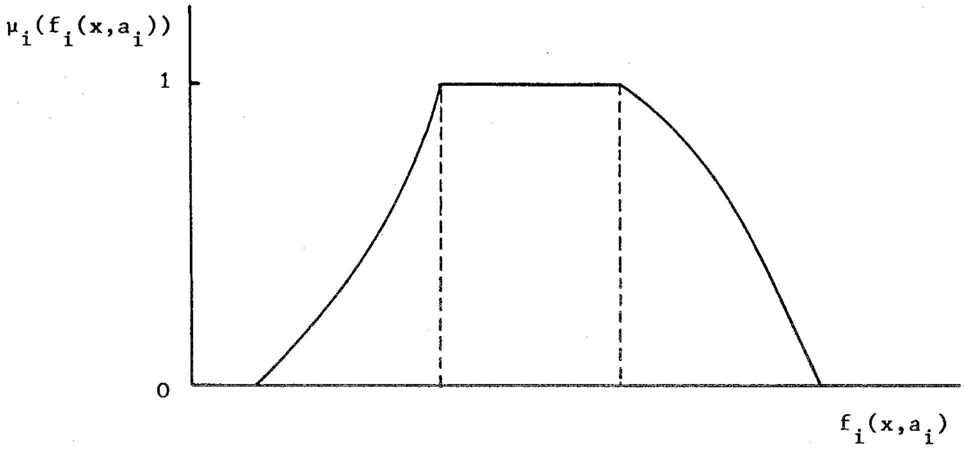

Fig. 4.2 Fuzzy equal membership function

When fuzzy equal is included in the fuzzy goals of the DM, it is desirable that $f_i(x,a_i)$ should be as close to C_i as possible. Consequently, the notion of (local) α-Pareto optimal solutions defined in terms of objective functions cannot be applied. For this reason, we introduce the concept of (local) M-α-Pareto optimal solutions which is defined in terms of membership functions instead of objective functions, where M refers to membership.

Definition 4.5 ((local) M-α-Pareto optimal solution)

$x^* \in X(b^*)$ is said to be a (local) M-α-Pareto optimal solution to the Gα-MONLP (4.5), if and only if there does not exist another $x \in X(b)$

($\cap$ $N(x^*;r)$), $(a,b) \in L_\alpha(\tilde{a},\tilde{b})$ ($\cap$ $N(a^*,b^*;r')$) such that $\mu_i(f_i(x,a_i)) \geq \mu_i(f_i(x^*,a_i^*))$, $i=1,\ldots,k$, with strict inequality holding for at least one i, where the corresponding values of parameters a^* and b^* are called α-level (local) optimal parameters.

Having elicited the membership functions $\mu_i(f_i(x,a_i))$, $i=1,\ldots,k$ from the DM for each of the objective functions $f_i(x,a_i)$, $i=1,\ldots,k$, if we introduce a general aggregation function

$$\mu_D(\ \mu(f(x,a)),\ \alpha\ ) = \mu_D(\ \mu_1(f_1(x,a_1)),\ldots,\ \mu_k(f_k(x,a_k)),\ \alpha\ ) \tag{4.6}$$

a general fuzzy α-multiobjective decision problem (Fα-MODP) can be defined by:

$$\max \quad \mu_D(\ \mu(f(x,a)),\ \alpha\ ) \tag{4.7}$$

subject to

$$(x,a,b) \in P(\alpha), \quad \alpha \in [\ 0,\ 1\ ] \tag{4.8}$$

where $P(\alpha)$ is the set of M-α-Pareto optimal solutions and corresponding α-level optimal parameters to the Gα-MONLP.

Probably the most crucial problem in the Fα-MODP is the identification of an appropriate aggregation function which well represents the human decision makers' fuzzy preferences. If $\mu_D(.)$ can be explicitly identified, then the Fα-MODP reduces to a standard mathematical programming problem. However, this rarely happens and as an alternative, it becomes evident that an interaction with the DM is necessary.

Throughout this section we make the following assumptions.

Assumption 4.1

The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions through the interaction with the DM.

Assumption 4.2

$\mu_D(.)$ exists and is known only implicitly to the DM, which means the DM cannot specify the entire form of $\mu_D(.)$, but he/she can provide local information concerning his/her preference. Moreover, it is strictly increasing and continuous with respect to $\mu_i(.)$ and α.

Assumption 4.3

All $f_i(x,a_i)$, $i=1,\ldots,k$ and all $g_j(x,b_j)$, $j=1,\ldots,m$ are continuously differentiable in their respective domains.

4.3.2 Minimax Problems

Having determined the membership functions for each of the objective functions, in order to generate a candidate for the (local) satisficing solution which is also (local) M-α-Pareto optimal, the DM is asked to specify the degree α of the α-level set and the reference levels of achievement of the membership functions, called the reference membership values. Observe that the idea of the reference membership values (e.g. Sakawa, Yumine and Yano 1984a,b; Sakawa and Yano 1984a,1985f) can be viewed as an obvious extension of the idea of the reference point of Wierzbicki (1979a).

For the DM's degree α and the reference membership values $\bar{\mu}_i$, $i=1,\ldots,k$, the following minimax problem is solved in order to generate the (local) M-α-Pareto optimal solution, which is in a sense close to his/her requirement or better than that if the reference membership values are attainable.

$$\min_{\substack{x \in X(b) \\ (a,b) \in L_\alpha(\tilde{a},\tilde{b})}} \max_{1 \leq i \leq k} \left(\bar{\mu}_i - \mu_i(f_i(x,a_i)) \right) \tag{4.9}$$

or equivalently

$$\min \quad v \tag{4.10}$$

$$\text{subject to} \quad \bar{\mu}_i - \mu_i(f_i(x,a_i)) \leq v, \quad i=1,\ldots,k \tag{4.11}$$

$$x \in X(b), \quad (a,b) \in L_\alpha(\tilde{a},\tilde{b}) \tag{4.12}$$

Fig. 4.3 illustrates a graphical description of the minimax problem in a membership space.

The relationships between the (local) optimal solutions of the minimax problem and the (local) M-α-Pareto optimal concept of the Gα-MONLP can be characterized by the following theorems.

Theorem 4.1

If (x^*, a^*, b^*) is a unique (local) optimal solution to the minimax problem for some $\bar{\mu}_i$, $i=1,\ldots,k$, then x^* is a (local) M-α-Pareto optimal solution and a^*, b^* are α-level (local) optimal parameters to the Gα-MONLP.

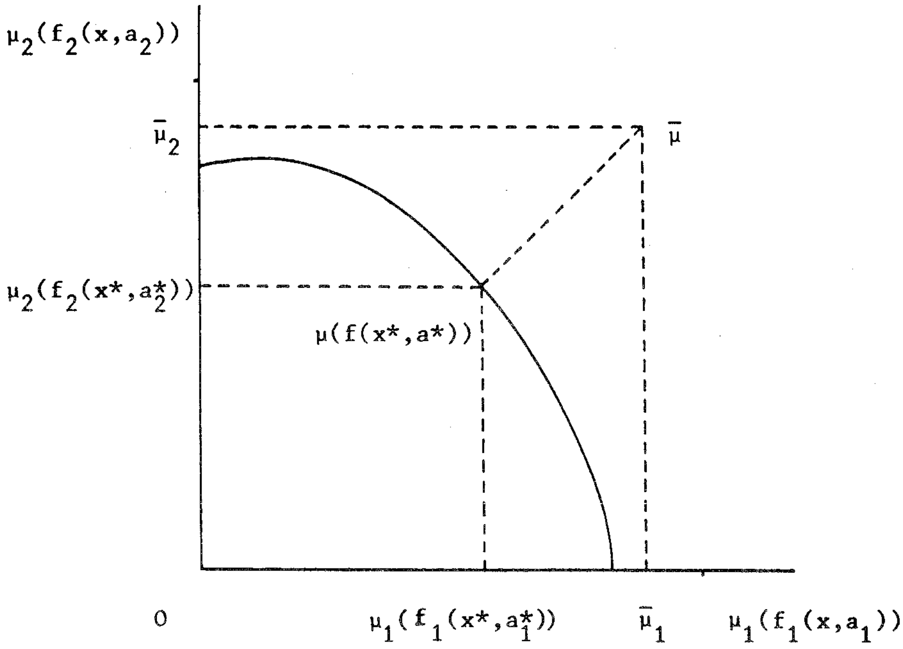

Fig. 4.3 Minimax problem

(Proof)

Assume that x^* is not a (local) M-α-Pareto optimal solution or a^*, b^* are not α-level (local) optimal parameters to the Gα-MONLP, then there exists $x \in X(b)(\cap N(x^*,r))$, $(a,b) \in L_\alpha(\tilde{a},\tilde{b})(\cap N(a^*,b^*;r'))$ such that $\mu(f(x,a)) \geq \mu(f(x^*,a^*))$. This implies that $\bar{\mu} - \mu(f(x,a)) \leq \bar{\mu} - \mu(f(x^*,a^*))$, where $\mu(f(x,a)) = (\mu_1(f_1(x,a_1)),\ldots,\mu_k(f_k(x,a_k)))$ and $\bar{\mu} = (\bar{\mu}_1,\ldots, \bar{\mu}_k)$. Then it holds that

$$\max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) \leq \max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i^*)))$$

which contradicts the fact that (x*,a*,b*) is a unique (local) optimal solution to the minimax problem. Hence x* is a (local) M-α-Pareto optimal solution and a*,b* are α-level (local) optimal parameters to the Gα-MONLP. Q.E.D.

Theorem 4.2

If x* is a (local) M-α-Pareto optimal solution and a*,b* are α-level (local) optimal parameters to the Gα-MONLP, then there exist $\bar{\mu}_i$, i=1,...,k such that (x*,a*,b*) is a (local) optimal solution to the minimax problem.

(Proof)

Assume that (x*,a*,b*) is not a (local) optimal solution to the minimax problem for any $\bar{\mu}_i$, i=1,...,k, satisfying

$$\bar{\mu}_1 - \mu_1(f_1(x*,a_1^*)) = \ldots = \bar{\mu}_k - \mu_k(f_k(x*,a_k^*)) .$$

Then there exists $x \in X(\cap N(x*,r))$ and $(a,b) \in L_\alpha(\tilde{a},\tilde{b})(\cap N(a*,b*;r'))$ such that

$$\max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i^*))) > \max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))).$$

This implies that

$$\max_{1\leq i\leq k} (\mu_i(f_i(x*,a_i^*)) - \mu_i(f_i(x,a_i))) < 0$$

hence

$$\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i)) < 0 \quad , \quad i=1,\ldots k$$

must hold, which contradicts the fact that x* is a (local) M-α-Pareto optimal solution and a*,b* are α-level (local) optimal parameters to the Gα-MONLP, and the theorem is proved. Q.E.D.

4.3.3 Augmented Minimax Problems

In order to circumvent the necessity to perform the (local) M-α-Pareto optimality test in the minimax problems, for the nonlinear case, it is reasonable to use augmented minimax problems instead of minimax problems. For the DM's degree α and the reference membership values $\bar{\mu}_i$, i=1,...,k, the following augmented minimax problem is solved for generating the (local) M-α-Pareto optimal solution, which is in a sense close to his/her requirement or better than that if the reference membership values are attainable.

$$\min_{\substack{x \in X(b) \\ (a,b) \in L_\alpha(\tilde{a},\tilde{b})}} \left\{ \max_{1 \leq i \leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) \right\} \tag{4.13}$$

or equivalently

$$\min \quad v + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) \tag{4.14}$$

$$\text{subject to} \quad \bar{\mu}_i - \mu_i(f_i(x,a_i)) \leq v \ , \ i=1,\ldots,k, \tag{4.15}$$

$$x \in X(b), \ (a,b) \in L_\alpha(\tilde{a},\tilde{b}). \tag{4.16}$$

Such an augmented minimax problem can be viewed as a modified fuzzy version of the augmented Tchebycheff norm problem of Steuer and Choo (1983) or Choo and Atkins (1983).

The relationships between the (local) optimal solutions of the augmented minimax problem and the (local) M-α-Pareto optimal concept of the Gα-MONLP can be characterized by the following theorems.

Theorem 4.3

If (x^*,a^*,b^*) is a (local) optimal solution to the augmented minimax problem for some $\bar{\mu}_i$, $i=1,\ldots,k$, then x^* is a (local) M-α-Pareto optimal solution and a^*,b^* are α-level (local) optimal parameters to the Gα-MONLP.

(Proof)

Assume that x^* is not a (local) M-α-Pareto optimal solution or a^*,b^* are not α-level (local) optimal parameters to the Gα-MONLP, then there exists $x \in X(b)(\cap N(x^*,r))$, $(a,b) \in L_\alpha(\tilde{a},\tilde{b})(\cap N(a^*,b^*;r'))$ such that $\mu(f(x,a)) \geq \mu(f(x^*,a^*))$. This implies that $\bar{\mu} - \mu(f(x,a)) \leq \bar{\mu} - \mu(f(x^*,a^*))$, where $\mu(f(x,a)) = (\mu_1(f_1(x,a_1)),\ldots,\mu_k(f_k(x,a_k)))$ and $\bar{\mu} = (\bar{\mu}_1,\ldots,\bar{\mu}_k)$. Then it holds that

$$\max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) \leq \max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x^*,a_i^*)))$$

This means that

$$\max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x,a_i)))$$

$$< \max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i*))) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i*)))$$

which contradicts the fact that (x*,a*,b*) is a (local) optimal solution to the augmented minimax problem. Hence x* is a (local) M-α-Pareto optimal solution and a*,b* are α-level (local) optimal parameters to the Gα-MONLP. Q.E.D.

Theorem 4.4

If x* is a (local) M-α-Pareto optimal solution and a*,b* are α-level (local) optimal parameters to the Gα-MONLP, then there exist $\bar{\mu}_i$, i=1,...,k such that (x*,a*,b*) is a (local) optimal solution to the augmented minimax problem for sufficiently small positive ρ.

(Proof)

Assume that (x*,a*,b*) is not a (local) optimal solution to the augmented minimax problem for any $\bar{\mu}_i$, i=1,...,k, satisfying

$$\bar{\mu}_1 - \mu_1(f_1(x*,a_1*)) = \dots = \bar{\mu}_k - \mu_k(f(x*,a_k*)) .$$

Then there exists $x \in X(\cap N(x*,r))$, $(a,b) \in L_\alpha(\tilde{a},\tilde{b})(\cap N(a*,b*;r'))$ such that

$$\max_{1\leq i\leq k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i*))) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x*,a_i*)))$$

$$> \max_{1 \le i \le k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) .$$

This implies that

$$\max_{1 \le i \le k} (\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i)))$$

$$+ \rho \sum_{i=1}^{k} (\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i))) < 0$$

Now if either any $\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i))$ is positive or all $\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i))$, $i=1,\ldots,k$, are zero, this inequality would be violated for sufficiently small positive ρ. Hence

$$\mu_i(f_i(x^*,a_i^*)) - \mu_i(f_i(x,a_i)) \le 0 \quad , \quad i=1,\ldots k$$

must hold, which contradicts the fact that x^* is a (local) M-α-Pareto optimal solution and a^*,b^* are α-level (local) optimal parameters to the Gα-MONLP, and the theorem is proved. Q.E.D.

As can be seen from the above proofs, it must be observed here that an obvious advantage of the augmented minimax problem over the usual minimax problem is that, because of the presence of the augmented term, (local) M-α-Pareto optimality is guaranteed without the uniqueness assumption for the solution.

Added insight can be obtained by comparing the isoquant of the augmented minimax problem

$$\bar{\mu}_i - \mu_i(f_i(x,a_i)) + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x,a_i))) = \text{constant},$$

$$i=1,\ldots,k \tag{4.17}$$

with the isoquant of the minimax problem

$$\bar{\mu}_i - \mu_i(f_i(x,a_i)) = \text{constant}, \quad i=1,\ldots,k \tag{4.18}$$

in the membership function space as depicted in Fig. 4.4.

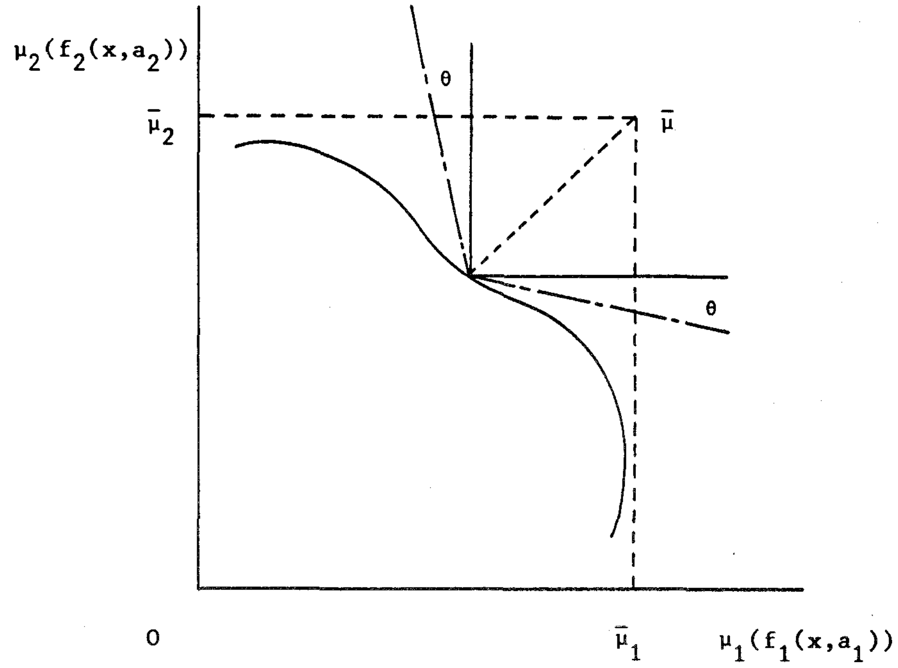

Fig. 4.4 Isoquants of the minimax problem and the augmented minimax problem

Observe that, in Fig. 4.4, the normal vectors of the isoquant of the augmented minimax problem and the minimax problem become $(-\rho,\ldots,-\rho,-1-\rho,-\rho,\ldots,-\rho)$ and $(0,\ldots,0,-1,0,\ldots,0)$ respectively, it trivially follows that the cosine of the angle θ between these two normal vectors is given by $\cos\theta = (1+\rho) / \sqrt{1+2\rho+k\rho^2}$. Hence we have

$$\theta = \tan^{-1}(\sqrt{k-1}\ \rho / (1+\rho)). \tag{4.19}$$

This relation shows that θ is monotone increasing with respect to ρ. Thus, for sufficiently small positive scalar, augmented minimax problems overcome the possibility to generate weak M-α-Pareto optimal solutions as was shown in Theorems 4.3 and 4.4. Hence augmented minimax problems are attractive for generating M-α-Pareto optimal solutions even if appropriate convexity assumptions are absent.

Naturally, ρ should be a sufficiently small, but computationally significant, positive scalar. However, for practical purposes, a computationally significant lower bound of ρ may be

$$\rho = 10^{(a-b)-c+1} \tag{4.20}$$

where a and b are the figures of $\max_{1 \leq i \leq k} (\bar{\mu}_i - \mu_i(f_i(x,a_i)))$ and $\sum_{i=1}^{k}(\bar{\mu}_i - \mu_i(f_i(x,a_i)))$ in the first and second terms in the augmented minimax problem (4.13) respectively and c is the precision figure of the computer. Usually, since the values of a and b are unknown in advance, if we roughly estimate a = b, then we have

$$\rho = 10^{-c+1} .$$

In most cases, a computationally significant value of $\rho = 10^{-3} \sim 10^{-5}$ should suffice.

4.3.4 Algorithm Using Augmented Minimax Problems

Now given the (local) M-α-Pareto optimal solution for the degree α and the reference membership values specified by the DM by solving the corresponding augmented minimax problem, the DM must either be satisfied with the current (local) M-α-Pareto optimal solution and α, or update the reference membership values and/or the degree α. In order to help the DM express his/her degree of preference, trade-off information between a standing membership function and each of the other membership functions as well as between the degree α and the membership functions is very useful. Fortunately, such a trade-off information is easily obtainable since it is closely related to the strict positive Lagrange multipliers of the augmented minimax problem.

To derive the trade-off information, we first define the Lagrangian function L for the augmented minimax problem (4.14)-(4.16) as follows:

$$L = v + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i(f_i(x, a_i)))$$
$$+ \sum_{i=1}^{k} \lambda_i^{\mu} (\bar{\mu}_i - \mu_i(f_i(x, a_i)) - v) + \sum_{j=1}^{m} \lambda_j^{g} g_j(x, b_j)$$
$$+ \sum_{i=1}^{k} \sum_{r=1}^{p_i} \lambda_{ir}^{a} (\alpha - \mu_{\tilde{a}_{ir}}(a_{ir})) + \sum_{j=1}^{m} \sum_{s=1}^{q_j} \lambda_{js}^{b} (\alpha - \mu_{\tilde{b}_{js}}(b_{js})) \quad (4.21)$$

In the following, for notational convenience, we denote the decision variables in the augmented minimax problem (4.14)-(4.16) by $y = (x, v, a, b)$ and let us assume that the augmented minimax problem has a unique local optimal solution y^* satisfying the following three assumptions.

Assumption 4.4

y^* is a regular point of the constraints of the augmented minimax problem.

Assumption 4.5

The second-order sufficiency conditions are satisfied at y^* .

Assumption 4.6

There are no degenerate constraints at y^* .

Then the following existence theorem, which is based on the implicit function theorem (Fiacco 1983), holds.

Theorem 4.5

Let $y* = (x*,v*,a*,b*)$ be a unique local solution of the augmented minimax problem (4.14)-(4.16) satisfying Assumptions 4.4, 4.5 and 4.6. Let $\lambda* = (\lambda^{\mu*},\lambda^{a*},\lambda^{b*},\lambda^{g*})$ denote the Lagrange multipliers corresponding to the constraints (4.15)-(4.16). Then there exist a continuously differentiable vector valued function $y(.)$ and $\lambda(.)$ defined on some neighborhood $N(\alpha*)$ so that $y(\alpha*) = y*$, $\lambda(\alpha*) = \lambda*$, where $y(\alpha)$ is a unique local solution of the augmented minimax problem for any $\alpha \in N(\alpha*)$ satisfying Assumptions 4.4, 4.5 and 4.6, and $\lambda(\alpha)$ is the Lagrange multiplier corresponding to the constraints (4.15)-(4.16).

In Theorem 4.5,

$$\inf_{x,v,a,b} \{ v + \rho \sum_{i=1}^{k} (\bar{\mu}_i - \mu_i (f_i (x,a_i))) \mid \bar{\mu}_i - \mu_i (f_i (x,a_i)) \leq v ,$$

$$i=1,\dots,k,\ x \in X(b),\ (a,b) \in L_{\alpha}(\tilde{a},\tilde{b}) \}$$

can be viewed as the optimal value function of the augmented minimax problem (4.14)-(4.16) for any $\alpha \in N(\alpha*)$. Therefore, the following theorem holds under the same assumptions in Theorem 4.5.

Theorem 4.6

If all the assumptions in Theorem 4.5 are satisfied, then the following relation holds on some neighborhood $N(\alpha*)$ of $\alpha*$.

$$\frac{\partial \left(v + \rho \sum_{i=1}^{k} \left(\bar{\mu}_i - \mu_i(f_i(x,a_i)) \right) \right)}{\partial \alpha}$$

$$= \frac{\partial L}{\partial \alpha} = \sum_{i=1}^{k} \sum_{r=1}^{p_i} \lambda_{ir}^{a} + \sum_{j=1}^{m} \sum_{s=1}^{q_j} \lambda_{js}^{b} \tag{4.22}$$

If all the constraints (4.15) of the augmented minimax problem are active, namely if $v(\alpha*) = \bar{\mu}_i - \mu_i(f_i(x(\alpha*),a_i(\alpha*)))$, $i=1,\ldots,k$, then the following theorem holds.

Theorem 4.7

Let all the assumptions in Theorem 4.6 be satisfied. Also assume that all the constraints (4.15) of the augmented minimax problem are active. Then it holds that

$$-\left.\frac{\partial \mu_i(f_i(x,a_i))}{\partial \alpha}\right|_{\alpha=\alpha*} = \frac{1}{1+\rho k} \left(\sum_{i=1}^{k} \sum_{r=1}^{p_i} \lambda_{ir}^{a*} + \sum_{j=1}^{m} \sum_{s=1}^{q_j} \lambda_{js}^{b*} \right), \quad i=1,\ldots,k . \tag{4.23}$$

Regarding a trade-off rate between $\mu_1(f_1(x,a_1))$ and $\mu_i(f_i(x,a_i))$ for each $i=2,\ldots,k$, by extending the results in Haimes and Chankong (1979), it can be proved that the following theorem holds (Yano and Sakawa 1985).

Theorem 4.8

Let all the assumptions in Theorem 4.7 be satisfied. Also assume that the constraints (4.15) are active. Then it holds that

$$- \left. \frac{\partial \mu_i(f_i(x,a_i))}{\partial \mu_1(f_1(x,a_1))} \right|_{y=y*} = \frac{\lambda_1^{\mu *} + \rho}{\lambda_i^{\mu *} + \rho}, \quad i=2,\ldots,k . \tag{4.24}$$

It should be noted here that in order to obtain the trade-off rate information from (4.23) and (4.24), all the constraints (4.15) of the augmented minimax problem must be active. Therefore, if there are inactive constraints, it is necessary to replace $\bar{\mu}_i$ for inactive constraints by $\mu_i(f_i(x*,a_i^*)) + v*$ and solve the corresponding augmented minimax problem for obtaining the Lagrange multipliers.

Following the above discussions, we can now construct the interactive algorithm in order to derive the (local) satisficing solution for the DM from among the (local) M-α-Pareto optimal solution set. The steps marked with an asterisk involve interaction with the DM.

Step 0 Calculate the (local) individual minimum f_i^{min} and maximum f_i^{max} of each objective function $f_i(x)$ under the given constraints for $\alpha = 0$ and $\alpha = 1$.

Step 1* Elicit a membership function $\mu_i(f_i(x,a_i))$ from the DM for each of the objective functions.

Step 2* Ask the DM to select the initial values of α ($0 \leqq \alpha \leqq 1$) and set the initial reference membership values $\bar{\mu}_i^{(1)} = 1$, $i=1,\ldots,k$. Set the iteration index r=1.

Step 3 Set $\bar{\mu}_i = \bar{\mu}_i^{(r)}$, $i=1,\ldots,k$, solve the corresponding augmented minimax problem to obtain the (local) M-α-Pareto optimal solution $x^{(r)}$, $f(x^{(r)},a^{(r)})$ and the membership function value $\mu(f(x^{(r)},a^{(r)}))$ together with the trade-off rate information between the membership functions and the degree α.

Step 4* If the DM is satisfied with the current levels of $\mu(f(x^{(r)},a^{(r)})$, $i=1,\ldots,k$ of the (local) M-α-Pareto optimal solution and α, stop. Then the current (local) M-α-Pareto optimal solution $f(x^{(r)},a^{(r)}) = (f_1(x^{(r)},a_1^{(r)}),\ldots, f_k(x^{(r)},a_k^{(r)}))$ is the (local) satisficing solution of the DM. Otherwise, ask the DM to update the current reference membership values $\bar{\mu}_i^{(r)}$ and/or the degree $\alpha^{(r)}$ to the new reference membership values $\bar{\mu}_i^{(r+1)}, i=1,\ldots,k$ and/or the degree $\alpha^{(r+1)}$ by considering the current values of the membership functions together with the trade-off rates between the membership functions and the degree α. Set $r=r+1$ and return to Step 3.

Here it should be stressed for the DM that (1) any improvement of one membership function can be achieved only at the expense of at least one of the other membership functions for some fixed degree α, and (2) the greater value of the degree α gives worse values of the membership functions for some fixed reference membership values.

4.4 Conclusion

As a nonlinear generalization of the previous two chapters, an interactive decision making method for multiobjective nonlinear programming problems with fuzzy parameters has been proposed in this chapter. Although the general conclusions of this chapter is essentially same as in chapters 2 and 3, the following is a brief summary of the desirable features of our proposed method.

(1) The experts' ambiguous understanding of the nature of the parameters in the problem-formulation process can be incorporated.

(2) The fuzzy goals of the DM can be quantified by eliciting the corresponding membership functions, which may be nonlinear.

(3) For the degree α and the reference membership values specified by the DM, the corresponding (local) M-α-Pareto optimal solution can be obtained by solving the augmented minimax problems based on the nonlinear programming method.

(4) With the augmented minimax problems, (local) M-α-Pareto optimality of the generated solution in each iteration is guaranteed.

(5) The trade-off information between the membership functions and the degree α is easily obtainable, since it is closely related to the Lagrange multipliers of the augmented minimax problems.

(6) The (local) satisficing solution of the DM can be derived efficiently from among (local) M-α-Pareto optimal solutions by updating his/her reference membership values and/or the degree α based on the current values of the (local) M-α-Pareto optimal solution together with the trade-off information between the membership functions and the degree α.

CHAPTER 5

INTERACTIVE COMPUTER PROGRAMS AND ILLUSTRATIVE NUMERICAL EXAMPLES

5.1 Computer Programs

Interactive decision making processes for multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters discussed thus far include eliciting a membership function for each of the objective functions together with reference membership values and degree α from the DM. Thus, mitigation and speed-up of computation works are indispensable to this approach, and interactive utilization of computer facilities is highly recommended. Based on the methods described in chapters 2, 3 and 4, we have developed corresponding interactive computer programs for solving multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters. The entire programs are written in FORTRAN, because FORTRAN language is popular among many scientists and also almost all computer facilities are available for loading the programs with minor changes.

Our programs include graphical representations by which the DM can visualize the shapes of his/her membership functions, and he/she can find incorrect assessments or inconsistent evaluations promptly, revise them immediately and proceed to the next stage more easily. Each of our computer programs is composed of one main program and several subroutines. The main program calls in and runs the subprograms with commands indicated by the user (DM). The major commands prepared in our computer programs are summarized as follows.

(1) MINMAX: Displays the calculated (local) individual minimum and maximum of each of the objective functions under the given constraints for $\alpha = 0$ and $\alpha = 1$.

(2) MF: Elicit a membership function from the DM for each of the objective functions.

(3) GRAPH: Depicts graphcally the shape of the membership function for each of the objective functions.

(4) GO: Derives the (local) satisficing solution for the DM from among the (local) M-α-Pareto optimal solution set by updating the reference membership values and/or the degree α.

(5) STOP: Exists from the program.

(6) SAVE: Saves all the necessary information, which has been put in, in a file.

(7) READ: Restores the information which was saved in the file.

In all of our computer programs, the fuzzy parameters, which reflect the experts' ambiguous understanding of the nature of parameters in the problem-formulation process, are assumed to be characterized by the fuzzy numbers whose membership functions are either linear or exponential as shown in Fig. 5.1 or Fig. 5.2 respectively.

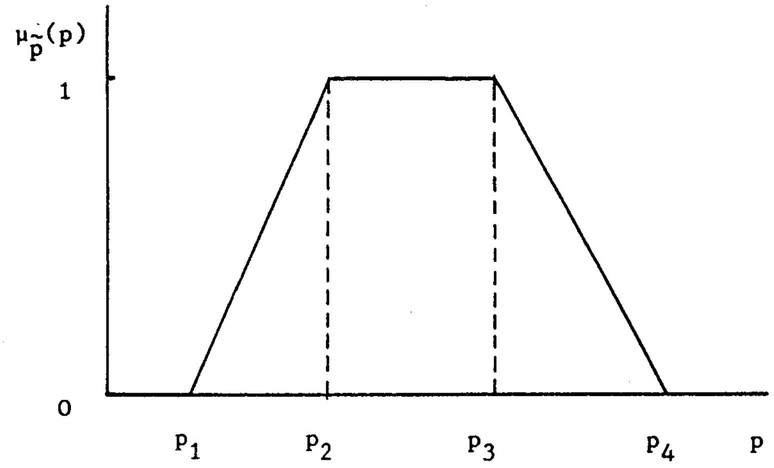

Fig. 5.1 Linear membership function

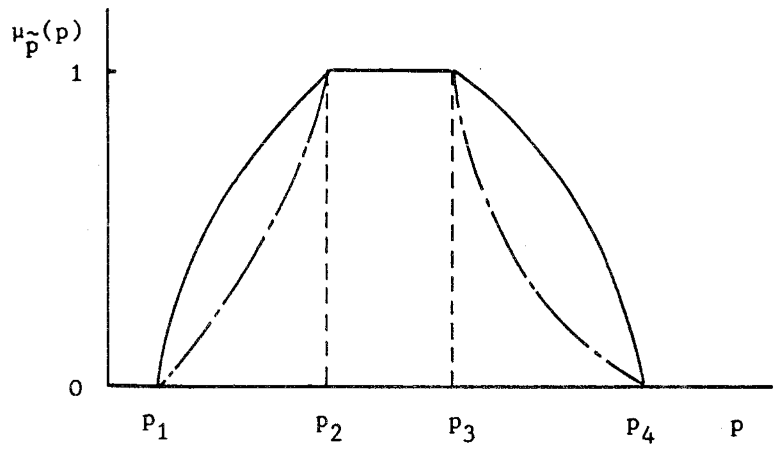

Fig. 5.2 Exponential membership function

Each of the membership functions for the fuzzy parameters can be determined by specifying the four points p_1, p_2, p_3, p_4 together with the types of its left and right functions (linear or exponential).

M-α-Pareto optimal solutions for multiobjective linear or linear fractional programming problems are calculated by solving the minimax problems on the basis of the simplex method of linear programming.

For multiobjective nonlinear programming problems, (local)M-α-Pareto optimal solutions are obtained by solving the augmented minimax problems instead of minimax problems using the revised version of the generalized reduced gradient (GRG) program (Lasdon, Fox and Ratner, 1974) called GRG2 (Lasdon, Waren and Ratner, 1980). In GRG2 there are two optimality tests, i.e.,

(1) to satisfy the Kuhn-Tucker optimality conditions, and

(2) to satisfy the fractional change condition

$$| \text{FM} - \text{OBJTST} | < \text{EPSTOP} \times | \text{OBJTST} |$$

for NSTOP times consecutive iterations. FM is the current objective value and OBJTST is the objective value at the start of the previous one dimensional search. NSTOP has a default value of 3.

In our computer programs, the DM can select his/her membership functions in a subjective manner by considering the rate of increase of membership of satisfaction from among the following five types of functions; linear, exponential, hyperbolic, hyperbolic inverse and piecewise linear functions. Then the parameter values are determined through the interaction with the DM. In the following discussions concerning membership functions, it is convenient to deal with the nonlinear case. Here, except for the hyperbolic functions, it is assumed that $\mu_i(f_i(x,a_i)) = 0$ if $f_i(x) \leq f_i^0$ and $\mu_i(f_i(x,a_i)) = 1$ if $f_i(x,a_i) \geq f_i^1$, where f_i^0 is an unacceptable level for $f_i(x,a_i)$, f_i^1 is a completely desirable level for $f_i(x,a_i)$, and f_i^a represents the value of $f_i(x,a_i)$ such that the degree of membership function $\mu_i(f_i(x,a_i))$ is a.

(1) Linear membership function

For each objective function, the corresponding linear membership function is defined as follows:

$$\mu_i(f_i(x,a_i)) = (f_i(x,a_i) - f_i^0) / (f_i^1 - f_i^0) . \tag{5.1}$$

The linear membership function can be determined by asking the DM to specify the two points, f_i^0 and f_i^1, within f_i^{max} and f_i^{min}. Fig. 5.3 illustrates the graph of the linear membership function.

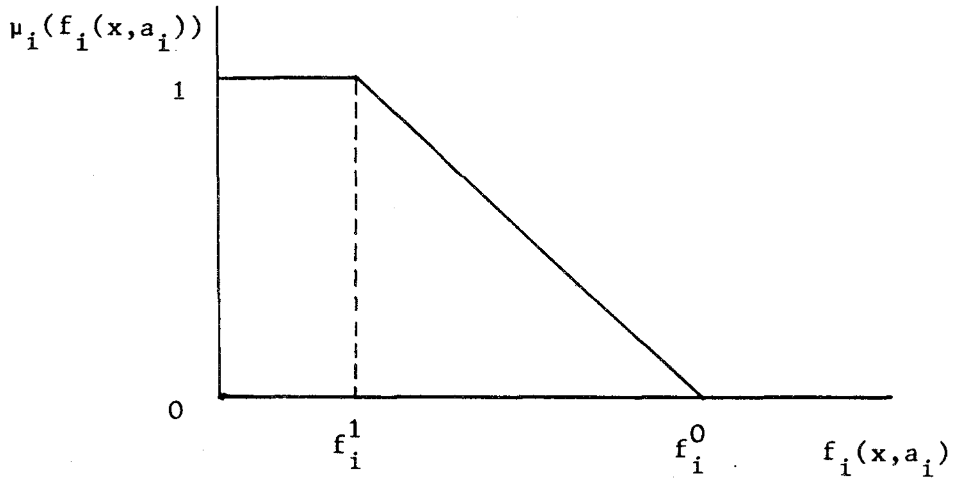

Fig. 5.3 Linear membership function

(2) Exponential membership function

For each objective function, the corresponding exponential membership function is defined by:

$$\mu_i(f_i(x,a_i)) = \alpha_i\{ 1 - \exp(-\beta_i (f_i(x,a_i) - f_i^0) / (f_i^1 - f_i^0))\}, \tag{5.2}$$

where $\alpha_i > 1$, $\beta_i > 0$ or $\alpha_i < 0$, $\beta_i < 0$.

The exponential membership function can be determined by asking the DM to specify the three points, f_i^0, $f_i^{0.5}$ and f_i^1, within f_i^{max} and f_i^{min}, where β_i is a shape parameter. Fig. 5.4 illustrates the graph of the exponential membership function.

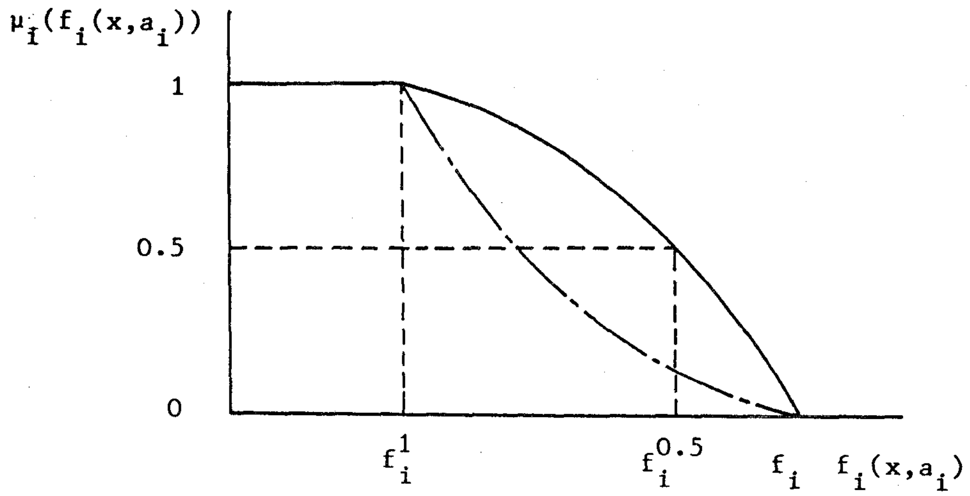

Fig. 5.4 Exponential membership function

(3) Hyperbolic membership function

For each objective function, the corresponding hyperbolic membership function is defined by:

$$\mu_i(f_i(x,a_i)) = (1/2) \tanh (\alpha_i(f_i(x,a_i) - \beta_i)) + (1/2), \qquad (5.3)$$

where $\alpha_i > 0$ or $\alpha_i < 0$.

The hyperbolic membership function can be determined by asking the DM to specify the two points, $f_i^{0.25}$ and $f_i^{0.5}$, within f_i^{max} and f_i^{min}, where α_i is a shape parameter and β_i is associated with the point of inflection. Fig. 5.5 illustrates the graph of the hyperbolic membership function.

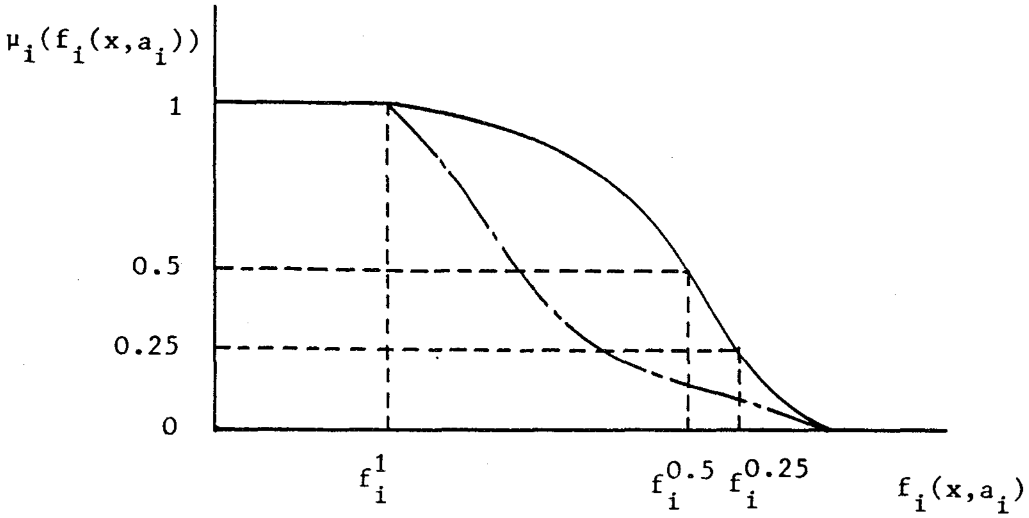

Fig. 5.5 Hyperbolic membership function

(4) Hyperbolic inverse membership function

For each objective function, the corresponding hyperbolic inverse membership function is defined by:

$$\mu_i(f_i(x,a_i)) = \alpha_i \tanh^{-1}(\beta_i(f_i(x,a_i) - \gamma_i)) + (1/2), \tag{5.4}$$

where $\alpha_i > 0$, $\beta_i > 0$ or $\beta_i < 0$.

The hyperbolic inverse membership function can be determined by asking the DM to specify the three points, f_i^0, $f_i^{0.25}$ and $f_i^{0.5}$, within f_i^{max} and f_i^{min}, where β_i is a shape parameter and γ_i is associated with the point of inflection. Fig. 5.6 illustrates the graph of the hyperbolic inverse membership function.

(5) Piecewise linear membership function

For each objective function, the corresponding piecewise linear membership function is defined by:

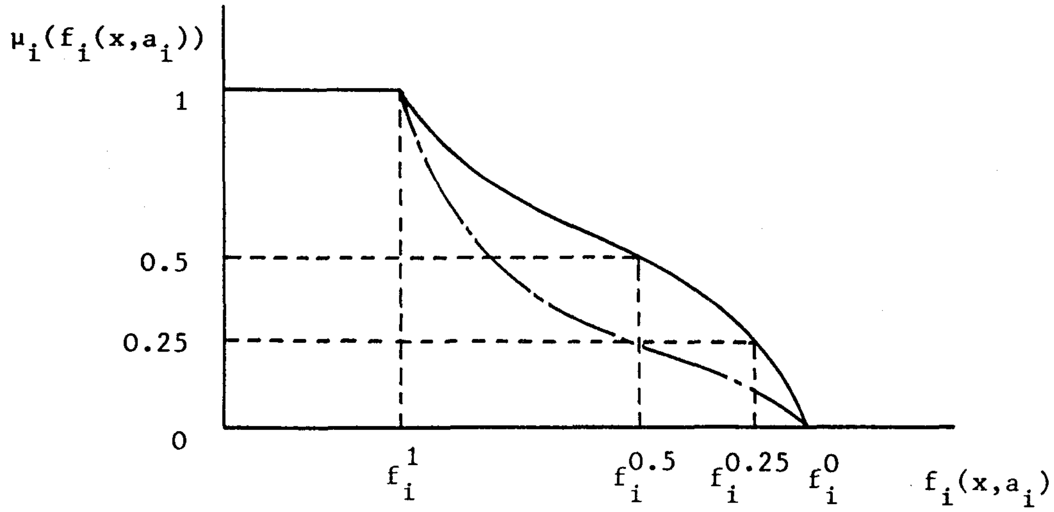

Fig. 5.6 Hyperbolic inverse membership function

$$\mu_i(f_i(x,a_i)) = \sum_{j=1}^{N_i} \alpha_{ij} \left| f_i(x,a_i) - g_{ij} \right| + \beta_i f_i(x,a_i) + \gamma_i \qquad (5.5)$$

where

$$\alpha_{ij} = (t_{i,j+1} - t_{ij})/2, \quad \beta_i = (t_{i,N_i+1} + t_{i1})/2,$$

$$\gamma_i = (s_{i,N_i+1} + s_{i1})/2. \qquad (5.6)$$

That is to say, it is assumed that $\mu_i(f_i(x,a_i)) = t_{ir}f_i(x,a_i) + s_{ir}$ for each segment $g_{i,r-1} \leq f_i(x,a_i) \leq g_{ir}$, where t_{ir} is the slope and s_{ir} is the y-intercept for the section of the curve initiated at $g_{i,r-1}$ and terminated at g_{ir}. The piecewise linear membership function can be determined by asking the DM to specify the degree of membership in each of several values of objective functions within f_i^0 and f_i^1. Fig. 5.7 illustrates the graph of the piecewise linear membership function.

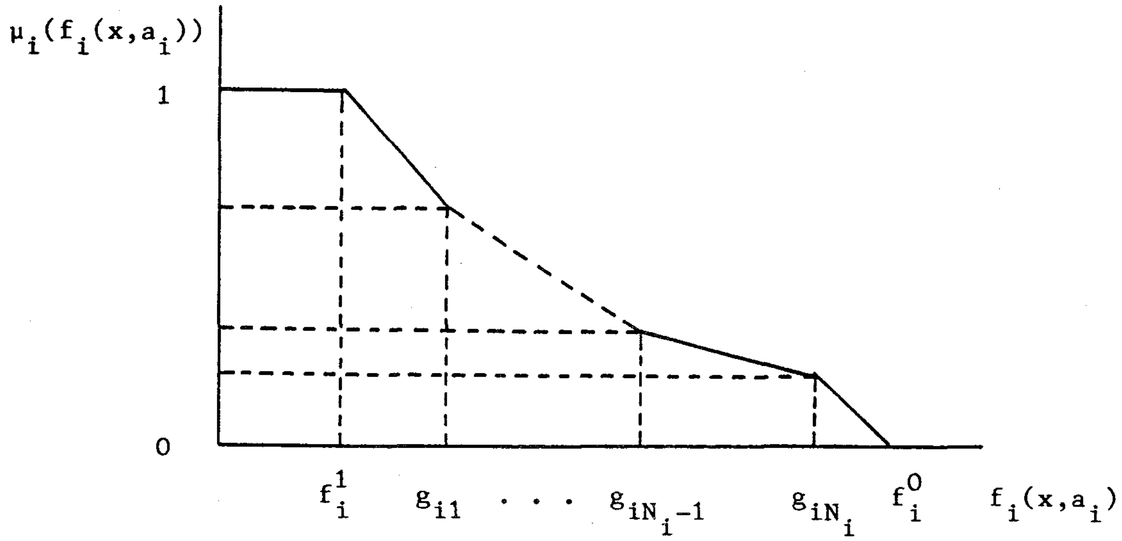

Fig. 5.7 Piecewise linear membership function

It should be noted here that for the fuzzy equal membership functions, the DM can select his/her left and right functions from among the same types of membership functions previously described above (excluding the hyperbolic ones).

5.2 Illustrative Examples with Computer Outputs

We now demonstrate the interaction processes for multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters using our corresponding computer programs by means of several illustrative numerical examples which are designed to test each of the programs.

Example 5.1 (Linear Problem)

Consider the following three objective linear programming problem with fuzzy parameters.

$$\text{fuzzy min} \quad z_1(x,\tilde{c}_1) = \tilde{c}_{11}x_1 - 4x_2 + x_3$$

$$\text{fuzzy max} \quad z_2(x,\tilde{c}_2) = -3x_1 + \tilde{c}_{22}x_2 + x_3$$

$$\text{fuzzy equal} \quad z_3(x,\tilde{c}_3) = 5x_1 + x_2 + \tilde{c}_{33}x_3$$

subject to

$$\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + 3x_3 \leq 12, \qquad x_1 + 2x_2 + \tilde{a}_{23}x_3 \leq \tilde{b}_2$$

$$x_i \geq 0, \quad i=1,2,3$$

The membership functions for the fuzzy numbers $\tilde{a}_{11},\ldots,\tilde{a}_{23}$, $\tilde{b}_2$, $\tilde{c}_{11},\ldots,\tilde{c}_{33}$ are explained in Table 5.1, where L and E represent respectively linear and exponential membership functions.

Table 5.1 Fuzzy numbers for Example 5.1

$\tilde{t}$	p_1	p_2	p_3	p_4	left	right
$\tilde{c}_{11}$	0.00,	2.00,	2.00,	2.50	L	E
$\tilde{c}_{22}$	-1.25,	-0.75,	-0.75,	-0.25	E	E
$\tilde{c}_{33}$	-0.25,	0.00,	0.00,	1.00	E	E
$\tilde{a}_{11}$	0.00,	3.00,	3.00,	4.00	L	L
$\tilde{a}_{12}$	0.50,	1.00,	1.00,	1.50	E	L
$\tilde{a}_{23}$	0.50,	1.00,	1.00,	1.50	E	E
$\tilde{b}_2$	8.00,	12.00,	12.00,	14.00	L	E

In applying our computer program to this problem, suppose that the interaction with the hypothetical DM establishes the following membership functions and the corresponding assessment values for three objective functions in the α-MOLP.

z_1 : exponential $(z_{1R}^{0}, z_{1R}^{0.5}, z_{1R}^{1}) = (30, -10, -25)$

z_2 : hyperbolic inverse $(z_{2L}^{0}, z_{2L}^{0.25}, z_{2L}^{0.5}) = (-8, -7.5, -6.5)$

$$z_3 \begin{cases} \text{left exponential } (z_{3L}^{0}, z_{3L}^{0.5}, z_{3L}^{1}) = (12, 14, 15) \\ \\ \text{right linear } (z_{3R}^{0}, z_{3R}^{1}) = (18, 15) \end{cases}$$

In the following illustrations, interaction processes using the time-sharing computer program under TSS of MELCOM COSMO 700S digital computer in the computer center of Kagawa University in Japan are explained through the aid of some of the computer outputs.

Illustration 5.1

Using the MINMAX command, the calculated individual minimum and maximum of each of the objective functions z_1, z_2 and z_3 for $\alpha = 0$ and $\alpha = 1$ are displayed.

```
COMMAND:
?MINMAX

         INDIVIDUAL MINIMUM AND MAXIMUM FOR ALFA=1

          I          MINIMUM          I          MAXIMUM
----------+---------------------------+-------------------------
  Z( 1)   I                -24.0000   I                 8.0000
  Z( 2)   I                -12.0000   I                 4.0000
  Z( 3)   I                   .0000   I                20.0000

         INDIVIDUAL MINIMUM AND MAXIMUM FOR ALFA=0

          I          MINIMUM          I          MAXIMUM
----------+---------------------------+-------------------------
  Z( 1)   I                -28.0000   I                35.0000
  Z( 2)   I                -42.0000   I                 4.0000
  Z( 3)   I                 -1.0000   I                70.0000
```

Illustration 5.2

The MF command is utilized to determine the membership functions for each of the objective functions z_1, z_2 and z_3 sequentially. Here the interaction with the hypothetical DM establishes that the first membership function should be exponential, the second hyperbolic inverse and the the third exponential and linear. For each type of membership functions, the corresponding assessment values are input in a subjective manner by considering the calculated individual minimum and maximum of each of the objective functions.

```
COMMAND:
?MF
INPUT THE OBJECTIVE FUNCTION NUMBER:
?1
INPUT FUZZY GOAL:
   (1) FUZZY MAX
   (2) FUZZY MIN
   (3) FUZZY EQUAL
?2
DO YOU WANT LIST OF MEMBERSHIP FUNCTION TYPE ?
?YES
LIST OF MEMBERSHIP FUNCTION TYPE
  (1) LINEAR
  (2) EXPONENTIAL
  (3) HYPERBOLIC
  (4) HYPERBOLIC INVERSE
  (5) PIECEWISE LINEAR
```

```
INPUT MEMBERSHIP FUNCTION TYPE:
?2
INPUT THREE POINTS(Z1,Z2,Z3) SUCH THAT
         M(Z1)=0.0  ( Z1 : UNACCEPTABLE LEVEL )
         M(Z2)=0.5
         M(Z3)=1.0  ( Z3 : TOTALLY DESIRABLE LEVEL )
?30 -10 -25
ANOTHER MSET ?
?YES
INPUT THE OBJECTIVE FUNCTION NUMBER:
?2
INPUT FUZZY GOAL:
  (1) FUZZY MAX
  (2) FUZZY MIN
  (3) FUZZY EQUAL
?1
DO YOU WANT LIST OF MEMBERSHIP FUNCTION TYPE ?
?NO
INPUT MEMBERSHIP FUNCTION TYPE:
?4
INPUT THREE POINTS(Z1,Z2,Z3) SUCH THAT
         M(Z1)=0.00  ( Z1 : UNACCEPTABLE LEVEL )
         M(Z2)=0.25
         M(Z3)=0.50
?-8 -7.5 -6.5
ANOTHER MSET ?
?YES
INPUT THE OBJECTIVE FUNCTION NUMBER:
?3
INPUT FUZZY GOAL:
  (1) FUZZY MAX
  (2) FUZZY MIN
  (3) FUZZY EQUAL
?3
DO YOU WANT LIST OF MEMBERSHIP FUNCTION TYPE ?
?NO
INPUT LEFT AND RIGHT TYPE:
?2 1
-------------------- LEFT --------------------
INPUT THREE POINTS(Z1,Z2,Z3) SUCH THAT
         M(Z1)=0.0  ( Z1 : UNACCEPTABLE LEVEL )
         M(Z2)=0.5
         M(Z3)=1.0  ( Z3 : TOTALLY DESIRABLE LEVEL )
?12 14 15
-------------------- RIGHT -------------------
INPUT TWO POINTS(Z1,Z2) SUCH THAT
         M(Z1)=0.0  ( Z1 : UNACCEPTABLE LEVEL )
         M(Z2)=1.0  ( Z2 : TOTALLY DESIRABLE LEVEL )
?18 15
ANOTHER MSET ?
?NO
```

Illustration 5.3

With the GRAPH command, the shape of the membership function for z_1 is shown graphically. Thus the DM can check the properties of his/her membership functions visually.

```
COMMAND:
?GRAPH
INPUT THE MEMBERSHIP FUNCTION NUMBER:
?1
GRAPH OF THE MEMBERSHIP FUNCTION ( NO. 1 )
  MEMBERSHIP FUNCTION TYPE --- EXPONENTIAL


1.00*---------|---------|---------|---------|---------+
    | *                                               |
    |                                                 |
    |                                                 |
    |                                                 |
    |   *                                             |
    |                                                 |
    |                                                 |
    |      *                                          |
    |                                                 |
 .50-        *                                        -
    |                                                 |
    |          *                                      |
    |                                                 |
    |            *                                    |
    |               *                                 |
    |                  *                              |
    |                    *                            |
    |                      *                          |
    |                         *                       |
 .00+---------|---------|------*-*|*--------|---------+
  -2.80      -.80      1.20      3.20      5.20      7.20
    (10E 1)
```

Illustration 5.4

Using the GO command, the minimax problem is solved for the initial reference membership values and the degree α , and the DM is supplied with the corresponding M-α-Pareto optimal solution and the trade-off rates between the membership functions and the degree α. Since the DM is not satisfied with the current values of the membership functions, the DM updates his/her reference membership values.

```
COMMAND:
?GO

----------------------< ITERATION 1 >----------------------

INITIATES AN INTERACTION WITH ALL THE INITIAL REFERENCE
MEMBERSHIP VALUES ARE 1

INPUT THE DEGREE ALFA OF THE ALFA LEVEL SETS
FOR THE FUZZY PARAMETERS:
?0.75

APPROXIMATE SOLUTION TO THE MINIMAX PROBLEM
FOR INITIAL REFERENCE MEMBERSHIP VALUES
------------------------------------------------------------
         MEMBERSHIP            I      OBJECTIVE FUNCTION
-------------------------------+----------------------------
   M(Z1) =             .4913   I   Z(1) =          -9.6425
   M(Z2) =             .4912   I   Z(2) =          -6.5401
   M(Z3) =             .4912   I   Z(3) =          13.9773
------------------------------------------------------------
   X( 1) =            2.0385     X( 2) =            3.5563
   X( 3) =            1.5249
------------------------------------------------------------

M-ALFA-PARETO OPTIMALITY TEST
  EPS( 1)=  .0000000000D+00
  EPS( 2)=  .0000000000D+00
  EPS( 3)=  .0000000000D+00

------------------------------------------------------------
TRADE-OFFS AMONG MEMBERSHIP FUNCTIONS
 -DM(Z2)/DM(Z1) =              .1589
 -DM(Z3)/DM(Z1) =              .3625
------------------------------------------------------------
TRADE-OFF BETWEEN ALFA AND MEMBERSHIP FUNCTION M(Z1)
 -DM(Z1)/DALFA =            8.4098
------------------------------------------------------------

ARE YOU SATISFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION ?
?NO

----------------------< ITERATION 2 >----------------------

CONSIDER THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION TOGETHER WITH
THE TRADE-OFFS AMONG THE MEMBERSHIP FUNCTIONS AND
THE TRADE-OFF BETWEEN ALFA AND M(Z1)
```

```
THEN INPUT YOUR REFERENCE MEMBERSHIP VALUES
FOR EACH OF THE MEMBERSHIP FUNCTIONS:
?0.45 0.5 0.55

INPUT THE DEGREE ALFA OF THE ALFA LEVEL SETS
FOR THE FUZZY PARAMETERS:
?0.75
```

In this example, at the 5th iteration, the satisficing solution of the DM is derived and the whole interactive processes are summarized in Table 5.2.

Table 5.2 Interactive processes for Example 5.2

Iteration	1	2	3	4	5
$\bar{\mu}_1$	1	0.45	0.4	0.4	0.4
$\bar{\mu}_2$	1	0.5	0.5	0.5	0.6
$\bar{\mu}_3$	1	0.55	0.55	0.55	0.65
α	0.75	0.75	0.75	0.65	0.65
μ_1	0.4913	0.4331	0.3888	0.4727	0.3773
μ_2	0.4912	0.4830	0.4888	0.5726	0.5772
μ_3	0.4912	0.5330	0.5388	0.6226	0.6272
z_1	-9.6425	-7.1239	-5.0258	-8.8649	-4.4564
z_2	-6.5401	-6.5773	-6.5513	-6.1726	-6.1526
z_3	13.9773	14.0832	14.0972	14.2933	14.3035
x_1	2.0385	2.1716	2.2677	2.1092	2.3074
x_2	3.5563	2.9893	2.5144	3.3520	2.3443
x_3	1.5249	1.5760	1.6302	1.8010	1.9210
$-\partial\mu_2/\partial\mu_1$	0.1589	0.1755	0.1901	0.0606	0.0722
$-\partial\mu_3/\partial\mu_1$	0.3625	0.4205	0.4593	0.1589	0.1894
$-\partial\mu_1/\partial\alpha$	8.4098	7.2504	6.3777	22.1895	17.3201

CPU time required in this interaction process was 9.791 seconds and the example session takes about 7 minutes.

Example 5.2 (Linear Fractional Problem)

Consider the following three objective linear fractional programming problem with fuzzy parameters.

$$\text{fuzzy min} \quad z_1(x,\tilde{c}_1,\tilde{d}_1) = (\tilde{c}_{11}x_1 + x_3 + 1) \,/\, (x_1 + \tilde{d}_{12}x_2 + \tilde{d}_{13})$$

$$\text{fuzzy max} \quad z_2(x,\tilde{d}_2) \quad = (3x_1 + x_2 + 1) \,/\, (-x_1 - 2x_3 + \tilde{d}_{23})$$

$$\text{fuzzy equal} \quad z_3(x,\tilde{c}_3,\tilde{d}_3) = (x_2 + \tilde{c}_{32}x_3 + 1) \,/\, (\tilde{d}_{31}x_1 + x_3 + \tilde{d}_{33})$$

subject to

$$x \in X(\tilde{a},\tilde{b}) = \{ (x_1,x_2,x_3) \mid x_1 + x_2 + \tilde{a}_{13}x_3 \le \tilde{b}_1,$$

$$\tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 + x_3 \le \tilde{b}_2, \quad \tilde{a}_{31}x_1 - x_2 \le 0, \quad x_i \ge 0,\ i=1,2,3 \}$$

The membership functions for the fuzzy numbers $\tilde{a}_{13},\ldots,\tilde{a}_{31}, \tilde{b}_1, \tilde{b}_2, \tilde{c}_{11},\tilde{c}_{32}, \tilde{d}_{12},\ldots,\tilde{d}_{33}$ in this example are explained in Table 5.3. where L and E represent respectively linear and exponential membership functions.

Table 5.3 Fuzzy numbers for Example 5.2

$\tilde{t}$	p_1	p_2	p_3	p_4	left	right
$\tilde{c}_{11}$	0.9 ,	1.1 ,	1.1 ,	1.3	L	E
$\tilde{c}_{32}$	4.0 ,	4.0 ,	4.5 ,	5.5	L	L

$\tilde{d}_{12}$	1.2 ,	1.4 ,	1.4 ,	1.6	E	E
$\tilde{d}_{13}$	2.7 ,	2.9 ,	2.9 ,	3.1	E	L
$\tilde{d}_{23}$	9. ,	11. ,	11. ,	13.	E	E
$\tilde{d}_{31}$	0.9 ,	1.1 ,	1.1 ,	1.3	L	L
$\tilde{d}_{33}$	4.8 ,	5.2 ,	5.2 ,	5.4	L	L
$\tilde{a}_{13}$	0.9 ,	1.1 ,	1.2 ,	1.3	L	L
$\tilde{a}_{21}$	1.8 ,	2.2 ,	2.2 ,	2.4	L	E
$\tilde{a}_{22}$	1.8 ,	2.2 ,	2.2 ,	2.4	E	E
$\tilde{a}_{31}$	0.9 ,	1.1 ,	1.1 ,	1.2	E	L
$\tilde{b}_{1}$	2.8 ,	3.2 ,	3.2 ,	3.4	E	L
$\tilde{b}_{2}$	3.8 ,	4.2 ,	4.2 ,	4.4	L	L

In applying our computer program to this problem, suppose that the interaction with the hypothetical DM establishes the following membership functions and the corresponding assessment values for the three objective functions in the α-MOLFP.

z_1 : exponential, $(z_1^0, z_1^{0.5}, z_1^1) = (1.25, 0.6, 0.25)$

z_2 : hyperbolic inverse, $(z_2^0, z_2^{0.25}, z_2^{0.5}) = (0.1, 0.15, 0.28)$

$$z_3 : \begin{cases} \text{left : exponential, } (z_3^0, z_3^{0.5}, z_3^1) = (0.5, 1.1, 1.4) \\ \\ \text{right : linear, } (z_3^0, z_3^1) = (1.6, 1.4) \end{cases}$$

In this example, at the 6th iteration, the satisficing solution of the DM is derived and the whole interactive processes are summarized in Table 5.4.

Table 5.4 Interactive processes for Example 5.2

Iteration	1	2	3	4	5	6
$\bar{\mu}_1$	1	0.5	0.6	0.63	0.63	0.63
$\bar{\mu}_2$	1	0.7	0.65	0.6	0.6	0.6
$\bar{\mu}_3$	1	0.9	0.8	0.75	0.75	0.79
α	0.9	0.9	0.9	0.9	0.8	0.8
μ_1	0.6179	0.4689	0.5624	0.6109	0.6412	0.6349
μ_2	0.6179	0.6689	0.6124	0.5809	0.6112	0.6049
μ_3	0.6179	0.8689	0.7624	0.7309	0.7612	0.7949
z_1	0.5023	0.6280	0.5468	0.5078	0.4844	0.4891
z_2	0.3506	0.3767	0.3476	0.3297	0.3469	0.3435
z_3	1.1848	1.3342	1.2751	1.2565	1.2744	1.2938
x_1	0.1961	0.2377	0.1679	0.1156	0.1125	0.0982
x_2	1.2153	0.9899	1.1569	1.2510	1.3226	1.3221
x_3	1.2149	1.6036	1.3997	1.3129	1.2711	1.3025
$-\partial\mu_2/\partial\mu_1$	0.9279	1.2308	0.9428	0.8357	0.9329	0.9183
$-\partial\mu_3/\partial\mu_1$	2.2354	2.7608	2.4766	2.3718	2.5674	2.6148
$-\partial\mu_1/\partial\alpha$	0.7945	0.8201	0.8168	0.8156	0.7269	0.7322

Example 5.3 (Nonlinear Problem)

Consider the following three objective nonlinear programming problem with fuzzy parameters.

$$\text{fuzzy min} \quad f_1(x,\tilde{a}_1) = (x_1 + 5)^2 + \tilde{a}_{11}x_2^2 + 2(x_3 - \tilde{a}_{12})^2$$

$$\text{fuzzy min} \quad f_2(x,\tilde{a}_2) = \tilde{a}_{21}(x_1 - 45)^2 + (x_2 + 15)^2 + 3(x_3 + \tilde{a}_{22})^2$$

$$\text{fuzzy equal} \quad f_3(x,\tilde{a}_3) = \tilde{a}_{31}(x_1 + 20)^2 + \tilde{a}_{32}(x_2 - 45)^2 + (x_3 + 15)^2$$

subject to

$$x \in X(\tilde{b}) = \{ (x_1, x_2, x_3) \mid g_1(x,\tilde{b}_1) = \tilde{b}_{11}x_1^2 + \tilde{b}_{12}x_2^2 + \tilde{b}_{13}x_3^2 \leq 100,$$
$$0 \leq x_i \leq 10 \text{ , } i=1,2,3 \}.$$

The membership functions for the fuzzy numbers $\tilde{a}_{11}, \ldots, \tilde{a}_{32}, \tilde{b}_{11}, \ldots, \tilde{b}_{13}$ in this example are explained in Table 5.5 where L and E represent respectively linear and exponential membership functions.

Table 5.5 Fuzzy numbers for Example 5.3

$\tilde{t}$	p_1	p_2	p_3	p_4	left	right
$\tilde{a}_{11}$	3.8 ,	4.0 ,	4.0 ,	4.3	L	E
$\tilde{a}_{12}$	48.5 ,	50.0 ,	50.0 ,	52.0	E	E
$\tilde{a}_{21}$	1.85,	2.0 ,	2.0 ,	2.2	E	L
$\tilde{a}_{22}$	18.2 ,	20.0 ,	20.0 ,	22.5	L	E
$\tilde{a}_{31}$	2.9 ,	3.0 ,	3.0 ,	3.15	E	L
$\tilde{a}_{32}$	4.7 ,	5.0 ,	5.0 ,	5.35	L	L
$\tilde{b}_{11}$	0.9 ,	1.0 ,	1.0 ,	1.1	E	E

$\tilde{b}_{12}$	0.8 ,	1.0 ,	1.0 ,	1.2	E	E
$\tilde{b}_{13}$	0.85,	1.0 ,	1.0 ,	1.15	E	L

In applying our computer program to this problem, suppose that the interaction with the hypothetical DM establishes the following membership functions and the corresponding assessment values for the three objective functions.

f_1 : exponential, $(f_1^0, f_1^{0.5}, f_1^1) = (5400, 5000, 3300)$

f_2 : exponential, $(f_2^0, f_2^{0.5}, f_2^1) = (6900, 4600, 3900)$

$$f_3 \begin{cases} \text{left : hyperbolic inverse,} & (f_3^0, f_3^{0.5}, f_3^1, b_3) \\ & = (7800, 8200, 10000, 9000) \\ \text{right : exponential,} & (f_3^0, f_3^{0.5}, f_3^1) = (13300, 11000, 10000) \end{cases}$$

In Fig. 5.8, the interaction processes using the time-sharing computer program under TSS of MELCOM COSMO 700S digital computer in the computer center of Kagawa University in Japan are explained especially for the first iteration through the aid of some of the computer outputs. M-α-Pareto optimal solutions are obtained by solving the augmented minimax problem using the revised version of the generalized reduced gradient (GRG) (Lasdon, Fox and Ratner 1974) program called GRG2 (Lasdon, Waren and Ratner 1980).

In this example, at the 5th iteration, the satisficing solution of the DM is derived. The whole interactive processes are summarized in Table 5.6. CPU time required in this interaction process was 114.7 seconds and the example session takes about 16 minutes.

```
COMMAND:
?GO

INPUT THE DEGREE ALFA OF THE ALFA LEVEL SETS
FOR THE FUZZY PARAMETERS:
?0.9

----------------------< ITERATION 1 >----------------------

INITIATES AN INTERACTION WITH ALL THE INITIAL REFERENCE
MEMBERSHIP VALUES ARE 1

( KUHN-TUCKER CONDITIONS SATISFIED )

M-ALFA-PARETO OPTIMAL SOLUTION
TO THE AUGEMENTED MINIMAX PROBLEM
FOR INITIAL REFERENCE MEMBERSHIP VALUES
------------------------------------------------------------
         MEMBERSHIP           I      OBJECTIVE FUNCTION
------------------------------+-----------------------------
   M(F1) =             .5678  I  F(1) =         4917.7079
   M(F2) =             .5678  I  F(2) =         4475.2311
   M(F3) =             .5678  I  F(3) =        10828.7795
------------------------------------------------------------
   X( 1) =            8.4567    X( 2) =            5.2592
   X( 3) =            1.8210
------------------------------------------------------------
TRADE-OFFS AMONG MEMBERSHIP FUNCTIONS
 -DM(F2)/DM(F1) =            .6433
 -DM(F3)/DM(F1) =           1.5303
------------------------------------------------------------
TRADE-OFFS BETWEEN ALFA AND MEMBERSHIPS
 -DM(F)/DALFA =              .2143
------------------------------------------------------------

ARE YOU SATIFIED WITH THE CURRENT MEMBERSHIP VALUES OF
THE M-ALFA-PARETO OPTIMAL SOLUTION ?
?NO
```

Fig. 5.8 Computer outputs for Example 5.3

Table 5.6 Interactive processes for Example 5.3

Iteration	1	2	3	4	5
$\bar{\mu}_1$	1	0.5	0.55	0.55	0.65
$\bar{\mu}_2$	1	0.6	0.50	0.50	0.55
$\bar{\mu}_3$	1	0.8	0.75	0.75	0.75
α	0.9	0.9	0.9	0.7	0.7
μ_1	0.5678	0.4558	0.5543	0.5979	0.6392
μ_2	0.5678	0.5558	0.5043	0.5479	0.5392
μ_3	0.5678	0.7558	0.7543	0.7979	0.7392
f_1	4917.71	5048.17	4934.94	4877.16	4816.81
f_2	4475.23	4496.28	4591.57	4510.35	4526.11
f_3	10828.78	10422.53	10425.39	10342.49	10455.02
x_1	8.4567	8.0461	7.9090	8.1024	8.1775
x_2	5.2592	6.0275	6.0594	6.0905	5.8785
x_3	1.8210	1.2689	1.8437	1.9837	2.2585
$-\partial\mu_2/\partial\mu_1$	0.6433	0.4845	0.5757	0.6824	0.7638
$-\partial\mu_3/\partial\mu_1$	1.5303	1.2998	1.6415	1.9187	2.1046
$-\partial\mu_i/\partial\alpha$	0.2143	0.2390	0.2164	0.2193	0.2074

CHAPTER 6

CONCLUSION

In this thesis, in order to deal with the imprecise or vague nature of the human judgements in the real-world decision situations involving multiple, noncommensurate and conflicting objectives, the theory of fuzzy set has been incorporated in multiobjective programming problems. The most important conclusions drawn from this thesis will be summarized in a set of brief statements:

(1) Considering the imprecise or fuzzy nature of the human judgements in multiobjective programming problems under fuzziness, two types of fuzziness of human judgements have been incorporated : one is the experts' ambiguous understanding of the nature of the parameters in the problem-formulation process, and the other is the fuzzy goals of the decision maker (DM) for each of the objective functions.

(2) To cope with both types of fuzziness, multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters have been formulated and the fuzzy goals of the DM for each of the objective functions are quantified by eliciting the corresponding membership functions.

(3) The concept of α-multiobjective linear, linear fractional and nonlinear programming together with (local) M-α-Pareto optimality have been introduced on the basis of the α-level sets of the fuzzy numbers.

(4) New interactive decision making methods for multiobjective linear, linear fractional and nonlinear programming problems have been presented to derive the satisficing solution of the DM efficiently from among (local) M-α-Pareto optimal solution set on the basis of his/her subjective precise or imprecise value-judgements.

(5) In our interactive schemes, the (local) satisficing solution of the DM can be derived efficiently by updating the reference membership values and/or the degree α based on the current values of the (local) M-α-Pareto optimal solution together with the trade-off rates between the membership functions and the degree α. Furthermore, (local) M-α-Pareto optimality of the generated solution in each iteration is guaranteed.

(6) Concerning multiobjective linear and linear fractional programming problems, for generating a candidate for the satisficing solution which is also M-α-Pareto optimal, minimax problems were adopted, and consequently, it was shown that the formulated minimax problems can be solved based mainly on the well-known linear programming method. This is a remarkable result, because the traditional linear programming is very popular among many scientists and also relatively large-scale problems may be solved compared with the nonlinear case.

(7) In the nonlinear case, however, since some nonlinear programming codes are necessary, it was recommended to adopt augmented minimax

problems instead of minimax problems for circumventing the necessity to perform the (local) M-α-Pareto optimality tests for the current solution to the minimax problems.

(8) On the basis of the proposed methods, the time-sharing computer programs for all the proposed methods have been written in FORTRAN to implement man-machine interactive procedures. Illustrative numerical examples for multiobjective linear, linear fractional and nonlinear programming problems with fuzzy parameters were shown along with the corresponding computer outputs. Interactive processes for the numerical examples demonstrated the feasibility and efficiency of both the proposed methods and the corresponding interactive computer programs by simulating the responses of the hypothetical DM. Although the actual DM for the numerical examples would of course select other (local) M-α-Pareto optimal solutions than the ones which were selected by the hypothetical DM used in this thesis, the way to iterate and calculate is essentially the same.

However, applications to the real-world problems must be carried out in cooperation with a person actually involved in decision making. From such experiences the proposed methods and the corresponding interactive computer programs must be revised. We hope that the proposed methods and their extensions will become efficient tools for man-machine interactive decision making under multiple conflict objectives and fuzziness.

Appendix

HYPERPLANE METHODS AND TRADE-OFF RATES

In this Appendix, as a generalization of the well-known existing scalarizing methods for multiobjective programming problems (e.g., Chankong and Haimes, 1983a,b or Lightner, 1979), a new scalarizing method called the hyperplane method is introduced by putting the special emphasis not only on generating Pareto optimal solutions but also on obtaining trade-off information.

A.1 Hyperplane Problems

Consider multiobjective nonlinear programming (MONLP) problems of the following form:

$$\min_{x \in X} f(x) \triangleq (f_1(x), f_2(x), \ldots, f_k(x)) \tag{A.1}$$

$$\text{subject to} \quad x \in X \triangleq \{ x \in E^n \mid g_j(x) \leqq 0, \ j=1,\ldots,m \},$$

where x is an n-dimensional vector of decision variables, $f_1(x),\ldots,f_k(x)$ are k distinct objective functions of the decision vector x, $g_1(x),\ldots,$ $g_m(x)$ are m inequality constraints, X is the feasible set of constrained decisions, and the functions $f_i(x)$, $i=1,\ldots,k$, $g_j(x)$, $j=1,\ldots,m$ are assumed to be twice continuously differentiable.

Associated with the MONLP (A.1), define the following generalized scalar optimization problem called the hyperplane problem.

$$\mathrm{HP}(\Lambda, t_{-k}) \qquad \min_{x \in X} \quad z \tag{A.2}$$

$$\text{subject to} \quad Q(\Lambda)F(f(x)) \leqq D(z, t_{-k})$$

where Λ is a closed convex cone in the k-dimensional real space, and the generators of Λ are denoted by $\{ q_1, \ldots, q_k \}$. To be more specific, Λ is defined by:

$$\Lambda = \sum_{i=1}^{k} \alpha_i q_i , \quad \alpha_i \geqq 0 , \; i=1,\ldots,k, \quad \sum_{i=1}^{k} \alpha_i > 0 \tag{A.3}$$

$$q_i \triangleq (q_{i1}, \ldots, q_{ik}) . \tag{A.4}$$

$Q(\Lambda)$ is the (k×k) dimensional square matrix whose i-th row vector is the i-th generator of Λ, i.e.,

$$Q(\Lambda) = \begin{bmatrix} q_1 \\ \cdots \\ q_k \end{bmatrix}. \tag{A.5}$$

$F(.)$ is the k-dimensional vector function defined by :

$$F(f(x)) \triangleq (F_1(f_1(x)), \ldots, F_k(f_k(x)))^T, \tag{A.6}$$

where $F_i(.)$, $i=1,\ldots,k$ is a strictly monotone increasing and continuously differentiable function on the range $\{ f_i(x) \mid x \in X \}$, i.e.,

$$\frac{\partial F_i(f_i(x))}{\partial f_i(x)} > 0 , \quad \text{for any } x \in X , \; i=1,\ldots,k. \tag{A.7}$$

$D(.)$ is the k-dimensional vector function defined by:

$$D(z, t_{-k}) \triangleq (D_1(z, t_{-k}), \ldots, D_k(z, t_{-k}))^T , \tag{A.8}$$

and $D_i(.)$, $i=1,\ldots,k$ is continuously differentiable on (z, t_{-k}).

Let

$$S \triangleq \{ (z, t_{-k}) \in R^k \mid \det A(z, t_{-k}) \neq 0 \} \tag{A.9}$$

where

$$A(z, t_{-k}) \triangleq \begin{bmatrix} \partial D_1/\partial z & \partial D_1/\partial t_1 & \cdots & \partial D_1/\partial t_{k-1} \\ \partial D_2/\partial z & \partial D_2/\partial t_1 & \cdots & \partial D_2/\partial t_{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ \partial D_k/\partial z & \partial D_k/\partial t_1 & \cdots & \partial D_k/\partial t_{k-1} \end{bmatrix} \tag{A.10}$$

is the Jacobian matrix of $D(.)$.

Then it is reasonable to assume that $D(z, t_{-k})$ satisfies the following two properties.

Property A.1

There exists $(z, t_{-k}) \in S$ satisfying

$$Q(\Lambda)F(f(x)) = D(z, t_{-k}), \quad \text{for any } x \in X \tag{A.11}$$

Property A.2

$$\frac{\partial D(z, t_{-k})}{\partial z} \geq 0, \quad \text{for any } (z, t_{-k}) \in S \tag{A.12}$$

Now, consider the case where the closed convex cone Λ in the $HP(\Lambda, t_{-k})$ is set as follows:

$$\Lambda = \Lambda^{\geqq} \triangleq \{ \sum_{i=1}^{k} \alpha_i e_i \mid \alpha_i \geqq 0, \ i=1,\ldots,k, \ \sum_{i=1}^{k} \alpha_i > 0 \} \tag{A.13}$$

where e_i, $i=1,\ldots,k$ is a k-dimensional identity vector.

Then, the $HP(\Lambda, t_{-k})$ becomes as follows.

$$HP(\Lambda^{\geqq}, t_{-k}) \quad \min_{x \in X} \quad z \tag{A.14}$$

$$\text{subject to} \quad F(f(x)) \leqq D(z, t_{-k})$$

The relationships between an optimal solution of the HP($\Lambda^{\geq}$,t_{-k}) and a Pareto optimal solution of the MONLP (A.1) can be characterized by the following theorems.

Theorem A.1

(1) If there exists some t_{-k} such that x* is a unique optimal solution to the HP($\Lambda^{\geq}$,t_{-k}), then x* is a Pareto optimal solution of the MONLP.

(2) If x* is a Pareto optimal solution of the MONLP, then there exists some t_{-k} such that x* is an optimal solution to the HP($\Lambda^{\geq}$,t_{-k}).

(Proof)

(1) Assume that x* is a unique optimal solution to the HP($\Lambda^{\geq}$,t_{-k}) and z* is a corresponding minimum value, then it holds that $F(f(x^*)) \leqq D(z^*, t_{-k})$. If x* is not a Pareto optimal solution to the MONLP, there exists $x \in X$ such that $f(x) \lneq f(x^*)$, or equivalently $F(f(x)) \lneq F(f(x^*))$. Therefore, it holds that $F(f(x)) \lneq F(f(x^*)) \leqq D(z^*, t_{-k})$. This contradicts the fact that x* is a unique optimal solution to the HP($\Lambda^{\geq}$,t_{-k}).

(2) Assume that x* is a Pareto optimal solution to the MONLP, then there exists $(z^*, t^*_{-k}) \in S$ such that $F(f(x^*)) = D(z^*, t^*_{-k})$ from Property A.1 and $x^* \in X$. If (x*,z*) is not an optimal solution to the HP($\Lambda^{\geq}$,t^*_{-k}), there exist z and $x \in X$ such that $z < z^*$ and $F(f(x)) \leqq D(z, t^*_{-k})$.

Therefore, it holds that $F(f(x)) \leq D(z,t^*_{-k}) \lneq D(z^*,t^*_{-k}) = F(f(x^*))$. This means that $f(x) \lneq f(x^*)$, which contradicts to the assumption that x^* is a Pareto optimal solution to the MONLP. Q.E.D.

To investigate the relationships between the hyperplane problem $HP(\Lambda^{\geq},t_{-k})$ and the minimax problems $MP(t_{-k})$ discussed in chapters 2, 3 and 4, assume that

$$D(z,t_{-k}) = D_{MP}(z,t_{-k}) \triangleq (z/t_1 ,\dots, z/t_k)^T, \quad t_i = t_j, i \neq j \qquad (A.15)$$

Then, by using the substitution

$$f_i(x) = -\mu_i(f_i(x)), \quad i=1,\dots,k, \qquad (A.16)$$

$$F_i(\mu_i(f_i(x))) = \bar{\mu}_i - \mu_i(f_i(x)), \quad i=1,\dots,k, \qquad (A.17)$$

it is evident that the $HP(\Lambda^{\geq},t_{-k})$ reduced to the minimax problems discussed in chapter 2,3 and 4.

It is now appropriate to show that $D_{MP}(z,t_{-k})$ satisfies the Properties A.1 and A.2. Let

$$S_{MP} = \{ (z,t_{-k}) \mid z \neq 0, \sum_{i=1}^{k} t_i = 1, t_i > 0, i=1,\dots,k \}$$

Then the Jacobian matrix $A_{MP}(.)$ for $D_{MP}(.)$ becomes as follows:

$$A_{MP}(z,t_{-k}) = \begin{bmatrix} t_1^{-1} & -zt_1^{-2} & & 0 & \\ t_2^{-1} & 0 & -zt_2^{-2} & \cdots\cdots & -zt_{k-1}^{-2} \\ t_k^{-1} & zt_k^{-2} & zt_k^{-2} & \cdots\cdots & zt_k^{-2} \end{bmatrix}$$

By applying the fundamental transformations for the matrix $A_{MP}(.)$ repeatedly, we have

$$\det A_{MP}(z, t_{-k}) = z^{k-1} \prod_{i=1}^{k} t_i^{-2}.$$

Clearly it holds that

$$\det A_{MP}(z, t_{-k}) \neq 0, \quad \text{for any } (z, t_{-k}) \in S_{MP}.$$

On the other hand, $\{ D_{MP}(z,t_{-k}) \mid (z,t_{-k}) \in S_{MP} \}$ is equivalent to the region $\{ D = (D_1, \dots, D_k) \in E^k \mid D_i > 0 \text{ or } D_i < 0,\ i=1,\dots,k \}$. This implies that $D_{MP}(.)$ satisfies Property A.1 if $F(f(X)) \subset E_k^+$ or $F(f(X)) \subset E_k^-$. Moreover, it holds that

$$\frac{\partial D_{MP}}{\partial z} = (t_1^{-1}, \dots, t_k^{-1}) > 0, \quad \text{for any } (z, t_{-k}) \in S_{MP}$$

Therefore, $D_{MP}(.)$ also satisfies Property A.2.

It should be emphasized here that for generating Pareto optimal solutions using Theorem A.1 the uniqueness of the solution must be verified. In order to circumvent the necessity to test the uniqueness of the solution to the $HP(\Lambda^{\geq}, t_{-k})$, we introduce the following hyperplane problems.

$$HP(\Lambda^{>}(\rho), t_{-k}) \qquad \min_{x \in X} \ z \tag{A.18}$$

$$\text{subject to} \quad Q(\Lambda^{>}(\rho))F(f(x)) \leq D(z, t_{-k})$$

where the closed convex cone $\Lambda^{>}(\rho)$ is defined as:

$$\Lambda^{>}(\rho) \triangleq \{ \sum_{i=1}^{k} \alpha_i q_i , \alpha = (\alpha_1 ,\dots, \alpha_k)^T \geq 0 \} \tag{A.19}$$

$$q_i = (\rho,\dots,\rho,1+\rho,\rho,\dots,\rho)^T \tag{A.20}$$

and ρ is the sufficiently small positive number.

Observe that the closed convex cone $\Lambda^{>}(\rho)$ is used instead of the open convex cone $\Lambda^{>}$. Then the HP($\Lambda^{>}(\rho),t_{-k}$) becomes as follows:

$$\min_{x \in X} \quad z \tag{A.21}$$

$$\text{subject to } F_i(f_i(x)) + \rho \sum_{j=1}^{k} F_j(f_j(x)) \leq D_i(z,t_{-k}), \; i=1,\dots,k.$$

The following theorem shows that the optimal solution of the HP($\Lambda^{>}(\rho),t_{-k}$) is Pareto optimal even if it is not unique.

Theorem A.2

(1) If $x*$ is an optimal solution to the HP($\Lambda^{>}(\rho),t_{-k}$) for a positive scalar ρ, then $x*$ is a Pareto optimal solution to the MONLP.

(2) If $x*$ is a Pareto optimal solution to the MONLP, then there exist a sufficiently small positive scalar ρ and t_{-k} such that $x*$ is an optimal solution to the HP($\Lambda^{>}(\rho),t_{-k}$).

(Proof)

(1) Let $x*$ be an optimal solution to the HP($\Lambda^{>}(\rho),t_{-k}$) and $z*$ is a corresponding optimal value. Then it holds that $Q(\Lambda^{>}(\rho))F(f(x*)) \leq D(z*,t_{-k})$. Assume that $x*$ is not a Pareto optimal solution, then there

exists $x \in X$ such that $f(x) \leq f(x*)$, or equivalently $F(f(x)) \leq F(f(x*))$. Since all of the elements of the matrix $Q(\Lambda^{>}(\rho))$ are positive, it holds that

$$Q(\Lambda^{>}(\rho))F(f(x)) < Q(\Lambda^{>}(\rho))F(f(x*)) \leqq D(z*,t_{-k}).$$

This inequality means that there exist z such that $z < z*$ and $Q(\Lambda^{>}(\rho))F(f(x)) \leqq D(z,t_{-k})$, due to the fact that at least one of $D_i(.)$, $i=1,...,k$ is continuously differentiable on z from Property A.2. This is a contradiction. Hence x* is a Pareto optimal solution to the MONLP.

(2) Assume that (x*,z*) is not an optimal solution to the $HP(\Lambda^{>}(\rho),t_{-k})$ for any positive scalar ρ and t_{-k} such that $Q(\Lambda^{>}(\rho))F(f(x*)) = D(z*,t_{-k})$. Then there exist $x \in X$ and $z < z*$ such that $Q(\Lambda^{>}(\rho))F(f(x)) \leqq D(z,t_{-k})$. Since it holds that $D(z*,t_{-k}) \geq D(z,t_{-k})$ from Property A.2 and $z* > z$, the following relation holds.

$$Q(\Lambda^{>}(\rho)) \{ F(f(x)) - F(f(x*)) \} \leq 0$$

If either any $\{ F_i(f_i(x)) - F_i(f_i(x*)) \}$ is positive or all $\{ F_i(f_i(x)) - F_i(f_i(x*)) \}$, $i=1,...,k$ are zero, this inequality would be violated for sufficiently small positive ρ. Hence

$$F_i(f_i(x)) - F_i(f_i(x*)) \leq 0, \quad i=1,...,k$$

must hold. Since by (A.7), we have $f_i(x) - f_i(x*) \leq 0$, which contradicts the fact that x* is a Pareto optimal solution. Q.E.D.

Now, by substituting $D_{MP}(.)$ for $D(.)$ in the $HP(\Lambda^{>}(\rho),t_{-k})$, we can immediately obtain the following augmented minimax problem.

$$AMP(t_{-k}) \qquad \min_{x \in X} \quad z \tag{A.22}$$

$$\text{subject to} \quad Q(\Lambda^{>}(\rho))F(f(x)) \leq D_{MP}(z,t_{-k})$$

Then, by using the same substitution as was shown in (A.16) and (A.17), it can be easily understood that the $HP(\Lambda^{>}(\rho),t_{-k})$ reduces to the augmented minimax problem discussed in chapter 4.

A.2. Trade-off Rates

In order to develop a meaningful formula which relates the trade-off rates within the Pareto optimal solution set to the Lagrange multipliers of the $HP(\Lambda,t_{-k})$ where $\Lambda = \Lambda^{\geq}$ or $\Lambda = \Lambda^{>}(\rho)$, define the following Lagrangian function $L(x,z,\lambda,t_{-k})$ for the $HP(\Lambda,t_{-k})$.

$$L(x,z,\lambda,t_{-k}) = z + \sum_{i=1}^{k} \lambda_i \{ \sum_{j=1}^{k} q_{ij} F_j (f_j(x)) - D_i(z,t_{-k}) \} \tag{A.23}$$

In the following let us assume that (x^*,z^*) is a unique local optimal solution of the $HP(\Lambda,t_{-k})$ and satisfies the following three assumptions.

Assumption A.1

The second-order sufficiency conditions are satisfied at (x^*,z^*).

Assumption A.2

(x^*, z^*) is a regular point of the constraint of the $HP(\Lambda, t_{-k})$, i.e. the gradients of the active constraints are linearly independent.

Assumption A.3

There are no degenerate constraints at (x^*, z^*), i.e. all active constraints have strictly positive corresponding Lagrange multipliers.

Then the following existence theorem (for details, see, e.g., Luenberger, 1973 or Fiacco, 1983) which is based on the implicit function theorem, holds.

Theorem A.3

Let (x^*, z^*) be a unique local optimal solution of the $HP(\Lambda, t^*_{-k})$ satisfying Assumptions A.1, A.2 and A.3. Let λ^* denote the Lagrange multipliers corresponding to the constraints of the $HP(\Lambda, t^*_{-k})$. Then there exist a continuously differentiable vector valued function $x(t_{-k})$, $z(t_{-k})$ and $\lambda(t_{-k})$ defined on some neighborhood $N(t^*_{-k})$ so that $x(t^*_{-k}) = x^*$, $z(t^*_{-k}) = z^*$, $\lambda(t^*_{-k}) = \lambda^*$, where $x(t_{-k})$, $z(t_{-k})$ is a unique local optimal solution of the $HP(\Lambda, t_{-k})$ for any $t_{-k} \in N(t^*_{-k})$ satisfying Assumptions A.1, A.2 and A.3, and $\lambda(t_{-k})$ is the Lagrange multiplier corresponding to the constraints of $HP(\Lambda, t_{-k})$.

In Theorem A.3,

$$z(t_{-k}) = \min_{x \in X} \{ z \mid \sum_{j=1}^{k} q_{ij} F_j(f_j(x)) - D_i(z, t_{-k}) \leq 0,\ i=1,\ldots,k \} \quad (A.24)$$

can be viewed as the optimal value function of the HP(Λ, t_{-k}) for any $t_{-k} \in N(t^*_{-k})$. Therefore, the following theorem holds under the same assumptions in Theorem A.3.

Theorem A.4

If all the assumptions in Theorem A.3 are satisfied, then the following relations hold on some neighborhood $N(t^*_{-k})$ of t^*_{-k}.

$$\frac{\partial z(t_{-k})}{\partial t_i} = \frac{\partial L}{\partial t_i} = - \sum_{j=1}^{k} \lambda_j(t_{-k}) \frac{\partial D_j(z(t_{-k}), t_{-k})}{\partial t_i}, \quad i=1,\ldots,k-1 \quad (A.25)$$

From Theorem A.4, if all the k constraints of HP(Λ, t_{-k}) are active, then the following theorem holds.

Theorem A.5.

Let all the assumptions in Theorem A.4 be satisfied. Also assume that all the k constraints of the HP(Λ, t_{-k}) are active. Then for the optimal value function $z(t_{-k})$ and the corresponding Lagrange multipliers $\lambda(t_{-k})$ for any $t_{-k} \in N(t^*_{-k})$, it holds that

$$\sum_{i=1}^{k} \lambda_i(t_{-k})\ d \{ D_i(z(t_{-k}), t_{-k}) \} = 0, \quad (A.26)$$

where d { $D_i(.)$ } denotes a total differential with respect to t_{-k}, i.e.,

$$d\{ D_i(z(t_{-k}),t_{-k}) \} = \sum_{j=1}^{k-1} \{ \frac{\partial D_i}{\partial z} \frac{\partial z}{\partial t_j} + \frac{\partial D_i}{\partial t_j} \} dt_j , \quad i=1,\dots,k \quad (A.27)$$

(Proof)

Observe that $\lambda(t_{-k}) > 0$ for $t_{-k} \in N(t^*_{-k})$, due to the active condition and Assumption A.3. Also from Assumption A.1, it holds that

$$\frac{\partial L}{\partial z} = 1 - \sum_{i=1}^{k} \lambda_i(t_{-k}) \frac{\partial D_i(z(t_{-k}),t_{-k})}{\partial z} = 0 \quad (A.28)$$

Using the Jacobian matrix $A(z,t_{-k})$ of $D(z,t_{-k})$, the total differential form of $D(z,t_{-k})$ becomes as follows.

$$\begin{bmatrix} d\{ D_1(z,t_{-k}) \} \\ \cdots\cdots \\ d\{ D_k(z,t_{-k}) \} \end{bmatrix} = A \begin{bmatrix} dz \\ dt_1 \\ \dots \\ dt_{k-1} \end{bmatrix} \quad (A.29)$$

Multiplying both side of (A.29) by $\lambda(t_{-k})$ yields

$$\sum_{i=1}^{k} \lambda_i(t_{-k})\, d\{ D_i(z(t_{-k}),t_{-k}) \}$$

$$= [\sum_{i=1}^{k} \lambda_i(t_{-k}) \frac{\partial D_i(z,t_{-k})}{\partial z} , \sum_{i=1}^{k} \lambda_i(t_{-k}) \frac{\partial D_i(z,t_{-k})}{\partial t_1} ,\dots,$$

$$\sum_{i=1}^{k} \lambda_i(t_{-k}) \frac{\partial D_i(z,t_{-k})}{\partial t_{k-1}}] \times [dz , dt_1 ,\dots, dt_{k-1}]^T .$$

In view of (A.25) and (A.28), for the optimal value function $z(t_{-k})$ for any $t_{-k} \in N(t_{-k})$, it follows that

$$\sum_{i=1}^{k} \lambda_i(t_{-k})\, d\{ D_i(z(t_{-k}),t_{-k}) \} = dz - \sum_{j=1}^{k-1} \frac{\partial z(t_{-k})}{\partial t_j} dt_j \quad (A.30)$$

Q.E.D.

From Theorem A.5, the following result immediately follows.

Theorem A.6

Let all the assumptions in Theorem A.5 be satisfied. Then for the optimal solution $x(t_{-k})$ and the corresponding Lagrange multipliers $\lambda(t_{-k})$ of the $HP(\Lambda, t_{-k})$ for any $t_{-k} \in N(t^*_{-k})$, it holds that

$$\sum_{j=1}^{k} \{ \sum_{i=1}^{k} (\lambda_i(t_{-k}) q_{ij}) \frac{\partial F_j}{\partial f_j} \} df_j(x(t_{-k})) = 0. \qquad (A.31)$$

(Proof)

By the active condition and Theorem A.3, it follows that

$$\sum_{j=1}^{k} q_{ij} F_j(f_j(x(t_{-k}))) = D_i(z(t_{-k}), t_{-k}) \quad . \qquad (A.32)$$

This implies

$$\sum_{i=1}^{k} \lambda_i(t_{-k}) \; d \{ D_i(z(t_{-k}), t_{-k}) \}$$

$$= \sum_{i=1}^{k} \lambda_i(t_{-k}) \{ \sum_{j=1}^{k} q_{ij} \frac{\partial F_j}{\partial f_j} df_j(x(t_{-k})) \}. \qquad \text{Q.E.D.}$$

It is now appropriate to consider the practical implications of Theorem A.6. Let

$$\bar{f}_j(t_{-k}) \triangleq f_j(x(t_{-k})) \; , \quad j=1,\ldots,k \qquad (A.33)$$

Since the continuous differentiability of $\bar{f}_j(.)$ follows from those of $f_j(.)$ and $x(.)$, (A.31) becomes as follows.

$$\sum_{j=1}^{k} \{ \sum_{i=1}^{k} (\lambda_i(t_{-k}) q_{ij}) \frac{\partial F_j}{\partial \bar{f}_j} d\bar{f}_j(t_{-k}) = 0 \qquad (A.34)$$

Let

$$\bar{P} = \{ \bar{f}(t_{-k}) \mid t_{-k} \in N(t^*_{-k}) \} ,$$

then $\bar{P}$ represents a surface in the k-dimensional space and the slope at its tangent plane can be obtained by (A.34). Namely, the trade-off rates between $\bar{f}_i, i=1,\dots,k-1$ and $\bar{f}_k$ on the surface $\bar{P}$ are represented by:

$$- \left. \frac{\partial \bar{f}_k(t_{-k})}{\partial \bar{f}_i(t_{-k})} \right|_{t_{-k} \in N(t^*_{-k})} = \frac{\sum_{j=1}^{k} (\lambda_j(t_{-k}) q_{ji}) \ \partial F_i(\bar{f}_i(t_{-k}))/\partial \bar{f}_i}{\sum_{j=1}^{k} (\lambda_j(t_{-k}) q_{jk}) \ \partial F_k(\bar{f}_k(t_{-k}))/\partial \bar{f}_k} \qquad i=1,\dots,k-1 \qquad (A.35)$$

In what follows, we shall show that the trade-off rates between f_i, $i=1,\dots,k-1$ and f_k on the Pareto surface

$$P = \{ f(x) \mid x \in X^P \}$$

where X^P is the Pareto optimal solutions set, coincide with the trade-off rates between $\bar{f}_i$, $i=1,\dots,k-1$ and $\bar{f}_k$. Namely, we shall prove the following result.

$$\left. \frac{\partial f_k(x)}{\partial f_i(x)} \right|_{\substack{x = x* \\ f \in P}} = \left. \frac{\partial \bar{f}_k(t_{-k})}{\partial \bar{f}_i(t_{-k})} \right|_{\substack{t_{-k} = t^*_{-k} \\ \bar{f} \in \bar{P}}} \qquad i=1,\dots,k-1 \qquad (A.36)$$

For that purpose, define the continuously differentiable function $\bar{D}_i(t_{-k})$, $i=1,\ldots,k$ by:

$$\bar{D}_i(t_{-k}) \quad = \quad D_i(z(t_{-k}),t_{-k}) \qquad i=1,\ldots,k \tag{A.37}$$

where $z(t_{-k})$ is the optimal value function. Let

$$\bar{D}_{-k}(t_{-k}) \quad = \quad (\ \bar{D}_1(t_{-k}) \ ,\ldots,\ \bar{D}_{k-1}(t_{-k})\ )^T, \tag{A.38}$$

$$\bar{D}(t_{-k}) \quad = \quad (\ \bar{D}_1(t_{-k}) \ ,\ldots,\ \bar{D}_k(t_{-k})\ )^T, \tag{A.39}$$

and define the $(k-1)\times(k-1)$ Jacobian matrix $B_{-k}(t_{-k})$ of $\bar{D}_{-k}(t_{-k})$ by:

$$B_{-k}(t_{-k}) \quad = \quad \begin{bmatrix} \dfrac{\partial \bar{D}_1}{\partial t_1} & \cdots & \dfrac{\partial \bar{D}_1}{\partial t_{k-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \bar{D}_{k-1}}{\partial t_1} & \cdots & \dfrac{\partial \bar{D}_{k-1}}{\partial t_{k-1}} \end{bmatrix} \tag{A.40}$$

Then the relationships between the $(k\times k)$ Jacobian matrix $A(z,t_{-k})$ and the $(k-1)\times(k-1)$ Jacobian matrix $B_{-k}(t_{-k})$ can be characterized by the following theorem.

Theorem A.7

Let all the assumptions in Theorem A.5 be satisfied. Then for any $t_{-k} \in N(t^*_{-k})$, it holds that

$$\det B_{-k}(t_{-k}) \quad = \quad (-1)^{k-1}\ \lambda_k \ \det A(z(t_{-k}),t_{-k}), \tag{A.41}$$

where $z(t_{-k})$ is the optimal value function and λ_k is the k-th Lagrange multiplier of the k-th constraint of $HP(\Lambda,t_{-k})$.

(Proof)

From the definition of $B_{-k}(t_{-k})$, it follows that

$$\det B_{-k}(t_{-k}) = \det \begin{bmatrix} \frac{\partial D_1}{\partial z}\frac{\partial z}{\partial t_1} + \frac{\partial D_1}{\partial t_1} & \cdots & \frac{\partial D_1}{\partial z}\frac{\partial z}{\partial t_{k-1}} + \frac{\partial D_1}{\partial t_{k-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial D_{k-1}}{\partial z}\frac{\partial z}{\partial t_1} + \frac{\partial D_{k-1}}{\partial t_1} & \cdots & \frac{\partial D_{k-1}}{\partial z}\frac{\partial z}{\partial t_{k-1}} + \frac{\partial D_{k-1}}{\partial t_{k-1}} \end{bmatrix}$$

$$= \{ \sum_{i=1}^{k-1} \frac{\partial D_i}{\partial z} \} \{ -\lambda_1 | A_{k1} | - (-1)^{k-2} \lambda_k | A_{11} | \}$$

$$+ \sum_{i=1}^{k-1} \det \begin{bmatrix} \frac{\partial D_i}{\partial t_1} & \cdots & \frac{\partial D_i}{\partial t_{k-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial D_{k-1}}{\partial z}\frac{\partial z}{\partial t_1} + \frac{\partial D_{k-1}}{\partial t_1} & \cdots & \frac{\partial D_{k-1}}{\partial z}\frac{\partial z}{\partial t_{k-1}} + \frac{\partial D_{k-1}}{\partial t_{k-1}} \end{bmatrix}$$

where $| A_{ij} |$ denotes the determinant of the $(k-1)\times(k-1)$ matrix without the ith row and the jth column of $A(z(t_{-k}), t_{-k})$. By similar operations, it follows that

$$\det B_{-k}(t_{-k})$$

$$= \{ \sum_{i=1}^{k-1} \frac{\partial D_i}{\partial z} \} \{ -\lambda_1 | A_{k1} | - (-1)^{k-2} \lambda_k | A_{11} | \}$$

$$+ \{ \sum_{i=2}^{k-1} \frac{\partial D_i}{\partial z} \} \{ \lambda_1 | A_{k1} | - \lambda_2 | A_{k1} | - (-1)^{k-3} \lambda_k | A_{21} | - (-1)^{k-3} \lambda_k | A_{11} | \}$$

$$+ \cdots\cdots\cdots\cdots$$

$$+ \{ \sum_{i=k-1}^{k-1} \frac{\partial D_i}{\partial z} \} \{ \lambda_{k-2} | A_{k1} | - \lambda_{k-1} | A_{k1} |$$

$$- (-1)^0 \lambda_k | A_{k-1,1} | - (-1)^0 \lambda_k | A_{k-2,1} | \}$$

$$+ | A_{k1} |$$

$$= \frac{\partial D_1}{\partial z} \{ -\lambda_1 | A_{k1} | - (-1)^{(k-1)-1} \lambda_k | A_{11} | \}$$

$$+ \quad \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{\partial D_{k-1}}{\partial z} \{ -\lambda_{k-1} | A_{k1} | - (-1)^{(k-1)-(k-1)} \lambda_k | A_{k-1,1} | \}$$

$$+ \quad | A_{k1} |$$

$$= (1 - \sum_{i=1}^{k-1} \lambda_i \frac{\partial D_i}{\partial z}) | A_{k1} | - \lambda_k \sum_{i=1}^{k-1} (-1)^{(k-1)-i} \frac{\partial D_i}{\partial z} | A_{i1} |$$

In view of (A.28), it holds that

$$\det B_{-k}(t_{-k})$$

$$= \lambda_k \frac{\partial D_k}{\partial z} | A_{k1} | - \lambda_k \sum_{i=1}^{k-1} (-1)^{(k-1)-i} \frac{\partial D_i}{\partial z} | A_{i1} |$$

$$= - \lambda_k \sum_{i=1}^{k} (-1)^{(k-1)-i} \frac{\partial D_i}{\partial z} | A_{i1} |$$

By expanding the det $A(z(t_{-k}),t_{-k})$ with respect to the first column, we obtain

$$\det A(z(t_{-k}),t_{-k}) = \sum_{i=1}^{k} (-1)^{i-1} \frac{\partial D_i}{\partial z} | A_{i1} | .$$

Therefore (A.41) has been established. Q.E.D.

The following theorem shows the relationships between Pareto optimal solutions in the decision space and the parameter $t_{-k} \in T$ in the $HP(\Lambda, t_{-k})$, where

$$T \triangleq \{ t_{-k} \in R^{k-1} \mid (z, t_{-k}) \in S \} .$$

Theorem A.8

Let all the assumptions in Theorem A.5 be satisfied. Then there exist a neighborhood $N(x*)$ of $x*$ and a continuously differentiable vector valued function $x(t_{-k})$ defined on some neighborhood $N(t^*_{-k})$ of $t^*_{-k} \in T$ such that

$$X^P \cap N(x*) \subset x(N(t^*_{-k})) \subset X^P \qquad (A.42)$$

(Proof)

From Theorem A.3, there exists a neighborhood $N(t^*_{-k})$ of t^*_{-k} such that the $HP(\Lambda, t_{-k})$ has a unique local solution $x(t_{-k})$ for any $t_{-k} \in N(t^*_{-k})$, and that $x(t_{-k})$ is a continuously differentiable function of t_{-k} defined on $N(t^*_{-k})$. Hence, it follows from Theorems A.1 and A.2

$$x(N(t_{-k})) \subset X^P .$$

To prove the first part of (A.42) we shall show that there exists a neighborhood $N(x*)$ of $x*$ such that

$$\exists\ t_{-k} \in N(t^*_{-k}) \text{ such that } \hat{x} = x(t_{-k}) \text{ for } \forall\ \hat{x} \in X^P \cap N(x*) .$$

Let

$$N(t^*_{-k}) = \{ t_{-k} \in T \mid \| t_{-k} - t^*_{-k} \|_{k-1} < \delta_t \} \qquad (A.43)$$

where $\delta_t > 0$ and $\| \cdot \|_{k-1}$ is an ℓ_p-norm in E^{k-1}. By the continuity of $Q_{-k}(\Lambda)F(f_i(.))$ in x, for any $\delta_d > 0$, there exists $\delta_x > 0$ such that

$$\| x - x* \|_n < \delta_x \rightarrow \| Q_{-k}(\Lambda)F(f(x)) - Q_{-k}(\Lambda)F(f(x*)) \|_{k-1} < \delta_d . \quad (A.44)$$

For such δ_x, define the neighborhood $N(x*)$ of $x*$ by

$$N(x*) \triangleq \{ x \mid \| x - x* \|_n < \delta_x \}. \tag{A.45}$$

Now, from Property A.1, for any $\hat{x} \in N(x*) \cap X^P$, there exists $(\hat{z}, \hat{t}_{-k})$ such that

$$Q(\Lambda)F(f(\hat{x})) = D(\hat{z}, \hat{t}_{-k}). \tag{A.46}$$

This directly follows

$$\| D_{-k}(\hat{z},\hat{t}_{-k}) - D_{-k}(z^*,t^*_{-k}) \|_{k-1} < \delta_d . \tag{A.47}$$

Now if we assume $\hat{z} \neq z(\hat{t}_{-k})$, where $(x(\hat{t}_{-k}), z(\hat{t}_{-k}))$ be an optimal solution to the $HP(\Lambda,\hat{t}_{-k})$, then following two cases arise.

(I) If $\hat{z} < z(\hat{t}_{-k})$, then, from (A.46) and $\hat{x} \in X^P$, it contradicts the fact that $z(\hat{t}_{-k})$ is an optimal value to the $HP(\Lambda,\hat{t}_{-k})$.

(II) If $\hat{z} > z(\hat{t}_{-k})$, then, from Property A.2, it holds that $D(\hat{z},\hat{t}_{-k}) \geq D(z(\hat{t}_{-k}),\hat{t}_{-k})$. Hence it follows from (A.46), $Q(\Lambda)F(f(\hat{x})) \geq Q(\Lambda)F(f(x(\hat{t}_{-k})))$. Moreover, since $\Lambda = \Lambda^{\geqq}$ or $\Lambda = \Lambda^{>}(\rho)$, we have $f(\hat{x}) \geq f(x(\hat{t}_{-k}))$, which contradicts the fact that $\hat{x} \in X^P$.

From (I) and (II), we conclude that $\hat{z} = z(\hat{t}_{-k})$. Consequently, by the definition of $\bar{D}_{-k}(.)$, it holds that

$$\| D_{-k}(\hat{z},\hat{t}_{-k}) - D_{-k}(z^*,t^*_{-k}) \|_{k-1}$$

$$= \| \bar{D}_{-k}(\hat{t}_{-k}) - \bar{D}_{-k}(t^*_{-k}) \|_{k-1} < \delta_d . \tag{A.48}$$

Now, from Theorem A.7 observe that $\det B_{-k}(t_{-k}) \neq 0$ for any $t_{-k} \in T$. From this fact and the implicit function theorem, for any $\delta_t > 0$ there exists δ_d such that

$$\| \bar{D}_{-k}(\hat{t}_{-k}) - \bar{D}_{-k}(t^*_{-k}) \|_{k-1} < \delta_d \rightarrow \| \hat{t}_{-k} - t^*_{-k} \|_{k-1} < \delta_t \qquad (A.49)$$

In view of (A.44) and (A.49), it follows that $\hat{t}_{-k} \in N(t^*_{-k})$. Moreover, by the definition of $N(t^*_{-k})$, the HP$(\Lambda, \hat{t}_{-k})$ has a unique local solution $x(\hat{t}_{-k})$. Also since $\hat{x} \in X^P$, $\hat{x}$ is a local solution to the HP$(\Lambda, \hat{t}_{-k})$ from Theorem A.1 and A.2. Therefore, we conclude that $\hat{x} = x(\hat{t}_{-k})$ as required.

Q.E.D.

From Theorem A.8 and (A.35), we can immediately obtain the following theorem.

Theorem A.9

Let all the assumptions in Theorem A.5 be satisfied. Then the trade-off rates between the objective functions $f_i(x*)$, $i=1,\ldots,k-1$, and $f_k(x*)$ at the optimal solution to the HP(Λ, t^*_{-k}) on the Pareto surface P in the objective space can be represented by

$$- \left. \frac{\partial f_k(x)}{\partial f_i(x)} \right|_{\substack{x=x* \\ f \in P}} = \frac{\sum_{j=1}^{k} \lambda^*_j \, q_{ji} \; \partial F_i(f_i(x*))/\partial f_i}{\sum_{j=1}^{k} \lambda^*_j \, q_{jk} \; \partial F_k(f_k(x*))/\partial f_k}, \quad i=1,\ldots,k-1. \qquad (A.50)$$

It should be noted here that the vector function D(.) in the $HP(\Lambda, t_{-k})$ does not explicitly appear in (A.50). In other words, the trade-off rates on the Pareto surface can be determined by the values of Lagrange multipliers and the elements of $Q(\Lambda)$ and $F(.)$. This means that the vector function D(.) influences the values of trade-off rates implicitly through the Lagrange multipliers.

We are now ready to demonstrate that the trade-off rates formula in chapters 2, 3 and 4 can be derived from the trade-off rates formula (A.50).

In the $HP(\Lambda, t_{-k})$, the $MP(t_{-k})$ is obtained by using the substitution shown in (A.16), (A.17) and

$$q_{ii} = 1 \ , \ i=1,\dots,k, \qquad q_{ij} = 0 \ , \ i \neq j \ .$$

Thus the trade-off rates formula for the $MP(t_{-k})$ is represented by:

$$-\frac{\partial \mu_k(f_k(x))}{\partial \mu_i(f_i(x))} = \frac{\lambda_i}{\lambda_k} \ , \quad i=1,\dots,k-1. \tag{A.51}$$

Similarly, by using the substitution,

$$q_{ii} = 1 + \rho \ , \ i=1,\dots,k, \qquad q_{ij} = \rho \ , \ i \neq j$$

in the $HP(\Lambda, t_{-k})$, the $AMP(t_{-k})$ is obtained. Therefore, the trade-off rates formula for the $AMP(t_{-k})$ is represented by:

$$-\frac{\partial \mu_k(f_k(x))}{\partial \mu_i(f_i(x))} = \frac{\lambda_i + \rho}{\lambda_k + \rho} \ , \quad i=1,\dots,k-1. \tag{A.52}$$

Observe that this trade-off rate formula (A.52) coincides with the trade-off rate formula in chapter 4.

So far, we have discussed the trade-off rates formula for the $HP(\Lambda,t_{-k})$ under Assumptions A.1, A.2 and A.3. However, for the linear $HP(\Lambda,t_{-k})$ in which all the objective functions and the constraints are linear, it should be emphasized here that the second-order sufficiency conditions are not satisfied because the Hessian matrix always becomes a zero matrix.

For the linear $HP(\Lambda,t_{-k})$, only with Assumption A.3, the following observations can be made concerning the trade-off rates formula (A.50). If Assumption A.3 is satisfied for the optimal solution to the linear $HP(\Lambda,t_{-k})$, then from the theory of the simplex method of linear programming, it is known that there exists the neighborhood $N(t_{-k})$ of t_{-k} such that

$$z(t_{-k}) = \sum_{i=1}^{k} \pi_i \; D_i(z(t_{-k}),t_{-k}), \tag{A.53}$$

where $z(t_{-k})$ is the optimal value function to the linear $HP(\Lambda,t_{-k})$ and π_i $i=1,\ldots,k$ are the simplex multipliers corresponding to the constraint in (A.2).

Since Theorem A.4 directly follows from (A.53), if all the constraints of the linear $HP(\Lambda,t_{-k})$ are active, then both Theorems A.5 and A.6 are satisfied for the linear $HP(\Lambda,t_{-k})$. Therefore, the following theorem holds.

Theorem A.10

Let $x*$ be a nondegenerate optimal solution to the linear $HP(\Lambda, t^*_{-k})$, and let π^*_i, $i=1,\ldots,k$ denote the simplex multipliers corresponding to the constraints of the linear $HP(\Lambda, t^*_{-k})$. Also assume that all the k constraints of the linear $HP(\Lambda, t^*_{-k})$ are active. Then for the linear $HP(\Lambda, t^*_{-k})$, the trade-off rates between the objective functions $f_i(x*)$, $i=1,\ldots,k-1$, and $f_k(x*)$ at the optimal solution to the $HP(\Lambda, t^*_{-k})$ on the Pareto surface P in the objective space can be represented by

$$- \left. \frac{\partial f_k(x)}{\partial f_i(x)} \right|_{\substack{x=x* \\ f \in P}} = \frac{\sum_{j=1}^{k} \pi^*_j \; q_{ji} \; \partial F_i(f_i(x*))/\partial f_i}{\sum_{j=1}^{k} \pi^*_j \; q_{jk} \; \partial F_k(f_k(x*))/\partial f_k}, \quad i=1,\ldots,k-1. \tag{A.54}$$

Now, let us investigate the relationships between the hyperplane problem $HP(\Lambda^{\geq}, t_{-k})$ and the linear constraint problems in chapters 2 and 3.

By replacing $D(z, t_{-k})$ by

$$D_{CP}(z, t_{-k}) \triangleq (t_1, \ldots, t_{k-1}, z)^T, \tag{A.55}$$

the $HP(\Lambda^{\geq}, t_{-k})$ reduces to the constraint problem $CP(t_{-k})$ itself (Haimes and Chankong 1979). To show that $D_{CP}(z, t_{-k})$ satisfies Properties 1 and 2, let $S_{CP} = E^k$. Then the Jacobian matrix $A_{CP}(.)$ for $D_{CP}(.)$ becomes as:

$$A_{CP}(z,t_{-k}) = \begin{bmatrix} 0 & & & & \\ \cdot & & & & \\ \cdot & & I & & \\ \cdot & & & & \\ 0 & & & & \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \qquad \text{(A.56)}$$

and

$$\det A_{CP}(z,t_{-k}) = (-1)^{k-1} \neq 0, \quad \text{for any } (z,t_{-k}) \in S_{CP}.$$

This means that $D_{CP}(.)$ satisfies Property 1. Furthermore, it holds that

$$\frac{\partial D_{CP}}{\partial z} = (\ 0, \ \ldots \ , 0, \ 1\ )^T \geq 0$$

This implies that $D_{CP}(.)$ satisfies Property 2.

From the above discussions, the $CP(t_{-k})$ can be regarded as one of the special cases of the $HP(\Lambda^{\geq},t_{-k})$.

Therefore, from Theorem A.10, the trade-off rates formula for the linear $CP(t_{-k})$ is represented as follows.

$$-\frac{\partial u_k(f_k(x))}{\partial u_i(f_i(x))} = \frac{\pi_i}{\pi_k}, \quad i=1,\ldots,k-1. \qquad \text{(A.57)}$$

However, since it always holds that $\pi_k = 1$ in the $CP(t_{-k})$ because of $\partial L/\partial z = 0$, (A.57) is represented as:

$$-\frac{\partial\mu_k(f_k(x))}{\partial\mu_i(f_i(x))} = \Pi_i, \quad i=1,\ldots,k-1. \tag{A.58}$$

This formula coincides with the trade-off rates formula in chapter 2. The trade-off rates formula in chapter 3 is much like the above discussion and thus is omitted.

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