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Remarks on Positive Interest Rates

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Abstract

We analyze the bond prices and the market price of risk by investigating integrated processes appearing on the potential approach by Flesaker and Hughston (1996) and Rutkowski (1999) for positive interest rates models.

JEL Classification Numbers: G12
Key Words: positive interest rates, potential, pricing kernel

1 Introduction

Interest rates models have been developed with several different spirits. The features to be considered in the modeling are (*i*) analytical calculation of bonds and options, (*ii*) positivity of rates, and (*iii*) easy calibration to the option markets. The affine term structure models have the merit in (*i*). For the positivity of rates for the representation of nominal rates, an appropriate approach is the potential approach by Flesaker and Hughston(1996). Rutkowski(1999) shows a rational lognormal model in line with Flesaker and Hughston(1996). For the purpose of (*iii*), Brace, Gatarek and Musiela(1997) develop a model (BGM model) which keeps the log-normality of forward LIBORs and thus has the strength in treating caps consisting of a portfolio of options on LIBOR. The BGM model is called the LIBOR market model and it is very widely used by practitioners since volatilities of caps and swaptions are easily observed in the markets of major currencies. As opposed to the LIBOR market model by Brace, Gatarek and Musiela(1997), Jamshidian(1997) formulates a market model based on forward swap rates.

Flesaker and Hughston propose a family of positive interest rate models under which the bond price can be written as

$$B(t,T) = \frac{\int_T^\infty h(u)M(t,u)du}{\int_t^\infty h(u)M(t,u)du}, \tag{1}$$

where h is a deterministic function and $M(t,T)$ is a positive martingale for each T. The bond price is derived from a pricing kernel

$$Z_t = \int_t^\infty h(u)M(t,u)du. \tag{2}$$

This pricing kernel induces a positive interest rate model. The converse problem is solved by Jin and Glasserman(2001). They show that any pricing kernel which is a potential and induces a positive interest rate model is written in the above form. Therefore, any positive interest rate models without bubble can be characterized by the above pricing kernel. This article provides remarks on the potential approach and an example of positive interest rates with a bubble term.

2 Positive Interest Rates

The uncertainty of the model is represented by a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ supporting a Brownian motion W_t. The probability measure P is an objective one. Suppose that we are given deterministic volatility $\sigma^M(t,T)$ and the initial bond prices $\{B(0,T)\}_{T\in\mathbb{R}_+}$ with $\lim_{T\to\infty} B(0,T) = 0$. Then we can define a family of strictly positive P-martingales $\{M(t,T)\}_T$ by

$$M(t,T) = \exp\Big(\int_0^{t\wedge T} \sigma^M(s,T)dW_s - \frac{1}{2}\int_0^{t\wedge T} \sigma^M(s,T)^2 ds\Big). \tag{3}$$

Note that for a fixed T, the process $M(t,T)$ satisfies a stochastic differential equation (SDE)

$$dM(t,T) = 1_{\{t\le T\}} M(t,T)\sigma^M(t,T)dW_t, \quad M(0,T) = 1.$$

For a family of processes $\{Y(t,T)\}_T$ and a deterministic and differentiable function ϕ, we define a process

$$\mathcal{A}_\phi(Y)(t,T) = \int_T^\infty \phi(u)Y(t,u)du. \tag{4}$$

We are interested in the SDEs of $\mathcal{A}_\phi(Y)(t,T)$ and $\mathcal{A}_\phi(Y)(t,t)$ when the SDE of $Y(t,T)$ is given. The next lemma shows the SDE.

Lemma 1. *Suppose that $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a deterministic and differentiable function and a family of processes $\{Y(t,T)\}_T$ is an Ito process satisfying*

$$dY(t,T) = \mu_Y(t,T)dt + \sigma_Y(t,T)dW_t.$$

Then for $t \in [0,T]$, it holds that

$$
\begin{aligned}
d\mathcal{A}_\phi(Y)(t,T) &= \mathcal{A}_\phi(\mu^Y)(t,T)dt + \mathcal{A}_\phi(\sigma^Y)(t,T)dW_t,\\
d\mathcal{A}_\phi(Y)(t,t) &= \Big(\mathcal{A}_\phi(\mu^Y)(t,t) - \phi(t)Y(t,t)\Big)dt + \mathcal{A}_\phi(\sigma^Y)(t,t)dW_t.
\end{aligned}
$$

Take any deterministic and twice differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$
\begin{aligned}
&0 \le f(u) < 1 \quad \forall u \in \mathbb{R}_+, \quad f'(u) \le 0 \quad \forall u \in \mathbb{R}_+,\\
&\lim_{u\to\infty} f(u) = 0, \quad \lim_{u\to\infty} f'(u) = 0.
\end{aligned}
$$

One of the example is $f(u) = aB(0,u)$ with any constant $0 \le a < 1$ if the initial bond price curve $B(0,u)$ is twice differentiable with respect to maturity u. Let a strictly positive function $g : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$
g(u) \equiv (1 - f(0))\left(-\frac{\partial B(0,u)}{\partial u}\right).
$$

The function g satisfies

$$
\lim_{u\to\infty} g(u) = 0, \quad \int_0^\infty g(u)du = 1 - f(0).
$$

By following Flesaker and Hughston(1996) and Rutkowski(1997), we construct the pricing kernel Z defined by

$$
Z_t \equiv f(t) + \int_t^\infty g(u)M(t,u)du = f(t) + \mathcal{A}_g(M)(t,t). \tag{5}
$$

All assets are priced with the pricing kernel under the objective measure P. The time t-price of zero coupon bond with maturity date T is given by

$$
B(t,T) \equiv \frac{E^P(Z_T|\mathcal{F}_t)}{Z_t} = \frac{f(T) + \int_T^\infty g(u)M(t,u)du}{f(t) + \int_t^\infty g(u)M(t,u)du} = \frac{f(T) + \mathcal{A}_g(M)(t,T)}{f(t) + \mathcal{A}_g(M)(t,t)}. \tag{6}
$$

It is easy to check the following properties as desired: (i) $B(T,T) = 1$ for any maturity T, (ii) $B(t,U) \le B(t,T)$ for every $0 \le t \le T \le U$, and (iii) for any maturity T

$$
B(0,T) = f(T) + \int_T^\infty g(u)du.
$$

By applying Lemma 1, we obtain the following results on the short rate r_t and the market price of risk λ_t easily.

Proposition 1. *It holds that*

$$
\begin{aligned}
dZ_t &= -r_t Z_t dt + Z_t \lambda_t dW_t, \quad Z_0 = 1,\\
r_t &= \frac{-f'(t) + g(t)M(t,t)}{f(t) + \mathcal{A}_g(M)(t,t)},\\
\lambda_t &= \frac{\mathcal{A}_g(M\sigma^M)(t,t)}{f(t) + \mathcal{A}_g(M)(t,t)},\\
dB(t,T) &= r_t B(t,T)dt\\
&\quad + B(t,T)\Big(\frac{\mathcal{A}_g(M\sigma^M)(t,T)}{f(T) + \mathcal{A}_g(M)(t,T)} - \frac{\mathcal{A}_g(M\sigma^M)(t,t)}{f(t) + \mathcal{A}_g(M)(t,t)}\Big)(dW_t - \lambda_t dt).
\end{aligned}
$$

Proposition 1 shows that the volatility of the zero coupon bond is given by

$$\frac{\mathcal{A}_g(M\sigma^M)(t,T)}{f(T) + \mathcal{A}_g(M)(t,T)} - \frac{\mathcal{A}_g(M\sigma^M)(t,t)}{f(t) + \mathcal{A}_g(M)(t,t)},$$

and that the Brownian motion W_t^Q under the risk-neutral measure is given by

$$dW_t^Q = dW_t - \lambda_t dt.$$

When the volatility $\sigma^M(t,T)$ does not depend on the maturity T, such situation implies a one-factor model and the above formula is reduced to the case of the so-called rational lognormal model

$$r_t = -\frac{a'(t) + b'(t)M(t)}{a(t) + b(t)M(t)}$$

as discussed in Rutkowski(1997).

Corollary 1. *If the volatility does not depend on* T, $\sigma^M(t,T) = \sigma(t)$, *then*

$$
\begin{aligned}
Z_t &= f(t) + (1 - f(0))B(0,t)M(t),\\
r_t &= \frac{-f'(t) - (1 - f(0))\frac{\partial B(0,t)}{\partial t}M(t)}{f(t) + (1 - f(0))B(0,t)M(t)},\\
\lambda_t &= \frac{\sigma(t)(1 - f(0))B(0,t)M(t)}{f(t) + (1 - f(0))B(0,t)M(t)}.
\end{aligned}
$$

where $M(t) = \lim_{u\to\infty} M(t,u)$. *Furthermore, if* $f \equiv 0$, *then the short rate and the market price of risk are deterministic,*

$$r_t = -\frac{\partial \ln B(0,t)}{\partial t}, \quad \lambda_t = \sigma(t).$$

3 Positive Interest Rates with Bubble

We will study whether a bubble on bond price is consistent with positive interest rates, i.e., whether the assumption $\lim_{T\to\infty} B(0,T) = 0$ can be replaced with $\lim_{T\to\infty} B(0,T) = b \geq 0$. This issue might be meaningless as long as we work in a "nominal world". However, it turns out meaningful if we need to make decisions based on the "real" quantities such as consumption and/or real money holding. Once a money is introduced into the model, the price of money is a rate to exchange a real commodity with money and therefore plays similar role as foreign exchange rates. The literature treats the price of money analogously as a foreign exchange rate. However, there is a different feature between them. A foreign exchange rate is a rate applied to the exchange of a unit price in one currency with positive interest rates into a price in another currency with positive interest rates. On the other hand, the price of money is an exchange rate a unit price on a real commodity with possibly negative real interest rates into a nominal price with positive interest rates. Therefore the behavior of the price of money will be different from one of the foreign exchange rate. A bubble on a nominal interest rate may be associated with such a model of the price of money.

Suppose that $B(0,T)$ is a decreasing function in T and

$$\lim_{T\to\infty} B(0,T) = b \quad (0 \leq b < 1), \quad \lim_{T\to\infty} \frac{\partial B(0,T)}{\partial T} = 0.$$

Take any deterministic and twice differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\begin{aligned}
&0 \leq f(u) < 1 \quad \forall u \in \mathbb{R}_+, \quad f'(u) \leq 0 \quad \forall u \in \mathbb{R}_+,\\
&\lim_{u\to\infty} f(u) = c, \quad \lim_{u\to\infty} f'(u) = 0,\\
&0 \leq c \leq b(1 - b(1 - f(0))).
\end{aligned}$$

A positive function $g : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$g(u) \equiv -(1 - f(0))\frac{\partial B(0,u)}{\partial u},$$

which satisfies

$$\lim_{u\to\infty} g(u) = 0, \quad \int_0^\infty g(u)du = (1 - f(0))(1 - b).$$

Then we call the pricing kernel a process Z, which is a supermartingale, defined by

$$\begin{aligned} Z_t &\equiv k\Big(f(t) + l + \int_t^\infty g(u)M(t,u)du\Big), \qquad (7) \\ \text{where} \qquad k &= \frac{1-b}{1-b(1-f(0))-c}, \quad l = \frac{b(1-b(1-f(0)))-c}{1-b}. \end{aligned}$$

The time t-price of zero coupon bond with maturity date T is given by

$$B(t,T) = \frac{E^P(Z_T|\mathcal{F}_t)}{Z_t} = \frac{f(T) + l + \mathcal{A}_g(M)(t,T)}{f(t) + l + \mathcal{A}_g(M)(t,t)} \qquad (8)$$

satisfying (i) $B(T,T) = 1$ for any maturity T, (ii) $B(t,U) \leq B(t,T)$ for every $0 \leq t \leq T \leq U$, (iii) for any maturity T

$$B(0,T) = \frac{f(T) + l + \int_T^\infty g(u)du}{f(0) + l + \int_0^\infty g(u)du},$$

and (iv) the bubble

$$\lim_{T\to\infty} B(0,T) = \frac{c+l}{f(0) + l + (1-f(0))(1-b)} = b.$$

Note that $Z(0) = 1$, $\lim_{t\to\infty} Z_t = b$, and as such, Z is not a potential, $\lim_{T\to\infty} E[Z_T] = b$, unless $b = 0$. For the process Z, we obtain the following results in a similar way.

Proposition 2. *It holds that*

$$\begin{aligned} dZ_t &= -r_t Z_t dt + Z_t \lambda_t dW_t, \quad Z_0 = 1, \\ r_t &= \frac{-f'(t) + g(t)M(t,t)}{f(t) + l + \mathcal{A}_g(M)(t,t)}, \\ \lambda_t &= \frac{\mathcal{A}_g(M\sigma^M)(t,t)}{f(t) + l + \mathcal{A}_g(M)(t,t)}, \\ dB(t,T) &= r_t B(t,T)dt \\ &\quad + B(t,T)\Big(\frac{\mathcal{A}_g(M\sigma^M)(t,T)}{f(T) + l + \mathcal{A}_g(M)(t,T)} - \frac{\mathcal{A}_g(M\sigma^M)(t,t)}{f(t) + l + \mathcal{A}_g(M)(t,t)}\Big)(dW_t - \lambda_t dt). \end{aligned}$$

4 Conclusion

We analyze the SDE of an integrated process appearing on the potential approach by Flesaker and Hughston(1996) and Rutkowski(1997) which will contribute to the future research of positive interest rates with a bubble term.

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