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Introduction. In the recent paper [5] Kusuoka-Strook gave a sufficient condition of hypoellipticity for degenerate elliptic operators of second order, as an application of the Malliavin calculus (see Theorem 8.13 of [5], cf. [4]). Their method is applicable even to infinitely degenerate elliptic operators which do not satisfy the famous sufficient condition given by Hörmander [2]. One of remarkable results by means of their condition is as follows: Let L be a differential operator of the form $\partial_{x_1}^2 + \partial_{x_2}^2 + \sigma(x_1)^2\partial_y^2$ in R^3, where $\sigma \in C^\infty$, $\sigma(0) = 0$, $\sigma(x_1) > 0$ ($x_1 \neq 0$), $\sigma(x_1) = \sigma(-x_1)$ and σ is non-decreasing in $[0,\infty)$. Then L is hypoelliptic in R^3 if σ satisfies

$$(*) \qquad \lim_{x_1 \downarrow 0} |x_1 \log \sigma(x_1)| = 0 \qquad \text{(Theorem 8.41 of [5]).}$$

The condition (*) allows the infinite degeneracy of σ at $x_1 = 0$. For example, if $\sigma(x_1) = \exp(-1/|x_1|^\delta)$ for $\delta > 0$ the condition (*) means $\delta < 1$. The main purpose of the present paper is to show the sufficiency of the condition (*) by using the theory of pseudodifferential operators. In [5] it is proved that the condition (*) is necessary for L to be hypoelliptic. The author [7] has given a simple proof of the necessity of (*) without using the Malliavin calculus. The arguments in [7] apply to degenerate elliptic operators of higher order (see Theorem 3 of [7]).

As to the operator L we remark that an operator $\partial_{x_1}^2 + \sigma(x_1)^2\partial_y^2$ ($= L - \partial_{x_2}^2$) is hypoelliptic in $R^2_{x_1,y}$ without the condition (*). This result is due to Fediǐ [1] (cf. [6]), who studied the criterion of hypoellipticity by means of apriori estimates. Such criteria have been investigated by Treves [9] and Oleinik-Radkevich [8]. Our proof of the hypoellipticity of L will be done by improving criteria studied by [8] and [1].

To explain the idea of the present paper we consider a simple case $\sigma(x_1) = \exp(-1/|x_1|^\delta)$, $\delta > 0$. Then L degenerates infinitely at $x_1 = 0$, and hence Hörmander's sufficient condition does not apply to L . In the proof of hypoellipticity by means of apriori estimates, the technical difficulty comes from the fact that for any $\kappa > 0$ subelliptic estimate

$$\|u\|_\kappa \leqq \text{Const.}(\|Lu\|_0 + \|u\|_0), \quad u \in C_0^\infty(R^3),$$

does not hold (see Theorem 1.2 of [6]), where $\|\cdot\|_s$ denotes the norm of the Sobolev space H_s for real s . However, by means of Poincaré's inequality we have the following estimate

$$\|(\log \langle D_y\rangle)^{2/\delta}u\|_0 \leqq \text{Const.}(\|Lu\|_0 + \|u\|_0), \quad u \in C_0^\infty(R^3),$$

(cf. Lemma 5.1 of Section 5),

where $\langle D_y\rangle = \sqrt{1 + D_y^2}$. The main idea is based on the fact that if $0 < \delta < 1$ then the repeated use of the above estimate with logarithmic regularity up gives the regularity up of polinomial order.

The plan of this paper is as follows: In Section 1 we state our main theorem, which is formulated for a differential operator $P = a(x,y,D_x) + g(x')b(x,y,D_y)$ in $R^n_{x,y}$, $x = (x',x'')$, having a slightly more general form than L (see Remark 2 of Theorem 1.1 in Section 1).

In Section 2 we give a new criterion of the hypoellipticity, which is composed of five apriori estimates. In Section 3 we show that P satisfies each estimate. Sections 4 and 5 are devoted to the proof of two lemmas, which play important roles in Section 3. The discussion of Section 4 is similar to the one of Section 5 of [6] and is employed to estimate the commutator between P and the cut off function of y variables. In Section 5 we estimate the commutator between P and the cut off function of x'' variables. For this estimation we need a condition similar to (*) (see (1.4) in Section 1).

As studied in [6], the method of this paper seems to be extendible to infinitely degenerate elliptic operators of higher order, which will be investigated in the future. Finally we remark that the method of the present paper does not apply to all results of [5], for example, the hypoellipticity of an operator $x_2^2 \partial_{x_1}^2 + \partial_{x_2}^2 + \sigma(x_1)^2(\partial_{y_1}^2 + y_1^2\partial_{y_2}^2)$. It is also future work to show the hypoellipticity for this operator by extending our method.

1. Main result

Let $P = p(x,y,D_x,D_y)$ be a differential operator of second order with C^∞-coefficients of the form

$$(1.1) \qquad P = a(x,y,D_x) + g(x')b(x,y,D_y) \quad \text{in} \quad R^n = R_x^{n_1} \times R_y^{n_2} ,$$

where $x = (x',x'') \in R_{x'}^{n_1'} \times R_{x''}^{n_1''}$. Assume the following:

1°) $a(x,y,D_x)$ and $b(x,y,D_y)$ are strongly elliptic with respect to x and y, respectively, that is,

$$(1.2) \qquad \operatorname{Re} a(x,y,\xi) \geqq c_1|\xi|^2 \quad \text{for large } |\xi| ,$$

$$(1.3) \qquad \operatorname{Re} b(x,y,\eta) \geqq c_2|\eta|^2 \quad \text{for large } |\eta| ,$$

where c_1 and c_2 are positive constants.

2°) $g(x')$ belongs to $C^\infty(R_{x'}^{n_1'})$, $g(0) = 0$ and $g(x') > 0$ for $x' \neq 0$.

<u>Theorem 1.1.</u> Let P satisfy 1°) and 2°). Assume that $g(x')$ satisfies

$$(1.4) \qquad \lim_{|x'| \to 0} |x'||\log g(x')| = 0 .$$

Then P is hypoelliptic in R^n. Namely, for any $u \in \mathcal{D}'(R^n)$ and for any open set Ω of R^n it follows that $Pu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

Remark 1. If x'' variables do not exist this theorem is included in Theorem 1.1 of [6].

Remark 2. Set

$$g_0(x') = \exp(-1/|x'|^{1/4})\sin^2(1/|x'|) + \exp(-1/|x'|^{1/2}).$$

Then $g_0(x')$ satisfies conditions 2°) and (1.4). In view of this function we see that Theorem 1.1 is slightly more general than Theorem 8.41 of [5] because $g_0(x')$ is not expressed in the form $g_0(x') = \sigma(x')^2$ for any non-negative C^∞-function σ (see Remark 2 of Theorem 1.1 of [6]).

In what follows we shall tacitly use the notation in [6] and Kumano-go [3]. For example, we often write $\phi \subset\subset \psi$ for $\phi, \psi \in C_0^\infty(R^n)$ if $\psi(x) = 1$ in a neighborhood of $\operatorname{supp} \phi$.

2. Criterion of hypoellipticity

In this section we shall give an improvement of the criterion of hypoellipticity studied by [8], [1] and refined by [6].

Let Ω be an open set of R^n and let $P(x,D_x)$ be a differential operator of order m with coefficients in $C^\infty(\Omega)$. We assume:

(I) For any compact set K of Ω and any $N > 0$ there exists a constant $C_1 = C_1(K,N)$ such that

(2.1) $$\|u\|_0 \leqq C_1(\|Pu\|_0 + \|u\|_{-N}) , \quad u \in C_0^\infty(K).$$

(II) For any compact set K of Ω, any β ($|\beta| \neq 0$), any $\mu > 0$ and any $N > 0$ there exists a constant $C_2 = C_2(K,\beta,\mu,N)$ such that

(2.2) $$\|P_{(\beta)}u\|_{-|\beta|} \leqq \mu \|Pu\|_0 + C_2 \|u\|_{-N} , \quad u \in C_0^\infty(K), \quad (|\beta| \neq 0),$$

where $p_{(\beta)}(x,\xi) = D_x^\beta p(x,\xi)$ and $D_x = -i\partial_x$.

(III) For any compact set K of Ω, any α and any $N > 0$ there exists a constant $C_3 = C_3(K,\alpha,N)$ such that

(2.3) $$\|P^{(\alpha)}u\|_0 \leqq C_3(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K) ,$$

where $p^{(\alpha)}(x,\xi) = \partial_\xi^\alpha p(x,\xi)$.

(IV) For any $x_0 \in \Omega$ and any neighborhood U of x_0 there exist $\phi(x)$ and $\psi(x) \in C_0^\infty(U)$ such that

$$\begin{cases} \phi(x) = 1 & \text{in some neighborhood of } x_0 , \\ \phi \subset\subset \psi & (\text{,that is, } \psi = 1 \text{ in a neighborhood of } \operatorname{supp}\phi), \end{cases}$$

and the estimate

$$(2.4) \qquad \| P\phi u \|_{\kappa} \leqq C_4(K,N,\phi,\psi)(\|\psi P u\|_{\kappa} + \|Pu\|_0 + \|u\|_{-N}), \; u \in C_0^{\infty}(K),$$

holds for any compact set K of Ω and any $N > 0$, where C_4 is a constant depending on K, N, ϕ and ψ. Here κ is a positive number smaller than 1, independent of K, N, ϕ and ψ.

(V) For any compact set K of Ω, any β ($|\beta| \neq 0$), any $\mu > 0$, any $N > 0$ and any $\psi(x) \in C_0^{\infty}(\Omega)$ there exists a constant $C_5 = C_5(K,\beta,\mu, N,\psi)$ such that

$$(2.5) \qquad \|(\psi P)_{(\beta)} u\|_{\kappa-|\beta|} \leqq \mu \|\psi P u\|_{\kappa} + C_5(\|Pu\|_0 + \|u\|_{-N}),$$

$$u \in C_0^{\infty}(K), (|\beta| \neq 0),$$

where κ is the same as in (IV).

<u>Theorem 2.1.</u> Assume that a differential operator $P = p(x,D_x)$ satisfies above conditions (I)-(V). Then for any $v \in \mathcal{D}'(\Omega)$, for any open set $\Omega' \subset\subset \Omega$ and for any real s it follows that

$Pv \in H_s^{loc}(\Omega')$ implies $v \in H_s^{loc}(\Omega')$.

Therefore, P is hypoelliptic in Ω.

As in §2 of [6] we employ a pseudodifferential operator $\Lambda_{s,k,\varepsilon}$ with a symbol $\langle\xi\rangle^{s}(1+\varepsilon\langle\xi\rangle)^{-k}$ for real s, $\varepsilon > 0$ and $k \geqq 0$. We denote $\Lambda_{s,0,\varepsilon}$ simply by Λ_s.

Lemmma 2.1 (cf. Lemma 2.10 of [6]). Let P satisfy the condition (II). Then, for any compact set K of Ω, any β ($|\beta| \neq 0$), any real s, any $\mu > 0$, $N > 0$, $\varepsilon > 0$ and $k \geqq 0$ there exists a constant $C = C(K,\beta,s,\mu,N,k)$ independent of ε such that

$$(2.6) \qquad \|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0 \leqq \mu\|\Lambda_{s,k,\varepsilon}Pu\|_0 + C\|u\|_{-N}, \quad u \in C_0^\infty(K).$$

Furthermore, for any K of Ω, any real s, s', any $\mu > 0$, $N > 0$, $\varepsilon > 0$ and $k \geqq 0$ there exists a constant $C' = C'(K,s,s',\mu,N,k)$ independent of ε such that

$$(2.7) \qquad \|[P,\Lambda_{s,k,\varepsilon}]u\|_{s'} \leqq \mu\|\Lambda_{s,k,\varepsilon}Pu\|_{s'} + C'\|u\|_{-N}, \quad u \in C_0^\infty(K).$$

Proof. The former assertion of the lemma is the same as in Lemma 2.10 of [6]. The estimate (2.7) easily follows from (2.6) and the expansion formula

$$(2.8) \qquad [P,\Lambda_{s,k,\varepsilon}] - \sum_{0<|\alpha|<s+m+N} \frac{(-1)^{|\alpha|}}{\alpha!}\Lambda_{s,k,\varepsilon}^{(\alpha)} P_{(\alpha)} \in S^{-N}.$$

Q.E.D.

Lemma 2.2. Let $\phi(x)$ belong to $C_0^\infty(\Omega)$ and let P satisfy conditions (II) and (III). Then, for any compact set K of Ω, any real s, any $\varepsilon > 0$, $k \geqq 0$ there exists a constant $C = C(K,s,N,k)$ independent of ε such that

$$(2.9) \qquad \|\Lambda_{s,k,\varepsilon}P\phi u\|_0 \leqq C(\|\Lambda_{s,k,\varepsilon}Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K).$$

Proof. When $s = k = 0$, the estimate (2.9) easily follows from the condition (III). In fact, noting the Leibniz formula $[P,\phi] = \sum_{0 < |\alpha| \leq m} \phi_{(\alpha)} P^{(\alpha)}/\alpha!$ we have

$$(2.10) \quad \begin{aligned} \|P\phi u\|_0 &\leq \|\phi P u\|_0 + \|[P,\phi]u\|_0 \\ &\leq C(\|Pu\|_0 + \sum_{0<|\alpha|\leq m} \|P^{(\alpha)}u\|_0) \\ &\leq C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K). \end{aligned}$$

Here and in what follows we denote by the same notation C different constants (independent of ε). In the general case, by means of (2.7) we have

$$\|\Lambda_{s,k,\varepsilon} P\phi u\|_0 \leq C(\|P\Lambda_{s,k,\varepsilon}\phi u\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K).$$

Using the expansion formula

$$(2.11) \qquad \Lambda_{s,k,\varepsilon}\phi \equiv \sum_{0 \leq |\alpha| < s+N+m} \phi_{(\alpha)} \Lambda_{s,k,\varepsilon}^{(\alpha)}/\alpha! \quad \text{mod} \quad S^{-N-m}$$

we have

$$\|P\Lambda_{s,k,\varepsilon}\phi u\|_0 \leq C(\sum_{0 \leq |\alpha| < s+N+m} \|P\phi_{(\alpha)}\Lambda_{s,k,\varepsilon}^{(\alpha)} u\|_0 + \|u\|_{-N}),$$

$$u \in C_0^\infty(K).$$

By the similar argument in the biginning of the proof of Lemma 2.10 of [6], it follows from (2.10) that

$$\|P\phi_{(\alpha)}\Lambda_{s,k,\varepsilon}^{(\alpha)} u\|_0 \leq C(\|P\Lambda_{s,k,\varepsilon}^{(\alpha)} u\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K).$$

By means of (2.6) and the expansion formula similar to (2.8) we have

$$\|P\Lambda_{s,k,\varepsilon}^{(\alpha)}u\|_0 \leqq C(\|\Lambda_{s-|\alpha|,k,\varepsilon}Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(K).$$

Combining above four estimate we have (2.9). Q.E.D.

Lemma 2.3. Let P satisfy conditions (II) and (V). Then, for any compact set K of Ω, any β ($|\beta| \neq 0$), any real s, any ε, μ, $N > 0$ and any $k \geqq 0$ there exists a constant $C = C(K,s,\mu,N,k)$ independent of ε such that

(2.12) $$\|\Lambda_{s-|\beta|,k,\varepsilon}(\psi P)_{(\beta)}u\|_0$$
$$\leqq \mu\|\Lambda_{s,k,\varepsilon}(\psi P)u\|_0 + C(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0 + \|u\|_{-N}),$$
$$u \in C_0^\infty(K),$$

where $\kappa > 0$ and $\psi \in C_0^\infty(\Omega)$ are the same as in the condition (V).

Proof. The lemma follows from the almost same way as in the proof of Lemma 2.10 in [6]. As in its beginning, from (2.5) we have

$$\|(\psi P)_{(\beta)}\Lambda_{s-\kappa,k,\varepsilon}u\|_{\kappa-|\beta|}$$
$$\leqq \mu\|(\psi P)\Lambda_{s-\kappa,k,\varepsilon}u\|_\kappa + C(\|P\Lambda_{s-\kappa,k,\varepsilon}u\|_0 + \|u\|_{-N}),$$
$$u \in C_0^\infty(K).$$

Replace the operator P and the term $\|u\|_{-N}$ in the proof of Lemma 2.10 of [6] by ψP and $\|P\Lambda_{s-\kappa,k,\varepsilon}u\|_0 + \|u\|_{-N}$, respectively.

Then it follows that

$$\|\Lambda_{s-|\beta|,k,\varepsilon}(\psi P)_{(\beta)}u\|_0$$

$$\leqq \mu\|\Lambda_{s,k,\varepsilon}(\psi P)u\|_0 + C(\|P\Lambda_{s-\kappa,k,\varepsilon}u\|_0 + \|u\|_{-N}),$$

$$u \in C_0^\infty(K) .$$

Using (2.7) for the term $\|P\Lambda_{s-\kappa,k,\varepsilon}u\|_0$ we obtain (2.12). Q.E.D.

By the same way as in getting the corollary of Lemma 2,10 of [6] we have

Corollary 2.4. Let P satisfy conditions (II) and (V). Then, for any compact set K of Ω, any real s, s', any $N > 0$, $\varepsilon > 0$ and $k \geqq 0$ there exists a constant $C = C(K,s,s',N,k)$ independent of ε such that

(2.13) $$\|[\psi P,\Lambda_{s,k,\varepsilon}]u\|_{s'}$$

$$\leqq C(\|\Lambda_{s+s',k,\varepsilon}\psi Pu\|_0 + \|\Lambda_{s+s'-\kappa,k,\varepsilon}Pu\|_0 + \|u\|_{-N}),$$

$$u \in C_0^\infty(K).$$

Lemma 2.5. Let P satisfy conditions (II)-(V). Then, for any compact set K of Ω, any real s, any $\varepsilon > 0$, $N > 0$ and $k \geqq 0$ there exists a constant $C = C(K,s,N,k)$ independent of ε such that

(2.14) $$\|\Lambda_{s+\kappa,k,\varepsilon}P\phi u\|_0$$

$$\leqq C(\|\Lambda_{s+\kappa,k,\varepsilon}\psi Pu\|_0 + \|\Lambda_{s,k,\varepsilon}Pu\|_0 + \|u\|_{-N}),\ u \in C_0^\infty(K),$$

where κ and $\phi, \psi \in C_0^\infty(\Omega)$ are the same as in the condition (IV).

Proof. It follows from (2.7) that

$$\|\Lambda_{s+\kappa,k,\varepsilon}P\phi u\|_0$$

$$\leqq C(\|P\Lambda_{s,k,\varepsilon}\phi u\|_\kappa + \|u\|_{-N}), \qquad u \in C_0^\infty(K).$$

In view of the expansion formula (2.11) we have

$$\|P\Lambda_{s,k,\varepsilon}\phi u\|_\kappa$$

$$\leqq C(\|P\phi\Lambda_{s,k,\varepsilon}u\|_\kappa + \sum_{0<|\alpha|<s+m+N+\kappa} \|P\phi_{(\alpha)}\Lambda^{(\alpha)}_{s,k,\varepsilon}u\|_\kappa + \|u\|_{-N}),$$

$$u \in C_0^\infty(K).$$

By means of (2.9) with $s=\kappa, k=0$ and (2.6) we have for $|\alpha| \neq 0$

$$\|P\phi_{(\alpha)}\Lambda^{(\alpha)}_{s,k,\varepsilon}u\|_\kappa \leqq C(\|P\Lambda^{(\alpha)}_{s,k,\varepsilon}u\|_\kappa + \|u\|_{-N})$$

$$\leqq C(\|\Lambda_{s-|\alpha|+\kappa,k,\varepsilon}Pu\|_0 + \|u\|_{-N}), \ u \in C_0^\infty(K).$$

The conjunction of above three estimates gives

(2.15) $$\| \Lambda_{s+\kappa,k,\varepsilon}P\phi u\|_0$$

$$\leqq C(\|P\phi\Lambda_{s,k,\varepsilon}u\|_\kappa + \|\Lambda_{s,k,\varepsilon}Pu\|_0 + \|u\|_{-N}), \ u \in C_0^\infty(K).$$

Substituting $\Lambda_{s,k,\varepsilon}u$ into (2.4) we obtain

(2.16) $$\|P\phi\Lambda_{s,k,\varepsilon}u\|_\kappa$$

$$\leqq C(\|\psi P\Lambda_{s,k,\varepsilon}u\|_\kappa + \|P\Lambda_{s,k,\varepsilon}u\|_0 + \|u\|_{-N}), \ u \in C_0^\infty(K).$$

Noting estimates (2.7) and (2.13), we obtain (2.14) from (2.15) and (2.16). Q.E.D.

Remark. Set $k = s_0+m+N+\kappa$ for $s_0 > 0$. Then, for any $v \in H_{-N} \cap \mathcal{E}'(K)$ the estimate

$$\|\Lambda_{s+\kappa,k,\varepsilon} P\phi v\|_0$$

$$\leqq C(\|\Lambda_{s+\kappa,k,\varepsilon}\psi P v\|_0 + \|\Lambda_{s,k,\varepsilon} P v\|_0 + \|v\|_{-N})$$

holds, where $s \leqq s_0$ and C is a constant independent of ε. This fact follows from (2.14) by the same way as in the remark of Lemma 2.11 in [6].

Proof of Theorem 2.1. Let x_0 be any fixed point in Ω' and let $\psi(x) \in C_0^\infty(\Omega')$ such that $\psi(x) \equiv 1$ in a neighborhood $U(x_0)$ of x_0. Then, for any natural number ℓ we can find finite sequences $\{\phi_j\}_{j=1}^{\ell}$, $\{\psi_j\}_{j=1}^{\ell} \subset C_0^\infty(\Omega')$ such that

$$\phi_1 \subset\subset \psi_1 \subset\subset \phi_2 \subset\subset \psi_2 \cdots \subset\subset \phi_\ell \subset\subset \psi_\ell \subset\subset \psi$$

and we have

$$\text{(2.17)} \quad \|P\phi_j u\|_\kappa \leqq C(K,N,\phi_j,\psi_j)(\|\psi_j P u\|_\kappa + \|Pu\|_0 + \|u\|_{-N}),$$

$$u \in C_0^\infty(K), \ (j = 1,\dots,\ell),$$

for any K of Ω and $N > 0$, where κ is some positive number. Indeed, from the condition (IV), we can take $\tilde{\phi}_1$, $\tilde{\psi}_1 \in C_0^\infty(U(x_0))$ such that $\tilde{\phi}_1 \subset\subset \tilde{\psi}_1$, $\tilde{\phi}_1 \equiv 1$ in some neighborhood $V(x_0)$ of x_0 and satisfies (2.4). For x_0 and the neighborhood $V(x_0)$ we can take again $\tilde{\phi}_2$, $\tilde{\psi}_2 \in C_0^\infty(V(x_0))$ such that $\tilde{\phi}_2 \subset\subset \tilde{\psi}_2$, $\tilde{\phi}_2 \equiv 1$

in some neighborhood of x_0 and satisfies (2.4). Repeating these steps ℓ times, we have sequences $\{\tilde{\phi}_j\}_{j=1}^{\ell}$, $\{\tilde{\psi}_j\}_{j=1}^{\ell} \subset C_0^\infty(\Omega')$. Set $\phi_j = \tilde{\phi}_{\ell-j+1}$, $\psi_j = \tilde{\psi}_{\ell-j+1}$ ($j = 1,\dots,\ell$). Then, $\{\phi_j\}_{j=1}^{\ell}$ and $\{\psi_j\}_{j=1}^{\ell}$ are desired sequences. As well-known, for $v \in E'$ there exists a $N > 0$ sucht that $v \in H_{-N}$. Let us choose ℓ bigger than $(s+m+N)/\kappa$. By means of Lemma 2.11 in [6] and its remark, for $\phi_1 v \in H_{-N} \cap E'(K)$ (, where $K = \text{supp}\,\psi$) the estimate

$$(2.18)\qquad \|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0 \leqq C(\|\Lambda_{s,k,\varepsilon}P\phi_1 v\|_0 + \|\phi_1 v\|_{-N})$$

holds for a constant C independent of ε and $k = s+m+N$. From (2.17) and the remark of Lemma 2.5 it is easy to see that if $k = s+m+N$, then for any $s' \leqq s$ the estimate

$$(2.19)\qquad \|\Lambda_{s',k,\varepsilon}P\phi_j v\|_0 = \|\Lambda_{s',k,\varepsilon}P\phi_j\phi_{j+1}v\|_0$$

$$\leqq C(\|\Lambda_{s',k,\varepsilon}\psi_j Pv\|_0 + \|\Lambda_{s'-\kappa,k,\varepsilon}P\phi_{j+1}v\|_0 + \|\psi v\|_{-N})$$

holds because of $\psi_j P\phi_{j+1} = \psi_j P$ and $\phi_{j+1}v = \phi_{j+1}\psi v$. From (2.18) and (2.19) we have

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0$$

$$\leqq C(\|\Lambda_{s,k,\varepsilon}\psi_1 Pv\|_0 + \|\Lambda_{s-\kappa,k,\varepsilon}P\phi_2 v\|_0 + \|\psi v\|_{-N})$$

Applying (2.19) to the second term of the right hand side, we have

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0 \leqq C(\|\Lambda_{s,k,\varepsilon}\psi_1 Pv\|_0 + \|\Lambda_{s-\kappa,k,\varepsilon}\psi_2 Pv\|_0$$

$$+ \|\Lambda_{s-2\kappa,k,\varepsilon}P\phi_3 v\|_0 + \|\psi v\|_{-N}).$$

Applying again (2.19) to the third term on the right hand side, and repeating the same procedure, we have

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0 \leqq C\Big(\sum_{j=1}^{\ell} \|\Lambda_{s-\kappa(j-1),k,\varepsilon}\psi_j P v\|_0$$

$$+ \|\Lambda_{s-\kappa\ell,k,\varepsilon} P\psi v\| + \|\psi v\|_{-N}\Big)$$

Since $\psi_j P v \in H_s$ from the hypothesis of the theorem, and since $\Lambda_{s-\kappa\ell,k,\varepsilon} P \in \mathcal{S}^{-N}$ for any $\varepsilon > 0$, we obtain from (2.17) of [6]

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0 \leqq C\Big(\sum_{j=1}^{\ell} \|\psi_j P v\|_{s-\kappa(j-1)} + \|\psi v\|_{-N}\Big)$$

$$\leqq C(\|\psi P v\|_s + \|\psi v\|_{-N}).$$

Letting ε tend to 0, we finally obtain

$$\|\phi_1 v\|_s \leqq C(\|\psi P v\|_s + \|\psi v\|_{-N}).$$

This shows v belongs to H_s in some neighborhood of x_0. Since x_0 is arbitrary point in Ω' we can complete the proof.

Q.E.D.

We end this section by the following corollary:

Corollary 2.6. Assume that for a constant C we have

(2.20) $p(x,\xi) \neq 0$ if $|\xi'| \leqq C|\xi''|$ and $|\xi''|$ large enough,

where $\xi = (\xi',\xi'')$. Then we can ameliorate Theorem 2.1. Namely, we can replace the multi-index $\beta = (\beta',\beta'')$ in conditions (II) and (V) by the multi-index β' with respect to x' variables ($x = (x',x'')$).

Proof. Take a symbol $\chi(\xi)$ in $S^0_{1,0}$ such that $\chi = 1$ on $\{|\xi'| \geqq C|\xi''|\} \cap \{|\xi| \geqq 2\}$ and $\chi = 0$ on $\{|\xi'| \leqq C|\xi''|/2\} \cap \{|\xi| \leqq 1\}$. If $Pv \in H^{loc}_s(\Omega')$ we have $(1-\chi(D_x))v \in H^{loc}_{s+m}(\Omega')$ and $P\chi v = Pv - P(1-\chi)v \in H^{loc}_s(\Omega')$ because it follows from (2.20) that there exists a microlocal parametrix of P on supp $(1-\chi)$. Since ξ and ξ' are equivalent on supp χ, we can replace the pseudodifferential operator $\Lambda_{s,k,\varepsilon}$ by a pseudodifferential operator with a symbol $(1+\varepsilon\langle\xi'\rangle)^{-k}\langle\xi'\rangle^s$, which permits the amelioration of Theorem 2.1. Q.E.D.

3. Proof of Theorem 1.1

Let $P = p(x,y,D_x,D_y) = A + gB = a(x,y,D_x) + g(x')b(x,y,D_y)$ denote the differential operator in Theorem 1.1. In view of Theorem 2.1, for the proof of Theorem 1.1 it suffices to show that P satisfies conditions (I)-(V) in Section 2. (Talking more accurately about the plan of the proof, we shall use Corollary 2.6 in checking (V)).

Since conditions (I)-(V) are stated for a compact set K of R^n, we may assume, without loss of generality, that $g(x')$ and coefficients of A and B belong to $\mathcal{B}^\infty(R^n)$, and $g(x')$ satisfy for any $\varepsilon > 0$

$$(3.1) \qquad g(x') \geqq C_\varepsilon > 0 \qquad \text{on} \quad \{|x'| \geqq \varepsilon\}.$$

Lemma 3.1. Set $\Omega_\varepsilon = \{(x,y) \in R^n ;\ |x'| < \varepsilon\}$. Then, for any $\varepsilon > 0$, any α, any real s and any $N > 0$ there exist constants $C(\varepsilon,s,N)$ and $C(\varepsilon,\alpha,s,N)$ such that

$$(3.2) \qquad \|u\|_s \leqq C(\varepsilon,s,N)(\ \|Pu\|_{s-2} + \|u\|_{-N}),$$

$$(3.3) \qquad \|P^{(\alpha)}u\|_s \leqq C(\varepsilon,\alpha,s,N)(\ \|Pu\|_{s-|\alpha|} + \|u\|_{-N}),$$

$$u \in C_0^\infty(R^n \setminus \Omega_\varepsilon).$$

Proof is the same as in Lemma 3.1 of [6].

Lemma 3.2. Let $\phi_0(x')$ be a function in $C^\infty(R^{n_1}_{x'})$ such that for any $\alpha \neq 0$, $\phi_{0(\alpha)} = 0$ on $\{|x'| \leqq \varepsilon\}$, where $\phi_{0(\alpha)} = D^\alpha_{x'}\phi_0$. Then, for any $\varepsilon > 0$, any real s and any $N > 0$ there exists a constant $C(\varepsilon,s,N)$ such that

$$(3.4) \quad \|[P,\phi_0]u\|_s \leqq C(\varepsilon,s,N)(\ \|Pu\|_{s-1} + \|u\|_{-N}),\ u \in C_0^\infty(R^n).$$

The lemma easily follows from Lemma 3.1 by the same way as in the proof of Lemma 3.2 of [6].

Lemma 3.3. For any $\mu > 0$ and any $N > 0$ there exists a constant $C(\mu,N)$ such that

(3.5) $$\|u\|_0 \leqq \mu\|Pu\|_0 + C(\mu,N)\,\|u\|_{-N}\ , \quad u \in C_0^\infty(R^n).$$

Proof. From conditions 1°) and 2°) for P it is easy to see

(3.6) $$\|\langle D_x\rangle u\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle u\|_0^2 \leqq C(\ \mathrm{Re}(Pu,u) + \|u\|_0^2\ )$$
$$\leqq C(\ \|Pu\|_0^2 + \|u\|_0^2\ ),\quad u \in C_0^\infty(R^n)\ .$$

Let $\phi_0(x') \in C_0^\infty(R_{x'}^{n_1'})$ such that $\mathrm{supp}\,\phi_0 \subset \{|x'| < \varepsilon, \phi_0(x') \equiv 1$ on $\{|x'| \leqq \varepsilon/2\}$. Then, on account of Poincaré's inequality we have

(3.7) $$\|\phi_0 u\|_0 \leqq \delta(\varepsilon)\|\ \langle D_{x'}\rangle\ u\|_0 \leqq \delta(\varepsilon)\|\ \langle D_x\rangle\ u\|_0\ ,\ u \in C_0^\infty(R^n)\ ,$$

where $\delta(\varepsilon) \downarrow 0 \quad (\varepsilon \downarrow 0)$. From (3.7) and the estimate obtained by setting $u = \phi_0 u$ in (3.6) we have

$$\|\phi_0 u\|_0 \leqq C\delta(\varepsilon)(\ \|P\phi_0 u\|_0 + \|\phi_0 u\|_0\ ),\quad u \in C_0^\infty(R^n).$$

Using this and the preceding two lemmas we obtain (3.5), by the similar way as in the proof of Lemma 3.3 of [6]. Q.E.D.

It follows from Lemma 3.3 that P satisfies the condition (I). Now, we shall check conditions (II) and (III).

Lemma 3.4. For any β $(|\beta| \neq 0\ )$, any μ and $N > 0$ there exists a constant $C(\beta,\mu,N)$ such that

$$(3.8) \quad \|P_{(\beta)}u\|_{-|\beta|} \leqq \mu\|Pu\|_0 + C(\beta,\mu,N)\|u\|_{-N}, \quad u \in C_0^\infty(R^n).$$

For any α and any $N > 0$ there exista a cinstant $C(\alpha,N)$ such that

$$(3.9) \quad \|P^{(\alpha)}u\|_0 \leqq C(\alpha,N)(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Proof. On account of (3.5) it suffices to check (3.8) for $|\beta| = 1$ and (3.9) for $|\alpha| = 1$. It is clear that for $|\alpha| = 1$ we have

$$\|P^{(\alpha)}u\|_0 \leqq C(\|\langle D_x\rangle u\|_0 + \|g(x')\langle D_y\rangle u\|_0), \quad u \in C_0^\infty(R^n).$$
$$(|\alpha| = 1).$$

From this and (3.6) we have (3.9) for $|\alpha| = 1$. Since $g(x')$ is non-negative function we have

$$|\partial_{x_j'} g(x')| \leqq C_1 \sqrt{g(x')} \quad \text{in a neighborhood of } x' = 0,$$
$$(j = 1, \ldots, n_1'),$$

for a constant C_1 (see Remark 1 of Theorem 1.1 of [6]). In view of this inequality we have for $|\beta| = 1$

$$\|P_{(\beta)}u\|_{-|\beta|} \leqq C(\|\langle D_x\rangle u\|_0 + \|g_{(\beta)}(x')\langle D_y\rangle u\|_0 + \|g(x')\langle D_y\rangle u\|_0)$$
$$\leqq C(\|\langle D_x\rangle u\|_0 + \|g(x')^{1/2}\langle D_y\rangle u\|_0),$$
$$u \in C_0^\infty(R^n).$$

Since we have

$$(3.6)' \quad \|\langle D_x\rangle u\|_0 + \|g(x')^{1/2}\langle D_y\rangle u\|_0 \leqq C(\mathrm{Re}(Pu,u) + \|u\|_0)$$
$$\leqq \mu\|Pu\|_0 + C_\mu\|u\|_0, \quad u \in C_0^\infty(R^n)$$

for any $\mu > 0$ and some constant C_μ, we get (3.8) for $|\beta| = 1$.

Q.E.D.

In order to check conditions (IV) and (V) we state two preparatory lemmas which will be proved in the following two sections.

Lemma 3.5. Let κ be equal to 1/3. For any $N > 0$ there exists a constant $C(N)$ such that

$$(3.10)\qquad \|g(x')\langle D_y\rangle^{1+\kappa}u\|_0 \leqq C(N)(\|Pu\|_0 + \|u\|_{-N}),\ u \in C_0^\infty(R^n).$$

Lemma 3.6. Let κ be 1/3 and let $\phi_1(x'')$ and $\psi_1(x'')$ be functions in $C_0^\infty(R_{x''}^{n_1''})$ such that $\phi_1 \subset\subset \psi_1$. Then, for any $N > 0$ there exists a constant $C(N)$ such that

$$(3.11)\qquad \|[P,\phi_1]u\|_\kappa \leqq C(N)(\|\psi_1 Pu\|_\kappa + \|Pu\|_0 + \|u\|_{-N}),\ u \in C_0^\infty(R^n).$$

We give a corollary to Lemma 3.5.

Corollary 3.7. Let κ be equal to 1/3 and let $\phi_2(y) \in C_0^\infty(R_y^{n_2})$. For any $N > 0$ there exists a constant $C(N)$ such that

$$(3.12)\qquad \|[P,\phi_2]u\|_\kappa \leqq C(N)(\|Pu\|_0 + \|u\|_{-N}),\quad u \in C_0^\infty(R^n).$$

Proof. Let $f(\xi,\eta)$ be a symbol in $S^0_{1,0}$ such that

$$(3.13)\qquad \begin{cases} f = 1 \text{ on } \{|\xi| \leqq |\eta|\} \cap \{|\xi| + |\eta| \geqq 1\}, \\ \operatorname{supp} f \subset \{|\xi| \leqq 2|\eta|\} \cap \{|\xi|+|\eta| \geqq 1/2\}. \end{cases}$$

Since P is microlocally elliptic on $\{|\xi| \geqq |\eta|\}$ it is easy to see

$$(3.14)\qquad \|(1-f)u\|_{1+\kappa} + \|[P,f]u\|_\kappa \leqq C(\|Pu\|_0 + \|u\|_{-N}),\quad u \in C_0^\infty(R^n).$$

In view of the microlocal ellipticity of P we also have

$$(3.15)\quad \|[P,\phi_2]fu\|_\kappa \leqq C(\|g(x')\langle D_y\rangle^{1+\kappa}u\|_0 + \|Pu\|_0 + \|u\|_{-N}).$$

Together with (3.14), estiamtes (3.15) and (3.10) give (3.12).

Q.E.D.

We shall show that P satisfies the condition (IV). Since $P = p(x,y,D_x,D_y)$ is elliptic except $x' = 0$ and the assumptions of p are invariant under the translation with respect to x'' and y variables, it suffices to check the condition (IV) for the origin and its arbitrary neighborhood $U = U_{x'}\times U_{x''}\times U_y$. Let $\phi(x,y)$ be a $C_0^\infty(U)$ function such that $\phi(x,y) = \phi_0(x')\phi_1(x'')\phi_2(y)$, where $\phi_0(x') \in C_0^\infty(U_{x'})$ satisfies $\phi_{0(\alpha)} = 0$ near $x' = 0$ for $|\alpha| \neq 0$. Note that for $u \in C_0^\infty(R^n)$

$$\begin{aligned} P\phi(x,y)u &= P\phi_0(x')\phi_1(x'')\phi_2(y)u \\ &= \phi_1 P\phi_0\phi_2 u + [P,\phi_1]\phi_0\phi_2 u \\ &= \phi_1\phi_0 P\phi_2 u + \phi_1[P,\phi_0]\phi_2 u + [P,\phi_1]\phi_0\phi_2 u \\ &= \phi P u + \phi_1\phi_0[P,\phi_2]u + \phi_1[P,\phi_0]\phi_2 u + [P,\phi_1]\phi_0\phi_2 u . \end{aligned}$$

Let $\psi(x,y)$ be a $C_0^\infty(U)$ function such that $\psi(x,y) = \psi_0(x')\psi_1(x'')\psi_2(y)$ and $\phi \subset\subset \psi$ (in particular $\phi_1 \subset\subset \psi_1$). Then it follows from Corollary 3.7, Lemma 3.2 and Lemma 3.6 that for $\kappa = 1/3$

$$(3.16)\quad \|P\phi u\|_\kappa \leqq C(\|\psi P u\|_\kappa + \|Pu\|_0 + \|u\|_{-N}),\quad u \in C_0^\infty(R^n).$$

Indeed, the estimate is obvious because we see by means of Lemma 2.2 that for any real s and any $\tilde{\phi}(x,y) \in C^\infty(R^n)$ there exists a constant $C(s,\tilde{\phi})$ such that

$$(3.17) \qquad \|P\tilde{\phi}u\|_s \leqq C(s,\tilde{\phi})(\|Pu\|_s + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

To complete the proof of Theorem 1.1 it remains to check the conditon (V). Note Corollary 2.6 at the end of Section 2 and the fact that $p(x,y,\xi,\eta) \neq 0$ if $0 < |\eta| \leqq |\xi|$ and $|\xi|$ large enough. Then it suffices to show, in place of the condition (V), that for any multi-index $\tilde{\beta} \neq 0$ with respect to only y variables, and for any $\mu > 0$ and any $N > 0$ there exists a constant $C = C(\tilde{\beta},\mu,N)$ such that

$$(3.18) \qquad \|(\psi P)_{(\tilde{\beta})}u\|_{\kappa-|\tilde{\beta}|} \leqq \mu\|(\psi P)u\|_\kappa + C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

From now on we shall prove the following estimate stronger than (3.18)

$$(3.19) \qquad \|(\psi P)_{(\tilde{\beta})}u\|_{\kappa-|\tilde{\beta}|} \leqq C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n), \quad (|\tilde{\beta}| \neq 0).$$

By means of (3.5), the estimate (3.19) is obvious if $|\tilde{\beta}| \geqq 3$.

Note that for $0 < |\tilde{\beta}| \leqq 2$

$$(\psi P)_{(\tilde{\beta})} = (\psi A)_{(\tilde{\beta})} + g(x')(\psi B)_{(\tilde{\beta})}.$$

It follows from Lemma 3.5 that for $\kappa = 1/3$ and $0 < |\tilde{\beta}| \leq 2$

$$\|g(x')(\psi B)_{(\tilde{\beta})}u\|_{\kappa-|\tilde{\beta}|} \leq \|g(x')<D_y>^{1+\kappa}u\|_0$$

$$\leq C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

On the other hand, for the case $|\tilde{\beta}| = 2$ it follows from (3.6) that

$$\|(\psi A)_{(\tilde{\beta})}u\|_{\kappa-|\tilde{\beta}|} \leq C\|<D_x>^\kappa u\|_0$$

$$\leq C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n),$$

and for $|\tilde{\beta}| = 1$ we have from the ellipticity of A with respect to x variables

$$\|(\psi A)_{(\tilde{\beta})}u\|_{\kappa-|\tilde{\beta}|} \leq C\|<D_x>^2 u\|_{\kappa-1}$$

$$\leq C(\|A<D_x,D_y>^{\kappa-1}u\|_0 + \|u\|_0)$$

$$\leq C(\|Au\|_{\kappa-1} + \|<D_x>^\kappa u\|_0 + \|u\|_0)$$

$$\leq C(\|Pu\|_{\kappa-1} + \|g(x')Bu\|_{\kappa-1} + \|<D_x>u\|_0 + \|u\|_0)$$

$$\leq C(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Here we used (3.5), (3.6), Lemma 3.5 and the fact that

$$\|[A, <D_x,D_y>^{\kappa-1}]u\|_0 \leq C\|<D_x>^2<D_x,D_y>^{\kappa-2}u\|_0$$

$$\leq C\|<D_x>^\kappa u\|_0 .$$

Thus we obtain (3.19), which completes the proof of Theorem 1.1.

4. Proof of Lemma 3.5

As stated in the proof of Lemma 3.4, from the property of non-negative function we have for any β with $|\beta| \leqq 2$

$$(4.1) \qquad |\partial_{x'}^{\beta} g(x')| \leqq C_\beta g(x')^{1-(1/2)|\beta|}$$

in a neighborhood of $x' = 0$,

where C_β is a constant depending on β. Since P is elliptic except for $x' = 0$ the estimate (3.10) holds for $u \in C_0^\infty(R^n \setminus \Omega_\varepsilon)$, where Ω_ε is the same as in Lemma 3.1. In view of Lemma 3.2 it suffices to show (3.10) for $u \in C_0^\infty(\Omega_\varepsilon)$. Therefore, we may assume that (4.1) holds for all x' by modifying $g(x')$ out of some neighborhood of $x' = 0$.

Let $\tilde{\phi}_0(t)$, $\tilde{\phi}_1(t)$ and $\tilde{\phi}_2(t)$ be C^∞-functions in $[0,\infty)$ such that

$$\begin{cases} \operatorname{supp} \tilde{\phi}_0(t) \subset [0,1), & \tilde{\phi}_0(t) = 1 \text{ on } [0,1/2], \\ \operatorname{supp} \tilde{\phi}_1(t) \subset [0,2), & \tilde{\phi}_1(t) = 1 \text{ on } [0,1], \\ \operatorname{supp} \tilde{\phi}_2(t) \subset (1,\infty), & \tilde{\phi}_2(t) = 1 \text{ in } [2,\infty) \end{cases}$$

and

$$(4.2) \qquad \tilde{\phi}_1 + \tilde{\phi}_2 = 1 \text{ in } [0,\infty).$$

Set $\lambda(\xi,\eta) = (|\xi|^6 + \langle\eta\rangle^4)^{1/6}$. Then $\lambda(\xi,\eta)$ satisfies inequalities (2.5) and (2.6) in [6], so it is a basic weight function associated with pseudodifferential operators.

Lemma 4.1. Set $\chi_j(x',\xi,\eta) = \tilde{\phi}_j(g(x')\lambda(\xi,\eta))$ ($j = 0,1,2$). Then $\chi_j(x',D_x,D_y)$ belongs to $S^0_{\lambda,\mathbb{1},\delta}$, where $\mathbb{1} = (1,\ldots,1)$ and $\delta = (\delta_1,\ldots,\delta_{n_1'},0,\ldots,0)$, $\delta_k = 1/2$. Furthermore we have

$$(4.3) \qquad \chi_1 + \chi_2 = I.$$

The lemma follows from (4.1) and (4.2), by the same way as in the proof of Proposition 5.1 of [6]. (About the definition of $S^m_{\lambda,\mathbb{1},\delta}$ see Definition 2.3 of [6]).

Lemma 4.2. There exists a constant C_0 such that

$$(4.4) \qquad \|g(x')\langle D_y\rangle^{1+\kappa}v_1\|_0 \leqq C_0(\|Pv_1\|_0 + \|v_1\|_0), \quad (\kappa = 1/3),$$

if $v_1 = \chi_1(x',D_x,D_y)u$ for $u \in C_0^\infty(R^n)$.

Proof. Let $\tilde{\phi}_3(t)$ be a C^∞- function in $[0,\infty)$ such that

$$\text{supp}\,\tilde{\phi}_3(t) \subset [0,3), \quad \tilde{\phi}_3(t) = 1 \text{ on } [0,2].$$

Set $\chi_3(x',\eta) = \tilde{\phi}_3(g(x')\lambda(0,\eta))$. Then, clearly we have $\chi_3(x',D_y)v_1 = v_1$. Using the fact that $g(x') \leqq 3\langle\eta\rangle^{-2/3}$ on supp $\chi_3(x',\eta)$ we have

$$|g(x')\langle\eta\rangle^{1+\kappa}\chi_3| \leqq \sqrt{3}g(x')^{1/2}\langle\eta\rangle.$$

From this and (3.6) we obtain (4.4). Q.E.D.

As in §5 of [6], we consider an operator $\tilde{p}(x,y,D_x,D_y)$ which is obtained by modifying $p(x,y,D_x,D_y)$ in " a neighborhood of $x' = 0$ " as follows: Set

$$\tilde{p}(x,y,\xi,\eta) = a(x,y,\xi) + (g(x')\lambda(\xi,\eta) + \chi_0(x',\xi,\eta))$$
$$\times\lambda(\xi,\eta)^{-1}b(x,y,\eta).$$

Then we have

Lemma 4.3. $\tilde{P} = \tilde{p}(x,y,D_x,D_y)$ belongs to $S^3_{\lambda,\mathbf{1},\delta}$ and $\tilde{p}(x,y,\xi,\eta)$ satisfies (H)-condition, in the following sense:

i) There exists a constant $c_0 > 0$ such that

(4.5) $$|\tilde{p}(x,y,\xi,\eta)| \geqq c_0\lambda(\xi,\eta)^2 \quad \text{for large } |\xi|+|\eta|.$$

ii) For any α and β there exists a constant $C_{\alpha\beta}$ such that

(4.6) $$|\tilde{p}^{(\alpha)}_{(\beta)}(x,y,\xi,\eta)/\tilde{p}(x,y,\xi,\eta)| \leqq C_{\alpha\beta}\lambda(\xi,\eta)^{\delta\cdot\beta-|\alpha|} \quad \text{for large } |\xi|+|\eta|,$$

where $\delta = (\delta_1,\dots,\delta_{n_1'},0,\dots,0)$, $\delta_k = 1/2$.

The proof is done by using (4.1), similarly as in the proof of Proposition 5.3 of [6].

By means of Proposition 2.7 of [6] and Lemma 4.3 we have a parametrix $Q \in S^{-2}_{\lambda,\mathbf{1},\delta}$ such that for $\tilde{P} \in S^2_{\lambda,\mathbf{1},\delta}$

(4.7) $$I = Q\tilde{P} + K, \quad K \in S^{-\infty},$$

furthermore

(4.8) $$\begin{cases} Q = Q_0Q_1, \quad Q_0 \in S^{-2}_{\lambda,\mathbf{1},\delta}, \quad Q_1 \in S^0_{\lambda,\mathbf{1},\delta}, \\ \sigma(Q_0) = \tilde{p}(x,y,\xi,\eta)^{-1} \quad \text{for large } |\xi|+|\eta|. \end{cases}$$

Lemma 4.4. Set $v_2 = \chi_2(x', D_x, D_y)u$ for $u \in C_0^\infty(R^n)$. Then, for any $N > 0$ there exists a constant $C(N)$ such that

$$(4.9) \quad \|\langle D_x, D_y\rangle^{1+\kappa} v_2\|_0 + \|g(x')\langle D_y\rangle^2 v_2\|_0$$

$$\leqq C(N)(\|Pv_2\|_0 + \|u\|_{-N}),$$

where $\kappa = 1/3$.

Proof. By checking symbols of $\langle D_x, D_y\rangle^{1+\kappa} Q_0$ and $g(x')\langle D_y\rangle^2 Q_0$ we see that they belong to $\mathcal{S}^0_{\lambda,1,\mathfrak{S}}$. Note that $\tilde{P}\chi_2 \equiv P\chi_2 \mod \mathcal{S}^{-\infty}$. In view of (4.7) and (4.8) we obtain (4.9) by means of L^2-boundedness of the operator belonging to $\mathcal{S}^0_{\lambda,1,\mathfrak{S}}$ (see Proposition 2.5 of [6]). Q.E.D.

Lemma 4.5. For any $N > 0$ there exists a constant $C(N)$ such that for $j = 1,2$ we have

$$(4.10) \quad \|[P,\chi_j]u\|_\kappa \leqq C(N)(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n),$$

where $\kappa = 1/3$.

Proof. It follows from Lemmas 4.1 and 4.3 that we have the expansion formula

$$[\tilde{P},\chi_j] = \sum_{0 < |\alpha+\beta| < N_0} (-1)^{|\alpha|} \chi_{j(\beta)}^{(\alpha)} \tilde{P}^{(\beta)}_{(\alpha)}/\alpha!\beta! \in \mathcal{S}^{3-N_0/2}_{\lambda,1,\mathfrak{S}} \subset \mathcal{S}^{-N}_{2/3,1/2}$$

(cf. (2.10) of [6]),

where $N_0 = 3N+6$. From (4.7) we have

$$[P, \chi_j] \equiv [\tilde{P}, \chi_j] Q \tilde{P}$$

$$\equiv \sum_{0 < |\alpha+\beta| < N_0} (-1)^{|\alpha|} \chi_{j(\beta)}^{(\alpha)} \tilde{p}_{(\alpha)}^{(\beta)} QP/\alpha!\beta! \qquad \mathrm{mod}\ \mathcal{S}^{-N}_{2/3,1/2}.$$

In view of (4.6) it is not difficult to see $\chi_{j(\beta)}^{(\alpha)} \tilde{p}_{(\alpha)}^{(\beta)} Q \in \mathcal{S}^{-1/2}_{\lambda,\mathbb{1},\mathcal{B}}$ for $|\alpha+\beta| \neq 0$. Noting $\Lambda_\kappa = \langle D_x, D_y\rangle^\kappa \in \mathcal{S}^{1/2}_{\lambda,\mathbb{1},0}$ we obtain (4.10).

Q.E.D.

Noting Lemma 3.3 and using the division $u = \chi_1 u + \chi_2 u$ for $u \in C_0^\infty(R^n)$, as in the proof of Lemma 4.1 of [6] we obtain (3.10) by means of Lemmas 4.2, 4.4 and 4.5. We have completed the proof of Lemma 3.5.

5. Proof of Lemma 3.6

In this section we denote $\phi_1(x'')$ and $\psi_1(x'')$ in Lemma 3.6 simply by $\phi(x'')$ and $\psi(x'')$, respectively. Let κ be 1/3 and use the division $u = \chi_1 u + \chi_2 u = v_1 + v_2$ for $u \in C_0^\infty(R^n)$ in the preceding section. By means of Lemmas 4.4 and 4.5 it is easy to see that for any $N > 0$ and any $\phi(x'') \in C_0^\infty(R_{x''}^{n_1''})$ there exists a constant $C(N,\phi)$ such that

(5.1) $\quad \|[P,\phi]v_2\|_\kappa \leqq C(N,\phi)(\|Pu\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n),$

because $[P,\phi]$ belongs to $S_{1,0}^1$. In view of Lemmas 3.3 and 4.5, for the proof of Lemma 3.6 it suffices to show for a constant $C(\phi,\psi)$

(5.2) $\quad \|[P,\phi]v_1\|_\kappa \leqq C(\phi,\psi)(\|\psi P v_1\|_\kappa + \|Pu\|_0 + \|u\|_0),$

provided that $\phi,\ \psi \in C_0^\infty(R_{x''}^{n_1''})$ satisfy $\phi \subset\subset \psi$.

It follows from the hypothesis (1.4) of Theorem 1.1 that for any $s \geqq 1$ there exists a $c_s > 0$ such that

(5.3) $\quad |x'| \leqq (s \log\langle\eta\rangle)^{-1} \quad$ on supp χ_1

$$\text{if } \langle\eta\rangle \geqq c_s ,$$

because $(x',\xi,\eta) \in$ supp χ_1 implies $g(x')\langle\eta\rangle^{2/3} \leqq 3$. Set $h_M(D_y) = \tilde{\phi}_2(M^{-1}\langle D_y\rangle)$ for a $M \geqq 3$, where $\tilde{\phi}_2(t) \in C^\infty(R^1)$ is the same as in §4. Let $f(\xi,\eta)$ be the symbol in $S_{1,0}^0$ defined by (3.13) in Section 3 .

Set

$$(5.4) \qquad w = h_M(D_y)\, \chi_1(x',D_x,D_y)f(D_x,D_y)u \quad \text{for} \quad u \in C_0^\infty(R^n)$$

and let $\tilde{\Lambda}_s$ denote an operator with a symbol $s \log \langle\eta\rangle$. To make clear the idea of this section, first we shall prove (5.2) by assuming that coefficients of P are independent of y.

<u>Lemma 5.1.</u> Assume that coefficients of P are independent of y. For any real $s \geqq 1$ there exists a $M_s \geqq 3$ such that for w defined by (5.4) with $M \geqq M_s$ we have

$$(5.5) \qquad \|\tilde{\Lambda}_s^2 w\|_0 + \|\tilde{\Lambda}_s \langle D_x\rangle w\|_0 \leqq C_0 \|Pw\|_0 ,$$

where C_0 is a constant independent of s, and moreover for any integer $k \geqq 0$ and any $\tilde{\psi}(x'') \in C_0^\infty(R_{x''}^{n_1''})$ the estimate

$$(5.6) \qquad \|\tilde{\Lambda}_s^{k+2}\tilde{\psi}\, w\|_0 + \|\tilde{\Lambda}_s^{k+1}\langle D_x\rangle\tilde{\psi} w\|_0 \leqq C_0 \|P\tilde{\Lambda}_s^k\, \tilde{\psi} w\|_0$$

holds with the same constant C_0 in (5.5) (independent of k and $\tilde{\psi}$).

<u>Proof.</u> By setting $u = w$ in (3.6) we have

$$(5.7) \qquad \|\langle D_x\rangle w\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle w\|_0^2$$
$$\leqq C(\ \mathrm{Re}(Pw,w) + \|w\|_0^2).$$

Here and in what follows we denote different constants independent of s by the same notation C. Since it follows that $[P,\tilde{\Lambda}_s] = 0$ from (5.7) we obtain

$$(5.8) \qquad \|\langle D_x\rangle w\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle w\|_0^2$$
$$\leqq C(\ \mathrm{Re}(P\tilde{\Lambda}_s^{-1}w,\tilde{\Lambda}_s w) + \|w\|_0^2).$$

From this we see that for any $\mu > 0$ there exists a constant C_μ independent of s such that

$$(5.9)\qquad \|<D_x>w\|_0^2 + \|g(x')^{1/2}<D_y>w\|_0^2 \leqq \mu\|\tilde{\Lambda}_s w\|_0^2 + C_\mu(\|P\tilde{\Lambda}_s^{-1}w\|_0^2 + \|w\|_0^2).$$

By means of Poincaré's inequalities it follows from (5.3) that for a constnat C independent of s we have

$$(5.10)\qquad \|(s \log <\eta>)\tilde{w}\|^2_{L^2(R_x^{n_1})} \leqq C\|D_{x_1}\tilde{w}\|^2_{L^2(R_x^{n_1})} \quad \text{if } M \geqq c_s ,$$

where $\tilde{w}$ is the Fourier transform of w with respect to y. In fact, in view of (5.3) and (5.4) we have

$$\operatorname{supp} \tilde{w} \subset \{(x,\eta) ;\ |x_1| \leqq (s \log<\eta>)^{-1}\} .$$

Integrating (5.10) with respect to η we obtain

$$(5.11)\qquad \|\tilde{\Lambda}_s w\|_0^2 \leqq C\|D_{x_1}w\|_0^2 \quad (\leqq C\|<D_x>w\|_0^2).$$

Set $M_s = \max(M_0, c_s, 3)$. Then, combining (5.9) and (5.11) we have

$$\|\tilde{\Lambda}_s w\|_0 + \|<D_x>w\|_0 \leqq C(\|P\tilde{\Lambda}_s^{-1}w\|_0 + \|w\|_0)$$

for w with $M \geqq M_s$. Replacing w by $\tilde{\Lambda}_s w$ we obtain (5.5) because $C\|\tilde{\Lambda}_s w\|_0$ is estimated above by $\|\tilde{\Lambda}_s^2 w\|_0/2$ if M_s is large enough such that $2C \leqq \log <\eta>$ on $\operatorname{supp}\tilde{w}$. The derivation of (5.5) is still valid even if we replace w by $\tilde{\Lambda}_s^k\tilde{\psi}\, w$ for any integer $k \geqq 0$ and any $\tilde{\psi} \in C_0^\infty(R_{x''}^{n_1''})$. So, we obtain (5.6). Q.E.D.

Lemma 5.2. Assume that coefficients of P are independent of y. Let $s \geqq 1$ and let w be defined by (5.4) with $M \geqq M_s$, where M_s is defined by Lemma 5.1. Let $\phi, \psi \in C_0^\infty(R^{n_1''}_{x''})$ satisfy $\phi \subset\subset \psi$. Then there exists a constant $C_1 = C_1(\phi,\psi)$ independent of s such that for any integer $N > 0$

$$(5.12) \quad \|\tilde{\Lambda}_s^{N+1}\langle D_x\rangle\phi w\|_0 \leqq 2C_1 \|\tilde{\Lambda}_s^N \psi P w\|_0 + C_1^{N+1} N^N(\|Pw\|_0 + \|w\|_0).$$

Proof. For any integer $N > 0$ there exists a sequence $\{ \psi_j(x'') \}_{j=1}^{N-1} \subset C_0^\infty(R^{n_1''}_{x''})$ such that

$$\phi \subset\subset \psi_1 \subset\subset \psi_2 \subset\subset \cdots \subset\subset \psi_{N-1} \subset\subset \psi$$

and for a fixed integer $\ell_0 \geqq 2$ we have

$$(5.13) \qquad |D^\beta_{x''} \psi_j| \leqq (C_2 N)^{|\beta|} \qquad \text{for } |\beta| \leqq \ell_0 ,$$

where $C_2 \equiv C_2(\phi,\psi,\ell_0)$ is a constant independent of j and N. In fact, we can find such a sequence by dividing N times a space between supp ϕ and the complement of $\{x'' ; \psi = 1 \}$ and by noting Lemma 1.1 of Chapter V of [10] (The constant C_2 is given in the form $C_2 = \tilde{C}_2 \ell_0$ for a constant $\tilde{C}_2$ independent of ℓ_0). In view of $[P,\tilde{\Lambda}_s] = 0$,it follows from (5.6) that

$$(5.14) \quad \|\tilde{\Lambda}_s^{N+1}\langle D_x\rangle\phi w\|_0 \leqq C_0 \|P\tilde{\Lambda}_s^N\phi w\|_0 = C_0 \|\tilde{\Lambda}_s^N P\phi\psi_1 w\|_0$$

$$\leqq C_0(\|\tilde{\Lambda}_s^N\phi P w\|_0 + \|\tilde{\Lambda}_s^N[P,\phi]\psi_1 w\|_0).$$

Noting that the estimate

$$\|\tilde{\Lambda}_s^N[P,\phi]\ \psi_1 w\|_0 \leqq C'\ \|\tilde{\Lambda}_s^N < D_x > \psi_1 w\|_0$$

holds for a constant C' independent of s and N, from (5.14) and (5.6) we have

$$(5.15)\quad \|\tilde{\Lambda}_s^{N+1} < D_x > \phi w\|_0 \leqq C_0\|\tilde{\Lambda}_s^N \phi P w\|_0 + C_0^2 C'\ \|P\tilde{\Lambda}_s^{N-1}\psi_1 w\|_0 .$$

By means of (5.13), there exists a constant C_2' independent of s, j and N such that

$$(5.16)\quad \|\tilde{\Lambda}_s^{N-j}[P,\psi_j]\ \psi_{j+1} w\|_0$$

$$\leqq (C_2'N)^2\ \|\tilde{\Lambda}_s^{N-j}\ \psi_{j+1} w\|_0 + C_2'N\ \|\tilde{\Lambda}_s^{N-j} < D_x > \psi_{j+1} w\|_0 .$$

Assume that for a fixed N we have $s \log <\eta> \geqq C_2'N$ on $\operatorname{supp} \tilde{w}$, where $\tilde{w}$ is the Fourier transform of w with respect to y. Then, since we have $C_2'N\ \|\tilde{\Lambda}_s^{N-j} w\|_0 \leqq \|\tilde{\Lambda}_s^{N-j+1} w\|_0$ it follows from (5.16) that

$$\|P\tilde{\Lambda}_s^{N-j}\psi_j w\|_0 \leqq \|\tilde{\Lambda}_s^{N-j}\psi_j P w\|_0$$

$$+ C_2'N(\ \|\tilde{\Lambda}_s^{N-j+1}\psi_{j+1} w\|_0 + \|\tilde{\Lambda}_s^{N-j} < D_x > \psi_{j+1} w\|_0).$$

Applying (5.6) to the second term of the right hand side we obtain

$$(5.17)\quad \|P\tilde{\Lambda}_s^{N-j}\psi_j w\|_0 \leqq \|\tilde{\Lambda}_s^{N-j}\psi_j P w\|_0 + C_0 C_2'N\ \|P\tilde{\Lambda}_s^{N-j-1}\psi_{j+1} w\|_0$$

$$(\ j = 1,\dots,N-1, \quad \psi_N = \psi\ ).$$

By means of (3.9) in Lemma 3.4 we have

(5.18) $\|[P,\psi]w\|_0 \leqq C''(\|Pw\|_0 + \|w\|_0)$,

where C'' is a constant independent of s and N. In view of (5.13), there exists a constant C_3 independent of s, j, and N such that

$$\|\psi_j w\|_0 \leqq C_3 \|\psi w\|_0 \quad (j = 0,\dots,N-1,\ \psi_0 = \phi).$$

Set $C_1 = \max(C_0C_3,\ C_0^2C'C_3,\ C_0C_2'C_3,\ C_0C_2'C'')$. Then it follows from (5.15), (5.17) and (5.18) that

(5.19) $$\|\tilde{\Lambda}_s^{N+1}\langle D_x\rangle\phi w\|_0 \leqq C_1\|\tilde{\Lambda}_s^N\psi Pw\|_0 + \sum_{j=1}^{N-1} C_1^j N^{j-1}\|\tilde{\Lambda}_s^{N-j}\psi Pw\|_0$$

$$+ C_1^N N^{N-1}(\|Pw\|_0 + \|w\|_0),$$

$$\text{if } s\log\langle\eta\rangle \geqq C_1N \text{ on } \operatorname{supp}\tilde{w}.$$

From this we obtain (5.12) if $s\log\langle\eta\rangle \geqq C_1N$ on $\operatorname{supp}\tilde{w}$ because $C_1^j N^j\|\tilde{\Lambda}_s^{N-j}\psi Pw\|_0 \leqq \|\tilde{\Lambda}_s^N\psi Pw\|_0$. We can remove the assumption $s\log\langle\eta\rangle \geqq C_1N$ on $\operatorname{supp}\tilde{w}$. In fact, if $s\log\langle\eta\rangle \leqq C_1N$ on $\operatorname{supp}\tilde{w}$ it follows from (5.6) that the estimate

$$\|\tilde{\Lambda}_s^{N+1}\langle D_x\rangle\phi w\|_0 \leqq C_3'(C_1N)^N\|\tilde{\Lambda}_s\langle D_x\rangle w\|_0 \leqq C_0C_3'(C_1N)^N\|Pw\|_0,$$

holds for some constant C_3' independent of s, j and N. Taking C_1 such that $C_1 \geqq C_0C_3'$ furthermore we can complete the proof of the lemma. Q.E.D.

Lemma 5.3. Assume that coefficients of P are independent of y. Let $s \geqq 1$ and let w, ϕ and ψ be the same as in Lemma 5.2. Then there exists a positive number τ ($\leqq 1/3$) independent of s and a constant $C(\phi,\psi)$ such that

$$(5.20) \quad \| \langle D_y \rangle^{s\tau} \langle D_x \rangle \phi w \|_0 \leqq C(\phi,\psi) (\| \langle D_y \rangle^{s\tau} \psi P w \|_0 + \| Pw \|_0 + \| w \|_0).$$

Proof. It follows from (5.12) that for any integer $N > 0$

$$(5.12)' \quad \| \tilde{\Lambda}_s^{N+1} \langle D_x \rangle \phi w \|_0^2 \leqq 8C_1^2 \| \tilde{\Lambda}_s^N \psi P w \|_0^2 + 2C_1^{2N+2} N^{2N} J^2,$$

where $J = \|Pw\|_0 + \|w\|_0$. Multiplying both sides by $\tau^{2N+2}/(2N+2)!$ for $0 < \tau \leqq 1/3$ and using the stirling formula $N^{2N} \leqq e^{2N}(2N)!$ we have

$$(5.21) \quad \iint \frac{(\log \langle \eta \rangle^{\tau s})^{2N+2}}{(2N+2)!} | \langle D_x \rangle \phi \tilde{w} |^2 \, dx d\eta$$

$$\leqq 8C_1^2 \iint \frac{(\log \langle \eta \rangle^{\tau s})^{2N}}{(2N)!} |\psi \widetilde{Pw}|^2 dx d\eta + 2C_1^2 (\tau C_1 e)^{2N} J^2, \quad N = 1,2,\ldots,$$

where $\widetilde{Pw}$ denotes the Fourier transform of Pw with respect to y. Let I_N denote the right hand side of (5.21). Multiplying both sides of (5.12)' by $\tau^{2N+1}/(2N+1)!$ again, we also see that

$$\iint (\log \langle \eta \rangle^{\tau s})^{2N+1} | \langle D_x \rangle \phi \tilde{w} |^2 \, dx d\eta / (2N+1)! \leqq I_N$$

because we have $\log \langle \eta \rangle^s \geqq 1$ by means of $M_s \geqq 3$. Hence we have

$$(5.22)\iint \left(\frac{(\log \langle\eta\rangle^{\tau s})^{2N+1}}{(2N+1)!} + \frac{(\log \langle\eta\rangle^{\tau s})^{2N+2}}{(2N+2)!}\right)|\langle D_x\rangle\phi\tilde{w}|^2 dx d\eta \leqq 2I_N$$

Note $\sum_{m=0}^{\infty}(\log\langle\eta\rangle^{\tau s})^m/m! = \langle\eta\rangle^{s\tau}$ and take τ small enough such that $\tau C_1 e < 1$. Then, summing (5.22) with respect to $N = 1,2,\ldots$ we obtain (5.20) because it follows from Lemma 5.1 that $\iint \sum_{m=0}^{2}(\log\langle\eta\rangle^{\tau s})^m|\langle D_x\rangle\phi\tilde{w}|^2 dx d\eta/m!$ is estimated above by the constant times of J^2. Q.E.D.

Assume that coefficients of P are independent of y. Since τ is independent of s we can choose $s \geqq 1$ such that $s\tau = \kappa = 1/3$. For s chosen above take M_s of Lemma 5.1. Then, since $\|\langle D_x\rangle\phi w\|_\kappa$ is estimated above by $C(\|\langle D_y\rangle^\kappa\langle D_x\rangle\phi w\|_0 + \|\langle D_x\rangle u\|_0)$ it follows from (5.20) and (3.6) that

$$\|\langle D_x\rangle\phi w\|_\kappa \leqq C(\|\psi P w\|_\kappa + \|Pw\|_0 + \|Pu\|_0 + \|u\|_0)$$

holds with w defined by (5.4) with $M \geqq M_s$. Since $(1 - h_M(D_y))f \in S^{-\infty}$ and P is microlocally elliptic on supp $(1 - f)$ it is easy to see that

$$(5.23)\quad \|(1-f)u\|_{1+\kappa} + \|(1-h_M)fu\|_{1+\kappa} + \|[P,f]u\|_\kappa + \|[P,h_M]fu\|_\kappa \leqq C(\|Pu\|_0 + \|u\|_0),\ u \in C_0^\infty(R^n).$$

By means of above two estimates we obtain

$$(5.24)\quad \|\langle D_x\rangle\phi v_1\|_\kappa \leqq C(\|\psi P v_1\|_\kappa + \|Pu\|_0 + \|u\|_0),$$

which shows that (5.2) holds. Indeed, (5.2) follws from (5.24) with ϕ replaced by $D_{x''}^\alpha\phi$ $(|\alpha| = 1)$.

From now on we shall consider the case when coefficients of P depend on y.

Lemma 5.4. There exists an integer $m_0 > 0$ depending only on the dimension n of (x,y) variables and satisfying the following: For a fixed integer $N > 0$ take a sequence $\{ \psi_j \}_{j=1}^{N-1} \subset C_0^\infty(R_{x''}^{n_1''})$ in the proof of Lemma 5.2 such that the integer ℓ_0 of (5.13) is sufficiently larger than m_0. Then, there exists a constant C_4 independent of j and N such that for any $s \geqq 1$ the estimate

$$(5.25) \quad \|[P,\tilde{\Lambda}_s^k]\psi_j w\|_0 \leqq C_4 N(\|<D_x>\tilde{\Lambda}_s^k\psi_j w\|_0 + \|g(x')^{1/2}<D_y>\tilde{\Lambda}_s^k\psi_j w\|_0) + (C_4 N)^2 \|\tilde{\Lambda}_s^k\psi_j w\|_0 + C_4 N^{m_0} \|\tilde{\Lambda}_s^k h_M u\|_{-1} ,$$

$$j,k \in \{0,1,\ldots,N\} , \quad \psi_0 = \phi , \quad \psi_N = \psi ,$$

holds with w defined for $u \in C_0^\infty(R^n)$ by (5.4).

Proof. Since each term of (5.25) has a common devisor s^k it suffices to show it when $s = 1$. Take a symbol $\tilde{f}(\xi,\eta)$ in $S_{1,0}^0$ such that

$$\begin{cases} \tilde{f} = 1 & \text{on supp } f \\ \text{supp } \tilde{f} \subset \{ |\xi| \leqq 3|\eta| \} \cap \{ |\xi|+|\eta| \geqq 1/3 \}. \end{cases}$$

Note

$$[P,\tilde{\Lambda}_1^k]\psi_j w = [P,\tilde{\Lambda}_1^k]\tilde{\Lambda}_1^{-k}\tilde{f}\tilde{\Lambda}_1^k\psi_j w + [P,\tilde{\Lambda}_1^k]\tilde{\Lambda}_1^{-k}(1-\tilde{f})\psi_j \chi_1 f\tilde{\Lambda}_1^k h_M u$$

$$\equiv Q\tilde{\Lambda}_1^k\psi_j w + R\tilde{\Lambda}_1^k h_M u ,$$

and set

$$q^0(\eta',x,y,\xi,\eta) = (\log<\eta'>)^k p(x,y,\xi,\eta)(\log<\eta>)^{-k}\tilde{f}(\xi,\eta).$$

Then we have the expansion of the symbol of Q

$$\sigma(Q) = -\sum_{1 \leqq |\alpha| \leqq 2} \partial_\eta^\alpha((\log \langle\eta\rangle)^k) D_y^\alpha p(x,y,\xi,\eta)(\log\langle\eta\rangle)^{-k}\tilde{f}(\xi,\eta)/\alpha! \tag{5.26}$$

$$- 3\sum_{|\gamma|=3}\int_0^1 \frac{(1-\theta)^2}{\gamma!}\Big\{ O_s\text{-}\iint e^{-iz\cdot\zeta}$$

$$\partial_\eta^\gamma D_y^\gamma q^0(\eta+\theta\zeta,x,y+z,\xi,\eta)dzd\zeta\Big\}\ d\theta$$

$$\equiv q_1(x,y,\xi,\eta) + q_2(x,y,\xi,\eta).$$

It is clear that $\|q_1(x,y,D_x,D_y)\psi_j\tilde{\Lambda}_1^k w\|_0$ is estimated above by the first three terms of the right hand side of (5.25) with $s = 1$. If $\tilde{q}_2(x,y,\xi,\eta)$ denotes the symbol of $q_2(x,y,D_x,D_y)\psi_j(x'')\chi_1(x,D_x,D_y)$ $f(D_x,D_y)\langle D_x,D_y\rangle$ then we have $\tilde{q}_2 \in S^0_{\lambda,\mathbb{1},\mathfrak{G}}$ and in view of (5.13) we see that semi-norms of $\tilde{q}_2$ defined by

$$|\tilde{q}_2|_\ell^{(0)} = \max_{|\alpha|+|\beta| \leqq \ell}\ \sup_{R^{2n}} \{|\tilde{q}_{2(\beta)}^{(\alpha)}(x,y,\xi,\eta)|\ \lambda(\xi,\eta)^{|\alpha|-\mathfrak{G}\beta}\}$$

$$(\ \ell = 0,1,2,\dots,\ell_0-2n\ )$$

are estimated above by $C_\ell N^{\ell+2n}$, where C_ℓ is a constant independent of N. Then it follows from Theorem 1.6 of Chapter 7 of [3] that $\|\tilde{q}_2(x,y,D_x,D_y)\langle D_x,D_y\rangle^{-1}\tilde{\Lambda}_1^k h_M(D_y)u\|_0$ is estimated above by the fourth term of the right hand side of (5.25) with $s = 1$. By noting the expansion of the symbol of R we can easily see $\|R\tilde{\Lambda}_1^k h_M u\|_0$ is estimated above by the same term. Q.E.D.

Lemma 5.5. Let m_0, ϕ, ψ and $\{ \psi_j \}_{j=1}^{N-1}$ be the same as in Lemma 5.4. Then for any $s \geqq 1$ there exists a constant $C_5 = C_5(\phi,\psi)$ independent of s, j and N such that the estimate

$$(5.27)\quad \|\tilde{\Lambda}_s^{k+2}\psi_j w\|_0 + \|\tilde{\Lambda}_s^{k+1}\langle D_x\rangle\psi_j w\|_0 + \|\tilde{\Lambda}_s^{k+1} g(x')^{1/2}\langle D_y\rangle \psi_j w\|_0$$
$$\leqq C_5(\|P\tilde{\Lambda}_s^{k}\psi_j w\|_0 + N^{m_0}\| \tilde{\Lambda}_s^{k+1} h_M u\|_{-1}) ,$$
$$j,k \in \{0,\dots,N\} , \quad \psi_0 = \phi , \quad \psi_N = \psi ,$$

holds for w defined by (5.4) with $M \geqq M_s$, where M_s is a constant independent of N and j.

Proof. By setting $u = \tilde{\Lambda}_s^k\psi_j w$ in (3.6) we have

$$(5.28)\quad \|\langle D_x\rangle\tilde{\Lambda}_s^k\psi_j w\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle \tilde{\Lambda}_s^k\psi_j w\|_0^2$$
$$\leqq C(\mathrm{Re}(P\tilde{\Lambda}_s^k\psi_j w,\ \tilde{\Lambda}_s^k\psi_j w) + \|\tilde{\Lambda}_s^k\psi_j w\|_0^2)$$
$$\leqq C(\mathrm{Re}(P\tilde{\Lambda}_s^{k-1}\psi_j w,\ \tilde{\Lambda}_s^{k+1}\psi_j w)$$
$$+ \mathrm{Re}([P,\tilde{\Lambda}_1]\tilde{\Lambda}_1^{-1}\tilde{\Lambda}_s^k\psi_j w,\tilde{\Lambda}_s^k\psi_j w) + \|\tilde{\Lambda}_s^k\psi_j w\|_0^2),$$

Here and in what follows we denote by the same notation C different constants independent of s, j and N. Note

$$[P,\tilde{\Lambda}_1]\tilde{\Lambda}_1^{-1}\tilde{\Lambda}_s^k\psi_j w$$
$$= [P,\tilde{\Lambda}_1]\tilde{\Lambda}_1^{-1}\tilde{f}\tilde{\Lambda}_s^k\psi_j w + [P,\tilde{\Lambda}_1]\tilde{\Lambda}_1^{-1}(1-\tilde{f})\psi_j\chi_1 f\tilde{\Lambda}_s^k h_M u ,$$
$$\equiv Q_0\tilde{\Lambda}_s^k\psi_j w + R_0\tilde{\Lambda}_s^k h_M u ,$$

where $\tilde{f}$ is the same as in the proof of Lemma 5.4. By noting the expansion of $\sigma(Q_0)$ as (5.26) we see that

$$\|Q_0\tilde{\Lambda}_s^k\psi_j w\|_0 \leqq C(\|\langle D_x\rangle\tilde{\Lambda}_s^k\psi_j w\|_0 + \|g(x')^{1/2}\langle D_y\rangle\tilde{\Lambda}_s^k\psi_j w\|_0 + N^{m_0}\|\tilde{\Lambda}_s^k h_M u\|_{-1}).$$

Similarly we obtain

$$\|R_0\tilde{\Lambda}_s^k h_M u\|_0 \leqq CN^{m_0}\|\tilde{\Lambda}_s^k h_M u\|_{-1} .$$

Consequently, for any $\mu > 0$ and some constant C_μ we obtain

$$(5.29) \quad \mathrm{Re}([P,\tilde{\Lambda}_1]\tilde{\Lambda}_1^{-1}\tilde{\Lambda}_s^k\psi_j w, \tilde{\Lambda}_s^k\psi_j w)$$
$$\leqq \mu(\|\langle D_x\rangle\tilde{\Lambda}_s^k\psi_j w\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle\tilde{\Lambda}_s^k\psi_j w\|_0^2$$
$$+ C_\mu(\underline{N}^{m_0}\|\tilde{\Lambda}_s^k h_M u\|_{-1}^2 + \|\tilde{\Lambda}_s^k\psi_j w\|_0^2).$$

It follows from (5.28) and (5.29) that for any $\mu > 0$ there exists a constant C'_μ such that

$$\|\langle D_x\rangle\tilde{\Lambda}_s^k\psi_j w\|_0^2 + \|g(x')^{1/2}\langle D_y\rangle\tilde{\Lambda}_s^k\psi_j w\|_0^2$$
$$\leqq \mu\|\tilde{\Lambda}_s^{k+1}\psi_j w\|_0^2 + C'_\mu(\|P\tilde{\Lambda}_s^{k-1}\psi_j w\|_0^2$$
$$+ \|\tilde{\Lambda}_s^k\psi_j w\|_0^2 + N^{m_0}\|\tilde{\Lambda}_s^k h_M u\|_{-1}^2)$$

(cf. (5.9)).

From this we obtain (5.27) with k replaced by $k-1$ because we have

$$C'_\mu\|\tilde{\Lambda}_s^k\psi_j w\|_0^2 \leqq (1/2)\|\tilde{\Lambda}_s^{k+1}\psi_j w\|_0^2$$

if M_s is large enough. Q.E.D.

<u>Lemma 5.6</u>. Let m_0, ϕ and ψ be the same as in Lemma 5.4. Let $s \geqq 1$ and let w be a function defined for $u \in C_0^\infty(R^n)$ by (5.4) with $M \geqq M_s$, where M_s is the same as in Lemma 5.5. Then, there exists constant $C_1 = C_1(\phi,\psi)$ and $C_6 = C_6(\phi,\psi)$ independent of s such that for any integer $N > 0$

$$(5.30)\quad \|\tilde{\Lambda}_s^{N+1} < D_x > \phi w\|_0 \leqq 2C_1 \|\tilde{\Lambda}_s^N \psi P w\|_0 + C_1^{N+1} N^N(\|Pw\|_0 + \|u\|_0) + C_6 N^{m_0+1} s^N N! \|u\|_0 .$$

<u>Proof</u>. For a fixed $N > 0$ assume that $s \log <\eta> \geqq 2C_4C_5N$ on supp $\tilde{w}$. Then it follows from Lemmas 5.4 and 5.5 that

$$(5.31)\quad \|[P,\tilde{\Lambda}_s^k]\psi_j w\|_0 \leqq 1/2 \|P\tilde{\Lambda}_s^k \psi_j w\|_0 + CN^{m_0} \|\tilde{\Lambda}_s^{k+1} h_M u\|_{-1}$$

because we have $2C_4C_5N \|\tilde{\Lambda}_s^k \psi_j w\|_0 \leqq \|\tilde{\Lambda}_s^{k+1} \psi_j w\|_0$ and we may assume $C_5 \geqq 1$. From (5.31) we have

$$(5.32)\quad \|P\tilde{\Lambda}_s^k \psi_j w\|_0 \leqq 2 \|\tilde{\Lambda}_s^k P \psi_j w\|_0 + 2CN^{m_0} \|\tilde{\Lambda}_s^{k+1} h_M u\|_{-1} .$$

By the similar way as in the proof of Lemma 5.4, it is easy to see

$$(5.33)\quad \begin{cases} \sum_{|\alpha|=1} \|[P^{(\alpha)},\tilde{\Lambda}_s^k]\psi_j w\|_0 \leqq C_4' N \|\tilde{\Lambda}_s^k \psi_j w\|_0 + C_4' N^{m_0} \|\tilde{\Lambda}_s^k h_M u\|_{-1} , \\ \sum_{|\alpha|=2} \|[P^{(\alpha)},\tilde{\Lambda}_s^k]\psi_j w\|_0 \leqq C_4' N^{m_0} \|\tilde{\Lambda}_s^k h_M u\|_{-1} . \end{cases}$$

In view of $[P,\psi_j] = \sum_{0 < |\alpha| \leqq 2} \psi_{j(\alpha)} P^{(\alpha)}/\alpha!$, it follows from (5.13) and (5.33) that the estimate

$$(5.34)\quad \|\tilde{\Lambda}_s^{N-j}[P,\psi_j]\psi_{j+1}w\|_0$$

$$\leqq (C_4''N)^2\|\tilde{\Lambda}_s^{N-j}\psi_{j+1}w\|_0 + C_4''N\|\tilde{\Lambda}_s^{N-j}<D_x>\psi_{j+1}w\|_0$$

$$+ C_4'' N^{m_0}\|\tilde{\Lambda}_s^k h_M u\|_{-1}$$

holds in place of (5.16). Then, by using (5.32) and (5.27) (instead of (5.6)), we can obtain (5.30) as in the proof of Lemma 5.2 because it follows that

$$C_1^j N^j\|\tilde{\Lambda}_s^{N-j}h_M u\|_{-1} \leqq \|\tilde{\Lambda}_s^N h_M u\|_{-1}$$

$$\leqq \|\tilde{\Lambda}_s^N <D_y>^{-1}h_M u\|_0$$

$$\leqq s^N N!\,\|h_M u\|_0$$

if $s\log<\eta> \geqq C_1N$ on $\operatorname{supp}\tilde{u}$,

where $\tilde{u}$ is the Fourier transform of u with respect to y . When $s\log<\eta> \leqq C_1N$ on $\operatorname{supp}\tilde{u}$ the estimate (5.30) is obvious.

Q.E.D.

As in the proof of Lemma 5.3 it follows from Lemma 5.6 that we have

$$(5.35)\quad \|<D_y>^{s\tau}<D_x>\phi w\|_0 \leqq C(\phi,\psi)(\|<D_y>^{s\tau}\psi Pw\|_0 + \|Pw\|_0 + \|u\|_0)$$

because we have, in view of $s\tau = \kappa = 1/3$,

$$\sum_{N=0}^{\infty} N^{2m_0+2}(s\tau)^{2N}(N!)^2/(2N)! < \infty \quad .$$

From (5.35) and (5.23) we also have (5.24) when coefficients of P depend also on y. Since (5.2) follows from (5.24), we have completed the proof of Lemma 3.6.

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