# Studies on Mathematical Methods for Asset Allocation Problems with Randomness and Fuzziness

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# Preface

Until now, in order to deal with uncertainty in the sense of the mathematical programming, many studies with respect to stochastic and fuzzy programming have been performed. Stochastic programming is a field of mathematical methods to deal with optimization problems under uncertainty expressed by the stochastic fluctuation. The application areas of stochastic programming include many fields (inventory, finance and marketing, etc.). In the real world, production companies and financial institutions consider the appropriate allocation such as funds, human and product resource, production time, facility location, etc.. Then, optimizing the asset allocation is one of very important decision makings in order to maintain growth and development of corporate activities. However, while the decision of asset allocation may create significant future profits, it may have the possibility of greater loss in the case that decision makers mistake to it. Therefore, there exist a lot of mathematical researches with respect to the asset allocation until now, and they greatly contribute to the rational decision of optimal asset allocation. Most recently, such as the bankruptcy of major financial institution, inflation of production price and the wild ups and downs of financial markets, there are various uncertain conditions that it is hard to predict future production price, customer's demand and economic trend. Under such uncertainty, the importance maintaining the stable economic growth with the future risk-aversion is re-acknowledged.

On the other hand, with respect to the ambiguity such as lack of reliable information and decision maker's subjectivity, they are assumed to be rather fuzziness than randomness. The centre of social behavior such as economy, investment and production is human behavior, and so it is obvious that psychological aspects of decision makers have a major impact on social behaviors. Then, it is also clear that some factors in historical data include the ambiguity. Therefore, in order to represent such ambiguity and subjectivity, a fuzzy number was introduced in some previous researches. The fuzzy number is roughly a number to represent the degrees of attribution and preference to objectives directly. Thus, the concept of fuzzy number is different from that of random variable. Many previous researches were dealt with random variables or fuzzy numbers in mathematical programming problems, separately. However, practical social systems obviously weave such randomness with fuzziness. Therefore, in the case that decision makers consider the present social problems as mathematical programming problems, they need to consider not only randomness but also fuzziness, simultaneously.

In this thesis, we describe several types of asset allocation problems under randomness and fuzziness. In Chapter 2, we introduce the basic theory of stochastic and fuzzy programming. With respect to the stochastic programming, we mainly focus on the four deterministic programming problems, E model, V model, probability fractile optimization model, and probability maximization model. With respect to the fuzzy programming, we mainly focus on the possibility and necessity programming. By introducing these programming approaches, we can solve stochastic and fuzzy programming problems analytically. Furthermore, considering both randomness and fuzziness simultaneously, we introduce fuzzy random variable and random fuzzy variable. Based on these programming approaches, we develop the analytical and efficient solution methods for various asset allocation problems in the following chapters.

First, we focus on the portfolio selection problem which is one of most important problems in finance and investment fields, and combined probability and optimization theory with the investment behavior. It is one of the most important problems in stochastic programming, and since portfolio theories have focused on risk management under random and ambiguous conditions, they play an important role as the basis of the various social problems for the risk-management. Thus, in the field of portfolio selection problems, researchers have studied models that include uncertainty, and have proposed efficient and versatile models of appropriate risk management. Furthermore, recently, some researchers have also considered applying the portfolio theory to general mathematical programming problems such as the asset allocation in production processes and logistics. Therefore, stochastic and fuzzy programming based on the portfolio theory has become an important field to the mathematical programming from the view point of theory as well as practice.

In Chapter 3, we propose new portfolio selection problems including random fuzzy variables. These proposed problems are initially not well-defined problems due to random and fuzzy variables. Therefore, we introduce chance constraint for the random fuzzy functions setting each confidence level of objects and constraints and introducing the fuzzy goals, and transform the original problems into the deterministic equivalent problems. However, since these problems are nonlinear programming problems, it is also difficult to solve them straightforward. Therefore, we manage to develop the analytical solution method by reducing to the previous efficient solution methods. Furthermore, in order to compare our proposed models with previous standard problems, we provide numerical examples, and show the efficiency of the proposed models under uncertainty.

In Chapter 4, we focus on the large-scale portfolio selection problems. In order to solve them, we deal with the compact factorization for portfolio selection problems proposed by Konno. In this chapter, we extend various standard portfolio models using historical data to fuzzy portfolio selection problems. Then, we show that our proposed models are equivalent to original portfolio models in the sense of the mathematical programming. By performing the extension, the proposed models may be versatile to apply to various uncertain social conditions.

In Chapter 5, we consider multi-scenario and robust portfolio selection problems. In the practical investment, investors receive a lot of information and predict the future return of each stock. Then, investors predict future returns envisaging various future situations such as a substantial fall or rise in stock prices, they usually assume not only one but several scenarios and expect a portfolio decision satisfying goals with respect to all scenarios. Therefore, we consider both randomness and multi-scenarios for future returns, and propose probability fractile optimization and probability maximization models for two situations; (a) the case that a decision maker sets a weight to each scenario based on statistical analysis of historical data and her or his subjectivity, and aggregate all objective function into one weighted function, and (b) the case maximizing the minimum aspiration level among all the scenarios. Furthermore, in order to consider the case that the decision maker often does not assume each weight to be a fixed value due to uncertainty derived from a lack of reliable information and the subjectivity of the decision maker considering the robustness of the portfolio, but assumes them to include an interval to each weight. Coping with these situations, we propose robust portfolio selection problems.

In Chapter 6, by extending the risk management methods used in the portfolio theory to product-mix decision problems, we propose new and versatile product-mix decision problems which are the most important problems in the manufactures. Particularly, applying Theory of Constraint (TOC) which plays an important role in the recent supply chain management, we propose several types of new product mix problems such as the reduction of uncertainty and the improvement of satisfaction of customers, workers, and decision makers. Furthermore, we develop the efficient solution method based on the linear programming approach by performing equivalent transformations to original problems.

In Chapter 7, we consider the general 0-1 programming problems. 0-1 programming problems are one of the most important problems in practical management fields such as project selection problems, scheduling and facility location problems. However, there are few researches including fuzzy random and random fuzzy variables. Therefore, we propose new 0-1 programming problems including fuzzy random and random fuzzy variables. Furthermore, we propose the efficient strict solution method by combining a hybrid method with 0-1 relaxation problem and branch-bound method, and show the analytical efficiency comparing with previous solution methods.

The proposed approaches in this thesis for asset allocation problems extend the previous standard models to more versatile models dealing with various uncertain practical conditions. We hope for the wide application of assets allocation problems and for the development of efficient analytical solution method to stochastic and fuzzy programming approaches.

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# List of Publications

## Journal Papers

1. **Takashi Hasuike** and Hiroaki Ishii, An efficient solution method of project selection problem considering randomness and flexibility, *Scientiae Mathematicae Japonicae* **68(1)**, pp. 117-127, 2008.
2. **Takashi Hasuike**, Hiroaki Ishii and Hideki Katagiri, An Efficient Solution Method for 0-1 Random Fuzzy Programming Problems Considering the Relaxation Problems, *The Open Operational Research Journal* **2**, pp.38-43, 2008.
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6. **Takashi Hasuike** and Hiroaki Ishii, Product mix problems considering several probabilistic conditions and flexibility of constraints, *Computers & Industrial Engineering*, 2009 (in press).
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# Contents

## Chapter 1

# Introduction to Asset Allocation Problems under Randomness and Fuzziness

In many asset allocation problems, the portfolio selection problem is one of the most important problems in financial and investment fields, and the various studies have been performed until now. As for the research history on mathematical approach, Markowitz (Markowitz [90]) has proposed the mean-variance analysis model. It has been central to research activity in the real financial field and numerous researchers have contributed to the development of modern portfolio theory (cf. Campbell.et al. [16], Elton and Gruber [31], Jorion [57], Luenberger [87]). The mean-variance model considers that the variance is assumed to be a risk factor and decision makers should minimize the total variance with satisfying the total profit more than the target value. This concept is same as the risk-aversion, and so the portfolio selection problem is also considered as one of the most important risk-management models. Furthermore, many researchers have proposed models of portfolio selection problems which extended Markowitz model; Capital Asset Pricing Model (CAPM) (Sharpe [108], Lintner [82], Mossin [94]), mean-absolute-deviation model (Konno [68], Konno, et al. [69]), semi-variance model (Bawa and Lindenberg [9]), safety-first model [31], Value at Risk and conditional Value at Risk model (Rockafellar [103]), etc.. As a result, nowadays it is common practice to extend these classical economic models of financial investment to various types of portfolio models. In practice, after Markowitz’s work, many researchers have been trying different mathematical approaches to develop the theory of portfolio selection.

In previous many researches, future returns have been treated as only random variables, and the expected returns and variances also have been assumed to be fixed values in many previous studies. However, since investors receive effective or ineffective information from the real world, ambiguous factors usually exist in it. Furthermore, there are investors who absolutely believe in the predictive ability of historical data. Then, in the case that investors predict future returns envisaging various future situations such as substantial drop or rise of stock price, they usually assume not only one scenario but also several scenarios and expect a portfolio decision satisfying goals with respect to all the scenarios. Consequently, we need to consider not only random conditions but also ambiguity and subjectivity for portfolio selection problems considering possible uncertainty conditions. Until now, portfolio theories have focused on risk management under random and ambiguous conditions, and some researches have recently considered portfolio selection problems under randomness and fuzziness. Guo and Tanaka [41], Inuiguchi and Tanino [53], Tanaka et al. [111,112, 113] and Watada [119] considered the portfolio selection based on fuzzy probabilities and possibility distributions.

Inuiguchi and Ramik [52] considered a fuzzy portfolio selection problem and compared it with a stochastic programming problem. Vercher et al. [117] considered the fuzzy portfolio optimization under downside risk measures. Furthermore, some researchers recently considered more versatile portfolio selection problems under both randomness and fuzziness. Katagiri et al. [60, 61], Huang [49] have represented such conditions as the fuzzy random environment, and considered fuzzy random variables are related with the ambiguity of the realization of a random variable and dealt with a fuzzy number that the center value occurs according to a random variable. Thus, they proposed fuzzy random portfolio selection problems. Yazenin (Yazenin [120, 121]) has considered some models for portfolio selection problems in the probabilistic-possibilistic environment that profits from financial assets are fuzzy random variables. Most recently, Li and Xu [81] has considered a portfolio selection problems under the hybrid uncertain environment including several randomness and fuzziness.

On the other hand, in previous portfolio selection models, the future return is usually assumed to be a random variable derived from statistical analysis based on historical data. Then, when a decision maker considers the case under randomness and fuzziness, she or he often assumes that the future return has ambiguous factors due to quantity of received information and her or his subjectivity is based on the long experience of the investment. Thus, the future return may be dealt with a random variable whose parameters are assumed to be fuzzy numbers based on the decision maker’s subjectivity. Therefore, in this thesis, we propose portfolio selection problems with each future return treated as a random fuzzy variable which Liu (Liu [83, 84]) defined. There are a few studies of random fuzzy programming problem (Katagiri et al. [58, 59], Huang [48]). Most recently, Huang has proposed a portfolio selection model including random fuzzy variables [47]. However, there is no study of the random fuzzy portfolio selection problem which is analytically solved and introducing a fuzzy goal considering decision makers’ intentions with respect to a target profit. Furthermore, we propose a bi-criteria random fuzzy portfolio selection problem including both goals of total future profit and probability fractile level to the target profit. In the real world, it is natural to consider maximizing the probability fractile level, just as maximizing the total profit. Therefore, this proposed model may be more versatile model applying to various investment situations than previous portfolio models.

Thus, comparing with previous and recent researches of portfolio selection problems, it is clear to make advance in various mathematical and practical studies considering randomness and fuzziness. On the other hand, most production companies face many decision-making problems, such as scheduling, logistics, data mining, and resource allocation. Particularly, the product-mix problem, which considers the appropriate decision to the production amount of products, is one of major and important problems in production companies, and recently many researchers also have proposed various models focused on minimizing the total cost arising from the production processes of firms (for example, Letmathe and Balakrishnan [78], Li and Tirupati [80], Morgan and Daniels [93], and Mula et al. [97]). Then, some recent articles have elaborated on studies of production planning problems that include such ambiguous situations (Mula et al. [95, 96], Vasant [116]). Furthermore, in order to cope with such changes in the manufacturing environment, production companies need to possess some degree of flexibility to remain competitive and profitable. Therefore, flexibility in manufacturing operations, particularly product-mix flexibility, becomes more important in order to respond more quickly to changes in the environment. It also makes it possible to deal with demand volatility. Since the study by Browne et al. [14] in classifying and distinguishing different flexibility types, many authors have provided different interpretations of flexibility types that are related to product-mix flexibility, job flexibility (Buzacott [15]), product flexibility (Browne et al. [14], Hyun and Ahn [50], Sethi and Sethi [107], and Son and Park [109]), process flexibility (Sethi and Sethi [107]), and product-mix flexibility (Berry and Cooper [11], Gerwin [35], Grubbström and Olhager [40], and Olhager [99]). Most recently, Gong and Hu [39] has developed a product-mix flexibility model that comprises labor flexibility, machine flexibility, routing flexibility, and information technology, with product-mix flexibility measured by an economic index.

Furthermore, some researchers have also considered applying the portfolio theory to problems in production processes. Lau and Lau [74] considered the inventory control problem based on the mean-variance approach. Gan et al. [34] defined a new concept of supply chain coordination based on the Pareto-optimality criterion and the flexibility to adjustment of some parameters, under the condition that one or more agents are risk-averse, and proposed efficient solution methods for coordinating the supply chain. Choi et al. [21] focused on the mean-variance analysis of a single supplier and a retailer supply chain under a return policy. Most recently, Choi et al. [22] considered channel coordination in the supply chain based on the mean-variance approach. Thus, the portfolio theory has been applied to some problems in production processes, particularly supply chain management. Supply chain management is one of the most important managerial problems. Ding and Chen [25] recently studied the coordination issue of a three-level supply chain selling short life cycle products in a single-period model, and constructed the so-called flexible return policy by setting the rules of pricing while postponing the determination of the final contract prices.

However, in most approaches to product-mix decision problems, randomness and fuzziness are considered separately; but to represent real product-mix decision cases under the changes of future customers' demands and a large amount of effective and ineffective information in the real market, it may not be valid to consider future profits as fixed values, random variables, or fuzzy variables. Rather, they should be considered as product-mix decision problems that integrate randomness and fuzziness. Furthermore, in most previous studies, the main focus is not on the concept of flexibility in responding to many different future scenarios. For example, we assume that decision makers consider product-mix decision problems by including various elements of randomness and fuzziness to represent uncertain situations in the real world. As a result, they decide on an unduly strict original product-mix decision. If an unpredictable situation occurs in the future, then they will not earn the profit predicted, due to the limitation of the constraint, even when randomness and fuzziness are included in the model. Therefore, it is important to introduce flexibilities such as considering several future scenarios and their levels of satisfaction, in terms of the target total profit and the upper values of constraints. At present, no model considers random and ambiguous situations, flexibility and the level of satisfaction for objective function and constraints simultaneously, particularly in the case of models that include probabilistic future returns. Therefore, in this thesis, we focus on product-mix decision problems, in order to take several constraints into account, including randomness, ambiguity, and flexibility. Under such uncertain conditions and flexibilities, if the original plan is to function appropriately and smoothly, then it is most important to undertake appropriate risk management, such as the reduction of uncertainty and the improvement of satisfaction of customers, workers, and decision makers.

In mathematical programming problems, these problems with randomness and fuzziness are called stochastic and fuzzy programming problems (for example, Liu [83, 84]), and are not well-defined problems due to the existence of random and fuzzy variables. Therefore, we need to set some criterion to solve these problems analytically in the sense of the deterministic equivalent mathematical programming. In this thesis, we introduce chance constraints setting the target values to objects and original constrains, and the original problems are usually transformed into nonlinear programming problems. Since it is almost impossible to obtain their global optimal solution directly, we construct the efficient solution method to obtain the global optimal solution by performing the equivalent transformation for several nonlinear programming problems.

# Chapter 2

# Basic Theories of Stochastic and Fuzzy Programming

## 2.1 Stochastic Programming

In the real world, there are many cases of decision making under randomness such as human and machine errors, statistical analysis based on historical data, etc.. Stochastic programming has been developed as the probabilistic generalization of mathematical programming and has played a important role in the mathematical programming in the sense of theory as well as practice such as agriculture, inventory, finance, production, etc.. In the sense of the mathematical programming, previous studies are dealt with stochastic programming approaches based on the probability theory in order to consider randomness. In the 1950s, stochastic programming has been set up independently by Beale [10], Dantzig [24], Charnes and Cooper [19] and others who have observed that for many linear programs to be solved, the values of coefficients are not known precisely. They have suggested to replacing the deterministic view by a stochastic one assuming that these unknown coefficients or parameters are random and their probability distribution is known and independent of the decision variables. Then, many basic models for the stochastic programming have been proposed by some researchers until now, such as the two stage problem, (Beale [10], Danzig [24], Everett and Ziemba [32], Walkup and Wets [118]), the multi-stage problem (Dupacova [30]), the chance constrained approach (Charnes and Cooper [19, 20], Prekopa [101]), the various types of stochastic problems connected with a game theory and an information theory. In the following discussion, we mainly focused on in the framework of deterministic models for the stochastic programming, which are closely related to this thesis.

The stochastic objective function of the stochastic programming problem is handled by its certainty equivalent in the framework of deterministic models in order to solve analytically. The stochastic programming problem including random variable $\boldsymbol{\xi}$ is formulated as follows:

$$\begin{aligned} &\text{Maximize } f\left(\boldsymbol{x},\boldsymbol{\xi}\right) \\ &\text{subject to } g_i\left(\boldsymbol{x},\boldsymbol{\xi}\right) \le 0,\ j=1,2,...,m \end{aligned}$$

With respect to this problem, several types of deterministic mathematical programming problems are considered.

### 2.1.1 Deterministic mathematical programming problems for the stochastic programming

(a) Expected value model

The first type of stochastic programming is called expected value model (E model), which optimizes some expected objective functions subject to some expected constraints. This problem is basically formulated as follows.

$$\begin{aligned}&\text{Maximize } E\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]\\&\text{subject to } E\left[g_i\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]\le 0,\ j=1,2,\ldots,m\end{aligned}$$

where $E\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]$ denotes the expectation of $f\left(\boldsymbol{x},\boldsymbol{\xi}\right)$. E models are indeed a popular method for dealing with stochastic optimization problems. However, in the practical decision making, it is argued that the decision maker need consider the variance, since it may not be desirable to optimize the expected value when its variance is very large.

(b) Variance model

In a way similar to E model, variance model (V model), which is dealt with one of important factors to random variables and optimizes some variance objective functions subject to some constraints, is considered by some research areas. This problem is basically formulated as follows.

$$\begin{aligned}&\text{Minimize } V\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]\\&\text{subject to } g_i\left(\boldsymbol{x}\right)\le 0,\ i=1,2,\ldots,m\end{aligned}$$

where $V\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]$ denotes the variance of $f\left(\boldsymbol{x},\boldsymbol{\xi}\right)$. V models are also indeed a popular method for dealing with stochastic optimization problems. Furthermore, as the natural extension to E model and V model, EV model with the following objective function incorporating two conflicting objects such as

$$\frac{E\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]}{V\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]},\ \mathrm{E}\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]-p\cdot V\left[f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\right]$$

where $p$ is the positive weight coefficient on the variance like a penalty cost.

(c) Stochastic chance constraint programming

E models and V models are very popular, but there are many situations in which E models and V models are not applicable. Therefore, as opposed to E models and V models, the second type of stochastic programming is considered, called chance-constrained programming (CCP). CCP offers a powerful tool of modeling stochastic decision systems with assumption that the stochastic constraints will hold at least the confidence level provided as an appropriate safety margin by the decision maker.

First, we introduce the following chance constraint to provide a confidence level $\alpha$ at which it is desired that the stochastic constraints hold:

$$\Pr\left\{g_i\left(\boldsymbol{x},\boldsymbol{\xi}\right)\le 0,\ i=1,2,\dots,m\right\}\ge\alpha$$

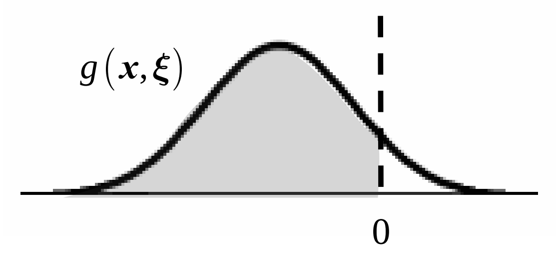


**Fig 2.1.** Stochastic constraint $\Pr\left\{g\left(\boldsymbol{x},\boldsymbol{\xi}\right)\le 0\right\}$

CCP is initialized by Charnes and Cooper [19, 20] and subsequently developed by many researchers. In stochastic environment, we introduce the following CCP model maximizing the target value:

$$\begin{aligned}&\text{Maximize } f\\&\text{subject to } \Pr\left\{f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\ge f\right\}\ge\beta\\&\qquad\qquad\ \Pr\left\{g_i\left(\boldsymbol{x},\boldsymbol{\xi}\right)\le 0,\ i=1,2,\dots,m\right\}\ge\alpha\end{aligned}$$

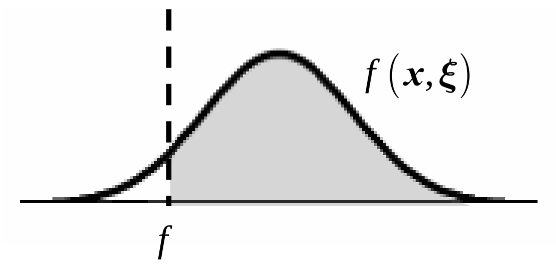


**Fig 2.2.** Stochastic constraint $\Pr\left\{f\left(\boldsymbol{x},\boldsymbol{\xi}\right)\ge f\right\}$

Particularly, as a special case of CCP, we have considered the following problem:

$$\begin{aligned}&\text{Maximize } f\\&\text{subject to } \Pr\left\{\sum_{j=1}^{n} r_j x_j\ge f\right\}\ge\beta\\&\qquad\qquad\ \sum_{j=1}^{n} a_{ij}x_j\le b_i,\ \left(i=1,2,\dots,m\right)\end{aligned}$$

where $r_j$ is the random variable to occur according to a normal distribution $N\left(\overline{r}_j,\sigma_j^2\right)$ with the mean $\overline{r}_j$ and the variance $\sigma_j^2$. In case that the covariance matrix is $\mathbf{V}$, the probability of that

$\sum_{j=1}^{n} r_j x_j$ is over $f$ is equivalently transformed into the following form using the property of normal distribution:

$$\Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f\right\} \geq \beta \quad \Leftrightarrow \quad \Pr\left\{\frac{\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}} \geq \frac{f - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}}\right\} \geq \beta$$

$$\Leftrightarrow \quad \Phi\left(\frac{\sum_{j=1}^{n} \bar{r}_j x_j - f}{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}}\right) \geq \beta$$

$$\Leftrightarrow \quad f \leq \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution $N(0,1)$, and $K_\beta$ is $\beta$-percentile point, i.e. $K_\beta = \Phi^{-1}(\beta)$. As is easily known, $\sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}$ is convex in the case of $\beta \geq 1/2$.

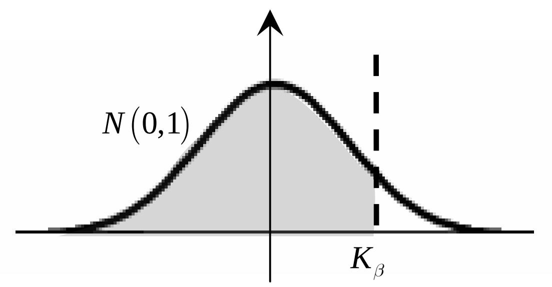


**Fig 2.3.** Standard normal distribution and $\beta$-percentile point

(d) Probability maximization model

In a way similar to probability fractile optimization model, we introduce the probability maximization model as follows:

$$\begin{aligned} &\text{Maximize } \Pr\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq f\} \\ &\text{subject to } \Pr\{g_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0,\ i = 1, 2, \ldots, m\} \geq \alpha \end{aligned}$$

The probability maximization model maximizes the probability that $f(\boldsymbol{x}, \boldsymbol{\xi})$ exceeds a given goal

$f$. If the objective function $f(\boldsymbol{x},\boldsymbol{\xi})$ is equal to be $\sum_{j=1}^{n} r_j x_j$ where $r_j$ is the random variable to occur according to a normal distribution $N(\overline{r}_j, \sigma_j^2)$, it is known that the maximization of this probability is equivalent to the maximization of the fractional function $\dfrac{f - \sum_{j=1}^{n} \overline{r}_j x_j}{\sqrt{\boldsymbol{x}^t \mathbf{V} \boldsymbol{x}}}$.

## 2.2 Fuzzy Programming

For centuries, probability theory and error calculus have been the only models to treat uncertainty. However, a lot of new models recently have been introduced for handling incomplete numerical and linguistic information, decision maker's subjectivity, etc.. For example, "old man", "reputable", "similar", "satisfactory", "large number", "approximately equal to 10". They are not tractable by the classical set theory nor probability theory. In order to deal with such uncertainty, Zadeh [122] first defined the fuzzy set theory, and the fuzzy set theory has been well developed and applied in a wide variety of real problems. The notion of the fuzzy set is non-statistical in nature and the concept provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership function rather than the presence of random variables. Concept of fuzzy set is as follows:

**Definition 2.1**

A fuzzy set $A$ in a universe $X$ is a mapping from $X$ to $[0,1]$. For any $x \in X$ the value $A(x)$ is called the degree of membership of $x$ in $A$. $X$ is called the carrier of the fuzzy set $A$. The degree of membership can also be represented by $x$ instead of $A(x)$. The class of all fuzzy sets in $X$ is denoted by $F(x)$.

The notion of fuzzy variable is first introduced by Kaufmann [65] and then it appeared in Zadeh [123, 124] and Nahmias [98].

**Definition 2.2**

A fuzzy variable is defined as a function from the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$ to the real line $R$. Subsequently, let $\Theta$ be a nonempty set, and $P(\Theta)$ be the power set of $\Theta$. For each $A \in P(\Theta)$, there is a nonnegative number $\mathrm{Pos}\{A\}$, called its possibility, such as

(i) $\mathrm{Pos}\{\phi\} = 0$, $\mathrm{Pos}\{\Theta\} = 1$, and

(ii) $\mathrm{Pos}\{\cup_k A_k\} = \sup_k \mathrm{Pos}\{A_k\}$ for any arbitrary collection $\{A_k\}$ in $P(\Theta)$.

**Definition 2.3**

Let $\xi$ be a fuzzy variable on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$. Then, its membership function is derived from the possibility measure Pos by

$$\mu(x) = \mathrm{Pos}\{\theta \in \Theta \,|\, \xi(\theta) = x\}$$

Furthermore, let $\tilde{a}_i$, $(i = 1, 2, ..., n)$ be fuzzy variables defined on the possibility space $(\Theta, P(\Theta), \mathrm{Pos})$, respectively. Their membership function are also derived from the possibility measures as follows:

$$\mu_{\tilde{a}_i}(x) = \mathrm{Pos}_i\{\theta \in \Theta_i \,|\, \tilde{a}_i(\theta) = x\},\ i = 1, 2, ..., n$$

From these definitions, the following theorem holds.

**Theorem 2.1**

Then, the membership function $\mu_{\tilde{a}}(x)$ of $\tilde{a} = f(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$ is derived from the membership functions $\mu_{\tilde{a}_i}(x)$, $i = 1, 2, ..., n$ as follows:

$$\mu_{\tilde{a}}(x) = \sup_{x_1, x_2, ..., x_n \in R} \left\{ \min_{1 \le i \le n} \mu_{\tilde{a}_i}(x_i) \,\middle|\, x = f(x_1, x_2, ..., x_n) \right\}$$

Definition 2.3 coincides with the extension principle of Zadeh. Now let us illustrate the operation on fuzzy variables.

**Example 2.1**

We introduce a trapezoidal fuzzy variable represented by $(r_1, r_2, r_3, r_4)$ of crisp numbers with $r_1 < r_2 < r_3 < r_4$, whose membership function can be denoted by

$$\mu(x) = \begin{cases} \dfrac{x - r_1}{r_2 - r_1} & (r_1 \le x \le r_2) \\ 1 & (r_2 \le x \le r_3) \\ \dfrac{r_4 - x}{r_4 - r_3} & (r_3 \le x \le r_4) \\ 0 & \text{otherwise.} \end{cases}$$

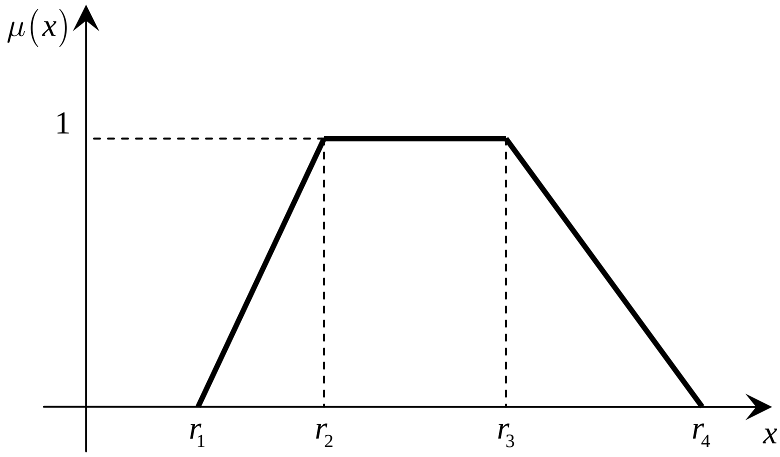


**Fig. 2.4.** Membership function of trapezoidal fuzzy number

Thus, the fuzzy variable is useful tools for the decision making under uncertain environment. Then, the fuzzy mathematical programming has been developed for treating uncertainty in the setting of optimization problems. Until now, there are many studies from the theoretical as well as from the computational point of view (for example, Dubois and Prade [27], Hasuike and Ishii [45], Inuiguchi et al. [51], Liu [83, 84], Luhandjura [88], Maeda [89], Sakawa [105], Zimmermann [126]). In fuzzy programming problems, the objective function is fuzzy-valued and there is no universal concept of optimal solutions to be accepted widely. Therefore, it is important to define some concepts for the fuzzy objective functions and constraints and to investigate their properties. With respect to this viewpoint, there are many ideas such as parametric linear programming problems in order to obtain a reasonable optimal solution (Tanaka et al [111, 112]), possibility and necessity programming problems (Inuiguchi & Ramik [52], Katagiri et al. [61, 62]), interactive programming problems (Katagiri et al. [63], Sakawa [105]), etc..

### 2.2.1 Possibility and necessity measure

In deterministic problems of the fuzzy programming, possibility and necessity measures are introduced. Possibility theory was first proposed by Zadeh [124], and developed by many researchers such as Dubois and Prade [28, 29]. Now, let a and b be fuzzy variables on the possibility spaces $\left(\Theta_1, P\left(\Theta_1\right), \mathrm{Pos}_1\right)$ and $\left(\Theta_2, P\left(\Theta_2\right), \mathrm{Pos}_2\right)$, respectively. Then, $\tilde{a} \leq \tilde{b}$ is a fuzzy event defined on the product possibility space $\left(\Theta, P\left(\Theta\right), \mathrm{Pos}\right)$, whose possibility is

$$\mathrm{Pos}\left\{\tilde{a} \leq \tilde{b}\right\} = \sup_{x,y \in R} \left\{\mu_{\tilde{a}}\left(x\right) \wedge \mu_{\tilde{b}}\left(x\right) \middle| x \leq y\right\}$$

where the abbreviation Pos represents possibility. This means that the possibility of $\tilde{a} \leq \tilde{b}$ is the largest possibility that there exists at least on pair of values $x, y \in R$ such that $x \le y$, and the values of $\tilde{a}$ and $\tilde{b}$ are $x$ and $y$, respectively.

More generally, the possibility of fuzzy event is provided as follows:

**Definition 2.4**

Let $\tilde{a}_i$, $(i = 1, 2, ..., n)$ be fuzzy variables, and $f_j : R^n \to R$, $(j = 1, 2, ..., m)$ be continuous functions. Then, the possibility of the fuzzy event characterized by $f_j(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le 0$, $(j = 1, 2, ..., m)$ is as follows:

$$\begin{aligned} &\mathrm{Pos}\left\{ f_j(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \le 0,\ j = 1, 2, ..., m \right\} \\ &= \sup_{x_1, x_2, ..., x_n \in R} \left\{ \min_{1 \le i \le n} \mu_{\tilde{a}_i}(x_i) \middle| f_j(x_1, x_2, ..., x_n) \le 0,\ (j = 1, 2, ..., m) \right\} \end{aligned}$$

In a similar way to the possibility measure, the necessity measure of a set A is defined as the impossibility of the opposite set Ac.

**Definition 2.5**

Let $(\Theta, P(\Theta), \mathrm{Pos})$ be a possibility space, and $A$ be a set in $P(\Theta)$. Then, the necessity measure of $A$ is defined by

$$\mathrm{Nec}\{A\} = 1 - \mathrm{Pos}\{A^c\}$$

Thus, the necessity measure is the dual of possibility measure, i.e. $\mathrm{Pos}\{A\} + \mathrm{Nec}\{A^c\} = 1$ for any $A \in P(\Theta)$.

**Example 2.2**

Let us consider a trapezoidal fuzzy variable $\xi = (r_1, r_2, r_3, r_4)$ and the target fixed value $\lambda$. From the definitions of possibility and necessity, we obtain the following results:

$$\mathrm{Pos}\{\xi \le \lambda\} = \begin{cases} 1 & r_2 < \lambda \\ \dfrac{\lambda - r_1}{r_2 - r_1} & r_1 \le \lambda \le r_2 \\ 0 & \text{otherwise} \end{cases}, \quad \mathrm{Nec}\{\xi \le \lambda\} = \begin{cases} 1 & r_4 < \lambda \\ \dfrac{r_3 - \lambda}{r_3 - r_4} & r_3 \le \lambda \le r_4 \\ 0 & \text{otherwise} \end{cases}$$

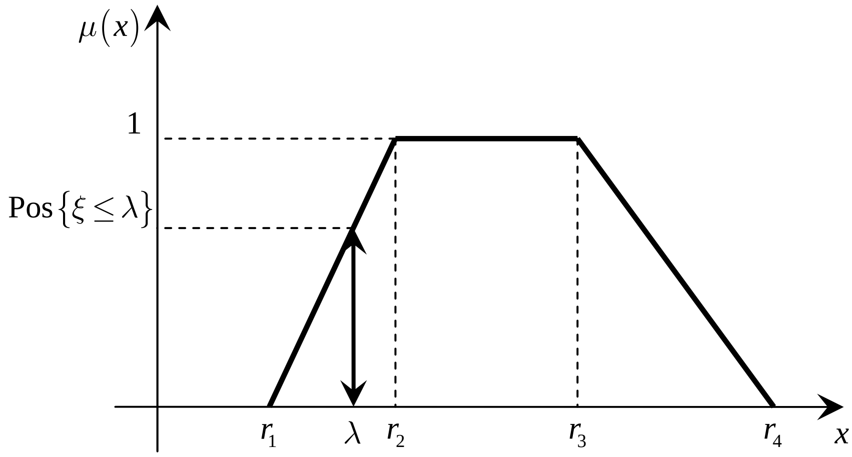


**Figure 2.5.** Possibility measure for the trapezoidal fuzzy number

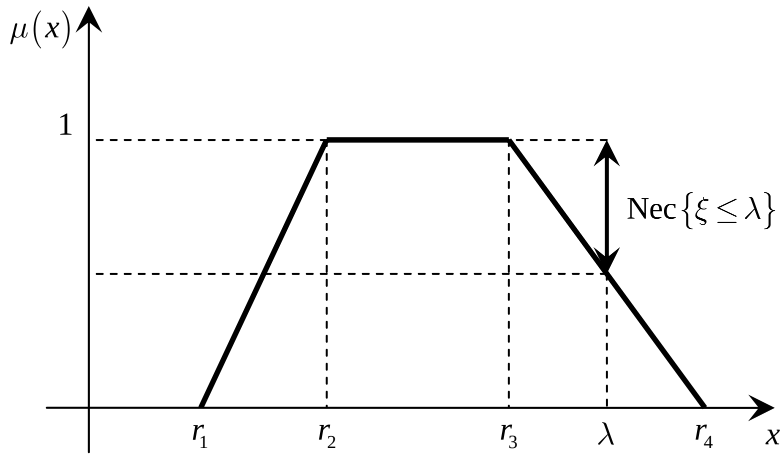


**Figure 2.6.** Necessity measure for the trapezoidal fuzzy number

### 2.2.2 Fuzzy chance constrained programming

In a way similar to the stochastic chance constraint, we introduce the fuzzy chance constraint. Assume that $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is a fuzzy vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is a return function, and

$g_j(\boldsymbol{x},\boldsymbol{\xi}),\ (j=1,2,...,m)$ are continuous functions. Since the fuzzy constraints $g_j(\boldsymbol{x},\boldsymbol{\xi})\le 0$ do not define a deterministic feasible set, a natural idea is to provide a possibility $\alpha$ where it is desired that the fuzzy constraints hold. Thus, we introduce the chance constraint as follows:

$$\mathrm{Pos}\left\{g_j(\boldsymbol{x},\boldsymbol{\xi})\le 0\right\}\ge\alpha,\ (j=1,2,...,m)$$

Using these fuzzy stochastic constraints and the similar manner of the stochastic programming, we first introduce the possibility fractile optimization model as follows:

$$\begin{aligned}&\text{Maximize } \bar{f}\\&\text{subject to } \mathrm{Pos}\left\{f(\boldsymbol{x},\boldsymbol{\xi})\ge\bar{f}\right\}\ge\beta\\&\qquad\qquad\ \mathrm{Pos}\left\{g_j(\boldsymbol{x},\boldsymbol{\xi})\le 0\right\}\ge\alpha,\ (j=1,2,...,m)\end{aligned}$$

where $\alpha$ and $\beta$ are the predetermined confidence levels. Then, in a similar way to this approach, we also introduce the following possibility maximization model.

$$\begin{aligned}&\text{Maximize } \mathrm{Pos}\left\{f(\boldsymbol{x},\boldsymbol{\xi})\ge\bar{f}\right\}\\&\text{subject to } \mathrm{Pos}\left\{g_j(\boldsymbol{x},\boldsymbol{\xi})\le 0\right\}\ge\alpha,\ (j=1,2,...,m)\end{aligned}$$

where $\bar{f}$ is the predetermined confidence level to the object. In order to solve these problems, there are various types of solution approaches using not only deterministic programming approaches but also genetic algorithm (GA), neural network (NN), and the other heuristics.

## 2.3 Fuzzy Random and Random Fuzzy Programming

In previous many researches considering uncertainty, randomness and fuzziness are separately-considered. However, in the real world, there are many cases which need to be considered both randomness and fuzziness, simultaneously, and so it is insufficient to deal with either random or fuzzy variables.

Fuzzy random and random fuzzy variables are mathematical descriptions for fuzzy stochastic phenomena, and are defined in several ways. The notion of fuzzy random variable is first introduced by Kwakernaak [72]. This concept is then developed by several researchers such as Puri and Ralescu [102], Kruse and Meyer [70], and Liu and Liu [85]. Roughly speaking, a fuzzy random variable is a measurable function from a probability space to a collection of fuzzy variables. In other words, the fuzzy random variable is a random variable taking fuzzy values. This mathematical definition is provided as follows:

**Definition 2.6**

Let $(\Omega, B, P)$ be a probability space, $F(\mathbf{R})$ the set of fuzzy numbers with compact supports and $X$ measurable mapping $\Omega \to F(\mathbf{R})$. Then, $X$ is a fuzzy random variable if and only if given $\omega \in \Omega$, $X_\alpha(\omega)$ is a random interval for any $\alpha \in [0,1]$, where $X_\alpha(\omega)$ is a $\alpha$-level set of the fuzzy set $X(\omega)$.

**Example 2.3**

With respect to the relation between weather conditions and sales volumes of drinks, let the weather condition; hot, warm or cool be a random variable. Then, let the sales volume of drinks to each weather condition be a fuzzy number. In this case, the predicted future volume of drinks considering these randomness and fuzziness is represented a fuzzy random variable as follows (see Figure 2.7).

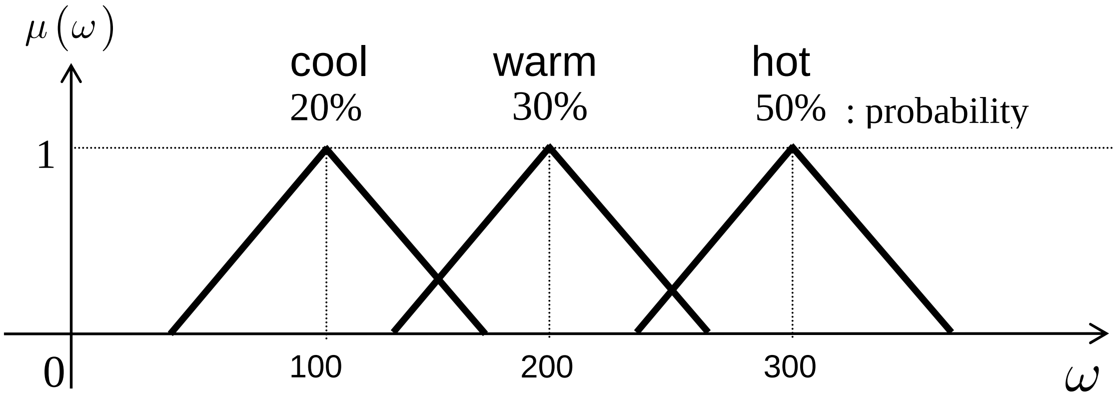


**Fig 2.7.** Example of fuzzy random variable

The above definition of fuzzy random variables corresponds to a special case of those given by Kwakernaak [72] and Puri and Ralesu [102]. The definitions of them are equivalent to the above case because a fuzzy number is a convex fuzzy set. Though it is a simple definition, it would be useful for various applications. Then, in a way similar to stochastic and fuzzy programming, we have introduced the chance constraint based on the study of Liu. Assume that $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is a fuzzy random vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is a return function, and $g_i(\boldsymbol{x}, \boldsymbol{\xi}),\ i = 1, 2, \ldots, m$ are constraint functions. Then, the following problem is a fuzzy random programming problem:

$$
\begin{aligned}
&\text{Maximize } f(\boldsymbol{x}, \boldsymbol{\xi}) \\
&\text{subject to } g_i(\boldsymbol{x}, \boldsymbol{\xi}) \le 0,\ j = 1, 2, \ldots, m
\end{aligned}
$$

Now, let us consider the chance of fuzzy random event. Recall that the probability of the random event and the possibility of the fuzzy event are defined as a real number. However, for a fuzzy random event, the primitive chance is defined as a function rather than a number.

**Definition 2.7**

Let $\boldsymbol{\xi}=(\xi_1,\xi_2,...,\xi_n)$ be a fuzzy random vector on the probability space $(\Omega, A, \mathrm{Pr})$, and $f_j : R^n \to R,\ (j=1,2,...,m)$ be continuous functions. Then, the primitive chance of fuzzy random event characterized by $f_j(\boldsymbol{\xi}),\ (j=1,2,...,m)$ is a function from [0,1] to [0,1], defined as

$$\begin{aligned}&\mathrm{Ch}\left\{f_j(\boldsymbol{\xi})\le 0,\ (j=1,2,...,m)\right\}(\alpha)\\&=\sup\left\{\beta\left|\mathrm{Pr}\left\{\boldsymbol{\omega}\in\Omega\left|\mathrm{Pos}\left\{\begin{matrix}f_j(\boldsymbol{\xi})\le 0,\\(j=1,2,...,m)\end{matrix}\right\}\ge\beta\right.\right\}\ge\alpha\right.\right\}\end{aligned}$$

Using these chance constraints, we introduce the following chance constrained programming for the fuzzy random programming:

$$\begin{aligned}&\text{Maximize } \bar{f}\\&\text{subject to } \mathrm{Ch}\left\{f(\boldsymbol{x},\boldsymbol{\xi})\ge\bar{f}\right\}(\gamma)\ge\delta,\\&\qquad\qquad\mathrm{Ch}\left\{g_i(\boldsymbol{x},\boldsymbol{\xi})\le 0\right\}(\alpha_j)\ge\beta_j,\ j=1,2,...,m\end{aligned}$$

where parameters $\alpha_j,\beta_j,\gamma$ and $\delta$ are predetermined confidence levels.

On the other hand, Liu [83, 84] initialized the concept of random fuzzy variable. The primitive chance measure of random fuzzy event was defined by Liu as a function form [0,1] to [0,1]. In detail, a random fuzzy variable is a fuzzy variable on the universal set of random variables, and the mathematical definition is provided by Liu as follows:

**Definition 2.8**

A random fuzzy variable is a function $\xi$ from a possibility space $(\Theta, P(\Theta), \mathrm{Pos})$ to collection of random variables $R$. An $n$-dimensional random fuzzy vector $\boldsymbol{\xi}=(\xi_1,\xi_2,...,\xi_n)$ is an $n$-tuple of random fuzzy variables $\xi_1,\xi_2,...,\xi_n$.

That is, a random fuzzy variable is a fuzzy set defined on a universal set of random variables.

**Example 2.4**

With respect to random variable $\xi$ which is assumed to be a normal distribution $N\left(m,\sigma^2\right)$, in the case that the mean value $m$ is assume to be a fuzzy number due to ambiguity, variable $\xi$ is a random fuzzy variable characterized by the following function in Fig. 2.8:

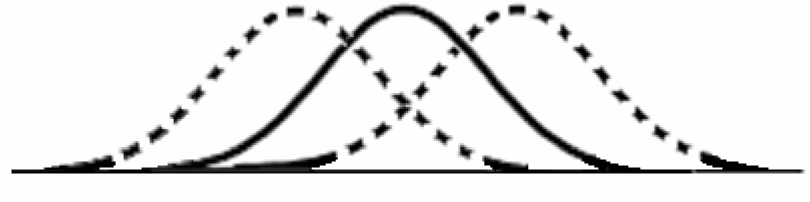


**Fig. 2.8.** Example of random fuzzy normal distribution

From the definition and example, random fuzzy variables not only include ambiguity and subjectivity into random variables, but also consider surrounding congeneric random distributions to the original random distribution, simultaneously.

Furthermore, the following random fuzzy arithmetic definition is introduced.

**Definition 2.9**

Let $\xi_1,\xi_2,...,\xi_n$ be random fuzzy variables, and $f: R^n \to R$ be a continuous function. Then, $\xi = f\left(\xi_1,\xi_2,...,\xi_n\right)$ is a random fuzzy variable on the product possibility space $\left(\Theta, P\left(\Theta\right), \mathrm{Pos}\right)$, defined as

$$\xi\left(\theta_1,\theta_2,...,\theta_n\right) = f\left(\xi_1\left(\theta_1\right),\xi_2\left(\theta_2\right),...,\xi_n\left(\theta_n\right)\right)$$

for all $\left(\theta_1,\theta_2,...,\theta_n\right)\in\Theta$.

From these definitions, the following theorem is derived.

**Theorem 2.2**

Let $\xi_i$ be random fuzzy variables with membership functions $\mu_i$, $i=1,2,...,n$, respectively, and $f: R^n \to R$ be a continuous function. Then, $\xi = f\left(\xi_1,\xi_2,...,\xi_n\right)$ is a random fuzzy variable whose membership function is

$$\mu\left(\eta\right) = \sup_{\eta_i\in R_i, 1\le i\le n}\left\{\min_{1\le i\le n}\mu_i\left(\eta_i\right)\middle|\eta = f\left(\eta_1,\eta_2,...,\eta_n\right)\right\}$$

for all $\eta \in R$, where $R = \left\{ f\left(\eta_1, \eta_2, ..., \eta_n\right) \middle| \eta_i \in R_i,\ i = 1, 2, ..., n \right\}$.

By introducing such fuzzy random and random fuzzy variables into the mathematical programming problems, we can deal with various types of practical social problems under both randomness and fuzziness. In the following chapters, we consider many asset allocation problems with random variables, fuzzy numbers, fuzzy random and random fuzzy variables, and develop the analytical solution method in the sense of the deterministic mathematical programming.

# Chapter 3

# Random Fuzzy Portfolio Selection Problems

Since the mean-variance analysis model proposed by Markowitz (Markowitz [90]), portfolio selection problems have been the centre of research activities in the real financial field, and numerous researchers have contributed to the development of modern portfolio theory. In most of these previous studies, future returns have been treated as only random variables, and the expected returns and variances also have been assumed to be fixed values. However, since investors receive reliable or unreliable information from the real world, ambiguous factors usually exist in it. Furthermore, there are investors who absolutely believe in the predictive ability of historical data. Consequently, we need to consider not only random conditions but also ambiguous and subjective conditions for portfolio selection problems. Until now, some researchers have proposed fuzzy portfolio selection problems considering such fuzziness (for example, Guo & Tanaka [41], Inuiguchi and Ramik [52], Tanaka et al. [111, 112], Watada [119]) Furthermore, there are recently some portfolio models to deal with both randomness and fuzziness as fuzzy random variables (Katagiri et al. [60, 61]). In the studies [60, 61], fuzzy random variables were related with the ambiguity of the realization of a random variable and dealt with a fuzzy number that the center value occurs according to a random variable. Then, Yazenin (Yazenin [120, 121]) considered some models for portfolio selection problems in the probabilistic-possibilistic environment, that profitabilities of financial assets are fuzzy random variables.

On the other hand, in previous portfolio selection models, the random variable to future return is basically derived from statistical analysis based on historical data. Then, when a decision maker considers the current trend in the financial market under randomness and fuzziness, she or he often assumes that the random distribution is unconfirmed and the future return has ambiguous factors because of quantity of received information and her or his subjectivity based on the long experience of the investment. Thus, the future return may be dealt with a random variable whose parameters are assumed to be fuzzy numbers due to the decision maker's subjectivity. Therefore, in this chapter, we propose portfolio selection problems with each future return to treat as a random fuzzy variable which Liu (Liu [83, 84]) defined. There are a few studies of random fuzzy programming problem (Katagiri et al. [58, 59], Huang [48]). Most recently, Huang has proposed a portfolio selection model including random fuzzy variables [47]. However, there is no study of the random fuzzy portfolio selection problem which is analytically solved and introducing a fuzzy goal considering decision makers' intentions with respect to a target profit. Furthermore, we propose a bi-criteria random fuzzy portfolio selection problem including both goals of total future profit and probability fractile level to the target profit. In the real world, it is natural to consider maximizing the probability fractile level just as maximizing the total profit. Therefore, this proposed model may be more versatile model applying to various investment situations than previous portfolio models.

With respect to these mathematical programming problems including randomness and fuzziness, since they are not well-defined problems due to randomness and fuzziness, it is necessary that we consider a certain optimization criterion so as to transform these problems into well-defined problems. In this chapter we formulate two problems using chance constraints: (a) possibility fractile optimization problem, (b) possibility maximization problem. However, since they are usually transformed into nonlinear programming problems, it is difficult to find a global optimal solution directly. Therefore, in this chapter, we construct an efficient solution method to obtain a global optimal solution of deterministic equivalent problem including more complicated constraints.

## 3.1 Formulation of the Random Fuzzy Portfolio Selection Problem

The previous studies on random and fuzzy portfolio selection problems often have considered not standard mean-variance model but safety first model introducing probability or fuzzy chance constraints based on a standard asset allocation problem. Therefore, in this chapter, we deal with the following portfolio selection problem involving the random fuzzy variable based on the standard asset allocation problem to maximize the total future return.

$$\begin{array}{ll} \text{Maximize} & \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{array} \tag{3.1}$$

where the parameters and variables are as follows:

$\bar{\tilde{r}}_j$ : Future return of the $j$ th financial asset assumed to be a random fuzzy variable

$a_j$ : Cost of investing the $j$ th financial asset

$b$ : Limited upper value with respect to fund budgeting

$\hat{b}_j$ : Limited upper value of each budgeting to the $j$ th financial asset

$n$ : Total number of assets

$x_j$ : Budgeting allocation to the $j$ th financial asset

In this chapter, we denote randomness and fuzziness of the coefficients by the "dash above" and "wave above", i.e., “ — ” and “~”, respectively. In portfolio selection problems, each future return is generally considered as a random variable distributed according to the normal distribution $N\left(m_j, \sigma_j^2\right)$. However, for the lack of effective information from the real market, we assume the case where the expected return includes an ambiguity and the probability distribution of each future return is represented with the following form based on the introduction obtained by Hasuike [43] and Katagiri [58]:

$$f_j(z) = \frac{1}{\sqrt{2\pi}\sigma_j}\exp\left(-\frac{\left(z-\tilde{M}_j\right)^2}{2\sigma_j^2}\right),\quad \mu_{\tilde{M}_j}(t) = \begin{cases} L\left(\dfrac{m_j - t}{\alpha_j}\right), & \left(m_j - \alpha_j \le t \le m_j\right) \\ R\left(\dfrac{t - m_j}{\beta_j}\right), & \left(m_j < t \le m_j + \beta_j\right) \\ 0 & \left(t < m_j - \alpha_j,\ m_j + \beta_j < t\right) \end{cases},\quad j = 1,\cdots,n \tag{3.2}$$

where $L(x)$ and $R(x)$ are strictly decreasing and continuous reference functions to satisfy $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ and the parameters $\alpha_j$ and $\beta_j$ represent the spreads corresponding to the left and the right sides, respectively, and both parameters are positive values. When the future return $\bar{\tilde{r}}_j$ is a random fuzzy variable characterized by formula (3.2), the membership function of $r_j$ is expressed as

$$\mu_{\bar{\tilde{r}}_j}\left(\bar{\gamma}_j\right) = \sup_{s_j}\left\{\mu_{\tilde{M}_j}\left(s_j\right)\middle|\bar{\gamma}_j \sim N\left(s_j, \sigma_j^2\right)\right\},\ \forall \bar{\gamma}_j \in \Gamma \tag{3.3}$$

where $\Gamma$ is a universal set of normal random variable. Each membership function value $\mu_{r_j}\left(\bar{\gamma}_j\right)$ is interpreted as a degree of possibility or compatibility that $\bar{\tilde{r}}_j$ is equal to $\bar{\gamma}_j$. Then, the objective function $\bar{\tilde{Z}} = \sum_{j=1}^{n}\bar{\tilde{r}}_j x_j$ is defined as a random fuzzy variable characterized by the following membership function on fixed the parameters $x_j$:

$$\mu_{\bar{\tilde{Z}}}\left(\bar{u}\right) = \sup_{\bar{\gamma}}\left\{\min_{1\le j\le n}\mu_{\bar{\tilde{r}}_j}\left(\bar{\gamma}_j\right)\middle|\bar{u} = \sum_{j=1}^{n}\bar{\gamma}_j x_j\right\},\ \forall \bar{u} \in Y \tag{3.4}$$

where $\bar{\gamma} = \left(\gamma_1, \dots, \gamma_n\right)$ and $Y$ is defined by

$$Y = \left\{\sum_{j=1}^{n}\bar{\gamma}_j x_j \middle| \bar{\gamma}_j \in \Gamma,\ j = 1,\dots,n\right\} \tag{3.5}$$

From definitions (3.4) and (3.5), we obtain

$$\begin{aligned}\mu_{\bar{\tilde{Z}}}(\bar{u}) &= \sup_{\bar{\gamma}}\left\{\min_{1\le j\le n}\mu_{\bar{\tilde{r}}_j}\left(\bar{\gamma}_j\right)\middle|\bar{u}=\sum_{j=1}^{n}\bar{\gamma}_j x_j\right\}\\ &= \sup_{s}\left\{\min_{1\le j\le n}\mu_{\tilde{M}_j}\left(s_j\right)\middle|\bar{\gamma}_j \sim N\left(s_j,\sigma_j^2\right),\ \bar{u}=\sum_{j=1}^{n}\bar{\gamma}_j x_j\right\}\\ &= \sup_{s}\left\{\min_{1\le j\le n}\mu_{\tilde{M}_j}\left(s_j\right)\middle|\bar{u}\sim N\left(\sum_{j=1}^{n}s_j x_j,\sum_{j=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\right)\right\}\end{aligned} \tag{3.6}$$

where $\boldsymbol{s}=\left(s_1,...,s_n\right)$.

Furthermore, we discuss the probability $\Pr\left\{\omega\middle|\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j \ge f\right\}$ which is a probability that the objective function value is greater than or equal to an aspiration level $f$. Since $\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j$ is represented with a random fuzzy variable, we express the probability $\Pr\left\{\omega\middle|\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j \ge f\right\}$ as a fuzzy set $\tilde{P}$ and define the membership function of $\tilde{P}$ as follows:

$$\begin{aligned}\mu_{\tilde{P}}(p) &= \sup_{\bar{u}}\left\{\mu_Z(\bar{u})\middle|p=\Pr\left\{\omega\middle|\bar{u}(\omega)\ge f\right\}\right\}\\ &= \sup_{s}\min_{1\le j\le n}\left\{\mu_{\tilde{M}_j}\left(s_j\right)\middle|p=\Pr\left\{\omega\middle|\bar{u}(\omega)\ge f\right\},\ \bar{u}\sim N\left(\sum_{j=1}^{n}s_j x_j,\sum_{j=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\right)\right\}\end{aligned} \tag{3.7}$$

Here, since original problem (3.1) is not a well-defined problem due to including random fuzzy variable returns, we need to set a criterion with respect to probability and possibility of future returns for the deterministic optimization. In general decision cases with respect to investment, an investor usually focused on maximizing either the goal of the total profit or that of achieve probability. Therefore, we propose three single criteria random fuzzy portfolio selection problems introducing a chance constraint; (a) expected return maximization model and mean-variance model, (b) possibility fractile optimization model, (c) possibility maximization model.

## 3.2 Single Criteria Optimization Models for Random Fuzzy Portfolio Selection Problems

### 3.2.1 Expected return optimization model and mean-variance model

In previous researches, mean-variance models for portfolio selection problems based on Markowitz model are introduced. In this chapter, we formally introduce the expected return maximization model for random fuzzy portfolio selection model as follows:

$$\begin{aligned} &\text{Maximize} \quad \tilde{E}(Z) \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{aligned} \tag{3.8}$$

In this problem, $\tilde{E}(Z)$ means an expected value derived from the following expression:

$$\begin{aligned} \mu_{\tilde{E}(Z)}(\eta) &= \sup_{s} \left\{ \min_{1\le j\le n} \mu_{\tilde{M}_j}(s_j) \,\middle|\, \overline{\gamma}_j \sim N(s_j, \sigma_j^2),\ \eta = E\left( \sum_{j=1}^{n} \overline{\gamma}_j x_j \right) \right\}, \\ &= \sup_{s} \left\{ \min_{1\le j\le n} \mu_{\tilde{M}_j}(s_j) \,\middle|\, \eta = \sum_{j=1}^{n} s_j x_j \right\} \end{aligned} \tag{3.9}$$

This means that $\tilde{E}(Z)$ is expressed with a fuzzy set. Consequently, problem (3.8) is a fuzzy optimization problem for portfolio selection problems and is solved by using results of previous studies on fuzzy portfolio selection models (For example, Carlsson et al. [17, 18] and Vercher et al. [117]). Furthermore, we formally introduce the following mean-variance model:

$$\begin{aligned} &\text{Minimize} \quad V(Z) \\ &\text{subject to} \quad \tilde{E}(Z) \succeq r_G, \\ &\qquad\qquad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{aligned} \tag{3.10}$$

where $r_G$ is a minimum target value of the total future profit and $\tilde{E}(Z) \succeq r_G$ means that the total expected return is approximately more than $r_G$. Then, $V(Z)$ means a variance and $V(Z) = \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$ due to not including random and fuzzy variables in each variance. Therefore, problem (3.10) is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Minimize} \quad \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ &\text{subject to} \quad \tilde{E}(Z) \succeq r_G, \\ &\qquad\qquad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{aligned} \tag{3.11}$$

This problem is also a fuzzy portfolio selection problem due to $\tilde{E}(Z)$ involving fuzzy numbers.

Therefore, we can take the fuzzy optimization approaches to obtain the optimal portfolio.

### 3.2.2 Possibility fractile optimization model with respect to the total profit

In the case that a decision maker sets the target values of probability fractile level $\beta$ and possibility fractile level $h$ using the chance constraint, maximizing the target future return $f$ is mainly considered. Therefore, in this section, we consider a possibility fractile optimization model with respect to the future returns. This model is formulated as the following form:

$$\begin{array}{ll} \text{Maximize} & f \\ \text{subject to} & \mu_{\tilde{P}}(p) \geq h, \quad p \geq \beta, \\ & \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1, 2, \cdots n \end{array} \tag{3.12}$$

In this problem, the constraint $\mu_{\tilde{P}}(p) \geq h,\ p \geq \beta$ is transformed into the following form:

$$\begin{aligned} & \mu_{\tilde{P}}(p) \geq h,\ p \geq \beta \\ \Leftrightarrow\quad & \sup_{\boldsymbol{s}} \min_{1 \leq j \leq n} \left\{ \mu_{\tilde{M}_j}(s_j) \middle| \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta,\ \bar{u} \sim N\left( \sum_{j=1}^{n} s_j x_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right) \right\} \geq h \\ \Leftrightarrow\quad & \exists \boldsymbol{s},\ \exists \bar{u} : \sup_{\boldsymbol{s}} \min_{1 \leq j \leq n} \mu_{\tilde{M}_j}(s_j) \geq h,\ \ \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta,\ \bar{u} \sim N\left( \sum_{j=1}^{n} s_j x_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right) \\ \Leftrightarrow\quad & \exists \boldsymbol{s},\ \exists \bar{u} : \min_{1 \leq j \leq n} \mu_{\tilde{M}_j}(s_j) \geq h,\ \ \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta,\ \bar{u} \sim N\left( \sum_{j=1}^{n} s_j x_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right) \\ \Leftrightarrow\quad & \exists \boldsymbol{s},\ \exists \bar{u} : \mu_{\tilde{M}_j}(s_j) \geq h,\ \ (j = 1, \ldots, n),\ \ \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta,\ \bar{u} \sim N\left( \sum_{j=1}^{n} s_j x_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right) \\ \Leftrightarrow\quad & \exists \bar{u} : \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta,\ \bar{u} \sim N\left( \sum_{j=1}^{n} \left( m_j + R^*(h) \alpha_j \right) x_j, \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \right) \end{aligned} \tag{3.13}$$

where $R^*(x)$ is a pseudo inverse function of $R(x)$ in membership function (3.2). Then, we transform problem (3.12) into the following problem:

$$\begin{array}{ll} \text{Maximize} & f \\ \text{subject to} & \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq \beta, \\ & \bar{u} \sim N\left( \sum_{j=1}^{n} \left( m_j + R^*(h) \alpha_j \right) x_j, \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \right) \\ & \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1, 2, \cdots, n \end{array} \tag{3.14}$$

Next, we consider the transformation of probability chance constraint $\Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\} \geq \beta$. Then, it follows that

$$\frac{\bar{u} - \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}}$$

is a random variable with the standard normal distribution. Therefore, we obtain the following transformation of probability chance constraint:

$$\begin{aligned}
&\Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\} \geq \beta \\
\Leftrightarrow \quad &\Pr\left\{\frac{\bar{u} - \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}} \geq \frac{f - \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}}\right\} \geq \beta \\
\Leftrightarrow \quad &\frac{\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}} \geq K_\beta \\
\Leftrightarrow \quad &\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - f \geq K_\beta \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \\
\Leftrightarrow \quad &\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - K_\beta \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \geq f
\end{aligned}$$

where $F(y)$ is the distribution function of the standard normal distribution and $K_\beta = F^{-1}(\beta)$. In this chapter, we consider $\beta \geq \frac{1}{2}$ due to the following assumptions:

(a) In the practical decision making, almost all decision makers do not select a portfolio whose achievement probability for the goal of total return is less than half.

(b) In mathematical programming, $\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - K_\beta \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}$ is a concave function in the case that $\beta \geq \frac{1}{2}$.

Accordingly problem (3.14) is transformed into the following equivalent deterministic problem:

$$
\begin{aligned}
&\text{Maximize} \quad f \\
&\text{subject to} \quad \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - K_\beta\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \geq f \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1,2,\cdots,n
\end{aligned}
\tag{3.15}
$$

In this problem, we find that the decision variable $f$ is involved only in first constraint and maximizing $f$ is equivalent to maximizing $\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - K_\beta\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j}$ . Therefore, problem (3.15) is also transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - K_\beta\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1,2,\cdots,n \\
\Leftrightarrow \quad &\text{Minimize} \quad -\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j + K_\beta\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1,2,\cdots,n
\end{aligned}
\tag{3.16}
$$

Since problem (3.16) is equivalent to a convex programming problem, its global optimal solution surely exists. However it is difficult to solve it directly because problem (3.16) includes the square root term. Therefore, we consider the equivalent transformation of problem (3.16).

First, for simplicity of the following discussion, we do transformations of variables as follows since a symmetric variance-covariance matrix is a positive definite matrix:

$$
\begin{aligned}
&K_\beta\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j = K_\beta \boldsymbol{x}^t V \boldsymbol{x} = \boldsymbol{y}^t \boldsymbol{y}, \\
&\boldsymbol{x}^t V \boldsymbol{x} = (Q\boldsymbol{x})^t \Lambda (Q\boldsymbol{x}), \ Q: \text{eigen vector of } V, \\
&\Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}, \ \lambda_i: \text{eigen value of } V, \ \sqrt{\Lambda} = \begin{pmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{pmatrix}, \\
&\boldsymbol{y} = \sqrt{K_\beta}\sqrt{\Lambda}Q\boldsymbol{x}, \\
&\boldsymbol{m}' = \frac{1}{\sqrt{K_\beta}}\left(\boldsymbol{m} + R^*(h)\boldsymbol{\alpha}\right)\left(\sqrt{\Lambda}\right)^{-1}Q, \ \boldsymbol{a}' = \frac{1}{\sqrt{K_\beta}}\boldsymbol{a}\left(\sqrt{\Lambda}\right)^{-1}Q, \\
&b' = \frac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1}Qb, \ \hat{\boldsymbol{b}}' = \frac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1}Q\hat{\boldsymbol{b}}
\end{aligned}
\tag{3.17}
$$

Furthermore, we reset $\boldsymbol{y} \to \boldsymbol{x},\ m_j' \to m_j,\ a_j' \to a_j,\ b' \to b,\ b_j' \to \hat{b}_j$. From these transformations of variables, problem (3.16) is transformed into the following problem:

$$\begin{aligned} &\text{Minimize} \quad -\sum_{j=1}^{n} m_j x_j + \sqrt{\sum_{j=1}^{n} x_j^2} \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j,\ j = 1,2,\cdots,n \end{aligned} \tag{3.18}$$

This problem still includes a square root term. Therefore, we introduce the following auxiliary problem using a parameter $R$ not including the square root term:

$$\begin{aligned} &\text{Minimize} \quad -R\sum_{j=1}^{n} m_j x_j + \frac{1}{2}\left(\sum_{j=1}^{n} x_j^2\right) \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j,\ j = 1,2,\cdots,n \end{aligned} \tag{3.19}$$

Since this problem is a quadratic convex programming problem, we solve problem (3.19) more easily than problem (3.18). This problem is equivalent to the model derived from study of Ishii and Nishida [54]. Therefore, in a similar way to the solution method of the study, we obtain the strict optimal portfolio of problem (3.18). Furthermore, in previous researches ([54], Katagiri et al. [60, 61]), each variance is considered to be independent. However, since we consider the variance-covariance matrix with respect to each variance in this chapter, we find that this model is the extended version of previous models and the solution method derived from the study [54] is extended to the general case of portfolio selection models.

### 3.2.3 Possibility maximization model with respect to the achievement probability

In Subsection 3.2.2, the possibility fractile maximization model with respect to the total profit has been considered. On the other hand, in the case that a decision maker sets the target value of total future profit $f$, they consider a portfolio selection maximizing the probability that they earn future returns more than the target value $f$. In this subsection, we consider the possibility maximization model with respect to the probability. First of all, we introduce a basic probability maximization model as follows:

$$\begin{aligned} &\text{Maximize} \quad \tilde{P} = \Pr\left\{\sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \ge f\right\} \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le \hat{b}_j,\ j = 1,2,...,n \end{aligned} \tag{3.20}$$

However, since this objective function includes random fuzzy variables, it is not a well-defined problem. Therefore, introducing the possibility chance constraint to the objective function, we set the target level $h$ of possibility and introduce the following problem:

$$
\begin{aligned}
&\text{Maximize} && \beta \\
&\text{subject to} && \mu_{\tilde{p}}(p) \geq h,\ p \geq \beta \\
& && \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1, 2, \cdots, n
\end{aligned}
\tag{3.21}
$$

In a manner similar to transformation (3.13), problem (3.21) is transformed into the following form:

$$
\begin{aligned}
&\text{Maximize} && \beta \\
&\text{subject to} && \Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\} \geq \beta, \\
& && \bar{u} \sim N\left(\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j, \sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j\right) \\
& && \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq \hat{b}_j, \quad j = 1, 2, \cdots, n
\end{aligned}
\tag{3.22}
$$

Since $\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j} > 0$ and $K_\beta$ is a strictly increasing value with respect to $\beta$, this problem is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize} && \frac{\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - f}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}} \\
&\text{subject to} && \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq \hat{b}_j,\ j = 1, 2, ..., n \\
\Leftrightarrow\ &\text{Minimize} && \frac{f - \sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}} \\
&\text{subject to} && \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq \hat{b}_j,\ j = 1, 2, ..., n
\end{aligned}
\tag{3.23}
$$

Furthermore, in a way similar to transformation in Subsection 3.2, we reset the variables, and problem (3.23) is transformed into the following form:

$$\text{Minimize} \quad \frac{f-\sum_{j=1}^{n} m_j x_j}{\sqrt{\sum_{i=1}^{n} x_j^2}} \tag{3.24}$$
$$\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq \hat{b}_j,\ \ j=1,2,...,n$$

Since this problem is equivalent to the model derived from the previous study of Ishii and Nishida [57], we apply the solution method in the study to this problem and obtain the strict optimal solution. In the study, each variance is assumed to be independent. In this chapter, we consider the variance-covariance matrix with respect to each variance, and so this model is the extended model of [57] and the solution method is extended to the general case of probability maximization models.

## 3.3 Bi-criteria Optimization Model for Random Fuzzy Portfolio Selection Problems

In Section 3.2, the goal of probability fractile level $\beta$ is assumed to be a fixed value. Then in Section 3.2, the goal of total future profit $f$ is assumed to be a fixed value. On the other hand, in practical situations, the relation between the target future return $f$ and probability $\beta$ is ambivalent, and a decision maker considers increasing the goal of total profit and that of probability, simultaneously. Furthermore, considering many real decision cases and taking account of the vagueness of human judgment and flexibility for the execution of a plan, a decision maker often has subjective and ambiguous goals with respect to $\beta$ and $f$ such as "The achievement probability is hopefully more than $\beta_1$.", and "Total future return $\sum_{j=1}^{n} \bar{\tilde{r}}_j x_j$ is approximately larger than $f_1$.", respectively. In this section, we propose the more flexible model considering the aspiration level to both goals of the total future return and achievement probability, simultaneously. In this chapter, we represent these subjective and ambiguous goals with respect to $\beta$ and $f$ as fuzzy goals characterized by a membership function. First, the fuzzy goal for minimum target value of total profit $f$ is given as follows:

$$\mu_G(f) = \begin{cases} 1, & f_1 \leq f \\ g_F(f), & f_0 \leq f \leq f_1 \\ 0, & f \leq f_0 \end{cases} \tag{3.25}$$

where $g_F(f)$ is a strictly increasing continuous function. In a similar way to the total profit, we introduce the fuzzy goal of achievement probability as the following form:

$$\mu_{\tilde{G}_p}(w) = \begin{cases} 0 & w \le p_0 \\ g_p(w) & p_0 \le w \le p_1 \\ 1 & p_1 \le w \end{cases} \tag{3.26}$$

where $g_p(w)$ is a strictly increasing continuous function. Furthermore, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\prod_{\tilde{P}}\left(\tilde{G}_p\right) = \sup_w \min\left\{\mu_{\tilde{P}}(w),\ \mu_{\tilde{G}_p}(w)\right\} \tag{3.27}$$

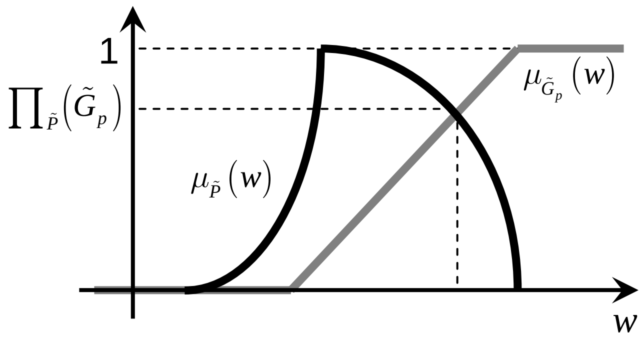


**Fig. 3.1.** The degree of possibility

From this possibility measure, we consider the following possibility maximization problem:

$$\begin{array}{ll} \text{Maximize} & \min\left\{\prod_{\tilde{P}}\left(\tilde{G}_p\right),\ \mu_G(f)\right\} \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{array} \tag{3.28}$$

$$\Leftrightarrow \begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & \prod_{\tilde{P}}\left(\tilde{G}_p\right) \ge h,\ \mu_G(f) \ge h \\ & \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{array} \tag{3.29}$$

In this problem, the constraint $\prod_{\tilde{P}}\left(\tilde{G}_p\right) \ge h$ and $\mu_G(f) \ge h$ are transformed into the following inequality:

$$
\begin{aligned}
&\prod\nolimits_{\tilde{P}}\left(\tilde{G}_p\right)\geq h,\ \mu_G\left(f\right)\geq h\\
\Leftrightarrow\quad & \sup_w \min\left\{\mu_{\tilde{P}}\left(w\right),\mu_{\tilde{G}_p}\left(w\right)\right\}\geq h,\ \mu_G\left(f\right)\geq h\\
\Leftrightarrow\quad & \exists w:\ \mu_{\tilde{P}}\left(w\right)\geq h,\ \mu_{\tilde{G}_p}\left(w\right)\geq h,\ f\geq g_F^{-1}\left(h\right)\\
\Leftrightarrow\quad & \exists w:\ \sup_{\boldsymbol{s}}\min_{1\leq j\leq n}\left\{\mu_{\tilde{M}_j}\left(s_j\right)\left|\begin{array}{l} w=\Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\},\\ \bar{u}\sim N\left(\sum_{j=1}^n s_j x_j,\sum_{j=1}^n\sum_{j=1}^n\sigma_{ij}x_i x_j\right)\end{array}\right.\right\}\geq h,\ \mu_{\tilde{G}_p}\left(w\right)\geq h\\
\Leftrightarrow\quad & \begin{array}{l}\exists w,\ \exists \boldsymbol{s},\ \exists\bar{u}:\sup_{\boldsymbol{s}}\min_{1\leq j\leq n}\mu_{\tilde{M}_j}\left(s_j\right)\geq h,\ w=\Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\},\\ \qquad\bar{u}\sim N\left(\sum_{j=1}^n s_j x_j,\sum_{j=1}^n\sum_{j=1}^n\sigma_{ij}x_i x_j\right),\ \mu_{\tilde{G}_p}\left(w\right)\geq h\end{array} \qquad (3.30)\\
\Leftrightarrow\quad & \begin{array}{l}\exists w,\ \exists \boldsymbol{s},\ \exists\bar{u}:\min_{1\leq j\leq n}\mu_{\tilde{M}_j}\left(s_j\right)\geq h,\ w=\Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\},\\ \qquad\bar{u}\sim N\left(\sum_{j=1}^n s_j x_j,\sum_{j=1}^n\sum_{j=1}^n\sigma_{ij}x_i x_j\right),\ \ w\geq g_p^{-1}\left(h\right)\end{array}\\
\Leftrightarrow\quad & \begin{array}{l}\exists \boldsymbol{s},\ \exists\bar{u}:\mu_{\tilde{M}_j}\left(s_j\right)\geq h,\ \left(j=1,...,n\right),\ \bar{u}\sim N\left(\sum_{j=1}^n s_j x_j,\sum_{j=1}^n\sum_{j=1}^n\sigma_{ij}x_i x_j\right),\\ \qquad\Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\}\geq g_p^{-1}\left(h\right)\end{array}\\
\Leftrightarrow\quad & \exists\bar{u}:\Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\}\geq g_p^{-1}\left(h\right),\ \bar{u}\sim N\left(\sum_{j=1}^n\left\{m_j+R^*\left(h\right)\alpha_j\right\}x_j,\sum_{j=1}^n\sum_{i=1}^n\sigma_{ij}x_i x_j\right)
\end{aligned}
$$

Therefore, we transform problem (3.29) into the following problem:

$$
\begin{aligned}
&\text{Maximize}\quad h\\
&\text{subject to}\quad \Pr\left\{\omega\left|\bar{u}\left(\omega\right)\geq g_F^{-1}\left(h\right)\right.\right\}\geq g_p^{-1}\left(h\right),\\
&\qquad\qquad \bar{u}\sim N\left(\sum_{j=1}^n\left\{m_j+R^*\left(h\right)\alpha_j\right\}x_j,\sum_{j=1}^n\sum_{i=1}^n\sigma_{ij}x_i x_j\right), \qquad (3.31)\\
&\qquad\qquad \sum_{j=1}^n a_j x_j\leq b,\quad 0\leq x_j\leq \hat{b}_j,\quad j=1,2,\cdots,n
\end{aligned}
$$

Furthermore, using the property of normal distribution, we do the transformation to the stochastic constraint and problem (3.31) is equivalent to the following equivalent deterministic problem:

$$
\begin{aligned}
&\text{Maximize}\quad h\\
&\text{subject to}\quad \sum_{j=1}^n\left\{m_j+R^*\left(h\right)\alpha_j\right\}x_j-K_{g_p^{-1}(h)}\sqrt{\sum_{j=1}^n\sum_{i=1}^n\sigma_{ij}x_i x_j}\geq g_F^{-1}\left(h\right), \qquad (3.32)\\
&\qquad\qquad \sum_{j=1}^n a_j x_j\leq b,\quad 0\leq x_j\leq \hat{b}_j,\quad j=1,2,\cdots,n
\end{aligned}
$$

It should be noted here that problem (3.32) is a nonconvex programming problem and it is not solved by the linear programming techniques or convex programming techniques. However, since a decision variable $h$ is involved only in first constraint, we introduce the following subproblem involving a parameter $q$:

$$
\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n}\left\{m_j + R^*(q)\alpha_j\right\}x_j - K_{g_p^{-1}(q)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_ix_j} \\
&\text{subject to} \quad \sum_{j=1}^{n}a_jx_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j=1,2,\cdots n \\
\Leftrightarrow \quad &\begin{aligned}
&\text{Minimize} \quad -\sum_{j=1}^{n}\left\{m_j + R^*(q)\alpha_j\right\}x_j + K_{g_p^{-1}(q)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_ix_j} \\
&\text{subject to} \quad \sum_{j=1}^{n}a_jx_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j=1,2,\cdots n
\end{aligned}
\end{aligned}
\tag{3.33}
$$

In the case that we fix the parameter $q$, problem (3.33) is equivalent to a convex programming problem. Furthermore, let $x(q)$ and $Z(q)$ be an optimal solution of problem (3.33) and its optimal value, respectively. Then, the following theorem holds.

**Theorem 3.1**

For $q$ satisfying $0<q<1$, $Z(q)$ is a strictly increasing function of $q$.

**Proof.**

We assume $q'<q$. Let $\boldsymbol{x}(q)$ and $\boldsymbol{x}(q')$ be optimal solutions of problem $P(q)$ and $P(q')$, respectively. From the relations that $R(q)$ is a decreasing function and $K_{g_p^{-1}(q)}$ is an increasing function with respect to $q$, the following inequality holds.

$$
\begin{aligned}
&-\sum_{j=1}^{n}\left\{m_j + R^*(q)\alpha_j\right\}x_j(q) + K_{g_p^{-1}(q)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i(q)x_j(q)} \\
&> -\sum_{j=1}^{n}\left\{m_j + R^*(q')\alpha_j\right\}x_j(q) + K_{g_p^{-1}(q)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i(q)x_j(q)} \\
&> -\sum_{j=1}^{n}\left\{m_j + R^*(q')\alpha_j\right\}x_j(q) + K_{g_p^{-1}(q')}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i(q)x_j(q)} \\
&> -\sum_{j=1}^{n}\left\{m_j + R^*(q')\alpha_j\right\}x_j(q') + K_{g_p^{-1}(q')}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i(q')x_j(q')}
\end{aligned}
$$

Therefore, this theorem holds. □

Let $\hat{q}$ denote $q$ satisfying $Z(q) = g_F^{-1}(q)$. Then the relation between problems (3.32) and (3.33) is derived as follows.

**Theorem 3.2**

Suppose that $0 < h^* < 1$ holds. Then $(x(\hat{q}), \hat{q})$ is equal to $(x^*, h^*)$.

**Proof.**

From Theorem 1, we obtain $g(x(q),\, q) = \sum_{j=1}^{n} \left\{ m_j + R^*(q)\alpha_j \right\} x_j(q) - K_{g_p^{-1}(q)} \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij} x_i(q) x_j(q)}$ is a strictly decreasing function of $q$. Therefore, in problem (3.32), if $g(x(q),\, q) > g_F^{-1}(q)$, ${}^{\exists} q' > q$, $g(x(q),\, q) > g(x(q'),\, q') \geq g_F^{-1}(q)$. Furthermore, ${}^{\exists} \hat{q} \geq q'$, $g(x(q'),\, q') \geq g(x(\hat{q}),\, \hat{q}) = g_F^{-1}(\hat{q})$. Consequently, this theorem holds. □

Therefore, since problem (3.33) is the same problem as problem (3.16), we develop the following solution procedure using the solution procedure in Section 3, Theorems 2 and 3 extending the results obtained.

**Solution procedure**

Step 1: Elicit the membership function of a fuzzy goal for the probability with respect to the objective function value.

Step 2: Set $q \leftarrow 1$ and solve problem (3.33). If the optimal objective value $Z(q)$ of problem (3.33) satisfies $Z(q) \leq g_F^{-1}(q)$, then terminate. In this case, the obtained current solution is an optimal solution of main problem.

Step 3: Set $q \leftarrow 0$ and solve problem (3.33). If the optimal objective value $Z(q)$ of problem (3.33) satisfies $Z(q) > g_F^{-1}(q)$, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for the probability or the aspiration level $f$.

Step 4: Set $U_q \leftarrow 1$ and $L_q \leftarrow 0$.

Step 5: Set $\gamma \leftarrow \dfrac{U_q + L_q}{2}$

Step 6: Solve problem (3.33) and calculate the optimal objective value $Z(q)$ of problem (3.33). If

$Z(q) > g_F^{-1}(q)$, then set $U_q \leftarrow q$ and return to Step 5. If $Z(q) \le g_F^{-1}(q)$, then set $L_q \leftarrow q$ and return to Step 5. If $Z(q) = g_F^{-1}(q)$, then terminate the algorithm. In this case, $\boldsymbol{x}^*(q)$ is equal to a global optimal solution of main problem.

Problem (3.33) includes various situations in the real world; for example, in the case that a decision maker does not consider the flexibility of the goal with respect to the total profit, i.e., $Z \ge f$, problem (3.33) degenerates to the following problem:

$$\begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & \sum_{j=1}^{n}\left\{m_j + R^*(h)\alpha_j\right\}x_j - K_{g_P^{-1}(h)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \ge f, \\ & \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le \hat{b}_j, \quad j = 1, 2, \cdots, n \end{array} \tag{3.34}$$

Since we apply the solution method to this problem by changing $g_F^{-1}(q)$ into $f$, we obtain the strict optimal solution of problem (3.34). Furthermore, in the case that all fuzziness and flexibilities are removed, problem (3.33) degenerates to a basic safety first model for portfolio selection problems. Consequently, we find that problem (3.33) is more flexible problem reflecting on practical situations and decision maker's subjectivity for making a choice of randomness, fuzziness and flexibility.

## 3.4 Numerical Example

In order to compare our proposed models with other models for portfolio selection problems, let us consider an example shown in Table 1 based on data introduced by Markowitz [90]. This data have been used in previous many studies concerning portfolio problems.

### 3.4.1 Markowitz 's historical data set

Let us assume that each return is distributed according to a normal distribution and that each mean is a symmetric triangle fuzzy number shown in Table 3.1.

Then, we introduce the asset allocation rate $x_j$, $(j = 1, 2, ..., 9)$ to each security, and its upper value is assumed to be 0.2. In this chapter, we compare our proposed model (3.29) in Subsection 3.4 with Carlsson et al. model [18] and Vercher et al. model [117]. In studies [18] and [117], this Markowitz data is used as a numerical example. These previous two models include the possibilistic mean value and variance. Each model is formulated as the following form with respect to the Markowitz historical data in Table 3.1.

Table 3.1. Sample data from Markowitz's historical data and their fuzzy number

| Returns | Sample mean | SD | Fuzzy number to sample mean |
|---|---|---|---|
| R1 | 0.066 | 0.238 | (0.066, 0.01) |
| R2 | 0.062 | 0.125 | (0.062, 0.02) |
| R3 | 0.146 | 0.301 | (0.146, 0.02) |
| R4 | 0.173 | 0.318 | (0.173, 0.08) |
| R5 | 0.198 | 0.368 | (0.198, 0.1) |
| R6 | 0.055 | 0.209 | (0.055, 0.01) |
| R7 | 0.128 | 0.175 | (0.128, 0.05) |
| R8 | 0.118 | 0.286 | (0.118, 0.08) |
| R9 | 0.116 | 0.290 | (0.116, 0.06) |

(Our proposed model)

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \sum_{j=1}^{n}\left\{m_j + R^*(h)\alpha_j\right\}x_j - K_{g_p^{-1}(h)}\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{ij}x_i x_j} \ge g_f^{-1}(h), \\
&\qquad\qquad \sum_{j=1}^{9} x_j = 1, \quad 0 \le x_j \le 0.2, \quad j = 1, 2, \cdots, 9
\end{aligned}$$

where each fuzzy goal is as follows:

$$\mu_G(f) = \begin{cases} 1 & (0.12 \le f) \\ g_F(f) = \dfrac{f - 0.08}{0.02} & (0.08 \le f \le 0.12) \\ 0 & (f \le 0.08) \end{cases}, \quad \mu_G(p) = \begin{cases} 1 & (0.8 \le p) \\ g_p(p) = \dfrac{p - 0.7}{0.1} & (0.7 \le p \le 0.8) \\ 0 & (p \le 0.7) \end{cases}$$

(Carlsson et al. model)

$$\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{9}\frac{1}{2}\left[a_j + b_j + \frac{1}{3}\left(\beta_j - \alpha_j\right)\right]x_j \\
& -\frac{0.0123}{4}\left(\sum_{j=1}^{9}\frac{1}{2}\left[b_j - a_j + \frac{1}{3}\left(\alpha_j + \beta_j\right)\right]x_j\right)^2 - \frac{0.0123}{72}\left(\sum_{j=1}^{9}\left(\alpha_j + \beta_j\right)x_j\right)^2 \\
\text{subject to} \quad & \sum_{j=1}^{9} x_j = 1, \quad 0 \le x_j \le 0.2, \quad j = 1, 2, \cdots, 9
\end{aligned}$$

(Vercher et al. model)

$$\begin{aligned}
\text{Minimize} \quad & \sum_{j=1}^{9} \frac{1}{2}\left[b_j - a_j + \frac{1}{3}(\alpha_j + \beta_j)\right] x_j \\
\text{subject to} \quad & \sum_{j=1}^{9} \frac{1}{2}\left[a_j + b_j + \frac{1}{3}(\beta_j - \alpha_j)\right] x_j \geq \rho, \\
& \sum_{j=1}^{9} x_j = 1, \quad 0 \leq x_j \leq 0.2, \quad j = 1, 2, \cdots, 9
\end{aligned}$$

where $\rho$ is the target value set by the decision maker, and $a_j,\ b_j,\ \alpha_j,\ \beta_j,\ (j = 1, 2, ..., 9)$ are parameters of trapezoidal fuzzy number $(a_j, b_j, \alpha_j \beta_j)$ and shown in Table 3.2 based on study [117] as follows:

Table 3.2. Each parameter value based on study [117]

| Returns | $a_j$ | $b_j$ | $\alpha_j$ | $\beta_j$ |
|---|---|---|---|---|
| R1 | 0.070 | -0.011 | 0.273 | 0.386 |
| R2 | 0.089 | 0.052 | 0.227 | 0.140 |
| R3 | 0.136 | 0.018 | 0.211 | 0.622 |
| R4 | 0.238 | 0.161 | 0.468 | 0.467 |
| R5 | 0.325 | 0.062 | 0.491 | 0.346 |
| R6 | 0.094 | -0.064 | 0.298 | 0.258 |
| R7 | 0.164 | 0.090 | 0.222 | 0.192 |
| R8 | 0.196 | 0.104 | 0.415 | 0.391 |
| R9 | 0.196 | 0.104 | 0.420 | 0.391 |

In this numerical example, we assume $\rho = 0.12$, and solve these problems and obtain optimal portfolios shown in Table 3.3.

Table 3.3. Each optimal solution with respect to three problems

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|---|---|---|---|---|---|---|---|---|---|
| Our model | 0 | 0.003 | 0.134 | 0.197 | 0.188 | 0.028 | 0.194 | 0.142 | 0.114 |
| Carlsson et al. | 0.002 | 0.024 | 0.190 | 0.166 | 0.199 | 0.012 | 0.115 | 0.123 | 0.169 |
| Vercher et al. | 0.145 | 0.200 | 0 | 0.200 | 0 | 0 | 0.200 | 0.200 | 0.055 |

From the optimal portfolios, we find that the optimal portfolio for our model selects higher return securities such as R4, R5 and R7 than the others. Then, it is similar to the optimal portfolio for Carlsson et al. model. This means that our model and Carlsson et al. model mainly consider maximizing the total profit, while Vercher et al. model considers minimizing the downside risk of the investment, measured by the mean-semi-absolute deviation. On the other hand, some rates of asset allocations for our model such as securities R4 and R7 are similar to those of Vercher et al. model.

Furthermore, we consider two cases; (a) Returns of all securities become more increasing up to 10 percents than the sample means in Table 3.1, (b) Returns of all securities become more decreasing up to 10 percents than the sample mean in Table 3.1. With respect to each case, we randomly generate 100 sample data according to the uniform distribution and calculate the total return of each model.

Table 3.4. Total return with respect to each model of the 10 percent increase case

| | Average value | Maximum value | Minimum value | Variance (100 samples) |
|---|---|---|---|---|
| Our model | 0.1545 | 0.1596 | 0.1503 | $3.683\times10^{-6}$ |
| Carlsson et al. | 0.1539 | 0.1593 | 0.1498 | $3.743\times10^{-6}$ |
| Vercher et al. | 0.1177 | 0.1210 | 0.1141 | $2.322\times10^{-6}$ |

Table 3.5. Total return with respect to each model of the 10 percent decrease case

| | Average value | Maximum value | Minimum value | Variance (100 samples) |
|---|---|---|---|---|
| Our model | 0.1401 | 0.1454 | 0.1359 | $3.693\times10^{-6}$ |
| Carlsson et al. | 0.1395 | 0.1449 | 0.1354 | $3.801\times10^{-6}$ |
| Vercher et al. | 0.1063 | 0.1214 | 0.1030 | $2.254\times10^{-6}$ |

From Tables 3.4 and 3.5, the total return of our model is the highest among three portfolio models even if returns become more increasing and decreasing than the expected returns. Furthermore, in the case of comparing our model with Carlsson et al. model, each average return is nearly equal, but variance of these 100 samples for our model is lower than that of Carlsson et al. model. On the other hand, Vercher et al. model achieves a diminution of the risk in Tables 3.4 and 3.5. However, the mean value of the total return of Vercher et al. model is much less than that of our proposed model. Consequently, we find that our proposed model is applied to other various cases in practical investments, particularly the case where the decision maker finds the way to earn maximum profits in stead of accepting some degree of risks.

### 3.4.2 Tokyo Stock Exchange data

Furthermore, in order to show the usefulness of our proposed model more clearly, we consider another numerical example based on the data of securities on the Tokyo Stock Exchange. Let us consider ten securities shown in Table 3.6, whose mean values and standard deviations are based on historical data in the decade between 1995 and 2004.

Table 3.6. Sample data from Tokyo Stock Exchange

| Returns | Sample mean | SD | Fuzzy number to sample mean |
|---|---|---|---|
| R1 | 0.055 | 0.445 | (0.055, 0.05) |
| R2 | 0.046 | 0.289 | (0.046, 0.03) |
| R3 | 0.035 | 0.306 | (0.035, 0.06) |
| R4 | 0.114 | 0.208 | (0.114, 0.08) |
| R5 | 0.043 | 0.253 | (0.043, 0.02) |
| R6 | 0.034 | 0.269 | (0.034, 0.01) |
| R7 | 0.018 | 0.230 | (0.018, 0.01) |
| R8 | 0.171 | 0.297 | (0.171, 0.10) |
| R9 | 0.087 | 0.388 | (0.087, 0.07) |
| R10 | 0.080 | 0.318 | (0.080, 0.02) |

In a way similar to subsection 3.3.1, parameters $a_j,\ b_j,\ \alpha_j,\ \beta_j,\ \left(j=1,2,...,10\right)$ of trapezoidal fuzzy numbers $\left(a_j, b_j, \alpha_j \beta_j\right)$ based on study [117] are shown in Table 3.7 using the historical data.

Table 3.7. Each parameter value based on the historical data

| Returns | $a_j$ | $b_j$ | $\alpha_j$ | $\beta_j$ |
|---|---|---|---|---|
| R1 | -0.123 | 0.005 | 0.239 | 0.868 |
| R2 | -0.069 | 0.069 | 0.301 | 0.467 |
| R3 | -0.129 | 0.025 | 0.200 | 0.713 |
| R4 | 0.005 | 0.177 | 0.198 | 0.235 |
| R5 | -0.082 | 0.114 | 0.217 | 0.323 |
| R6 | -0.052 | 0.108 | 0.290 | 0.382 |
| R7 | -0.056 | 0.067 | 0.236 | 0.369 |
| R8 | -0.060 | 0.193 | 0.310 | 0.456 |
| R9 | -0.093 | 0.130 | 0.312 | 0.626 |
| R10 | 0.009 | 0.236 | 0.563 | 0.264 |

Using these data in Tables 3.6 and 3.7 and introducing the following fuzzy goals;

$$\mu_G(f)=\begin{cases} 1 & (0.06\le f) \\ g_F(f)=\dfrac{f-0.03}{0.03} & (0.03\le f\le 0.06) \\ 0 & (f\le 0.03) \end{cases},\quad \mu_G(p)=\begin{cases} 1 & (0.8\le p) \\ g_p(p)=\dfrac{p-0.7}{0.1} & (0.7\le p\le 0.8) \\ 0 & (p\le 0.7) \end{cases}$$

we solve problems in subsection 3.3.1. Then, we set the parameter $\rho$ in the Vercher et al. model as $\rho=0.06$, and obtain optimal portfolios for three models shown in Table 3.8, respectively.

Table 3.8. Each optimal solution with respect to three problems

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Our model | 0.051 | 0.078 | 0.083 | 0.200 | 0.095 | 0.055 | 0.029 | 0.200 | 0.108 | 0.102 |
| Carlsson et al. | 0 | 0 | 0 | 0.200 | 0 | 0.200 | 0 | 0.200 | 0.200 | 0.200 |
| Vercher et al. | 0 | 0 | 0 | 0.200 | 0.200 | 0.200 | 0.200 | 0.200 | 0 | 0 |

Subsequently, we consider the case where an investor purchases securities at the end of 2004 according to each portfolio shown in Table 3.8. Then, the total return of three models at term ends of 2005 and 2007 become the following values shown in Table 3.9, respectively.

Table 3.9. Total return at term ends of 2005 and 2007

| | Term end of 2005 | Term end of 2007 |
|---|---|---|
| Our model | 0.3058 | 0.3318 |
| Carlsson et al. | 0.2194 | 0.2395 |
| Vercher et al. | 0.2852 | 0.2743 |

Table 3.9 obviously shows that our proposed model earns much higher profits than Carlsson et al. model and Vercher et al. model. Then, from Table 3.8, the optimal portfolio of our proposed model is well-decentralized compared to previous standard fuzzy models. Therefore, our proposed model may be an appropriate model in the sense of the general risk management that investors should keep a diversified portfolio. Consequently, we find that our proposed model is much useful than the previous standard fuzzy models in current investment markets.

Furthermore, we consider several investor’s subjectivities such as optimistic, pessimistic and neutral. Using the numerical example in Tables 3.6 and 3.7, we assume that each investor has the different sample mean, and consider the following three types of investors as Table 3.10.

Table 3.10. Sample mean of each subjectivity

| | Pessimistic | Neutral | Optimistic |
|---|---|---|---|
| R1 | (0.030, 0.05) | (0.055, 0.05) | (0.070, 0.05) |
| R2 | (0.031, 0.03) | (0.046, 0.03) | (0.061, 0.03) |
| R3 | (0.005, 0.06) | (0.035, 0.06) | (0.065, 0.06) |
| R4 | (0.074, 0.08) | (0.114, 0.08) | (0.154, 0.08) |
| R5 | (0.033, 0.02) | (0.043, 0.02) | (0.053, 0.02) |
| R6 | (0.029, 0.01) | (0.034, 0.01) | (0.039, 0.01) |
| R7 | (0.013, 0.01) | (0.018, 0.01) | (0.023, 0.01) |
| R8 | (0.121, 0.10) | (0.171, 0.10) | (0.221, 0.10) |
| R9 | (0.052, 0.07) | (0.087, 0.07) | (0.121, 0.07) |
| R10 | (0.070, 0.02) | (0.080, 0.02) | (0.090, 0.02) |

Subsequently, in the case of optimistic investor, we deal with the possibility measure in our proposed model. On the other hand, in the case of pessimistic investor, we deal with the necessity measure. In these assumptions, we solve them using the our proposed approaches and obtain the following optimal portfolios.

Table 3.11. Each optimal solution with respect to three subjectivities

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Pessimistic | 0.033 | 0.080 | 0.018 | 0.200 | 0.110 | 0.087 | 0.060 | 0.200 | 0.071 | 0.141 |
| Neutral | 0.051 | 0.078 | 0.083 | 0.200 | 0.095 | 0.055 | 0.029 | 0.200 | 0.108 | 0.102 |
| Optimistic | 0.045 | 0.081 | 0.091 | 0.200 | 0.088 | 0.048 | 0.024 | 0.200 | 0.109 | 0.114 |

Then, in a similar manner to Table 3.9, we consider the case where an investor purchases securities at the end of 2004 according to each portfolio shown in Table 3.11. Then, the total return of three models at term ends of 2005 and 2007 become the following values shown in Table 3.12, respectively.

Table 3.12. Total return at term ends of 2005 and 2007

| | Term end of 2005 | Term end of 2007 |
|---|---|---|
| Pessimistic | 0.2504 | 0.2712 |
| Neutral | 0.3058 | 0.3318 |
| Optimistic | 0.3174 | 0.3434 |

From Table 3.12, we find that the optimistic investor earn the highest future return among three subjectivities. Therefore, this result may mean that the trend of the financial market in this term is booming.

## 3.5 Random Fuzzy CAPM Model

In portfolio theories, Capital Asset Pricing Model (CAPM) proposed by Sharpe [108], Lintner [82], and Mossin [94] has been used in many practical investment cases by not only researchers but also practical investors. The main advantage of CAPM is easily-handled since the relation between returns of each asset and market portfolio such as NASDAQ and TOPIX can be represented as the following linear equation.

$$r_j = d_j^1 + d_j^2 r_m$$

where $r_m$ is the return of market portfolio, and $d_j^1$ and $d_j^2$ are inherent values derived from historical data in investment fields. However, market portfolio $r_m$ is not entirely equal to NASDAQ and TOPIX, and so it is almost impossible to observe $r_m$ exactly in the investment field. Furthermore, in the case that the decision maker predicts the future return using CAPM, it is obvious that market portfolio $r_m$ also occurs according to a random distribution. Therefore, considering these situations, we propose a random fuzzy CAPM model which is assumed $r_m$ to be a random fuzzy variables.

First, we mainly reconsider the following basic random fuzzy programming problem:

$$\begin{aligned} &\text{Maximize } \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \\ &\text{subject to } \boldsymbol{x} \in X \triangleq \left\{ \boldsymbol{x} \in R^n \left| \begin{array}{l} \sum_{j=1}^{n} a_j x_j \le b, \\ x_j \ge 0,\ j = 1, 2, ..., n \end{array} \right. \right\} \end{aligned} \tag{3.35}$$

Subsequently, using CAPM, let us assume that coefficients of objective functions $\bar{\tilde{r}}_j = d_j^1 + \bar{\tilde{t}} d_j^2$ where $d_j^1$ and $d_j^2$ are constants, and $\bar{\tilde{t}}$ is a random variable with variance $\sigma_j^2$ and mean $\tilde{m}_j$ characterized by the following membership functions:

$$\mu_{\tilde{M}}(\xi) = \begin{cases} L\left(\dfrac{m-\xi}{\alpha}\right) & (m \ge \xi) \\ R\left(\dfrac{\xi - m}{\beta}\right) & (m < \xi) \end{cases} \tag{3.36}$$

where $L(x)$ and $R(x)$ are nonincreasing reference functions to satisfy $L(0) = R(0) = 1$, $L(1) = R(1) = 0$ and the parameters $\alpha_i$ and $\beta_i$ represent the spreads corresponding to the left and the right sides, respectively, and both parameters are positive values. In this case that the return coefficient $\bar{\tilde{r}}_j$ is a random fuzzy variable characterized by (3.36), the membership function of $\bar{\tilde{r}}_j$ is expressed as

$$\mu_{\bar{\tilde{r}}_j}(\bar{\gamma}_j) = \sup_{s_j} \left\{ \mu_{\tilde{M}_j}(s_j) \left| \begin{array}{l} \bar{\gamma}_j \sim T_j(s_j, \sigma_j^2), \\ s_j = d_j^1 + \tilde{m} d_j^2 \end{array} \right. \right\},\ \forall \bar{\gamma}_j \in \Gamma \tag{3.37}$$

where $\Gamma$ is a universal set of normal random variable and $T_j(m_j, \sigma_j^2)$ is a probability density function for the random variable with mean $m_j$ and variance $\sigma_j^2$. Then, the objective function $Z=\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j$ is defined as a random fuzzy variable characterized by the following membership function:

$$\begin{aligned}
&\mu_{\bar{\tilde{Z}}}(\bar{u})=\sup_{\bar{\gamma}_j}\left\{\min_{1\le j\le n}\mu_{\bar{\tilde{r}}_j}(\bar{\gamma}_j)\left|\begin{array}{l}\bar{u}=\sum_{j=1}^{n}\bar{\gamma}_j x_j,\\ \bar{\gamma}_j\sim T_j(s_j,\sigma_j^2),\ s_j=d_j^1+\tilde{m}d_j^2\end{array}\right.\right\},\\
&\forall\bar{u}\in Y\triangleq\left\{\sum_{j=1}^{n}\bar{\gamma}_j x_j\,\middle|\,\bar{\gamma}_j\in\Gamma,\ j=1,\dots,n\right\}
\end{aligned}\tag{3.38}$$

where $\bar{\gamma}=(\gamma_1,\dots,\gamma_n)$. Furthermore, we discuss probabilities $\Pr\left\{\omega\middle|\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j\ge f\right\}$ satisfying that the objective function value is greater than or equal to an aspiration level $f$. Since $\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j$ is represented with a random fuzzy variable, we express each probability $\Pr\left\{\omega\middle|\sum_{j=1}^{n}\bar{\tilde{r}}_j x_j\ge f\right\}$ as a fuzzy set $\tilde{P}$ and define the membership function of $\tilde{P}$ as follows:

$$\begin{aligned}
&\mu_{\tilde{P}}(p)\\
&=\sup_{\bar{u}}\left\{\mu_Z(\bar{u})\middle|p=\Pr\left\{\omega\middle|\bar{u}(\omega)\ge f\right\}\right\}\\
&=\sup_{s}\min_{1\le j\le n}\left\{\mu_{\tilde{M}_j}(s_j)\left|\begin{array}{l}p=\Pr\left\{\omega\middle|\bar{u}(\omega)\ge f\right\},\ \bar{u}=\sum_{j=1}^{n}\bar{\gamma}_j x_j,\\ \bar{\gamma}_j\sim T_j(s_j,\sigma_j^2),\ s_j=d_j^1+\tilde{m}d_j^2\end{array}\right.\right\}
\end{aligned}\tag{3.39}$$

Subsequently, since problem (3.35) is not a well-defined problem due to including random fuzzy variable returns, we need to set a criterion with respect to probability and possibility of future returns for the deterministic optimization. In general decision cases with respect to investment, an investor usually focused on maximizing either the goal of the total profit or that of achieve probability. Therefore, in this subsection, we propose a possibility maximization model for probability maximization model in random fuzzy programming problems.

### 3.5.1 RFP model: possibility maximization model for probability maximization model

In this subsection, we mainly consider the case that a decision maker sets the target value $f$ and she or he considers a multi-criteria programming problem maximizing the probability such as the objective function is more than the target value $f$. First of all, we introduce a basic probability maximization model introducing the probability chance constraint as follows:

$$\begin{aligned} &\text{Maximize} \quad \tilde{P} = \Pr\left\{ \sum_{j=1}^{n} \bar{\tilde{r}}_j x_j \geq f \right\} \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{3.40}$$

However, a decision maker usually has a goal that she or he would like to earn the probability more than $p_1$. Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan in many real decision cases, we give a fuzzy goal to the target probability as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of probability $\mu_{\tilde{G}}(p)$ which is represented by,

$$\mu_{\tilde{G}}(\omega) = \begin{cases} 0 & \omega \leq p_0 \\ g_p(\omega) & p_0 \leq \omega \leq p_1 \\ 1 & p_1 \leq \omega \end{cases} \tag{3.41}$$

where $g_p(\omega)$ is a strictly increasing continuous function. Furthermore, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\prod_{\tilde{P}}\left(\tilde{G}\right) = \sup_{p} \min\left\{ \mu_{\tilde{P}}(p),\ \mu_{\tilde{G}}(p) \right\} \tag{3.42}$$

Using this degree of possibility, we consider the following possibility maximization model for the probability maximization model:

$$\begin{aligned} &\text{Maximize} \quad \Pi_{\tilde{P}}\left(\tilde{G}\right) \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{3.43}$$

This problem is equivalently transformed into the following problem by introducing a parameter *h*:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \Pi_{\tilde{P}}\left(\tilde{G}\right) \geq h,\ \boldsymbol{x} \in X \end{aligned} \tag{3.44}$$

In this problem, each constraint $\Pi_{\tilde{P}}(G) \geq h$ is transformed into the following inequality:

$$
\begin{aligned}
& \prod\nolimits_{\tilde{P}}\left(\tilde{G}\right) \geq h \\
\Leftrightarrow\quad & \sup_{p} \min\left\{\mu_{\tilde{P}}\left(p\right), \mu_{\tilde{G}}\left(p\right)\right\} \geq h \\
\Leftrightarrow\quad & \exists p:\ \mu_{\tilde{P}}\left(p\right) \geq h,\ \mu_{\tilde{G}}\left(p\right) \geq h \\
\Leftrightarrow\quad & \exists p: \sup_{\boldsymbol{s}} \min_{1\leq j\leq n}\left\{\mu_{\tilde{M}_j}\left(s_j\right) \left| \begin{array}{l} p = \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\},\ \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j, \\ \bar{\gamma}_j \sim T_j\left(s_j, \sigma_j^2\right),\ s_j = d_j^1 + \tilde{m} d_j^2 \end{array}\right.\right\} \geq h, \\
& \quad \mu_{\tilde{G}}\left(p\right) \geq h \\
& \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \sup_{\boldsymbol{s}} \min_{1\leq j\leq n} \mu_{\tilde{M}_j}\left(s_j\right) \geq h,\ s_j = d_j^1 + \tilde{m} d_j^2, \\
\Leftrightarrow\quad & \qquad p = \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\},\ \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j\left(s_j, \sigma_j^2\right), \\
& \qquad \mu_{\tilde{G}}\left(p\right) \geq h \\
& \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \min_{1\leq j\leq n} \mu_{\tilde{M}_j}\left(s_j\right) \geq h,\ s_j = d_j^1 + \tilde{m} d_j^2, \\
\Leftrightarrow\quad & \qquad p = \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\},\ \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j\left(s_j, \sigma_j^2\right), \\
& \qquad p \geq g_p^{-1}\left(h\right) \\
& \exists p,\ \exists \boldsymbol{s},\ \exists \bar{u}: \mu_{\tilde{M}_j}\left(s_j\right) \geq h,\ \left(j = 1, \ldots, n\right),\ s_j = d_j^1 + \tilde{m} d_j^2, \\
\Leftrightarrow\quad & \qquad \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T_j\left(s_j, \sigma_j^2\right), \\
& \qquad \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\} \geq g_p^{-1}\left(h\right) \\
& \exists \bar{u}: \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\} \geq g_p^{-1}\left(h\right), \\
\Leftrightarrow\quad & \quad \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T\left(d_j^1 + \left(m + R^*\left(h\right)\alpha\right) d_j^2, \sigma_j^2\right)
\end{aligned}
\tag{3.45}
$$

where $L^*\left(h\right)$ is a pseudo inverse function defined as $L^*\left(h\right) = \sup\left\{t \left| L\left(t\right) \geq h\right.\right\}$.

From this transformation, problem (11) is equivalently transformed into the following problem:

$$
\begin{aligned}
\text{Maximize} \quad & h \\
\text{subject to} \quad & \Pr\left\{\omega \left| \bar{u}\left(\omega\right) \geq f\right.\right\} \geq g_p^{-1}\left(h\right), \\
& \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j,\ \bar{\gamma}_j \sim T\left(d_j^1 + \left(m + R^*\left(h\right)\alpha\right) d_j^2, \sigma_j^2\right), \\
& \boldsymbol{x} \in X
\end{aligned}
\tag{3.46}
$$

Furthermore, we do the transformation to the stochastic constraint as follows:

$$
\begin{aligned}
&\Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\} \geq g_p^{-1}(h) \\
\Leftrightarrow\quad &\Pr\left\{\omega \middle| \begin{array}{l} \sum_{j=1}^{n}\left(d_j^1 + \bar{t}(\omega) d_j^2\right) x_j \geq f, \\ \bar{t} \sim T\left(m + R^*(h)\alpha, \sigma_j^2\right) \end{array}\right\} \geq g_p^{-1}(h) \\
\Leftrightarrow\quad &\Pr\left\{\omega \middle| \begin{array}{l} \bar{t}(\omega) \geq \dfrac{f - \boldsymbol{d}^1\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}, \\ \bar{t} \sim T\left(m + R^*(h)\alpha, \sigma_j^2\right) \end{array}\right\} \geq g_p^{-1}(h) \\
\Leftrightarrow\quad &\Pr\left\{\omega \middle| \begin{array}{l} \bar{t}(\omega) - R^*(h)\alpha \geq \dfrac{f - \boldsymbol{d}^1\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}} - R^*(h)\alpha, \\ \bar{t} - R^*(h)\alpha \sim T\left(m, \sigma_j^2\right) \end{array}\right\} \geq g_p^{-1}(h) \\
\Leftrightarrow\quad &\Pr\left\{\omega \middle| \begin{array}{l} \bar{t}'(\omega) \geq \dfrac{f - \left(\boldsymbol{d}^1 + R^*(h)\alpha \boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}, \\ \bar{t}' \sim T\left(m, \sigma_j^2\right) \end{array}\right\} \geq g_p^{-1}(h) \\
\Leftrightarrow\quad &T\left(\frac{f - \left(\boldsymbol{d}^1 + R^*(h)\alpha \boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}\right) \leq 1 - g_p^{-1}(h)
\end{aligned}
\tag{3.47}
$$

Consequently, problem (3.46) is transformed into the following problem:

$$
\begin{aligned}
\text{Maximize} \quad & h \\
\text{subject to} \quad & T\left(\frac{f - \left(\boldsymbol{d}^1 + R^*(h)\alpha \boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}}\right) \leq 1 - g_p^{-1}(h), \\
& \boldsymbol{x} \in X
\end{aligned}
\tag{3.48}
$$

This problem is transformed into the following problem by introducing the pseudo inverse function $T_i^*$ for $T_i$:

$$
\begin{aligned}
\text{Maximize} \quad & h \\
\text{subject to} \quad & \frac{f - \left(\boldsymbol{d}^1 + R^*(h)\alpha \boldsymbol{d}^2\right)\boldsymbol{x}}{\boldsymbol{d}^2\boldsymbol{x}} \leq T^*\left(1 - g_p^{-1}(h)\right), \\
& \boldsymbol{x} \in X
\end{aligned}
\tag{3.49}
$$

Problem (3.49) is a nonconvex programming problem, and so an optimal solution of this problem is not necessarily obtained by usual nonlinear programming approaches. However, in the case that the value of parameter *h* is fixed, constraints of this problem are reduced to a set of linear inequalities. This means that an optimal solution of this problem is obtained by using simplex method of linear programming approach and bisection method for parameter *h*.

Consequently, we construct the following analytical solution method.

**Solution procedure**

STEP1: Elicit the membership function of a fuzzy goal for the probability with respect to the objective function value.

STEP2: Solve problem (3.49) in the case $h=1$. If an optimal solution is obtained, it is a strict optimal solution of main problem and terminate this algorithm. Otherwise, go to STEP3.

STEP3: Solve problem (3.49) in the case $h=0$. If there is no feasible solution, terminate this algorithm. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for the probability. Otherwise, go to STEP4.

STEP4: Set $U_h \leftarrow 1$ and $L_h \leftarrow 0$.

STEP5: Set $k \leftarrow k+1$ and $h_k \leftarrow \dfrac{L_h + U_h}{2}$

STEP6: If $\left|h_{k+1} - h_k\right| < \varepsilon$ for the sufficiently small number $\varepsilon$, go to STEP8. Otherwise, go to STEP7

STEP7: Solve problem (3.49). If an optimal solution is obtained $x(k)$, reset $L_h \leftarrow h$ and return to STEP5. If there is no feasible solution, reset $U_h \leftarrow h$ and return to STEP5.

STEP8: $x(k)$ is an optimal solution and terminate this algorithm.

### 3.5.2 Numerical example

In order to illustrate our considering situation that the proposing solution method is applied, we give a numerical example. We assume that there are four decision variables, three constraints and three scenarios to all parameters. Then, the constant parameters values are given as the following Tables 3.13. In this numerical example, we assume that random fuzzy variables $\bar{\tilde{t}}_i$ occur according to uniform distributions. Then, mean values and interval values for distribution function of $\bar{\tilde{t}}_i$ are given as the following Table 3.14. Subsequently, each mean value $\tilde{m}_i$ is a symmetric triangle fuzzy numbers $\langle \gamma, \delta \rangle$ where $\gamma$ is the center value and $\delta$ is the spread value.

Table 3.13. Parameters values of coefficients in objective functions

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $d_{i1}^1$ | 0.3 | 0.5 | 0.1 |
| $d_{i2}^1$ | 0.1 | 0.2 | 0.2 |
| $d_{i3}^1$ | 0.2 | 0.1 | 0.4 |
| $d_{i4}^1$ | 0.2 | 0.2 | 0.3 |
| $d_{i1}^2$ | 0.4 | 0.3 | 0.3 |
| $d_{i2}^2$ | 0.4 | 0.5 | 0.4 |
| $d_{i3}^2$ | 0.2 | 0.1 | 0.2 |
| $d_{i4}^2$ | 0.4 | 0.3 | 0.4 |

Table 3.14. Mean values and distribution functions of random fuzzy variables

| | Scenario 1 | Scenario 2 | Scenario 3 |
|---|---|---|---|
| $\tilde{m}_i$ | $\langle 0.3, 0.1 \rangle$ | $\langle 0.5, 0.2 \rangle$ | $\langle 0.2, 0.1 \rangle$ |
| Interval | 0.1 | 0.15 | 0.05 |

Furthermore, each fuzzy goal for the probability to each scenario is given as follows:

$$\mu_{\tilde{G}_i}(\omega) = \begin{cases} 0 & \omega < 0.7 \\ \dfrac{\omega - 0.7}{0.2} & 0.7 \le \omega < 0.9, \ i = 1,2,3 \\ 1 & 0.9 \le \omega \end{cases}$$

From these data in Tables 3.13 and 14, we set the target value of total profit which is 0.42, and solve the mini-max problem based on Problem (3.49). Then, we obtain the following optimal solution

Table 3.15. Optimal solution of problem (3.49)

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|
| 0.080 | 0.110 | 0 | 0.810 |

Subsequently, comparing the proposed model with the case not including fuzzy variables, each optimal solution is given as the following Table 3.16.

Table 3.16. Optimal solutions of three problems

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| Not Fuzzy | 0.140 | 0 | 0.158 | 0.702 |
| Problem (3.49) | 0.080 | 0.110 | 0 | 0.810 |

From the result in Table 3.16, we find that the rate of portfolios between $x_2$ and $x_3$ in Problem (3.49) is opposite to the basic stochastic problem including fuzzy numbers. This means that decision variables to be the higher rate of $d_{ij}^2$ tend to be chosen in our proposed model. Therefore, by setting the random fuzzy variable as the market portfolios which is represented the investor's subjectivity, we find that the optimal portfolio largely changes and our proposed model may provide the more appropriate portfolio suited to each investor.

## 3.6 Conclusion

In this chapter, we have considered portfolio selection problems involving ambiguous expected returns distributed according to the normal distribution, and proposed several models of random fuzzy portfolio selection problems; (a) Single criteria optimization model, (b) Bi-criteria optimization model introducing a fuzzy goal to the probability and the target future returns. Since each problem is equivalent to a parametric nonlinear programming problem, we have constructed each efficient solution method involving the procedure of solving a parametric convex programming problem to find a global optimal solution. Then, by comparing the proposed model with other standard fuzzy portfolio models using two numerical examples, we have found that the proposed model has been applied to more flexibly and changeable cases than two previous models. Furthermore, we have considered the random fuzzy CAPM model which had been a useful tool of the investment, and found that the proposed model was solved analytically and efficiently by using the linear programming approach and the bisection algorithm on the parameter.

In the future, we will apply these random fuzzy portfolio selection problem and solution methods to other asset allocation problems, combinational optimization models and multi-period problems. Nonetheless, the new proposed models of portfolio selection problems and their efficient solution methods will allow us to solve more complicate problems in real situations under more random and ambiguous conditions.

# Chapter 4

# Fuzzy Extension for Large-Scale Portfolio Selection Problems

In Chapter 3, we focused on various random fuzzy portfolio selection problems. These problems are based on mean-variance models or safety first models extending the Markowitz model. On the other hand, in the mathematical programming, the mean-variance model is generally formulated as a quadratic programming problem to minimize the variances or maximize the total profit, and so the use of large-scale mean-variance models is restricted to the stock portfolio selection in spite of the recent development in computational and modeling technologies in financial engineering. Therefore, in order to solve the mean-variance model more efficiently, Konno and his research group [68, 69] have proposed the mean-absolute derivation model by introducing the compact factorization based on historical data. This model has been approximately formulated as a linear programming problem and been solved in shorter calculation time than a corresponding mean-variance model. After Konno's studies, many researchers have applied the compact factorization approach to standard portfolio selection problems such as safety-first models, Value-at Risk (VaR), etc.. However, in these previous researches, the subjectivities of investors and ambiguity of received information are not considered. In practice investment cases, it is necessary to take various uncertain conditions such as not only randomness but also fuzziness into consideration. Due to such uncertainty, since it is difficult that investors know precise information, they need to make appropriate decisions to optimize the portfolio under uncertainty based on their own sense of the present market.

Therefore, the main object in this chapter is to extend previous large-scale portfolio selection problems using historical data analysis such as the compact factorization to under stochastic and such fuzzy environments. The proposed models include many previous stochastic and fuzzy portfolio selection problems by considering the parameters setting, and so the proposed models may become versatile to apply various investment situations. However, in the sense of the deterministic mathematical programming, these mathematical models are not well-defined problems due to random and fuzzy variables, and so we need to set a criterion for each objective or constraint involving random and fuzzy variables in order to solve them analytically using the mathematical programming. In this chapter, using the hybrid programming approach to the stochastic approach such as the compact factorization and fuzzy programming approach such as the possibility programming, we perform the deterministic equivalent transformations to main problems. Furthermore, deterministic equivalent problems derived from such stochastic and fuzzy programming problems are generally complicate problems, and so it is often difficult to apply the standard programming approaches to these problems. Therefore, we develop the efficient and analytical solution method to each proposed model considering the analytical strictness and simple usage in practical investment situations.

## 4.1 Formulation of Portfolio Selection Problems Based on Historical Data

In this chapter, the following parameters and variables are used:

$\bar{\boldsymbol{r}}$ : Mean value of *n*-dimensional Gaussian random variable row vectors

$\mathbf{V}$ : Gaussian random variance

$r_G$ : Minimum value of the goal for expected total return

$a_{ij}$ : Cost coefficient of *j*th decision variable to *i*th constraint

$b_i$ : Maximum value to *i*th constraint

$\boldsymbol{x}$ : *n*-dimensional decision variable column vector of the budgeting allocation

Then, we assume that the main object in the following problems based on the Markowitz model is minimizing the total risk, which generates in the case that a decision maker earns the total profit of assets or products more than the target value. Then, in simplify of the following discussion, we also assume that the coefficients of parameter $r_j$ are only fuzzy numbers. In the case that cost coefficients of constraints are assumed to be fuzzy or random variables, we discuss to the problems including such fuzzy or random variables in the same manner.

### 4.1.1 Portfolio selection problem based on mean-variance theory

First, we introduce the mean-variance model proposed by Markowitz [90], which is one of the traditional and important mathematical approaches for portfolio selection problems. This model has proposed the following mathematical programming problem as a portfolio selection problem:
(Mean-variance model)

$$\begin{array}{ll} \text{Minimize} & V(\boldsymbol{x}) = \boldsymbol{x}^t \mathbf{V} \boldsymbol{x} \\ \text{subject to} & E(\boldsymbol{x}) = \bar{\boldsymbol{r}}^t \boldsymbol{x} \geq r_G \\ & \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \mathbf{0} \end{array} \tag{4.1}$$

The mean-variance model has long served as the basis of financial theory. This problem is a quadratic programming problem, and so we find the optimal solution using standard convex or nonlinear programming approaches. However, it is not efficient to solve the large scale quadratic programming problem directly. Furthermore, in the case that a decision maker expects the future return of each product, she or he doesn't consider only one scenario of the future return, but often several scenarios based on the historical data and the statistical analysis.

In this regard, in order to perform the re-formulation of original mean-variance model (4.1), we introduce multi-scenario to the future returns and the following compact factorization approach. Let $r_{sj}$ be the realization of random variable $R_s$ about the scenario $s$, $(s = 1,2,...,S)$, which we assume to be available from historical data and from subjective prediction of decision makers. Then, the return vector of scenario *i* is as follows;

$$\boldsymbol{r}_s = \left(r_{s1}, r_{s2}, ..., r_{sn}\right), \ \left(s = 1,2,...,S\right) \tag{4.2}$$

where *n* is the number of total asset. We introduce the probabilities for each scenario as follows:

$$p_s = \Pr\left\{\boldsymbol{r} = \boldsymbol{r}_S\right\}, \ \left(s = 1, 2..., S\right) \tag{4.3}$$

We also assume that the expected value of the random variable can be approximated by the average derived from these data. Particularly, we let

$$\overline{r}_j \equiv E\left[R_j\right] = \sum_{s=1}^{S} p_s r_{sj} \tag{4.4}$$

Then, the mean value $E(\boldsymbol{x})$ and variance $V(\boldsymbol{x})$ derived from the data are as follows:

$$E\left(\boldsymbol{x}\right) = \sum_{j=1}^{n} \overline{r}_j x_j = \sum_{j=1}^{n} \left(\sum_{s=1}^{S} p_s r_{sj}\right) x_j \tag{4.5}$$

$$\begin{aligned} V\left(\boldsymbol{x}\right) &= \sum_{s=1}^{S} p_s \left(\sum_{j=1}^{n} r_{sj} x_j - E\left(\boldsymbol{x}\right)\right)^2 \\ &= \sum_{s=1}^{S} p_s \left(\sum_{j=1}^{n} r_{sj} x_j - \sum_{j=1}^{n}\left(\sum_{s=1}^{S} p_s r_{sj}\right) x_j\right)^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} \sigma_{jk} x_j x_k \end{aligned} \tag{4.6}$$

For simplify of the following discussion, we assume each probability $p_s$ to become same value $\frac{1}{S}$. From above parameters, we transformed mean-variance model (4.1) into the following problem using the compact factorization approach from (4.2) to (4.6):

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S}\left(\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j\right)^2 \\
&\text{subject to} \quad \overline{r}_j = \frac{1}{S}\sum_{s=1}^{S} r_{sj} \\
&\qquad\qquad \sum_{j=1}^{n}\overline{r}_j x_j \ge r_G,\ \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned} \tag{4.7}$$

Furthermore, introducing parameters $z_s = \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j,\ \left(s=1,2,...,S\right)$, we equivalently transformed problem (4.7) into the following problem:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S} z_s^2 \\
&\text{subject to} \quad z_s - \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j = 0,\ \left(s=1,2,...,S\right) \\
&\qquad\qquad \overline{r}_j = \frac{1}{S}\sum_{s=1}^{S} r_{sj}, \\
&\qquad\qquad \sum_{j=1}^{n}\overline{r}_j x_j \ge r_G,\ \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned} \tag{4.8}$$

Since this problem is a quadratic programming problem not to include the variances, we set each parameter and solve it more efficiently than original model (4.1). However, this problem is not a linear programming problem which is easily solved using the efficient linear programming approaches.

### 4.1.2 Portfolio selection problem based on mean-absolute deviation theory

In order to solve mean-variance model (4.1) efficiently, Konno [68] considered the mean-absolute deviation model for portfolio selection problems. This model considers an absolute deviation instead of the variance as the risk factor. Now, let the absolute deviation (AD)

$$AD \equiv \frac{1}{S}\sum_{s=1}^{S}\left|\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j\right| \equiv \frac{1}{S}\sum_{s=1}^{S}\left|r_s-\overline{r}_P\right| \tag{4.9}$$

By introducing this definition of absolute deviation, he proposed the following portfolio selection problem:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S}\left|\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j\right| \\
&\text{subject to} \quad \overline{r}_j=\frac{1}{S}\sum_{i=1}^{m}r_{sj} \\
&\qquad \sum_{j=1}^{n}\overline{r}_j x_j \ge r_G,\ \sum_{j=1}^{n}a_j x_j \le b,\ \boldsymbol{x}\ge\boldsymbol{0}
\end{aligned} \tag{4.10}$$

Then, introducing parameters $z_s=\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j$, problem (10) is transformed into the following problem based on the result of the previous study of Konno [68]:

$$\begin{aligned}
&\text{Minimize} \quad \frac{2}{S}\sum_{s=1}^{S}z_s \\
&\text{subject to} \quad z_s+\sum_{j=1}^{n}r_{sj}x_j \ge \sum_{j=1}^{n}\overline{r}_j x_j,\ \left(s=1,2,\ldots,S\right) \\
&\qquad \overline{r}_j=\frac{1}{S}\sum_{s=1}^{S}r_{sj}, \\
&\qquad \sum_{j=1}^{n}\overline{r}_j x_j \ge r_G,\ \sum_{j=1}^{n}a_j x_j \le b,\ \boldsymbol{x}\ge\boldsymbol{0}
\end{aligned} \tag{4.11}$$

This problem is a linear programming problem, and so it is efficiently solved by using the standard linear programming approaches such as the Simplex method and the Interior point method. Furthermore, it is essentially equivalent to the model based on the mean-variance theory if the rate of the return of assets is multivariate normally distributed. Consequently, by using the mean-absolute deviation model, investors can solve large scale portfolio selection problems easily.

### 4.1.3 Portfolio selection problem based on mean-variance theory

With respect to portfolio selection problems based on mean-variance and mean-absolute deviation theories, the main concept is that the risk-management is equal to minimizing the total variance and absolute deviation, respectively. However, the nonfulfillment probability, which means that the total return is less than the target value, is also considered as the factor of risk-management. In this regard, many researchers have proposed safety first models with respect to portfolio selection problems, Roy model [104], Kataoka model [64], Telser model [114], etc..

First, we introduce the Roy model. This model has been formulated as the following mathematical programming problem.

(Roy model)

$$\begin{aligned} \text{Minimize} \quad & \Pr\left\{ R(\boldsymbol{x}) \le r_f \right\} \\ \text{subject to} \quad & \bar{\boldsymbol{r}}^t \boldsymbol{x} \ge r_G \\ & \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0} \end{aligned} \tag{4.12}$$

where $R(\boldsymbol{x})$ is the total profit and $r_f$ is the goal of $R(\boldsymbol{x})$. Furthermore, we introduce other safety first models, Kataoka model and Telser model, as follows:

(Kataoka model)

$$\begin{aligned} \text{Maximize} \quad & r_f \\ \text{subject to} \quad & \Pr\left\{ R(\boldsymbol{x}) \le r_f \right\} \le \alpha, \\ & \bar{\boldsymbol{r}}^t \boldsymbol{x} \ge r_G,\ \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0} \end{aligned} \tag{4.13}$$

where $\alpha$ is a goal of nonfulfillment probability.

(Telser model)

$$\begin{aligned} \text{Maximize} \quad & \bar{\boldsymbol{r}}^t \boldsymbol{x} \\ \text{subject to} \quad & \Pr\left\{ R(\boldsymbol{x}) \le r_f \right\} \le \alpha, \\ & \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0} \end{aligned} \tag{4.14}$$

The safety first models such as Roy, Kataoka and Telser models also have long served as the basis of financial theory as well as mean-variance and mean-absolute deviation models. In mathematical programming, these problems are stochastic programming problems, and so we obtain the optimal portfolio as stochastic programming approaches.

In some previous researches, each future return is assumed to be a random variable including the fixed expected return and variance. However, it is hard to observe variances of each asset in real market accurately and determine them as fixed values. Furthermore, in the case that decision makers expect the future return of each product, they don't consider the only one scenario of the future return, but often several scenarios. Therefore, in a way similar to mean-variance and mean-absolute deviation models, we introduce the scenario of return vector (4.2) and the occurrence probability (4.3). Then, we assume each probability to be $p_s = \frac{1}{S}$. Using these parameters and the compact factorization approach, we transform these safety first models into the following problems introducing the parameters $z_s \in \{0,1\}, (s = 1,2,..., S)$, respectively:

(Roy model)

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S} z_s \\
&\text{subject to} \quad \sum_{j=1}^{n} r_{sj}x_j + M\cdot z_s \geq r_f,\ \left(s=1,2,...,S\right) \\
&\qquad\qquad \bar{\boldsymbol{r}}^t\boldsymbol{x} \geq r_G,\ \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned} \tag{4.15}$$

(Kataoka model)

$$\begin{aligned}
&\text{Maximize} \quad r_f \\
&\text{subject to} \quad \frac{1}{S}\sum_{s=1}^{S} z_s \leq \alpha, \\
&\qquad\qquad \sum_{j=1}^{n} r_{sj}x_j + M\cdot z_s \geq r_f,\ \left(s=1,2,...,S\right) \\
&\qquad\qquad \bar{\boldsymbol{r}}^t\boldsymbol{x} \geq r_G,\ \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned} \tag{4.16}$$

(Telser model)

$$\begin{aligned}
&\text{Maximize} \quad \bar{\boldsymbol{r}}^t\boldsymbol{x} \\
&\text{subject to} \quad \frac{1}{S}\sum_{s=1}^{S} z_s \leq \alpha, \\
&\qquad\qquad \sum_{j=1}^{n} r_{sj}x_j + M\cdot z_s \geq r_f,\ \left(s=1,2,...,S\right) \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned} \tag{4.17}$$

where $M$ is a sufficiently large number satisfying $M \geq r_G - \min\left\{\sum_{j=1}^{n} r_{sj}x_j\right\}$. These models are 0-1 mixed linear programming problems. Therefore, we obtain these optimal portfolios using integer programming approaches such as Branch-bound method.

## 4.2 Fuzzy Extension to the Large-Scale Portfolio Selection Problems Based on Historical Data

### 4.2.1 Portfolio selection problem based on the fuzzy extension of mean-variance theory

With respect to previous portfolio models in Subsection 4.1, each return of scenarios is considered as a fixed value derived from a random variable. However, considering the psychological aspects investors, it is difficult to predict the future return as the fixed value. Therefore, we need to consider that the future return includes ambiguous factors. Therefore, we propose the risk-management models based on the portfolio theory using scenarios where the return is ambiguous. In this chapter, the return including the ambiguity is assumed to the following triangular fuzzy number:

$$\tilde{r}_{sj} = \left\langle \bar{r}_{sj}, \alpha_j, \alpha_j \right\rangle = \left\langle \bar{r}_{sj}, \alpha_j \right\rangle \tag{4.18}$$

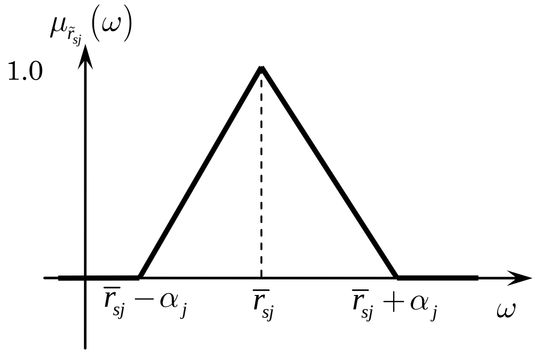


**Figure 4.1.** Shape of the membership function $\mu_{\tilde{r}_{sj}}(\omega)$

In this chapter, for simplify of the following discussion, we assume $\bar{r}_{sj} - (1-h)\alpha_j$ to be a positive value. Since $\tilde{r}_{sj}$ is a fuzzy variable, the objective function and the parameters including $\tilde{r}_{sj}$ in the basic mean-variance model are also assumed to be fuzzy variables. Therefore, we can not optimize this problem without transforming the objective function into another form.

In previous researches, some criteria with respect to fuzzy portfolio problems have been proposed. For example, Liu [83, 84], and Huang [49] have proposed a portfolio selection problems using fuzzy or hybrid (fuzzy and random) expected value and its variance. Katagiri [61] has proposed a portfolio selection problem using possibility measure and probability measure. Carlsson [17] has proposed a portfolio selection problem using the possibility mean value. In, this chapter, we assume the following cases:

(a) Since main object is minimizing the total variance and the decision maker considers that she or he manages to minimize it as small as possible even if the aspiration level becomes smaller.

(b) On the other hand, it is clear that the decision maker also considers that she or he never fails to earn the total return more than the goal in the variance constraint.

Therefore, we consider fuzzy portfolio selection problems for probabilistic expected value and variance. Then, we introduce a possibility measure for total variance based on assumption (a) and a necessity measure for the expected total return based on assumption (b). Then, we convert basic mean-variance model (4.7) into the following problem including the chance-constraint:

$$\begin{aligned}
&\text{Minimize} \quad \sigma_G \\
&\text{subject to} \quad \mathrm{Pos}\left\{\tilde{V}(\boldsymbol{x}) \le \sigma_G\right\} \ge h, \\
&\qquad\qquad \mathrm{Nec}\left\{\tilde{E}(\boldsymbol{x}) \ge r_G\right\} \ge h, \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned} \tag{4.19}$$

where $\mathrm{Pos}\left\{\tilde{V}(\boldsymbol{x}) \le \sigma_G\right\}$ is a possibility measure and this means $\mathrm{Pos}\left\{\tilde{V}(\boldsymbol{x}) \le \sigma_G\right\} = \sup_{\sigma \le \sigma_G} \mu_{\tilde{V}(\boldsymbol{x})}(\sigma)$, then $\mathrm{Nec}\left\{\tilde{E}(\boldsymbol{x}) \ge r_G\right\}$ is a necessity measure and this means $\mathrm{Nec}\left\{\tilde{E}(\boldsymbol{x}) \ge r_G\right\} = 1 - \sup_{r \ge r_G} \mu_{\tilde{E}(\boldsymbol{x})}(r)$. In this problem, membership functions $\mu_{\tilde{E}(\boldsymbol{x})}(r)$ and $\mu_{\tilde{V}(\boldsymbol{x})}(\sigma)$ are assumed to be the following forms using fuzzy extension principle. First, fuzzy number $\tilde{E}(\boldsymbol{x})$ is given as the following triangular fuzzy numbers:

$$\tilde{E}(\boldsymbol{x}) = \left\langle \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j, \sum_{j=1}^{n}\alpha_j x_j \right\rangle \tag{4.20}$$

Therefore, we obtain membership function $\mu_{\tilde{E}(\boldsymbol{x})}(r)$ as follows:

$$\mu_{\tilde{E}(\boldsymbol{x})}(\omega) = \begin{cases} \dfrac{\frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - \omega}{\sum_{j=1}^{n}\alpha_j x_j} & \left(q^- \le \omega \le q\right) \\ \dfrac{\omega - \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j}{\sum_{j=1}^{n}\alpha_j x_j} & \left(q \le \omega \le q^+\right) \\ 0 & \left(\omega < q^-,\ q^+ < \omega\right) \end{cases} \tag{4.21}$$

$$q^- = \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - \sum_{j=1}^{n}\alpha_j x_j,\ q^+ = \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j + \sum_{j=1}^{n}\alpha_j x_j,\ q = \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j$$

Next, we consider membership function $\mu_{\tilde{V}(\boldsymbol{x})}(\sigma)$. In a way similar to $\mu_{\tilde{E}(\boldsymbol{x})}(r)$, we obtain the fuzzy number for $\tilde{z}_s = \sum_{j=1}^{n} \left( \tilde{r}_{sj} - \tilde{r}_j \right) x_j = \sum_{j=1}^{n} \tilde{r}_{sj} x_j - \tilde{E}\left( \boldsymbol{x} \right)$ as the following triangular fuzzy number:

$$\tilde{z}_s = \left\langle \sum_{j=1}^{n} \overline{r}_{sj} x_j - \frac{1}{S} \sum_{j=1}^{n} \left( \sum_{s=1}^{S} \overline{r}_{sj} \right) x_j , 2\sum_{j=1}^{n} \alpha_j x_j \right\rangle = \left\langle z_s , 2\sum_{j=1}^{n} \alpha_j x_j \right\rangle \tag{4.22}$$

Therefore, we obtain membership function $\mu_{\tilde{z}_s}\left(\omega\right)$ as follows:

$$\mu_{\tilde{z}_s}\left(\omega\right) = \begin{cases} \dfrac{z_s - \omega}{2\sum_{j=1}^{n} \alpha_j x_j} & \left( z_s - 2\sum_{j=1}^{n} \alpha_j x_j \le \omega \le z_s \right) \\ \dfrac{\omega - z_s}{2\sum_{j=1}^{n} \alpha_j x_j} & \left( z_s \le \omega \le z_s + 2\sum_{j=1}^{n} \alpha_j x_j \right) \\ 0 & \left( \omega < z_s - 2\sum_{j=1}^{n} \alpha_j x_j ,\ z_s + 2\sum_{j=1}^{n} \alpha_j x_j < \omega \right) \end{cases} \tag{4.23}$$

Furthermore, we consider the membership function for $\tilde{S}_s = \left( \sum_{j=1}^{n} \left( r_{sj} - \overline{r}_j \right) x_j \right)^2$. In general cases using fuzzy numbers, membership functions often become much complicate functions. Therefore, with respect to $\mu_{\tilde{z}_s}\left(\omega\right)$, we introduce the $h$-cut of this membership function in order to represent membership functions briefly:

$$\tilde{z}_s = \left[ \mu_{Z_s}^{(L)}\left(h\right), \mu_{Z_s}^{(R)}\left(h\right) \right] = \left[ \mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(L)}\left(h\right) - \mu_{E(\boldsymbol{x})}^{(R)}\left(h\right), \mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(R)}\left(h\right) - \mu_{E(\boldsymbol{x})}^{(L)}\left(h\right) \right] \tag{4.24}$$

where

$$\begin{aligned} \tilde{\boldsymbol{r}}^t \boldsymbol{x} &= \left[ \mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(L)}\left(h\right), \mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(R)}\left(h\right) \right] \\ &= \left[ \sum_{j=1}^{n} \overline{r}_{sj} x_j - \left(1-h\right) \sum_{j=1}^{n} \alpha_j x_j , \sum_{j=1}^{n} \overline{r}_{sj} x_j + \left(1-h\right) \sum_{j=1}^{n} \alpha_j x_j \right] \\ \tilde{E}\left(\boldsymbol{x}\right) &= \left[ \mu_{E(\boldsymbol{x})}^{(L)}\left(h\right), \mu_{E(\boldsymbol{x})}^{(R)}\left(h\right) \right] \\ &= \left[ \frac{1}{S} \sum_{j=1}^{n} \left( \sum_{s=1}^{S} \overline{r}_{sj} \right) x_j - \left(1-h\right) \sum_{j=1}^{n} \alpha_j x_j , \frac{1}{S} \sum_{j=1}^{n} \left( \sum_{s=1}^{S} \overline{r}_{sj} \right) x_j + \left(1-h\right) \sum_{j=1}^{n} \alpha_j x_j \right] \end{aligned}$$

From this $h$-cut of this membership function, that of membership function $\mu_{\tilde{S}_s}(\omega)$ is the following form:

$$\tilde{S}_s = \left[\mu_{S_s}^{(L)}(h), \mu_{S_s}^{(R)}(h)\right]$$
$$\begin{cases} \mu_{S_s}^{(L)}(h) = \min\left\{0, \left(\mu_{Z_s}^{(L)}(h)\right)^2, \left(\mu_{Z_s}^{(R)}(h)\right)^2\right\} \\ \mu_{S_s}^{(R)}(h) = \max\left\{\left(\mu_{Z_s}^{(L)}(h)\right)^2, \left(\mu_{Z_s}^{(R)}(h)\right)^2\right\} \end{cases} \tag{4.25}$$

Therefore, the membership function $\mu_{\tilde{V}(\boldsymbol{x})}(\sigma)$ is given as follows.

$$\tilde{V}(\boldsymbol{x}) = \left[\frac{1}{S}\sum_{s=1}^{S}\mu_{S_s}^{(L)}(h), \frac{1}{S}\sum_{s=1}^{S}\mu_{S_s}^{(R)}(h)\right] \tag{4.26}$$

By using these membership functions, we transform the problem (4.19) into the following problem:

$$\begin{aligned} \text{Minimize} \quad & \sigma_G \\ \text{subject to} \quad & \frac{1}{S}\sum_{s=1}^{S}\mu_{S_s}^{(L)}(h) \le \sigma_G, \\ & \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - h\sum_{j=1}^{n}\alpha_j x_j \ge r_G, \\ & \sum_{j=1}^{n} a_j x_j \le b, \ \boldsymbol{x} \ge \boldsymbol{0} \end{aligned} \tag{4.27}$$

where the left part of membership function $\mu_{S_s}^{(L)}(h)$ is as follows:

$$\begin{aligned} \mu_{S_s}^{(L)}(h) &= \left(\sum_{j=1}^{n}\overline{r}_{sj}x_j \pm (1-h)\sum_{j=1}^{n}\alpha_j x_j\right)^2 + \left(\frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j \mp (1-h)\sum_{j=1}^{n}\alpha_j x_j\right)^2 \\ &\quad -2\left(\sum_{j=1}^{n}\overline{r}_{sj}x_j \pm (1-h)\sum_{j=1}^{n}\alpha_j x_j\right)\left(\frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j \mp (1-h)\sum_{j=1}^{n}\alpha_j x_j\right) \\ &= \left(\sum_{j=1}^{n}\overline{r}_{sj}x_j - \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j\right)^2 - 4(1-h)\left(\sum_{j=1}^{n}\alpha_j x_j\right)\left(\sum_{j=1}^{n}\overline{r}_{sj}x_j\right) \end{aligned} \tag{4.28}$$

Then, introducing parameters $z_s^{(h)}$, problem (4.27) is equivalently transformed into the following problem:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\left(z_i^2 - 4(1-h)\left(\sum_{j=1}^{n}\alpha_j x_j\right)\left(\sum_{j=1}^{n}\overline{r}_{sj}x_j\right)\right) \\
&\text{subject to} \quad z_s - \sum_{j=1}^{n}\overline{r}_{sj}x_j + \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j = 0, \ (s = 1,2,...,S) \\
&\qquad\qquad \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - h\sum_{j=1}^{n}\alpha_j x_j \geq r_G, \\
&\qquad\qquad \sum_{j=1}^{n}a_j x_j \leq b, \ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned} \tag{4.29}$$

In problem (4.29), the objective function is a convex quadratic function, and so problem (4.29) is equivalent to a convex quadratic programming problem. Therefore, we obtain a global optimal solution by using standard convex programming approaches. Furthermore, in the case that each return does not include fuzziness, i.e., each $\alpha_j = 0, (j = 1,2,...,n)$, problem (4.29) is degenerated to the following problem:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S}z_s^2 \\
&\text{subject to} \quad z_s - \sum_{j=1}^{n}\overline{r}_{sj}x_j + \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j = 0, \ (s = 1,2,...,S) \\
&\qquad\qquad \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j \geq r_G, \\
&\qquad\qquad \sum_{j=1}^{n}a_j x_j \leq b, \ \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned} \tag{4.30}$$

This problem is equivalent to basic mean-variance portfolio selection problem (4.8). Consequently, we find that problem (4.30) is a fuzzy extended model for basic mean-variance model.

### 4.2.2 Portfolio selection problem based on the fuzzy extension of mean-absolute deviation theory

In a way similar to the mean-variance model, we consider the fuzzy extension of the mean-absolute deviation model. First, we rewrite the mean-absolute deviation model:

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S}\left|\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j\right| \\
&\text{subject to} \quad \overline{r}_j=\frac{1}{S}\sum_{i=1}^{m}r_{sj} \\
&\qquad \sum_{j=1}^{n}\overline{r}_j x_j \geq r_G,\ \sum_{j=1}^{n}a_j x_j \leq b,\ \boldsymbol{x}\geq \mathbf{0}
\end{aligned} \tag{4.31}$$

In this problem, the objective function and parameters including fuzzy variables $\tilde{r}_{sj}$ are also assumed to be fuzzy variables. Therefore, we convert problem (4.31) into the following problem by using chance constraints:

$$\begin{aligned}
&\text{Minimize} \quad \sigma_G \\
&\text{subject to} \quad \text{Pos}\left\{\tilde{D}(\boldsymbol{x})\leq \sigma_G\right\}\geq h, \\
&\qquad \text{Nec}\left\{\tilde{E}(\boldsymbol{x})\geq r_G\right\}\geq h, \\
&\qquad \sum_{j=1}^{n}a_j x_j \leq b,\ \boldsymbol{x}\geq \mathbf{0}
\end{aligned} \tag{4.32}$$

The same kind of membership functions is given in Section 3, so we consider membership function $\mu_{\tilde{D}(\boldsymbol{x})}(\omega)$. The $h$-cut of membership function becomes the following form in a way similar to the mean-variance model:

$$\tilde{D}(\boldsymbol{x})=\left[\mu_D^{(L)}(h),\mu_D^{(R)}(h)\right]=\left[\frac{1}{S}\sum_{s=1}^{S}\mu_{Z_s'}^{(L)}(h),\frac{1}{S}\sum_{s=1}^{S}\mu_{Z_s'}^{(R)}(h)\right] \tag{4.33}$$

where

$$\mu_{Z_s'}^{(L)}(h)=\begin{cases}\mu_{Z_s}^{(L)}(h) & \left(\mu_{Z_s}^{(L)}(h)\geq 0\right)\\ 0 & \left(\mu_{Z_s}^{(L)}(h)<0\leq \mu_{Z_s}^{(R)}(h)\right)\\ -\mu_{Z_s}^{(R)}(h) & \left(\mu_{Z_s}^{(R)}(h)<0\right)\end{cases}$$

$$\mu_{Z_s'}^{(R)}(h)=\begin{cases}\mu_{Z_s}^{(R)}(h) & \left(\mu_{Z_s}^{(L)}(h)\geq 0\right)\\ \max\left\{0,-\mu_{Z_s}^{(L)}(h),\mu_{Z_s}^{(R)}(h)\right\} & \left(\mu_{Z_s}^{(L)}(h)<0\leq \mu_{Z_s}^{(R)}(h)\right)\\ -\mu_{Z_s}^{(L)}(h) & \left(\mu_{Z_s}^{(R)}(h)<0\right)\end{cases}$$

and

$$
\begin{aligned}
\tilde{z}_s &= \left[\mu_{Z_s}^{(L)}(h), \mu_{Z_s}^{(R)}(h)\right] \\
&= \left[\mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(L)}(h) - \mu_{E(\boldsymbol{x})}^{(R)}(h), \mu_{\boldsymbol{r}^t\boldsymbol{x}}^{(R)}(h) - \mu_{E(\boldsymbol{x})}^{(L)}(h)\right] \\
&= \left[z_s - 2(1-h)\sum_{j=1}^{n}\alpha_j x_j, z_s + 2(1-h)\sum_{j=1}^{n}\alpha_j x_j\right] \\
z_s &= \sum_{j=1}^{n}\overline{r}_{sj}x_j - \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j, \ (s = 1, 2, ..., S)
\end{aligned}
$$

Using this membership function, problem (4.32) is transformed into the following problem:

$$
\begin{aligned}
&\text{Minimize} \quad \sigma_G \\
&\text{subject to} \quad \frac{1}{S}\sum_{s=1}^{S}\mu_{Z_s'}^{(L)}(h) \le \sigma_G \\
&\qquad \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - h\sum_{j=1}^{n}\alpha_j x_j \ge r_G, \\
&\qquad \sum_{j=1}^{n}a_j x_j \le b, \ \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned}
\tag{4.34}
$$

In order to solve problem (4.34) analytically, we introduce the following parameters:

$$
\begin{aligned}
&z_s = v_s - v_s^- + v_s^+, \\
&v_s^- = \overline{\alpha}(h) - z_s, \ \left(v_s^+ = 0, z_s \ge \alpha(h)\right), \\
&v_s^+ = z_s - \left(-\overline{\alpha}(h)\right), \ \left(v_s^- = 0, z_s \le -\alpha(h)\right), \\
&-\overline{\alpha}(h) \le v_s \le \overline{\alpha}(h), \ \overline{\alpha}(h) = 2(1-h)\left(\sum_{j=1}^{n}\alpha_j x_j\right)
\end{aligned}
$$

By using these parameters, problem (4.34) is equivalently transformed into the following problem based on the previous study of King [66]:

$$
\begin{aligned}
&\text{Minimize} \quad \frac{1}{S}\sum_{s=1}^{S}\left(2z_s - 2\overline{\alpha}(h)\right) \\
&\text{subject to} \quad z_s = v_s - v_s^- + v_s^+, \ -\overline{\alpha}(h) \le v_s \le \overline{\alpha}(h), \\
&\qquad z_s - \alpha(h) \ge 0, \, z_s + \alpha(h) \ge 0, \\
&\qquad \frac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j - h\sum_{j=1}^{n}\alpha_j x_j \ge r_G, \\
&\qquad \sum_{j=1}^{n}a_j x_j \le b, \ \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned}
\tag{4.35}
$$

Since this problem is a linear programming problem, we obtain the global optimal solution by using linear programming approaches such as Simplex method and the Interior point method. Furthermore, in a way similar to the proposed fuzzy mean-variance model in Section 3, in the case that each return does not include fuzziness, i.e., each $\alpha_j = 0, (j = 1,2,...,n)$, problem (4.35) is degenerated to the following problem:

$$\begin{array}{ll} \text{Minimize} & \dfrac{2}{S}\sum_{s=1}^{S}\left(2z_s - 2\overline{\alpha}(h)\right) \\ \text{subject to} & z_s - \left(\sum_{j=1}^{n}\overline{r}_{sj}x_j - \dfrac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j\right) \ge 0, \\ & \left(\sum_{j=1}^{n}\overline{r}_{sj}x_j - \dfrac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j\right) - z_s \ge 0, \\ & \dfrac{1}{S}\sum_{j=1}^{n}\left(\sum_{s=1}^{S}\overline{r}_{sj}\right)x_j \ge r_G, \\ & \sum_{j=1}^{n}a_jx_j \le b,\ \mathbf{x} \ge \mathbf{0} \end{array} \tag{4.36}$$

This problem is equivalent to basic mean-absolute deviation model (4.11). Consequently, we find that problem (36) is a fuzzy extended model for basic mean-absolute deviation model.

### 4.2.3 Portfolio selection problem based on the fuzzy extension of safety-first theory

In previous researches using scenario models with respect to future returns, the each return is considered as the fixed value derived from a random variable. However, considering the psychological aspect of decision makers and the uncertainty of given information, we need to consider that the future return has the ambiguity since it is difficult to predict the future return as the fixed value. Therefore, we propose the portfolio selection problem using the scenarios involving the ambiguous to the returns. In this chapter, each return is assumed to the triangle fuzzy number (4.18). These problems (4.15), (4.16) and (4.17) include fuzzy numbers in objective functions and constraints; for example, each membership function of the objective function $\tilde{f}_s = \sum_{j=1}^{n}\tilde{r}_{sj}x_j + M \cdot z_s$, $(s = 1,2,...,S)$ is given as follows:

$$\tilde{f}_s = \left\langle \sum_{j=1}^{n}\overline{r}_{sj}x_j + M \cdot z_s, \sum_{j=1}^{n}\alpha_j x_j \right\rangle \tag{4.37}$$

In the case that we introduce these membership functions in problems (4.15), (4.16) and (4.17), directly, these problems are not well-defined problems due to fuzzy numbers, and we need to transform these problems into deterministic equivalent problems to solve using mathematical programming approaches. In the case that a decision maker deals with each safety first model, it is obvious that she or he focuses on its objective function more greatly than the other constraint. With respect to its objective function, she or he considers that the object is realized as well as possible even if the possibility decreases a little. Then, with respect to constraint $\bar{\boldsymbol{r}}^t \boldsymbol{x} \geq r_G$, we introduce the possibility mean function based on the result of previous research [18].

$$
\begin{aligned}
& \bar{\boldsymbol{r}}^t \boldsymbol{x} \geq r_G \\
\Leftrightarrow\quad & \int_0^1 \gamma \left( \sum_{j=1}^{n} \left( \frac{1}{S} \sum_{s=1}^{S} \bar{r}_{sj} - (1-\gamma)\alpha_j \right) x_j \right) d\gamma + \int_0^1 \gamma \left( \sum_{j=1}^{n} \left( \frac{1}{S} \sum_{s=1}^{S} \bar{r}_{sj} + (1+\gamma)\beta_j \right) x_j \right) d\gamma \geq r_G \\
\Leftrightarrow\quad & \sum_{j=1}^{n} \left( \frac{1}{S} \sum_{s=1}^{S} \bar{r}_{sj} x_j \right) - \left( \int_0^1 \gamma(1-\gamma)\, d\gamma \right) \sum_{j=1}^{n} \alpha_j x_j + \left( \int_0^1 \gamma(1-\gamma)\, d\gamma \right) \sum_{j=1}^{n} \beta_j x_j \geq r_G \\
\Leftrightarrow\quad & \sum_{j=1}^{n} \left( \frac{1}{S} \sum_{s=1}^{S} \bar{r}_{sj} x_j \right) - \frac{1}{6} \sum_{j=1}^{n} \alpha_j x_j + \frac{1}{6} \sum_{j=1}^{n} \beta_j x_j \geq r_G
\end{aligned}
\tag{4.38}
$$

Furthermore, if target value $r_G$ is sufficiently small, the maximum value of aspiration level for each object is sufficiently large. However, a decision maker often has a goal to each object such as "Total future return $\tilde{f}_s$ is approximately larger than the goal $f_1$." and "The goal of nonfulfillment probability is less than about $p_1$.". Furthermore, in a similar way to each return involving the decision maker's subjectivity and ambiguity, taking account of the vagueness of human judgment and flexibility for the execution of a plan, the decision maker has some subjectivity and ambiguity with respect to each goal. In this chapter, we introduce such subjective goals for the total return $r_G$, nonfulfillment probability $p = \frac{1}{S}\sum_{s=1}^{S} z_s$ and expected total return $r_G$ as fuzzy goals. First, the fuzzy goal of $r_G$ is assumed to be the following membership function:

$$
\mu_G(r_G) = \begin{cases} 1 & (r_G \leq r_{G1}) \\ \dfrac{r_{G0} - r_G}{r_{G0} - r_{G1}} & (r_{G1} < r_G \leq r_{G0}) \\ 0 & (r_{G0} < r_G) \end{cases} \tag{4.39}
$$

In a similar way to $r_G$, fuzzy goals of $p$ and $r$ are assumed to be the following forms:

$$\mu_G(p)=\begin{cases} 1 & (p \le p_1) \\ \dfrac{p_0-p}{p_0-p_1} & (p_1 < p \le p_0) \\ 0 & (p_0 < p) \end{cases} \tag{4.40}$$

$$\mu_G(r)=\begin{cases} 1 & (r_1 \le r) \\ \dfrac{r-r_0}{r_1-r_0} & (r_o \le r < r_1) \\ 0 & (r < r_0) \end{cases} \tag{4.41}$$

Then, we introduce the following possibility measure $\Pi_{\tilde{f}_s}(\tilde{G})$ and necessity measure $N_{\tilde{f}_s}(\tilde{G})$.

$$\begin{aligned} \Pi_{\tilde{f}_s}(\tilde{G}) &= \sup_{\omega} \min\left\{\mu_{\tilde{f}_s}(\omega), \mu_G(\omega)\right\} \\ N_{\tilde{f}_s}(\tilde{G}) &= \inf_{\omega} \max\left\{1-\mu_{\tilde{f}_s}(\omega), \mu_G(\omega)\right\} \end{aligned} \tag{4.42}$$

We assumed the following cases:

(a) In the case that the object is maximizing the total return, we deal with possibility measure $\Pi_{\tilde{f}_s}(\tilde{G})$ since a decision maker strongly considers that she or he manages to earn the total return as much as possible.

(b) In the case that the object is not maximizing the total return but other objects such as minimizing the nonfilfullment probability, we deal with necessity measure $N_{\tilde{f}_s}(\tilde{G})$ since a decision maker consider that the total return is sure to be hold more than the goal satisfying the other object.

From these assumptions, in this chapter, we possibility maximization models with respect to each objective function of basic safety first model satisfying this aspiration level becomes more than the other aspiration level. Therefore, safety first models (4.15), (4.16) and (4.17) are transformed into the following problems introducing parameter $h$:

(Fuzzy Roy model)

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \mu_G(p) \ge N_{\tilde{f}_s}(\tilde{G}) \ge h, \ (s=1,2,\dots,S) \\ &\qquad \mu_G(p) \ge \mu_G(r) \ge h, \\ &\qquad \sum_{j=1}^{n} a_j x_j \le b, \ \boldsymbol{x} \ge \mathbf{0}, \ z_s \in \{0,1\} \end{aligned} \tag{4.43}$$

(Fuzzy Kataoka model)

$$\begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & \Pi_{\tilde{f}_s}\left(\tilde{G}\right) \geq \mu_G\left(p\right) \geq h,\ \left(s=1,2,...,S\right) \\ & \Pi_{\tilde{f}_s}\left(\tilde{G}\right) \geq \mu_G\left(r\right) \geq h,\ \left(s=1,2,...,S\right) \\ & \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \boldsymbol{0},\ z_s \in \left\{0,1\right\} \end{array} \tag{4.44}$$

(Fuzzy Telser model)

$$\begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & \mu_G\left(r\right) \geq \mu_G\left(r_G\right) \geq h, \\ & \mu_G\left(r\right) \geq N_{\tilde{f}_s}\left(\tilde{G}\right) \geq h,\ \left(s=1,2,...,S\right) \\ & \sum_{j=1}^{n} a_j x_j \leq b,\ \boldsymbol{x} \geq \boldsymbol{0},\ z_s \in \left\{0,1\right\} \end{array} \tag{4.45}$$

In these problems, each constraint is transformed into the following inequality:

$$\begin{array}{lll} \mu_G\left(p\right) \geq h & \Leftrightarrow & p \leq \left(1-h\right) p_0 + h p_1 \\ \mu_G\left(r\right) \geq h & \Leftrightarrow & r \geq \left(1-h\right) r_0 + h r_1 \\ \Pi_{\tilde{f}_s}\left(\tilde{G}\right) \geq h & \Leftrightarrow & \sup_{\omega} \min\left\{\mu_{\tilde{f}_s}\left(\omega\right), \mu_G\left(\omega\right)\right\} \geq h \\ & \Leftrightarrow & \mu_{\tilde{f}_s}\left(\omega\right) \geq h,\ \mu_G\left(\omega\right) \geq h \\ & \Leftrightarrow & \sum_{j=1}^{n} \overline{r}_{sj} x_j + R^*\left(h\right) \sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \geq g_{\tilde{f}_s}^{-1}\left(h\right) \\ N_{\tilde{f}_s}\left(\tilde{G}\right) \geq h & \Leftrightarrow & \inf_{\omega} \min\left\{1-\mu_{\tilde{f}_s}\left(\omega\right), \mu_{r_G}\left(\omega\right)\right\} \geq h \\ & \Leftrightarrow & {}^{\exists}h,\ 1-\mu_{\tilde{f}_s}\left(\omega\right) \leq h \Rightarrow \mu_{r_G}\left(\omega\right) \geq h \\ & \Leftrightarrow & \sum_{j=1}^{n} \overline{r}_{sj} x_j - L^*\left(1-h\right) \sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \geq g_{\tilde{f}_s}^{-1}\left(h\right) \end{array} \tag{4.46}$$

Furthermore, introducing a parameter $\overline{h}$ into problems (4.43), (4.44) and (4.45), we also obtain the similar inequalities with respect to $\overline{h}$ using the same manner to (4.46). From these inequalities, we transform these safety first models into the following problems:

(Fuzzy Roy model)

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \frac{1}{S}\sum_{s=1}^{S} z_s \le \left(1-\bar{h}\right) p_0 + \bar{h} p_1, \\
&\quad \sum_{j=1}^{n} \bar{r}_{sj} x_j - L^*\left(1-h\right)\sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \ge g_{f_s}^{-1}\left(h\right),\ \left(s=1,2,\ldots,S\right) \\
&\quad \sum_{j=1}^{n} \bar{r}_{sj} x_j - L^*\left(1-\bar{h}\right)\sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \le g_{f_s}^{-1}\left(\bar{h}\right),\ \left(s=1,2,\ldots,S\right) \\
&\quad \left(1-h\right) r_0 + h r_1 \le \sum_{j=1}^{n}\left(\frac{1}{S}\sum_{s=1}^{S} \bar{r}_{sj} x_j\right) - \frac{1}{6}\sum_{j=1}^{n} \alpha_j x_j + \frac{1}{6}\sum_{j=1}^{n} \beta_j x_j \le \left(1-\bar{h}\right) r_0 + \bar{h} r_1, \\
&\quad \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0},\ z_s \in \{0,1\}
\end{aligned} \tag{4.47}$$

(Fuzzy Kataoka model)

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \left(1-\bar{h}\right) p_0 + \bar{h} p_1 \le \frac{1}{S}\sum_{s=1}^{S} z_s \le \left(1-h\right) p_0 + h p_1, \\
&\quad \sum_{j=1}^{n} \bar{r}_{sj} x_j + R^*\left(\bar{h}\right)\sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \ge g_{f_s}^{-1}\left(\bar{h}\right),\ \left(s=1,2,\ldots,S\right) \\
&\quad \left(1-h\right) r_0 + h r_1 \le \sum_{j=1}^{n}\left(\frac{1}{S}\sum_{s=1}^{S} \bar{r}_{sj} x_j\right) - \frac{1}{6}\sum_{j=1}^{n} \alpha_j x_j + \frac{1}{6}\sum_{j=1}^{n} \beta_j x_j \le \left(1-\bar{h}\right) r_0 + \bar{h} r_1, \\
&\quad \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0},\ z_s \in \{0,1\}
\end{aligned} \tag{4.48}$$

(Fuzzy Telser model)

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \left(1-\bar{h}\right) p_0 + \bar{h} p_1 \le \frac{1}{S}\sum_{s=1}^{S} z_s \le \left(1-h\right) p_0 + h p_1, \\
&\quad \sum_{j=1}^{n} \bar{r}_{sj} x_j - L^*\left(1-h\right)\sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \ge g_{f_s}^{-1}\left(h\right),\ \left(s=1,2,\ldots,S\right) \\
&\quad \sum_{j=1}^{n} \bar{r}_{sj} x_j - L^*\left(1-\bar{h}\right)\sum_{j=1}^{n} \alpha_j x_j + M \cdot z_s \le g_{f_s}^{-1}\left(\bar{h}\right),\ \left(s=1,2,\ldots,S\right) \\
&\quad \left(1-h\right) r_0 + h r_1 \le \sum_{j=1}^{n}\left(\frac{1}{S}\sum_{s=1}^{S} \bar{r}_{sj} x_j\right) - \frac{1}{6}\sum_{j=1}^{n} \alpha_j x_j + \frac{1}{6}\sum_{j=1}^{n} \beta_j x_j, \\
&\quad \sum_{j=1}^{n} a_j x_j \le b,\ \boldsymbol{x} \ge \boldsymbol{0},\ z_s \in \{0,1\}
\end{aligned} \tag{4.49}$$

These problems are 0-1 mixed nonlinear programming problems, and so it is almost impossible to solve them directly. However, in the case that parameters $h$ and $\bar{h}$ is fixed such as $h = h_f$ and

$\bar{h} = \bar{h}_f$, by considering the following problems;

(Fuzzy Roy model)

$$\begin{array}{ll} \text{Maximize} & \dfrac{1}{S}\sum_{s=1}^{S} z_s \\ \text{subject to} & \sum_{j=1}^{n}\bar{r}_{sj}x_j - L^*\left(1-h_f\right)\sum_{j=1}^{n}\alpha_j x_j + M\cdot z_s \geq g_{f_s}^{-1}\left(h_f\right),\ \left(s=1,2,...,S\right) \\ & \sum_{j=1}^{n}\bar{r}_{sj}x_j - L^*\left(1-\bar{h}_f\right)\sum_{j=1}^{n}\alpha_j x_j + M\cdot z_s \leq g_{f_s}^{-1}\left(\bar{h}_f\right),\ \left(s=1,2,...,S\right) \\ & \left(1-h_f\right)r_0 + h_f r_1 \leq \sum_{j=1}^{n}\left(\dfrac{1}{S}\sum_{s=1}^{S}\bar{r}_{sj}x_j\right) - \dfrac{1}{6}\sum_{j=1}^{n}\alpha_j x_j + \dfrac{1}{6}\sum_{j=1}^{n}\beta_j x_j \leq \left(1-\bar{h}_f\right)r_0 + \bar{h}_f r_1, \\ & \sum_{j=1}^{n}a_j x_j \leq b,\ \boldsymbol{x}\geq\boldsymbol{0},\ z_s\in\{0,1\} \end{array} \tag{4.50}$$

(Fuzzy Kataoka model)

$$\begin{array}{ll} \text{Maximize} & \bar{r}_G \\ \text{subject to} & \left(1-\bar{h}_f\right)p_0 + \bar{h}_f p_1 \leq \dfrac{1}{S}\sum_{s=1}^{S} z_s \leq \left(1-h_f\right)p_0 + h_f p_1, \\ & \sum_{j=1}^{n}\bar{r}_{sj}x_j + R^*\left(\bar{h}_f\right)\sum_{j=1}^{n}\alpha_j x_j + M\cdot z_s \geq \bar{r}_G,\ \left(s=1,2,...,S\right) \\ & \left(1-h_f\right)r_0 + h_f r_1 \leq \sum_{j=1}^{n}\left(\dfrac{1}{S}\sum_{s=1}^{S}\bar{r}_{sj}x_j\right) - \dfrac{1}{6}\sum_{j=1}^{n}\alpha_j x_j + \dfrac{1}{6}\sum_{j=1}^{n}\beta_j x_j \leq \left(1-\bar{h}_f\right)r_0 + \bar{h}_f r_1, \\ & \sum_{j=1}^{n}a_j x_j \leq b,\ \boldsymbol{x}\geq\boldsymbol{0},\ z_s\in\{0,1\} \end{array} \tag{4.51}$$

(Fuzzy Telser model)

$$\begin{array}{ll} \text{Maximize} & \sum_{j=1}^{n}\left(\dfrac{1}{S}\sum_{s=1}^{S}\bar{r}_{sj}x_j\right) - \dfrac{1}{6}\sum_{j=1}^{n}\alpha_j x_j + \dfrac{1}{6}\sum_{j=1}^{n}\beta_j x_j \\ \text{subject to} & \left(1-\bar{h}_f\right)p_0 + \bar{h}_f p_1 \leq \dfrac{1}{S}\sum_{s=1}^{S} z_s \leq \left(1-h_f\right)p_0 + h_f p_1, \\ & \sum_{j=1}^{n}\bar{r}_{sj}x_j - L^*\left(1-h_f\right)\sum_{j=1}^{n}\alpha_j x_j + M\cdot z_s \geq g_{f_s}^{-1}\left(h_f\right),\ \left(s=1,2,...,S\right) \\ & \sum_{j=1}^{n}\bar{r}_{sj}x_j - L^*\left(1-\bar{h}_f\right)\sum_{j=1}^{n}\alpha_j x_j + M\cdot z_s \leq g_{f_s}^{-1}\left(\bar{h}_f\right),\ \left(s=1,2,...,S\right) \\ & \sum_{j=1}^{n}a_j x_j \leq b,\ \boldsymbol{x}\geq\boldsymbol{0},\ z_s\in\{0,1\} \end{array} \tag{4.52}$$

and evaluating whether each objective function is equal to $g_p^{-1}\left(\bar{h}_f\right) = \left(1-\bar{h}_f\right)p_0 + \bar{h}_f p_1$,

$g_{f_s}^{-1}\left(\overline{h}_f\right)$ and $g_r^{-1}\left(\overline{h}_f\right)$ using the bisection algorithm for parameter $h$, respectively, we obtain each optimal solution each global optimal solution. Consequently, the following solution method is constructed with respect to the risk-management model based on Roy model.

**Solution method**

STEP 1: Elicit the membership function of a fuzzy goal with respect to total profit, nonfulfillment probability and expected total return.

STEP 2: Set $h \leftarrow 1$ and solve problem (4.50). If the optimal objective value $Z(h)$ of the problem satisfies $Z(h) \le g_p^{-1}(h)$ and its feasible solution including constraints exists, then terminate. In this case, the obtained current solution is an optimal solution of main problem. Otherwise, go to STEP 3.

STEP 3: Set $h \leftarrow 0$ and solve problem (4.50). If the optimal objective value $Z(h)$ of the problem satisfies $Z(h) > g_p^{-1}(h)$ or the feasible solution including constrains does not exist, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for each objective function. Otherwise, go to STEP 4.

STEP 4: Set $U_h \leftarrow 1$ and $L_h \leftarrow 0$.

STEP 5: Set $h \leftarrow \dfrac{U_h + L_h}{2}$

STEP 6: Solve problem (4.50) and calculate the optimal objective value $Z(h)$ of the problem. If $Z(h) \le g_p^{-1}(h)$, then set $U_h \leftarrow h$ and return to STEP 5. If $Z(h) < g_p^{-1}(h)$, then set $L_h \leftarrow h$ and return to STEP 5. If $Z(h) = g_p^{-1}(h)$, then terminate the algorithm. In this case, $\boldsymbol{x}^*(h)$ is equal to a global optimal solution of main problem.

In a similar way to this solution method for the Roy model, the solution methods for the Kataoka model and the Telser model are constructed. Furthermore, in problems (4.50), (4.51) and (4.52), the decision maker considers that all the fuzzy numbers and fuzzy goals is not included in each problem, each problem is degenerated to basic safety first model (4.15), (4.16) or (4.17), respectively. Therefore, we find that our proposed models (4.50), (4.51) and (4.52) include many previous safety first models.

## 4.3 Numerical Example

In order to compare our proposal models with one of the most important portfolio model, mean-variance model, let us consider an example shown in Table 4.1. We assume that there are nine decision variables whose returns are assumed to be symmetric triangle fuzzy numbers involving spread $\alpha$. Then, we assumed that the number of scenarios with respect to each return is 10 and spread $\alpha$ for each return is equal among all scenarios.

Table 4.1. Numerical example in the case of nine decision variables and ten return scenarios

| | R1 | R2 | R3 | R4 | R5 | R6 | R7 | R8 | R9 |
|---|---|---|---|---|---|---|---|---|---|
| Scenario1 | 0.082 | 0.061 | 0.162 | 0.179 | 0.191 | 0.053 | 0.129 | 0.136 | 0.122 |
| Scenario2 | 0.072 | 0.062 | 0.141 | 0.163 | 0.210 | 0.062 | 0.142 | 0.121 | 0.119 |
| Scenario3 | 0.075 | 0.055 | 0.153 | 0.177 | 0.204 | 0.060 | 0.119 | 0.111 | 0.125 |
| Scenario4 | 0.070 | 0.051 | 0.149 | 0.182 | 0.205 | 0.054 | 0.128 | 0.127 | 0.113 |
| Scenario5 | 0.084 | 0.070 | 0.143 | 0.162 | 0.189 | 0.067 | 0.135 | 0.120 | 0.115 |
| Scenario6 | 0.066 | 0.064 | 0.152 | 0.159 | 0.195 | 0.049 | 0.144 | 0.118 | 0.107 |
| Scenario7 | 0.076 | 0.071 | 0.146 | 0.163 | 0.191 | 0.053 | 0.135 | 0.125 | 0.126 |
| Scenario8 | 0.072 | 0.058 | 0.140 | 0.167 | 0.202 | 0.046 | 0.118 | 0.107 | 0.119 |
| Scenario9 | 0.065 | 0.053 | 0.155 | 0.181 | 0.190 | 0.069 | 0.138 | 0.123 | 0.115 |
| Scenario10 | 0.078 | 0.072 | 0.164 | 0.169 | 0.188 | 0.064 | 0.133 | 0.111 | 0.124 |
| Spread $\alpha$ | 0.01 | 0.02 | 0.02 | 0.05 | 0.1 | 0.02 | 0.04 | 0.06 | 0.08 |

In this numerical example Table 4.1, we assume that the upper rate of purchasing volume to each asset is 0.2. In this case, we solve the following three problems; (P1) Basic mean-variance model, (P2) Fuzzy mean-variance model, (P3) Fuzz mean-absolute deviation model, (P4) Fuzzy Roy model, (P5) Fuzzy Kataoka model and (P6) Fuzzy Telser model.

Then, membership functions of fuzzy goals $\mu_G(r_G)$, $\mu_G(p)$ and $\mu_G(r)$ are assumed to be the following linear functions.

$$\mu_G(r_G)=\begin{cases}1\\ \dfrac{r_G-0.1}{0.01}\\ 0\end{cases},\quad \mu_G(p)=\begin{cases}1\\ \dfrac{0.2-p}{0.1}\\ 0\end{cases},\quad \mu_G(r)=\begin{cases}1\\ \dfrac{r-0.1}{0.01}\\ 0\end{cases}$$

Consequently, we solve these problems using standard approaches or the proposed solution method, and obtain each optimal portfolio shown in Table 4.2:

Table 4.2. Optimal solutions with respect to each problem

| Problem | R1 | R2 | R3 | R4 | R5 | R6 | R7 | R8 | R9 |
|---|---|---|---|---|---|---|---|---|---|
| P1 | 0.075 | 0.120 | 0.002 | 0.197 | 0.200 | 0.001 | 0.200 | 0.195 | 0.010 |
| P2 | 0 | 0 | 0.200 | 0.200 | 0.200 | 0 | 0.200 | 0 | 0.200 |
| P3 | 0 | 0 | 0.200 | 0.200 | 0.200 | 0 | 0.166 | 0.034 | 0.200 |
| P4 | 0.135 | 0 | 0 | 0.200 | 0.179 | 0 | 0.086 | 0.200 | 0.200 |
| P5 | 0.178 | 0.200 | 0 | 0.200 | 0.200 | 0 | 0.200 | 0.022 | 0 |
| P6 | 0.085 | 0 | 0 | 0.200 | 0.200 | 0 | 0.115 | 0.200 | 0.200 |

From these optimal portfolios in this numerical example, we find that assets involving the higher future returns tend to be selected in the case that we also consider the possibility of that total future return. Then, with respect to assets R7 and R8, the expected return of R7 is lager than that of R8, but spread $\alpha_8$ derived from the ambiguity is larger than spread $\alpha_7$. Possibility maximization model considers the case that decision makers try to earn the total profit as much as possible and it is possible that the return of asset R8 is larger than that of asset R7. Therefore, we find that asset R8 is selected in possibility maximization model P3.

Furthermore, we find that the optimal portfolio of fuzzy Roy model is similar to that of fuzzy Telser model in this case. On the other hand, we also find that the optimal portfolio of fuzzy Kataoka model is similar to that of basic Mean-variance model. In this chapter, we deal with necessity measure with respect to Roy model and Telser model, and possibility measure with respect to Kataoka model. Therefore, we consider that this tendency appears. Thus, by introducing fuzziness in general safety first models of portfolio selection problems, portfolios considering some trends such as positive
or negative are constructed. Then, our proposed model can be appropriately applied in response to the tendency of each investor.

## 4.4 Conclusion

In this chapter, we have proposed the fuzzy extension to several standard approaches for the large-scale portfolio selection problems based on multi-scenario model, and considered the multi-scenario fuzzy portfolio selection problem maximizing the total future profit. First, we have considered mean-variance and mean-absolute deviation models based on the compact factorization. In order to solve them more efficiently, we have transformed the original problems into deterministic equivalent problems and constructed their efficient solution method. From these final deterministic problems, we found that these problems were equivalent to the original large-scale models in the sense of the mathematical programming. Next, we have proposed fuzzy extension models with respect to safety first models for portfolio selection problems based on the multi-scenario, and considered models maximizing the aspiration level for nonfulfillment probability, total profit and expected total return, respectively. Furthermore, in order to solve them more efficiently, we have transformed main problems into the deterministic equivalent 0-1 mixed linear programming problems, which are equal to the original problems in the sense of the mathematical programming and analytically solved using standard integer programming approaches. We may be able to apply the proposed solution method to the cases including not only fuzziness but also both randomness and fuzziness which are called to fuzzy random variable or random fuzzy variable.

As future studies, we will consider general asset allocation problems in the sense of risk management problems under these uncertainty conditions. In general, portfolio selection problems consider only budget constraint. However, the asset allocation problems consider various types of constraints such as human, resource, facility, delivery time, etc.. Then, it is clear that such constraints of asset allocation problems include much uncertainty than portfolio selection problems. Therefore, we will extend the proposed approaches in this chapter to the general asset allocation problems, and construct versatile risk management models.

# Chapter 5

# Multi-Scenario and Robust Models for Portfolio Selection Problems

In traditional portfolio selection problems, the authors focused on only an expected return or its variance, and maximizing the total profit or minimizing the total variance are considered separately, i.e. if one is considered as a main object, the other is a constraint. However, since the future return is treated as an original random variable, and both expected returns and variances are important factors in portfolio selection problems, we need to consider the model including both expected return and variance to the objective function, simultaneously. Furthermore, in case that investors predict future returns envisaging various future situations such as a substantial fall or rise in stock prices, they usually assume not just one but several scenarios and expect a portfolio decision satisfying goals with respect to all the scenarios. Hence, we need to consider not only randomness but also multi-scenarios for future returns.

The proposed problem is initially formulated as a multiobjective programming problem. The multiobjective programming problem is one of the important mathematical programming problems, and many researchers have proposed the solution method such as the weight-scalarization method, the norm minimization method and the interactive method (for example, Miettinen [91], Sakawa [105], Sawaragi et al. [106]). Furthermore, the proposed models are also formulated as multiobjective stochastic or goal programming problems due to including random variables. In the mathematical programming approach to portfolio selection problems, there are some basic studies using a stochastic programming approach (a recent one. Aouni et al. [5]), a goal programming approach (Kumar et al. [71], Lee and Chesser [75], Levary and Avery [79]), a multi-criteria linear goal approach (Abdelaziz et al. [2, 3, 4], Ziemba and Mulvey [125]) and a multiobjective mixed fuzzy-stochastic programming approach (Mohan and Nguyen [92]). Furthermore, considering ambiguous situations such as receiving effective or ineffective information from the real world, there are some portfolio selection problems to treat ambiguous factors as fuzzy sets (Vajda [115], Watada [119], Tanaka et al. [1111, 112], Inuiguchi and Ramik [52], Leon et al. [77]), or to have both randomness and fuzziness as fuzzy random variables (Katagiri et al. [60, 61]). More recently, Abdelaziz et al. [1] proposed multi-objective stochastic programming for portfolio selection. Pflug and Wozabal [100] discussed an approach that explicitly considered the ambiguity such as uncertainty with respect to the possible probability model. However, there are few researches who consider multi-scenario portfolio selection models with possible probability scenarios including their aspiration levels of satisfaction functions to all scenarios and also solve them analytically.

On the other hand, from the standpoint of reducing the uncertainty and keeping the robustness of the selected portfolio, it is important that we consider the probability maximization model that the total future return is more than or equal to a goal set by the decision maker. Therefore, we propose several types of probability maximization models for multi-scenario portfolio selection problems and consider the robustness of the appropriate portfolio and their aspiration levels. Since the proposed problems including randomness are usually transformed into nonlinear programming problems, it is difficult to find a global optimal solution efficiently. Furthermore, since our proposed models are multi-criteria stochastic programming problems, it is almost impossible to solve them directly. Therefore, in addition to new types of portfolio selection problems, we manage to construct efficient solution methods for them using the equivalent transformations to the main problem based on the properties of random variable and satisfaction function.

## 5.1 Probability Fractile Maximization Model

First, we formulate the basic mathematical programming problem for an asset allocation maximizing the total future profit as follows.

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} r_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ \left(j = 1, 2, \cdots, n\right) \end{aligned} \tag{5.1}$$

where the notations of parameters and variables are as follows:

$r_j$ : Future return of the $j$ th financial asset

$a_j$ : Cost of investing the $j$ th financial asset

$b$ : Limited upper value with respect to fund budgeting

$b_j$ : Limited upper value of each budgeting to the $j$ th financial asset

$n$ : Total number of assets

$x_j$ : Budgeting allocation to the $j$ th financial asset

This problem is a linear programming problem in the case that all parameters are constant. Therefore, we find an optimal portfolio efficiently by using linear programming approaches. However, since it is difficult for decision makers to predict future returns as a constant value due to uncertainties in the real world, they need to consider them as random variables. Furthermore, in the case that they predict future returns, they often need to consider the following situations:

(a) In economic conditions, there are various trends such as buoyant, depressed, moderate, wilds ups and downs, etc.. Then, we obtain some scenarios of future returns based on historical data derived from each condition. These future returns occur according to different random distributions every economic condition. Therefore, we need to consider setting some scenarios to future returns with random variables.

(b) In the real world, there are many veteran investors and economists. Recently, by developing the information technology rapidly, it is easy to receive a lot of information from them. Then, in the case that we classify them by the investment stance, we obtain some classifications such as optimistic, pessimistic, neutral, etc.. Furthermore, in each group, every investor has her or his own prediction to future returns. By performing statistical analysis of data derived from all investors, we obtain random variables to future returns in each group. Therefore, in the case that we consider there groups as scenarios to future returns with random variables.

Therefore, decision makers usually assume not only one scenario but also several possible scenarios for future returns. In order to represent these situations, we assume the scenarios for future returns to be the following multivariate random vector $\boldsymbol{r}_i$ with the variance-covariance matrix $\mathbf{V}_i$ :

$$\boldsymbol{r} = \begin{cases} \boldsymbol{r}_1 = \{r_{11}, r_{11}, ..., r_{1n}\} \\ \boldsymbol{r}_2 = \{r_{21}, r_{22}, ..., r_{2n}\} \\ \vdots \\ \boldsymbol{r}_m = \{r_{m1}, r_{m2}, ..., r_{mn}\} \end{cases} \tag{5.2}$$

where the notations of parameters and variables are as follows:

$r_{ij}$ : A future return of the *j*th asset under the *i*th scenario

$m$ : Total number of scenarios

This multiple scenarios for the future return may correspond to the ambiguity in probability distribution concerning the future return (for example, Bewley [12]). Subsequently, we assume that each $r_{ij}$ occurs according to a normal distribution $N\left(\overline{r}_{ij}, \sigma_{ij}^2\right)$, where $\overline{r}_{ij}$ is the mean value and $\sigma_{ij}^2$ is the variance. In many previous studies concerning portfolio selection problems, each future return is generally considered as a random variable distributed according to the normal distribution. Recently, there are some researches of portfolio selection problems with non-normal distributions. However, from the standpoint of the study of portfolio models based on the modern portfolio theory, we deal with the normal distribution $N\left(\overline{r}_{ij}, \sigma_{ij}^2\right)$. Under these assumptions, we formulate a portfolio selection problem under this multi-scenario as follows.

$$\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} r_{1j} x_j \\
&\text{Maximize} \quad \sum_{j=1}^{n} r_{2j} x_j \\
&\quad \vdots \\
&\text{Maximize} \quad \sum_{j=1}^{n} r_{mj} x_j \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ \left(j = 1, 2, \cdots, n\right)
\end{aligned} \tag{5.3}$$

This problem is a multi-objective programming problem with random variables. Therefore, it is not a well-defined problem due to random variables, and so in the sense of the deterministic mathematical programming, it is almost impossible to solve problem (5.3) directly without setting a criterion for each object with random variables. In this chapter, since we consider the case where a decision maker earns the maximum total profit even if each future return changes randomly, we deal with the probability of the total profit more than or equal to its target value *f*, i.e. the safety first model as a risk measure. Then, we consider two cases; (a) the case that a decision maker decides each weight to scenarios considering each occurrence probability and possibility of scenarios, (b) the case maximizing the minimum aspiration level among all the scenarios.

## 5.2 Portfolio Selection Problem Using the Weighted-Scalarization Approach

In practical investments, many investors need to consider randomness of future returns and the occurrence probabilities and possibilities of scenarios to future returns, simultaneously. In this section, we consider the case in which a decision maker sets a weight to each scenario based on statistical analysis of historical data and her or his subjectivity, and aggregate all objective function into one weighted function. Then, we assume each weight $w_i$ to be a positive weight value to scenario *i*th scenario, and formulate this model as the following form.

$$\begin{aligned}
&\text{Maximize} \quad \sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ \left(j = 1, 2, \cdots, n\right)
\end{aligned} \tag{5.4}$$

**Case 1.**

In the case that a weight value $w_{i'} = 1$ and the other weight $w_i = 0,\ \left(i = 1, 2, ..., m,\ i \ne i'\right)$, this problem is transformed into the probability maximization model considering particular scenario $i'$. Therefore, problem (5.4) includes the model considering one particular scenario.

**Case 2.**

With respect to $w_i$, in the case that a decision maker considers the weight value based on the occurrence probability of each scenario which is assumed to be $p_i$, $w_i$ is assumed to be $p_i$.

Problem (5.4) is a single-criterion programming problem. Then, if each future return $r_{ij}$ is assumed to be a fixed value, this problem is a linear programming problem. However, since $r_{ij}$ is usually not fixed and considered to be a random variable, problem (5.4) is not a well-defined problem. Therefore, we need to transform it into the other form in order to solve problem (5.4) analytically. In this chapter, for introducing a chance constraint to objective function, we consider its probability fractile optimization model. This model is formulated as follows.

$$\begin{aligned} &\text{Maximize} \quad f \\ &\text{subject to} \quad \Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \geq f\right\} \geq \beta, \\ &\qquad\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad (j = 1, 2, ..., n) \end{aligned} \tag{5.5}$$

where $f$ is the target value of objective function and $\beta$ is the probability fractile level. In this problem, since each $r_{ij}$ occurs according to the normal distribution $N\left(\bar{r}_{ij}, \sigma_{ij}^2\right)$, the chance constraint $\Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \geq f\right\} \geq \beta$ is transformed into the following deterministic equivalent inequality using the property of the normal distribution:

$$\begin{aligned} \Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \geq f\right\} \geq \beta \;&\Leftrightarrow\; \Pr\left\{\frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \geq \frac{f - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}}\right\} \geq \beta \\ &\Leftrightarrow\; \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \geq K_\beta \\ &\Leftrightarrow\; \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j - K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})} \geq f \end{aligned} \tag{5.6}$$

where $\mathbf{V}_i$ is the variance covariance matrix for future returns in the $i$ th scenario, and $F(y)$ is the distribution function of the standard normal distribution and $K_\beta = F^{-1}(\beta)$. From this transformation, we equivalently transform problem (5.5) into the following form.

$$\begin{aligned} &\text{Maximize} \quad f \\ &\text{subject to} \quad \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})} \geq f, \\ &\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad (j = 1, 2, ..., n) \end{aligned} \tag{5.7}$$

Furthermore, we find that the decision variable $f$ is involved only in first constraint, and maximizing $f$ is equivalent to maximizing $\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}$. Therefore, problem (5.7) is equivalently transformed as follows.

$$\begin{aligned} &\text{Maximize} \quad \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})} \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad (j = 1, 2, ..., n) \\ \Leftrightarrow \quad &\text{Minimize} \quad -\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})} \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad (j = 1, 2, ..., n) \end{aligned} \tag{5.8}$$

In this chapter, we consider $\beta \geq \frac{1}{2}$ due to the following assumptions:

(a) In the practical decision making, almost all decision makers do not select a portfolio whose achievement probability for the goal of total return is less than half.

(b) In mathematical programming, $-\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_\beta \sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}$ is a convex function in the case that $\beta > \frac{1}{2}$.

Since all the constraints in problem (5.8) are linear constraints, problem (5.8) is a convex programming problem, and we find a global optimal solution using convex programming approaches such as the gradient method. However, since this problem includes square root terms, it is difficult to solve problem (5.8) efficiently using general solvers.

Subsequently, we consider the following auxiliary problem introducing a parameter $R$:

$$\begin{array}{ll} \text{Minimize} & -R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j + \frac{K_\beta}{2}\left(\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)\right) \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le b_j \quad \left(j = 1, 2, \ldots, n\right) \end{array} \tag{5.9}$$

Previous researches (For example, Ishii and Nishida [54] and Katagiri et al. [60, 61]) to analytically solve problem (5.9) have considered that each random variable was assumed to be independent each other in order to solve it analytically, i.e., the element $\sigma_{ij}$ of covariance matrix $\mathbf{V}_i$ is assumed to be $\sigma_{ij} = 0, \ \left(i \ne j\right)$. However, in practical decision making, there exist many cases that a decision maker considers the relation among all decision variables. Therefore, in this chapter, we extend these previous analytical approaches to the more standard analytical approach.

First, with respect to a relation between this auxiliary problem and main problem (5.8), the following theorem holds.

**Theorem 5.1**

Let $\boldsymbol{x}^*$ be an optimal solution of problem (5.9). If $R = \sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}$ is satisfied, $\boldsymbol{x}^*$ is also an optimal solution of main problem (5.8).

**Proof.**

We comparing Karush-Kuhn-Tucker (KKT) condition of problem (5.9) with that of problem (5.8):

$$\begin{array}{l} -\sum_{i=1}^{m} w_i \overline{r}_{ij} + K_\beta \dfrac{\sum_{i=1}^{m} \left(\mathbf{V}_i\right)_j \left(w_i \boldsymbol{x}^*\right)}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}} + \lambda a_j + u_j - v_j = 0 \\ \lambda\left(\sum_{j=1}^{n} a_j x_j - b\right) = 0, \ u_j\left(x_j - b_j\right) = 0, \ v_j x_j = 0, \ \left(j = 1, 2, \ldots, n\right) \end{array} \tag{5.10}$$

$$\begin{array}{l} -R_1 \sum_{i=2}^{m} w_i \overline{r}_{ij} + K_\beta \sum_{i=1}^{m} \left(\mathbf{V}_i\right)_j \left(w_i \boldsymbol{x}^*\right) + \lambda' a_j + u'_j - v'_j = 0, \\ \lambda'\left(\sum_{j=1}^{n} a_j x_j - b\right) = 0, \ u'_j\left(x_j - b_j\right) = 0, \ v'_j x_j = 0, \ \left(j = 1, 2, \ldots, n\right) \end{array} \tag{5.11}$$

With respect to these KKT conditions, we set

$$\lambda = \frac{\lambda'}{\sqrt{\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}},\ u_j = \frac{u'_j}{\sqrt{\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}},\ v_j = \frac{v'_j}{\sqrt{\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}}.$$

Then, satisfying $R = \sqrt{\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}$, KKT condition of problem (5.9) is same as that of problem (5.10). Therefore, this theorem holds. □

Furthermore, let $g(R) = R - \sqrt{\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^*\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^*\right)}$, and then the following theorem holds.

**Theorem 5.2**

Let the optimal solution to auxiliary problem be $\boldsymbol{x}^*$ and the optimal value of parameter $R$ be $R^*$. With respect to $R^*$, the following relationship holds.

$$\begin{aligned} R^* > R \quad &\Leftrightarrow \quad g(R) > 0 \\ R^* = R \quad &\Leftrightarrow \quad g(R) = 0 \\ R^* < R \quad &\Leftrightarrow \quad g(R) < 0 \end{aligned}$$

**Proof.**

First, we show the following two lemmas.

**Lemma 5.2.1**

With respect to $R$, $-\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j$ is a decreasing function of $R$.

**Proof.**

We assume $R' < R$. If $\boldsymbol{x}^R$ is assumed to be an optimal solution of the problem (5.9), it is unique from the convexity and so the following inequality is derived.

$$\begin{aligned} &-R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) < -R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'} + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) \\ \Leftrightarrow\ &-R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) + R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'} - \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) < 0 \end{aligned}$$

Similarly, we assume $\boldsymbol{x}^{R'}$ is an optimal solution in the case that $R = R'$ of the problem (5.9). Then we derived the following inequality.

$$-R'\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) + R'\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'} - \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) > 0$$

Therefore, from the deference of these two inequalities, the following inequality holds:

$$\left(R - R'\right)\left(\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'}\right) > 0$$

Here, for $R - R' > 0$, it holds that

$$\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'} > 0 \quad \Leftrightarrow \quad -\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R < -\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'}$$

Therefore, the monotonous decreasing is derived. □

**Lemma 5.2.2**

$\sum_{i=1}^{m}\left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)$ is an increasing function of $R$.

**Proof.**

We assume $R' < R$. $\boldsymbol{x}^{R'}$ is assumed to be the optimal solution in the case that $R = R'$ of the problem (5.9), and so we find the following inequality.

$$-R'\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) + R'\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'} - \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) > 0$$

$$\Leftrightarrow \quad R'\left(-\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'}\right) + \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) - \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) > 0$$

Subsequently, from Lemma 5.2.1, $R'\left(-\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^R + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^{R'}\right) < 0$ and the following inequality is derived:

$$\frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^R\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^R\right)\right) - \frac{K_\beta}{2}\left(\sum_{i=1}^{m}\left(w_i \boldsymbol{x}^{R'}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}^{R'}\right)\right) > 0$$

Then, for $K_\beta > 0$, we also derive the following inequality.

$$\left(\sum_{i=1}^{m}\left(w_i\boldsymbol{x}^R\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}^R\right)\right)-\left(\sum_{i=1}^{m}\left(w_i\boldsymbol{x}^{R'}\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}^{R'}\right)\right)>0$$
$$\Leftrightarrow\quad\left(\sum_{i=1}^{m}\left(w_i\boldsymbol{x}^R\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}^R\right)\right)>\left(\sum_{i=1}^{m}\left(w_i\boldsymbol{x}^{R'}\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}^{R'}\right)\right)$$

Therefore, Lemma 5.2.2 holds. □

Furthermore, since the feasible region of problem (5.9) is same as that of problem (5.8) and it is a bounded region, it holds that $-\sum_{i=1}^{m}w_i\sum_{j=1}^{n}\bar{r}_{ij}x_j>-\infty$ and $\sum_{i=1}^{m}\left(w_i\boldsymbol{x}\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}\right)<\infty$ with respect to the optimal solution $\boldsymbol{x}^R$ of problem (5.9) for a fixed parameter $R$. $\boldsymbol{x}^R$ is a continuous function to $R$, and so $\sum_{i=1}^{m}\left(w_i\boldsymbol{x}^R\right)^t\mathbf{V}_i\left(w_i\boldsymbol{x}^R\right)$ is also a continuous function to $R$.

Consequently, form Lemmas 5.2.1 and 5.2.2, mean value theorem and uniqueness of $\boldsymbol{x}$, Theorem 5.2 is derived. □

With respect to problem (5.9), we set $\mathbf{V}=\sum_{i=1}^{m}w_i^2\mathbf{V}_i$. Since each $\mathbf{V}_i$ is a covariance matrix and each $w_i$ is a positive value, it is obvious that $\mathbf{V}$ is a symmetric positive definite matrix. Then, problem (5.9) is transformed into the following quadratic programming problem:

$$\begin{aligned}&\text{Minimize}\quad -R\sum_{i=1}^{m}w_i\sum_{j=1}^{n}\bar{r}_{ij}x_j+\frac{K_\beta}{2}\left(\boldsymbol{x}^t\mathbf{V}\boldsymbol{x}\right)\\&\text{subject to}\quad \sum_{j=1}^{n}a_jx_j\le b,\quad 0\le x_j\le b_j\quad\left(j=1,2,...,n\right)\end{aligned}\tag{5.12}$$

Furthermore, for simplicity, we perform the following transformations of variables:

$$\begin{aligned}&K_\beta\boldsymbol{x}^t\mathbf{V}\boldsymbol{x}=\boldsymbol{y}^t\boldsymbol{y}\\&\mathbf{V}=Q'\Lambda Q,\ Q:\text{eigen vector of }\mathbf{V},\\&\sqrt{\Lambda}=\begin{pmatrix}\sqrt{\lambda_1}&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&\sqrt{\lambda_n}\end{pmatrix},\ \lambda_j:\text{eigen value of }\mathbf{V},\\&\boldsymbol{y}=\sqrt{K_\beta}\sqrt{\Lambda}Q\boldsymbol{x},\ \bar{\boldsymbol{r}}'=\frac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1}Q\bar{\boldsymbol{r}},\\&\boldsymbol{a}'=\frac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1}Q\boldsymbol{a},\ b'=\frac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1}Qb,\ \mathbf{b}'=\left(\sqrt{\Lambda}\right)^{-1}Q'\mathbf{b}\end{aligned}\tag{5.13}$$

Then, we reset $r' \to r,\ a'_j \to a_j,\ b' \to b,\ b'_j \to b_j$. From these variable transformations, problem (5.12) is equivalently transformed into the following form:

$$\begin{array}{ll} \text{Minimize} & -R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} y_j + \frac{1}{2}\left(\boldsymbol{y}^t \boldsymbol{y}\right) \\ \text{subject to} & \sum_{j=1}^{n} a_j y_j \le b, \quad 0 \le y_j \le b_j \quad \left(j = 1, 2, \ldots, n\right) \end{array} \tag{5.14}$$

Therefore, from Lagrange function and Karush-Kuhn-Tucker (KKT) condition of problem (5.14),
(Lagrange function)

$$L = -R\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} y_j + \frac{1}{2}\sum_{j=1}^{n}\left(\boldsymbol{y}^t \boldsymbol{y}\right) + \lambda\left(\sum_{j=1}^{n} a_j y_j - b\right) + \sum_{j=1}^{n} u_j \left(y_j - b_j\right) - \sum_{j=1}^{n} v_j y_j \tag{5.15}$$

(KKT condition)

$$\begin{array}{l} \dfrac{\partial L}{\partial y_j} = -R\sum_{i=1}^{m} w_i \overline{r}_{ij} + y_j + a_j\lambda + u_j - v_j = 0 \\ \lambda\left(\sum_{j=1}^{n} a_j y_j - b\right) = 0,\ u_j\left(y_j - b_j\right) = 0,\ v_j y_j = 0,\ \left(j = 1, \cdots, n\right) \end{array} \tag{5.16}$$

From KKT condition (5.16), we find an optimal solution of problem (5.14) as follows:

$$y_j^* = \begin{cases} b_j & \left(\lambda \le R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j \Big/ a_j\right) \\ R\sum_{i=1}^{m} w_i \overline{r}_{ij} - a_j \lambda & \left(R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j \Big/ a_j \le \lambda \le R\sum_{i=1}^{m} w_i \overline{r}_{ij} \Big/ a_j\right),\ \left(j = 1, \cdots, n\right) \\ 0 & \left(R\sum_{i=1}^{m} w_i \overline{r}_{ij} \Big/ a_j \le \lambda\right) \end{cases} \tag{5.17}$$

In this optimal solution, if we properly determine the parameter $\lambda$ , we obtain a strict optimal solution. Therefore, we consider the range including $\lambda$ with respect to $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij}}{a_j}$ and $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j}{a_j}$, $\left(j = 1, 2, \ldots, n\right)$. In this case, since $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij}}{a_j}$ and $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j}{a_j}$ are fixed if $R$ is fixed, we settle the ordering of them. Then, we arrange them in a nondecreasing order and let the result be as follows:

$$\omega_1 < \omega_2 < \cdots < \omega_{m(R)}$$

where each $\omega_l$ corresponding to either $\dfrac{\sum_{i=1}^{m} w_i \overline{r}_{ij}}{a_j}$ or $\dfrac{\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j}{a_j}$, and $m(R)$ is the number of $\omega$ taking different values. Therefore, we obtain $m(R) \leq 2n$. Furthermore, let $R_{kj}$ denote $R$ such that $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ik}}{a_k} = \dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j}{a_j}$, $(k \neq j)$ or $\dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ik} - b_k}{a_k} = \dfrac{R\sum_{i=1}^{m} w_i \overline{r}_{ij} - b_j}{a_j}$, $(k \neq j)$, and we arrange them in the following nondecreasing order:

$$R_0 \triangleq 0 < R_1 < R_2 < \cdots < R_N < R_{N+1} \triangleq M$$

where $M$ is a sufficient large number. Since $N$ is the number of their different values of $R_{kj}$, we obtain $N \leq \frac{1}{2} n(3n+1)$. Furthermore, for considering optimal solution (5.17) in the case of $\lambda' = R_l$ and the value of $\sum_{j=1}^{n} a_j y_j - b$ in the sequence, we determine the optimal range of $\lambda'$, and since $\boldsymbol{y} = \sqrt{K_\beta}\sqrt{\Lambda} Q\boldsymbol{x}$, we obtain the optimal solution $\boldsymbol{x} = \dfrac{1}{\sqrt{K_\beta}}\left(\sqrt{\Lambda}\right)^{-1} Q\boldsymbol{y}$.

Consequently, we develop the following solution algorithm.

**Solution Algorithm 5.1**

STEP1: Set the variables and introduce problem (5.14).

STEP2: Set $k \leftarrow 0$.

STEP3: Set $S \leftarrow (R_k, R_{k+1}]$.

STEP4: Set $l \leftarrow 1$.

STEP5: For $R \in S$ and $\lambda \in [\omega_l, \omega_{l+1}]$, find the optimal solution $y_j^*$ of problem (5.14), solve

$\sum_{j=1}^{n} a_j y_j^* = b$ with respect to $\lambda$. Let this solution $\lambda_l^{(k)}$.

STEP6: Solve $\omega_l \leq \lambda_l^{(k)} \leq \omega_{l+1}$ with respect to $R$ and find its solution set $S_l^{(k)}$.

STEP7: If $S_l^{(k)} = \phi$ and $l \neq m_{(R)}$, set $l \leftarrow l+1$ and return to STEP5. If $S_l^{(k)} = \phi$ and

$l = m_{(R)}$, set $k \leftarrow k+1$ and return to STEP3. If $S_l \neq \phi$, go to STEP8.

STEP8: Find $R = \hat{R}$ such that $\hat{R} = \sqrt{\sum_{j=1}^{n} \left(y_j^*\right)^2}$. If $\hat{R} \notin S$, set $l \leftarrow l+1$ and return to STEP5.

If $\hat{R} \in S$, $y_j^*$ is the optimal solution and terminate this algorithm.

Consequently, we have extended the previous models and solution method to obtain the optimal portfolio to the model considering not only each variance but also the covariance. Thereby, the proposed model is more versatile and practical than some previous models.

## 5.3 Maximization of the Minimum Aspiration Level of Objective Values among All Scenarios

In Section 5.2, all scenarios have been weighted using occurrence probabilities and the decision maker's subjectivity. However, there often exists the case that all scenarios are treated equally due to the lack of enough and reliable information and uncertain economic conditions, i.e. all the occurrence probabilities are assumed to have the same values and the decision maker makes the same assessments with respect to all the scenarios. In this case, she or he often sets the goal of the total future profit to each scenario and consider its aspiration level to achievement of the goal. Furthermore, considering the vagueness of human judgment and flexibility for the execution of a plan, each aspiration level is assumed to be a satisfaction function. Then, investors need to consider the condition that they satisfy the aspiration level more than a goal even if any scenarios occur.

Therefore, in this section, we consider the portfolio selection decision that maximizes the minimum aspiration level of objective value among all objectives in portfolio selection problems. First, we introduce the satisfaction function $\mu_i\left(Z_i\right)$ to each $Z_i = \sum_{j=1}^{n} r_{ij} x_j$ as follows:

$$\mu_i\left(Z_i\right) = \begin{cases} 1 & f_{i1} \leq Z_i \\ g_i\left(Z_i\right) & f_{i0} \leq Z_i \leq f_{i1}, \\ 0 & Z_i \leq f_{i0} \end{cases} \quad \left(i = 1, 2, \cdots, m\right) \tag{5.18}$$

where each $g_i\left(Z_i\right)$ is a monotonous increasing function. This satisfaction function means that a decision maker is entirely agreeable with respect to the portfolio satisfying objective function $Z_i$ is more than the target value $f_{i1}$, but is never agreeable to the portfolio satisfying $Z_i$ is less than

$f_{i0}$. Then, it also means that she or he is partially agreeable depending on her or his psychological aspect and subjectivity if $Z_i$ is between $f_{i0}$ and $f_{i1}$. For example, in the case that the decision maker is risk-averse, $g_i(Z_i)$ may be a concave function. Particularly, in this chapter, we consider the case that $g_i(Z_i)$ is assumed to be a linear function, i.e. $g_i(Z_i)=\frac{Z_i - f_{i0}}{f_{i1}-f_{i0}}$. Using these satisfaction functions, we formulate this model as the following max-min programming problem:

$$\begin{aligned} &\text{Maximize} \quad \min_i \left[\mu_i(Z_i),\ (i=1,2,\cdots,m)\right] \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.19}$$

This type of problem is generally called the robust control in economic and finance literature (for example, Gilboa and Scemedler [36]). Then, it is possible that this problem is transformed into the following form introducing a parameter $h$:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \mu_i(Z_i) \ge h,\ (i=1,2,\cdots,m) \\ &\qquad\qquad\quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.20}$$

In this problem, the constraint $\mu_i(Z_i) \ge h$ is transformed into the following inequality:

$$\begin{aligned} \mu_i(Z_i) \ge h \quad &\Leftrightarrow \quad \frac{\sum_{j=1}^{n} r_{ij} x_j - f_{i0}}{f_{i1} - f_{i0}} \ge h \\ &\Leftrightarrow \quad \sum_{j=1}^{n} r_{ij} x_j \ge (f_{i1} - f_{i0})h + f_{i0},\ (i=1,2,...,m) \end{aligned}$$

Therefore, problem (5.20) is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \sum_{j=1}^{n} r_{ij} x_j \ge (f_{i1} - f_{i0})h + f_{i0},\ (i=1,2,\cdots,m) \\ &\qquad\qquad\quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.21}$$

However, since each $\sum_{j=1}^{n} r_{ij} x_j$ includes random variables $r_{ij}$, problem (5.21) is not a well-defined.

Therefore, in a way similar to Section 3, we introduce chance constraints as follows:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \Pr\left\{ \sum_{j=1}^{n} r_{ij} x_j \geq \left( f_{i1} - f_{i0} \right) h + f_{i0} \right\} \geq \beta, \ \left( i = 1, 2, \cdots, m \right) \\ &\qquad \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ \left( j = 1, 2, \cdots, n \right) \end{aligned} \tag{5.22}$$

Then, since each stochastic constraint is transformed into the following inequality using the transformation (5.6);

$$\Pr\left\{ \sum_{j=1}^{n} r_{ij} x_j \geq \left( f_{i1} - f_{i0} \right) h + f_{i0} \right\} \geq \beta \quad \Leftrightarrow \quad \sum_{j=1}^{n} \overline{r}_{ij} x_j - K_{\beta} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}} \geq \left( f_{i1} - f_{i0} \right) h + f_{i0}$$

problem (5.22) is transformed into the following deterministic equivalent problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \sum_{j=1}^{n} \overline{r}_{ij} x_j - K_{\beta} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}} \geq \left( f_{i1} - f_{i0} \right) h + f_{i0}, \ \left( i = 1, 2, \cdots, m \right) \\ &\qquad \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ \left( j = 1, 2, \cdots, n \right) \end{aligned} \tag{5.23}$$

$$\begin{aligned} &\text{Maximize} \quad h \\ i.e. \quad &\text{subject to} \quad \left( f_{i1} - f_{i0} \right) h + f_{i0} - \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_{\beta} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}} \leq 0, \ \left( i = 1, 2, \cdots, m \right) \\ &\qquad \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ \left( j = 1, 2, \cdots, n \right) \end{aligned} \tag{5.24}$$

Since each $f_i\left( \boldsymbol{x}, h \right) = \left( f_{i1} - f_{i0} \right) h + f_{i0} - \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_{\beta} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}}$ is a convex function, this problem is a convex programming problem. However, each constraint includes a square root term, and so it is hard to solve this problem using KKT conditions analytically due to the multiple square root terms. Therefore, we construct an efficient solution method using the following mean-absolute deviation.

$$W\left[ R_i\left( \boldsymbol{x} \right) \right] = E\left[ \left| \sum_{j=1}^{n} r_{ij} x_j - \sum_{j=1}^{n} \overline{r}_{ij} x_j \right| \right] \tag{5.25}$$

With respect to the relation between this mean-absolute deviation $W\left[ R_i\left( \boldsymbol{x} \right) \right]$ and the variance $\sigma_i^2\left( \boldsymbol{x} \right) = \boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}$, the following theorem holds based on the result obtained by Konno [71].

**Theorem 5.3**

In the case that each return occurs according to the normal distribution,

$$\sigma_i^2(\boldsymbol{x}) = \frac{\pi}{2}\left(W\left[R_i(\boldsymbol{x})\right]\right)^2$$

holds.

From this theorem, problem (5.24) is transformed into the following problem:

$$\begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & (f_{i1} - f_{i0})h + f_{i0} - \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_\beta \sqrt{\frac{\pi}{2}\left(W\left[R_i(\boldsymbol{x})\right]\right)^2} \leq 0, \ (i = 1, 2, \cdots, m) \\ & \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ (j = 1, 2, \cdots, n) \end{array} \tag{5.26}$$

$$i.e. \quad \begin{array}{ll} \text{Maximize} & h \\ \text{subject to} & (f_{i1} - f_{i0})h + f_{i0} - \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_\beta \sqrt{\frac{\pi}{2}}\left(W\left[R_i(\boldsymbol{x})\right]\right) \leq 0, \ (i = 1, 2, \cdots, m) \\ & \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ (j = 1, 2, \cdots, n) \end{array} \tag{5.27}$$

Furthermore, in order to solve problem (5.27) more efficiently, we introduce the following return set derived from each normal distribution $r_{ij}$:

$$\boldsymbol{r}_i^{(t)} = \left\{\left(r_{i1}^{(t)}, r_{i2}^{(t)}, ..., r_{in}^{(t)}\right), \ (t = 1, 2, ..., T)\right\}, \ (i = 1, 2, ..., m) \tag{5.28}$$

where $T$ is the total number of return sets and the occurrence probability $p_i^{(t)}$ is defined as the following form:

$$p_i^{(t)} = \Pr\left\{\boldsymbol{r}_i^{(t)} = \left(r_{i1}^{(t)}, r_{i2}^{(t)}, ..., r_{in}^{(t)}\right)\right\}, \ (t = 1, 2, ..., T) \tag{5.29}$$

From this return set, in the case that $T$ is sufficiently large, we equivalently transform $W\left[R_i(\boldsymbol{x})\right]$ into

$$\begin{aligned} W\left[R_i(\boldsymbol{x})\right] &= E\left[\left|\sum_{j=1}^{n} r_{ij} x_j - \sum_{j=1}^{n} \overline{r}_{ij} x_j\right|\right] \\ &= E\left[\left|\sum_{j=1}^{n} \left(r_{ij} - \overline{r}_{ij}\right) x_j\right|\right] = \sum_{t=1}^{T} p_i^{(t)} \left|\sum_{j=1}^{n} \left(r_{ij}^{(t)} - \overline{r}_{ij}\right) x_j\right|, \ (i = 1, 2, ..., m) \end{aligned} \tag{5.30}$$

Therefore, problem (5.27) is transformed into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \left(f_{i1}-f_{i0}\right)h+f_{i0}-\sum_{j=1}^{n}\overline{r}_{ij}x_j+K_{\beta}\sqrt{\frac{\pi}{2}}\left(\sum_{t=1}^{T}p_i^{(t)}\left|\sum_{j=1}^{n}\left(r_{ij}^{(t)}-\overline{r}_{ij}\right)x_j\right|\right)\le 0,\ \left(i=1,2,\cdots,m\right) \\
&\qquad\qquad \sum_{j=1}^{n}a_jx_j\le b,\ 0\le x_j\le b_j,\ \left(j=1,2,\cdots,n\right)
\end{aligned} \tag{5.31}$$

In this problem, since it includes the absolute-deviation, it is not easy to solve it analytically due to the inclusion of indifferentiable points in the feasible region. On the other hand, we introduce the following subproblem;

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \left(f_{i1}-f_{i0}\right)h+f_{i0}-\sum_{j=1}^{n}\overline{r}_{ij}x_j+K_{\beta}\sqrt{\frac{\pi}{2}}\left(\sum_{t=1}^{T}p_i^{(t)}y_i^{(t)}\right)\le 0,\ \left(i=1,2,\cdots,m\right) \\
&\qquad\qquad y_i^{(t)}-\sum_{j=1}^{n}\left(r_{ij}^{(t)}-\overline{r}_{ij}\right)x_j\ge 0, \\
&\qquad\qquad y_i^{(t)}+\sum_{j=1}^{n}\left(r_{ij}^{(t)}-\overline{r}_{ij}\right)x_j\ge 0, \\
&\qquad\qquad \sum_{j=1}^{n}a_jx_j\le b,\ 0\le x_j\le b_j,\ \left(j=1,2,\cdots,n\right)
\end{aligned} \tag{5.32}$$

and the following theorem is derived wit respect to the relationship between problems (5.31) and (5.32).

**Theorem 5.4**

The optimal solution of problem (5.32) is equal to that of problem (5.31).

**Proof**

First, we introduce the following two auxiliary problems to each scenario $i$:

$$\begin{aligned}
&\text{Minimize} \quad \sum_{t=1}^{T}p_i^{(t)}\left|\sum_{j=1}^{n}\left(r_{ij}^{(t)}-\overline{r}_{ij}\right)x_j\right| \\
&\text{subject to} \quad \sum_{j=1}^{n}\overline{r}_{ij}x_j=\rho, \\
&\qquad\qquad \sum_{j=1}^{n}a_jx_j\le b,\ 0\le x_j\le b_j,\ \left(j=1,2,\cdots,n\right)
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
&\text{Minimize} \quad \sum_{t=1}^{T} p_i^{(t)} y_i^{(t)} \\
&\text{subject to} \quad y_i^{(t)} - \sum_{j=1}^{n} \left( r_{ij}^{(t)} - \overline{r}_{ij} \right) x_j \geq 0, \\
&\qquad y_i^{(t)} + \sum_{j=1}^{n} \left( r_{ij}^{(t)} - \overline{r}_{ij} \right) x_j \geq 0, \\
&\qquad \sum_{j=1}^{n} \overline{r}_{ij} x_j = \rho,\ \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq b_j,\ \left( j = 1, 2, \cdots, n \right)
\end{aligned} \tag{5.34}$$

Then, it is well known that the optimal solution of problem (5.34) is equal to that of problem (5.33) based on the result obtained by Konno [71]. Furthermore, with respect to the parameter $h$ of problem (5.31), from the constraint $\left( f_{i1} - f_{i0} \right) h + f_{i0} - \sum_{j=1}^{n} \overline{r}_{ij} x_j + K_\beta \sqrt{\frac{\pi}{2}} \left( \sum_{t=1}^{T} p_i^{(t)} y_i^{(t)} \right) \leq 0$, we easily obtain that maximizing $h$ is equivalent to minimizing the $\sum_{t=1}^{T} p_i^{(t)} y_i^{(t)}$. Consequently, theorem 4 holds. □

Problem (5.32) is a linear programming problem, and so we solve it easily and efficiently using Simplex method or Interior point method. Therefore, we also find the optimal solution of main problem (5.24) efficiently and obtain the following solution algorithm.

**Solution Algorithm 5.2**

STEP1: Set the satisfaction function to each $Z_i = \sum_{j=1}^{n} r_{ij} x_j,\ \left( i = 1, 2, ..., m \right)$.

STEP2: Set the return sets $\boldsymbol{r}_i^{(t)} = \left\{ \left( r_{i1}^{(t)}, r_{i2}^{(t)}, ..., r_{in}^{(t)} \right),\ \left( t = 1, 2, ..., T \right) \right\},\ \left( i = 1, 2, ..., m \right)$ and each occurrence probability $p_i^{(t)}$.

STEP3: Solve the problem (5.32) and find its optimal solution $\boldsymbol{x}_h^* \left( t \right)$. This optimal solution is equal to that of main problem (24).

In the case that we solve the main problem (5.24) using this solution method, if the number of scenarios $S$ is sufficiently large, this solution method is a strict solution method. Furthermore, even if the number of scenarios is not sufficiently large, we may obtain the sufficiently appropriate optimal portfolio using this solution method, since the large scale of return sets is easily generated using the random simulations such as Monte Carlo simulation. Therefore, we apply this virtually strict solution method to various real investment cases.

## 5.4 Probability Maximization Model for the Multi-Criteria Stochastic Programming Problem

### 5.4.1 Portfolio selection problem setting constant weights

In this subsection, we propose the probability maximization model to the original problem (5.3) as follows:

$$\begin{array}{ll} \text{Maximize} & \Pr\left\{\sum_{j=1}^{n} r_{1j}x_j \geq f_1\right\} \\ \text{Maximize} & \Pr\left\{\sum_{j=1}^{n} r_{2j}x_j \geq f_2\right\} \\ \quad\vdots & \\ \text{Maximize} & \Pr\left\{\sum_{j=1}^{n} r_{mj}x_j \geq f_m\right\} \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j \leq b,\ 0 \leq x_j \leq b_j,\ \left(j=1,2,\cdots,n\right) \end{array} \tag{5.35}$$

Then, in a similar way to subsection 5.3, we first consider the case a decision maker sets a weight to each scenario based on statistical analysis of historical data and her or his subjectivity, and aggregate all objective function into one weighted function. Then, we assume weight $w_i$ to be a positive weight value to $i$th scenario and $\sum_{i=1}^{m} w_i = 1$. Using these weights, we introduce the following single-criteria probability maximization model of portfolio selection problem introducing a probability chance constraint to the objective function and its target value $f$ :

$$\begin{array}{ll} \text{Maximize} & \Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij}x_j \geq f\right\} \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j=1,2,...,n\right) \end{array} \tag{5.36}$$

In the case that a weight value $w_{i'} = 1$ and the other weight $w_i = 0,\ \left(i=1,2,...,m,\ i \neq i'\right)$, this problem is transformed into the probability maximization model considering particular scenario $i'$. Therefore, problem (5.36) includes the model considering one particular scenario. Furthermore, in the case that a decision maker considers the weight value based on the occurrence probability of each scenario which is assumed to be $p_i$, $w_i$ is assumed to be $p_i$.

In problem (5.36), since the whole vector $\left(r_{i1}, r_{i2}, \ldots, r_{in}\right)$ occurs according to a normal distribution $N\left(\overline{r}_{ij}, \sigma_{ij}^2\right)$, the objective function $\Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \geq f\right\}$ is transformed into the following deterministic equivalent objective function.

$$\begin{aligned}
&\Pr\left\{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j \geq f\right\} \\
\Leftrightarrow\quad &\Pr\left\{\frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)}} \geq \frac{f - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)}}\right\} \\
\Leftrightarrow\quad &\Phi\left(\frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)}}\right)
\end{aligned} \tag{5.37}$$

where $\mathbf{V}_i$ is a variance-covariance matrix to future returns in $i$ th scenario, and $\Phi\left(\cdot\right)$ is a distribution function of the standard normal distribution. From this transformation, we equivalently transform problem (5.36) into the following form.

$$\begin{aligned}
&\text{Maximize} \quad \Phi\left(\frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)}}\right) \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j = 1, 2, \ldots, n\right)
\end{aligned} \tag{5.38}$$

Furthermore, since $\Phi\left(\cdot\right)$ is a increasing function, this problem is equivalently transformed as follows.

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} \left(w_i \boldsymbol{x}\right)^t \mathbf{V}_i \left(w_i \boldsymbol{x}\right)}} \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j = 1, 2, \ldots, n\right)
\end{aligned} \tag{5.39}$$

Previous researches (For example, Ishii and Nishida [57] and Katagiri [63, 64]) to analytically solve problem (5.39) have considered that each random variable was assumed to be independent each other in order to solve it analytically, i.e., the element $\sigma_{ij}$ of covariance matrix $\mathbf{V}_i$ is assumed to be $\sigma_{ij} = 0, \ (i \neq j)$. However, in practical decision making, there exist many cases that a decision maker considers the relation among all decision variables. Therefore, in this chapter, we extend these previous analytical approaches to the more standard analytical approach.

Subsequently, we set $\mathbf{V} = \sum_{i=1}^{m} w_i^2 \mathbf{V}_i$. Since each $\mathbf{V}_i$ is a covariance matrix and each $w_i$ is a positive value, it is obvious that $\mathbf{V}$ is a symmetric positive definite matrix. In a way similar to performing the transformation (5.13), problem (5.39) is equivalently transformed into the following form:

$$\begin{aligned} &\text{Maximize} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} y_j - f}{\sqrt{\sum_{j=1}^{n} y_j^2}} \\ &\text{subject to} \quad \sum_{j=1}^{n} a_j y_j \leq b, \quad 0 \leq y_j \leq b_j \quad (j = 1, 2, \ldots, n) \end{aligned} \tag{5.40}$$

In this chapter, it is assumed that there exists a feasible solution satisfying $\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} y_j > f$. This assumption implies the probability that total future profit exceed prescribed value $f$, is greater than $1/2$. Problem (5.40) is equivalent to the model derived from the previous study (Ishii and Nishida [57]). Therefore, we apply the solution method in Ishii's study to this problem and obtain the optimal solution analytically.

Consequently, we have extended the previous models and solution method to obtain the optimal portfolio to the model considering not only each variance but also the covariance. Thereby, the proposed model is more versatile and practical than some previous models.

### 5.4.2 Portfolio selection problem considering the flexibility of all weights

In Subsection 5.4.1, all weights $w_i$ for scenarios are assumed to be constant. However, a decision maker often does not assume each weight to be a fixed value due to uncertainty derived from a lack of reliable information and the subjectivity of the decision maker considering the robustness of the portfolio, but assumes them to include an interval to each weight. In this chapter, this interval of weight is given as $w_i^L \le w_i \le w_i^U$, where the lower value $w_i^L$ and upper value $\overline{w}_i$ are assumed to be constant. Thereby, it is possible that the decision maker constructs the versatile portfolio selection model involving various practical conditions by considering the interval of weight. Then, in this subsection, we particularly focus on a minimax portfolio selection problem in which the decision maker considers the case that the total profit is more than a target value under all the situations including the interval of weights. This means that the proposed model is a robust portfolio model applied to various future cases such as more substantial changes of future profits. Therefore, we propose the following robust portfolio selection problem extending main problem (5.39):

$$\begin{aligned}
&\text{Maximize} \quad \min_{\boldsymbol{w}\in W} \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le b_j \quad (j = 1, 2, ..., n) \\
&\qquad W = \left\{ \boldsymbol{w} \,\middle|\, w_i^L \le w_i \le w_i^U, \ (i = 1, 2, ..., m), \ \sum_{i=1}^{m} w_i = 1 \right\}
\end{aligned} \tag{5.41}$$

Subsequently, we introduce a parameter $h$ in order to perform following equivalent transformation to the objective function:

$$\min_{\boldsymbol{w}\in W} \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \quad \Leftrightarrow \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \ge h, \ (\forall \boldsymbol{w} \in W) \tag{5.42}$$

Parameter $h$ indicates the minimum value of all functions $\dfrac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}}$. Therefore, problem (5.41) is equivalently transformed into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\sum_{i=1}^{m} (w_i \boldsymbol{x})^t \mathbf{V}_i (w_i \boldsymbol{x})}} \ge h, \ (\forall \boldsymbol{w} \in W) \\
&\qquad \sum_{j=1}^{n} a_j x_j \le b, \quad 0 \le x_j \le b_j \quad (j = 1, 2, ..., n)
\end{aligned} \tag{5.43}$$

However, this problem includes a square root term; therefore, it is difficult to solve this problem efficiently even if we use nonlinear programming approaches such as the gradient method. Therefore, we need to construct the more efficient and analytical solution method rather than previous nonlinear programming approaches. Subsequently, we introduce the following mean-absolute deviation for random variable $R(\boldsymbol{x})=\sum_{i=1}^{m} w_i \sum_{j=1}^{n} r_{ij} x_j$

$$W\left[R(\boldsymbol{x})\right]=E\left[\left|\sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \overline{r}_j x_j\right)\right|\right] \tag{5.44}$$

By using a property of normal distribution with respect to the relationship between this mean-absolute deviation $W\left[R(\boldsymbol{x})\right]$ and the variance $\sigma^2(\boldsymbol{x})=\boldsymbol{x}^t\mathbf{V}\boldsymbol{x}$, $\sigma^2(\boldsymbol{x})=\frac{\pi}{2}\left(W\left[R(\boldsymbol{x})\right]\right)^2$ holds based on the result in Section 5.3. From this theorem, problem (5.43) is transformed into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\frac{\pi}{2}\left(W\left[R(\boldsymbol{x})\right]\right)^2}} \geq h, \ \left(\forall \boldsymbol{w} \in W\right) \\
&\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j=1,2,...,n\right)
\end{aligned} \tag{5.45}$$

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\textit{i.e.} \quad \text{subject to} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\frac{\pi}{2}}W\left[R(\boldsymbol{x})\right]} \geq h, \ \left(\forall \boldsymbol{w} \in W\right) \\
&\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j=1,2,...,n\right)
\end{aligned} \tag{5.46}$$

In general, decision makers obtain historical data of future returns from practical investment markets. Then, if each return occurs according to a random distribution, the return sets are clearly sample values of the multivariate random variable. Therefore, it is natural that return sets based on historical data as well as subjectivity of the decision maker are introduced into problem (5.46). In this chapter, we introduce the following return set derived from each $r_{ij}$ according to the multivariate normal distribution:

$$\boldsymbol{r}_i^{(k)}=\left\{\left(r_{i1}^{(k)}, r_{i2}^{(k)}, ..., r_{in}^{(k)}\right), \ \left(k=1,2,...,S\right)\right\}, \ \left(i=1,2,...,m\right) \tag{5.47}$$

where $S$ is the total number of scenarios and the occurrence probability $p_i^{(k)}$ is defined as the following form:

$$p^{(k)} = \Pr\left\{ \boldsymbol{r}^{(k)} = \left( \boldsymbol{r}_1^{(k)}, \boldsymbol{r}_2^{(k)}, ..., \boldsymbol{r}_m^{(k)} \right) \right\}, \ \left( k = 1,2,...,S \right), \ \sum_{k=1}^{S} p^{(k)} = 1 \tag{5.48}$$

In this chapter, we use the following expression introducing the return sets (5.47) and occurrence probabilities (5.48) in place of the absolute deviation.

$$\begin{aligned} W\left[R\left(\boldsymbol{x}\right)\right] &= E\left[\left|\sum_{i=1}^{m} w_i \left( \sum_{j=1}^{n} r_{ij} x_j - \sum_{j=1}^{n} \overline{r}_{ij} x_j \right)\right|\right] \\ &= E\left[\left|\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij} - \overline{r}_{ij} \right) x_j \right|\right] \\ &= \sum_{k=1}^{S} p^{(k)} \left| \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \overline{r}_{ij} \right) x_j \right| \end{aligned} \tag{5.49}$$

where $\overline{r}_{ij}$ in (5.49) is redefined as the arithmetic mean derived from return sets (5.47) and the probability (5.48). Subsequently, it is obvious that $W\left[R\left(\boldsymbol{x}\right)\right]$ is strictly equal to $\sum_{k=1}^{S} p^{(k)} \left| \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \overline{r}_{ij} \right) x_j \right|$ in the case of $S \to \infty$. Then, $S$ is sufficiently large, expression (5.49) is sufficiently similar to the absolute deviation. Therefore, problem (5.46) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\frac{\pi}{2}} \sum_{k=1}^{S} p^{(k)} \left| \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \overline{r}_{ij} \right) x_j \right|} \geq h, \ \left( \forall \boldsymbol{w} \in W \right) \\ &\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left( j = 1,2,...,n \right) \end{aligned} \tag{5.50}$$

Since this problem includes some absolute values, it is not easy to solve it analytically due to the inclusion of indifferentiable points in the feasible region. On the other hand, we introduce the following subproblem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \frac{\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j - f}{\sqrt{\frac{\pi}{2} \sum_{k=1}^{S} p^{(k)} \xi^{(k)}}} \geq h, \ \left(\forall \boldsymbol{w} \in W\right) \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \leq b, \quad 0 \leq x_j \leq b_j \quad \left(j = 1, 2, ..., n\right) \\
&\qquad\qquad \xi^{(k)} - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left(r_{ij}^{(k)} - \overline{r}_{ij}\right) x_j \geq 0, \\
&\qquad\qquad \xi^{(k)} + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left(r_{ij}^{(k)} - \overline{r}_{ij}\right) x_j \geq 0, \ \left(k = 1, 2, ..., S\right)
\end{aligned} \tag{5.51}$$

Then, the following theorem is derived with respect to the relationship between problems (5.50) and (5.51).

**Theorem 5.5**

The solution sets of problems (5.50) and (5.51) coincide.

**Proof**

First, we introduce the following two auxiliary problems to each scenario $i$:

$$\begin{aligned}
&\text{Minimize} \quad \sum_{k=1}^{S} p_i^{(k)} \left| \sum_{j=1}^{n} \left(r_{ij}^{(k)} - \overline{r}_{ij}\right) x_j \right| \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ \left(j = 1, 2, \cdots, n\right)
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
&\text{Minimize} \quad \sum_{k=1}^{S} p_i^{(k)} \theta_i^{(k)} \\
&\text{subject to} \quad \theta_i^{(k)} - \sum_{j=1}^{n} \left(r_{ij}^{(k)} - \overline{r}_{ij}\right) x_j \geq 0, \\
&\qquad\qquad \theta_i^{(k)} + \sum_{j=1}^{n} \left(r_{ij}^{(k)} - \overline{r}_{ij}\right) x_j \geq 0, \ \left(k = 1, 2, ..., S\right) \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \leq b, \ 0 \leq x_j \leq b_j, \ \left(j = 1, 2, \cdots, n\right)
\end{aligned} \tag{5.53}$$

Then, it is well known that the optimal solution of problem (5.52) is also optimal for problem (5.53) based on the result obtained by Konno [68]. Furthermore, by the optimality condition with respect to optimal solution $\boldsymbol{x}$, an optimal solution of the following problem (5.54) is equivalent to that of the following problem (5.55):

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} \bar{r}_{ij} x_j - f_i}{\sum_{k=1}^{S} p_i^{(k)} \left| \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x_j \right|} \\
&\text{subject to} \quad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ \left( j = 1, 2, \cdots, n \right)
\end{aligned} \tag{5.54}$$

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} \bar{r}_{ij} x_j - f_i}{\sum_{k=1}^{S} p_i^{(k)} \theta_i^{(k)}} \\
&\text{subject to} \quad \theta_i^{(k)} - \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x_j \ge 0, \\
&\qquad\qquad \theta_i^{(k)} + \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x_j \ge 0,\ \left( k = 1, 2, ..., S \right) \\
&\qquad\qquad \sum_{j=1}^{n} a_j x_j \le b,\ 0 \le x_j \le b_j,\ \left( j = 1, 2, \cdots, n \right)
\end{aligned} \tag{5.55}$$

Then, since it is easy to extend these problems to minimax programming problems holding the optimality conditions, it is clear that Theorem 5.5 holds. □

Problem (5.51) is a linear fractional programming problem. Therefore, by introducing the following parameter $\eta$;

$$\frac{1}{\sum_{k=1}^{S} p^{(k)} \xi^{(k)}} = \eta,\ \xi'^{(k)} = \eta \xi^{(k)},\ x'_j = \eta x_j \tag{5.56}$$

problem (5.51) is equivalently transformed into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x'_j - f\eta \ge h,\ \left( \forall \mathbf{w} \in W \right) \\
&\qquad \sum_{j=1}^{n} a_j x'_j \le b\eta, \quad 0 \le x'_j \le b_j \eta \quad \left( j = 1, 2, ..., n \right) \\
&\qquad \xi'^{(k)} - \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x'_j \ge 0, \\
&\qquad \xi'^{(k)} + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x'_j \ge 0,\ \left( k = 1, 2, ..., S \right) \\
&\qquad \sqrt{\frac{\pi}{2}} \sum_{k=1}^{S} p^{(k)} \xi'^{(k)} = 1
\end{aligned} \tag{5.57}$$

Since we perform the deterministic equivalent and sufficiently approximate transformations from problems (5.41) to (5.57), we efficiently obtain an optimal portfolio by solving the simple linear programming problem (5.57). In general, probability maximization model using the finite return sets is formulated as a mixed integer programming problem, and it is difficult to solve it rapidly due to NP-hard. However, in the case that future returns occur according to normal distributions, in order to perform the transformations from (5.41) to (5.57), we can use the sufficient approximate model as the linear programming problem (5.57) in place of the probability maximization model. Consequently, we can obtain a sufficiently practical portfolio much rapidly by solving problem (5.57) than general mixed integer programming problems even if S is very large.

Subsequently, all constraints in problem (5.57) are linear constraints, but problem (5.57) is a semi-infinite programming problem (SIP) due to the inclusion of infinite inequalities $\sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x'_j - f\eta \geq h, \quad \xi'^{(k)} \pm \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x'_j \geq 0, \ \left( \forall \boldsymbol{w} \in W \right)$. Therefore, we need to construct the more efficient solution method as it cannot be solved directly using basic linear programming approaches. However, in the case that we fix the parameter $\boldsymbol{w}$, problem (5.57) is equivalent to a finite linear programming problem. In order to construct the solution method, we introduce the following subproblem SP( $E$ ) where $E$ is the index set defined by $E := \left\{ \boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}, \ldots, \boldsymbol{w}^{(T)} \right\}$:

$$
\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \sum_{i=1}^{m} w_i^{(t)} \sum_{j=1}^{n} \bar{r}_{ij} x'_j - f\eta \geq h, \ \left( \forall \boldsymbol{w}^{(t)} \in E, \ \boldsymbol{w}^{(t)} \in W \right) \\
&\qquad \sum_{j=1}^{n} a_j x'_j \leq b\eta, \quad 0 \leq x'_j \leq b_j \eta \quad \left( j = 1, 2, \ldots, n \right) \\
&\qquad \xi'^{(k)} - \sum_{i=1}^{m} w_i^{(l)} \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x'_j \geq 0, \\
&\qquad \xi'^{(k)} + \sum_{i=1}^{m} w_i^{(l)} \sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x'_j \geq 0, \ \left( k = 1, 2, \ldots, S \right) \\
&\qquad \sqrt{\frac{\pi}{2}} \sum_{k=1}^{S} p^{(k)} \xi'^{(k)} = 1
\end{aligned}
\tag{5.58}
$$

Since the number of constraints in SP( $E$ ) is finite, this problem is equivalent to a standard linear programming problem. Therefore, we construct the following solution method for the main robust programming problem (5.41).

**Solution algorithm 5.3**

STEP1: Set all intervals of weights $w_i^L \leq w_i \leq w_i^U, \ \left( i = 1, 2, \ldots, m \right)$, return sets $\boldsymbol{r}_i^{(k)}, \ \left( k = 1, 2, \ldots, S \right)$, and the occurrence probabilities $p^{(k)}, \ \left( k = 1, 2, \ldots, S \right)$.

STEP2: Set a finite number $t$ and initial weight sets $E^0 = \left\{ \boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}, ..., \boldsymbol{w}^{(t)} \right\}$, $\left( \boldsymbol{w}^{(t)} \in W \right)$ and solve problem (5.58). Then, set the optimal solution to be $\boldsymbol{x}^0, t^0, h^0$, and $d \leftarrow 0$.

STEP3: If $\min_{\boldsymbol{w} \in W} \left\{ \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^d - f\eta^d - h^d \right\} \geq 0$, $\boldsymbol{x}^d$ is an optimal solution of problem (5.41) and terminate this algorithm. If not, reset the values as follows:

$$\boldsymbol{w}^{(t+d+1)} := \left\{ \boldsymbol{w} \in W \middle| \min_{\boldsymbol{w} \in W} \left\{ \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \overline{r}_{ij} x_j^d - f\eta^d - h^d \right\} \right\} \text{ and } \overline{E}^{d+1} := E^d \cup \left\{ \boldsymbol{w}^{(t+d+1)} \right\}$$

STEP4: Solve the primal problem SP($\overline{E}^{d+1}$) and obtain optimal solutions $\boldsymbol{x}^{d+1}, \eta^{d+1}$ and $h^{d+1}$. Then set $d \leftarrow d+1$ and return to STEP 3.

However, in this solution method, constraints of problem (5.58) are increase each iteration from Step 3 to Step 4, and so this algorithm is not always efficient. Therefore, we improve this solution algorithm using the dual problem to problem (5.58). First, using coefficient matrix $A$ of problem (5.58) defined by

$$\mathbf{A} = \begin{pmatrix} -\boldsymbol{w}_L \boldsymbol{r} & f\mathbf{1}_L & \mathbf{1}_L & \mathbf{0} \\ \boldsymbol{a}^t & -b & 0 & \mathbf{0} \\ \mathbf{I}_n & -\boldsymbol{b} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{w}_L \boldsymbol{r}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{w}_L \boldsymbol{r}_S & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{(S)} \\ -\boldsymbol{w}_L \boldsymbol{r}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ -\boldsymbol{w}_L \boldsymbol{r}_S & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{(S)} \\ \mathbf{0} & 0 & 0 & \sqrt{\frac{\pi}{2}}\boldsymbol{p} \end{pmatrix} \tag{5.59}$$

$$\boldsymbol{w}_L = \begin{pmatrix} w_1^{(1)} & \cdots & w_m^{(1)} \\ \vdots & \ddots & \vdots \\ w_1^{(L)} & \cdots & w_m^{(L)} \end{pmatrix}, \ \boldsymbol{r} = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \cdots & r_{mn} \end{pmatrix}, \ \boldsymbol{r}_k = \begin{pmatrix} r_{11}^{(k)} - \overline{r}_{11} & \cdots & r_{1n}^{(k)} - \overline{r}_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1}^{(k)} - \overline{r}_{m1} & \cdots & r_{mn}^{(k)} - \overline{r}_{mn} \end{pmatrix},$$

$$\boldsymbol{a} = (a_1, a_2, ..., a_n), \ \boldsymbol{p} = \left( p^{(1)}, p^{(2)}, ..., p^{(S)} \right)$$

where Then, $\mathbf{I}_n$ is the $n \times n$ unit matrix, $\mathbf{1}_L$ is the L-dimensional vector whose all elements are

1, and each $\mathbf{1}_{(k)}$ is the $L \times S$ matrix whose elements of the kth column are 1 and the other elements are 0. $\mathbf{A} \in \mathbf{R}^{(n+(2S+1)L+2)\times(n+S+2)}$. Then, the dual problem DSP($E$) to SP($E$) is given as follows:

$$\begin{array}{ll} \text{Minimize} & \zeta \\ \text{subject to} & \mathbf{A}^t (\boldsymbol{y}, \zeta) \geq (\mathbf{0}, 1)^t \\ & \boldsymbol{y} \geq \mathbf{0} \end{array} \tag{5.60}$$

where $\boldsymbol{y}$ is the $(n+(2S+1)L+1)$-dimensional decision variable vector of DSP($E$), and so $(\boldsymbol{y}, \zeta)$ is the $(n+(2S+1)L+2)$-dimensional decision variable vector. This dual problem DSP ($E$) is also a linear programming problem. Therefore, it is easy to obtain an optimal solution of DSP($E$). Thereby, using these problems SP($E$) and DSP($E$), we develop the following efficient solution algorithm for problem (5.41) extending the solution method of Lai and Wu [73].

**Solution Algorithm 5.4**

STEP1: Set all intervals of weights $w_i^L \leq w_i \leq w_i^U$, $(i = 1, 2, ..., m)$, return sets $\boldsymbol{r}_i^{(k)}$, $(k = 1, 2, ..., S)$, and the occurrence probabilities $p^{(k)}$, $(k = 1, 2, ..., S)$.

STEP2: Set initial weight sets $E^0 = \{1, 2, ..., \gamma\}$, $\left(\forall l \in E^0,\ \boldsymbol{w}^{(l)} \in W\right)$ and solve problem (5.58). Then, set the optimal solution to be $\boldsymbol{x}^0, t^0, h^0$, and $d \leftarrow 0$.

STEP3: If $\min_{\boldsymbol{w} \in W} \left\{ \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j^d - f\eta^d - h^d \right\} \geq 0$, $\boldsymbol{x}^{(d)}$ is an optimal solution of problem (5.41) and terminate this algorithm. If not, reset the values as follows:

$$l_{new}^d := \arg\min_{\boldsymbol{w} \in W} \left\{ \sum_{i=1}^{m} w_i \sum_{j=1}^{n} \bar{r}_{ij} x_j^d - f\eta^d - h^d \right\} \text{ and } \bar{E}^{d+1} := E^d \cup \left\{ l_{new}^d \right\}$$

STEP4: Solve the primal problem SP($\bar{E}^{d+1}$) and dual problem DSP ($\bar{E}^{d+1}$), and obtain optimal solutions $\boldsymbol{x}^{d+1}$ and $v_{d+1}(l)$

STEP5: Reset $E^{d+1} := \left\{ l \in \bar{E}^{d+1} \middle| v_{d+1}(l) > 0 \right\}$ and $d \leftarrow d + 1$. Then, return to STEP 3.

Since this solution algorithm is based on the study of Lai and Wu [73], it is obvious that the convergence of this algorithm is obtained in a way similar to the study of Lai and Wu [73].

## 5.5 Maximization of the Minimum Aspiration Level of Objective Values among All Scenarios for the Probability Maximization Model

In Subsection 5.4, all scenarios have been weighted using occurrence probabilities and the decision maker's subjectivity. However, there often exists the case that all scenarios are treated equally due to the lack of enough information and uncertain economic conditions, i.e., all the occurrence probabilities are estimated to have the same values and the decision maker makes the same assessments with respect to all the scenarios. In this case, she or he often sets the goal of the total future profit for each scenario and consider its aspiration level for the achievement of the probability more than the goal. Furthermore, considering the vagueness and subjectivity of human judgment and flexibility for the execution of a plan, each aspiration level is assumed to be a satisfaction function. Then, investors need to consider the condition that they satisfy the aspiration level more than a goal even if any scenario occurs.

Therefore, in this section, we consider the portfolio selection decision that maximizes the minimum aspiration level of probabilities in portfolio selection problems with respect to each scenario of future return. First, we introduce the following satisfaction function $\mu_i\left(Z_i\right)$ to each $Z_i = \Phi\left(\frac{\sum_{j=1}^{n}\bar{r}_{ij}x_j - f_i}{\sqrt{\boldsymbol{x}^t\mathbf{V}_i\boldsymbol{x}}}\right)$ as follows:

$$\mu_i\left(Z_i\right) = \begin{cases} 1 & p_{i1} \le Z_i \\ g_i\left(Z_i\right) & p_{i0} \le Z_i \le p_{i1}\,, \\ 0 & Z_i \le p_{i0} \end{cases} \quad \left(i = 1, 2, \cdots, m\right) \tag{5.61}$$

where each $g_i\left(Z_i\right)$ is a monotonous increasing function. This satisfaction function means that a decision maker is entirely agreeable with respect to the portfolio satisfying objective function $Z_i$ is more than the target value $p_{i1}$, but is never agreeable to the portfolio satisfying $Z_i$ is less than $p_{i0}$. Then, it also means that she or he is partially agreeable depending on her or his psychological aspect and subjectivity if $Z_i$ is between $p_{i0}$ and $p_{i1}$. Using these satisfaction functions, we formulate this model as the following minimax programming problem:

$$\begin{aligned} &\text{Maximize} \quad \min_i \left[\mu_{p_i}(Z_i),\ (i=1,2,\cdots,m)\right] \\ &\text{subject to} \quad \sum_{j=1}^n a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.62}$$

Then, this problem is transformed into the following form:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \mu_{p_i}(Z_i) \ge h,\ (i=1,2,\cdots,m) \\ &\qquad\qquad \sum_{j=1}^n a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.63}$$

In this problem, the constraint $\mu_{p_i}(Z_i) \ge h$ is transformed into the following inequality:

$$\begin{aligned} \mu_i(Z_i) \ge h \quad &\Leftrightarrow \quad g_i(Z_i) \ge h \\ &\Leftrightarrow \quad \Phi\left(\frac{\sum_{j=1}^n \bar{r}_{ij} x_j - f_i}{\sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}}}\right) \ge g_i^{-1}(h) \\ &\Leftrightarrow \quad \frac{\sum_{j=1}^n \bar{r}_{ij} x_j - f_i}{\sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}}} \ge K_{g_i^{-1}(h)}, \\ &\Leftrightarrow \quad \sum_{j=1}^n \bar{r}_{ij} x_j - f_i \ge K_{g_i^{-1}(h)} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}},\ (i=1,2,...,m) \end{aligned} \tag{5.64}$$

where $K_\beta$ is the $\beta$-quantile to the normal distribution. Therefore, problem (5.63) is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad f_i - \sum_{j=1}^n \bar{r}_{ij} x_j + K_{\mu_{p_i}^*(h)} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}} \le 0,\ (i=1,2,\cdots,m) \\ &\qquad\qquad \sum_{j=1}^n a_j x_j \le b,\ 0 \le x_j \le b_j,\ (j=1,2,\cdots,n) \end{aligned} \tag{5.65}$$

Since each $f_i(\boldsymbol{x}, h) = f - \sum_{j=1}^n \bar{r}_{ij} x_j + K_{\mu_{p_i}^*(h)} \sqrt{\boldsymbol{x}^t \mathbf{V}_i \boldsymbol{x}}$ is a convex function, this problem is a convex programming problem and we obtain a global optimal solution using convex programming approaches. However, its constraint includes a square root term; therefore, it is difficult to solve this problem analytically. Therefore, in a way similar as described in Subsection 5.5, we construct an efficient solution method using the following mean-absolute deviation.

$$W\left[R_i\left(\boldsymbol{x}\right)\right] = E\left[\left|\sum_{j=1}^{n} r_{ij}x_j - \sum_{j=1}^{n} \overline{r}_{ij}x_j\right|\right] \tag{5.66}$$

Using this mean-absolute deviation, problem (5.65) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad f_i - \sum_{j=1}^{n} \overline{r}_{ij}x_j + K_{\mu^*_{p_i}(h)}\sqrt{\frac{\pi}{2}\left(W\left[R_i\left(\boldsymbol{x}\right)\right]\right)^2} \le 0, \ \left(i = 1, 2, \cdots, m\right) \\ &\sum_{j=1}^{n} a_j x_j \le b, \ 0 \le x_j \le b_j, \ \left(j = 1, 2, \cdots, n\right) \end{aligned} \tag{5.67}$$

$$\begin{aligned} &\text{Maximize} \quad h \\ \textit{i.e.} \quad &\text{subject to} \quad f_i - \sum_{j=1}^{n} \overline{r}_{ij}x_j + K_{\mu^*_{p_i}(h)}\sqrt{\frac{\pi}{2}}\left(W\left[R_i\left(\boldsymbol{x}\right)\right]\right) \le 0, \ \left(i = 1, 2, \cdots, m\right) \\ &\sum_{j=1}^{n} a_j x_j \le b, \ 0 \le x_j \le b_j, \ \left(j = 1, 2, \cdots, n\right) \end{aligned} \tag{5.68}$$

Furthermore, in a way similar to Subsection 5.5, we introduce return sets that are identical to formulas (5.47), (5.48), and (5.49), and problem (5.68) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad f_i - \sum_{j=1}^{n} \overline{r}_{ij}x_j + K_{\mu^*_{p_i}(h)}\sqrt{\frac{\pi}{2}}\left(\sum_{k=1}^{S} p_i^{(k)}\left|\sum_{j=1}^{n}\left(r_{ij}^{(k)} - \overline{r}_{ij}\right)x_j\right|\right) \le 0, \\ &\sum_{j=1}^{n} a_j x_j \le b, \ 0 \le x_j \le b_j, \ \left(i = 1, \ldots, m; \ j = 1, 2, \cdots, n\right) \end{aligned} \tag{5.69}$$

Then, we introduce the following subproblem including parameters $\theta_i^{(k)}$ in a way similar to problem (5.58) in Section 3:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad f_i - \sum_{j=1}^{n} \overline{r}_{ij}x_j + K_{\mu^*_{p_i}(h)}\sqrt{\frac{\pi}{2}}\left(\sum_{k=1}^{S} p_i^{(k)}\theta_i^{(k)}\right) \le 0, \ \left(i = 1, 2, \cdots, m\right) \\ &\theta_i^{(k)} - \sum_{j=1}^{n}\left(r_{ij}^{(k)} - \overline{r}_{ij}\right)x_j \ge 0, \ \theta_i^{(k)} + \sum_{j=1}^{n}\left(r_{ij}^{(k)} - \overline{r}_{ij}\right)x_j \ge 0, \\ &\sum_{j=1}^{n} a_j x_j \le b, \ 0 \le x_j \le b_j, \ \left(j = 1, 2, \cdots, n\right) \end{aligned} \tag{5.70}$$

Then, the following theorem holds with respect to the relation between problems (5.69) and (5.70).

**Theorem 5.6**

The optimal solution of problem (5.70) is also optimal for problem (5.69).

**Proof**

In a similar manner to Proof of Theorem 5.5, this theorem obviously holds. □

Since this problem is a nonlinear and nonconvex programming problem due to $K_{\mu^*_{p_i}(h)}$, it is difficult to solve it directly and analytically. However, if we fix the parameter $h$, we consider the existence of the feasible solution $\boldsymbol{x}_{\bar{h}}$ involving the following set:

$$\boldsymbol{x}_{\bar{h}} \in S_{\bar{h}} = \left\{ \boldsymbol{x} \left| \begin{array}{l} f_i - \bar{\boldsymbol{r}}^t \boldsymbol{x} + K_{\mu^*_{p_i}(\bar{h})} \sqrt{\dfrac{\pi}{2}} \left( \displaystyle\sum_{k=1}^{S} p_i^{(k)} \theta_i^{(k)} \right) \le 0, \ \left( i = 1, 2, \cdots, m \right) \\ \theta_i^{(k)} \pm \displaystyle\sum_{j=1}^{n} \left( r_{ij}^{(k)} - \bar{r}_{ij} \right) x_j \ge 0, \\ \displaystyle\sum_{j=1}^{n} a_j x_j \le b, \ 0 \le x_j \le b_j, \ \left( j = 1, 2, \cdots, n \right) \end{array} \right. \right\} \tag{5.71}$$

where $\bar{h}$ is a fixed value satisfying $0 \le \bar{h} \le 1$. It is easy to obtain a feasible solution using linear programming approaches such as the Simplex method or Interior point method and so we solve it efficiently. Therefore, we also find the optimal solution of main problem (5.63) efficiently and analytically. Thus, we obtain the following solution algorithm.

**Solution Algorithm 5.5**

STEP1: Set the initial value $h_l \leftarrow 0, \ h_u \leftarrow 1, \ k \leftarrow 1$.

STEP2: Set the satisfaction function to each $Z_i = \sum_{j=1}^{n} r_{ij} x_j, \ \left( i = 1, 2, ..., m \right)$.

STEP3: Set the return set $\boldsymbol{r}_i^{(k)} = \left\{ \left( r_{i1}^{(k)}, r_{i2}^{(k)}, ..., r_{in}^{(k)} \right), \ \left( k = 1, 2, ..., S \right) \right\}, \ \left( i = 1, 2, ..., m \right)$ and each occurrence probability $p_i^{(k)}$.

STEP4: Set $h_k \leftarrow \dfrac{h_l + h_u}{2}$.

STEP5: Solve the problem (5.70) and find its feasible solution $\boldsymbol{x}_h^*(k)$.

STEP6: If the feasible solution exists in STEP5, $h_u \leftarrow h_k$ and return to STEP4. Else, if the feasible solution does not exist, $h_l \leftarrow h_k$ and return to STEP4. Then, if a feasible solution exists in

STEP5 and $\left|h_k - h_{k-1}\right| < \varepsilon$ with respect to the sufficient small value $\varepsilon$, $\boldsymbol{x}_h^*(k)$ is an optimal solution and terminate this algorithm.

## 5.6 Numerical Example

In order to compare our proposed models with basic models, i.e. the Markowitz model and basic probability maximization model, let us consider an example shown in Table 1 based on data provided by Markowitz [90]. We assume that there are nine financial assets whose returns are distributed according to normal distributions, and to simplify the discussion, we assume each return to be an independent normal distribution. Furthermore, we assume three scenarios with respect to expected returns of all the securities shown in Table 5.1.

Table 5.1. Sample data of returns and SD from Markowitz's historical data

| Returns | Scenario 1 | Scenario 2 | Scenario 3 | SD |
|---|---|---|---|---|
| R1 | 0.066 | 0.077 | 0.058 | 0.238 |
| R2 | 0.062 | 0.070 | 0.055 | 0.125 |
| R3 | 0.146 | 0.164 | 0.126 | 0.301 |
| R4 | 0.173 | 0.191 | 0.148 | 0.318 |
| R5 | 0.198 | 0.211 | 0.177 | 0.368 |
| R6 | 0.055 | 0.067 | 0.051 | 0.209 |
| R7 | 0.128 | 0.130 | 0.120 | 0.175 |
| R8 | 0.118 | 0.130 | 0.109 | 0.286 |
| R9 | 0.116 | 0.127 | 0.109 | 0.290 |

$$\mathbf{V} = \begin{pmatrix} 0.056644 & 0.022908 & 0.030804 & 0.052222 & 0.017517 & 0.034322 & 0.025823 & 0.030631 & 0.038651 \\ 0.022908 & 0.015625 & 0.019941 & 0.025838 & 0.00874 & 0.010711 & 0.015531 & 0.019663 & 0.022113 \\ 0.030804 & 0.019941 & 0.090601 & 0.066045 & 0.04763 & 0.01384 & 0.011062 & 0.052512 & 0.044518 \\ 0.052222 & 0.025838 & 0.066045 & 0.101124 & 0.005501 & 0.030573 & 0.02226 & 0.06912 & 0.038732 \\ 0.017517 & 0.00874 & 0.04763 & 0.055001 & 0.135424 & 0.013844 & 0.02254 & 0.077884 & 0.048024 \\ 0.034322 & 0.010711 & 0.01384 & 0.030573 & 0.013844 & 0.043681 & 0.01207 & 0.022714 & 0.023032 \\ 0.025823 & 0.015531 & 0.011062 & 0.02226 & 0.02254 & 0.01207 & 0.030625 & 0.022523 & 0.018778 \\ 0.030631 & 0.019663 & 0.052512 & 0.06912 & 0.077884 & 0.022714 & 0.022523 & 0.081796 & 0.040641 \\ 0.038651 & 0.022113 & 0.044518 & 0.038732 & 0.048024 & 0.023032 & 0.018778 & 0.040641 & 0.0841 \end{pmatrix}$$

### 5.6.1 Probability fractile optimization model

First of all, we consider the following portfolio selection problem to maximize total expected returns of scenario 1:

(Problem P)

$$\begin{aligned}&\text{Maximize} \quad 0.066x_1+0.062x_2+0.146x_3+0.173x_4+0.198x_5+0.055x_6+0.128x_7+0.118x_8+0.116x_9\\&\text{subject to} \quad \sum_{j=1}^{9}x_j=1,\ 0\le x_k\le 0.2,\ \left(k=1,\cdots,9\right)\end{aligned}$$

We solve this problem and obtain optimal portfolios shown in Table 5.2.

Table 5.2. Optimal portfolios of problem P

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | Objective value |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0.2 | 0 | 0.1526 |

Problem P is a basic linear knapsack problem, and so Table 2 obviously shows that the decision maker should purchase assets with the high expected returns. This result corresponds to the property for optimal solution of the basic linear knapsack problem. Second, we consider the probability fractile optimization model considering both expected values and variances for Scenario 1. In the case that we assume the probability level $\beta$ to 0.7, this model is given as follows:

(Problem P1)

$$\begin{aligned}&\text{Maximize} \quad 0.066x_1+0.062x_2+0.146x_3+0.173x_4+0.198x_5+0.055x_6+0.128x_7+0.118x_8+0.116x_9\\&\qquad\qquad -K_{0.7}\sqrt{\mathbf{x}^t\mathbf{V}\mathbf{x}}\\&\text{subject to} \quad \sum_{j=1}^{9}x_j=1,\ 0\le x_j\le 0.2,\ \left(j=1,\cdots,9\right)\end{aligned}$$

We solve this problem and obtain the optimal portfolio shown in Table 3.

Table 5.3. Optimal portfolios of problem P1

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | Objective value |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.136 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.064 | 0.0388 |

By considering the variance of each asset, the rate of optimal portfolios changes from problem P, particularly with respect to assets R2, R8 and R9. Table 5.3 shows that assets with not only high expected returns but also low variances tend to be selected in problem P1. Similarly done to problem P1, we consider the probability fractile optimization models for Scenario 2 and Scenario 3, respectively.

(Problem P2 based on Scenario 2)

$$\begin{aligned}
\text{Maximize}\quad & 0.077x_1+0.055x_2+0.164x_3+0.148x_4+0.211x_5+0.051x_6+0.130x_7+0.109x_8+0.127x_9 \\
& -K_{0.7}\sqrt{\boldsymbol{x}^t\mathbf{V}\boldsymbol{x}} \\
\text{subject to}\quad & \sum_{j=1}^{9}x_j=1,\ 0\le x_j\le 0.2,\ \left(j=1,\cdots,9\right)
\end{aligned}$$

(Problem P3 based on Scenario 3)

$$\begin{aligned}
\text{Maximize}\quad & 0.058x_1+0.070x_2+0.126x_3+0.191x_4+0.177x_5+0.067x_6+0.120x_7+0.130x_8+0.109x_9 \\
& -K_{0.7}\sqrt{\boldsymbol{x}^t\mathbf{V}\boldsymbol{x}} \\
\text{subject to}\quad & \sum_{j=1}^{9}x_j=1,\ 0\le x_j\le 0.2,\ \left(j=1,\cdots,9\right)
\end{aligned}$$

We solve these problems and obtain optimal portfolios shown in Tables 5.4 and 5.5.

Table 5.4. Optimal portfolios of problem P2

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | Objective value |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.2 | 0.0416 |

Table 5.5. Optimal portfolios of problem P3

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | Objective value |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.2 | 0.093 | 0.2 | 0.2 | 0.083 | 0.2 | 0 | 0.024 | 0.0352 |

Furthermore, we consider the portfolio model weighted to each scenario in Section 3. This model is formulated as follows:

(Problem $P\left(\boldsymbol{w}\right)$)

$$\begin{aligned}
\text{Maximize}\quad & W_1\left(0.066x_1+0.062x_2+0.146x_3+0.173x_4+0.198x_5+0.055x_6+0.128x_7+0.118x_8+0.116x_9\right) \\
& +W_2\left(0.077x_1+0.055x_2+0.164x_3+0.148x_4+0.211x_5+0.051x_6+0.130x_7+0.109x_8+0.127x_9\right) \\
& +W_3\left(0.058x_1+0.070x_2+0.126x_3+0.191x_4+0.177x_5+0.067x_6+0.120x_7+0.130x_8+0.109x_9\right) \\
& -K_{0.7}\sqrt{\sum_{i=1}^{3}\left(W_i\boldsymbol{x}\right)^t\mathbf{V}\left(W_i\boldsymbol{x}\right)} \\
\text{subject to}\quad & \sum_{j=1}^{9}x_j=1,\ 0\le x_j\le 0.2,\ \left(j=1,\cdots,9\right)
\end{aligned}$$

In the case that we solve this problem with respect to five types of weights, we obtain optimal portfolios and optimal objective values shown in Tables 5.6 and 5.7.

Table 5.6. Optimal portfolios with respect to each weight of problem $P(\boldsymbol{w})$

| $W_1$ | $W_2$ | $W_3$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.6 | 0.2 | 0.2 | 0 | 0.131 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.069 |
| 0.2 | 0.6 | 0.2 | 0 | 0.065 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.135 |
| 0.2 | 0.2 | 0.6 | 0 | 0.2 | 0.153 | 0.2 | 0.2 | 0.003 | 0.2 | 0 | 0.044 |
| 0.6 | 0.3 | 0.1 | 0 | 0.101 | 0.2 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.099 |
| 0.6 | 0.1 | 0.3 | 0 | 0.177 | 0.181 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.042 |

Table 5.7. Optimal objective values with respect to each weight of problem $P(w)$

| $W_1$ | $W_2$ | $W_3$ | Objective value |
|---|---|---|---|
| 0.6 | 0.2 | 0.2 | 0.0380 |
| 0.2 | 0.6 | 0.2 | 0.0388 |
| 0.2 | 0.2 | 0.6 | 0.0361 |
| 0.6 | 0.3 | 0.1 | 0.0387 |
| 0.6 | 0.1 | 0.3 | 0.0373 |

From Table 5.6, we find that the rates of optimal portfolios to assets R2, R3 and R9 are largely different by each weight set $W_1$, $W_2$ and $W_3$ to three scenarios. On the other hand, we also find that the rates of optimal portfolios to assets R4, R5 and R8 do not change with respect to all weight sets. Therefore, we find that the purchase rate of assets R4, R5 and R8 tends to be unaffected by the several random changes of future returns.

Finally, the model introducing the satisfaction function to all scenarios in Subsection 5.4 is considered. In this chapter, we assume each satisfaction function to be the following linear function:

$$\mu_1(Z_1)=\begin{cases} 1 & 0.039 \le Z_1 \\ \dfrac{Z_1-0.035}{0.039-0.035} & 0.035 \le Z_1 \le 0.039 \\ 0 & Z_1 \le 0.035 \end{cases}, \quad \mu_2(Z_2)=\begin{cases} 1 & 0.041 \le Z_2 \\ \dfrac{Z_2-0.037}{0.041-0.037} & 0.037 \le Z_2 \le 0.041 \\ 0 & Z_2 \le 0.037 \end{cases},$$

$$\mu_3(Z_3)=\begin{cases} 1 & 0.035 \le Z_3 \\ \dfrac{Z_3-0.035}{0.035-0.03} & 0.03 \le Z_3 \le 0.035 \\ 0 & Z_3 \le 0.03 \end{cases}$$

Using these satisfaction functions, we introduce the following problem maximizing the minimum aspiration level among all satisfaction functions:

(Problem P4)

$$
\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad 0.004h+0.035-\left(0.066x_1+0.062x_2+0.146x_3+0.173x_4+0.198x_5+0.055x_6+0.128x_7+0.118x_8+0.116x_9\right)+K_{0.7}\sqrt{\mathbf{x}^t\mathbf{V}\mathbf{x}}\le 0 \\
&\qquad 0.004h+0.037-\left(0.077x_1+0.055x_2+0.164x_3+0.148x_4+0.211x_5+0.051x_6+0.130x_7+0.109x_8+0.127x_9\right)+K_{0.7}\sqrt{\mathbf{x}^t\mathbf{V}\mathbf{x}}\le 0 \\
&\qquad 0.005h+0.03-\left(0.058x_1+0.070x_2+0.126x_3+0.191x_4+0.177x_5+0.067x_6+0.120x_7+0.130x_8+0.109x_9\right)+K_{0.7}\sqrt{\mathbf{x}^t\mathbf{V}\mathbf{x}}\le 0 \\
&\qquad \sum_{j=1}^{9}x_j=1,\ 0\le x_j\le 0.2,\ \left(j=1,\cdots,9\right)
\end{aligned}
$$

Table 5.8. Optimal portfolios of problem P4

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | Aspiration level |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.149 | 0.190 | 0.2 | 0.2 | 0 | 0.2 | 0 | 0.061 | 0.709 |

The solution shown in Table 5.8 is an optimal solution of problem P4. We find that this optimal portfolio is similar to that of Scenario 1 shown in Table 3. Then, the rate of optimal portfolio for asset R3 is not 0.2 such as portfolios of problems P1 and P2, but is similar to problem $P\left(0.6, 0.1, 0.3\right)$ in Table 5.6. Therefore, Table 5.8 shows that problem P4 in this numerical example plays a role of the intermediate model with both properties of problems P1 and P3.

### 5.6.2 Probability maximization model

In a way similar to the probability fractile maximization model, in this numerical example, the mean-variance model (P5) and probability maximization model (P6) are given as follows:

(P5: Mean-variance model)

$$
\begin{aligned}
&\text{Minimize} \quad \sum_{j=1}^{9}\sigma_{1j}^2x_j^2 \\
&\text{subject to} \quad \sum_{j=1}^{9}r_{1j}x_j\ge 0.05, \\
&\qquad \sum_{j=1}^{9}x_j=1,\quad 0\le x_j\le 0.3,\quad \left(j=1,2,...,n\right)
\end{aligned}
$$

(P6: Probability maximization model)

$$
\begin{aligned}
&\text{Maximize} \quad \Pr\left\{\sum_{j=1}^{9}r_{1j}x_j\ge 0.05\right\} \\
&\text{subject to} \quad \sum_{j=1}^{9}x_j=1,\quad 0\le x_j\le 0.3,\quad \left(j=1,2,...,n\right)
\end{aligned}
$$

Then, we introduce our proposal models in Sections 7 and 8 as the following problems P3, P4 and P9:

(P7: Probability maximization model setting constant weights)

$$\begin{aligned}&\text{Maximize} \quad \Pr\left\{\sum_{i=1}^{3} w_i \sum_{j=1}^{n} r_{ij} x_j \geq 0.05\right\}\\&\text{subject to} \quad \sum_{j=1}^{9} x_j = 1, \quad 0 \leq x_j \leq 0.3, \quad (j = 1, 2, ..., n)\\&\text{where } \ w_1 = 0.5,\ w_2 = 0.25,\ w_3 = 0.25\end{aligned}$$

(P8: Probability maximization model considering flexibility of weights)

$$\begin{aligned}&\text{Maximize} \quad \Pr\left\{\min_{\boldsymbol{w}\in W}\left\{\sum_{i=1}^{3} w_i \sum_{j=1}^{9} r_{ij} x_j\right\} \geq 0.05\right\}\\&\text{subject to} \quad \sum_{j=1}^{9} x_j = 1, \quad 0 \leq x_j \leq 0.3, \quad (j = 1, 2, ..., n)\\&\qquad\qquad W = \left\{\boldsymbol{w} \middle| 0.3 \leq w_1 \leq 0.7,\ 0.1 \leq w_2 \leq 0.4,\ 0.1 \leq w_3 \leq 0.4\right\}\end{aligned}$$

(P9: Maximization model of aspiration levels)

$$\begin{aligned}&\text{Maximize} \quad h\\&\text{subject to} \quad \mu_{p_i}(Z_i) \geq h, \quad (i = 1, 2, 3)\\&\qquad\qquad \sum_{j=1}^{9} x_j = 1, \quad 0 \leq x_j \leq 0.3, \quad (j = 1, 2, \cdots, n)\end{aligned}$$

where each target value is $f_1 = 0.05,\ f_2 = 0.055,\ f_3 = 0.045$ and each satisfaction function with respect to the probability is as follows:

$$\mu_{p_i}(Z_i) = \begin{cases} 1 & 0.9 \leq Z_i \\ \dfrac{Z_i - 0.7}{0.2} & 0.7 \leq Z_i \leq 0.9, \quad (i = 1, 2, 3) \\ 0 & Z_i \leq 0.7 \end{cases}$$

We solve these problems and obtain the following optimal solutions shown in Table 5.9.

Table 5.9. Optimal portfolio with respect to each problem

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|---|---|---|---|---|---|---|---|---|---|
| P5 | 0.098 | 0.300 | 0.062 | 0.055 | 0.041 | 0.128 | 0.182 | 0.068 | 0.066 |
| P6 | 0.032 | 0.088 | 0.122 | 0.140 | 0.126 | 0.013 | 0.293 | 0.096 | 0.090 |
| P7 | 0.034 | 0.090 | 0.121 | 0.138 | 0.124 | 0.018 | 0.287 | 0.096 | 0.092 |
| P8 | 0.028 | 0.106 | 0.113 | 0.144 | 0.118 | 0.025 | 0.279 | 0.100 | 0.087 |
| P9 | 0.018 | 0.138 | 0.096 | 0.154 | 0.106 | 0.042 | 0.258 | 0.108 | 0.080 |

From Table 5.9, in mean-variance model P1, the assets having lower variances are lower tend to be selected. On the other hand, in basic probability maximization models and our proposed models, assets with not only low variances but also high expected return tend to be selected, considering the balance between both factors. For example, the variance of Asset 2 is the lowest of all assets, but its expected return is lower than the others; therefore, the rate of budgeting allocation in our proposed models is less than that of the mean-variance model. On the other hand, the variance of Asset 4 is higher than the others, but its expected return is also higher. Therefore, the rate of budgeting allocation in our proposed models is higher than that of the mean-variance model. Consequently, Table 5.9 shows that our proposed models tend to select optimal portfolios considering the balance between minimizing the total variance and maximizing the total profit.

Furthermore, with respect to Assets 1 and 2 whose expected returns are very similar but variances are very different, the optimal portfolio of problem P4 including the intervals of weights shows that the budgeting allocation of Asset 2 with the lower variance tend to increase more than that of problems P2 and P3 including fixed weights. Then, with respect to Assets 8 and 9 whose not only the expected returns but also the variances are very similar, we find the same trend as that in case of Assets 1 and 2. Therefore, problem P4 considering robustness tends to select assets with the lower variances in the case of portfolio selection in assets with similar characteristics. Then, we notably find this fact in the budgeting allocation of problem P5. Particularly, the budgeting rates of Assets 3, 7 and 9 are intermediate values between the mean-variance model and probability maximization model. On the other hand, we also find the aspect in which the rate of the asset with certain low variances but lower expected return such as Assets 1 and 6 is lower than that of the other models. Therefore, the maximization model of aspiration levels P5 may be a robust model involving the properties of both the mean-variance model and probability maximization model.

## 5.7 Conclusion

In this chapter, we have proposed two solution approaches for probability fractile optimization models in portfolio selection problems considering possible scenarios with respect to multivariate random future returns. First, we have aggregated multi-objective functions into one weighted function, i.e. the single objective problem, and proposed its analytical solution method. Since weights are flexibly decided for the conditions in real market, we may apply this model to several types of portfolio selection problems under uncertain situations. Second, we have proposed the robust portfolio model maximizing the minimum objective value among all the scenarios and constructed its efficient solution method using the compact factorization. Since this model considers versatile and robust cases in terms of setting aspiration levels according to the decision maker's subjectivity and maximizing the minimum aspiration level, we apply this model to the several risk management problems including multi probabilistic and ambiguous situations.

Next, we have proposed some probability maximization models of portfolio selection problems. First, we have aggregated multi-objective functions into one weighted function, i.e. the single-objective problem, and proposed its efficient solution method using the deterministic equivalent transformations of the main problem. Furthermore, we have proposed a robust portfolio model that considers the interval of each weight and obtains the appropriate portfolio that can apply to all weights including the interval. Since this model considers various conditions in the practical investment by using weights including flexibility, we may apply this model to several types of portfolio selection problems under uncertain situations. Second, we have proposed the model maximizing the minimum aspiration level of probability among all the scenarios and constructed its solution method. Since it is also the model considering the robust case, we apply this model to several problems including multi-probabilistic and ambiguous situations. This problem includes more wide ranging conditions of portfolio selection problems.

As future remarks, we need to consider the case that optimal solutions are restricted to integers and multi-period portfolio selection problems. Then, we also need to consider the more versatile portfolio model to reflect various economic factors considering the tendencies of the real market. Nonetheless, in this chapter, we have developed multiple approaches to portfolio selection problems. Therefore, our approach in this chapter may be helpful in the development of efficient solution methods with respect to these future problems.

# Chapter 6

# Product Mix Problems under Randomness and Fuzziness

In this chapter, we suggest that to maximize the total profit, the product-mix problem should be addressed from multiple perspectives. This thesis proposes using stochastic and fuzzy modeling to address probabilistic and ambiguous factors, flexibility to deal with demand volatility and readiness to make various changes from the original product-mix decision, and the theory of constraints to identify bottlenecks and portfolio selection approaches to deal with risk management.

The most important point at which it is necessary to improve the production process and supply chain is a bottleneck constraint. Unless machinery or human capacity at the bottleneck is improved, it is almost impossible for the production company to increase its profit, and for decision makers to apply an optimal production plan. Therefore, it is very important to consider product-mix decision problems by focusing on bottleneck constraints. At present, some researches have focused on bottleneck constraints in production processes, and this is known as the theory of constraints (TOC), as proposed by Goldratt [37, 38]. Focusing on several previous studies, Balakrishnan and Cheng [7], Finch and Luebbe [33], and Luebbe and Finch [86] have considered comparing TOC and linear programming problems (LP), and have shown that LP is a useful tool in TOC analysis. Lee and Plenert [76] examined the case of the introduction of a new product. Coman and Ronen [23] formulated a production-outsourcing problem as an LP problem, and identified an analytical solution. Aryanezhad and Komijan [6] and Köksal [67] proposed improvements to a TOC-based algorithm. Souren et al. [110] discussed some premises on which to generate optimal product-mix decisions using a TOC-based approach.

Most previous researchers using a TOC-based approach have applied TOC to some such type of product-mix decision problems, but they have not focused on random, ambiguous, and flexible conditions existing at the time when TOC-based product-mix decision problems arise. Given recent uncertainty in the practical markets of products and production processes, an approach to product-mix decision problems through TOC should take into account considerations such as uncertain conditions and decision makers' level of satisfaction, in order to make the product-mix decision robust, and to provide flexibility in responding to many future scenarios and intervals of goals. In approaching product-mix decision problems through TOC, we consider several types, particularly the following: (a) the type that includes problems associated with adding new product alternatives to an existing production line; (b) the type that includes problems concerning more than one bottleneck in which the algorithm cannot converge to the feasible optimum solution.

Bhattacharya and Vasant [13] compiled these product-mix decision problems using TOC, and proposed a fuzzy product-mix decision problem to extend these previous models.

However, in most approaches to product-mix decision problems through TOC, randomness and fuzziness are considered separately; but to represent real product-mix decision cases under the changes of future customers' demands and a large amount of efficient and inefficient information in the real market, it may not be valid to consider future profits as fixed values, random variables, or fuzzy variables. Rather, they should be considered as product-mix decision problems that integrate randomness and fuzziness. Furthermore, in most previous studies, the main focus is not on the concept of flexibility in responding to many different future scenarios. For example, we assume that decision makers consider product-mix decision problems by including various elements of randomness and fuzziness to represent uncertain situations in the real world. As a result, they decide on an unduly strict original product-mix decision. If an unpredictable situation occurs in the future, they will then not earn the profit predicted, due to the limitation of the constraint, even when randomness and fuzziness are included in the model. Therefore, it is important to introduce flexibilities such as considering several future scenarios and their levels of satisfaction, in terms of the target total profit and the upper values of constraints. At present, no model considers random and ambiguous situations, and flexibility and level of satisfaction for objective function and constraints simultaneously, particularly in the case of models that include probabilistic future returns. Therefore, in this chapter, we focus on product-mix decision problems, in order to take into account several constraints, including randomness, ambiguity, and flexibility. Under such uncertain conditions and flexibilities, if the original plan is to function appropriately and smoothly, then it is most important to undertake appropriate risk management, such as the reduction of uncertainty and the improvement of satisfaction of customers, workers, and decision makers.

Therefore, by extending the risk management methods used in the portfolio theory to product-mix decision problems, we propose new and versatile product-mix decision problems. In particular, we propose the following flexible models under randomness, fuzziness, and flexibility: (a) a probability fractile optimization model of total future profits, and (b) a probability maximization model of total future profits. These mathematical programming problems with randomness and fuzziness are called stochastic and fuzzy programming problems (for example, Liu [83, 84]), and are usually transformed into nonlinear programming problems by setting the target values and using chance constraints. Since it is almost impossible to obtain their global optimal solution directly, we construct the efficient solution method to obtain the global optimal solution by performing the equivalent transformation for several nonlinear programming problems.

## 6.1 The Formulation of Proposed Models Considering Uncertainty and Flexibility

The following notations are used in the chapter:

$x_j$ : Volume of production units of $j$ th product

$r_j$ : Future return of $j$ th product

$a_{kj}$ : Coefficient of $j$ th product of $k$ th constraint; i.e., resource constraint, time constraint, and personnel constraint.

$b_k$ : Maximum value with respect to constraint $k$

$c_j$ : Fund cost of each product $j$

$p_j$ : Maximum volume of $j$ th production units

$w$ : Maximum value of total available fund

$i, j$ : Index of products

$k$ : Index of constraints

$m$ : Total number of constraints

$n$ : Total number of products

In this chapter, we mainly focus on maximizing the total profit under several constraints in production processes. Generally, a basic product-mix decision model maximizing the total future profit is formulated as follows.

$$\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} r_j x_j \\
&\text{subject to} \quad \sum_{j=1}^{n} a_{kj} x_j \le b_k, \quad k = 1, 2, \cdots, m \\
&\qquad\qquad \sum_{j=1}^{n} c_j x_j \le w, \quad 0 \le x_j \le p_j, \quad j = 1, 2, \cdots, n
\end{aligned} \tag{6.1}$$

This problem is a linear programming problem. Therefore, we efficiently find an optimal solution by using linear programming approaches such as the simplex method or interior point method. However, in practical production processes, there are some probabilistic conditions such as breakdown of machines and change of customer's demand surrounding the real market, as well as ambiguous conditions derived from lack of reliable information and intuition of veteran workers. Hence, considering these situations, coefficients of objective functions and constraints are not fixed values but values including randomness and fuzziness.

In this chapter, we assume that each future return $r_j$ occurs according to a normal distribution $N\left(\bar{r}_j, \sigma_j^2\right)$ where $\bar{r}_j$ are the mean values of $r_j$ and $\sigma_j^2$ are its variances. Then, we represent the $ij$th element of a variance-covariance matrix for the returns $r_j$ as $\sigma_{ij}$. Furthermore, we assume each $a_{kj}$ and $c_j$ to be the following L fuzzy numbers:

$$\mu_{\tilde{a}_{kj}} = L\left(\frac{a_{kj}-\omega}{d_{kj}}\right),\ \mu_{\tilde{c}_j} = L\left(\frac{c_j-\omega}{d_{wj}}\right)$$

where $L(x)$ is a nonincreasing and nonnegative function on $[0,\infty)$ satisfying $L(0)=1$ and $L(1)=0$. Then, $d_{kj}$ and $d_{wj}$ represent a spread of the fuzzy numbers, and these are positive values. Furthermore, in the case that decision makers consider the production plan, they generally set goals for the total return and the total cost, respectively. Then, if the expected total profit is more than the setting goal, it is obvious that they are sufficiently satisfied with the production plan. On the other hand, even if the expected total profit is a little less than the setting goal, they may be accordingly satisfied with the production plan. In the real world, the setting goal often has a flexibility to execute the production planning smoothly and is represented as an interval value. This aspect is equivalently applied to the total cost. In this chapter, we introduce a level of satisfaction considering the flexibility of the production plan and the subjectivity of the decision maker. Therefore, we introduce the following satisfaction functions based on fuzzy programming approaches:

$$\mu_f\left(\sum_{j=1}^n r_j x_j\right) = \begin{cases} 1, & f_1 \le \sum_{j=1}^n r_j x_j \\ f\left(\sum_{j=1}^n r_j x_j\right), & f_0 \le \sum_{j=1}^n r_j x_j \le f_1, \\ 0, & \sum_{j=1}^n r_j x_j \le f_0 \end{cases}$$

$$\mu_{g_w}\left(\sum_{j=1}^n c_j x_j\right) = \begin{cases} 1, & \sum_{j=1}^n a_{kj} x_j \le w_1 \\ g_w\left(\sum_{j=1}^n c_j x_j\right), & w_1 \le \sum_{j=1}^n c_j x_j \le w_0, \\ 0, & w_0 \le \sum_{j=1}^n c_j x_j \end{cases} \tag{6.2}$$

$$\mu_{g_k}\left(\sum_{j=1}^n a_{kj} x_j\right) = \begin{cases} 1, & \sum_{j=1}^n a_{kj} x_j \le b_{k1} \\ g_k\left(\sum_{j=1}^n a_{kj} x_j\right), & b_{k1} \le \sum_{j=1}^n a_{kj} x_j \le b_{k0},\ k=1,2,...,m \\ 0, & b_{k0} \le \sum_{j=1}^n a_{kj} x_j \end{cases}$$

where $f(x)$ is a strictly monotonous increasing function and $g_k(x), (k=1,\cdots,m,w)$ are strictly monotonous decreasing functions. Using these membership functions, in this section, we consider the following mini-max mathematical programming problem to maximize the minimum level of satisfaction among all objectives and constraints as much as possible:

$$\begin{aligned}&\text{Maximize} \quad \min\left[\mu_f\left(\sum_{j=1}^{n} r_j x_j\right),\ \mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right), (k=1,2,\cdots,m,w)\right]\\&\text{subject to} \quad 0 \le x_j \le p_j,\ j=1,2,\cdots,n\end{aligned} \tag{6.3}$$

In the case that a minimum level of satisfaction is set as parameter $h$, whereby a decision maker has the degree of satisfaction such as she or he approximately earns the total profit rather than a target profit, the objective function in problem (6.3) $\min\left[\mu_f\left(\sum_{j=1}^{n} r_j x_j\right),\ \mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right), (k=1,2,\cdots,m,w)\right]$ means that all membership functions are larger than $h$. Therefore, problem (6.3) is equivalently transformed into the following problem:

$$\begin{aligned}&\text{Maximize} \quad h\\&\text{subject to} \quad \mu_f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h,\ \mu_w\left(\sum_{j=1}^{n} c_j x_j\right) \ge h,\ \mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h,\ (k=1,2,\cdots,m)\\&\qquad\qquad 0 \le x_j \le p_j,\ j=1,2,\cdots,n\end{aligned} \tag{6.4}$$

Subsequently, in order to solve problem (6.4) analytically using mathematical programming approaches, the constraints of problem (6.4) are transformed into the following forms based on fuzzy programming approaches.

$$\begin{aligned}&\mu_f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h \iff f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h \iff \sum_{j=1}^{n} r_j x_j \ge f^{-1}(h),\\&\mu_{g_w}\left(\sum_{j=1}^{n} c_j x_j\right) \ge h \iff g_w\left(\sum_{j=1}^{n} c_j x_j\right) \ge h \iff \sum_{j=1}^{n} c_j x_j \le g_w^{-1}(h),\\&\mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h \iff g_k\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h \iff \sum_{j=1}^{n} a_{kj} x_j \le g_k^{-1}(h),\ (k=1,2,\cdots,m)\end{aligned} \tag{6.5}$$

where $f^{-1}(h)$ and $g_k^{-1}(h)$ are inverse functions of $f$ and $g$, respectively. Using these inequalities, we equivalently transform problem (6.4) into the following problem:

$$\begin{aligned}&\text{Maximize} \quad h\\&\text{subject to} \quad \sum_{j=1}^{n} r_j x_j \ge f^{-1}(h),\ \sum_{j=1}^{n} c_j x_j \le g_w^{-1}(h),\\&\qquad\qquad \sum_{j=1}^{n} a_{kj} x_j \le g_k^{-1}(h),\ (k=1,2,\cdots,m)\\&\qquad\qquad 0 \le x_j \le p_j,\ j=1,2,\cdots,n\end{aligned} \tag{6.6}$$

Consequently, problem (6.6) is a product-mix decision problem with interval values to the target profit and the upper values of constraints. Subsequently, in the case that the decision maker does not consider the membership functions, i.e., $h=1$, problem (6.6) is degenerated to a basic product-mix decision problem (6.1). Therefore, problem (6.6) is a versatile model including previous product-mix decision problems.

However, problem (6.6) is not a well-defined problem due to the inclusion of random variables $r_j$ and fuzzy numbers $a_{kj}$ and $c_j$. Thus, in order to control uncertain factors as much as possible and solve this problem analytically, we need to set a criterion with respect to probability and possibility for objective function and constraints, and transform the original problem into the deterministic optimization model. Therefore, as main flexible product-mix decision problems, we consider the following stochastic programming problems in Section 6.2.

## 6.2 Stochastic Programming Problems of Flexible Product Mix Decision Problem

Mathematical programming problems considering randomness are generally called stochastic programming problems, and there are two standard models using chance constraints in stochastic programming problems: (a) probability fractile optimization model to total future profits, and (b) probability maximization model to total future profits. We discuss each of these problems below.

### 6.2.1 Probability fractile optimization model

In this problem, a decision maker considers maximizing the goal for total future profit. Subsequently, we assume that parameter $\beta$ means the probability that the total profit is more than or equal to target value $f^{-1}(h)$, and parameter $\alpha$ means the possibility that each total cost is less than or equal to target value $g_k^{-1}(h)$ as far as possible based on the fuzzy theory and the fuzzy programming approach. From these assumptions, we introduce chance constraints as follows.

$$
\begin{aligned}
&\Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f^{-1}(h)\right\} \geq \beta, \quad \text{for } \sum_{j=1}^{n} r_j x_j \geq f^{-1}(h) \\
&\operatorname{Pos}\left\{\sum_{j=1}^{n} a_{kj} x_j \leq g_k^{-1}(h)\right\} \geq \alpha, \quad \text{for } \sum_{j=1}^{n} a_{kj} x_j \leq g_k^{-1}(h),\ k=1,2,\cdots,m \qquad (6.7) \\
&\operatorname{Pos}\left\{\sum_{j=1}^{n} c_j x_j \leq g_w^{-1}(h)\right\} \geq \alpha, \quad \text{for } \sum_{j=1}^{n} c_j x_j \leq g_w^{-1}(h)
\end{aligned}
$$

By introducing these chance constraints into problem (6.6), previous product-mix decision problems are extended to the following probability fractile maximization model, whereby the decision maker make an optimal product-mix decision to maximize the total profit as much as possible, by controlling randomness derived from a statistical analysis of received data and considering the possibility of reducing cost coefficients in the constraints:

$$\begin{aligned}
&\text{Maximize} \quad f^{-1}(h) \\
&\text{subject to} \quad \Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f^{-1}(h)\right\} \geq \beta, \\
&\qquad \mathrm{Pos}\left\{\sum_{j=1}^{n} c_j x_j \leq g_w^{-1}(h)\right\} \geq \alpha, \\
&\qquad \mathrm{Pos}\left\{\sum_{j=1}^{n} a_{kj} x_j \leq g_k^{-1}(h)\right\} \geq \alpha, \ (k = 1, 2, \cdots, m) \\
&\qquad 0 \leq x_j \leq p_j, \ j = 1, 2, \cdots, n
\end{aligned} \tag{6.8}$$

In order to solve this problem using an analytical solution method and general solvers, we need to equivalently transform stochastic and possibilistic constraints into deterministic inequalities. First, the stochastic chance constraint is transformed into the following inequality based on the property of normal distribution:

$$\begin{aligned}
&\Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f^{-1}(h)\right\} \geq \beta \Leftrightarrow \Pr\left\{\frac{\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \geq \frac{f^{-1}(h) - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}}\right\} \geq \beta \\
&\Leftrightarrow \frac{\sum_{j=1}^{n} \bar{r}_j x_j - f^{-1}(h)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \geq K_\beta \Leftrightarrow \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \geq f^{-1}(h)
\end{aligned} \tag{6.9}$$

where $F(y)$ is the distribution function of the standard normal distribution and $K_\beta = F^{-1}(\beta)$.

Furthermore, each chance constraint for possibility is also transformed into the following inequalities based on the studies of possibility theory (for example, Inuiguchi and Ramik [52], Katagiri et al. [63]):

$$\begin{aligned}
&\mathrm{Pos}\left\{\sum_{j=1}^{n} c_j x_j \leq g_w^{-1}(h)\right\} \geq \alpha \Leftrightarrow \sum_{j=1}^{n} c_j x_j - L^*(\alpha)\sum_{j=1}^{n} d_{wj} \leq g_w^{-1}(h), \\
&\mathrm{Pos}\left\{\sum_{j=1}^{n} a_{kj} x_j \leq g_k^{-1}(h)\right\} \geq \alpha \Leftrightarrow \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \leq g_k^{-1}(h), \ k = 1, 2, \cdots, m
\end{aligned} \tag{6.10}$$

where $L^*(\alpha)$ is the pseudo inverse function of $L$. From these chance constraints, we transform problem (6.9) into the following problem:

$$
\begin{aligned}
\text{Maximize} \quad & f^{-1}(h) \\
\text{subject to} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \geq f^{-1}(h), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \leq g_w^{-1}(h), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \leq g_k^{-1}(h), \ (k = 1, 2, \cdots, m) \\
& 0 \leq x_j \leq p_j, \ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.11}
$$

This problem includes some basic previous models. For example, in the case that decision makers do not consider randomness, i.e., each $\sigma_{ij} = 0$, problem (6.11) degenerates to a fuzzy linear programming problem such as some previous models (Mula et al. [95, 96], Vasant [116]). Then, if they also do not consider fuzziness, i.e., $\alpha = 1$, problem (6.11) is equivalent to a basic product-mix decision problem (6.6). Therefore, problem (6.11) is a versatile model including many previous product-mix decision problems.

Furthermore, in the case that $f^{-1}(h)$ and all $g_k^{-1}(h)$ are linear functions, problem (6.11) is equivalent to a convex programming problem. Therefore, we solve this problem using convex programming approaches. Particularly, in the case that only one constraint in problem (6.11) is an active constraint, i.e.,

$$
\begin{aligned}
& \sum_{j=1}^{n} a_{k'j} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{k'j} = g_{k'}^{-1}(h), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h), \ (k = 1, 2, ..., m; \ k \neq k')
\end{aligned}
,
$$

problem (6.11) is equivalent to a mathematical programming problem for portfolio selection models. Therefore, we obtain the strict optimal solution by applying the efficient solution method based on the previous studies (Hasuike [42]) of this problem.

However, in practical decision making of each production volume, it would take more computational time to solve convex programming problems even if the ability of calculation machines were to be improved greatly. In the case that decision makers use problem (6.11), it is important that they consider not only including some uncertain situations but also constructing a new rapid and efficient solution method to deal with the real decision cases practically.

In Section 2, we have assumed that each future return $r_j$ occurred according to a normal distribution. Using the property of normal distribution, we introduce the following mean-absolute deviation.

$$W\left[R\left(\boldsymbol{x}\right)\right]=E\left[\left|\sum_{j=1}^{n}r_{j}x_{j}-\sum_{j=1}^{n}\overline{r}_{j}x_{j}\right|\right] \tag{6.12}$$

With respect to the relation between this mean-absolute deviation $W[R(\boldsymbol{x})]$ and the variance $\sigma^2\left(\boldsymbol{x}\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j$ , $\sigma^2\left(\boldsymbol{x}\right)=\frac{\pi}{2}\left(W\left[R\left(\boldsymbol{x}\right)\right]\right)^2$ holds based on the result obtained by Konno (Konno [68]). This means that problem (6.11) is equivalently transformed into the following problem not including the square root term $\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j}$ :

$$\begin{aligned}
&\text{Maximize } f^{-1}(h)\\
&\text{subject to } \sum_{j=1}^{n}\overline{r}_jx_j-K_\beta\sqrt{\frac{\pi}{2}}W\left[R\left(\boldsymbol{x}\right)\right]\geq f^{-1}(h),\\
&\qquad\sum_{j=1}^{n}c_jx_j-L^*(\alpha)\sum_{j=1}^{n}d_{wj}\leq g_w^{-1}(h),\\
&\qquad\sum_{j=1}^{n}a_{kj}x_j-L^*(\alpha)\sum_{j=1}^{n}d_{kj}\leq g_k^{-1}(h),\ k=1,2,...,m\\
&\qquad 0\leq x_j\leq p_j,\ j=1,2,...,n,
\end{aligned} \tag{6.13}$$

Furthermore, in order to consider the case that decision makers generally receive historical data in the real world and solve problem (6.13) more efficiently, we introduce the following return set $\boldsymbol{r}_s$ derived from each normal distribution $r_j$ :

$$\boldsymbol{r}_s=\left(r_{s1},r_{s2},...,r_{sn}\right),\ \left(s=1,2,...,S\right) \tag{6.14}$$

where $S$ is a total number of scenarios and each occurrence probability $\Pr\left\{\boldsymbol{r}_s=\left(r_{s1},r_{s2},...,r_{sn}\right)\right\}$ is assumed to be $\frac{1}{S}$, respectively. From this return set, we obtain

$$W\left[R\left(\boldsymbol{x}\right)\right]=E\left[\left|\sum_{j=1}^{n}\left(r_j-\overline{r}_j\right)x_j\right|\right]=\frac{1}{S}\sum_{s=1}^{S}\left|\sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j\right| \tag{6.15}$$

Therefore, problem (6.13) is transformed into the following problem:

$$\begin{aligned}
&\text{Maximize } f^{-1}(h) \\
&\text{subject to } \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\frac{\pi}{2}} \left( \frac{1}{S} \sum_{s=1}^{S} \left| \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \right| \right) \geq f^{-1}(h), \\
&\qquad \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \leq g_w^{-1}(h), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \leq g_k^{-1}(h), \ (k = 1, 2, ..., m) \\
&\qquad 0 \leq x_j \leq p_j, \ j = 1, 2, ..., n
\end{aligned} \tag{6.16}$$

In this problem, since it includes absolute deviations $\left| \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \right|, \ (s = 1, 2, ..., S)$, it is not easy to solve it directly. Subsequently, we consider the following subproblem introducing parameters $y_s$ to remove the absolute deviations:

$$\begin{aligned}
&\text{Maximize } f^{-1}(h) \\
&\text{subject to } \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\frac{\pi}{2}} \left( \frac{1}{S} \sum_{s=1}^{S} y_s \right) \geq f^{-1}(h), \\
&\qquad y_s - \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \geq 0, \ y_s + \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \geq 0, \ \ (s = 1, 2, ..., S) \\
&\qquad \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \leq g_w^{-1}(h), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \leq g_k^{-1}(h), \ (k = 1, 2, ..., m) \\
&\qquad 0 \leq x_j \leq p_j, \ j = 1, 2, ..., n
\end{aligned} \tag{6.17}$$

and the following theorem is derived with respect to the relation between problems (6.16) and (6.17) by extending the study of Konno (Konno [68]).

**Theorem 6.1**
The optimal solution of problem (6.16) is also optimal for problem (6.17).

This theorem means that we obtain the optimal product-mix decision of the main problem (6.8) by solving problem (6.17) due to equivalent transformations from (6.9) to (6.17). In problem (6.17), in the case that $f^{-1}(h)$ and all $g_k^{-1}(h)$ are linear functions, problem (6.17) is equivalent to a linear programming problem. Therefore, by using various types of efficient solution methods for linear programming, we find that it is easier to obtain an optimal solution efficiently than the main convex programming problem (6.11).

Furthermore, membership functions (6.2) decided by decision makers are often nonlinear functions. This case means that problem (6.17) is not a linear programming problem of that form. However, even if $f^{-1}(h)$ and all $g_k^{-1}(h)$ are nonlinear functions, by fixing parameter $h$ and introducing the following problem:

$$
\begin{aligned}
&\text{Maximize } \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\frac{\pi}{2}} \left( \frac{1}{S} \sum_{s=1}^{S} y_s \right) \\
&\text{subject to } y_s - \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \geq 0,\ y_s + \sum_{j=1}^{n} \left( r_{sj} - \bar{r}_j \right) x_j \geq 0, \quad (s = 1,2,\ldots,S) \\
&\qquad \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \leq g_w^{-1}\left(\bar{h}\right), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \leq g_k^{-1}\left(\bar{h}\right),\ (k = 1,2,\ldots,m) \\
&\qquad 0 \leq x_j \leq p_j,\ j = 1,2,\ldots,n
\end{aligned}
\tag{6.18}
$$

we construct the following analytical solution method. Since problem (6.18) is a linear programming problem, it is also easy to obtain the optimal solution of problem (6.18). Then, using a bisection algorithm with respect to parameter $h$, in the case that the value of objective function in problem (6.18) is equal to $f^{-1}\left(\bar{h}\right)$ in problem (6.17), the optimal solution of problem (6.18) is equal to that of problem (17). Consequently, we construct the following solution method.

**Solution method 6.1**

STEP 1: Elicit the membership functions of fuzzy goals for the objective function and cost constraints.

STEP 2: If all membership functions are linear functions, solve problem (6.17) and obtain the optimal solution. Otherwise, go to STEP 3.

STEP 3: Set $L_h \leftarrow 0$ and $U_h \leftarrow 0$.

STEP 4: Set $h \leftarrow \dfrac{L_h + U_h}{2}$.

STEP 5: Calculate $f^{-1}(h)$ and $g_k^{-1}(h)$, $(k = 1,2,\ldots,m)$.

STEP 6: Solve problem (6.18) on each $g_k^{-1}(h)$, and calculate the optimal objective value of problem (6.18) $Z^*(h)$.

STEP 7: If $Z^*(h) < f^{-1}(h)$, $L_h \leftarrow h$ and go to STEP 4. If $Z^*(h) > f^{-1}(h)$, $U_h \leftarrow h$ and go to STEP 4. If $Z^*(h) = f^{-1}(h)$, its optimal solution is a global optimal solution of the main problem and terminates this algorithm.

In the case that we solve the main problem (6.8) using this solution method, if the number of scenarios $S$ is sufficiently large, this solution method is a strict solution method. Furthermore, even if $S$ is not sufficiently large, by satisfying the following conditions and using equivalent transformations, this solution method becomes a strict solution method. This approach may use the case that decision makers manage to obtain the more detailed product-mix decision in order to draw up an accurate and important future plan.

As steps for obtaining the detailed product-mix decision, we introduce the following problem using a similar way to the transformation from problem (6.17) to (6.18) with respect to problem (6.11):

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} = g_k^{-1}(h),\ (k \in K), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h),\ (k \notin K), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
& 0 \le x_j \le p_j,\ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.19}
$$

where $K$ is an index set of bottleneck constraints obtained by solving problem (6.17). However, due to including a square root term in the objective function, it is also difficult to solve problem (6.19) efficiently using general solvers. Therefore, in order to perform equivalent transformations into problems solving more easily than problem (6.19), we introduce the following auxiliary problem by introducing a parameter $R$:

$$
\begin{aligned}
\text{Maximize} \quad & R \sum_{j=1}^{n} \bar{r}_j x_j - \frac{K_\beta}{2} \left( \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \right) \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} = g_k^{-1}(h),\ (k \in K), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h),\ (k \notin K), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
& 0 \le x_j \le p_j,\ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.20}
$$

Since the objective function of this problem is a quadratic function, this problem is a quadratic convex programming problem. Consequently, we easily find a global optimal solution. Subsequently, with respect to the relation between problem (6.19) and problem (6.20), the following theorem holds.

**Theorem 6.2**

Let $x^*$ be an optimal solution of problem (6.20). If $R^* = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^* x_j^*}$ is satisfied, $x^*$ is also an optimal solution of problem (6.19).

**Proof**

By comparing the Karush-Kuhn-Tucker (KKT) condition of problem (6.19) with that of problem (6.20) and simply adjusting each Lagrange multiple, it is clear that both KKT conditions are same in the case $R^* = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i^* x_j^*}$.

This theorem means that the strict optimal solution of the main problem (6.8) is equal to the solution of the following KKT condition for problem (6.20) considering that the form of optimal solutions is determined using Solution method 1, i.e., $x_j = p_j$, $x_j = p'_j,\ \left(0 < p'_j < p_j\right)$ or $x_j = 0$:

(KKT condition)

$$
\begin{aligned}
& R\bar{r}_j - K_\beta \sum_{i=1}^{n}\sigma_{ij}x_i + \sum_{k=1}^{m}\lambda_k a_{kj} + \xi c_j = 0, \\
& \lambda_k\left(\sum_{j=1}^{n}a_{kj}x_j - L^*(\alpha)\sum_{j=1}^{n}d_{kj} - g_k^{-1}(h)\right) = 0,\ \lambda_k > 0,\ (k \in K) \\
& \xi\left(\sum_{j=1}^{n}c_j x_j - L^*(\alpha)\sum_{j=1}^{n}d_{wj} - g_w^{-1}(h)\right) = 0, \\
& R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\
& j \in \left\{w \middle| x_w = p'_w,\ 0 < p'_w < p_w\right\},\ k \in K
\end{aligned}
\tag{6.21}
$$

where parameters $\lambda_k,\ \xi,\ u_j, v_j$ are Lagrange multiples of problem (6.20). All of these equations except for $R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j$ in KKT conditions are linear, and $R^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j$ is also a quadratic equation. Therefore, it is easy to solve these equations, and we analytically obtain the strict solution method of the main problem (6.13).

### 6.2.2 Probability maximization model

In this subsection, we consider the main problem (6.8) as a probability maximization model for total future profits, maximizing the level of satisfaction. This model is mainly focused on the objective that decision makers control reducing randomness as much as possible under the case that future returns randomly change and earn a total profit that is more than the target value. Therefore, this model will be used in the case that decision makers consider controlling the total risk. In a similar way to Subsection 3.1, we formulate this model as the following form:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} \bar{r}_j x_j - f^{-1}(h)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \\
&\text{subject to} \quad \sum_{j=1}^{n} c_j x_j - L^*(\alpha)\sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h), \ \ (k = 1,2,\cdots,m) \\
&\qquad 0 \le x_j \le p_j, \ j = 1,2,\cdots,n
\end{aligned} \tag{6.22}$$

This problem is a bi-criteria programming problem, and so this solution generally becomes a Pareto solution. Subsequently, we consider the relation between the level of decision maker's satisfaction and the risk management through maximizing the accomplishment probability under uncertain conditions. Then, in this chapter, we consider the probability maximization model that a decision maker sets the desired level of satisfaction. The level of satisfaction $h$ is assumed to be $\bar{h}$ and problem (6.22) is transformed into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad \frac{\sum_{j=1}^{n} \bar{r}_j x_j - f^{-1}(\bar{h})}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \\
&\text{subject to} \quad \sum_{j=1}^{n} c_j x_j - L^*(\alpha)\sum_{j=1}^{n} d_{wj} \le g_w^{-1}(\bar{h}), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(\bar{h}), \ \ (k = 1,2,\cdots,m) \\
&\qquad 0 \le x_j \le p_j, \ j = 1,2,\cdots,n
\end{aligned} \tag{6.23}$$

With respect to this problem, if fuzziness is also not considered, i.e., $\alpha = 1$, problem (6.23) degenerates into a standard probability maximization model. Therefore, problem (6.23) is a versatile model including some previous product mix problems. In mathematical programming, problem (6.23) is a nonlinear fractional programming problem. Particularly, in the case that the only constraint in problem (6.23) is an active constraint, i.e.,

$$\begin{aligned}
&\sum_{j=1}^{n} a_{k'j} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{k'j} = g_{k'}^{-1}(\bar{h}), \\
&\sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} < g_k^{-1}(\bar{h}), \ \ (k = 1,2,\ldots,m;\ k \ne k')
\end{aligned} ,$$

problem (6.23) is equivalent to a fractional continuous knapsack problem. Therefore, we obtain the strict optimal solution by applying the efficient solution method based on the previous studies (for example, Ishii and Nishida [54]) of this problem.

However, this case is rare in production processes and there often exist several bottleneck constraints. Then, generally speaking, it is difficult to solve the present form of problem (6.23) directly and efficiently using general solvers. Subsequently, in a way similar to Subsection 6.2.1, we introduce a mean-absolute deviation in order to solve problem (6.23) more efficiently. Then, problem (6.23) is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize } \frac{\sum_{j=1}^{n}\overline{r}_j x_j - f^{-1}\left(\overline{h}\right)}{\sqrt{\frac{\pi}{2}\left(\frac{1}{S}\sum_{s=1}^{S} y_s\right)}} \\
&\text{subject to } y_s - \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j \geq 0,\ y_s + \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j \geq 0,\quad \left(s=1,2,...,S\right) \\
&\qquad \sum_{j=1}^{n} c_j x_j - L^*\left(\alpha\right)\sum_{j=1}^{n} d_{wj} \leq g_w^{-1}\left(\overline{h}\right), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*\left(\alpha\right)\sum_{j=1}^{n} d_{kj} \leq g_k^{-1}\left(\overline{h}\right),\ \left(k=1,2,...,m\right) \\
&\qquad 0 \leq x_j \leq p_j,\ j=1,2,...,n
\end{aligned}
\tag{6.24}
$$

This problem is also a linear fractional programming problem. Subsequently, we introduce the following parameter to remove the fractional function:

$$
\xi = \frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{S}\sum_{s=1}^{S} y_s\right)}},\ x_j' = \xi x_j,\ \ y_s' = \xi y_s
$$

Using these parameters, problem (6.24) is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize } \sum_{j=1}^{n}\overline{r}_j x_j' - f^{-1}\left(\overline{h}\right)\xi \\
&\text{subject to } y_s' - \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j' \geq 0,\ y_s' + \sum_{j=1}^{n}\left(r_{sj}-\overline{r}_j\right)x_j' \geq 0,\quad \left(s=1,2,...,S\right) \\
&\qquad \sqrt{\frac{\pi}{2}\left(\frac{1}{S}\sum_{s=1}^{S} y_s'\right)} = 1, \\
&\qquad \sum_{j=1}^{n} c_j x_j - L^*\left(\alpha\right)\sum_{j=1}^{n} d_{wj} \leq g_w^{-1}\left(\overline{h}\right), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j' - \left(L^*\left(\alpha\right)\sum_{j=1}^{n} d_{kj}\right)\xi \leq g_k^{-1}\left(\overline{h}\right)\xi,\ \left(k=1,2,...,m\right) \\
&\qquad 0 \leq x_j' \leq p_j\xi,\ j=1,2,...,n
\end{aligned}
\tag{6.25}
$$

Using these equivalent transformations from (6.23) to (6.25), it is shown that the decision maker obtains the optimal product-mix decision of the main problem (6.22) by solving problem (6.25). This problem is a linear programming problem, and so we easily and efficiently obtain the optimal solution in the case that the value of the level of satisfaction is $\bar{h}$. Consequently, we construct the following solution method for the probability maximization model.

**Solution method 6.2**

STEP 1: Elicit the membership functions of fuzzy goals for the objective function and cost constraints.

STEP 2: Ask the decision maker to set the level of satisfaction $h$

STEP 3: Calculate $f^{-1}(h)$ and $g_k^{-1}(h)$, $(k=1,2,...,m)$

STEP 4: Solve problem (6.25) and calculate the current optimal value of probability in problem (6.24)

STEP 5: If the decision maker is satisfied with the current probability in STEP 4 and its level of satisfaction $h$, then terminate this algorithm. The current optimal solution is a satisfying solution for the decision maker. Otherwise, go to STEP 6.

STEP 6: If the decision maker considers that the current probability is less than her or his expected value or the level of satisfaction can be set higher due to sufficiently high current probability, update the level of satisfaction $h$ and return to STEP 3.

Furthermore, in a way similar to Subsection 3.1, if the number of scenarios $S$ is sufficiently large, this solution method is a strict solution method. Furthermore, even if $S$ is not sufficiently large, by considering the following problem, this solution method becomes a strict solution method.

With respect to problem (6.23), we introduce the following subproblem using a parameter $\lambda$ based on the previous study of Dinkelbach [26].

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - f^{-1}(h) - \lambda \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} = g_k^{-1}(h), \ (k \in K), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h), \ (k \notin K), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
& 0 \le x_j \le p_j, \ j = 1,2,\cdots,n
\end{aligned}
\tag{6.26}
$$

Dinkelbach [26] showed that the optimal solution of problem (6.26) was equal to that of problem (23). Then, this problem is similar to problem (6.19) and the main model in the previous study of Ishii and Nishida [54]. Therefore, by combining the solution method 1 in Subsection 3.1 and the extended method of Ishii and Nishida [54], we can analytically obtain the more detailed optimal product-mix decision of the main problem (6.22).

## 6.3 The Case Considering Improvement of Bottleneck Constraints for the Fund Injection

In Subsections 6.2.1 and 6.2.2, we constructed the solution method to find the present optimal product-mix decision and bottleneck constraints. In the case that there are some surplus funds in the original product-mix decision and decision makers consider the fund injection to the bottleneck constraints, these constraints may be improved and the decision maker may earn more profits. In contrast, as a result of the failure of improvement, it may be possible that profits decrease due to more cost and time. In this chapter, we consider that this situation occurs randomly and the model under this condition is formulated as the following two forms.

### 6.3.1 Improvement of maximum limited value of bottleneck constraint

First, we consider the case that a decision maker manages to improve the maximum value of each bottleneck constraint by performing fund injections. Subsequently, we set the fund injection for $k$th bottleneck constraint as $y_k,\ k \in K$, and assume the improved value $b_{y_k}$ by the fund injection $y_k$ occurs according to a normal distribution $N\left(\bar{b}_{y_k}, \sigma_{y_k}^2\right)$ where $\bar{b}_{y_k}$ is the expected value and $\sigma_{y_k}^2$ is its variance.

Furthermore, we assume that its expected value $\bar{b}_{y_k}$ and variance $\sigma_{y_k}^2$ depend on the fund injection $y_k$. In this chapter, functions of the expected value and variance are given as $\bar{b}_{y_k} = B(y_k)$ and $\sigma_{y_k}^2 = \sigma(y_k)$, respectively. Subsequently, $B(y_k)$ and $\sigma(y_k)$ are monotonously increasing. For the sake of simplifying our discussion, we assume $B(y_k)$ and $\sigma(y_k)$ are following functions; $B(y_k) = b_k y_k$, $\sigma(y_k) = \sigma_k^2 y_k^2$, respectively; and parameters $b_k$ and $\sigma_k^2$ are constant.

Then, we formulate this model as follows.

$$
\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j} \\
&\text{subject to} \quad \sum_{j=1}^{n} c_j x_j + \sum_{k\in K} y_k \le w, \\
&\qquad \sum_{j=1}^{n} a_{kj}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) + b_{y_k},\ (k=1,2,\cdots,m) \\
&\qquad 0 \le x_j \le p_j,\ j=1,2,\cdots,n \\
&\text{where } b_{y_k} = \begin{cases} b_{y_k}, & k\in K \\ 0, & k \notin K \end{cases}
\end{aligned}
\tag{6.27}
$$

In this problem, each constraint $\sum_{j=1}^{n} a_{kj}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) + b_{y_k}$ includes the normal distribution variable $b_{y_k}$. Therefore, in a similar way to transformation (6.9), we introduce a goal of probability $\beta_{y_k}$ set by the decision maker and transform this constraint into the following form introducing a chance constraint:

$$
\begin{aligned}
&\Pr\left\{\sum_{j=1}^{n} a_{ij}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{ij} \le g_i^{-1}(h) + b_{y_k}\right\} \ge \beta_{y_k} \\
&\Leftrightarrow \quad \Pr\left\{b_{y_k} \ge \sum_{j=1}^{n} a_{ij}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{ij} - g_i^{-1}(h)\right\} \ge \beta_{y_k} \\
&\Leftrightarrow \quad \sum_{j=1}^{n} a_{ij}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{ij} - g_i^{-1}(h) - \bar{b}_{y_k} \le -K_{\beta_{y_k}}\sigma_{y_k}
\end{aligned}
$$

Then, problem (6.27) is equivalently transformed into the following problem.

$$
\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j} \\
&\text{subject to} \quad \sum_{j=1}^{n} c_j x_j + \sum_{k\in K} y_k \le w, \\
&\qquad \sum_{j=1}^{n} a_{ij}x_j - L^*(\alpha)\sum_{j=1}^{n} d_{ij} \le g_i^{-1}(h) + \left(b_k - K_{\beta_{y_k}}\sigma_k\right) y_k, \\
&\qquad 0 \le x_j \le p_j,\ j=1,2,\cdots,n \\
&\text{where } y_k = \begin{cases} y_k, & k\in K \\ 0, & k \notin K \end{cases}
\end{aligned}
\tag{6.28}
$$

This problem is similar to the problem (6.19). Consequently, using the solution method similar to problem (6.19) in Subsection 6.2.1, we find an optimal solution.

### 6.3.2 Improvement of each coefficient of bottleneck constraint

Next, in a similar way to Subsection 6.3.1, we consider the improvement of each coefficient in bottleneck constraints by performing the fund injection; i.e., the production time of each product is shorter and the cost in production processes is smaller with the fund injection than the initial values. Subsequently, we assume each improved value of coefficient $b_{y_{kj}}$ for the fund injection occurs according to a normal distribution $N\left(\bar{b}_{y_{kj}}, \sigma_{y_{kj}}^2\right)$. Furthermore, expected value $\bar{b}_{y_{kj}}$ and variance $\sigma_{y_{kj}}^2$ occur according to the following functions: $\bar{b}_{y_{kj}} = b_{kj} y_k$ and $\sigma_{y_{kj}}^2 = \sigma_{kj}^2 y_k^2$, respectively. Then, this model is formulated as follows.

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} \left(a_{kj} - b_{y_{kj}}\right) x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h), \ \left(k \in K\right), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h), \ \left(k \notin K\right), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} + \sum_{k \in K} y_k \le g_w^{-1}(h), \\
& 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.29}
$$

Furthermore, we focus on the following constraint including random variables:

$$
\sum_{j=1}^{n} \left(a_{kj} - b_{y_{kj}}\right) x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h)
$$

This constraint includes random variables $b_{y_{kj}}$. Therefore, in order to solve problem (6.29) analytically, we introduce a chance constraint and transform this inequality into the following form introducing goals $\beta_k, \ k \in K$ for stochastic constraints in a way similar to transformation (6.9):

$$
\begin{aligned}
& \Pr\left\{ \sum_{j=1}^{n} \left(a_{kj} - b_{y_{kj}}\right) x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) \right\} \ge \beta_k \\
\Leftrightarrow \quad & \Pr\left\{ \sum_{j=1}^{n} b_{y_{kj}} x_j \ge \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} - g_k^{-1}(h) \right\} \ge \beta_k \\
\Leftrightarrow \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} - g_k^{-1}(h) - \sum_{j=1}^{n} \bar{b}_{y_{kj}} x_j \ge K_{\beta_k} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}^{(y)} x_i x_j}
\end{aligned}
$$

Therefore, introducing assumptions $\bar{b}_{y_{kj}} = b_{kj} y_k$ and $\sigma_{y_{kj}}^2 = \sigma_{kj}^2 y_k^2$, we transform the problem (6.29) into the following problem:

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) + y_k \left( \sum_{j=1}^{n} b_{kj} x_j - K_{\beta_k} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}^{(k)} x_i x_j} \right), \ (k \in K), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h), \ (k \notin K), \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} + \sum_{k \in K} y_k \le g_w^{-1}(h), \\
& 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.30}
$$

In this problem, each constraint $\sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) + y_k \left( \sum_{j=1}^{n} \bar{b}_{kj} x_j - K_{\beta_k} \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}^{(k)} x_i x_j} \right)$ is a convex region, and so problem (6.30) is a convex programming problem. Therefore, we obtain a global optimal solution to solve the problem using convex programming approaches. Furthermore, in order to solve this problem (6.30) more efficiently using general solvers, we consider the following equivalent problem for the problem by introducing return scenarios $\boldsymbol{r}_s^{(k)}$ and parameters $y_s^{(k)}$ in a way similar to setting future scenarios (6.14):

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\frac{\pi}{2}} \left( \frac{1}{S} \sum_{s=1}^{S} y_s \right) \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) + y_k \left( \sum_{j=1}^{n} b_{kj} x_j - K_{\beta_f} \sqrt{\frac{\pi}{2}} \left( \frac{1}{S} \sum_{s=1}^{S} y_s^{(k)} \right) \right), \ (k \in K) \\
& y_s^{(k)} - \sum_{j=1}^{n} \left( r_{sj}^{(k)} - \bar{r}_j^{(k)} \right) x_j \ge 0, \ y_s^{(k)} + \sum_{j=1}^{n} \left( r_{sj}^{(k)} - \bar{r}_j^{(k)} \right) x_j \ge 0, \ (s = 1, 2, ..., S) \\
& \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} + \sum_{k \in K} y_k \le g_w^{-1}(h), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{kj} < g_k^{-1}(h), \ (k \notin K) \\
& 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n
\end{aligned}
\tag{6.31}
$$

This problem is equivalent to a quadratic programming problem in the case parameter $h$ is fixed or each function $g_k^{-1}(h)$ is a linear function. Therefore, it is easier to solve problem (6.31) than problem (6.30).

Particularly, in the case that all fund injections for bottleneck constraints are equal; i.e., $y_k = y,\ k \in K$ , problem (6.31) is similar to problem (6.18) by fixing parameter $y$ . Therefore, by combining the solution method in Subsection 6.2.1 and bisection algorithm with respect to parameter $y$ , we construct the analytical solution method to obtain the optimal product-mix.

## 6.4 The Case Including the Changes of Expected Proportion of Product Mix

In this section, we particularly propose a model in which a decision maker considers changing the proportion of initial expected product-mix for excessive effort to the high-value added product or some serious accident in production processes in the future. If the coefficients of time, resources and cost constraints are changed, the expected profits may decrease due to the change of proportion of the initial expected product-mix and bottleneck constraints. Therefore, we need to consider the product mix problem including all such cases. Coping with this situation, first we consider the case that the coefficient of the constraints about one product changes. Then, we formulate this model as follows.

$$\begin{aligned}
&\text{Maximize} && \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
&\text{subject to} && \sum_{j=1}^{n} c_j x_j - L^*(\alpha)\sum_{j=1}^{n} d_{wj} + \sum_{k\in K} y_k \le g_w^{-1}(h), \\
& && \sum_{\substack{j=1\\ j\neq i}}^{n} a_{kj} x_j + \left(a_{ki} + c'_{ki}\right) x_i - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h),\ \ (k = 1,2,\cdots,m) \\
& && 0 \le x_j \le p_j,\ \ j = 1,2,\cdots,n
\end{aligned} \tag{6.32}$$

Subsequently, we assume each $c'_{ki}$ is a random variable according to a random distribution $T_i(\omega)$ in universal sets of random variables. Then, by setting a goal of probability to each constraint, we introduce a chance constraint and transform each constraint as follows:

$$\begin{aligned}
&\Pr\left\{ \sum_{\substack{j=1\\ j\neq i}}^{n} a_{kj} x_j + \left(a_{ki} + c'_{ki}\right) x_i - L^*(\alpha)\sum_{j=1}^{n} d_{kj} \le g_k^{-1}(h) \right\} \ge \beta_k \\
\Leftrightarrow\quad &\Pr\left\{ c'_{ki} \le \frac{g_k^{-1}(h) + L^*(\alpha)\sum_{j=1}^{n} d_{kj} - \sum_{j=1}^{n} a_{kj} x_j}{x_i} \right\} \ge \beta_k \\
\Leftrightarrow\quad & g_k^{-1}(h) + L^*(\alpha)\sum_{j=1}^{n} d_{kj} - \sum_{j=1}^{n} a_{kj} x_j \le T_k^*(\beta_k)\, x_i
\end{aligned}$$

where $T_k^*(\beta_k)$ is a pseudo inverse function of the distribution function $T_k(\omega)$. Consequently, we transform problem (6.32) into the following problem.

$$\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} + \sum_{k \in K} y_k \le g_w^{-1}(h), \\
& g_k^{-1}(h) + L^*(\alpha) \sum_{j=1}^{n} d_{kj} - \sum_{j=1}^{n} a_{kj} x_j \le T_k^*(\beta_k) x_i, \ (k = 1, 2, \cdots, m) \\
& 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n
\end{aligned} \tag{6.33}$$

If the optimal product-mix decision of problem (6.33) is equivalent to that of problem (6.8), a decision maker does not need to consider the change of product-mix. However, this case is rare and so they usually become different product-mix decisions. The more significantly the change from initial expected product-mix is, the more significant an adverse effect on the production there may be. On the contrary, it is possible that a small change from expected initial product-mix affects profits. Therefore, we consider the case whereby (i) minimizing the change between both initial product-mix decision and that after coefficients of constraints have been changed, and (ii) maximizing the total future profits occur simultaneously. In this chapter, we formulate this model as the following two-stage problem.

$$\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \\
\text{Minimize} \quad & \sum_{j=1}^{n} \hat{p}_j \left| x_j - y_j^* \right| \\
\text{subject to} \quad & \sum_{j=1}^{n} c_j x_j - L^*(\alpha) \sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
& \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha) \sum_{j=1}^{n} d_{ij} \le g_i^{-1}(h), \ (k = 1, \cdots, m) \\
& 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n, \ \mathbf{y}^* \in Y, \\
& Y = \left\{ \begin{array}{l}
\text{Max} \quad \sum_{j=1}^{n} \overline{r}_j y_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} y_i y_j} \\
\text{s.t.} \ \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{kj} y_j + (a_{ki} + c_{ki}) y_i - L^*(\alpha) \sum_{j=1}^{n} d_{ij} \le g_i^{-1}(h), \\
\quad \sum_{j=1}^{n} c_j y_j \le w, \quad 0 \le y_j \le p_j, \\
\quad j = 1, 2, \cdots, n, \ k = 1, \cdots, T
\end{array} \right\}
\end{aligned} \tag{6.34}$$

where $\hat{p}_j$ is a penalty value of each product due to change of product-mix decision. Since this problem is a bi-criteria programming problem, the solution of problem (6.34) generally becomes a Pareto solution. In this chapter, for finding one deterministic solution analytically, we consider the following single objective programming problem.

$$\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} - q\sum_{j=1}^{n} \hat{p}_j \left| x_j - y_j^* \right| \\
&\text{subject to} \quad \sum_{j=1}^{n} c_j x_j - L^*(\alpha)\sum_{j=1}^{n} d_{wj} \le g_w^{-1}(h), \\
&\qquad \sum_{j=1}^{n} a_{kj} x_j - L^*(\alpha)\sum_{j=1}^{n} d_{ij} \le g_i^{-1}(h), \ (k = 1, \cdots, m) \\
&\qquad 0 \le x_j \le p_j, \ j = 1, 2, \cdots, n, \ \mathbf{y}^* \in Y
\end{aligned} \tag{6.35}$$

where $q$ is a constant weight to $\sum_{j=1}^{n} \hat{p}_j \left| x_j - y_j^* \right|$, which should be decided by a decision maker. In this problem, its objective function is similar to the previous problems (6.19) and (6.26). Therefore, this problem is equivalently transformed into a problem similar to that in Section 6.2. Consequently, we obtain its global optimal solution using Solution method 1 in Subsection 6.2.1.

## 6.5 Product Mix Problem with the Preference Ranking to Each Fuzzy Goal

In Sections 6.3 and 6.4, we have considered the maximization of minimum aspiration level in all the fuzzy goals. On the other hand, a decision maker often has a preference ranking among all the fuzzy goals. For example, in the case that she or he particularly focuses on minimizing total costs, the aspiration level of cost constraint is the highest of all the aspiration levels. Consequently, she or he often ranks each aspiration level. In this section, we consider such a model considering the preference ranking with respect to each fuzzy goal. Particularly, in the case that the aspiration level of future returns is higher than the minimum aspiration levels of all the constraints, this model is formulated as the following problem:

$$\begin{aligned}
&\text{Maximize} \ h_1 \\
&\text{subject to} \ \mu_{g_m}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge \mu_{g_m}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge \cdots \ge \mu_{g_m}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h_1, \\
&\qquad \mu_f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h_1, \\
&\qquad 0 \le x_j \le p_j, \ j = 1, 2, ..., n
\end{aligned} \tag{6.37}$$

Furthermore, we introduce parameters $h_k, \ (k = 2, 3, ..., m)$. Each $h_k$ means a target aspiration level for cost function $\sum_{j=1}^{n} a_{kj} x_j$ . Then, problem (6.34) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize } h_1 \\ &\text{subject to } h_{k+1} \ge \mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h_k, (k=1,2,...,m-1) \\ &\qquad \mu_{g_m}\left(\sum_{j=1}^{n} a_{mj} x_j\right) \ge h_m, \mu_f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h_1, \\ &\qquad 0 \le x_j \le p_j, j=1,2,...,n \end{aligned} \tag{6.38}$$

In this problem, constraints including the membership function $\mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right), (k=1,2,...,m)$ and $\mu_f\left(\sum_{j=1}^{n} r_j x_j\right)$ are transformed as follows:

$$\begin{aligned} &h_{k+1} \ge \mu_{g_k}\left(\sum_{j=1}^{n} a_{kj} x_j\right) \ge h_k \quad \Leftrightarrow \quad g_k^{-1}(h_{k+1}) \le \sum_{j=1}^{n} a_{kj} x_j \le g_k^{-1}(h_k) \\ &\mu_f\left(\sum_{j=1}^{n} r_j x_j\right) \ge h_1 \quad \Leftrightarrow \quad \sum_{j=1}^{n} r_j x_j \ge f^{-1}(h_1) \end{aligned} \tag{6.39}$$

Consequently, we transform problem (6.35) into the following deterministic equivalent problem:

$$\begin{aligned} &\text{Maximize } h_1 \\ &\text{subject to } g_k^{-1}(h_{k+1}) \le \sum_{j=1}^{n} a_{kj} x_j \le g_k^{-1}(h_k), \ (k=1,2,...,m) \\ &\qquad \sum_{j=1}^{n} r_j x_j \ge f^{-1}(h_1), \\ &\qquad 0 \le x_j \le p_j, j=1,2,...,n \end{aligned} \tag{6.40}$$

In the case that each $g_k(w)$ is a linear function, problem (6.37) is transformed into

$$\begin{aligned} &\text{Maximize } h_1 \\ &\text{subject to } h_k \le \frac{b_{k0} - \sum_{j=1}^{n} a_{kj} x_j}{b_{k0} - b_{k1}} \le h_{k+1}, \ (k=1,2,...,m) \\ &\qquad \sum_{j=1}^{n} r_j x_j \ge f^{-1}(h_1), \\ &\qquad 0 \le x_j \le p_j, j=1,2,...,n \end{aligned} \tag{6.41}$$

i.e.,

$$
\begin{aligned}
&\text{Maximize } h_1 \\
&\text{subject to } \frac{b_{k0}-\sum_{j=1}^{n} a_{kj}x_j}{b_{k0}-b_{k1}} \le \frac{b_{k+1,0}-\sum_{j=1}^{n} a_{k+1,j}x_j}{b_{k+1,0}-b_{k+1,1}},\ \left(k=1,2,\dots,m-1\right) \\
&\qquad\qquad \frac{b_{k0}-\sum_{j=1}^{n} a_{kj}x_j}{b_{k0}-b_{k1}} \ge h_1,\ \ \sum_{j=1}^{n} r_j x_j \ge f^{-1}\left(h_1\right), \\
&\qquad\qquad 0 \le x_j \le p_j,\ j=1,2,\dots,n
\end{aligned}
\tag{6.42}
$$

In this problem, $\frac{b_{k0}-\sum_{j=1}^{n} a_{kj}x_j}{b_{k0}-b_{k1}} \le \frac{b_{k+1,0}-\sum_{j=1}^{n} a_{k+1,j}x_j}{b_{k+1,0}-b_{k+1,1}}$ are linear constraints. Therefore, in the case that we consider randomness to coefficients $r_j$ and fuzziness to coefficients $a_{kj}$, problem (6.42) is similar to problem (6.8) in Section 6.2 by introducing stochastic, possibility and necessity constraints. Consequently, the proposal preference ranking model is equivalently transformed into a linear programming problem in a way similar to Section 6.3, and we obtain its global optimal solution efficiently by using the solution method in Section 63.

## 6.6 Numerical Example

In this section, in order to illustrate the applicability of our proposal models, we consider a numerical example modified from the data in Hsu and Chung [46] as Table 1.

Table 6.1. Return and load per unit of each product

| | Load per unit of product | | | |
|---|---|---|---|---|
| Resource | A | B | C | D |
| return (dollar) | 80 | 60 | 50 | 30 |
| R1 (unit) | 20 | 10 | 10 | 5 |
| R2 (unit) | 5 | 10 | 5 | 15 |
| R3 (unit) | 10 | 5 | 10 | 10 |
| R4 (unit) | 0 | 30 | 15 | 5 |
| R5 (unit) | 5 | 5 | 20 | 5 |
| R6 (unit) | 5 | 5 | 5 | 15 |
| R7 (unit) | 20 | 5 | 10 | 0 |
| fund (dollar) | 10 | 10 | 5 | 5 |

The example of Table 1 shows how results are brought for solving various proposed approaches in product-mix decision problems in a multiple-constraint case. In this numerical example, we assume that four product types $A, B, C,$ and $D$ are produced using seven different resources, R1-R7, each with an available upper capacity of 2400 units and a fund cost constraint with an upper value of 2000 dollars. The future return and load of each resource for producing one unit of product $A, B, C,$ and $D$ can be collected as shown in Table 6.1.

From Table 6.1, the basic product-mix decision problem maximizing the total profit is formulated as follows.

(Problem P)

$$\begin{array}{ll} \text{Maximize} & Z = 80A + 60B + 50C + 30D \\ \text{subject to} & 20A + 10B + 10C + 5D \le 2400, \\ & 5A + 10B + 5C + 15D \le 2400, \\ & 10A + 5B + 10C + 10D \le 2400, \\ & 0A + 30B + 15C + 5D \le 2400, \\ & 5A + 5B + 20C + 5D \le 2400, \\ & 5A + 5B + 5C + 15D \le 2400, \\ & 20A + 5B + 10C + 0D \le 2400, \\ & 10A + 10B + 5C + 5D \le 2000, \\ & A \le 70,\ B \le 60,\ C \le 50,\ D \le 150 \end{array}$$

We solve this linear programming problem and obtain this optimal solution as Table 6.2.

Table 6.2. Optimal solution of basic model

| | Optimal product unit value | | | |
|---|---|---|---|---|
| Z | A | B | C | D |
| 11873.3 | 50.67 | 38.17 | 50 | 101 |

In this optimal case, each value of constraints is as follows.

Table 6.3. Values of constraints

| | Load for a unit product | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | Value |
| R1 | 20 | 10 | 10 | 5 | 2400 |
| R2 | 5 | 10 | 5 | 15 | 2400 |
| R3 | 10 | 5 | 10 | 10 | 2207.5 |
| R4 | 0 | 30 | 15 | 5 | 2400 |
| R5 | 5 | 5 | 20 | 5 | 1949.2 |
| R6 | 5 | 5 | 5 | 15 | 2209.2 |
| R7 | 20 | 5 | 10 | 0 | 1704.2 |
| fund | 10 | 10 | 5 | 5 | 1643.3 |

Consequently, we find that the bottleneck constraints are R1, R2, and R4. Furthermore, we consider the case that each return is a random variable and target values for the total profit and cost constraints are assumed to be fuzzy goals such as identified in Section 6.2. For the sake of simplifying our discussion, we assume each return of products to occur according to the following normal distribution:

$$
\begin{aligned}
&r_A \sim N(80,10),\ r_B \sim N(60,6),\\
&r_C \sim N(50,5),\ r_D \sim N(30,3)
\end{aligned}
$$

Then, fuzzy goals for the total profit and cost constraints are as the following linear functions:

$$
\begin{aligned}
&\mu_f\left(\sum_{j=1}^{4} r_j x_j\right)=\begin{cases} 1, & 12000 \le \sum_{j=1}^{4} r_j x_j \\ \dfrac{\sum_{j=1}^{4} r_j x_j - 10000}{2000}, & 10000 \le \sum_{j=1}^{4} r_j x_j < 12000 \\ 0, & \sum_{j=1}^{4} r_j x_j < 10000 \end{cases}\\
&\mu_{g_w}\left(\sum_{j=1}^{n} c_j x_j\right)=\begin{cases} 1, & \sum_{j=1}^{n} a_{kj} x_j \le 2000 \\ \dfrac{2200-\sum_{j=1}^{n} c_j x_j}{200}, & 2000 < \sum_{j=1}^{n} c_j x_j \le 2200 \\ 0, & 2200 < \sum_{j=1}^{n} c_j x_j \end{cases}\\
&\mu_{g_k}\left(\sum_{j=1}^{4} a_{kj} x_j\right)=\begin{cases} 1, & \sum_{j=1}^{4} a_{kj} x_j \le 2400 \\ \dfrac{2600-\sum_{j=1}^{4} a_{kj} x_j}{200}, & 2400 < \sum_{j=1}^{4} a_{kj} x_j \le 2600 \quad (k=1,2,\cdots,7) \\ 0, & 2600 < \sum_{j=1}^{4} a_{kj} x_j \end{cases}
\end{aligned}
\tag{6.43}
$$

In this problem, the target value for probability constraint $\beta$ is assumed to be 0.9. Using these parameters and fuzzy goals, we formulate the model based on the probability fractile optimization model described in Subsection 6.2.1 as follows:

(Problem P1)

$$
\begin{array}{ll}
\text{Maximize} & 10000+2000h \\
\text{subject to} & 80A+60B+50C+30D-K_{0.9}\sqrt{10A^2+6B^2+5C^2+3D^2}\geq 10000+2000h, \\
& 20A+10B+10C+5D\leq 2600-200h, \\
& 5A+10B+5C+15D\leq 2600-200h, \\
& 10A+5B+10C+10D\leq 2600-200h, \\
& 0A+30B+15C+5D\leq 2600-200h, \\
& 5A+5B+20C+5D\leq 2600-200h, \\
& 5A+5B+5C+15D\leq 2600-200h, \\
& 20A+5B+10C+0D\leq 2600-200h, \\
& 10A+10B+5C+5D\leq 2200-200h, \\
& A\leq 70,\ B\leq 60,\ C\leq 50,\ D\leq 150
\end{array}
$$

We solve this problem and find the following optimal solution.

Table 6.4. Optimal solution of P1

| | Optimal product unit value | | | |
|---|---|---|---|---|
| h | A | B | C | D |
| 0.780 | 61.91 | 58.62 | 12.06 | 97.67 |

In a way similar to Problem P1, we consider the probability maximization model in Subsection 6.2.2. In this numerical example, we fix the level of satisfaction $h$ to 0.78 in order to compare with the result of problem P1 in Table 6.4. Thereby, the probability maximization model is as follows:

(Problem P2)

$$
\begin{array}{ll}
\text{Maximize} & \dfrac{80A+60B+50C+30D-(10000+2000\cdot 0.78)}{\sqrt{10A^2+6B^2+5C^2+3D^2}} \\
\text{subject to} & 20A+10B+10C+5D\leq 2600-200\cdot 0.78, \\
& 5A+10B+5C+15D\leq 2600-200\cdot 0.78, \\
& 10A+5B+10C+10D\leq 2600-200\cdot 0.78, \\
& 0A+30B+15C+5D\leq 2600-200\cdot 0.78, \\
& 5A+5B+20C+5D\leq 2600-200\cdot 0.78, \\
& 5A+5B+5C+15D\leq 2600-200\cdot 0.78, \\
& 20A+5B+10C+0D\leq 2600-200\cdot 0.78, \\
& 10A+10B+5C+5D\leq 2200-200\cdot 0.78, \\
& A\leq 70,\ B\leq 60,\ C\leq 50,\ D\leq 150
\end{array}
$$

We solve this problem and find the following optimal solution.

Table 6.5. Optimal solution of P2

| Optimal product unit value | | | |
|---|---|---|---|
| A | B | C | D |
| 59.50 | 54.20 | 21.23 | 99.88 |

Subsequently, we compare these two product-mix decision problems P1 and P2 with basic product-mix decision problem P. If future returns are assumed to be expected returns, each total earning profit is as given in Table 6.6.

Table 6.6. The case of expected future returns

| | | Optimal product unit value | | | |
|---|---|---|---|---|---|
| | Z | A | B | C | D |
| P | 11873.3 | 50.67 | 38.17 | 50 | 101 |
| P1 | 12003.1 | 61.91 | 58.62 | 12.06 | 97.67 |
| P2 | 12069.9 | 59.50 | 54.20 | 21.23 | 99.88 |

From Table 6.6, it may be noticed that the total profits of problems P1 and P2 considering flexibility for the target values are larger than the basic product-mix problem P. Then, we find that the optimal product-mix, particularly for products B and C, is different and production volumes of products with high returns tend to increase in the models considering randomness and fuzzy goals.

Furthermore, we consider three cases for future returns: (C1) the case that all future returns are smaller than expected returns, (C2) the case that all future returns are larger than expected returns, and (C3) the case that returns of two products A and B are larger than expected returns and those of C and D are smaller than expected returns. In this numerical example, we assume these data to be as given in Table 6.7.

Table 7. The case of future returns change

| | Values of actual future returns | | | |
|---|---|---|---|---|
| Case | A | B | C | D |
| C1 | 70 | 54 | 45 | 27 |
| C2 | 90 | 66 | 55 | 33 |
| C3 | 90 | 66 | 45 | 27 |

Then, the total profit earnings for each case are as shown in Table 6.8.

Table 6.8. Total profits of three models

| | Total profit | | |
|---|---|---|---|
| Case | P | P1 | P2 |
| C1 | 10585.1 | 10679.0 | 10743.9 |
| C2 | 13162.5 | 13327.2 | 13395.9 |
| C3 | 12056.5 | 12620.6 | 12584.3 |

From Table 6.8 it may be observed that total profits of problems P1 and P2 are larger than that of problem P in all cases. Therefore, we find that the model considering randomness and fuzzy goals is more flexible. Furthermore, we also find that the total profit of problem P1 is larger than that of problem P2 in case C3. This means that decision makers can use these models as future business scenarios.

On the other hand, the available fund for the basic product-mix decision problem P is 356.7 derived from the result in Table 6.3. Therefore, we consider the case of fund injections for bottleneck constraints such as described in Section 6.3. First, we focus on the case described in Subsection 4.1. We assume that the available fund uses bottleneck constraints R1, R2, and R4 and each fund injection is assumed to be $y_1, y_2$, and $y_4$, respectively. Then, the improved value $b_{y_k}$ in $k$th constraint occurs according to a normal distribution $N(10y_k, 3)$. Considering these assumptions, we formulate this model as follows.

(Problem P3)

$$
\begin{array}{ll}
\text{Maximize} & 80A+60B+50C+30D-K_{0.9}\sqrt{10A^2+6B^2+5C^2+3D^2} \\
\text{subject to} & 20A+10B+10C+5D \le 2400+(10-1.645\times 3)y_1 \\
& 5A+10B+5C+15D \le 2400+(10-1.645\times 3)y_2 \\
& 10A+5B+10C+10D \le 2400 \\
& 0A+30B+15C+5D \le 2400+(10-1.645\times 3)y_4 \\
& 5A+5B+20C+5D \le 2400 \\
& 5A+5B+5C+15D \le 2400 \\
& 20A+5B+10C+0D \le 2400 \\
& 10A+10B+5C+5D \le 2000 \\
& A \le 70,\ B \le 60,\ C \le 50,\ D \le 150
\end{array}
$$

We solve this problem and find the following optimal solution and its fund injections.

Table 6.9. Optimal solution of P2

| | Optimal product unit value | | | |
|---|---|---|---|---|
| Z | A | B | C | D |
| 13108.5 | 70 | 48.13 | 50 | 87.91 |

Table 6.10. Fund injections to bottleneck constraints

| R1($y_1$) | R2($y_2$) | R4($y_4$) |
|---|---|---|
| 83.09 | 0 | 46.07 |

From Tables 6.9 and 10, by performing fund injection to bottleneck constraints, particularly R1 and R4, the production volume of product A becomes large compared to that of the basic problem P. Therefore, we find that total future profits are increasing significantly.

Next, we consider the case that the coefficients of constraints, particularly bottleneck constraints, change, such as in the model described in Subsection 6.3.2. In this numerical example, we assume that the values of coefficients change as follows:

Table 6.11. Changing values of coefficients

| Bottleneck | Changing values |
|---|---|
| R1 | $N\left(0.3, 0.01^2\right)$ |
| R2 | $N\left(0.2, 0.01^2\right)$ |
| R4 | $N\left(0.5, 0.01^2\right)$ |

Using these values, we formulate this model as follows.

(Problem P4)

$$
\begin{aligned}
\text{Maximize} \quad & 80A + 60B + 50C + 30D - K_{0.9}\sqrt{10A^2 + 6B^2 + 5C^2 + 3D^2} \\
\text{subject to} \quad & (20 - 0.3\cdot\sigma\cdot y)A + (10 - 0.3\cdot\sigma\cdot y)B \\
& + (10 - 0.3\cdot\sigma\cdot y)C + (5 - 0.3\cdot\sigma\cdot y)D \le 2400, \\
& (15 - 0.2\cdot\sigma\cdot y)A + (10 - 0.2\cdot\sigma\cdot y)B \\
& + (5 - 0.2\cdot\sigma\cdot y)C + (15 - 0.2\cdot\sigma\cdot y)D \le 2400, \\
& 10A + 5B + 10C + 10D \le 2400, \\
& (30 - 0.5\cdot\sigma\cdot y)B + (15 - 0.5\cdot\sigma\cdot y)C + (5 - 0.5\cdot\sigma\cdot y)D \le 2400, \\
& 5A + 5B + 20C + 5D \le 2400, \\
& 5A + 5B + 5C + 15D \le 2400, \\
& 20A + 5B + 10C + 0D \le 2400, \\
& 10A + 10B + 5C + 5D \le 2000, \\
& A \le 70,\ B \le 60,\ C \le 50,\ D \le 150,\ \sigma = K_{0.9}\cdot 0.01
\end{aligned}
$$

In a way similar to the previous discussion, we solve this problem and find the following optimal solution.

Table 6.12. Optimal solution of P3

| | Optimal product unit value | | | |
|---|---|---|---|---|
| Z | A | B | C | D |
| 13740.2 | 70 | 60 | 50 | 85.79 |

Table 6.12 means that production volumes of products with high returns, particularly products A and B, increase.

Finally, we consider the case that the coefficients of constraints, particularly bottleneck constraints, change as described in Section 6.4. We assume that the changes of coefficients occur as follows.

$$T_1^*(\beta_1)=10,\ T_2^*(\beta_2)=10,\ T_4^*(\beta_4)=5$$

Using these values, we formulate this model as follows.
(Problem P5)

$$\begin{array}{ll}
\text{Maximize} & 80A+60B+50C+30D-K_{0.9}\sqrt{10A^2+6B^2+5C^2+3D^2} \\
\text{subject to} & 30A+10B+10C+5D\le 2400 \\
& 15A+10B+5C+15D\le 2400 \\
& 10A+5B+10C+10D\le 2400 \\
& 5A+30B+15C+5D\le 2400 \\
& 5A+5B+20C+5D\le 2400 \\
& 5A+5B+5C+15D\le 2400 \\
& 20A+5B+10C+0D\le 2400 \\
& 10A+10B+5C+5D\le 2000 \\
& A\le 70,\ B\le 60,\ C\le 50,\ D\le 150
\end{array}$$

In a way similar to the previous discussion, we solve this problem and find the following optimal solution.

Table 6.13. Optimal solution of P3

| Optimal product unit value | | | |
|---|---|---|---|
| A | B | C | D |
| 38 | 35 | 50 | 82 |

In the case that this product-mix is used, if coefficients of bottleneck constraints do not change, each value of the constraints of problem P is as follows.

Table 6.14. Values of constraints

| | Load for a unit product | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | Value |
| R1 | 20 | 10 | 10 | 5 | 2020 |
| R2 | 5 | 10 | 5 | 15 | 2020 |
| R3 | 10 | 5 | 10 | 10 | 1875 |
| R4 | 0 | 30 | 15 | 5 | 2210 |
| R5 | 5 | 5 | 20 | 5 | 1775 |
| R6 | 5 | 5 | 5 | 15 | 1845 |
| R7 | 20 | 5 | 10 | 0 | 1435 |
| fund | 10 | 10 | 5 | 5 | 1390 |

From Table 6.4, all the constraints have available capacities. Therefore, a decision maker decides a further flexible product-mix based on the optimal product-mix of P5 considering information from specialists and employees.

Consequently, by involving flexibility for goals and upper values of constraints, decision makers can earn a larger profit than that of the product-mix problem with constant values even if future returns are measurably changed from the expected value. Furthermore, by budgeting surplus funds to bottleneck constraints and predicting future troubles in advance, decision makers obtain optimal decisions including flexibility such as buffers to be able to apply the additional product-mix decisions and earn a larger profit.

Finally, we consider the preference ranking model in Section 6.6. Particularly, we focus on the aspiration level of budget constraint and it is the highest of all the aspiration levels. In this model, we assume the following fuzzy numbers of coefficients in all the constraints:

$$\mu_{\tilde{a}_{1j}}(\omega)=\langle a_{1j},0.5\rangle,\ \mu_{\tilde{a}_{2j}}(\omega)=\langle a_{2j},1\rangle,\ \mu_{\tilde{a}_{3j}}(\omega)=\langle a_{3j},2\rangle,\ \mu_{\tilde{a}_{4j}}(\omega)=\langle a_{4j},2\rangle,$$
$$\mu_{\tilde{a}_{5j}}(\omega)=\langle a_{5j},0.5\rangle,\ \mu_{\tilde{a}_{6j}}(\omega)=\langle a_{6j},1\rangle,\ \mu_{\tilde{a}_{7j}}(\omega)=\langle a_{7j},0.5\rangle,\ \mu_{\tilde{a}_{fj}}(\omega)=\langle a_{fj},2\rangle,\quad (j=1,2,3,4)$$

From this assumption, this model is given as the following problem based on problem (6.42):

Problem P6:

$$\begin{aligned}
&\text{Maximize } h\\
&\text{subject to } 80A+60B+50C+30D-K_{0.8}\sqrt{50A^2+20B^2+30C^2+10D^2}\geq 10000+2000h,\\
&\frac{2600-(19.9A+9.9B+9.9C+4.9D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(4.8A+9.8B+4.8C+14.8D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(9.6A+4.6B+9.6C+9.6D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(0A+29.6B+14.6C+4.6D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(4.9A+4.9B+19.9C+4.9D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(4.8A+4.8B+4.8C+14.8D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&\frac{2600-(19.9A+4.9B+9.9C+0D)}{200}\leq\frac{2400-(9.6A+9.6B+4.6C+4.6D)}{200},\\
&9.6A+9.6B+4.6C+4.6D\leq 2400-200h,\\
&0\leq A\leq 70,\quad 0\leq B\leq 60,\quad 0\leq C\leq 50,\quad 0\leq D\leq 150
\end{aligned}$$

We solve this problem and obtain its optimal solution in Table 7.

Table 6.15. Optimal solution of problem $P_4$

| Optimal product unit value and aspiration level | | | | |
|---|---|---|---|---|
| A | B | C | D | $h$ |
| 54.274 | 44.273 | 50 | 90.422 | 0.7179 |

In Table 6.15, optimal volumes of products A and B are same as that of problem $P_1$. On the other hand, the optimal volume of product C is equal to that of basic problem $P$. Therefore, preference ranking model P6 is an intermediate model considering various types of conditions.

## 6.7 Conclusion

In this chapter, we have proposed several types of product-mix decision models, many of which include several randomness and fuzziness. First, we have considered a probability fractile optimization model as the main problem and a probability maximization model to total future profits, and developed an analytical and efficient solution method to find a global optimal solution. Second, we have considered several cases of product-mix problems that occur in practical market of products and production processes scenarios, particularly the models including changes of constraints. All these proposed models under randomness, fuzziness, and flexibility are equivalent to linear programming problems or quadratic convex programming problems. Therefore, we can efficiently solve these problems. Furthermore, we have proposed a preference ranking model to each fuzzy goal. Since this model represents many situations for changing the preference ranking, it may be applicable to more flexible and complicated problems in the real world.

Thus, since a decision maker decides the parameters arbitrarily, these models are capable of applying to models under many situations of uncertainty and flexibility. For example, in the case that price and demand ranges are intense and managers consider maximizing the total profit as much as possible by controlling such uncertainty, they will choose models in Section 6.3 and perform a flexible plan so that they can correspond to some moves. Furthermore, in the case that there are surplus funds in the original flexible plan and managers allow them to bottleneck constraints, they will choose models in Section 6.4 and draw up better product-mix decisions to earn larger profits. Then, if a manager is pessimistic and predicts future dangers such as machine breakdowns as far as possible, she or he will choose models in Section 6.5 and decide for an optimal flexible product-mix involving the capability of adding a new product-mix plan to earn larger profit. Then, we may be able to apply these solution methods not only to maximize the level of satisfaction of the total future profit, but also to minimize total costs and optimization under more complicated situations in the real world. Thus, our proposal models can be applied to various managerial situations by including flexibility.

## Chapter 7

# 0-1 Programming Problems under Randomness and Fuzziness

0-1 programming problems are one of the most important problems in practical management fields such as project selection problems, scheduling and facility location problems, and there are many previous researches (recent studies, Balev [8], Jahanshahloo [56]). Their solution method mainly divides two types; (a) strict solution methods such as dynamic programming and branch-bound method, (b) approximate solution methods such as genetic algorithm, heuristic methods, etc..

In previous standard mathematical programming problems involving 0-1 programming problems, the coefficients of objective functions or constraints are assumed to be completely known. However, in practical systems, they are rather uncertain than constant. In order to deal with such uncertainty, stochastic programming problem (for example, Beale [10], Dantzig [24], Vajda [115]) and fuzzy programming problem (for example, Inuiguchi [51], Sakawa [105], Zimmermann [126]) have been considered. Furthermore, Katagiri [62] has considered 0-1 programming problem considering both random and fuzzy conditions, i.e. fuzzy random 0-1 programming problem. However, in this research, fuzzy numbers have been assumed to be triangle fuzzy numbers and random variables have been assumed to be discrete random distributions.

This chapter particularly considers the more general stochastic and fuzzy 0-1 programming problem maximizing the objective function involving fuzzy random or random fuzzy variables considering both the objectivity derived from statistical analysis of data and decision maker's subjectivity such as the institution which comes from wide-ranging experiences, simultaneously. In this chapter, we deal fuzzy numbers with L-R fuzzy numbers and random variables with continuous random distribution, particularly normal distributions.

In the mathematical programming, 0-1 programming problems considering randomness and fuzziness are more complicate than the previous problems due to including both random variables and fuzzy numbers. Then, since this problem is not a well-defined knapsack problem, it is hard to solve it directly. Therefore, we need to set the target values for stochastic and fuzzy constraints and construct its efficient solution method. In this chapter, we transform main problems into deterministic equivalent integer programming problems using chance constraints, possibility measure and fuzzy goals based on both stochastic and fuzzy programming approaches. Furthermore, through the development of information technology and improvement of computers, we solve 0-1 programming problems more quickly using not only approximate solution methods but also strict solution methods even if they are little bit large scale problems. Therefore, in this chapter, we propose the efficient strict solution method based on a mixed method with 0-1 relaxation problem and branch-bound method, and show the analytical efficiency comparing with previous solution methods.

## 7.1 The Formulation of 0-1 Knapsack Problem under Several Random and Ambiguous Situations

In this section, with respect to knapsack problems or project selection problems which are formulated as the 0-1 programming problem, in order to consider problems in real world more widely and flexibly, we proposed a model considering random future returns, fuzzy coefficients of the constraint and flexibility of objective value and maximum value of constraint.

First of all, each notation in this chapter means as follows:

$n$ : Total number of projects

$r_j$ : Future return of project $j$ ,

$c_j$ : Capital budgeting of project $j$

$f$ : Goal of total future returns

$b$ : Upper limited value of total capital budgeting

$x_j$ : Decision variable satisfying $x_j = \begin{cases} 1 & \text{select project } j \\ 0 & \text{not select project } j \end{cases}$

A basic 0-1 knapsack problem maximizing the total profit is generally formulated as follows:

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} r_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} c_j x_j \le b, \\ &\qquad\qquad\quad x_j \in \{0,1\},\ j = 1,2,\cdots,n \end{aligned} \tag{7.1}$$

With respect to this problem, we obtain a strict optimal solution using dynamic programming method or branch-bound method. However, in the case that we assume each return $r_j$ as a random variable, problem (7.1) is not well-defined problem since the objective function also becomes a random variable. Therefore, in this chapter, introducing a chance constraint with respect to the objective function, we consider a probability fractile optimization model with respect to the total profit.

### 7.1.1 The formulation of probability fractile optimization model

We apply probability fractile optimization model to problem (7.1). This problem is formulated as the following form using the chance constraint and its probability level $\beta$:

$$\begin{aligned} &\text{Maximize} \quad f \\ &\text{subject to} \quad \Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f\right\} \geq \beta, \\ &\qquad\qquad \sum_{j=1}^{n} c_j x_j \leq b, \quad x_j \in \{0,1\}, \quad j=1,2,\cdots,n \end{aligned} \tag{7.2}$$

In this problem, we assume each future return $r_j$ occurs according to a normal distribution $N\left(\bar{r}_j, \sigma_j^2\right)$ where $\bar{r}_j$ is the mean value of $r_j$ and $\sigma_j^2$ is its variance. In this chapter, since each coefficient of the objective function is assumed to be independent of other variables, i.e.

$$\sigma_{ij} = \begin{cases} \sigma_j^2, & i=j \\ 0, & i \neq j \end{cases}, \quad i,j=1,2,\cdots,n$$

Under these assumptions, its stochastic constraint is transformed into the following inequality:

$$\begin{aligned} &\Pr\left\{\sum_{j=1}^{n} r_j x_j \geq f\right\} \geq \beta \Leftrightarrow \Pr\left\{\frac{\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \geq \frac{f - \sum_{j=1}^{n} \bar{r}_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}}\right\} \geq \beta \\ &\Leftrightarrow \frac{\sum_{j=1}^{n} \bar{r}_j x_j - f}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}} \geq K_\beta \Leftrightarrow \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} \geq f \end{aligned} \tag{7.3}$$

where $F(y)$ is the distribution function of the standard normal distribution and $K_\beta = F^{-1}(\beta)$.

Therefore, problem (7.2) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad f \\ &\text{subject to} \quad \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^2} \geq f, \\ &\qquad\qquad \sum_{j=1}^{n} c_j x_j \leq b, \quad x_j \in \{0,1\}, \quad j=1,2,\cdots,n \end{aligned} \tag{7.4}$$

Since each decision variable $x_j$ satisfies $x_j \in \{0,1\}$, we obtain $x_j^2 = x_j$, and so problem (7.4) is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize} \quad f \\
&\text{subject to} \quad \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \geq f \\
&\qquad \sum_{j=1}^{n} c_j x_j \leq b, \quad x_j \in \{0,1\}, \quad j = 1,2,\cdots,n
\end{aligned}
\tag{7.5}
$$

### 7.1.2 Introducing of fuzzy numbers and fuzzy goals

Furthermore, we consider the upper limited value of capital budgeting constraint. Since there is a lack of information, we assume each coefficient $c_j$ of the constraint to be the following L-fuzzy number:

$$\mu_{\tilde{c}_j}(\omega) = L\left(\frac{\omega - \overline{c}_j}{d_j}\right), \quad j = 1,2,\ldots,n$$

where $L(x)$ is a continuous nonincreasing nonnegative function satisfying $L(0) = 1,\ L(1) = 0$.

Therefore $\sum_{j=1}^{n} \tilde{c}_j x_j$ is also a fuzzy variable, and so the constraint of problem (7.5) is not a well-defined constraint. Hence, for the transformation into the deterministic equivalent constraint, we introduce possibility measure. The membership function with respect to $\sum_{j=1}^{n} \tilde{c}_j x_j$ is as follows:

$$\mu_Y(y) = L\left(\frac{y - \sum_{j=1}^{n} \overline{c}_j x_j}{\sum_{j=1}^{n} d_j x_j}\right) \tag{7.6}$$

Furthermore, we assume that the upper limited value $b$ of total capital budgeting includes flexibility. Generally speaking, it is possible to increase maximum capital budget $b$ a little in order to increase the goal of total future profits. On the other hand, If $b$ is increased too much, we consider that an aspiration of decision maker is decreasing greatly. Considering these situations, we introduce the following fuzzy goal with respect to $b$:

$$\mu_G(\omega)=\begin{cases}1, & \omega\le b_1\\ g(\omega), & b_1\le\omega\le b_0\\ 0, & b_0\le\omega\end{cases} \tag{7.7}$$

where $\mu_G(\omega)$ is assumed to be the following membership function using the monotonically decreasing function $g_b(\omega)$. Then, we consider the following possibility measure:

$$\Pi_Y(G)=\sup_y \min\{\mu_Y(y),\mu_G(y)\} \tag{7.8}$$

In a way similar to $b$, we also introduce the fuzzy goal of the total profit as follows:

$$\mu_F(z)=\begin{cases}1, & f_1\le z\\ f(z), & f_0\le z\le f_1,\\ 0, & z\le f_0\end{cases}\quad \left(z=\sum_{j=1}^{n}\overline{r}_j x_j-K_\beta\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}\right) \tag{7.9}$$

where $\mu_F(z)$ is assumed to be the following membership function using the monotonically increasing function $f(z)$. Using these possibility measure and fuzzy goal, we propose the following maximization model of minimum aspiration levels as the reformulation of problem (7.5):

$$\begin{aligned}&\text{Maximize} \quad \min\{\mu_F(z),\ \Pi_Y(G)\}\\ &\text{subject to} \quad x_j\in\{0,1\},\quad j=1,2,\cdots,n\end{aligned} \tag{7.10}$$

$$\begin{aligned}&\text{Maximize} \quad h\\ &\text{subject to} \quad \mu_F(z)\ge h,\quad \Pi_Y(G)\ge h,\\ &\qquad\qquad\quad x_j\in\{0,1\},\quad j=1,2,\cdots,n\end{aligned} \tag{7.11}$$

In this problem, constraint of possibility measure $\Pi_y(G)\ge h$ is equivalently transformed into the following form:

$$\begin{aligned}
&\Pi_Y(G) \geq h \\
\Leftrightarrow\quad &\sup_y \min\left\{\mu_Y(y), \mu_G(y)\right\} \geq h \\
\Leftrightarrow\quad &\mu_Y(y) \geq h,\ \mu_G(y) \geq h \\
\Leftrightarrow\quad &\sum_{j=1}^{n} \bar{c}_j x_j - L^*(h) \sum_{j=1}^{n} d_j x_j \leq y,\ y \leq g_b^{-1}(h) \\
\Leftrightarrow\quad &\sum_{j=1}^{n} \bar{c}_j x_j - L^*(h) \sum_{j=1}^{n} d_j x_j \leq g_b^{-1}(h)
\end{aligned} \tag{7.12}$$

Then, in a way similar to transformation (7.10), constraint $\mu_F(z) \geq h$ is transformed into the following form:

$$\mu_F(z) \geq h \quad \Leftrightarrow \quad \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \geq f^{-1}(h) \tag{7.13}$$

Therefore, we equivalently transform problem into the following problem:

$$\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \sum_{j=1}^{n} \bar{r}_j x_j - K_\beta \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \geq f^{-1}(h), \\
&\qquad\qquad\quad \sum_{j=1}^{n} \bar{c}_j x_j - L^*(h) \sum_{j=1}^{n} d_j x_j \leq g^{-1}(h), \\
&\qquad\qquad\quad x_j \in \{0,1\}, \quad j = 1, 2, \cdots, n
\end{aligned} \tag{7.14}$$

### 7.1.3 The efficient solution method of proposed 0-1 knapsack problem

Main problem (7.14) in Subsection 7.1.2 is a nonlinear 0-1 knapsack problem, and so it is hard to solve it directly using the standard solution method to solve discrete mathematical programming problems. However, in the case that parameter $h$ is fixed, constraint $\sum_{j=1}^{n} \bar{c}_j x_j - L^*(h) \sum_{j=1}^{n} d_j x_j \leq g^{-1}(h)$ is equivalent to a linear constraint with respect to $\boldsymbol{x}$. Furthermore, to solve problem (7.14) efficiently, we introduce the following subproblem:

$$\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n}\bar{r}_j x_j - K_\beta\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}\\
&\text{subject to} \quad \sum_{j=1}^{n}\bar{c}_j x_j - L^*\left(\bar{h}\right)\sum_{j=1}^{n}d_j x_j \le g^{-1}\left(\bar{h}\right),\\
&\qquad\qquad x_j \in \{0,1\}, \quad j=1,2,\cdots,n
\end{aligned} \tag{7.15}$$

$\bar{h}$ is a fixed value of parameter $h$ satisfying $0 \le \bar{h} \le 1$. With respect to the relation between subproblem (7.15) and problem (7.14), the following theorem holds.

**Theorem 7.1**

Let an optimal solution of subproblem (7.15) be $\boldsymbol{x}_{\bar{h}}$ and its optimal value $Z\left(\boldsymbol{x}_{\bar{h}}\right)$. Furthermore, let the optimal value of problem (7.14) be $h^*$. Then the following relation holds:

$$\begin{cases}
Z\left(\boldsymbol{x}_{\bar{h}}\right) > f^{-1}\left(\bar{h}\right) & \Leftrightarrow \quad \bar{h} < h^*\\
Z\left(\boldsymbol{x}_{\bar{h}}\right) = f^{-1}\left(\bar{h}\right) & \Leftrightarrow \quad \bar{h} = h^*\\
Z\left(\boldsymbol{x}_{\bar{h}}\right) < f^{-1}\left(\bar{h}\right) & \Leftrightarrow \quad \bar{h} > h^*
\end{cases}$$

From this theorem, we obtain that the optimal solution $\boldsymbol{x}_{\bar{h}}$ of subproblem (7.15) is equivalent to that of problem (7.14) in the case that $Z\left(\boldsymbol{x}_{\bar{h}}\right) = f^{-1}\left(\bar{h}\right)$. Furthermore, we consider the following auxiliary problem to subproblem (7.15) introducing a parameter $R$:

$$\begin{aligned}
&\text{Maximize} \quad R\sum_{j=1}^{n}\bar{r}_j x_j - K_\beta\left(\sum_{j=1}^{n}\sigma_j^2 x_j\right)\\
&\text{subject to} \quad \sum_{j=1}^{n}\bar{c}_j x_j - L^*\left(\bar{h}\right)\sum_{j=1}^{n}d_j x_j \le g^{-1}\left(\bar{h}\right),\\
&\qquad\qquad x_j \in \{0,1\}, \quad j=1,2,\cdots,n
\end{aligned} \tag{7.16}$$

With respect to the relation between problems (7.15) and (7.16), the following theorem holds based on previous research (Ishii et al. [55]).

**Theorem 7.2**

Let an optimal solution of problem (7.16) be $\boldsymbol{x}^*$. If $R = 2\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^*}$ is satisfied, $\boldsymbol{x}^*$ is also an optimal solution of problem (7.15).

Problem (7.16) is a parametric 0-1 knapsack problem. In previous researches, a solution method based on the parametric dynamic programming approach has been proposed. However, in this solution method, a dynamic programming is repeatedly used. Therefore, this solution method is not efficient. We introduce the following 0-1 relaxation problems with respect to problem (7.16);

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \\
\text{subject to} \quad & \sum_{j=1}^{n} \overline{c}_j x_j - L^*\left(\overline{h}\right) \sum_{j=1}^{n} d_j x_j \le g^{-1}\left(\overline{h}\right), \\
& 0 \le x_j \le 1, \quad j = 1, 2, \cdots, n
\end{aligned} \tag{7.17}
$$

and its auxiliary problem:

$$
\begin{aligned}
\text{Maximize} \quad & R \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \left( \sum_{j=1}^{n} \sigma_j^2 x_j \right) \\
\text{subject to} \quad & \sum_{j=1}^{n} \overline{c}_j x_j - L^*\left(\overline{h}\right) \sum_{j=1}^{n} d_j x_j \le g^{-1}\left(\overline{h}\right), \\
& 0 \le x_j \le 1, \quad j = 1, 2, \cdots, n
\end{aligned} \tag{7.18}
$$

Then, the following theorem holds with respect to the relation between problems (7.17) and (7.18) based on previous research [55].

**Theorem 7.3**

For $g(R) = R - \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^{*2}}$ , the following relation holds:

$$
\begin{aligned}
R^* > R \quad &\Leftrightarrow \quad g(R) > 0 \\
R^* = R \quad &\Leftrightarrow \quad g(R) = 0 \\
R^* < R \quad &\Leftrightarrow \quad g(R) < 0
\end{aligned}
$$

From this theorem, the optimal solution of problem (7.17) becomes equal to that of problem (7.18). In previous studies [55], parameter $R$ is repeatedly modified using bisection algorithm in order to solve problem (7.16) using dynamic programming. However, this solution method is not efficiently. Therefore, we propose a new solution method introducing the 0-1 relaxation problem and its optimal solution.

First, in order to construct the efficient solution method, the following lemmas are derived.

**Lemma 7.4.1**

With problem (7.16), there exists the ranges $\left[R_k, R_{k+1}\right]$, $\left(k=1,2,\ldots,K\right)$ that the optimal solution of problem (7.16) is unique in the case of $R$ including in $\left[R_k, R_{k+1}\right]$.

**Proof**

From the discreteness of decision variable, it is obvious that this theorem holds. □

**Lemma 7.4.2**

We set a range $\left[R_L, R_U\right]$ satisfying $R^* \in \left[R_L, R_U\right]$. Let the optimal solution of problem (7.16) be $\overline{\boldsymbol{x}}$. Then, $\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) \le 0$ holds.

**Proof**

Let $R^* = \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^*}$ and $\overline{R} = \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}$. In the case that $\overline{R} < R^*$, with respect to $\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)$,

$$\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) = \left(R_U - \overline{R}\right) > \left(R_U - R^*\right) > 0$$

holds. Furthermore, from the monotonous function $R - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}$ of $R$ and $R^* - \overline{R} > 0$, it is obvious that $\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) < 0$. Therefore, $\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) \le 0$ holds.

In the case that $\overline{R} > R^*$, with respect to $\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)$,

$$\begin{aligned}\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) &= \left(R_L - \overline{R}\right)\left(R_U - \overline{R}\right) \\ &\le \left(R_L - R^*\right)\left(R_U - R^*\right)\end{aligned}$$

holds. Then, since we obtain $\left(R_L - R^*\right) < 0$ and $\left(R_U - R^*\right) > 0$, $\left(R_L - R^*\right)\left(R_U - R^*\right) < 0$ and $\left(R_L - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right)\left(R_U - \sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j}\right) \le 0$ hold. Consequently, this theorem holds. □

**Lemma 7.4.3**

In the case that $T(R) = R - 2\sqrt{\sum_{j=1}^{n} \sigma_j^2 \overline{x}_j} = 0$, $\overline{x}_j$ is an optimal solution of main problem (14).

**Proof**

From Theorem , it is obvious that this theorem holds. □

Using these lemmas, the following theorem holds.

**Theorem 7.4**

Let the optimal solution of problem (7.18) be $\boldsymbol{x}^*$ and $R^* = \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^*}$. Then, the optimal solution of the following problem;

$$
\begin{aligned}
&\text{Maximize} \quad R^* \sum_{j=1}^{n} \overline{r}_j x_j - K_\beta \left( \sum_{j=1}^{n} \sigma_j^2 x_j \right) \\
&\text{subject to} \quad \sum_{j=1}^{n} \overline{c}_j x_j - L^*\left(\overline{h}\right) \sum_{j=1}^{n} d_j x_j \le g^{-1}\left(\overline{h}\right), \\
&\qquad\qquad 0 \le x_j \le 1, \quad j = 1, 2, \cdots, n
\end{aligned}
\tag{7.19}
$$

is equivalent to that of problem (7.16).

Consequently, in the case that we fix the parameter $h$ of main problem (7.14), introducing 0-1 relaxation problem and finding its optimal solution, we obtain an optimal solution without using dynamic programming repeatedly. Therefore, this solution method is more efficient than previous parametric dynamic programming approach in that the number of using dynamic programming is significantly decreasing. Then, we construct the following efficient solution method to solve main 0-1 nonlinear knapsack problem (7.14).

**Solution Method 7.1**

STEP 1: Elicit the membership function of a fuzzy goal for with respect to the total profit and maximum budget.

STEP 2: Set $h \leftarrow 1$ and solve problem (7.18). If the optimal objective value $Z(h)$ of problem (7.18) satisfies $Z(h) \ge f^{-1}(h)$ and its feasible solution including constraints exists, then terminate. In this case, the obtained current solution is an optimal solution of main problem.

STEP 3: Set $h \leftarrow 0$ and solve problem (7.18). If the optimal objective value $Z(h)$ of problem (7.18) satisfies $Z(h) < f^{-1}(h)$ or the feasible solution including constrains does not exist, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for the probability or the aspiration level $f$.

Step 4: Set $U_h \leftarrow 1$ and $L_h \leftarrow 0$.

Step 5: Set $h \leftarrow \dfrac{U_h + L_h}{2}$

Step 6: Solve problem (7.18) and calculate the optimal objective value $Z(h)$ of problem (7.18). If $Z(h) > f^{-1}(h)$, then set $L_h \leftarrow h$ and return to Step 5. If $Z(h) \leq f^{-1}(h)$, then set $U_h \leftarrow h$ and return to Step 5. If $Z(h) = f^{-1}(h)$, then terminate the algorithm. In this case, $\boldsymbol{x}^*(h)$ is equal to a global optimal solution of main problem.

## 7.2 Fuzzy Random 0-1 Programming Problem and the Efficient Solution Method

### 7.2.1 Formulation of fuzzy random 0-1 programming problem

A fuzzy random variable (Kwakernaak [72], Liu [83], Puri and Ralescu [102]) is one of the mathematical concepts dealing with randomness and fuzziness simultaneously. In this chapter, we deal with the fuzzy random variable based on Liu [83] and consider the following 0-1 programming problem:

$$\begin{aligned} &\text{Maximize} \quad \tilde{\bar{\mathbf{C}}}\boldsymbol{x} \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \leq \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.20}$$

where each notation is as follows:

$\mathbf{A}$ : $m \times n$ coefficient matrix

$\mathbf{b}$ : $m$-dimensional column vector

$\boldsymbol{x}$ : $n$-dimensional decision variable column vector

The coefficient vector of objective function is $\tilde{\bar{\mathbf{C}}} = \left(\tilde{\bar{c}}_1, \tilde{\bar{c}}_2, \ldots, \tilde{\bar{c}}_n\right)$ and each $\tilde{\bar{c}}_j$ is a fuzzy random variable characterized by the following membership function:

$$\mu_{\tilde{\bar{c}}_j}(\omega)=\begin{cases} L\left(\dfrac{\bar{c}_j-\omega}{\alpha_j}\right), & \left(\bar{c}_j-\alpha_j\le\omega\le\bar{c}_j\right) \\ R\left(\dfrac{\omega-\bar{c}_j}{\beta_j}\right), & \left(\bar{c}_j\le\omega\le\bar{c}_j+\beta_j\right) \\ 0, & \left(\omega\le\bar{c}_j-\alpha_j,\ \bar{c}_j+\beta_j\le\omega\right) \end{cases} \tag{7.21}$$

where $L(x)$ and $R(x)$ are nonincreasing reference functions to satisfy $L(0)=R(0)=1$, $L(1)=R(1)=0$ and the parameters $\alpha_j$ and $\beta_j$ represent the spreads corresponding to the left and the right sides, respectively, and $\bar{c}_j$ is a random variable according to a normal distribution $\bar{c}_j \sim N\left(m_j,\sigma_j^2\right)$. Problem (7.20) is a fuzzy random 0-1 programming problem due to including fuzzy random variables. Then, its objective function $\tilde{\bar{Y}}=\tilde{\bar{\mathbf{C}}}\boldsymbol{x}$ is the following fuzzy random variable using fuzzy extension principle:

$$\mu_{\tilde{\bar{c}}_j}(\omega)=\begin{cases} L\left(\dfrac{\sum_{j=1}^{n}\bar{c}_jx_j-\omega}{\sum_{j=1}^{n}\alpha_jx_j}\right), \\ R\left(\dfrac{\omega-\sum_{j=1}^{n}\bar{c}_jx_j}{\sum_{j=1}^{n}\beta_jx_j}\right), \\ 0, \end{cases} \tag{7.22}$$

Therefore, problem (7.20) is not a well-defined problem due to fuzzy random variables, and so it is necessary to interpret the problem from some point of view and to transform the problem into the deterministic equivalent problem. In this chapter, we consider the case where a decision maker prefers to maximize the degree of possibility that the objective function value satisfies the fuzzy goal, based on previous research Katagiri [62]. A fuzzy goal for the objective function is characterized by the following membership function:

$$\mu_{\tilde{G}}(y)=\begin{cases} 1, & \left(y>h_1\right) \\ g(y), & \left(h_0\le y\le h_1\right) \\ 0, & \left(y<h_0\right) \end{cases} \tag{7.23}$$

where $g(y)$ is a monotonous increasing function. Then, the degree of possibility that the objective function value satisfying a fuzzy goal $\tilde{G}$ is as follows:

$$\Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right) = \sup_{y} \min\left\{\mu_{\tilde{\bar{Y}}}(y), \mu_{\tilde{G}}(y)\right\} \tag{7.24}$$

Consequently, problem (7.20) is transformed into the following problem:

$$\begin{aligned} &\text{Maximize} \quad \Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right) \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.25}$$

Since $\Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right)$ varies randomly due to the randomness of $\tilde{\bar{c}}_j$, this problem is a stochastic programming problem. Therefore, problem (7.25) is also not a well defined problem. In stochastic programming, there are typical models such as the expectation optimization model, probability maximization model, and so on. In this chapter, we focus on the possibility fractile optimization model maximizing the degree of possibility. This problem is formulated as the following form:

$$\begin{aligned} &\text{Maximize} \quad h \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n, \\ &\qquad\qquad\quad \Pr\left\{\Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right) \ge h\right\} \ge t, \end{aligned} \tag{7.26}$$

### 7.2.2 Deterministic equivalent transformation of the proposed model

In problem (7.26), constraint $\Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right) \ge h$ is transformed into the following form:

$$\begin{aligned} \Pi_{\tilde{\bar{Y}}}\left(\tilde{G}\right) \ge h \quad &\Leftrightarrow \quad \sup_{y} \min\left\{\mu_{\tilde{\bar{Y}}}(y), \mu_{\tilde{G}}(y)\right\} \ge h \\ &\Leftrightarrow \quad \exists y: \ \mu_{\tilde{\bar{Y}}}(y) \ge h, \ \mu_{\tilde{G}}(y) \ge h \\ &\Leftrightarrow \quad \exists y: \ R\left(\frac{y - \sum_{j=1}^{n} \bar{c}_j x_j}{\sum_{j=1}^{n} \beta_j x_j}\right) \ge h, \ y \ge g^{-1}(h) \\ &\Leftrightarrow \quad \exists y: \ y \le \sum_{j=1}^{n} \bar{c}_j x_j + R^*(h) \sum_{j=1}^{n} \beta_j x_j, \ y \ge g^{-1}(h) \\ &\Leftrightarrow \quad \sum_{j=1}^{n} \bar{c}_j x_j + R^*(h) \sum_{j=1}^{n} \beta_j x_j \ge g^{-1}(h) \end{aligned} \tag{7.27}$$

where $R^*(h)$ is a pseudo inverse function of $R(x)$. From this inequality, problem (7.26) is equivalently transformed into the following problem:

$$
\begin{array}{ll}
\text{Maximize} & h \\
\text{subject to} & \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n, \\
& \Pr\left\{\sum_{j=1}^{n} \bar{c}_j x_j + R^*(h)\sum_{j=1}^{n}\beta_j x_j \ge g^{-1}(h)\right\} \ge t
\end{array}
\tag{7.28}
$$

Furthermore, with respect to stochastic constraint $\Pr\left\{\sum_{j=1}^{n} \bar{c}_j x_j + R^*(h)\sum_{j=1}^{n}\beta_j x_j \ge g^{-1}(h)\right\} \ge t$, by using the feature of normal distribution $\bar{c}_j \sim N\left(m_j, \sigma_j^2\right)$, this constraint is equivalently transformed into the following form:

$$
\begin{aligned}
& \Pr\left\{\sum_{j=1}^{n} \bar{c}_j x_j + R^*(h)\sum_{j=1}^{n}\beta_j x_j \ge g^{-1}(h)\right\} \ge t \\
\Leftrightarrow \ & \Pr\left\{\sum_{j=1}^{n} \bar{c}_j x_j \ge g^{-1}(h) - R^*(h)\sum_{j=1}^{n}\beta_j x_j\right\} \ge t \\
\Leftrightarrow \ & \Pr\left\{\frac{\sum_{j=1}^{n} \bar{c}_j x_j - \sum_{j=1}^{n} m_j x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}} \ge \frac{g^{-1}(h) - R^*(h)\sum_{j=1}^{n}\beta_j x_j - \sum_{j=1}^{n} m_j x_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}}\right\} \ge t \\
\Leftrightarrow \ & \frac{\sum_{j=1}^{n} m_j x_j - \left(g^{-1}(h) - R^*(h)\sum_{j=1}^{n}\beta_j x_j\right)}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2}} \ge K_t \\
\Leftrightarrow \ & \sum_{j=1}^{n} m_j x_j - \left(g^{-1}(h) - R^*(h)\sum_{j=1}^{n}\beta_j x_j\right) \ge K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2} \\
\Leftrightarrow \ & \sum_{j=1}^{n} m_j x_j + R^*(h)\sum_{j=1}^{n}\beta_j x_j - K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2} \ge g^{-1}(h)
\end{aligned}
\tag{7.29}
$$

where $F(z)$ is the distribution function of the standard normal distribution and $K_t = F^{-1}(t)$. Therefore, we equivalently transform problem (7.28) into the following problem:

$$
\begin{array}{ll}
\text{Maximize} & h \\
\text{subject to} & \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n, \\
& \sum_{j=1}^{n} m_j x_j + R^*(h)\sum_{j=1}^{n}\beta_j x_j - K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^2} \ge g^{-1}(h)
\end{array}
\tag{7.30}
$$

Furthermore, each decision variable $x_j$ satisfies $x_j \in \{0,1\}$, we obtain $x_j^2 = x_j$, and so problem (7.30) is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximize} \quad h \\
&\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n, \\
&\qquad \sum_{j=1}^{n} m_j x_j + R^*(h) \sum_{j=1}^{n} \beta_j x_j - K_t \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \ge g^{-1}(h)
\end{aligned}
\tag{7.31}
$$

It should be noted here that problem (7.31) is a nonconvex programming problem and it is not solved by the linear programming techniques or convex programming techniques. However, since a decision variable $h$ is involved only in constraint $\sum_{j=1}^{n} m_j x_j + R^*(h) \sum_{j=1}^{n} \beta_j x_j - K_t \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \ge g^{-1}(h)$, we introduce the following subproblem involving a parameter $q$:

$$
\begin{aligned}
&\text{Maximize} \quad \sum_{j=1}^{n} m_j x_j + R^*(q) \sum_{j=1}^{n} \beta_j x_j - K_t \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j} \\
&\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n,
\end{aligned}
\tag{7.32}
$$

In the case that we fix the parameter $q$, problem (7.32) is equivalent to a convex programming problem. Furthermore, let $x(q)$ and $Z(q)$ be an optimal solution of problem (7.32) and its optimal value, respectively. Then, the following theorem is derived from the study (Hasuike [45]).

**Theorem 7.5**

For $q$ satisfying $0 < q < 1$, $Z(q)$ is a strictly increasing function of $q$.

Furthermore, Let $\hat{q}$ denote $q$ satisfying $Z(\hat{q}) = g^{-1}(\hat{q})$. Then the relation between problems (7.31) and (7.32) is derived as follows.

**Theorem 7.6**

Suppose that $0 < \hat{q} < 1$ holds. Then $\left(x(\hat{q}), \hat{q}\right)$ is equal to $\left(x^*, h^*\right)$.

From these theorems, by using bisection algorithm for parameter $q$ and comparing objective function $Z(q)$ with $g^{-1}(q)$, we repeatedly solve problem (7.32) for each $q$ using branch-bound method, and finally obtain the optimal solution. This solution method is assured that its calculation times are infinite. However, it is not efficient due to increasing calculation times voluminously with the increase of parameters and decision variables. Therefore, we need to construct the more efficient solution method.

### 7.2.3 Construction of the efficient strict solution method

In order to construct the efficient solution method for problem (7.32), first of all, we introduce the following 0-1 relaxation problem of problem (7.32):

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} m_j x_j + R^*\left(\bar{h}\right)\sum_{j=1}^{n}\beta_j x_j - K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j} \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad 0 \le x_j \le 1, \quad \left(j = 1, 2, ..., n\right) \end{aligned} \tag{7.33}$$

Since this problem is a nonlinear programming problem due to including a square root term $\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}$ and it is difficult to solve it, we also consider the following auxiliary problem for problem (7.33) introducing a parameter $\gamma$ :

$$\begin{aligned} &\text{Maximize} \quad \gamma\left(\sum_{j=1}^{n} m_j x_j + R^*\left(\bar{h}\right)\sum_{j=1}^{n}\beta_j x_j\right) - K_t\left(\sum_{j=1}^{n}\sigma_j^2 x_j\right) \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad 0 \le x_j \le 1, \quad \left(j = 1, 2, ..., n\right) \end{aligned} \tag{7.34}$$

With respect to the relation between problems (7.33) and (7.34), the following theorem holds based on previous research (Ishii [55]).

**Theorem 7.7**

Let an optimal solution of problem (7.34) be $\boldsymbol{x}^*$. If $\gamma = 2\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^*}$ is satisfied, $\boldsymbol{x}^*$ is also an optimal solution of problem (7.33).

Auxiliary problem (7.34) is a linear programming problem in the case that $\gamma$ is fixed, and so we efficiently obtain the optimal solution using linear programming approaches and bisection algorithm for parameter $\bar{h}$. Furthermore, let the optimal value of parameter $\bar{h}$ in problem (7.34) be $\bar{h}^*$. Then, the following lemmas hold:

**Lemma 7.8.1**

With problem (7.32), there exists the ranges $\left[h_k, h_{k+1}\right]$, $\left(k = 1, 2, ...\right)$ that the optimal solution of problem (7.32) is unique in the case of $\bar{h}^*$ including in $\left[h_k, h_{k+1}\right]$.

**Lemma 7.8.2**

We set a range $\left[h_L, h_U\right]$ satisfying $\overline{h}^* \in \left[h_L, h_U\right]$. Let the optimal solution of problem (7.32) be $\boldsymbol{x}^*$. Then, with respect to any parameter $h \in \left[h_L, h_U\right]$, $\boldsymbol{x}^*$ is unique.

From these lemmas, the following theorem holds:

**Theorem 7.8**

Let the optimal solution of problem (7.34) be $\boldsymbol{x}^*$ and the optimal value of parameter $\overline{h}$ be $\overline{h}^*$. Then, the optimal solution of the following problem;

$$\begin{aligned} &\text{Maximize} \quad \sum_{j=1}^{n} m_j x_j + R^*\left(\overline{h}^*\right)\sum_{j=1}^{n}\beta_j x_j - K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j} \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \mathbf{b}, \quad \boldsymbol{x} \in \{0,1\}^n, \end{aligned} \tag{7.35}$$

is equivalent to that of problem (7.32). Consequently, in the case that we solve 0-1 relaxation problem (7.34) and obtain its optimal solution $\overline{h}^*$, we obtain an optimal solution without using branch-bound method repeatedly. Furthermore, from the optimal solution of problem (7.34), upper and temporary lower values of objective function $\sum_{j=1}^{n} m_j x_j + R^*\left(\overline{h}^*\right)\sum_{j=1}^{n}\beta_j x_j - K_t\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}$ are given. Therefore, by using these values effectively, this solution method is more efficient than previous parametric 0-1 programming approach in that the number of using branch-bound method is significantly decreasing. Consequently, we construct the following efficient solution method to solve main 0-1 programming problem (7.26).

**Solution method 7.2**

STEP 1: Elicit the membership function of a fuzzy goal for with respect to the total profit and maximum budget.

STEP 2: Set $h \leftarrow 1$ and solve problem (7.32). If the optimal objective value $Z(h)$ of problem (13) satisfies $Z(h) \ge g^{-1}(h)$ and its feasible solution including constraints exists, then terminate. In this case, the obtained current solution is an optimal solution of main problem. Otherwise, go to STEP 2.

STEP 3: Set $h \leftarrow 0$ and solve problem (7.32). If the optimal objective value $Z(h)$ of problem (13) satisfies $Z(h) < g^{-1}(h)$ or the feasible solution including constrains does not exist, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal for the probability or the aspiration level $f$. Otherwise, go to STEP4.

STEP 4: Solve problem (7.34) and obtain the optimal solution $\boldsymbol{x}^*(h)$ and optimal value $h$. Then, solve problem (7.32) using branch-bound method.

### 7.2.4 Numerical example

In order to illustrate the efficiency of our proposed model, we provide the numerical example. In this numerical example, we consider 10 decision variables and 3 constraints as Table 7.1, and we assume the all fuzzy numbers are triangle fuzzy numbers whose values of spreads are as Table 7.1.

Table 7.1. Sample data of coefficients in objective function and constraints

| | R1 | R2 | R3 | R4 | R5 | R6 | R7 | R8 | R9 | R10 | Upper |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Mean | 4 | 5 | 2 | 3 | 8 | 6 | 3 | 5 | 10 | 7 | |
| Spread | 1 | 2 | 0.5 | 1 | 3 | 0.5 | 2 | 1.5 | 1 | 2 | |
| Variance | 0.7 | 1.5 | 0.2 | 0.4 | 2.2 | 1.8 | 0.3 | 1.2 | 3 | 1.3 | |
| Const.1 | 2 | 3 | 1 | 1 | 5 | 4 | 2 | 3 | 5 | 3 | 15 |
| Const.2 | 3 | 2 | 2 | 3 | 5 | 4 | 3 | 1 | 6 | 3 | 20 |
| Const.3 | 2 | 2 | 3 | 1 | 4 | 3 | 2 | 1 | 4 | 5 | 15 |

In this numerical example, we solve general 0-1 programming problems using the previous solution method and our proposed method, and obtain the same optimal solution to each solution method as Table 7.2. Subsequently, in order to solve them, we use Mathematica 5.0 under the computer environment; Pentium 4 CPU 3.20GHz and 2.00GB RAM.

Table 7.2. Optimal solution of fuzzy random 0-1 programming problem

| | R1 | R2 | R3 | R4 | R5 | R6 | R7 | R8 | R9 | R10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Optimal | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |

Then, each calculation time to find the optimal solution is as Table 7.3.

Table 7.3. Calculation time

| | Proposed method | Parametric approach |
|---|---|---|
| Calculation time (sec.) | 2.984 | 3.078 |

From the result of Table 7.3, we find that the sample problem is solved in a shorted calculation time by our proposed approach.

Furthermore, with respect to previous parametric integer programming approach for fuzzy random

0-1 programming problems, integer programming approaches, such as the Dynamic programming and the Branch-bound method, have been used to every parameter value set by using bisection method. Therefore, the computational time complexity has been $(K_{IPM})\cdot(\log n)^2$, where $K_{IPM}$ is the computational time complexity of integer programming approach. However, with respect to our proposed approach, we do not use integer programming approach repeatedly. Therefore, the computational time complexity is $n^4 \log n + K_{IPM}$. Consequently, our proposed solution approach may be more useful in the case that the number of decision variables and constraints are larger than this numerical example.

## 7.3 Random Fuzzy 0-1 Programming Problem

### 7.3.1 Formulation of the random fuzzy 0-1 programming problem

Then, we formally introduce the following 0-1 programming problem:

$$\begin{array}{l} \text{Maximum } \tilde{\bar{\boldsymbol{r}}}\boldsymbol{x} \\ \text{subject to } \mathbf{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n \end{array} \tag{7.36}$$

where each notation is as follows:

**A**: $m \times n$ coefficient matrix

**b**: $m$-dimensional column vector

***x***: $n$-dimensional decision column vector (Decision variable)

The coefficient vector of objective function is $\tilde{\bar{\boldsymbol{r}}} = (\tilde{\bar{r}}_1, \tilde{\bar{r}}_2, \ldots, \tilde{\bar{r}}_n)$ and each $\tilde{\bar{r}}_j$ is a random fuzzy variable according to a normal distribution $N(\tilde{m}_j, \sigma_j^2)$ where $\tilde{m}_j$ is a mean value and $\sigma_j^2$ is a variance. Then, we represent the *ij*th element of variance-covariance matrix as $\sigma_{ij}$. Furthermore, we assume that $\tilde{m}_j$ is a fuzzy variable characterized by the following membership function:

$$\mu_{\tilde{m}_j}(\omega) = \begin{cases} L\left(\dfrac{m_j - \omega}{\alpha_j}\right) & (m_j - \alpha_j \le \omega \le m_j) \\ R\left(\dfrac{\omega - m_j}{\beta_j}\right) & (m_j < \omega \le m_j + \beta_j) \\ 0 & (\omega < m_j - \alpha_j,\ m_j + \beta_j < \omega) \end{cases} \tag{7.37}$$

where $L(x)$ and $R(x)$ are nonincreasing reference function to satisfy $L(0)=R(0)=1$, $L(1)=R(1)=0$ and the parameters $\alpha_j$ and $\beta_j$ represent the spreads corresponding to the left and the right sides, respectively. Problem (7.36) is a random fuzzy 0-1 programming problem due to including random fuzzy variables. Then, its objective function $\tilde{\bar{Z}} = \tilde{\bar{\boldsymbol{r}}}\boldsymbol{x}$ is defined as a random fuzzy variable by the following membership function introducing a marameter $\bar{\gamma}_j$ and an universal set of normal random variable $\Gamma$:

$$\mu_{\tilde{\bar{Z}}}(\bar{u}) = \sup_{\bar{\gamma}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{\bar{r}}_j}(\bar{\gamma}_j) \middle| \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j \right\}, \forall \bar{u} \in Y \tag{7.38}$$

where $\bar{\boldsymbol{\gamma}} = (\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_n)$, $\mu_{\tilde{\bar{r}}_j}(\bar{\gamma}_j)$ is defined by

$$\mu_{\tilde{\bar{r}}_j}(\bar{\gamma}_j) = \sup_{s_j} \left\{ \mu_{\tilde{M}_j}(s_j) \middle| \bar{\gamma}_j \sim N(s_j, \sigma_j^2) \right\}, \forall \bar{\gamma}_j \in \Gamma \tag{7.39}$$

and $Y$ is defined by

$$Y = \left\{ \sum_{j=1}^{n} \bar{\gamma}_j x_j \middle| \bar{\gamma}_j \in \Gamma, j = 1, 2, \ldots, n \right\} \tag{7.40}$$

From these settings, we obtain

$$\begin{aligned} \mu_{\tilde{\bar{Z}}}(\bar{u}) &= \sup_{\bar{\gamma}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{\bar{r}}_j}(\bar{\gamma}_j) \middle| \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j \right\} \\ &= \sup_{\boldsymbol{s}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{M}_j}(s_j) \middle| \bar{\gamma}_j \sim N(s_j, \sigma_j^2), \bar{u} = \sum_{j=1}^{n} \bar{\gamma}_j x_j \right\} \\ &= \sup_{\boldsymbol{s}} \left\{ \min_{1 \le j \le n} \mu_{\tilde{M}_j}(s_j) \middle| \bar{u} \sim N\left( \sum_{j=1}^{n} s_j x_j, V(\boldsymbol{x}) \right) \right\} \end{aligned} \tag{7.41}$$

where $V(\boldsymbol{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$ Furthermore, we discuss the probability that the objective function value is greater than or equal to an aspiration level $f$. Then, we represent the probability as $\Pr\left\{ \omega \middle| \sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \ge f \right\}$. Since $\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j$ is represented with a random fuzzy variable, we express the probability $\Pr\left\{ \omega \middle| \sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \ge f \right\}$ as a fuzzy set $\tilde{P}$ and defined the membership function of $\tilde{P}$ as follows:

$$\begin{aligned}\mu_{\tilde{P}}(p) &= \sup_{\bar{u}}\left\{\mu_{\bar{\tilde{Z}}}(\bar{u}) \,\middle|\, p = \Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\}\right\} \\ &= \sup_{\boldsymbol{s}} \min_{1 \leq j \leq n}\left\{\mu_{\tilde{M}_j}(s_j) \,\middle|\, \begin{array}{l} p = \Pr\left\{\omega \middle| \bar{u}(\omega) \geq f\right\}, \\ \bar{u} \sim N\left(\sum_{j=1}^{n} s_j x_j, V(\boldsymbol{x})\right) \end{array}\right\}\end{aligned} \tag{7.42}$$

where $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$. Due to these randomness and fuzziness, problem (7.36) is not a well-defined problem, and so it is necessary to interpret the problem from some point of view and to transform the problem into the deterministic equivalent problem. In this chapter, we consider the case where a decision maker prefers maximizing the degree of possibility for the probability that the value of objective function satisfies the fuzzy goal, based on previous research Katagiri [62] and Hasuike [45]. A fuzzy goal for the probability is characterized by the following membership function:

$$\mu_{\tilde{G}_P}(p) = \begin{cases} 1 & p_1 < p \\ g(p) & p_0 \leq p \leq p_1 \\ 0 & p < p_0 \end{cases} \tag{7.43}$$

where $g(y)$ is a monotonous increasing function. Then, using a concept of possibility measure, the degree of possibility to the objective function value satisfying a fuzzy goal $G$ is as follows:

$$\prod\nolimits_P(G) = \sup_p \min\left\{\mu_{\tilde{P}}(p), \mu_{\tilde{G}_P}(p)\right\} \tag{7.44}$$

Consequently, problem (7.36) is transformed into the following problem:

$$\begin{array}{ll} \text{Maximum} & \prod\nolimits_P(G) \\ \text{subject to} & \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n \end{array} \tag{7.45}$$

This problem is equivalently transformed into the following problem introducing a parameter $h$.

$$\begin{array}{ll} \text{Maximum} & h \\ \text{subject to} & \prod\nolimits_P(G) \geq h, \\ & \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n \end{array} \tag{7.46}$$

### 7.3.2 Deterministic equivalent transformations to the proposed model

In problem (7.46), constraint $\prod_P(G) \geq h$ is transformed into the following inequality based on the result obtained by Katagiri [58] and Hasuike [45]:

$$
\begin{aligned}
&\textstyle\prod_P (G) \geq h \\
\Leftrightarrow \quad & \sup_p \min \left\{ \mu_{\tilde{P}}(p), \mu_{\tilde{G}_P}(p) \right\} \geq h \\
\Leftrightarrow \quad & \exists p : \mu_{\tilde{P}}(p) \geq h, \mu_{\tilde{G}_P}(p) \geq h \\
\Leftrightarrow \quad & \exists p : \sup_{\boldsymbol{s}} \min_{1 \leq j \leq n} \left\{ \mu_{\tilde{M}_j}(s_j) \left| \begin{array}{l} p = \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\}, \\ \bar{u} \sim N\left( \sum_{j=1}^n s_j x_j, V(\boldsymbol{x}) \right) \end{array} \right. \right\} \geq h, \ \mu_{\tilde{G}_P}(p) \geq h \\
\Leftrightarrow \quad & \begin{array}{l} \exists p, \exists \boldsymbol{s}, \exists \bar{u} : \sup_{\boldsymbol{s}} \min_{1 \leq j \leq n} \mu_{\tilde{M}_j}(s_j) \geq h, \ p = \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\}, \\ \qquad \bar{u} \sim N\left( \sum_{j=1}^n s_j x_j, V(\boldsymbol{x}) \right), \mu_{\tilde{G}_P}(p) \geq h \end{array} \\
\Leftrightarrow \quad & \begin{array}{l} \exists p, \exists \boldsymbol{s}, \exists \bar{u} : \min_{1 \leq j \leq n} \mu_{\tilde{M}_j}(s_j) \geq h, \ p = \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\}, \\ \qquad \bar{u} \sim N\left( \sum_{j=1}^n s_j x_j, V(\boldsymbol{x}) \right), p \geq g^{-1}(h) \end{array} \\
\Leftrightarrow \quad & \begin{array}{l} \exists p, \exists \boldsymbol{s}, \exists \bar{u} : \mu_{\tilde{M}_j}(s_j) \geq h, \ (j = 1, 2, ..., n) \\ \qquad \bar{u} \sim N\left( \sum_{j=1}^n s_j x_j, V(\boldsymbol{x}) \right), \ \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq g^{-1}(h) \end{array} \\
\Leftrightarrow \quad & \exists \bar{u} : \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq g^{-1}(h), \ \bar{u} \sim N\left( \sum_{j=1}^n \left( m_j + R^*(h) \alpha_j \right) x_j, V(\boldsymbol{x}) \right)
\end{aligned}
\tag{7.47}
$$

where $R^*(x)$ is a pseudo inverse function of $R(x)$. From this inequality, problem (7.46) is equivalently transformed into the following problem:

$$
\begin{aligned}
&\text{Maximum } \ h \\
&\text{subject to } \ \Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq g^{-1}(h), \\
&\qquad \bar{u} \sim N\left( \sum_{j=1}^n \left( m_j + R^*(h) \alpha_j \right) x_j, V(\boldsymbol{x}) \right), \\
&\qquad \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \ \boldsymbol{x} \in \{0,1\}^n
\end{aligned}
\tag{7.48}
$$

Furthermore, with respect to stochastic constraint $\Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq g^{-1}(h)$, by using the property of normal distribution, this constraint is equivalently transformed into the following form:

$$
\begin{aligned}
&\Pr\left\{ \omega \middle| \bar{u}(\omega) \geq f \right\} \geq g^{-1}(h) \\
\Leftrightarrow \quad & \frac{\sum_{j=1}^n \left( m_j + R^*(h) \alpha_j \right) x_j - f}{\sqrt{V(\boldsymbol{x})}} \geq K_{g^{-1}(h)}
\end{aligned}
\tag{7.49}
$$

where $F(z)$ is the distribution function of the standard normal distribution and $K_t = F^{-1}(t)$.

Furthermore, each decision variable $x_j$ satisfies $x_j \in \{0,1\}$, we obtain $x_j^2 = x_j$ and assume that each vairiance is independent, i.e.,

$$\sigma_{ij} = \begin{cases} \sigma_j^2 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

Consequently, problem (7.48) is equivalently transformed into the following problem:

$$\begin{aligned} &\text{Maximum } h \\ &\text{subject to } \frac{\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}} \geq K_{g^{-1}(h)}, \\ &\qquad\qquad \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.50}$$

It should be noted here that problem (7.50) is a nonconvex integer programming problem and it is not solved by the linear programming techniques or convex programming techniques. However, since a decision variable *h* is involved only in first constraint, we introduce the following subproblem involving a parameter *q*:

$$\begin{aligned} &\text{Maximum } \frac{\sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}} \\ &\text{subject to } \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.51}$$

In the case that we fix the parameter *q*, problem (7.51) is equivalent to a convex integer programming problem. Furthermore, let $\boldsymbol{x}(q)$ and $Z(q)$ be an optimal solution of problem (7.51) and its optimal value, respectively. Then, the following theorem is derived from previous study [7, 13].

**Theorem 7.9**

For $q$ satisfying $0 < q < 1$, $Z(q)$ is a strictly increasing function of $q$.

Furthermore, let $\hat{q}$ denote $q$ satisfying $Z(\hat{q}) = g^{-1}(\hat{q})$ and the optimal solutions of main problem (7.50) be $(\boldsymbol{x}^*, h^*)$.Then the relation between problems (7.50) and (7.51) is derived as follows derived from previous study [60].

**Theorem 7.10**

Suppose that $0 < \hat{q} < 1$ holds. Then $(x(\hat{q}), \hat{q})$ is equal to $(\boldsymbol{x}^*, h^*)$.

From these theorems, by using bisection algorithm for parameter *q* and comparing objective function $Z(q)$ with $g^{-1}(q)$, we repeatedly solve problem (7.51) for each *q* using branch-bound method, and finally obtain the optimal solution. This solution method is assured that its calculation times are infinite. However, it is not efficient due to increasing computational times voluminously with the increase of parameters and decision variables. Therefore, we need to construct the more efficient solution method.

### 7.3.3 Construction of the efficient strict solution method

In order to construct the efficient strict solution method for problem (7.50), first of all, we introduce the following 0-1 relaxation problem of problem (7.50):

$$
\begin{aligned}
&\text{Maximum } h \\
&\text{subject to } \frac{\sum_{j=1}^{n}\left(m_j + R^*(h)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}} \geq K_{g^{-1}(h)}, \\
&\qquad \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, 0 \leq x_j \leq 1,\ j = 1, 2, \ldots, n
\end{aligned} \tag{7.52}
$$

In a way similar to problem (7.50), this problem is also a nonconvex programming problem and it is not solved by the linear programming techniques or convex programming techniques. Subsequently, we introduce the following subproblem:

$$
\begin{aligned}
&\text{Maximum } \frac{\sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}} \\
&\text{subject to } \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, 0 \leq x_j \leq 1,\ j = 1, 2, \ldots, n
\end{aligned} \tag{7.53}
$$

In this subsection, it is assumed that there exists a feasible solution satisfying $\sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)x_j > f$. This means that the probability that total future profit is more than target value $f$ is greater than 1/2. Furthermore, problem (7.53) is equivalent to the following problem:

$$\begin{aligned}
&\text{Minimum} \quad \frac{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}}{\sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)x_j - f} \\
&\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \boldsymbol{b},\, 0 \le x_j \le 1,\ j = 1,2,\ldots,n
\end{aligned} \tag{7.54}$$

In the case we fix the parameter $q$ since problem (7.54) is a nonlinear fractional programming problem due to including a square root term $\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}$ in the objective function, it is difficult to solve this original problem directly. Therefore, we introduce the following parameters;

$$t = 1\Bigg/\left(\sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)x_j - f\right),\ \boldsymbol{y} = t\boldsymbol{x}$$

and we do the transformation into the following deterministic equivalent problem:

$$\begin{aligned}
&\text{Minimum} \quad \sqrt{\sum_{j=1}^{n}\sigma_j^2 y_j} \\
&\text{subject to} \quad \sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)y_j - ft = 1, \\
&\qquad\qquad\quad \mathbf{A}\boldsymbol{y} \le \boldsymbol{b}t,\, 0 \le y_j \le t,\ j = 1,2,\ldots,n
\end{aligned} \tag{7.55}$$

Since objective function $\sqrt{\sum_{j=1}^{n}\sigma_j^2 y_j}$ is a monotonous increasing function, this problem is equivalently transformed into the following problem:

$$\begin{aligned}
&\text{Minimum} \quad \sum_{j=1}^{n}\sigma_j^2 y_j \\
&\text{subject to} \quad \sum_{j=1}^{n}\left(m_j + R^*(q)\alpha_j\right)y_j - ft = 1, \\
&\qquad\qquad\quad \mathbf{A}\boldsymbol{y} \le \boldsymbol{b}t,\, 0 \le y_j \le t,\ j = 1,2,\ldots,n
\end{aligned} \tag{7.56}$$

Problem (7.56) is a linear programming problem in the case that $q$ is fixed, and so we efficiently obtain the optimal solution using linear programming approaches and bisection algorithm for parameter $q$.

Furthermore, let the optimal value of parameter $h$ in problem (7.50) be $h^*$. Then, the following lemmas hold:

**Lemma 7.11.1**

With respect to problem (7.50), there exists ranges $[h_k, h_{k+1}]$, $(k = 1, 2, ...,)$ that the optimal solution of problem (7.50) is unique for any *h* including in $[h_k, h_{k+1}]$.

**Proof**

From the continuity of parameter *h* and discreteness of decision variable $\boldsymbol{x}^*$, this lemma clearly holds.

**Lemma 7.11.2**

Let the optimal value of problem (7.52) be $\bar{h}$, the optimal solution of problem (4) be ***x**** and the optimal value be $h^*$. Then in the case that we set a range $[h_L, h_U]$ satisfying $\bar{h} \in [h_L, h_U]$, $h^* \in [h_L, h_U]$ holds.

**Proof**

We consider the case that $h^* \notin [h_L, h_U]$ and $h^* \in [h'_L, h'_U]$. If $h'_U < h_L$, there exists the optimal solution $\boldsymbol{x}'$ and the optimal solution $h'$ satisfying $h' \in [h_L, h_U]$. This contradicts the optimality of parameter *h**. In a way similar to $h'_U < h_L$, if $h'_L > h_U$, we obviously find that $h'_L > h^*$. This means that the optimal value of discrete problem is larger than that of continuous problem, and contradicts the optimality of parameter $\bar{h} > h^*$. Consequently, this lemma holds.

From these lemmas, the following theorem to the relation between problems (7.50) and (7.52) holds:

**Theorem 7.11**

Let the optimal solution of problem (7.52) be $\boldsymbol{x}^*(\bar{h})$ and the optimal value of parameter *h* be $\bar{h}$. Then, the optimal solution of the following problem;

$$\begin{aligned} &\text{Maximum} \quad \frac{\sum_{j=1}^{n}\left(m_j + R^*\left(\bar{h}\right)\alpha_j\right)x_j - f}{\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j}} \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.57}$$

is equivalent to that of problem (7.50).

Consequently, in the case that we solve 0-1 relaxation problem (7.52) and obtain its optimal solution $h^*$, we obtain an optimal solution more efficiently than previous parametric approaches due to not using branch-bound method every value of parameter *h* repeatedly. However, since the objective function of problem (7.57) is a nonlinear function, it is not easy to deal with several efficient solution methods for integer programming approaches. Therefore, in order to have the more general versatility for our proposed model, we consider the other deterministic equivalent transformations for main problem.

First, we equivalently transform main problem (7.50) into the following problem;

$$\begin{aligned} &\text{Maximum} \quad h \\ &\text{subject to} \quad \sum_{j=1}^{n}\left(m_j + R^*\left(h\right)\alpha_j\right)x_j - K_{g^{-1}(h)}\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j} \ge f, \\ &\qquad \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.58}$$

and introduce this 0-1 relaxation problem as follows:

$$\begin{aligned} &\text{Maximum} \quad h \\ &\text{subject to} \quad \sum_{j=1}^{n}\left(m_j + R^*\left(h\right)\alpha_j\right)x_j - K_{g^{-1}(h)}\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j} \ge f, \\ &\qquad \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}, 0 \le x_j \le 1,\ j = 1,2,...,n \end{aligned} \tag{7.59}$$

This problem is a nonlinear programming problem. However, this problem is much similar to problem (7.52). Therefore, in order to solve problem (7.59) analytically, we introduce the following subproblem in a way similar to the transformation from problem (7.52) into problem (7.53):

$$\begin{aligned} &\text{Maximum} \quad \sum_{j=1}^{n}\left(m_j + R^*\left(h\right)\alpha_j\right)x_j - K_{g^{-1}(h)}\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j} \\ &\text{subject to} \quad \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}, 0 \le x_j \le 1,\ j = 1,2,...,n \end{aligned} \tag{7.60}$$

Then, we consider the following auxiliary problem:

$$\begin{array}{ll} \text{Maximum} & \gamma\sum_{j=1}^{n}\left(m_j+R^*\left(h\right)\alpha_j\right)x_j-K_{g^{-1}(h)}\left(\sum_{j=1}^{n}\sigma_j^2 x_j\right) \\ \text{subject to} & \mathbf{A}\boldsymbol{x}\le\boldsymbol{b},\ 0\le x_j\le 1,\ j=1,2,\ldots,n \end{array} \tag{7.61}$$

In the case that we fix parameter *h*, with respect to the relation between problems (7.60) and (7.61), the following theorem holds based on the previous research of Ishii [55].

**Theorem 7.12**

Let the optimal solution of problem (7.60) be $\boldsymbol{x}\left(h\right)$. Then, in the case $\gamma=2\sqrt{\sum_{j=1}^{n}\sigma_j^2 x_j^*}$, the optimal solution of problem (7.61) is equal to $\boldsymbol{x}\left(h\right)$.

From Theorem 4, in the case that parameter *h* is fixed, we obtain the optimal solution $\boldsymbol{x}\left(h\right)$. Furthermore, we consider the following problem to deal with optimal value $\bar{h}$ of problem (7.52):

$$\begin{array}{ll} \text{Maximum} & \gamma\sum_{j=1}^{n}\left(m_j+R^*\left(\bar{h}\right)\alpha_j\right)x_j-K_{g^{-1}(\bar{h})}\left(\sum_{j=1}^{n}\sigma_j^2 x_j\right) \\ \text{subject to} & \mathbf{A}\boldsymbol{x}\le\boldsymbol{b},\ 0\le x_j\le 1,\ j=1,2,\ldots,n \end{array} \tag{7.62}$$

Let this optimal solution be $\boldsymbol{x}\left(h^*\right)$. Subsequently, the following lemma with respect to each optimal solution for problems (7.50) and (7.57) holds.

**Lemma 7.13.1**
The optimal solution of problem (7.57) is equal to that of problem (7.50).

**Proof**
Since each problem is the deterministic equivalent problem for main problem (7.49), this lemma obviously holds.

Therefore, we obtain $\boldsymbol{x}\left(\bar{h}\right)=\boldsymbol{x}^*$. Then, the following theorem holds extending previous research of Hasuike.

**Theorem 7.13**

With respect to $\gamma^* = 2\sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^*}$ , the optimal solution of the following problem;

$$\begin{aligned} &\text{Maximum } \gamma^* \sum_{j=1}^{n} \left( m_j + R^* \left( \bar{h} \right) \alpha_j \right) x_j - K_{g^{-1}\left(\bar{h}\right)} \left( \sum_{j=1}^{n} \sigma_j^2 x_j \right) \\ &\text{subject to } \mathbf{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \in \{0,1\}^n \end{aligned} \tag{7.63}$$

is equal to that of problem (7.49).

From Theorem 7.13, we finally solve this linear 0-1 programming problem. It is more efficient to obtain its optimal solution of problem (7.57) using some efficient solution methods for integer programming approaches than that of problem (7.50). Furthermore, in the case using branch-bound method, we find that upper limited value for main problem becomes

$$\sum_{j=1}^{n} \left( m_j + R^* \left( \bar{h} \right) \alpha_j \right) x_j \left( \bar{h} \right) - K_{g^{-1}\left(\bar{h}\right)} \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j \left( \bar{h} \right)}$$

substituting optimal solution $\boldsymbol{x}\left(\bar{h}\right)$ and optimal value $\bar{h}$ of problem (7.52) and lower limited value becomes *f*. Therefore, by using these values in branch-bound method efficiently, we obtain the optimal solution of main problem more easily and rapidly. Consequently, we construct the following solution method.

**Solution method 7.4**

STEP 1: Elicit the membership function of a fuzzy goal for with respect to the probability and set each parameter.

STEP 2: Solve 0-1 relaxation problem (7.52), and find the optimal solution $\boldsymbol{x}^*$ and optimal value $\bar{h}$ .

STEP 3: Solve 0-1 programming problem (7.63) by using integer programming approaches such as branch-bound method.

## 7.4 Conclusion

In this Chapter, we have proposed new models of general 0-1 programming problems with fuzzy random and random fuzzy variables. First, we have considered the 0-1 knapsack problem including randomness of future returns and flexible goals for available budget and total return. Since our proposed model has been a nonlinear 0-1 knapsack problem by introducing the chance constraint and doing the transformation into the deterministic equivalent problems, we have constructed the efficient solution method. We have dealt with the 0-1 relaxation problem and its optimal value and found that the number of using dynamic programming in our proposed method is much less than that of previous parametric dynamic programming. This solution method is applicable to the general integer programming problems, particularly portfolio selection problem. Next, we have proposed general fuzzy random and random fuzzy 0-1 programming problems considering both random and fuzzy conditions. Since our proposed model has been a nonlinear 0-1 programming problem by introducing the chance constraint and doing the transformation into the deterministic equivalent problems, we have constructed the efficient strict solution method by dealing with some 0-1 relaxation problems. Consequently, we have found that the number of using branch-bound method in our proposed method is much less than that in previous parametric solution methods.

This solution method may be applicable to the general integer programming problems because our proposal model includes some previous models not considering randomness or fuzziness. However, in the case there are many decision variables and parameters, it takes much computational time to solve this 0-1 random fuzzy programming problem even if we use this solution method due to the nonpolynomial time algorithm to branch-bound method. Therefore, as future studies, we need to construct its efficient solution method using not only strict solution method such as branch-bound method but also approximation methods such as genetic algorithm and heuristic approaches. Furthermore, we will consider the multidimensional 0-1 and integer random fuzzy programming problems.

## Chapter 8

# Conclusion

In this thesis, we have proposed various types of asset allocation problems under randomness and fuzziness, particularly based on portfolio selection problems and product mix problems. Furthermore, we have developed the efficient analytical solution method to obtain the global optimal solution under uncertainty.

In Chapter 3, we have proposed portfolio selection problems involving ambiguous expected returns distributed according to normal distribution, and proposed several models of random fuzzy portfolio selection problems; (a) Single criteria optimization model, (b) Bi-criteria optimization model introducing a fuzzy goal to the probability and the target future returns. Since each problem is equivalent to a parametric nonlinear programming problem, we have constructed each analytical solution method involving the procedure of solving a parametric quadratic programming problem to find a global optimal solution. Then, by comparing the proposed model with other standard fuzzy portfolio models using two numerical examples, we have found that the proposed model has been applied to more flexible and changeable cases than two previous models. Furthermore, we have proposed the random fuzzy CAPM model which is one of the standard approaches in the investment fields. Then, the random fuzzy CAPM model has been equivalently transformed into the deterministic linear programming problem, and we have constructed the efficient solution method using the standard linear programming approach and bi-section algorithm. These random fuzzy portfolio models include various situations in the investment fields such as not only statistical approaches based on historical data but also the investor's subjectivity. Therefore, these models may be versatile investment models in the future.

In Chapter 4, we have considered the large-scale portfolio selection problems using the compact factorization approach and extended these previous models to fuzzy models. Furthermore, all proposed fuzzy extension models are transformed into the deterministic equivalent problems which are equivalent to the original problems in the sense of the mathematical programming. We may be able to apply the proposed solution method to the cases including not only fuzziness but also both randomness and fuzziness which are called to fuzzy random variable or random fuzzy variable. Then, we will also apply our proposed models to the other portfolio selection problems and be able to extend all portfolio selection problems to the models considering various uncertainty situations.

In Chapter 5, we have discussed some probability maximization models of portfolio selection problems considering possibility scenarios with respect to multivariate random future returns. First, we have aggregated multi-objective functions into one weighted function, i.e. the single-objective problem, and proposed its efficient solution method using the deterministic equivalent transformations of the main problem. Furthermore, we have proposed a robust portfolio model that considers the interval of each weight and obtains the appropriate portfolio that can apply to all weights including the interval. Since this model considers various conditions in the practical investment by using weights including flexibility, we may apply this model to several types of portfolio selection problems under uncertainty. Second, we have proposed the model maximizing the minimum aspiration level of probability among all the scenarios and constructed its solution method. Since these models consider the robust case that each investor can set the weight to each possible scenario of future returns and aspiration levels according to the decision maker's subjectivity, we apply this model to several problems including multi-probabilistic and ambiguous conditions. These proposed models include more wide ranging conditions of portfolio selection problems.

In Chapter 6, we have proposed several types of product-mix decision models, many of which include several randomness and fuzziness. First, we have considered a probability fractile optimization model as the main problem and a probability maximization model to total future profits, and developed an analytical and efficient solution method to find a global optimal solution. Second, we have considered several cases of product-mix problems that occur in practical market of products and production processes scenarios, particularly the models including changes of constraints. All these proposed models under randomness, fuzziness, and flexibility are equivalent to linear programming problems or quadratic convex programming problems. Therefore, we can efficiently solve these problems. Thus, since a decision maker decides the parameters arbitrarily, these models are capable of applying to models under many situations of uncertainty and flexibility, such as the cases considering the intensity of price and demand ranges, surplus funds to bottleneck constraints in the original flexible plan and the prediction of future dangers such as machine breakdowns. Therefore, we may be able to apply these proposed models and the efficient solution methods not only to maximize the level of satisfaction of the total future profit, but also to minimize total costs and optimization under more complicated situations in the real world. Thus, our proposal models can be applied to various managerial situations by including flexibility.

Finally, In Chapter 7, we have considered the general 0-1 programming problem, and proposed fuzzy random and random fuzzy 0-1 programming problems under uncertainty. Since each proposed model has been a nonlinear 0-1 programming problem by introducing the chance constraint and doing the transformation into the deterministic equivalent problems, we have developed the efficient strict solution method by dealing with some 0-1 relaxation problems. Consequently, we have found that the number of using branch-bound method in our proposed method is much less than that in previous parametric solution methods. This solution method may be applicable to the general integer programming problems because our proposal model includes some previous models not considering randomness or fuzziness.

In the real world, many real life problems are faced with randomness and fuzziness, and such randomness and fuzziness exists not separately, but simultaneously. Therefore, it is important to investigate stochastic and fuzzy programming approaches for the several asset allocation problems. For example, with respect to random fuzzy portfolio selection problems in Chapter 3, our proposed model is represented various subjectivities of investors by setting and changing the membership functions and fuzzy goals to each investor. Then, comparing optimal portfolios and received total profit under these subjectivities of investors with present market trends and factors to represent favorable or poor market conditions, it may be possible that investors find the more appropriate market trend and obtain the knowledge of more suitable investment stance under the present market. However, it is difficult to obtain an analytical knowledge which stance investors should keep in order to not only suit their investment style but also receive the maximum future profit. Therefore, we need to analysis the theoretical relation between our proposed model and these subjectivities of investors under several market trends as well as the empirical and numerical analyses in the future works.

Furthermore, considering application of these problems with uncertainty to the practical social decision problems, it is important to have the versatility as well as consider uncertainty. For example, in Chapter 6, we have represented that our proposed models are applicable to models under many situations of uncertainty and flexibility since a decision maker decides the parameters arbitrarily. Then, we have considered the possibility to apply our proposed models and solution methods to various social problems under more complicated situations in the real world. However, in production processes, there are various problems such as not only the product mix problem but also scheduling, logistics and inventory problems, and these problems are not separated, but closely connected each other. Therefore, in the future works, we need to deal with these problems as the integrated important problem, i.e. supply chain problems. Furthermore, from the general versatility of our proposed models, they will be applied to more global and world-wide practical social problems.

Thus, these proposed models and solution method includes versatility and possibility to their further extensions, and so the author hopes that the works contained in this thesis will contribute to the further development of general asset allocation problems dealing with stochastic and fuzzy programming.

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