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NASH G MANIFOLD STRUCTURES OF COMPACT OR COMPACTIFIABLE $C^\infty G$ MANIFOLDS

TOMOHIRO KAWAKAMI
NASH $G$ MANIFOLD STRUCTURES OF COMPACT OR COMPACTIFIABLE $C^\infty G$ MANIFOLDS

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1. Introduction.

Let $G$ be a compact affine Nash group. We say that a $C^\infty G$ manifold $X$ admits a (resp. an affine, a nonaffine) Nash $G$ manifold structure if there exists a (resp. an affine, a nonaffine) Nash $G$ manifold $Y$ such that $X$ is $C^\infty G$ diffeomorphic to $Y$. In the present paper we consider Nash $G$ manifold structures of compact or compactifiable $C^\infty G$ manifolds.

We have the following when $X$ is compact.

Theorem 1. Let $G$ be a compact affine Nash group and let $X$ be a compact $C^\infty G$ manifold with $\dim X \geq 1$.

(1) $X$ admits exactly one affine Nash $G$ manifold structure up to Nash $G$ diffeomorphism.

(2) If $G$ acts on $X$ transitively then a Nash $G$ manifold structure of $X$ is unique up to Nash $G$ diffeomorphism.

(3) If $X$ is connected and the action on $X$ is not transitive, then $X$ admits a continuum number of nonaffine Nash $G$ manifold structures.

In the non-equivariant category, M. Shiota in [4] proved that any compactifiable $C^\infty$ manifold $X$ admits a continuum number of nonaffine Nash manifold structures. When $X$ is not compact but compactifiable, an affine Nash compactification of $X$ is not unique, and the number of affine ones can be investigated by the cardinality of the Whitehead torsion of $X$ [6]. Here an affine Nash compactification of $X$ means an affine Nash manifold $Y$ with boundary so that $X$ is $C^\infty$ diffeomorphic to the interior of $Y$.

We say that a $C^\infty G$ manifold $X$ is compactifiable as a $C^\infty G$ manifold if there exists a compact $C^\infty G$ manifold $Y$ with boundary so that $X$ is $C^\infty G$ diffeomorphic to the interior of $Y$. We obtain the following.

Theorem 2. Let $G$ be a compact affine Nash group and let $X$ be a non-compact compactifiable $C^\infty G$ manifold with $\dim X \geq 1$.

(1) $X$ admits an affine Nash $G$ manifold structure.

(2) $X$ admits a continuum number of nonaffine Nash $G$ manifold structures.

This paper consists of two parts. The first half is to investigate Nash $G$ manifold structures of compact $C^\infty G$ manifolds. We consider Nash $G$ manifold structures of compactifiable (not compact) $C^\infty G$ manifolds in the latter half.
In this paper all Nash $G$ manifolds and all Nash $G$ maps are of class $C^\infty$ unless otherwise stated.

**Acknowledgement.** I would like to thank Professor M. Shiota for many useful conversations and suggestions. Theorem 1 (2) is due to the cooperation of Professor M. Shiota.


First of all we recall the definition of Nash groups.

*Definition 2.1.* A group is called a (resp. an affine) Nash group if it is a (resp. an affine) Nash manifold and that the multiplication $G \times G \to G$, the inversion $G \to G$ are Nash maps.

We remark that one-dimensional Nash groups are classified by J.J. Madden and C.M. Stanton [2]. Let $G$ be an affine Nash group. In this paper, a *representation* of $G$ means a Nash group homomorphism $G \to GL(\mathbb{R}^n)$ for some $\mathbb{R}^n$. Here a Nash group homomorphism means a group homomorphism which is a Nash map. We use a representation as a representation space.

*Definition 2.2.* Let $G$ be an affine Nash group

1. An affine Nash submanifold in some representation of $G$ is called an affine Nash $G$ submanifold if it is $G$ invariant. A Nash manifold $X$ with $G$ action is said to be a Nash $G$ manifold if the action map $G \times X \to X$ is a Nash map.

2. Let $X$ and $Y$ be Nash $G$ manifolds. A Nash map $f : X \to Y$ is called a Nash $G$ map if it is a $G$ map. We say that $X$ is Nash $G$ diffeomorphic to $Y$ if there exist Nash $G$ maps $f : X \to Y, h : Y \to X$ so that $f \circ h = id, h \circ f = id$.

3. A Nash $G$ manifold $X$ is said to be affine if there exists an affine Nash $G$ submanifold $Y$ so that $X$ is Nash $G$ diffeomorphic to $Y$.

Tubular neighborhood theorem and collaring theorem are well known in the smooth equivariant category. They are proved in the Nash category by M. Shiota (Lemma 1.3.2 [7], Lemma 6.1.6 [7]). Since M. Shiota's proofs work in the equivariant Nash category, the following two propositions are obtained.

**Proposition 2.3.** Let $G$ be a compact affine Nash group and let $X$ be an affine Nash $G$ submanifold in a representation $\Omega$ of $G$. Then there exists a Nash $G$ tubular neighborhood $(U, p)$ of $X$ in $\Omega$, namely, $U$ is an affine Nash $G$ submanifold in $\Omega$ and the orthogonal projection $p : U \to X$ is a Nash $G$ map. □

**Proposition 2.4.** Let $G$ be a compact affine Nash group. Any compact affine Nash $G$ manifold $X$ with boundary $\partial X$ admits a Nash $G$ collar, that is, there exists a Nash $G$ imbedding $\phi : \partial X \times [0, 1] \to X$ so that $\phi|_{\partial X \times 0} = t\partial X$, where the action on the closed unit interval $[0, 1]$ is trivial. □

3. Compact $C^\infty G$ manifolds.

Recall a theorem proved by K.H. Doerrmann, M. Masuda, and T. Petrie [1], which is a partial solution of the equivariant Nash conjecture.
Theorem 3.1 [1]. Let \( G \) be a compact affine Nash group and let \( X \) be a compact \( C^\infty \) manifold so that \( X \) is \( G \) cobordant to a nonsingular algebraic \( G \) set. Then \( X \) is \( C^\infty \) diffeomorphic to a nonsingular algebraic \( G \) set. Here an algebraic \( G \) set means a \( G \) invariant algebraic subset of some representation of \( G \).

Proof of Theorem 1. The disjoint union \( X \cdot X \) is null cobordant. By Theorem 3.1, \( X \cdot X \) is \( C^\infty \) diffeomorphic to a nonsingular algebraic \( G \) set in some representation \( \Omega \) of \( G \). Since a \( G \) invariant collection of connected components of a nonsingular algebraic \( G \) set is an affine Nash \( G \) submanifold in \( \Omega \), \( X \) admits an affine Nash \( G \) manifold structure \( Y \subset \Omega \). Let \( Z \) be another affine Nash \( G \) manifold structure of \( X \) in \( \Omega' \). We have to prove \( Y \) is Nash \( G \) diffeomorphic to \( Z \). Let \( f \) be a \( C^\infty \) diffeomorphism from \( Y \) to \( Z \). Let \( F \) denote the composition of \( f \) with the inclusion \( Z \to \Omega' \). By [1] \( F \) can be approximated by a polynomial \( G \) map \( q : Y \to \Omega' \). By Proposition 2.3, we have a Nash \( G \) tubular neighborhood \( (U, \rho) \) of \( Z \) in \( \Omega' \). Since \( Y \) is compact, if the approximation is close then the image of \( q \) lies in \( U \). Thus \( k := \rho \circ q \) is an approximation of \( f \). If the approximation is close then a Nash \( G \) map \( k : Y \to Z \) is a Nash \( G \) diffeomorphism. Therefore (1) is proved.

Next we prove (2). Let \( X_1, X_2 \) be two Nash \( G \) manifold structures (may not be affine) of \( X \) and let \( k \) be a \( C^\infty \) diffeomorphism from \( X_1 \) to \( X_2 \). Fix \( x_1 \in X_1 \), and let \( x_2 = k(x_1) \). Then the map \( f_i : G \to X_i : f_i(g) = gx_i \) is a surjective Nash \( G \) map because \( G \) acts on \( X_i \) (\( i = 1, 2 \)) transitively, and \( f_2 = k \circ f_1 \).

To prove \( k \) is a Nash map, it is enough to show \( k \) is a \( C^0 \) Nash map. By [4] we can find a \( C^0 \) Nash imbedding \( I_i \) from \( X_i \) to some Euclidean space \( \mathbb{R}^n \) (\( i = 1, 2 \)). Let \( X'_i = I_i(X_i) \) (\( i = 1, 2 \)), \( f'_i = I_i \circ f_i \) (\( i = 1, 2 \)) and \( k' = I_2 \circ k \circ I_1^{-1} \). Then \( f'_i : G \to X'_i \) (\( i = 1, 2 \)) is a \( C^0 \) Nash map. Since \( G \) and \( X_i \) (\( i = 1, 2 \)) are affine, there exists a finite semialgebraic open covering \( \{O_i \}_i \) of \( G \) such that each \( f'_i \mid O_i \) is semialgebraic. Therefore \( f'_i \) (\( i = 1, 2 \)) is semialgebraic. Since \( k' \) is \( C^0 \) Nash if and only if \( k \) is \( C^0 \) Nash, we have only to show that \( k' \) is \( C^0 \) Nash.

Since \( f'_i \) (\( i = 1, 2 \)) is a \( C^0 \) Nash map, there exist finite systems of coordinate neighborhoods \( \{\phi_i : W_i \to \mathbb{R}^n \} \) of \( G \), \( \{\psi_j : U_j \to \mathbb{R}^n \} \) of \( X_1 \), and \( \{\varphi_l : V_l \to \mathbb{R}^n \} \) of \( X_2 \) such that, for any \( i, j \) and \( l \), \( \phi_i((f'_i)^{-1}(U_j) \cap W_i), \varphi_l((f'_i)^{-1}(V_l) \cap W_i) \) are semialgebraic, and that \( \psi_j \circ f'_i \circ \phi_i^{-1} \) is \( \phi_i((f'_i)^{-1}(U_j) \cap W_i) \to \mathbb{R}^n \). \( \varphi_l \circ f'_i \circ \phi_i^{-1} \) is \( \phi_i((f'_i)^{-1}(V_l) \cap W_i) \to \mathbb{R}^n \) are \( C^0 \) Nash maps, where \( m \) (resp. \( n \)) denotes the dimension of \( G \) (resp. \( X_1 \)). We have only to show that each \( \phi_i \circ k' \circ \psi_j^{-1} \) is semialgebraic. For a map \( h \), let \( graph(h) \) denote the graph of \( h \). For \( j \) and \( l \), let

\[
K = \bigcup_i graph(\psi_j \circ f'_i \circ \phi_i^{-1}) \times graph(\varphi_l \circ f'_i \circ \phi_i^{-1}).
\]

Then \( K \) is semialgebraic in \( (\mathbb{R}^m \times \mathbb{R}^n) \times (\mathbb{R}^m \times \mathbb{R}^n) \), hence the image \( K' \) of \( K \) by the projection \( (\mathbb{R}^m \times \mathbb{R}^n) \times (\mathbb{R}^m \times \mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n \) is semialgebraic in \( \mathbb{R}^n \times \mathbb{R}^n \). Since \( f'_i \) (\( i = 1, 2 \)) is surjective and \( f'_2 = k' \circ f'_1 \), \( graph(\varphi_l \circ k' \circ \psi_j^{-1}) = K' \). Thus each \( \varphi_l \circ k' \circ \psi_j^{-1} \) is semialgebraic. Hence \( k' \) is a \( C^0 \) Nash. Therefore \( k \) is a Nash \( G \) diffeomorphism.

Now we prove (3). By (1) we can assume that \( X \) is an affine Nash \( G \) submanifold of a representation \( \Omega \) of \( G \). For any \( x \in X \), the orbit \( G(x) \) of \( x \) is a \( C^\infty \) submanifold of \( \Omega \) because \( G \) is compact. Moreover \( G(x) \) is a semialgebraic set. Hence \( G(x) \)
is an affine Nash $G$ submanifold in $\Omega$. Since the action on $G$ is not transitive and by Proposition 2.3, there exists some Nash $G$ tubular neighborhood $(U', p)$ of some orbit $G(x)$ in $\Omega$ with $X \neq U := U' \cap X$.

For $0 < c < 1$, set
\[
  a = 2^{2.5}(1 + c)^2/(1 - c)^2,
\]
\[
  d = 2 + 2^{0.5}\alpha + a^2 - (a + \sqrt{2})\sqrt{a^2 + 2^{2.5}a}.
\]

Then $a > 2^{2.5}, 1 < d < 2$. Suppose $k$ is a Nash function satisfying
\[
\sqrt{2}(x + k(x)) = (x - k(x))^2/a.
\]

The graph of $k$ comes to a rotation of the graph of $y = x^2/a$ with center at the origin. It follows from this and $a > 2^{2.5}$ that $k$ and its Nash extension $k'$ to
\[
[1 - 2^{-0.25}\sqrt{a}, 1 + 2^{-0.25}\sqrt{a}] \supset (-1, 3)
\]
is well-defined, and that $k'$ satisfies
\[
k'[1 - 2^{-0.25}\sqrt{a}, 1 + 2^{-0.25}\sqrt{a}] = [1 - 2^{-0.25}\sqrt{a}, 1 + 2^{-0.25}\sqrt{a}],
\]
the derivative of $k'$ is negative, $k' \circ k' = id$.

Let
\[
N_1 = (-\infty, d), N_2 = (0, \infty), N_3 = (0, 1).
\]

Define the Nash maps $h_1 : N_3 \to N_1, h_2 : N_3 \to N_2$ by
\[
h_1(t) = t^2 + k(t)^2 \text{ and } h_2(t) = 2t - t^2.
\]

Then $h_1$ and $h_2$ are Nash imbeddings so that $h_1(N_3) = (0, d), h_2(N_3) = (0, 1)$. We can extend $h_1$ to
\[
h_1' : [1 - 2^{-0.25}\sqrt{a}, 1 + 2^{-0.25}\sqrt{a}] \to \mathbb{R}
\]
as a Nash function such that the derivative vanishes at only 0 and that $h_1' = h_1' \circ k'$ because the derivative of $k'$ is negative and $k' \circ k' = id$.

Applying Proposition 2.3 to the boundary $\partial U$ of the closure $\overline{U}$ of $U$ in $X$, there exists a Nash $G$ collar $\phi : \partial U \times [0, 1] \to \overline{U}$. Let $D(\varepsilon) (0 < \varepsilon < 1)$ denote $\phi(\partial \overline{U} \times (0, \varepsilon))$. Take a Nash diffeomorphism $f : \mathbb{R} \to (0, 1)$ (e.g. the inverse map of the composition of $f : (0, 1) \to (-1, 1) : f(x) = 2x - 1$ with $h : (-1, 1) \to \mathbb{R} : h(x) = x/(1 - x^2)$). Set
\[
U_1 = D(f(d)), U_2 = X - \overline{D(f(0))}, U_3 = D(f(1)) - \overline{D(f(0))}.
\]

Then each $U_i$ is an open affine Nash $G$ submanifold of $X$. Let
\[
H_1 = \phi \circ (id \times (f \circ h_1 \circ f^{-1})) \circ \phi^{-1} : U_3 \to U_1,
\]
\[
H_2 = \phi \circ (id \times (f \circ h_2 \circ f^{-1})) \circ \phi^{-1} : U_3 \to U_2.
\]

We define $X_c$ by the quotient topological space of the disjoint union $\coprod_{i=1}^3 U_i$, and the equivalence relation $x \sim H_1(x) \sim H_2(x)$ for $x \in U_3$ on the union. Then one can check that $X_c$ is a Nash $G$ manifold which is $C^\infty G$ diffeomorphic to $X$. Next we prove $X_c$ is nonaffine. To prove this, we use the following lemma.
Lemma 3.2 (c.f. Remark 1.2.2.15 [7]). Let $f$ be a locally semialgebraic $C^\infty$ map from a Nash manifold $M$ to a Nash manifold $N$. If $N$ is affine then $f$ is a Nash map. 

Fix $0 < c < 1$ and $z \in S(f(1))$, where $S(f(1))$ denotes $\phi(\partial U \times \{f(1)\})$. Let $\psi_c : (f(0), f(1)) \to X_c$ be the composition

$$(f(0), f(1)) \to S(f(1)) \times (f(0), f(1)) \to U_3 \to X_c,$$

where the first map is $x \mapsto (z, x)$, the second is the natural Nash $G$ diffeomorphism from $S(f(1)) \times (f(0), f(1))$ to $U_3$, and the third is the natural imbedding from $U_3$ into $X_c$. Then $\psi_c$ is an imbedding. We extend $\psi_c$ as follows. Let $l_{ci} (i = 1, 2, 3)$ be the natural imbedding $U_i \to X_c$ and let $V_{ci} (i = 1, 2, 3)$ denote its image. Then

$$p \circ k_1^{-1} \circ l_1^{-1} \circ \psi_c = f \circ h_1 \circ f^{-1}, p \circ k_2^{-1} \circ l_2^{-1} \circ \psi_c = f \circ h_2 \circ f^{-1} \text{ on } (f(0), f(1)),$$

where $p$ denotes the projection $\partial U_3 \times (f(0), f(d)) \to (f(0), f(d))$ and $k_i (i = 1, 2)$ stands for the natural imbedding $\partial U_3 \times (f(0), f(d)) \to U_i$. We extend $\psi_c$ to $(f(0), f(1 + \epsilon))$ for small positive $\epsilon$. It suffices to consider $p \circ k_2^{-1} \circ l_2^{-1} \circ \psi_c$ because the image of $\psi_c$ lies in $V_{c2}$ and $\lim_{t \to f(1)} \psi_c(t) \in V_{c2}$. Now $p \circ k_2^{-1} \circ l_2^{-1} \circ \psi_c = f(2f^{-1}(t) - (f^{-1}(t))^2)$ on $(f(0), f(1))$. Thus $p \circ k_2^{-1} \circ l_2^{-1} \circ \psi_c$ and $\psi_c$ are extendible to $(f(0), f(2))$ and

$$p \circ k_2^{-1} \circ l_2^{-1} \circ \psi_c(t) = f(2f^{-1}(t) - (f^{-1}(t))^2) \text{ on } [f(1), f(2)).$$

Clearly we can extend $\psi_c$ to $[f(0), f(1)]$, and $\psi_c([f(0), f(2)] \subset \psi_c([f(0), f(1)]).$ Hence

$$\psi_c^{-1} \circ \psi_c(t) = f(2 - f^{-1}(t)) \text{ on } [f(1), f(2)),$$

$f(1)$ is the only and nondegenerate critical point, where $\psi_{c0}$ denotes the homeomorphism $\psi_c : [f(0), f(1)] \to [f(0), f(1)]$. In the same way, $\psi_c$ can be defined on $(f(k'(1)), f(0))$ satisfying

$$\psi_{c0}^{-1} \circ \psi_c(t) = f(k'(f^{-1}(t))) \text{ for } t \in (f(k'(1)), f(0)),$$

and the critical point is only $f(0)$ and nondegenerated. Repeating this argument, $\psi_c$ is extendible on

$$(f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a})),$$

and $\psi_c$ is locally semialgebraic, the image of $\psi_c$ is $\psi_c([f(0), f(1)])$, and that for any $\epsilon \in (f(0), f(1))$, $(\psi_{c0}^{-1} \circ \psi_c)^{-1}(\epsilon)$ is discrete and consists of infinitely many elements. The set of critical points of $\psi_c$ is $(\psi_{c0}^{-1} \circ \psi_c)^{-1}(f(0)) \cup (\psi_{c0}^{-1} \circ \psi_c)^{-1}(f(1))$, and they are nondegenerate ones. Since $\psi_c$ is locally semialgebraic and not semialgebraic and by Lemma 3.2, $X_c$ is not affine.

Finally we prove that $X_c$ is not Nash $G$ diffeomorphic to $X_{c'}$ if $0 < c, c' < 1, \alpha = \log f(c')/\log f(c)$ is irrational. Assume that there exists a Nash $G$ diffeomorphism $u : X_c \to X_{c'}$. Then we have to prove $\log f(c')/\log f(c)$ is rational. Set

$$a = 2^{2.5}(1 + c)^2/(1 - c)^2, a' = 2^{2.5}(1 + c')^2/(1 - c')^2,$$

$$\psi_c : (f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a})) \to X_c,$$

$$\psi_{c'} : (f(1 - 2^{-0.25} \sqrt{a'}), f(1 + 2^{-0.25} \sqrt{a'})) \to X_{c'}.$$
We also write

\[ \psi_{c_0} = \psi_c([f(0), f(1)] : [f(0), f(1)] \rightarrow \psi_c([f(0), f(1)]), \]

\[ \psi_{c'} = \psi_{c'}([f(0), f(1)] : [f(0), f(1)] \rightarrow \psi_{c'}([f(0), f(1)]), \]

Let \( L_1 \) be the composition of the diffeomorphism \( S(f(1)) \times (f(-10d), f(10d)) \rightarrow D(f(10d)) - D(f(-10d)) \) with the projection \( D(f(10d)) \rightarrow X_{c'}, \) and let \( L_2 \) be the projection \( S(f(1)) \times (f(-10d), f(-10d)) \rightarrow S(f(1)). \) By Lemma 3.2 and the infinite vibration of \( \psi_c, L_2 \circ L_1^{-1} \circ u \circ \psi_c \) is constant. Let \( z' \) denote this constant. Clearly the images of \( \phi(z' \times f(N_2)) \) and \( \phi(z' \times (f(1), f(d))) \) via \( \pi_{c'} \) in \( X_{c'} \) are affine Nash \( G \) submanifolds. Let \( k_c \) be the natural homeomorphism from \( U_3 \) into \( X_c. \)

Thus \( u \circ k_c \circ \phi(z \times [f(0), f(1)]) \) is not contained in these affine Nash \( G \) submanifolds because the image of \( \psi_c \) is not affine. This implies that

\[ u \circ k_c \circ \phi(z \times [f(0), f(1)]) \subset k_{c'} \circ \phi(z' \times [f(0), f(1)]). \]

Applying the same argument to \( u^{-1} \), we have

\[ u^{-1} \circ k_{c'} \circ \phi(z' \times [f(0), f(1)]) \subset k_c \circ \phi(z \times [f(0), f(1)]). \]

Therefore

\[ u \circ k_c \circ \phi(z \times [f(0), f(1)]) = k_{c'} \circ \phi(z' \times [f(0), f(1)]). \]

For any \( e \in (f(0), f(1)), \) let \( (\psi_{c_0}^{-1} \circ \psi_c)^{-1}(e) = \{e_i\}_{i \in \mathbb{Z}}, \) \( (\psi_{c_0}^{-1} \circ \psi_{c'})^{-1}(e) = \{e_i'\}_{i \in \mathbb{Z}}. \) Then

\[ \lim_{i \to \infty} (f(1 + 2^{-0.25} \sqrt{a}) - e_{-i-2})/(f(1 + 2^{-0.25} \sqrt{a}) - e_{-i}) = f(c), \]

\[ \lim_{i \to \infty} (f(1 + 2^{-0.25} \sqrt{a'}) - e'_{-i-2})/(f(1 + 2^{-0.25} \sqrt{a'}) - e'_{-i}) = f(c'), \]

are obtained as follows. The map \( t \rightarrow f(k'(2 - f^{-1}(t))) \) has fixed points only at the end of the interval, it repels from \( f(1 + 2^{-0.25} \sqrt{a}), \) attracts to \( f(1 - 2^{-0.25} \sqrt{a}) \) and its derivatives at the latter point is \( f(c). \) Thus \( (f(1 + 2^{-0.25} \sqrt{a}) - e_{-i-2})/(f(1 + 2^{-0.25} \sqrt{a}) - e_{-i}) \) converges \( f(c) \) because \( e_{-i-2} = f(k'(2 - f^{-1}(e_{-i}))). \) Hence we have [3.1]. A similar argument shows [3.2]. Since

\[ u \circ k_c \circ \phi(z \times [f(0), f(1)]) = k_{c'} \circ \phi(z' \times [f(0), f(1)]), \]

for a pair

\[ e_0 \in (f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a})) \] and

\[ e_0' \in (f(1 - 2^{-0.25} \sqrt{a'}), f(1 + 2^{-0.25} \sqrt{a'})) \]

with

\[ \psi_{c'}(e_0') = u \circ \psi_c(e_0) \]

there exits a homeomorphism

\[ \tau : (f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a})) \rightarrow (f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a})) \]
so that $\tau(e_0) = e'_0$ and $\psi_c' \circ \tau = u \circ \psi_c$ on $(f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a}))$. Remember that all critical points of $\psi_c, \psi_c'$ are nondegenerate. This shows that $\tau$ is of class $C^\omega$. Therefore, by Lemma 3.2, $\tau$ is a Nash diffeomorphism. Set

$$\psi_{c_0}^{-1} \circ \psi_c(e_0) = e, \psi_{c_0'}^{-1} \circ \psi_{c'}(e_0) = e',$$

$$(\psi_{c_0}^{-1} \circ \psi_c)^{-1}(e) = \{e_i\}_{i \in Z},$$

$$(\psi_{c_0'}^{-1} \circ \psi_{c'})^{-1}(e') = \{e'_i\}_{i \in Z}.$$

Then $\tau$ satisfies

$$\tau(e_i) = e'_i \text{ for any } i \in Z \text{ or },$$

$$\tau(e_i) = e'_{-i} \text{ for any } i \in Z.$$

A map $f \circ (\text{translation}) \circ f^{-1}$ takes $(f(1 - 2^{-0.25} \sqrt{a}), f(1 + 2^{-0.25} \sqrt{a}))$ to $(f(0), f(2^{0.75} \sqrt{a}))$, and a similar map $f \circ (\text{translation}) \circ f^{-1}$ takes $(f(1 - 2^{-0.25} \sqrt{a'}), f(1 + 2^{-0.25} \sqrt{a'}))$ to $(f(0), f(2^{0.75} \sqrt{a'}))$, we may suppose that $e_i$ and $e'_i$ converge to 0 as $i \to \infty$. Assume that $e$ and $e'$ lie in $(f(0), f(1))$. Then it follows from [3.1] and [3.2] that

$$\lim_{i \to \infty} e_{-i-2}/e_{-i} = f(c),$$

$$\lim_{i \to \infty} e'_{-i-2}/e'_{-i} = f(c').$$

Let $Z$ denote the Zariski closure of $\text{graph}(\tau)$. This is of dimension 1 because $\tau$ is semialgebraic. It is clear that $Z$ contains all $(e_i, e'_i)$. Let $P(x, y) = \sum S_{j=1}^s \delta_j x^{\beta_j} y^{\gamma_j}$ ($\delta_j \in \mathbb{R}, \beta_j, \gamma_j \in \mathbb{N}$) be a defining polynomial of $Z$. Then

$$P(e_i, e'_i) = 0 \text{ for any } i \in Z.$$

Since $\alpha$ is irrational,

$$\beta_i + \alpha \gamma_i \neq \beta_j + \alpha \gamma_j \text{ for } i \neq j.$$  

Set

$$P_i(x, y) = y^i.$$  

For each $n \in \mathbb{Z}$, let $E(n)$ denote the $s \times s$-matrix whose $(i, j)$ entry is

$$P_i(e_{-n-2+i} + e'_{-n-2+j+1}).$$

Then

$$(\delta_1, \ldots, \delta_l)E(n) = (P(e_{-n-1} + e'_{-n-1}), \ldots, P(e_{-n-2s+1} + e'_{-n-2s+1})) = 0.$$

In particular $\det E(n) = 0$. On the other hand, we have

$$\det E(n) = \prod_{i=1}^S P_i(e_{-n-1} + e'_{-n-1}) \det F(n),$$
where $F(n)$ is the $s \times s$-matrix whose $(i,j)$ entry is

$$P_i(e_{n-2j+1}, e'_{n-2j+1})/P_i(e_{n-1}, e'_{n-1}).$$

Now [3.3] and [3.4] mean that each entry of $F(n)$ converges to

$$(e^\beta_i (e')^\gamma_i)^{j-1} = e^{\beta_i + \alpha \gamma_i (j-1)}$$

as $n \to \infty$. Thus det $F(n)$ converges to a Vandermonde's determinant equals

$$\prod_{i<j} (e^{\beta_i + \alpha \gamma_i} - e^{\beta_j + \alpha \gamma_j}) \neq 0,$$

by [3.5]. Therefore det $E(n) \neq 0$ for large $n$. This proves the result. \qed

4. **Compactifiable $C^\infty G$ manifolds.**

The same argument of the proof of Theorem 1 (3) proves Theorem 2 (2). To prove Theorem 2 (1), we show a relative version of Theorem 3.1. After proving Theorem 4.2, we give a proof of Theorem 2 (1).

**Definition 4.1.** (1) An algebraic subset of a representation of $G$ is said to be an algebraic $G$ set if it is $G$ invariant. Moreover we call it a nonsingular algebraic $G$ set if it is nonsingular.

(2) Let $X$ be a $C^\infty G$ manifold and let $X'$ be a $C^\infty G$ submanifold of $X$. A pair $(X, X')$ is called algebraically $G$ cobordant if there exist a nonsingular algebraic $G$ set $Y$, a nonsingular algebraic $G$ subset $Y'$ of $Y$, a $G$ cobordism $N$ between $X$ and $Y$, and a $G$ cobordism $N'$ between $X'$ and $Y'$ such that $N'$ is a $C^\infty G$ submanifold of $N$.

**Theorem 4.2.** Let $G$ be a compact affine Nash group, $X$ a compact $C^\infty G$ manifold, and $X'$ a compact $C^\infty G$ submanifold of $X$. If the pair $(X, X')$ is algebraically $G$ cobordant then there exist a nonsingular algebraic $G$ set $Z$ in $X \times \Omega$ for some representation $\Omega$ of $G$, a nonsingular algebraic subset $Z'$ of $Z$, and a $C^\infty G$ diffeomorphism $\phi : X \to Z$ with $\phi(X') = Z'$.

For any $C^\infty G$ manifold $X$ and $C^\infty G$ submanifold $X'$ of $X$, the pair $(X \coprod X', X' \coprod X')$ is algebraically $G$ cobordant. Therefore we have the next corollary because a $G$ invariant collection of connected components of a nonsingular algebraic $G$ set is an affine Nash $G$ submanifold in some representation of $G$.

**Corollary 4.3.** Let $G$ be a compact affine Nash group, $X$ a compact $C^\infty G$ manifold, and $X'$ a compact $C^\infty G$ submanifold of $X$. Then there exist an affine Nash $G$ manifold $Y$, an affine Nash $G$ submanifold $Y'$ of $Y$, and a $C^\infty G$ diffeomorphism $\phi : X \to Y$ so that $\phi(X') = Y'$. \qed

**Proof of Theorem 4.2.** By the proof of Theorem 1.3 [1], $X'$ is $G$ isotopic to a nonsingular algebraic $G$ subset $Z'$ of $X \times \Omega$ by an arbitrarily small isotopy, for some representation of $G$. Extending this isotopy, we may assume that it maps $X \times 0$ to some $C^\infty G$ manifold $M$ in $X \times \Omega$ so that $M - X \times 0$ has compact closure and that the composition of the inclusion $M \to X \times \Omega$ with the projection
$X \times \Omega \rightarrow X$ is a $C^\infty G$ diffeomorphism. In particular $Z' \subset M$. Since $Z'$ is compact and by Lemma 4.7 [1], one can find a proper $G$ invariant polynomial $\rho$ such that $\rho^{-1}(0) = Z'$. Let $\alpha : X \rightarrow \Omega$ be a $C^\infty G$ map with compact support so that

$$M = \{(x, y) \in X \times \Omega | y = \alpha(x)\}.$$ 

Take a $G$ invariant $C^\infty$ function $\beta : X \times \Omega \rightarrow [0, 1]$ with compact support with $\beta(x, y) = 1$ when $|y| < 2|\alpha(x)|$. Let $\gamma : X \times \Omega \rightarrow \Omega$ be

$$\gamma(x, y) = \beta(x, y)(y - \alpha(x)) + (1 - \beta(x, y))\rho^2(x, y)y.$$ 

Then 0 is a regular value of $\gamma$, $\gamma^{-1}(0) = M$, and $\gamma$ is equal to the polynomial $\rho^2(x, y)y$ outside of a $G$ invariant compact set. By Lemma 5.1 [1], one can $C^1$ approximates $\gamma(x, y) - \rho^2(x, y)y$ by an equivariant entire rational map $u : (X \times \Omega, Z') \rightarrow (\Omega, 0)$. Here an entire rational map means a fraction of polynomial maps with nowhere vanishing denominator. This approximation is close on all $X \times \Omega$. Thus

$$w(x, y) = u(x, y) + \rho^2(x, y)y$$

is $C^1$ approximation of $\gamma$ on $X \times \Omega$. Since $\rho$ is proper and by equivariant Morse theory, there exists a $C^\infty G$ diffeomorphism from $Z := w^{-1}(0)$ to $M = \gamma^{-1}(0)$ fixing $Z'$. \Box

Proof of Theorem 2 (1). Since $X$ is compactifiable, there exists a $C^\infty G$ manifold $X'$ with boundary $\partial X$ so that $X$ is $C^\infty G$ diffeomorphic to the interior of $X'$. Let $Y$ be the double of $X'$. Applying Corollary 4.3 to the pair $(Y, \partial X')$, one can find a representation $\Omega$ of $G$ and a $C^\infty G$ imbedding $F : Y \rightarrow \Omega$ such that $F(Y)$ and $F(\partial X')$ are affine Nash $G$ manifolds. Hence $F(X)$ is an affine Nash $G$ manifold. Therefore $X$ admits an affine Nash $G$ manifold structure. \Box

On the other hand, T. Petrie [3] proved that any nonsingular algebraic $G$ set is compactifiable as a $C^\infty G$ manifold when $G$ is an algebraic group. A similar proof shows the next theorem, because the number of connected components of the zeros of a Nash map is finite.

Theorem 4.4. Let $G$ be a compact affine Nash group. Then every affine Nash $G$ manifold is compactifiable as a $C^\infty G$ manifold. \Box

M. Shiota studied compactifications of Nash manifolds as either $C^\infty$ manifolds [4] or Nash manifolds [5].

By Theorem 2 (1) and Theorem 4.4, we have the following.

Theorem 4.5. Let $G$ be a compact affine Nash group. Then a $C^\infty G$ manifold is compactifiable if and only if it admits an affine Nash $G$ manifold structure. \Box

REFERENCES

1. K.H. Dovermann, M. Masuda and T. Petrie, Fixed point free algebraic actions on varieties diffeomorphic to $\mathbb{R}^n$, Progress in Math. 80, Birkhäuser (1990), 49-80.