Strong unique continuation property for some second order elliptic systems

（ある２階楕円型偏微分方程式系に対する
強一意接続性定理について）

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Contents

Chapter 1

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1.1 Introduction

In this chapter we prove the strong unique continuation property for some second order systems. There are many results for second order single equation (for example [4], [5] and [10]). Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ containing the origin, and let

$$p(x,\partial) = \sum_{1\le j,k\le n} a_{j,k}(x)\partial_j\partial_k$$

be an elliptic differential operator in Ω such that $a_{j,k}(0)$ is real and $a_{j,k}(x)$ is Lipschitz continuous in Ω. In [5], he proved that if $u \in H^1_{loc}(\Omega)$ satisfies

$$|p(x,\partial)u| \le C_0|x|^{-2+\epsilon}|u| + C_1|x|^{-1+\epsilon}|\nabla u|, \quad \epsilon > 0$$

and

$$\lim_{\rho\to 0}\rho^{-\beta}\int_{|x|\le\rho}|u|^2dx = 0$$

for any positive β, then u is identically zero in Ω.

In [10], he improved Hörmander's result. He proved that the same result holds for inequality of the form

$$|p(x,\partial)u| \le C_0|x|^{-2}|u| + C_1|x|^{-1}|\nabla u|$$

if C_1 is sufficiently small.

We are interested in second order systems, that is, $a_{j,k}(x)$ is of matrix valued.

1.2 Main Results 1

Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ containing the origin, and let

$$P(x,\partial) = \sum_{1\le j,k\le n} A_{j,k}(x)\partial_j\partial_k \tag{1.2.1}$$

be an elliptic differential operator in Ω where $A_{j,k}$ is an $N\times N$ matrix valued function with the entries which are Lipschitz continuous in Ω for any $1\le j,k\le n$. We assume that $P(x,\partial)$ satisfies the following properties;

$$A^*_{j,k}A_{l,m} = A_{l,m}A^*_{j,k} \tag{1.2.2}$$

for any $1\le j,k,l,m\le n$, and there exist an elliptic differential operator $p(\partial) = \sum_{1\le j,k\le n} a_{j,k}\partial_j\partial_k$ with real coefficients and complex numbers λ_j $j=1,2\dots,N$ such that

$$P(0,\partial) = \mathrm{diag}\begin{pmatrix} \lambda_1 p(\partial) & & \\ & \ddots & \\ & & \lambda_N p(\partial) \end{pmatrix}. \tag{1.2.3}$$

Then it follows the following theorem.

Theorem 1.2.1. *There exists a positive constant C^* depending only on $p(\partial)$ such that if $u\in\{H^1_{loc}(\Omega)\}^N$ satisfies*

$$|P(x,\partial)u| \le C_0|u|/|x|^2 + C_1|\nabla u|/|x| \tag{1.2.4}$$

with $C_1 < C^*$ and

$$\lim_{\rho\to 0}\rho^{-\beta}\int_{|x|\le\rho}|u|^2dx = 0 \tag{1.2.5}$$

for any positive β, then u is identically zero in some neighborhood of the origin.

Remark 1.2.1. In [2], they proved S.U.C.P fails if C_1 is not small.

Remark 1.2.2. In [1], he proved the following theorem:

Let $P(x_1,x_2,\tilde{x},\partial_{x_1},\partial_{x_2},\partial_{\tilde{x}})$ be a m-th order ($m\ge 2$) elliptic differential operator defined in a neighborhood of the origin in $\mathbb{R}^n$ and let P_m be the principal symbol. If $P_m(0,0,0,1,\eta,0)$ has two simple, non real and non conjugate roots, then there exist a neighborhood V of the origin, two functions a, $u\in C^\infty(V)$, both satisfying (1.2.5), such that $Pu-au=0$ in V and $\{0\}\in \mathrm{supp}u$. Therefore the assumption (1.2.3) is essential. In fact, let $p(\partial)$ and $q(\partial)\ne\lambda p(\partial)$ be second order elliptic operators with real coefficients and let

$$P(0,\partial) = \mathrm{diag}\begin{pmatrix} \lambda_1 p(\partial) & & \\ & \lambda_2 q(\partial) & \\ & & \ddots \end{pmatrix}.$$

By Alinhac's theorem there exist a neighborhood V of the origin, two functions a, $u \in C^\infty(V)$, both satisfying (1.2.5), such that $p(\partial)q(\partial)u - au = 0$ in V and $\{0\} \in \mathrm{supp} u$. Setting $U = (q(\partial)u, u, \cdots)$ it follows that

$$\begin{aligned}|P(0,\partial)U| &= |(\lambda_1 p(\partial)q(\partial)u, \lambda_2 q(\partial)u, \cdots)| \\ &= |(\lambda_1 au, \lambda_2 q(\partial)u, \cdots)| \\ &\le C|U|,\end{aligned}$$

U satisfies (1.2.5) and $\{0\} \in \mathrm{supp} U$.

1.3 Proof of Theorem 1.2.1

In this section, we shall prove Theorem 1.2.1. The letter C stands for a generic constant whose value may vary from line to line.

After a linear transform, we may assume that $p(\partial) = \triangle$. Considering $\tilde{u} = (\lambda_1^{-1} u_1, \dots, \lambda_N^{-1} u_N)$, without loss of generality, it suffices to prove the theorem assuming $P(0, \partial) = \triangle I_N$.

In [10] Regbaoui proved the following result.

Lemma 1.3.1. *There exists a positive constant C such that*

$$\sum_{|\alpha| \le 2} \beta^{2-2|\alpha|} \int |x|^{-2\beta+2|\alpha|-3} |\partial_x^\alpha u|^2 dx \le C \int |x|^{-2\beta+1} |\triangle u|^2 dx \tag{1.3.1}$$

for any $\beta \in \{j + 1/2; j \in \mathbb{N}\}$ and any $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

Remark 1.3.1. The estimate (1.3.1) in Lemma remains valid if we assume $u \in \{H^2_{loc}(\Omega)\}^N$ with compact support satisfies (1.2.5).

Proposition 1.3.2. *Let $u \in \{H^1_{loc}(\Omega)\}^N$ satisfy* (1.2.4) *and* (1.2.5). *Then $u \in \{H^2_{loc}(\Omega)\}^N$ and there exist positive C_2 and C_3 such that*

$$\sum_{|\alpha| \le 2} \int_{B(\rho)} |\partial_x^\alpha u|^2 dx \le C_2 \exp(-C_3 \rho^{-1})$$

for any small positive ρ and for any $|\alpha| \le 2$ where $B(\rho) = \{x; |x| \le \rho\}$.

Proof. First we shall prove that $u \in H^2_{loc}(\Omega)$, and satisfies

$$\lim_{\rho \to 0} \rho^{-\beta} \int_{B(\rho)} |\partial_x^\alpha u|^2 dx = 0 \tag{1.3.2}$$

for any positive β and $|\alpha| \le 2$. By regularising and using Friedrichs's lemma and ellipticity of $P(x, \partial)$, we get without difficulties $u \in H^2_{loc}(\Omega \setminus \{0\})$. Following Hörmander [6] (Corollary 17. 1. 4.) we obtain

$$\lim_{\rho \to 0} \rho^{-\beta} \int_{B(2\rho) \setminus B(\rho)} |\partial_x^\alpha u|^2 dx = 0 \tag{1.3.3}$$

for any positive β and $|\alpha| \le 2$. Hence u is the sum of a function in $H^2_{loc}(\Omega)$ and a distribution with support at $\{0\}$. But no distribution with support at $\{0\}$ is in L^2_{loc}. It follows that $u \in H^2_{loc}(\Omega)$. Since $u \in H^2_{loc}(\Omega)$ it is clear that from (1.3.3) we have also (1.3.2).

Let $\chi(r) \in C_0^\infty([0, \infty))$ be a nonnegative function such that $\chi(r) = 1$ when $0 \le r \le 1$, $\chi(r) = 0$ when $2 \le r$ and $|\chi'| \le C$. Setting $\tilde{u}(x) = \chi(\epsilon^{-1}\beta|x|)u(x)$ where ϵ is a small positive parameter which will be determined later.

By Lemma 1.3.1, Remark 1.3.1 and (1.2.3) we have

$$\begin{aligned}\sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx &\le C\int |x|^{-2\beta+1}|\triangle \tilde{u}|^2 dx\\ &\le C\int |x|^{-2\beta+1}|P(0,\partial)\tilde{u}|^2 dx. \qquad (1.3.4)\end{aligned}$$

Since $A_{j,k}$ is Lipschitz continuous and $|x|\le 2\epsilon\beta^{-1}$, it follows that

$$\begin{aligned}\int |x|^{-2\beta+1}|P(0,\partial)\tilde{u}|^2 dx &\le \int |x|^{-2\beta+1}|(P(x,\partial)-P(0,\partial))\tilde{u}-P(x,\partial)\tilde{u}|^2 dx\\ &\le 2\int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx\\ &\quad + 2\sum_{|\alpha|\le 2}\int |x|^{-2\beta+3}|\partial_x^\alpha \tilde{u}|^2 dx \qquad (1.3.5)\end{aligned}$$

and

$$\sum_{|\alpha|\le 2}\int |x|^{-2\beta+3}|\partial_x^\alpha \tilde{u}|^2 dx \le 4\epsilon^2\beta^{-2}\sum_{|\alpha|\le 2}\int |x|^{-2\beta+1}|\partial_x^\alpha \tilde{u}|^2 dx. \qquad (1.3.6)$$

Fixing ϵ such that $1-8C\epsilon^2>0$, we obtain

$$\sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx \le C\int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx \qquad (1.3.7)$$

by (1.3.4), (1.3.5) and (1.3.6).

On the other hand, from (1.2.4) and $\epsilon\beta^{-1}\le |x|\le 2\epsilon\beta^{-1}$ if $x\in B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})$, we have

$$\begin{aligned}\int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx &\le \int_{B(\epsilon\beta^{-1})} |x|^{-2\beta+1}|P(x,\partial)u|^2 dx\\ &\quad + \int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx\\ &\le 2\int_{B(\epsilon\beta^{-1})} |x|^{-2\beta-1}(C_0^2|x|^{-2}|u|^2+C_1^2|\nabla u|^2)dx\\ &\quad + C\sum_{|\alpha|\le 2}\int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha u|^2 dx.\end{aligned} \qquad (1.3.8)$$

By (1.3.7) and (1.3.8), if C_1 is small enough, then we have

$$\begin{aligned}&\sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx\\ &\qquad \le C\sum_{|\alpha|\le 2}\int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha u|^2 dx\end{aligned}$$

for any large $\beta \in \{j+1/2; j \in \mathbb{N}\}$. It follows that

$$\begin{aligned}\beta^{-2}(\epsilon\beta^{-1}/2)^{-2\beta+1} &\sum_{|\alpha|\le 2} \int_{B(\epsilon\beta^{-1}/2)} |\partial_x^\alpha u|^2 dx \\ &\le \sum_{|\alpha|\le 2} \beta^{2-2|\alpha|} \int_{B(\epsilon\beta^{-1}/2)} |x|^{-2\beta+2|\alpha|-3} |\partial_x^\alpha u|^2 dx \\ &\le \sum_{|\alpha|\le 2} \beta^{2-2|\alpha|} \int |x|^{-2\beta+2|\alpha|-3} |\partial_x^\alpha \tilde{u}|^2 dx\end{aligned}$$

and

$$\begin{aligned}\sum_{|\alpha|\le 2} &\int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |x|^{-2\beta+2|\alpha|-3} |\partial_x^\alpha u|^2 dx \\ &\le (\epsilon\beta^{-1})^{-2\beta-3} \sum_{|\alpha|\le 2} \int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |\partial_x^\alpha u|^2 dx.\end{aligned}$$

Therefore there exist positive C_3 and C_4 such that

$$\begin{aligned}\sum_{|\alpha|\le 2} \int_{B(\epsilon\beta^{-1}/2)} |\partial_x^\alpha u|^2 dx &\le C(1/2)^{2\beta-1}\beta^6 \sum_{|\alpha|\le 2} \int_{B(2\epsilon\beta^{-1})\setminus B(\epsilon\beta^{-1})} |\partial_x^\alpha u|^2 dx \\ &\le C_3 \exp(-C_4\beta). \end{aligned} \tag{1.3.9}$$

Setting $R_j = \epsilon/(2j+1)$, from (1.3.9) we have

$$\sum_{|\alpha|\le 2} \int_{B(R_j)} |\partial_x^\alpha u|^2 dx \le C_3 \exp(-C_4\epsilon/(2R_j)), \tag{1.3.10}$$

for any large j.

On the other hand, for any small R there exists a positive j such that

$$R_{j+1} < R < R_j.$$

Since $R_j \le 2R_{j+1}$ for any $j > 1$, we have

$$R_{j+1} < R < R_j < 2R_{j+1}.$$

By (1.3.10) it follows that

$$\begin{aligned}\sum_{|\alpha|\le 2} \int_{B(R)} |\partial_x^\alpha u|^2 dx &\le \sum_{|\alpha|\le 2} \int_{B(R_j)} |\partial_x^\alpha u|^2 dx \\ &\le C_3 \exp(-C_4\epsilon/(2R_j)) \\ &\le C_3 \exp(-C_4\epsilon/(4R_{j+1})) \\ &\le C_3 \exp(-C_4\epsilon/(4R),\end{aligned}$$

which proves the desired conclusion. □

Proposition 1.3.2 allows us to use $e^{\gamma/2(\log|x|)^2}$ rather than the usual weight $|x|^{-\gamma}$. Using the similar method as [10], we can get the following Carleman estimate under the hypothesis of Theorem 1.2.1.

Proposition 1.3.3. *Under the hypothesis of Theorem 1.2.1 there exists a positive C such that*

$$\gamma\int_{\tilde{\Omega}}|x|^2\varphi_\gamma^2|\nabla u|^2|x|^{-n}dx+\gamma^3\int_{\tilde{\Omega}}\varphi_\gamma^2|u|^2|x|^{-n}dx \leq C\int_{\tilde{\Omega}}|x|^4\varphi_\gamma^2|P(x,\partial)u|^2|x|^{-n}dx \tag{1.3.11}$$

for any large γ and any $u\in C_0^\infty(\tilde{\Omega}\setminus\{0\})$ with a sufficiently small $\tilde{\Omega}$ where $\varphi_\gamma=\exp(\gamma/2(\log|x|)^2)$.

Remark 1.3.2. The estimate (1.3.11) in Proposition 1.3.3 remains valid if we assume $u\in H^2_{loc}(\tilde{\Omega})$ with compact support satisfies

$$\lim_{\rho\to 0}\exp(\beta(\log|\rho|)^2)\int_{|x|\leq\rho}|\partial_x^\alpha u|^2dx=0$$

for any positive β and any $|\alpha|\leq 2$.

Proof. Let's introduce polar coordinates in $\mathbb{R}^n\setminus\{0\}$ by setting $x=e^t\omega$, with $t\in\mathbb{R}$ and $\omega=(\omega_1,\dots,\omega_n)\in S^{n-1}$. For $k=1,\dots,n$, we set $D_k=\Omega_k$ and $D_0=\partial_t$. We have then

$$\partial_{x_j}=e^{-t}(\omega_j\partial_t+\Omega_j)$$

where Ω_j is a vector field on S^{n-1}.

The vector fields Ω_j have the properties

$$\sum_{j=1}^n\omega_j\Omega_j=0\quad\text{and}\quad\sum_{j=1}^n\Omega_j\omega_j=n-1. \tag{1.3.12}$$

The adjoint of Ω_j as an operator in $L^2(S^{n-1})$ is

$$\Omega_j^*=(n-1)\omega_j-\Omega_j. \tag{1.3.13}$$

Then the operator $P(x,\partial)$ takes the form

$$\begin{aligned}
e^{2t}P(x,\partial)&=e^{2t}P(0,\partial)+e^{2t}(P(x,\partial)-P(0,\partial))\\
&=(\partial_t^2+(n-2)\partial_t+\triangle_\omega)I_N\\
&\quad+\sum_{j,k}(A_{j,k}(e^t\omega)-A_{j,k}(0))(\omega_j(\partial_t-1)+\Omega_j)(\omega_k\partial_t+\Omega_k)\\
&:=(\partial_t^2+(n-2)\partial_t+\triangle_\omega)I_N+\sum_{j+|\alpha|\leq 2}B_{j,\alpha}(t,\omega)\partial_t^j\Omega^\alpha,
\end{aligned}$$

where $B_{j,\alpha}$ are $N \times N$ valued matrices such that $B_{j,\alpha} = O(e^t)$ as t tends to $-\infty$, and $D_k B_{j,\alpha} = O(e^t)$ as t tends to $-\infty$, for any $k \in \{0, 1, \dots, n\}$ and $\triangle_\omega = \sum_{j=1}^n \Omega_j^2$ is the Laplace-Beltrami operator in S^{n-1}.

We note

$$B_{j,\alpha}^* B_{k,\beta} = B_{k,\beta} B_{j,\alpha}^* \tag{1.3.14}$$

for any j, k, α and β thanks to the hypothesis (1.2.2).

Setting $u = e^{-\gamma t^2/2} v$ and $P_\gamma v = e^{\gamma t^2/2} P(e^{-\gamma t^2/2} v)$, P_γ can be written

$$\begin{aligned}
e^{2t} P_\gamma v &= (\partial_t - \gamma t)^2 v + (n-2)(\partial_t - \gamma t) v + \triangle_\omega v \\
&+ \sum_{j+|\alpha| \le 2} B_{j,\alpha} (\partial_t - \gamma t)^j \Omega^\alpha v \\
&= \partial_t^2 v + (n - 2 - 2\gamma t)\partial_t v + (\gamma^2 t^2 - \gamma - (n-2)\gamma t + \triangle_\omega) v \\
&+ \sum_{j+|\alpha| \le 2} B_{j,\alpha} (\partial_t - \gamma t)^j \Omega^\alpha v \\
&= a_1 + a_2 + a_3 + a_4
\end{aligned}$$

where $a_1 := \partial_t^2 v$, $a_2 := (n - 2 - 2\gamma t)\partial_t v$, $a_3 := (\gamma^2 t^2 - \gamma - (n-2)\gamma t + \triangle_\omega) v$ and $a_4 := \sum_{j+|\alpha| \le 2} B_{j,\alpha} (\partial_t - \gamma t)^j \Omega^\alpha v$.

The estimate (1.3.11) in Theorem 1.2.1 is then equivalent to

$$C \int |e^{2t} P_\gamma v|^2 dt d\omega \ge \gamma^3 \int |v|^2 dt d\omega + \gamma \int |\partial_t v|^2 dt d\omega + \gamma \sum_{j=1}^n \int |\Omega_j v|^2 dt d\omega \tag{1.3.15}$$

for any $v \in C_0^\infty((-\infty, T_0) \times S^{n-1})$.

We shall prove (1.3.15). Set

$$\begin{aligned}
e^{2t} P_\gamma^- v &= (-\partial_t - \gamma t)^2 v + (n-2)(-\partial_t - \gamma t) v + \triangle_\omega v - 2\gamma v \\
&+ \sum_{j+|\alpha| \le 2} B_{j,\alpha}^* (-\partial_t - \gamma t)^j (\Omega^*)^\alpha v \\
&= \partial_t^2 v - (n - 2 - 2\gamma t)\partial_t v + (\gamma^2 t^2 - \gamma - (n-2)\gamma t + \triangle_\omega) v \\
&+ \sum_{j+|\alpha| \le 2} B_{j,\alpha}^* (-\partial_t - \gamma t)^j (\Omega^*)^\alpha v \\
&= a_1 - a_2 + a_3 + a_5, \\
D(\gamma, v) &= \int |e^{2t} P_\gamma v|^2 dt d\omega - \int |e^{2t} P_\gamma^- v|^2 dt d\omega
\end{aligned}$$

and

$$S(\gamma, v) = \int |e^{2t} t^{-1} P_\gamma v|^2 dt d\omega + \int |e^{2t} t^{-1} P_\gamma^- v|^2 dt d\omega$$

where

$$a_5 := \sum_{j+|\alpha| \le 2} B_{j,\alpha}^* (-\partial_t - \gamma t)^j (\Omega^*)^\alpha v.$$

Thus we have

$$\begin{aligned}D(\gamma, v) &= (a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4)\\&- (a_1 - a_2 + a_3 + a_5, a_1 - a_2 + a_3 + a_5)\\&= 4\mathrm{Re}\{(a_1, a_2) + (a_2, a_3)\}\\&+ 2\mathrm{Re}\{(a_4, a_1 + a_2 + a_3) - (a_1 - a_2 + a_3, a_5)\} + \|a_4\|^2 - \|a_5\|^2\\S(\gamma, v) &= \|t^{-1}a_1 + t^{-1}a_2 + t^{-1}a_3 + t^{-1}a_4\| + \|t^{-1}a_1 - t^{-1}a_2 + t^{-1}a_3 + t^{-1}a_5\|\\&\geq \|t^{-1}(a_1 + a_2 + a_3)\|^2/2 + \|t^{-1}(a_1 - a_2 + a_3)\|^2/2\\&- \|t^{-1}a_4\|^2 - \|t^{-1}a_5\|^2\\&= \|t^{-1}a_1\|^2 + \|t^{-1}a_2\|^2 + \|t^{-1}a_3\|^2\\&+ 2\mathrm{Re}(t^{-1}a_1, t^{-1}a_3) - \|t^{-1}a_4\|^2 - \|t^{-1}a_5\|^2\end{aligned}$$

where $(\cdot, \cdot)$ is the L^2 inner product and $\|\cdot\|$ is the L^2 norm. Using integration by parts, we have

$$\begin{aligned}2\mathrm{Re}(a_1, a_2) &= 2\mathrm{Re}(\partial_t^2 v, (n - 2 - 2\gamma t)\partial_t v)\\&= (\partial_t^2 v, (n - 2 - 2\gamma t)\partial_t v) + ((n - 2 - 2\gamma t)\partial_t v, \partial_t^2 v)\\&= 2\gamma \int |\partial_t v|^2 dt d\omega.\end{aligned}$$

From (1.3.12) and (1.3.13) it follows that

$$\triangle_\omega = -\sum_{j=1}^{n} \Omega_j^* \Omega_j. \tag{1.3.16}$$

Using integration by parts and (1.3.16) we have

$$\begin{aligned}2\mathrm{Re}(a_2, a_3) &= 2\mathrm{Re}((n - 2 - 2\gamma t)\partial_t v, (\gamma^2 t^2 - \gamma - (n - 2)\gamma t + \triangle_\omega)v)\\&= ((n - 2 - 2\gamma t)\partial_t v, (\gamma^2 t^2 - \gamma - (n - 2)\gamma t)v)\\&+ ((\gamma^2 t^2 - \gamma - (n - 2)\gamma t)v, (n - 2 - 2\gamma t)\partial_t v)\\&+ ((n - 2 - 2\gamma t)\partial_t v, \triangle_\omega v) + (\triangle_\omega v, (n - 2 - 2\gamma t)\partial_t v)\\&= (v, -\partial_t\{(n - 2 - 2\gamma t)(\gamma^2 t^2 - \gamma - (n - 2)\gamma t)\}v)\\&- \sum_{j=1}^{n} \{((n - 2 - 2\gamma t)\partial_t v, \Omega_j^* \Omega v) + (\Omega_j^* \Omega v, (n - 2 - 2\gamma t)\partial_t v)\}\\&= (v, -\partial_t\{(n - 2 - 2\gamma t)(\gamma^2 t^2 - \gamma - (n - 2)\gamma t)\}v)\\&- \sum_{j=1}^{n} \{(\Omega_j \partial_t v, (n - 2 - 2\gamma t)\Omega_j v) + (\Omega_j v, (n - 2 - 2\gamma t)\Omega_j \partial_t v)\}\\&= 1/2 \int f^2 |v|^2 dt d\omega - 2\gamma \sum_{j=1}^{n} \int |\Omega_j v|^2 dt d\omega,\end{aligned}$$

$$\begin{aligned}\|t^{-1}a_3\|^2 &= \int |(\gamma^2 t - \gamma t^{-1} - (n-2)\gamma + t^{-1}\triangle_\omega)v|^2 dt d\omega \\ &= \int |(\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v|^2 dt d\omega + \int |t^{-1}\triangle_\omega v|^2 dt d\omega \\ &+ 2\mathrm{Re}((\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v, t^{-1}\triangle_\omega v) \end{aligned} \tag{1.3.17}$$

$$\begin{aligned}&2\mathrm{Re}((\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v, t^{-1}\triangle_\omega v) \\ &\quad = -\sum_{j=1}^{n}((\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v, t^{-1}\Omega_j^*\Omega_j v) \\ &\quad - \sum_{j=1}^{n}(t^{-1}\Omega_j^*\Omega_j v, (\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v) \\ &\quad = -\sum_{j=1}^{n}((\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)\Omega_j v, t^{-1}\Omega_j v) \\ &\quad - \sum_{j=1}^{n}(t^{-1}\Omega_j v, (\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)\Omega_j v) \\ &\quad = -2\sum_{j=1}^{n}\int (\gamma^2 - \gamma t^{-2} - (n-2)\gamma t^{-1})|\Omega_j v|^2 dt d\omega \end{aligned} \tag{1.3.18}$$

and

$$\begin{aligned}&2\mathrm{Re}(t^{-1}a_1, t^{-1}a_3) \\ &= (\partial_t^2 v, (\gamma^2 - \gamma t^{-2} - (n-2)\gamma t^{-1})v) + ((\gamma^2 - \gamma t^{-2} - (n-2)\gamma t^{-1})v, \partial_t^2 v) \\ &- \sum_{j=1}^{n}(\Omega_j \partial_t^2 v, t^{-2}\Omega_j v) - \sum_{j=1}^{n}(t^{-2}\Omega_j v, \Omega_j \partial_t^2 v) \\ &= \int (-2\gamma^2 + 2\gamma t^{-2} + 2(n-2)\gamma t^{-1})|\partial_t v|^2 dt d\omega \\ &+ (\partial_t v, (-2\gamma t^{-3} - (n-2)\gamma t^{-2})v) \\ &+ ((-2\gamma t^{-3} - (n-2)\gamma t^{-2})v, \partial_t v) + 2\sum_{j=1}^{n}\int t^{-2}|\Omega_j \partial_t v|^2 dt d\omega \\ &- \sum_{j=1}^{n}(\Omega_j \partial_t v, 2t^{-3}\Omega_j v) - \sum_{j=1}^{n}(2t^{-3}\Omega_j v, \Omega_j \partial_t v) \\ &= \int (-2\gamma^2 + 2\gamma t^{-2} + 2(n-2)\gamma t^{-1})|\partial_t v|^2 dt d\omega \\ &+ \int (-6\gamma t^{-4} - 2(n-2)\gamma t^{-3})|v|^2 dt d\omega \\ &+ 2\sum_{j=1}^{n}\int t^{-2}|\Omega_j \partial_t v|^2 dt d\omega - 6\sum_{j=1}^{n}\int t^{-4}|\Omega_j v|^2 dt d\omega \end{aligned} \tag{1.3.19}$$

where

$$f^2 = 12\gamma^3 t^2 - 12(n-2)\gamma^2 t - 4\gamma^2 + 2(n-2)^2\gamma.$$

Combining (1.3.17), (1.3.18) and (1.3.19) it follows that

$$\begin{aligned}
\|t^{-1}a_1\|^2 + &\|t^{-1}a_2\|^2 + \|t^{-1}a_3\|^2 + 2\mathrm{Re}(t^{-1}a_1, t^{-1}a_3) \\
&= \int t^{-2}|\partial_t^2 v|^2 dt d\omega + \int ((n-2)t^{-1} - 2\gamma)^2 |\partial_t v|^2 dt d\omega \\
&+ \int |(\gamma^2 t - \gamma t^{-1} - (n-2)\gamma)v|^2 dt d\omega + \int |t^{-1}\triangle_\omega v|^2 dt d\omega \\
&- 2\sum_{j=1}^n \int (\gamma^2 - \gamma t^{-2} - (n-2)\gamma t^{-1})|\Omega_j v|^2 dt d\omega \\
&+ \int (-2\gamma^2 + 2\gamma t^{-2} + 2(n-2)\gamma t^{-1})|\partial_t v|^2 dt d\omega \\
&+ \int (-6\gamma t^{-4} - 2(n-2)\gamma t^{-3})|v|^2 dt d\omega \\
&+ 2\sum_{j=1}^n \int t^{-2}|\Omega_j \partial_t v|^2 dt d\omega - 6\sum_{j=1}^n \int t^{-4}|\Omega_j v|^2 dt d\omega \\
&= \int t^{-2}|\partial_t^2 v|^2 dt d\omega + \int t^{-2}|\triangle_\omega v|^2 dt d\omega + 2\sum_{j=1}^n \int t^{-2}|\partial_t \Omega_j v|^2 dt d\omega \\
&+ \int h^2 |v|^2 dt d\omega + \int g^2 |\partial_t v|^2 dt d\omega - \sum_{j=1}^n \int l^2 |\Omega_j v|^2 dt d\omega
\end{aligned}$$

where

$$\begin{aligned}
g^2 &= (-2\gamma + (n-2)t^{-1})^2 - 2\gamma^2 + 2(n-2)\gamma t^{-1} + 2\gamma t^{-2}, \\
h^2 &= (\gamma^2 t - (n-2)\gamma - \gamma t^{-1})^2 - 2(n-2)\gamma t^{-3} - 6\gamma t^{-4}
\end{aligned}$$

and

$$l^2 = 2(\gamma^2 - (n-2)\gamma t^{-1} - \gamma t^{-2}) + 6t^{-4}.$$

On the other hand, by the definition of a_4 and a_5 it follows that

$$\|t^{-1}a_4\| + \|t^{-1}a_5\| \le \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \int |B_1 D^\alpha v|^2 dt d\omega$$

where $B_1 = B_1(t,\omega)$ satisfies $B_1(t,\omega) = O(te^t)$ as t tends to $-\infty$.

To prove Proposition 1.3.3 we need the following similar result as [10] (see Lemma 2.3 in [10]).

Lemma 1.3.4. *Let $\tilde{D}_0 = \partial_t - \gamma t$ and $\tilde{D}_j = \Omega_j$ for $j = 1, \ldots, n$, and let $A(t,\omega)$ be an $N \times N$ matrix valued function such that $A = O(e^t)$ as t tends to $-\infty$,* *and $D_k A = O(e^t)$ as t tends to $-\infty$. Then there exists $B(t,\omega)$ such that for any $v \in C_0^\infty(I \times S^{n-1})$, and for any $\alpha, \beta \in \mathbb{N}^{n+1}$, with $|\alpha|, |\beta| \le 2$ we have*

$$(A(t,\omega)\tilde{D}^\alpha v, \tilde{D}^\beta v) - (A(t,\omega)(\tilde{D}^*)^\beta v, (\tilde{D}^*)^\alpha v) \le \sum_{|\alpha|\le 2} \gamma^{3-2|\alpha|} \int |BD^\alpha v|^2 dtd\omega$$

and $B = O(t^2 e^{t/2})$ as t tends to $-\infty$.

Now, we proceed to the proof of Proposition 1.3.3. By Lemma 1.3.4 and (1.3.14) there exists a function $B_2(t,\omega)$ such that $B_2(t,\omega) = O(t^2 e^{t/2})$ as t tends to $-\infty$ and

$$\begin{aligned} &2\mathrm{Re}\{(a_4, a_1 + a_2 + a_3) - (a_1 - a_2 + a_3, a_5)\} + \|a_4\|^2 - \|a_5\|^2 \\ &\qquad\qquad \le \sum_{|\alpha|\le 2} \gamma^{3-2|\alpha|} \int |B_2 D^\alpha v|^2 dtd\omega. \end{aligned}$$

Thus we have

$$\begin{aligned} D(\gamma, v) \ge{}& 4\gamma \int |\partial_t v|^2 dtd\omega + \int f^2 |v|^2 dtd\omega - 4\gamma \sum_{j=1}^n \int |\Omega_j v|^2 dtd\omega\omega \\ &- \sum_{|\alpha|\le 2} \gamma^{3-2|\alpha|} \int |B_2 D^\alpha v|^2 dtd\omega \end{aligned}$$

and

$$\begin{aligned} S(\gamma, v) \ge{}& \int t^{-2} |\partial_t^2 v|^2 dtd\omega + \int t^{-2} |\triangle_\omega v|^2 dtd\omega + 2\sum_{j=1}^n \int t^{-2} |\partial_t \Omega_j v|^2 dtd\omega \\ &+ \int h^2 |v|^2 dtd\omega + \int g^2 |\partial_t v|^2 dtd\omega - \sum_{j=1}^n \int l^2 |\Omega_j v|^2 dtd\omega \\ &- \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \int |B_1 D^\alpha v|^2 dtd\omega. \end{aligned}$$

Therefore we have

$$\begin{aligned} \gamma D(\gamma, v) + S(\gamma, v) \ge{}& \int t^{-2} |\partial_t^2 v|^2 dtd\omega + \int t^{-2} |\triangle_\omega v|^2 dtd\omega \\ &+ 2\sum_{j=1}^n \int t^{-2} |\partial_t \Omega_j v|^2 dtd\omega + 4\gamma^2 \int |\partial_t v|^2 dtd\omega \\ &+ \int g^2 |\partial_t v|^2 dtd\omega + \int h^2 |v|^2 dtd\omega + \gamma \int f^2 |v|^2 dtd\omega \\ &- \sum_{j=1}^n \int l^2 |\Omega_j v|^2 dtd\omega - 4\gamma^2 \sum_{j=1}^n \int |\Omega_j v|^2 dtd\omega \\ &- \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \int |BD^\alpha v|^2 dtd\omega \end{aligned}$$

where $B = O(t^2 e^{t/2})$ as t tends to $-\infty$.
From (1.3.16) we have

$$\begin{aligned}
&\sum_{j=1}^{n} \int l^2 |\Omega_j v|^2 dt d\omega + 4\gamma^2 \sum_{j=1}^{n} \int |\Omega_j v|^2 dt d\omega \\
&= 2((l^2 + 4\gamma^2)v, \triangle_\omega v) \\
&\leq \epsilon^{-1} \int (l^2/2 + 2\gamma^2)^2 t^2 |v|^2 dt d\omega + \epsilon \int t^{-2} |\triangle_\omega v|^2 dt d\omega
\end{aligned} \tag{1.3.20}$$

for any positive ϵ. If $|T_0|$ and γ are large enough we have

$$\gamma f^2 + h^2 - \epsilon^{-1}(l^2/2 + 2\gamma^2)^2 t^2 \geq (12 - 9\epsilon^{-1})\gamma^4 t^2 \tag{1.3.21}$$

for any $t \in (-\infty, T_0)$. Fixing ϵ such that $12 - 9\epsilon^{-1} > 0$ and $0 < \epsilon < 1$, we have

$$\begin{aligned}
\gamma D(\gamma, v) + S(\gamma, v) \geq &\int t^{-2} |\partial_t^2 v|^2 dt d\omega + (1 - \epsilon) \int t^{-2} |\triangle_\omega v|^2 dt d\omega \\
&+ 2 \sum_{j=1}^{n} \int t^{-2} |\partial_t \Omega_j v|^2 dt d\omega + 4\gamma^2 \int |\partial_t v|^2 dt d\omega \\
&+ \int g^2 |\partial_t v|^2 dt d\omega + (12 - 9\epsilon^{-1})\gamma^4 \int t^2 |v|^2 dt d\omega \\
&- \sum_{|\alpha| \leq 2} \gamma^{4-2|\alpha|} \int |BD^\alpha v|^2 dt d\omega
\end{aligned}$$

By ellipticity of $\triangle_\omega$ there exists a positive constant C such that

$$\int t^{-2} |\triangle_\omega v|^2 dt d\omega \geq C \sum_{|\alpha|=2} \int t^{-2} |\Omega^\alpha v|^2 dt d\omega. \tag{1.3.22}$$

From (1.3.16) we have

$$\begin{aligned}
\gamma^2 \sum_{j=1}^{n} \int |\Omega_j v|^2 dt d\omega &= -\gamma^2 (v, \triangle_\omega v) \\
&\leq \gamma^4/2 \int t^2 |v|^2 dt d\omega + 1/2 \int t^{-2} |\triangle_\omega v|^2 dt d\omega.
\end{aligned} \tag{1.3.23}$$

By (1.3.22) and (1.3.23) there exists a positive constant such that

$$\begin{aligned}
(1 - \epsilon) \int t^{-2} |\triangle_\omega v|^2 dt d\omega + (12 - 9\epsilon^{-1})\gamma^4 \int t^2 |v|^2 dt d\omega \\
\geq C \sum_{|\alpha| \leq 2} \gamma^{4-2|\alpha|} \int t^{2-2|\alpha|} |\Omega^\alpha v|^2 dt d\omega.
\end{aligned}$$

Therefore we obtain

$$\gamma D(\gamma,v)+S(\gamma,v)\geq C\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\int t^{2-2|\alpha|}|D^\alpha v|^2dtd\omega$$
$$-\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\int |BD^\alpha v|^2dtd\omega$$

Since $B=O(t^2e^{t/2})$ as t tends to $-\infty$ if $|T_0|$ is sufficiently large, we get

$$\gamma D(\gamma,v)+S(\gamma,v)\geq C\sum_{|\alpha|\leq 2}\gamma^{4-2|\alpha|}\int t^{2-2|\alpha|}|D^\alpha v|^2dtd\omega$$

for any $v\in C_0^\infty((-\infty,T_0)\times S^{n-1})$.

On the other hand, by definitions of $D(\gamma,v)$ and $S(\gamma,v)$

$$\begin{aligned}\gamma D(\gamma,v)+S(\gamma,v)&\leq\gamma\int|e^{2t}P_\gamma v|^2dtd\omega-\gamma\int|e^{2t}P_\gamma^- v|^2dtd\omega\\&\quad+\int|e^{2t}t^{-1}P_\gamma v|^2dtd\omega+\int|e^{2t}t^{-1}P_\gamma^- v|^2dtd\omega\\&\leq(\gamma+1)\int|e^{2t}P_\gamma v|^2dtd\omega.\end{aligned}$$

Therefore we get

$$\int|e^{2t}P_\gamma v|^2dtd\omega\geq C\sum_{|\alpha|\leq 2}\gamma^{3-2|\alpha|}\int t^{2-2|\alpha|}|D^\alpha v|^2dtd\omega$$

for any $v\in C_0^\infty((-\infty,T_0)\times S^{n-1})$, which is a better estimate than the desired one (1.3.15). □

By Proposition 1.3.2 and Proposition 1.3.3 we can see Theorem 1.2.1 in the standard manner. In the rest of this section we prove Theorem 1.2.1.

Proof. Let $0<R_1<R_0$ where R_0 is sufficiently small so that Proposition 1.3.3 holds for $\tilde{\Omega}=B(R_0)$, and let $\chi(r)\in C^\infty([0,\infty))$ be a cut-off function such that $\chi(r)=1$ if $0\leq r\leq R_1$, $\chi(r)=0$ if $R_0\leq r$ and $|\chi'|\leq C$.

Applying Proposition 1.3.3 with $\tilde{u}(x)=\chi(|x|)u(x)$, we have

$$\gamma\int_{B(R_0)}|x|^2\varphi_\gamma^2|\nabla\tilde{u}|^2|x|^{-n}dx+\gamma^3\int_{B(R_0)}\varphi_\gamma^2|\tilde{u}|^2|x|^{-n}dx$$
$$\leq C\int_{B(R_0)}|x|^4\varphi_\gamma^2|P(x,\partial)\tilde{u}|^2|x|^{-n}dx$$

From (1.2.4) we have

$$
\begin{aligned}
&\int_{B(R_0)} |x|^4\varphi_\gamma^2|P(x,\partial)\tilde{u}|^2|x|^{-n}dx\\
&\le C\int_{B(R_0)} |x|^4\varphi_\gamma^2|\chi P(x,\partial)u|^2|x|^{-n}dx\\
&+C\int_{B(R_0)\setminus B(R_1)} |x|^4\varphi_\gamma^2(|x|^{-1}|u|+|\nabla u|)^2|x|^{-n}dx\\
&\le C\int_{B(R_0)} |x|^4\varphi_\gamma^2|\chi|^2(C_0|x|^{-2}|u|+C_1|x|^{-1}|\nabla u|)^2|x|^{-n}dx\\
&+C\int_{B(R_0)\setminus B(R_1)} |x|^4\varphi_\gamma^2(|x|^{-1}|u|+|\nabla u|)^2|x|^{-n}dx.\\
&\le C\int_{B(R_0)} |x|^4\varphi_\gamma^2(|x|^{-4}|\tilde{u}|^2+|x|^{-2}|\nabla\tilde{u}|^2)|x|^{-n}dx\\
&+C\int_{B(R_0)\setminus B(R_1)} |x|^4\varphi_\gamma^2(|x|^{-1}|u|+|\nabla u|)^2|x|^{-n}dx.
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
(\gamma-C)\int_{B(R_0)} |x|^2\varphi_\gamma^2|\nabla\tilde{u}|^2|x|^{-n}dx+(\gamma^3-C)\int_{B(R_0)}\varphi_\gamma^2|\tilde{u}|^2|x|^{-n}dx\\
\le C\int_{B(R_0)\setminus B(R_1)} |x|^4\varphi_\gamma^2(|x|^{-2}|u|^2+|\nabla u|^2)|x|^{-n}dx.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&e^{\gamma(\log R_1)^2}R_1^{2-n}(\gamma-C)\int_{B(R_1)}|\nabla u|^2dx+e^{\gamma(\log R_1)^2}R_1^{-n}(\gamma^3-C)\int_{B(R_1)}|u|^2dx\\
&\le(\gamma-C)\int_{B(R_1)}|x|^2\varphi_\gamma^2|\nabla u|^2|x|^{-n}dx+(\gamma^3-C)\int_{B(R_1)}\varphi_\gamma^2|u|^2|x|^{-n}dx\\
&\le(\gamma-C)\int_{B(R_0)}|x|^2\varphi_\gamma^2|\nabla \tilde{u}|^2|x|^{-n}dx+(\gamma^3-C)\int_{B(R_0)}\varphi_\gamma^2|\tilde{u}|^2|x|^{-n}dx\\
&\le C\int_{B(R_0)\setminus B(R_1)}|x|^4\varphi_\gamma^2(|x|^{-2}|u|^2+|\nabla u|^2)|x|^{-n}dx\\
&\le Ce^{\gamma(\log R_1)^2}R_1^{2-n}\int_{B(R_0)\setminus B(R_1)}(|u|^2+|\nabla u|^2)dx
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
(\gamma-C)\int_{B(R_1)}|\nabla u|^2dx+(\gamma^3-C)\int_{B(R_1)}|u|^2dx\\
\le C\int_{B(R_0)\setminus B(R_1)}(|u|^2+|\nabla u|^2)dx<\infty.
\end{aligned}
$$

Therefore letting γ tend to ∞, we conclude that $u=0$ in $B(R_1)$. □

Chapter 2

Strong unique continuation property for some second order elliptic systems with two independent variables

2.1 Introduction

In this chapter we prove the strong unique continuation property for some second order systems with two independent variables. As far as we know, there are few results for second order systems. On the other hand, there are many results for first order systems (for example [3], [7], [8] and [9]). In [7], Hile and Protter obtained an interesting result. They considered a system of the form

$$|\partial_x u + N(x,y)\partial_y u| \leq M|u| \quad \text{for all} \ \ (x,y) \in \Omega \tag{2.1.1}$$

where Ω is a nonempty open connected subset of $\mathbb{R}^2$ containing the origin and $N(x,y)$ is an $n \times n$ matrix with complex entries of the class $C^1(\Omega)$. They proved, roughly speaking, that if N is a normal elliptic matrix, any u satisfying (2.1.1) and

$$\lim_{r\to 0} \exp\,(x^2+y^2)^{-\beta/2} u(x,y) = 0 \quad \text{for all} \ \ \beta \geq 0 \tag{2.1.2}$$

vanishes in Ω where $r = \sqrt{x^2+y^2}$.

Okaji improved (2.1.2) in [9]. He proved that: Suppose that all the eigenvalues of $N(0,0)$ are ζ or $\bar{\zeta}$ with a non-real complex number ζ. Then there is a positive constant M_0 such that if $u \in C^1$ satisfies the inequality

$$|\partial_x u + N(x,y)\partial_y u| \leq M|u|/r \quad \text{for all} \ \ (x,y) \in \Omega \tag{2.1.3}$$

with $M < M_0$ and vanishes of infinite order at the origin, then u is identically zero.

Assuming that u verifies (2.1.3) and vanishes of infinite order at the origin he derives a stronger vanishing of u at the origin. Therefore he could use a stronger weight function than the usual weight $r^{-\beta}$.

We study the strong unique continuation property of solutions to some second order elliptic systems verifying (2.1.2) or vanishing of infinite order at the origin. In both cases we reduce our system to a first order system. In particular, in the case that u vanishes of infinite order at the origin we use a similar method as [9]. Then we shall apply Grammatico's result in [4].

We emphasize that there is no regularity assumptions on the eigenvalues of N as well as in [7] and [9].

2.2 Main Results 2

Let Ω be a nonempty open connected subset of $\mathbb{R}^2$ containing the origin. We denote by r the distance between (x, y) and the origin. $X^1(\Omega)$ denotes the class of functions f defined on Ω satisfying the following properties (2.2.1) and (2.2.2):

$$f(x,y) \in C^0(\bar{\Omega}) \cap C^1(\Omega \setminus \{0\}) \tag{2.2.1}$$

where $\bar{\Omega}$ is the closure of Ω and

$$|\nabla f(x,y)| = O(r^{-1}) \tag{2.2.2}$$

where we shall use the notation $g(x, y) = O(h(x, y))$ if

$$\lim_{\rho\to 0} \sup_{0\le r\le\rho} |g(x,y)/h(x,y)| < \infty.$$

$X^{1,\kappa}(\Omega)$ denotes the class of functions $f \in X^1(\Omega)$ satisfying the following properties (2.2.3) and (2.2.4): $f(x, y)$ is Hölder continuous of order κ, that is, there exists a positive C such that

$$|f(x,y) - f(x',y')| \le C|(x,y) - (x',y')|^\kappa \tag{2.2.3}$$

for all $(x, y), (x', y') \in \Omega$ and

$$|\nabla f(x,y)| = o(r^{-1}) \tag{2.2.4}$$

where we shall use the notation $g(x, y) = o(h(x, y))$ if

$$\lim_{\rho\to 0} \sup_{0\le r\le\rho} |g(x,y)/h(x,y)| = 0.$$

Put $L^{(k)} = \partial_x + N_k(x, y)\partial_y, k = 1, 2$ where $N_k(x, y)$ is an $n \times n$ normal matrix with complex entries of the class $X^1(\Omega)$. Moreover we shall assume that there exists a positive number δ such that

$$|\mathrm{Im}\lambda_j^{(k)}(x,y)| \ge \delta \quad j = 1, 2, \ldots, n$$

for all $(x, y) \in \Omega$ where $\lambda_j^{(k)}(x, y), j = 1, 2, \ldots, n$, are the eigenvalues of $N_k(x, y)$.

Theorem 2.2.1. *Let $L = L^{(1)}L^{(2)}$. Let $u \in H^1_{loc}(\Omega; \mathbb{C}^n)$ satisfy*

$$|Lu| \le C_0 r^{-\beta_0}|u| + C_1 r^{-1}|\nabla u| \tag{2.2.5}$$

with some $C_0, C_1 \ge 0$ and $\beta_0 \in \mathbb{R}$. If u satisfies

$$\lim_{r\to 0} \exp(r^{-\beta}) \int_{B(r)} (|u|^2 + |\nabla u|^2) dxdy = 0 \quad \textit{for all } \beta > 0, \tag{2.2.6}$$

then u is identically zero in Ω, where $B(\rho) = \{(x, y); x^2 + y^2 \le \rho^2\}$.

Corollary 2.2.2. *Let $L = \partial_x^2 + A(x,y)\partial_y^2$ where $A(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^1(\Omega)$. Let $\mu_j(x,y), j = 1,2,\ldots,n$, be the eigenvalues of $A(x,y)$ and suppose that there exists a positive number δ such that*

$$\mathrm{dist}(\mu_j(x,y),(-\infty,0]) \geq \delta \qquad j = 1,2,\ldots,n$$

for all $(x,y) \in \Omega$. Let $u \in H^1_{loc}(\Omega;\mathbb{C}^n)$ satisfy (2.2.5) *with some $C_0, C_1 \geq 0$ and $\beta_0 \in \mathbb{R}$. If u satisfies* (2.2.6)*, then u is identically zero in Ω.*

Corollary 2.2.3. *Let $L = \partial_x^2 + 2B(x,y)\partial_{xy}^2 + A(x,y)\partial_y^2$ where $A(x,y)$ and $B(x,y)$ are $n \times n$ Hermitian matrices with complex entries of the class $X^1(\Omega)$ and satisfy $AB = BA$. Suppose that L is elliptic, that is, there exists a positive δ such that*

$$((\xi^2 + 2B(x,y)\xi\eta + A(x,y)\eta^2)v, v) \geq \delta(\xi^2+\eta^2)^{1/2}|v|^2 \tag{2.2.7}$$

for any $(\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and any $v \in \mathbb{C}^n$. Let $u \in H^1_{loc}(\Omega;\mathbb{C}^n)$ satisfy (2.2.5) *with some $C_0, C_1 \geq 0$ and $\beta_0 \in \mathbb{R}$. If u satisfies* (2.2.6)*, then u is identically zero in Ω.*

Remark 2.2.1. For $L = L^{(1)}L^{(2)}\cdots L^{(m)}$, we obtain a similar result as Theorem 2.2.1 if $N_k(x,y), k = 1,2,\ldots,m$, belong to the class $C^m(\Omega)$.

Next we relax the assumption (2.2.6). In this case, we consider the system of differential operators $L = \partial_x^2 + N(x,y)^2\partial_y^2$ where $N(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$. Let $\lambda_j(x,y), j = 1,2,\ldots,n$, be eigenvalues of $N(x,y)$. We suppose that there exists a positive number δ such that

$$|\mathrm{Re}\lambda_j(x,y)| \geq \delta \quad j = 1,2,\ldots,n$$

for all $(x,y) \in \Omega$. Moreover we suppose that there exists $a \in \mathbb{R}$ such that

$$\lambda_j(0,0) = a \ \text{ or } \ -a \qquad j = 1,2,\ldots,n.$$

Theorem 2.2.4. *Let $u \in H^1_{loc}(\Omega;\mathbb{C}^n)$ satisfy*

$$|Lu| \leq C_0 r^{-2}|u| + C_1 r^{-1}|\nabla u| \tag{2.2.8}$$

with $C_0 \geq 0$ and $0 \leq C_1 < \min\{1,|a|\}/\sqrt{2}$. If u satisfies

$$\lim_{r\to 0} r^{-\beta}\int_{B(r)}(|u|^2 + |\nabla u|^2)dxdy = 0 \ \text{ for all } \beta > 0, \tag{2.2.9}$$

then u is identically zero in Ω.

Corollary 2.2.5. *Let $L = \partial_x^2 + A(x,y)\partial_y^2$, where $A(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$. Let $\mu_j(x,y)$ be eigenvalues of $A(x,y)$ and suppose that there exists a positive number δ such that*

$$\mathrm{dist}(\mu_j(x,y),(-\infty,0]) \geq \delta \qquad j = 1,2,\ldots,n$$

for all $(x, y) \in \Omega$. *Moreover we suppose that there exists a positive number* a *such that*

$$\mu_j(0,0) = a \qquad j = 1, 2, \dots, n.$$

Let $u \in H^1_{loc}(\Omega; \mathbb{C}^n)$ *satisfy (2.2.8) with* $C_0 \geq 0$ *and* $0 \leq C_1 < \min\{1, \sqrt{a}\}/\sqrt{2}$. *If* u *satisfies (2.2.9), then* u *is identically zero in* Ω.

2.3 Proof of Theorem 2.2.1

In this section, we shall prove Theorem 2.2.1. The letter C stands for a generic constant whose value may vary from line to line.

Let $P = \partial_x + M(x,y)\partial_y$ where $M(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $C^0(\bar{\Omega}) \cap C^1(\Omega)$ and

$$|\mathrm{Im}(\text{eigenvalues of } M(x,y))| \geq \delta.$$

Then in [7], they proved the following estimate.

Proposition 2.3.1. (*Hile and Protter* [7]) *There exists a positive C such that*

$$C \iint_\Omega e^{2\varphi}|Pu|^2 r^{-1} dxdy \geq \beta^2 \iint_\Omega e^{2\varphi} r^{-\beta-2}|u|^2 r^{-1} dxdy \tag{2.3.1}$$

for any $u \in C_0^1(\Omega)$ and any large β where $\varphi = r^{-\beta}$.

Remark 2.3.1. In [7], they assume $M(x,y) \in C^0(\bar{\Omega}) \cap C^1(\Omega)$. We obtain the same result if $M(x,y) \in X^1(\Omega)$.

In order to prove Proposition 2.3.1, we require the following elliptic estimate.

Lemma 2.3.2. *There exists a positive C such that*

$$\iint |\nabla u|^2 dxdy \leq C \iint (|L^{(2)}u|^2 + |u|^2) dxdy$$

for any $u \in C_0^1(\Omega \setminus \{(0,0)\})$.

Proof. Using a partition of unity, we reduce the problem to the case of finite number of constant matrices $\{N_2(x_j,y_j)\}_{j=1}^N$. Then the assertion can be easily verified in the standard manner. □

Applying Lemma 2.3.2 with $u = r^\gamma u$, we have the following elliptic estimate with weight function.

Lemma 2.3.3. *There exists a positive C such that*

$$\iint r^{2\gamma}|\nabla u|^2 dxdy \leq C \iint (r^{2\gamma}|L^{(2)}u|^2 + \gamma^2 r^{2\gamma-2}|u|^2) dxdy$$

for any $u \in C_0^1(\Omega \setminus \{(0,0)\})$ and any $\gamma \in \mathbb{R}$.

Proof. Applying Lemma 2.3.2 with $u = r^\gamma u$, we have

$$\begin{aligned}\iint |\nabla(r^\gamma u)|^2 dxdy &\le C \iint (|L^{(2)}(r^\gamma u)|^2 + |r^\gamma u|^2) dxdy \\ &\le C \iint r^{2\gamma}|L^{(2)}u|^2 dxdy \\ &+ C \iint (\gamma^2 r^{2\gamma-2}|xr^{-1}u|^2 + \gamma^2 r^{2\gamma-2}|yr^{-1}u|^2) dxdy \\ &+ C \iint r^{2\gamma}|u|^2 dxdyv \\ &\le C \iint (r^{2\gamma}|L^{(2)}u|^2 + \gamma^2 r^{2\gamma-2}|u|^2) dxdy.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\iint |\nabla(r^\gamma u)|^2 dxdy &= \iint (|r^\gamma \partial_x u + \gamma r^{\gamma-2} xu|^2 + |r^\gamma \partial_y u + \gamma r^{\gamma-2} yu|^2) dxdy \\ &\ge 1/2 \iint r^{2\gamma}|\nabla u|^2 dxdy - \iint \gamma^2 r^{2\gamma-2}|u|^2 dxdy\end{aligned}$$

we have a desire estimate. □

We shall show the proof of Proposition 2.3.1 with $M(x,y) \in X^1(\Omega)$.

Proof. Introduce the polar coordinates $(x,y) = (r\cos\theta, r\sin\theta)$. Then the operator $P(x,\partial)$ takes the form

$$P = T_1\partial_r + r^{-1}M_1\partial_\theta$$

where

$$T_1 = \cos\theta + \sin\theta M(r\cos\theta, r\cos\theta)$$

and

$$M_1 = -\sin\theta + \cos\theta M(r\cos\theta, r\sin\theta). \tag{2.3.2}$$

Since $M(x,y)$ is a normal matrix, there exists a unitary matrix U suth that

$$U^*MU = \mathrm{diag}\begin{pmatrix}\mu_1 & & \\ & \ddots & \\ & & \mu_n\end{pmatrix} \tag{2.3.3}$$

and

$$U^*T_1U = \mathrm{diag}\begin{pmatrix}\cos\theta + \sin\theta\mu_1 & & \\ & \ddots & \\ & & \cos\theta + \sin\theta\mu_n\end{pmatrix}. \tag{2.3.4}$$

Since $|\mathrm{Im}\mu_j|$ is positive, $|\det T_1| = |\det U^*T_1U|$ is positive. Here we set $P_2u = T_1^{-1}Pu$ and $\tilde{M} = T^{-1}M_1$.

Lemma 2.3.4. *$\tilde{M} = (\tilde{m}_{j,k})$ is also a normal matrix with eigenvalues κ_j satisfying*

$$|\mathrm{Im}\kappa_j| \geq \delta' > 0.$$

th entries satisfies $\tilde{m}_{j,k} \in C^0(\bar{\Omega}) \cap C^1(\Omega \setminus \{(0,0)\})$ and $|r\partial_r \tilde{m}_{j,k}|, |\partial_\theta \tilde{m}_{j,k}| = O(r^{-1})$.

Proof. By (2.3.3) and (2.3.4) we have

$$\begin{aligned}
U^*\tilde{M}U &= U^*T_1^{-1}UU^*M_1U \\
&= (U^*T_1U)^{-1}U^*M_1U \\
&= \mathrm{diag}\begin{pmatrix} (\cos\theta + \sin\theta\mu_1)^{-1} & & \\ & \ddots & \\ & & (\cos\theta + \sin\theta\mu_n)^{-1} \end{pmatrix} \\
&\quad \times \mathrm{diag}\begin{pmatrix} -\sin\theta + \cos\theta\mu_1 & & \\ & \ddots & \\ & & -\sin\theta + \cos\theta\mu_n \end{pmatrix}.
\end{aligned} \tag{2.3.5}$$

Therefore

$$\begin{aligned}
|\mathrm{Im}\kappa_j| &= |\mathrm{Im}(-\sin\theta + \cos\theta\mu_j)(\cos\theta + \sin\theta\mu_j)^{-1}| \\
&= |\mathrm{Im}\mu_j\{(\cos\theta + (\mathrm{Re}\mu_j)\sin\theta)^2 + (\mathrm{Im}\mu_j)^2\sin^2\theta\}^{-1}| \\
&\geq \delta\{(\cos\theta + (\mathrm{Re}\mu_j)\sin\theta)^2 + (\mathrm{Im}\mu_j)^2\sin^2\theta\}^{-1}.
\end{aligned}$$

Since

$$\begin{aligned}
U^*\tilde{M}^*U &= \mathrm{diag}\begin{pmatrix} (\cos\theta + \sin\theta\bar{\mu}_1)^{-1} & & \\ & \ddots & \\ & & (\cos\theta + \sin\theta\bar{\mu}_n)^{-1} \end{pmatrix} \\
&\quad \times \mathrm{diag}\begin{pmatrix} -\sin\theta + \cos\theta\bar{\mu}_1 & & \\ & \ddots & \\ & & -\sin\theta + \cos\theta\bar{\mu}_n \end{pmatrix}
\end{aligned}$$

and (2.3.5), $\tilde{M}$ is a normal matrix. Using the relation

$$|\nabla_{x,y} f|^2 = |\partial_r f|^2 + |r^{-1}\partial_\theta f|^2 \tag{2.3.6}$$

we can easily see $\tilde{m}_{j,k} \in C^0(\bar{\Omega}) \cap C^1(\Omega \setminus \{(0,0)\})$ and $|r\partial_r \tilde{m}_{j,k}|, |\partial_\theta \tilde{m}_{j,k}| = O(r^{-1})$. □

Setting $v = e^\varphi u$ and $\tilde{P}v = e^\varphi P_2 e^{-\varphi} v$ we have

$$\begin{aligned}
\tilde{P}v &= \partial_r v + r^{-1}\tilde{M}\partial_\theta v - \varphi' v \\
&= \partial_r v + r^{-1}(\tilde{M} + \tilde{M}^*)\partial_\theta v/2 + r^{-1}(\tilde{M} - \tilde{M}^*)\partial_\theta v/2 - \varphi' v \\
&= \partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2 + r^{-1}Q\partial_\theta v - \varphi' v - r^{-1}S_\theta v/2
\end{aligned}$$

where $S = (\tilde{M} + \tilde{M}^*)/2$ and $Q = (\tilde{M} - \tilde{M})/2$. Then S and Q satisfy $|rS_r|$, $|rQ_r|$, $|S_\theta|$, $|Q_\theta| = O(1)$. Therefore we have

$$\begin{aligned}
\int |\tilde{P}v|^2 drd\theta &= \int |\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2|^2 drd\theta \\
&+ \int |r^{-1}Q\partial_\theta v - \varphi' v - r^{-1}S_\theta v/2|^2 drd\theta \\
&+ 2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, r^{-1}Q\partial_\theta v - \varphi' v - r^{-1}S_\theta v/2)_{L^2} \\
&\geq 2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, r^{-1}Q\partial_\theta v - \varphi' v - r^{-1}S_\theta v/2)_{L^2} \\
&= 2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, -\varphi' v)_{L^2} \\
&+ 2\mathrm{Re}(\partial_r v, r^{-1}Q\partial_\theta v)_{L^2} + 2\mathrm{Re}(r^{-1}S\partial_\theta v, r^{-1}Q\partial_\theta v)_{L^2} \\
&+ 2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, -r^{-1}S_\theta v/2)_{L^2} \\
&+ 2\mathrm{Re}(r^{-1}S_\theta v/2, r^{-1}Q\partial_\theta v)_{L^2} \\
&\geq 2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, -\varphi' v)_{L^2} \\
&+ 2\mathrm{Re}(\partial_r v, r^{-1}Q\partial_\theta v)_{L^2} + 2\mathrm{Re}(r^{-1}S\partial_\theta v, r^{-1}Q\partial_\theta v)_{L^2} \\
&- C\int (|\partial_r v| + |r^{-1}\partial_\theta v| + |r^{-1}v|)|r^{-1}v| drd\theta \qquad (2.3.7)
\end{aligned}$$

Using integration by parts, $S = S^*$, $Q = -Q^*$ and $SQ = QS$ we have

$$\begin{aligned}
&2\mathrm{Re}(\partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2, -\varphi' v)_{L^2} \\
&= (v, \varphi'' v + \varphi' \partial_r v) + (\partial_\theta v, -\varphi' r^{-1}Sv) + (v, -\varphi' r^{-1}S_\theta v/2) \\
&+ (-\varphi' v, \partial_r v + r^{-1}S\partial_\theta v + r^{-1}S_\theta v/2) \\
&= (v, \varphi'' v), \qquad (2.3.8)
\end{aligned}$$

$$\begin{aligned}
2\mathrm{Re}(\partial_r v, r^{-1}Q\partial_\theta v)_{L^2} &= (v, -r^{-1}Q_r\partial_\theta v + r^{-2}Q\partial_\theta v) + (v, r^{-1}Q_\theta \partial_r v) \\
&\geq -C\int |r^{-1}v|(|r^{-1}\partial_\theta v| + |\partial_r v|) drd\theta \qquad (2.3.9)
\end{aligned}$$

and

$$\begin{aligned}
2\mathrm{Re}(r^{-1}S\partial_\theta v, r^{-1}Q\partial_\theta v)_{L^2} &= (-r^{-2}QS\partial_\theta v, \partial_\theta v) + (r^{-2}SQ\partial_\theta v, \partial_\theta v) \\
&= 0. \qquad (2.3.10)
\end{aligned}$$

Combining (2.3.7), (2.3.8), (2.3.9) and (2.3.10) we have

$$\begin{aligned}
\int |\tilde{P}v|^2 drd\theta &\geq \int \varphi'' |v|^2 drd\theta - C\int (|\partial_r v| + |r^{-1}\partial_\theta v| + |r^{-1}v|)|r^{-1}v| drd\theta \\
&\geq \int \varphi'' |v|^2 drd\theta - C\int r^{-2}|v|^2 drd\theta \\
&- C\int \beta^{-1}r^\beta(|\partial_r v|^2 + |r^{-1}\partial_\theta v|^2) drd\theta - C\int \beta r^{-\beta}|r^{-1}v|^2 drd\theta
\end{aligned}$$

From $\varphi''(r) = \beta(\beta+1)r^{-\beta-2}$ we obtain

$$\begin{aligned}\int |\tilde{P}v|^2 drd\theta \geq &\int (\beta(\beta+1)r^{-\beta-2} - Cr^{-2} - C\beta r^{-\beta-2}))|v|^2 drd\theta \\ &- C\int \beta^{-1}r^{\beta}(|\partial_r v|^2 + |r^{-1}\partial_\theta v|^2)drd\theta.\end{aligned}$$

Since β is a large parameter and $r \leq 1$, we may assume

$$\beta(\beta+1)r^{-\beta-2} - Cr^{-2} - C\beta r^{-\beta-2} \geq \beta^2/2.$$

By (2.3.6) it follows that

$$\int |\tilde{P}v|^2 drd\theta \geq \beta^2/2 \int r^{-\beta-2}|v|^2 drd\theta - C\int \beta^{-1}r^{\beta}|\nabla v|^2 r^{-1}dxdy.$$

Applying Lemma 2.3.3 with $L^{(2)}v = \tilde{P}v + \varphi' v$ and $\gamma = (\beta-1)/2$ we have

$$\begin{aligned}\int \beta^{-1}r^{\beta}|\nabla v|^2 r^{-1}dxdy &\leq C\beta^{-1}\int (r^{\beta}|\tilde{P}v + \varphi' v|^2 + \beta^2|v|^2 r^{\beta-2})r^{-1}dxdy \\ &= C\beta^{-1}\int (r^{\beta}|\tilde{P}v - \beta r^{-\beta-1}v|^2 + \beta^2|v|^2 r^{\beta-2})r^{-1}dxdy \\ &\leq C\beta^{-1}\int r^{\beta}|Pv|^2 r^{-1}dxdy + C\beta\int r^{-\beta-2}|v|^2 r^{-1}dxdy\end{aligned}$$

Thus it follows that

$$\begin{aligned}\int |\tilde{P}v|^2 drd\theta \geq &\beta^2/2\int r^{-\beta-2}|v|^2 drd\theta - C\beta^{-1}\int r^{\beta}|Pv|^2 r^{-1}dxdy \\ &- C\beta\int r^{-\beta-2}|v|^2 r^{-1}dxdy\end{aligned}$$

Since β is a large parameter and $r \leq 1$, we have

$$C\int |\tilde{P}v|^2 r^{-1}dxdy \geq \beta^2\int r^{-\beta-2}|v|^2 r^{-1}dxdy$$

Since

$$\begin{aligned}C\int e^{2\varphi}|Pu|^2 r^{-1}dxdy &\geq \int e^{2\varphi}|P_2 u|^2 r^{-1}dxdy \\ &= \int |\tilde{P}v|^2 r^{-1}dxdy,\end{aligned}$$

we have

$$\begin{aligned}C\int e^{2\varphi}|Pu|^2 r^{-1}dxdy &\geq \beta^2\int r^{-\beta-2}|v|^2 r^{-1}dxdy \\ &= \beta^2\int e^{2\varphi}r^{-\beta-2}|u|^2 r^{-1}dxdy,\end{aligned}$$

which is the desire estimate in Proposition 2.3.1. $\square$

By Proposition 2.3.1 we have the following Carleman estimate.

Lemma 2.3.5. *There exists a positive C such that*

$$C\iint e^{2\varphi}r^{-2\gamma}|L^{(2)}u|^2dxdy \geq \beta^2\iint e^{2\varphi}r^{-\beta-2-2\gamma}|u|^2dxdy$$

for any large β and any $u \in C_0^1(\Omega\setminus\{(0,0)\})$ where $\varphi = r^{-\beta}$ and γ is a linear function of β.

Proof. Applying (2.3.1) with $P = L^{(2)}$ and $u = r^{-\gamma+1/2}u$, we have

$$C\iint e^{2\varphi}|L^{(2)}(r^{-\gamma+1/2}u)|^2r^{-1}dxdy \geq \beta^2\iint e^{2\varphi}r^{-\beta-2-2\gamma}|u|^2dxdy.$$

Since $L^{(2)}$ is a first order operator

$$|L^{(2)}(r^{-\gamma+1/2}u)|^2 \leq Cr^{-2\gamma+1}|L^{(2)}u|^2 + C(\gamma-1/2)^2r^{-2\gamma-1}|u|^2.$$

Therefore we obtain the desired estimate in Lemma 2.3.5 if β is large enough. $\square$

In order to prove Theorem 2.2.1 we require the following elliptic estimate with weight function.

Lemma 2.3.6. *There exists a positive C such that*

$$\iint e^{2\varphi}r^{-2\gamma}|\nabla u|^2dxdy \leq C\iint e^{2\varphi}r^{-2\gamma}(|L^{(2)}u|^2+\beta^2r^{-2\beta-2}|u|^2)dxdy$$

for any $u \in C_0^1(\Omega\setminus\{(0,0)\})$ and any large β where $\gamma = \gamma_0\beta+\gamma_1$ with $\gamma_0, \gamma_1 \in \mathbb{R}$.

Proof. Applying Lemma 2.3.2 with $u = e^{\varphi}r^{-\gamma}u$, we have

$$\begin{aligned}
\iint |\nabla(e^{\varphi}r^{-\gamma}u)|^2dxdy &\leq C\iint (|L^{(2)}(e^{\varphi}r^{-\gamma}u)|^2 + |e^{\varphi}r^{-\gamma}u|^2)dxdy \\
&\leq C\iint e^{2\varphi}r^{-2\gamma}|L^{(2)}u|^2dxdy \\
&+ C\iint \gamma^2e^{2\varphi}r^{-2\gamma-2}(|xr^{-1}u|^2+|yr^{-1}u|^2)dxdy \\
&+ C\iint \beta^2e^{2\varphi}r^{-2\gamma-2\beta-2}(|xr^{-1}u|^2+|yr^{-1}u|^2)dxdy \\
&+ C\iint e^{2\varphi}r^{-2\gamma}|u|^2dxdyv \\
&\leq C\iint (e^{2\varphi}r^{-2\gamma}|L^{(2)}u|^2+\beta^2e^{2\varphi}r^{-2\gamma-2\beta-2}|u|^2)dxdy.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}\iint |\nabla(e^{\varphi}r^{-\gamma}u)|^2dxdy &= \iint |e^{\varphi}r^{-\gamma}(\partial_x u-\gamma r^{-2}xu-\beta r^{-\beta-2}xu)|^2dxdy\\ &+\iint |e^{\varphi}r^{-\gamma}(\partial_y u-\gamma r^{-2}yu-\beta r^{-\beta-2}yu)|^2dxdy\\ &\geq 1/2\iint e^{2\varphi}r^{-2\gamma}|\nabla u|^2dxdy\\ &-\iint \gamma^2e^{2\varphi}r^{-2\gamma-2}|u|^2dxdy\\ &-\iint \beta^2e^{2\varphi}r^{-2\gamma-2\beta-2}|u|^2dxdy\\ &\geq 1/2\iint e^{2\varphi}r^{-2\gamma}|\nabla u|^2dxdy\\ &-C\iint \beta^2e^{2\varphi}r^{-2\gamma-2\beta-2}|u|^2dxdy\end{aligned}$$

we have a desire estimate. □

In order to prove Theorem 2.2.1, we prove the following Carleman estimate.

Proposition 2.3.7. *There exists a positive number C such that*

$$\begin{aligned}\int_\Omega e^{2\varphi}|Lu|^2dxdy \geq C\beta^2\int_\Omega e^{2\varphi}r^{-2}|\nabla u|^2dxdy\\ +C\beta^4\int_\Omega e^{2\varphi}r^{-2\beta-4}|u|^2dxdy \qquad (2.3.11)\end{aligned}$$

for all large $\beta \geq 0$ and any $u \in C_0^2(\Omega\setminus\{(0,0)\})$.

Remark 2.3.2. The estimate (2.3.11) in Proposition 2.3.7 remains valid if we assume $u \in H^2_{loc}(\Omega)$ with compact support satisfies

$$\lim_{\rho\to 0}\exp(\rho^{-\beta})\int_{|x|\leq\rho}|\partial_x^\alpha u|^2dx = 0$$

for any positive β and any $|\alpha| \leq 2$.

Proof. Applying (2.3.1) with $P = L^{(1)}$ and $u = L^{(2)}u$, we have

$$C\iint e^{2\varphi}|Lu|^2dxdy \geq \beta^2\iint e^{2\varphi}r^{-\beta-2}|L^{(2)}u|^2dxdy. \qquad (2.3.12)$$

By Lemma 2.3.5 we have

$$\beta^2\iint e^{2\varphi}r^{-\beta-2}|L^{(2)}u|^2dxdy \geq \beta^4\iint e^{2\varphi}r^{-2\beta-4}|u|^2dxdy. \qquad (2.3.13)$$

On the other hand, by Lemma 2.3.6 we have

$$\begin{aligned}\beta^2 \iint e^{2\varphi} r^{-\beta-2} |L^{(2)} u|^2 dxdy &\geq \beta^2 \iint e^{2\varphi} r^{-2} |L^{(2)} u|^2 dxdy \\ &\geq C\beta^2 \iint e^{2\varphi} r^{-2} |\nabla u|^2 dxdy \\ &- C\beta^4 \iint e^{2\varphi} r^{-2\beta-4} |u|^2 dxdy. \qquad (2.3.14)\end{aligned}$$

Combining (2.3.12), (2.3.13) and (2.3.14), we obtain the desired estimate in Proposition 2.3.7. □

Now we can prove Theorem 2.2.1.

Proof. Let $0 < R_1 < R_0 < 1$ and let $\chi(r) \in C^\infty([0,\infty))$ be a cut-off function such that $\chi(r) = 1$ if $0 \leq r \leq R_1$ and $\chi(r) = 0$ if $R_0 \leq r$ and $|\chi'| \leq C$. Applying Proposition 2.3.7 with $\tilde{u}(x) = \chi(|x|)u(x)$, we have

$$\begin{aligned}C\beta^2 \int_{B(R_0)} e^{2\varphi} r^{-2} |\nabla \tilde{u}|^2 dxdy + C\beta^4 \int_{B(R_0)} e^{2\varphi} r^{-2\beta-4} |\tilde{u}|^2 dxdy \\ \leq \int_{B(R_0)} e^{2\varphi} |L\tilde{u}|^2 dxdy\end{aligned}$$

From (2.2.5) we have

$$\begin{aligned}\int_{B(R_0)} e^{2\varphi} |L\tilde{u}|^2 dxdy &\leq \int_{B(R_0)} e^{2\varphi} |\chi|^2 (C_0 r^{-\beta_0} |u| + C_1 r^{-1} |\nabla u|)^2 dxdy \\ &+ C \int_{B(R_0)\setminus B(R_1)} e^{2\varphi} (r^{-2}|u|^2 + |\nabla u|^2) dxdy \\ &\leq 2C_0^2 \int_{B(R_0)} e^{2\varphi} r^{-2\beta_0} |\tilde{u}|^2 dxdy \\ &+ 2C_1^2 \int_{B(R_0)} e^{2\varphi} r^{-2} |\nabla \tilde{u}|^2 dxdy \\ &+ C \int_{B(R_0)\setminus B(R_1)} e^{2\varphi} (r^{-2}|u|^2 + |\nabla u|^2) dxdy\end{aligned}$$

Therefore it follows that

$$\begin{aligned}&C \int_{B(R_0)\setminus B(R_1)} e^{2\varphi} (r^{-2}|u|^2 + |\nabla u|^2) dxdy \\ &\geq (C\beta^2 - 2C_1^2) \int_{B(R_0)} e^{2\varphi} r^{-2} |\nabla \tilde{u}|^2 dxdy \\ &+ C\beta^4 \int_{B(R_0)} e^{2\varphi} r^{-2\beta-4} |\tilde{u}|^2 dxdy - 2C_0^2 \int_{B(R_0)} e^{2\varphi} r^{-2\beta_0} |\tilde{u}|^2 dxdy.\end{aligned}$$

Thus for any large $\beta \geq 2\beta_0 + 2$ we have

$$C\int_{B(R_0)\setminus B(R_1)} e^{2\varphi}(r^{-2}|u|^2 + |\nabla u|^2)dxdy \geq \beta^2 \int_{B(R_0)} e^{2\varphi} r^{-2}|\nabla \tilde{u}|^2 dxdy + \beta^4 \int_{B(R_0)} e^{2\varphi} r^{-2\beta-4}|\tilde{u}|^2 dxdy$$

Thus it follows that

$$\begin{aligned}
&\beta^2 R_1^{-2} e^{2R_1^{-\beta}} \int_{B(R_1)} |\nabla u|^2 dxdy + \beta^4 R_1^{-2\beta-4} e^{2R_1^{-\beta}} \int_{B(R_1)} |u|^2 dxdy \\
&\leq \beta^2 \int_{B(R_1)} e^{2\varphi} r^{-2}|\nabla u|^2 dxdy + \beta^4 \int_{B(R_1)} e^{2\varphi} r^{-2\beta-4}|u|^2 dxdy \\
&\leq \beta^2 \int_{B(R_0)} e^{2\varphi} r^{-2}|\nabla \tilde{u}|^2 dxdy + \beta^4 \int_{B(R_0)} e^{2\varphi} r^{-2\beta-4}|\tilde{u}|^2 dxdy \\
&\leq C\int_{B(R_0)\setminus B(R_1)} e^{2\varphi}(r^{-2}|u|^2 + |\nabla u|^2)dxdy \\
&\leq Ce^{2R_1^{-\beta}} R_1^{-2} \int_{B(R_0)\setminus B(R_1)} (|u|^2 + |\nabla u|^2)dxdy.
\end{aligned}$$

From $u \in H^1_{loc}(\Omega)$ we have

$$\beta^2 \int_{B(R_1)} |\nabla u|^2 dxdy + \beta^4 \int_{B(R_1)} |u|^2 dxdy \leq C.$$

Therefore letting β tend to ∞, we conclude that $u = 0$ in $B(R_1)$. $\square$

Next we shall prove Corollary 2.2.2.

Proof. We define

$$B(x,y) = (2\pi i)^{-1} \oint_\Gamma \sqrt{\zeta}\ (\zeta - A(x,y))^{-1} d\zeta$$

where Γ is a closed curve in $\mathbb{C}\setminus(-\infty, 0]$ enclosing $\mu_j(x,y)$ $(j = 1, 2, \dots, n)$, symmetric with respect to the real axis and $\sqrt{\zeta}$ means $r^{1/2}e^{\theta/2}$ when $\zeta = re^{\theta}$. Then applying Theorem 2.2.1 with $N_1(x,y) = iB(x,y)$ and $N_2(x,y) = -iB(x,y)$, we can prove Corollary 2.2.2. In fact, from the first resolvent equation

$$(z - A)^{-1}(\zeta - A)^{-1} = \{(z - A)^{-1} - (\zeta - A)^{-1}\}/(\zeta - z),$$

we have

$$\begin{aligned}
B^2 &= (2\pi i)^{-2} \oint_\Gamma \sqrt{z}\ (z - A)^{-1} dz \oint_{\Gamma^-} \sqrt{\zeta}\ (\zeta - A)^{-1} d\zeta \\
&= (2\pi i)^{-2} \oint_\Gamma \sqrt{z}\ (z - A)^{-1} \{\oint_{\Gamma^-} \sqrt{\zeta}\ /(\zeta - z) d\zeta\} dz \\
&+ (2\pi i)^{-2} \oint_{\Gamma^-} \sqrt{\zeta}\ (\zeta - A)^{-1} \{\oint_\Gamma \sqrt{z}\ /(z - \zeta) dz\} d\zeta
\end{aligned}$$

where Γ^- is a closed curve inside Γ and satisfies the same conditions as Γ. From

$$\oint_{\Gamma^-} \sqrt{\zeta}\ /(\zeta - z)d\zeta = 0 \quad \text{and} \quad (2\pi i)^{-1} \oint_{\Gamma} \sqrt{z}\ /(z-\zeta)dz = \sqrt{\zeta},$$

it verifies

$$B(x,y)^2 = (2\pi i)^{-1} \oint_{\Gamma^-} \zeta(\zeta - A)^{-1} d\zeta = A(x,y).$$

In what follows, we denote this $B(x,y)$ by $\sqrt{A(x,y)}$. Since Γ is symmetric, we have

$$\sqrt{A(x,y)}^* = (2\pi i)^{-1} \oint_{\Gamma} \sqrt{\zeta}\ (\zeta - A(x,y)^*)^{-1} d\zeta.$$

Hence it easily follows that $\sqrt{A(x,y)}$ is a normal matrix. Moreover it is easy to see that the eigenvalues of $\sqrt{A(x,y)}$ are $\sqrt{\mu_j}$ and entries of $\sqrt{A(x,y)}$ belong to $X^1(\Omega)$. $\square$

In the rest of this section, we shall prove Corollary 2.2.3.

Proof. From our hypothesis, there exists a unitary matrix $U(x,y)$ such that

$$U^*AU = \operatorname{diag}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{and} \quad U^*BU = \operatorname{diag}\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}.$$

Hence

$$U^*(A - B^2)U = \operatorname{diag}\begin{pmatrix} \lambda_1 - \mu_1^2 & & \\ & \ddots & \\ & & \lambda_n - \mu_n^2 \end{pmatrix}.$$

By (2.2.7) we see that $\lambda_j(x,y) - \mu_j(x,y)^2 \geq \delta$ $(j = 1, 2, \ldots, n)$ for any $(x,y) \in \Omega$. Repeating the same arguments as the proof of Corollary 2.2.2 we can define $\sqrt{A(x,y) - B(x,y)^2}$. Since eigenvalues of $\sqrt{A - B^2}$ are $\sqrt{\lambda_j - \mu_j^2}$ and $\mu_j \in \mathbb{R}$, we have

$$\begin{aligned} |\mathrm{Im}(\text{eigenvalues of } B \pm i\sqrt{A - B^2})| &= |\mathrm{Im}(\mu_j \pm i\sqrt{\lambda_j - \mu_j^2})| \\ &= \sqrt{\lambda_j - \mu_j^2} \geq \delta. \end{aligned}$$

Applying Theorem 2.2.1 with $N_1(x,y) = B(x,y) + i\sqrt{A(x,y) - B(x,y)^2}$ and $N_2(x,y) = B(x,y) - i\sqrt{A(x,y) - B(x,y)^2}$, we obtain the desired conclusion of Corollary 2.2.3. $\square$

2.4 Proof of Theorem 2.2.4

First we shall give the proof of Theorem 2.2.4 with $a = 1$. We consider

$$L_0 = \partial_x^2 u + N(0,0)^2 \partial_y^2 u.$$

Then the first result we will show is the Carleman estimate of L_0.

Proposition 2.4.1. *For an arbitrary positive $B < 1$, there exists a positive number $\beta_0(B)$ such that if $\beta \geq \beta_0$ with $\beta \in \mathbb{N} + 1/2$ then*

$$\begin{aligned}(1+\epsilon) &\iint r^{-2\beta+2} |L_0 u|^2 dxdy \\ &\geq B/2 \iint r^{-2\beta} |\nabla u|^2 dxdy + \epsilon\beta^2/4 \iint r^{-2\beta-2} |u|^2 dxdy.\end{aligned}$$

for any $u \in C_0^2(\Omega \setminus \{(0,0)\})$ and any positive ϵ.

Proof. By our hypothesis there exists a unitary matrix U_0 such that $U_0^{-1} L_0 U_0 = (\partial_x^2 + \partial_y^2) I$. Introduce the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and making the change of variables $z = \log r$ we see the following.

Lemma 2.4.2. *For arbitrary $B < 1$ and $B' < 1$, there exists a positive $\beta_0 = \beta_0(B, B')$ such that if $\beta \geq \beta_0$ with $\beta \in \mathbb{N} + 1/2$ then*

$$\begin{aligned}(1+\epsilon) \iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta &\geq \alpha B \iint e^{-2\beta z} |\partial_\theta u|^2 dz d\theta \\ + (1-\alpha) B' \iint e^{-2\beta z} |\partial_z u|^2 dz d\theta &+ \epsilon\beta^2/4 \iint e^{-2\beta z} |u|^2 dz d\theta\end{aligned}$$

for any positive $\epsilon > 0$, any $\alpha \in [0,1]$ and $u \in C_0^2(\Omega \setminus \{(0,0)\})$.

Proof. We use the same method as [4]. We show it briefly (see [4] in detail). Putting $u = e^{\beta z} v$, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta = \iint |\partial_z^2 v + 2\beta \partial_z v + \beta^2 v + \partial_\theta^2 v|^2 dz d\theta.$$

By integration by parts, it follows that

$$\begin{aligned}2\mathrm{Re}(\partial_z^2 v, \partial_z v) = 2\mathrm{Re}(\partial_z v, v) = 2\mathrm{Re}(\partial_z v, \partial_\theta^2 v) &= 0, \\ 2\mathrm{Re}(\partial_z^2 v, v) &= -2\|\partial_z v\|^2, \\ 2\mathrm{Re}(\partial_z^2 v, \partial_\theta^2 v) &= 2\|\partial_{z,\theta}^2 v\|^2, \\ 2\mathrm{Re}(v, \partial_\theta^2 v) &= -2\|\partial_\theta v\|^2,\end{aligned}$$

where $(\cdot,\cdot)$ is the L^2 inner product, and $\|\cdot\|$ is the L^2 norm. Therefore, we have

$$\iint e^{-2\beta z} |(\partial_z^2 + \partial_\theta^2) u|^2 dz d\theta \geq 2\beta^2 \|\partial_z v\|^2 + \|\partial_\theta^2 v\|^2 - 2\beta^2 \|\partial_\theta v\|^2 + \beta^4 \|v\|^2.$$

We use Fourier series expansion of $v(z,\cdot) \in L^2(\mathrm{S}^1)$:

$$v(z,\theta) = \sum_{k\in\mathbb{Z}} v_k(z)e^{ik\theta}, \quad \int_0^{2\pi} |v|^2 d\theta = \sum_{k\in\mathbb{Z}} |v_k(z)|^2.$$

Note that

$$\partial_\theta v(z,\theta) = \sum_{k\in\mathbb{Z}} ikv_k(z)e^{ik\theta}, \quad \int_0^{2\pi} |\partial_\theta v|^2 d\theta = \sum_{k\in\mathbb{Z}} k^2|v_k(z)|^2,$$

$$\partial_\theta^2 v(z,\theta) = \sum_{k\in\mathbb{Z}} (-k^2)v_k(z)e^{ik\theta}, \quad \int_0^{2\pi} |\partial_\theta^2 v|^2 d\theta = \sum_{k\in\mathbb{Z}} k^4|v_k(z)|^2.$$

Thus, we have

$$\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2 dzd\theta \geq 2\beta^2\|\partial_z v\|^2 + \sum_{k\in\mathbb{Z}}(\beta^2-k^2)^2\int |v_k|^2dz.$$

For any positive $B<1$, there exists $\beta_0(B)$ such that if $\beta \geq \beta_0(B)$ with $\beta \in \mathbb{N}+1/2$, we have

$$\sum_{k\in\mathbb{Z}}(\beta^2-k^2)^2\int |v_k|^2dz \geq B\sum_{k\in\mathbb{Z}}k^2\int |v_k|^2dz = B\|\partial_\theta v\|^2.$$

Hence, we have

$$\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2 dzd\theta \geq B\iint e^{-2\beta z}|\partial_\theta u|^2dzd\theta. \tag{2.4.1}$$

On the other hand, for any positive B', there exists $\beta_1(B')$ such that if $\beta \geq \beta_1(B')$ with $\beta \in \mathbb{N}+1/2$, we have

$$\sum_{k\in\mathbb{Z}}(\beta^2-k^2)^2\int |v_k|^2dz \geq B'\beta^2\sum_{k\in\mathbb{Z}}\int |v_k|^2dz = B'\beta^2\|v\|^2.$$

Hence, we have

$$\begin{aligned}\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2 dzd\theta &\geq B'(\beta^2\|v\|^2 + \|\partial_z v\|^2)\\ &\geq B'\|\partial_z v+\beta v\|^2\\ &= B'\iint e^{-2\beta z}|\partial_z u|^2dzd\theta. \end{aligned}\tag{2.4.2}$$

Combining (2.4.1) and (2.4.2), for any positive $B>1$ and $B'>1$ there exists $\beta_0(B,B')$ such that if $\beta \geq \beta_0(B,B')$ with $\beta \in \mathbb{N}+1/2$, we have

$$\begin{aligned}&\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2 dzd\theta\\ &\qquad \geq \alpha B\iint e^{-2\beta z}|\partial_\theta u|^2dzd\theta + (1-\alpha)B'\iint e^{-2\beta z}|\partial_z u|^2dzd\theta \end{aligned}\tag{2.4.3}$$

for any $\alpha \in [0,1]$. We recall that the inequality

$$\iint e^{-2\beta z}|(\partial_z^2+\partial_\theta^2)u|^2 dz d\theta \geq \beta^2/4 \iint e^{-2\beta z}|u|^2 dz d\theta \tag{2.4.4}$$

holds (see the appendix of [4]). (2.4.3) and (2.4.4) show the desired conclusion of Lemma 2.4.2. □

Now, we proceed to the proof of Proposition 2.4.1. From Lemma 2.4.2 with $B = B'$ and $\alpha = 1/2$, it follows

$$\begin{aligned}(1+\epsilon) &\iint r^{-2\beta+2}|(\partial_x^2+\partial_y^2)u|^2 dxdy \\ &\geq B/2 \iint r^{-2\beta}|\nabla u|^2 dxdy + \epsilon\beta^2/4 \iint r^{-2\beta-2}|u|^2 dxdy,\end{aligned}$$

which proves the desired result. □

Proposition 2.4.1 and (2.2.3) give the following Carleman inequality with a remainder term.

Proposition 2.4.3. *For arbitrary $B < 1$, there exists a positive $\beta_0 = \beta_0(B)$ such that if $\beta \geq \beta_0$ with $\beta \in \mathbb{N}+1/2$ then*

$$\begin{aligned}(1+\epsilon)(1+\delta) \iint r^{-2\beta+2}|Lu|^2 dxdy + C(1+\epsilon)(1+\delta^{-1}) \iint r^{-2\beta+2+2\kappa}|\partial_y^2 u|^2 dxdy \\ \geq B/2 \iint r^{-2\beta}|\nabla u|^2 dxdy + \epsilon\beta^2/4 \iint r^{-2\beta-2}|u|^2 dxdy \end{aligned} \tag{2.4.5}$$

for any positive ϵ, δ and any $u \in C_0^2(\Omega \setminus \{(0,0)\})$.

Proof. We can write

$$Lu = L_0 u + (N(x,y)^2 - N(0,0)^2)\partial_y^2 u,$$

and

$$|N(x,y)^2 - N(0,0)^2| \leq C r^\kappa$$

because of their Hölder continuity. Using

$$\begin{aligned}&|Lu - (N(x,y)^2 - N(0,0)^2)\partial_y^2 u|^2 \\ &\qquad \leq (1+\delta)|Lu|^2 + C(1+\delta^{-1})|(N(x,y)^2 - N(0,0)^2)\partial_y^2 u|^2,\end{aligned}$$

the proof is clear. □

We require the following elliptic estimate.

Lemma 2.4.4. *There exists a positive constant C such that*

$$\iint_\Omega (|\partial_x^2 u|^2 + |\partial_y^2 u|^2) dxdy \leq C \iint_\Omega (|Lu|^2 + |\nabla u|^2 + |u|^2) dxdy$$

for any $u \in C_0^2(\Omega)$.

Applying Lemma 2.4.4 with $u = r^{-\beta}u$, we have

Lemma 2.4.5. *There exists a positive constant C such that*

$$\iint_{\Omega} r^{-2\beta}(|\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy \\ \leq C\iint_{\Omega} r^{-2\beta}(|Lu|^2 + \beta^2 r^{-2}|\nabla u|^2 + \beta^4 r^{-4}|u|^2)dxdy$$

for any $u \in C_0^2(\Omega \setminus \{(0,0)\})$.

Proposition 2.4.6. *Under the assumption of Theorem 2.2.4, there exist positive constants C_2 and C_3 such that*

$$\iint_{0\leq R(x,y)\leq\rho} (|u|^2 + |\nabla u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy \leq C_2 \exp(-C_3\rho^{-\kappa})$$

for any small positive ρ.

Proof. Let $\chi(r)$ be a nonnegative function belonging to $C_0^1([0,2))$ such that $\chi(r) = 1$ when $0 \leq r < 1$. We shall consider $\tilde{u}(x,y) = \chi(M\beta^{1/\kappa}r)u(x,y)$. Here, M is a large positive parameter, which will be determined later. By Proposition 2.4.3 and Lemma 2.4.5, we have

$$\begin{aligned}(B/2 - C/K)\iint r^{-2\beta}|\nabla\tilde{u}|^2 dxdy + (\epsilon/4 - C/K)\beta^2\iint r^{-2\beta-2}|\tilde{u}|^2 dxdy \\ + (K\beta^2)^{-1}\iint r^{-2\beta+2}(|\partial_x^2\tilde{u}|^2 + |\partial_y^2\tilde{u}|^2)dxdy \\ \leq \{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\}\iint r^{-2\beta+2}|L\tilde{u}|^2 dxdy \\ + C(1+\epsilon)(1+\delta^{-1})\iint r^{-2\beta+2+2\kappa}|\partial_y^2\tilde{u}|^2 dxdy \qquad (2.4.6)\end{aligned}$$

where K is a large parameter which will be determined later. On the other hand, for all positive ϵ_1 we have

$$\iint r^{-2\beta+2}|L\tilde{u}|^2 dxdy \leq (1+\epsilon_1)\iint r^{-2\beta+2}|\chi Lu|^2 dxdy + C(1+\epsilon_1^{-1})\times \\ \times \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2}(M^2\beta^{2/\kappa}|\nabla u|^2 + M^4\beta^{4/\kappa}|u|^2)dxdy. \qquad (2.4.7)$$

because of

$$1 \leq M^2\beta^{2/\kappa}r^2 \leq 4 \quad \text{if } (x,y) \in B(2M^{-1}\beta^{-1/\kappa}) \setminus B(M^{-1}\beta^{-1/\kappa}).$$

From (2.2.8), we have

$$\begin{aligned}
&\iint r^{-2\beta+2}|\chi Lu|^2 dxdy \\
&\le (1+\epsilon_2)(1+\epsilon_3)C_1^2 \iint r^{-2\beta}|\nabla\tilde{u}|^2 dxdy \\
&+ C(1+\epsilon_2)(1+\epsilon_3^{-1})C_1^2 M^2\beta^{2/\kappa} \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|u|^2 dxdy \\
&+ (1+\epsilon_2^{-1})C_0^2 \iint r^{-2\beta-2}|\tilde{u}|^2 dxdy
\end{aligned} \tag{2.4.8}$$

for all positive ϵ_2 and ϵ_3. Combining (2.4.6), (2.4.7) and (2.4.8), we see that

$$\begin{aligned}
&T_1 \iint r^{-2\beta}|\nabla\tilde{u}|^2 dxdy + T_2\beta^2 \iint r^{-2\beta-2}|\tilde{u}|^2 dxdy \\
&+ (K\beta^2)^{-1} \iint r^{-2\beta+2}(|\partial_x^2\tilde{u}|^2 + |\partial_y^2\tilde{u}|^2) dxdy \\
&\le T_3 \iint r^{-2\beta+2+2\kappa}|\partial_y^2\tilde{u}|^2 dxdy \\
&+ T_4 \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2}(M^2\beta^{2/\kappa}|\nabla u|^2 + M^4\beta^{4/\kappa}|u|^2) dxdy \\
&+ T_5 M^2\beta^{2/\kappa} \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|u|^2 dxdy,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= B/2 - C/K - (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\}C_1^2, \\
T_2 &= (\epsilon/4 - C/K) - (1+\epsilon_1)(1+\epsilon_2^{-1})\{(1+\epsilon)(1+\delta) + C(K\beta^2)^{-1}\}C_0^2\beta^{-2}
\end{aligned}$$

and T_3, T_4, T_5 are positive constants depending only on δ, ϵ_1 and ϵ_3. Take $\epsilon, \delta, \epsilon_1, \epsilon_2$ and ϵ_3 to be small enough. Moreover taking K to be large enough, by our assumption, T_1 and T_2 are positive if β is large enough. Choose M such that $T_3 M^{-2\kappa} < 1/(8K)$. Then it holds that $T_3 r^{2\kappa} \le 1/(2K\beta^2)$ if $(x,y) \in B(2M^{-1}\beta^{-1/\kappa})$. Then it follows that

$$\begin{aligned}
&T_1 \iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|\nabla u|^2 dxdy + T_2\beta^2 \iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta-2}|u|^2 dxdy \\
&+ (2K\beta^2)^{-1} \iint_{B(1/2M^{-1}\beta^{-1/\kappa})} r^{-2\beta+2}(|\partial_x^2 u|^2 + |\partial_y^2 u|^2) dxdy \\
&\le C \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta}|\nabla u|^2 dxdy \\
&+ C \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} r^{-2\beta-2}|u|^2 dxdy.
\end{aligned}$$

Therefore, we conclude that

$$\iint_{B(1/2M^{-1}\beta^{-1/\kappa})} (|\nabla u|^2 + |u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy$$
$$\leq C2^{-2\beta+2}(M\beta^{1/\kappa})^4\beta^2 \times$$
$$\times \iint_{B(2M^{-1}\beta^{-1/\kappa})\setminus B(M^{-1}\beta^{-1/\kappa})} (|\nabla u|^2 + |u|^2 + |\partial_x^2 u|^2 + |\partial_y^2 u|^2)dxdy$$

for any large $\beta \in \mathbb{N} + 1/2$. This gives the conclusion of Proposition 2.4.6. □

Now we recall an estimate in the case of a first order system. Let $P = \partial_x + M(x,y)\partial_y$ where $M(x,y)$ is an $n \times n$ normal matrix with complex entries of the class $X^{1,\kappa}(\Omega)$ and

$$|\mathrm{Im}(\text{eigenvalues of } M(x,y))| \geq \delta.$$

Moreover suppose that all the eigenvalues of $M(0,0)$ are ζ or $\bar{\zeta}$ with a non-real complex number ζ. Then in [9], he proved the following estimate.

Proposition 2.4.7. *(Okaji [9]) For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$C\iint_{\tilde{\Omega}} e^{\beta(\log r)^2}|Pu|^2 r^{-1}dxdy \geq \beta\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy$$

for any $u \in C_0^1(\Omega \setminus \{(0,0)\})$ and any large β.

By Proposition 2.4.7 with $u = r^{-1}|\log r|^{1/2}u$ we have the following estimate.

Lemma 2.4.8. *For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|Pu|^2 r^{-1}dxdy$$
$$\geq C\beta\iint_{\tilde{\Omega}} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy$$

for any $u \in C_0^1(\Omega \setminus \{(0,0)\})$ and any large β.

Thus, we have the following Carleman estimate with a stronger weight function.

Proposition 2.4.9. *For a sufficiently small $\tilde{\Omega}$, there exists a positive C independent of $\tilde{\Omega}$ such that*

$$\int_{\tilde{\Omega}} e^{\beta(\log r)^2}|Lu|^2 r^{-1}dxdy \geq C\beta\iint_{\tilde{\Omega}} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy$$
$$+ C\beta^2\iint_{\tilde{\Omega}} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy$$

for any $u \in C_0^2(\Omega \setminus \{(0,0)\})$ and any large β.

Proof. Putting

$$\tilde{L} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \partial_x + \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix} \partial_y \quad \text{and} \quad U = \begin{pmatrix} u \\ u \end{pmatrix},$$

it follows that

$$\tilde{L}U = \begin{pmatrix} \partial_x u + N(x,y)\partial_y u \\ \partial_x u - N(x,y)\partial_y u \end{pmatrix}, \tilde{L}^2 U = \begin{pmatrix} Lu + A(x,y)u \\ Lu + B(x,y)u \end{pmatrix}$$

where $A(x,y)u = -N_x \partial_y u + N N_y \partial_y u$ and $B(x,y)u = N_x \partial_y u + N N_y \partial_y u$. Since $|A(x,y)u|, |B(x,y)u| \le Cr^{-1}|\nabla u|$ we have

$$\begin{aligned} \int e^{\beta(\log r)^2} |Lu|^2 r^{-1} dxdy \ge C \int e^{\beta(\log r)^2} |\tilde{L}(\tilde{L}U)|^2 r^{-1} dxdy \\ - C \int e^{\beta(\log r)^2} r^{-2} |\nabla u|^2 r^{-1} dxdy. \end{aligned}$$

By Proposition 2.4.7 with $P = \tilde{L}$ and $u = \tilde{L}U$ we have

$$C \iint_{\tilde{\Omega}} e^{\beta(\log r)^2} |\tilde{L}(\tilde{L}U)|^2 r^{-1} dxdy \ge \beta \iint_{\tilde{\Omega}} r^{-2} |\log r| e^{\beta(\log r)^2} |\tilde{L}U|^2 r^{-1} dxdy$$

for a sufficiently small $\tilde{\Omega}$. Moreover applying Lemma 2.4.8 with $P = \tilde{L}$ and $u = U$ we have

$$\begin{aligned} &\iint_{\tilde{\Omega}} r^{-2} |\log r| e^{\beta(\log r)^2} |\tilde{L}U|^2 r^{-1} dxdy \\ &\qquad \ge C\beta \iint_{\tilde{\Omega}} r^{-4} |\log r|^2 e^{\beta(\log r)^2} |U|^2 r^{-1} dxdy. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\iint_{\tilde{\Omega}} r^{-2} |\log r| e^{\beta(\log r)^2} |\tilde{L}U|^2 r^{-1} dxdy \\ &\qquad \ge C \iint_{\tilde{\Omega}} r^{-2} |\log r| e^{\beta(\log r)^2} |\nabla u|^2 r^{-1} dxdy \end{aligned}$$

from

$$\begin{aligned} |\tilde{L}U|^2 &= |\partial_x u + N(x,y)\partial_y u|^2 + |\partial_x u - N(x,y)\partial_y u|^2 \\ &= 2|\partial_x u|^2 + 2|N(x,y)\partial_y u|^2 \\ &\ge 2\min\{1, \delta^2\} |\nabla u|^2. \end{aligned}$$

Thus we obtain the desired estimate in Proposition 2.4.9. □

Theorem 2.2.4 with $a = 1$ follows from Proposition 2.4.6 and 2.4.9.

Proof. Suppose that R_0 is sufficiently small so that Proposition 2.4.9 holds for $\tilde{\Omega} = B(R_0)$. Fix $0 < R_1 < R_0$ and take $\delta > 0$ and a smooth function $\chi_\delta \in C_0^\infty(0, R_0)$ such that

$$\chi_\delta(r) = \begin{cases} 1 & \text{if} \quad \delta \le r \le R_1 \\ 0 & \text{if} \quad r \le \delta/2 \end{cases}, \quad |\chi_\delta'(r)| = \begin{cases} C\delta^{-1} & \text{if} \quad \delta/2 \le r \le \delta \\ C & \text{if} \quad R_1 \le r \le R_0 \end{cases}$$

and

$$|\chi_\delta''(r)| = \begin{cases} C\delta^{-2} & \text{if} \quad \delta/2 \le r \le \delta \\ C & \text{if} \quad R_1 \le r \le R_0 \end{cases}$$

for a positive constant C. By Proposition 2.4.9 it follows that

$$\begin{aligned} & C\beta \iint_{B(R_1)\setminus B(\delta)} r^{-2}|\log r| e^{\beta(\log r)^2} |\nabla u|^2 r^{-1} dxdy \\ & + C\beta^2 \iint_{B(R_1)\setminus B(\delta)} r^{-4}|\log r|^2 e^{\beta(\log r)^2} |u|^2 r^{-1} dxdy \\ & \le C\beta \iint_{B(R_0)} r^{-2}|\log r| e^{\beta(\log r)^2} |\nabla(\chi_\delta u)|^2 r^{-1} dxdy \\ & + C\beta^2 \iint_{B(R_0)} r^{-4}|\log r|^2 e^{\beta(\log r)^2} |\chi_\delta u|^2 r^{-1} dxdy \\ & \le \iint_{B(R_0)} e^{\beta(\log r)^2} |L(\chi_\delta u)|^2 r^{-1} dxdy. \end{aligned}$$

From (2.2.8) we have

$$\begin{aligned} & \iint_{B(R_0)} e^{\beta(\log r)^2} |L(\chi_\delta u)|^2 r^{-1} dxdy \\ & \le C \iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2} (|Lu|^2 + |\nabla u|^2 + |u|^2) r^{-1} dxdy \\ & + \iint_{B(R_1)\setminus B(\delta)} e^{\beta(\log r)^2} |Lu|^2 r^{-1} dxdy \\ & + C \iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2} (|Lu|^2 + \delta^{-2}|\nabla u|^2 + \delta^{-4}|u|^2) r^{-1} dxdy \\ & \le C \iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2} (r^{-2}|\nabla u|^2 + r^{-4}|u|^2) r^{-1} dxdy \\ & + C \iint_{B(R_1)\setminus B(\delta)} e^{\beta(\log r)^2} (r^{-2}|\nabla u|^2 + r^{-4}|u|^2) r^{-1} dxdy \\ & + C \iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2} (\delta^{-2} r^{-2}|\nabla u|^2 + \delta^{-4} r^{-4}|u|^2) r^{-1} dxdy. \end{aligned}$$

Therefore we have

$$\begin{aligned}
&(C\beta - C)\iint_{B(R_1)\setminus B(\delta)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy \\
&+ (C\beta^2 - C)\iint_{B(R_1)\setminus B(\delta)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy \\
&\leq C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2 + r^{-4}|u|^2)r^{-1}dxdy \\
&+ C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(\delta^{-2}r^{-2}|\nabla u|^2 + \delta^{-4}r^{-4}|u|^2)r^{-1}dxdy.
\end{aligned}$$

Since

$$\begin{aligned}
&C\iint_{B(\delta)\setminus B(\delta/2)} e^{\beta(\log r)^2}(\delta^{-2}r^{-2}|\nabla u|^2 + \delta^{-4}r^{-4}|u|^2)r^{-1}dxdy \\
&\quad\leq Ce^{\beta(\log\delta/2)^2}\delta^{-4}\iint_{B(\delta)}|\nabla u|^2dxdy + Ce^{\beta(\log\delta/2)^2}\delta^{-8}\iint_{B(\delta)}|u|^2dxdy,
\end{aligned}$$

this integral tend to zero if $\delta \to 0$ by Proposition 2.4.6. Hence letting δ tend to zero it follows that

$$\begin{aligned}
&(C\beta - C)\iint_{B(R_1)} r^{-2}|\log r|e^{\beta(\log r)^2}|\nabla u|^2 r^{-1}dxdy \\
&+ (C\beta^2 - C)\iint_{B(R_1)} r^{-4}|\log r|^2 e^{\beta(\log r)^2}|u|^2 r^{-1}dxdy \\
&\leq C\iint_{B(R_0)\setminus B(R_1)} e^{\beta(\log r)^2}(r^{-2}|\nabla u|^2 + r^{-4}|u|^2)r^{-1}dxdy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&(C\beta - C)R_1^2|\log R_1|\iint_{B(R_1)}(|\nabla u|^2 + |u|^2)dxdy \\
&\qquad\qquad\leq C\iint_{B(R_0)\setminus B(R_1)}(|\nabla u|^2 + |u|^2)dxdy < \infty.
\end{aligned}$$

Letting β large enough, we have that u is identically zero in $B(R_1)$. By Theorem 2.2.1 with $N_1(x,y) = iN(x,y)$ and $N_2(x,y) = -iN(x,y)$ we have that u is identically zero in Ω. □

Next we prove Theorem 2.2.4 with $a \in \mathbb{R}$.

Proof. Setting $v(x,y) = u(x,ay)$ it follows that

$$\begin{aligned}
&|(\partial_x^2 + a^{-2}N(x,ay)^2\partial_y^2)v(x,y)| = |L(u(x,ay))| \\
&\leq C_0r^{-2}|u(x,ay)| + C_1r^{-1}|(\nabla u)(x,ay)| \\
&\leq C_0r^{-2}|v(x,y)| + C_1r^{-1}\max\{1,|a|^{-1}\}|\nabla v(x,y)|.
\end{aligned}$$

By Theorem 2.2.4 with $a = 1$, v is identically zero in Ω if $C_1 < \min\{1, |a|\}/\sqrt{2}$. Therefore u is identically zero in Ω. $\square$

Finally we shall prove Corollary 2.2.5.

Proof. We define $\sqrt{A(x, y)}$ in the same way as the proof of Corollary 2.2.2. Then $\sqrt{A(x, y)}$ satisfies the assumptions of Theorem 2.2.4 because the eigenvalues of $\sqrt{A(x, y)}$ are $\sqrt{\mu_j(x, y)}$. Hence, by Theorem 2.2.4 with $N(x, y) = \sqrt{A(x, y)}$ the proof is complete. $\square$

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