TRANSIENT WAVES IN LAMINATED COMPOSITES

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Preface

The present dissertation work has been carried out under the direction of Professor T. Hayashi in his laboratory at Osaka University during 1977-1982.

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Transient Waves in Laminated Composites

Contents

ABSTRACT

The transient wave propagation in laminated composites parallel to the layers is treated, and the wave shape modification is investigated by using the diffusing continuum theory. Concerning the transient waves in laminated materials, the far-field solutions were obtained for the elastic laminates by using the head-of-the-pulse approximation. But, the near-field behavior of wave where the wave front changes its shape as it travels has not been investigated. Especially concerning the viscoelastic laminates, no analytical method for the transient waves has been proposed up to the present. In the present analysis, the transient waves in the elastic or viscoelastic laminates are considered, and behaviors of stress waves in the near-field are investigated. In order to obtain an analytical solution in the near-field, the multi-wave-fronts expansion method is proposed which is useful to treat the transient waves with two wave fronts like as the wave in a laminated medium. The near-field solutions for the transient waves in the elastic or viscoelastic laminates are obtained for the cases of the two kinds of compressive boundary conditions. The experimental studies on the transient waves in laminated media are performed, and the comparison between experimental and theoretical results is described. The results show that the theoretical treatment used in the present work can predict all the important qualitative features of the transient wave, and also the quantitative agreement is good enough.

Transient Waves in Laminated Composites

CHAPTER 1

Introduction

Increasing use of composite materials in modern structural elements has recently stimulated considerable interest in the dynamic response of composite materials, and wave propagation problems in those materials are a subject of interest in numerous fields of engineering. It is well known that stress wave in a composite material shows geometric dispersion as a result of scattering induced by the inhomogeneities, and its shape is modified as it propagates through such a medium.

The investigations on stress waves in composite materials can be classified into two kinds of studies, one on a dispersion relation for harmonic waves and the other on a modification of transient wave. In the study on a harmonic wave propagation parallel to the layers in an elastic laminated composite, the exact dispersion relations were obtained by Rytov(1), and by Sun, Achenbach and Herrman(2). Those exact theories, however, are very complicated in mathematical treatment, and so approximate models(2-6) which exhibit geometric dispersion, and are less complex than exact formulations, were proposed by several authors. One analytical model satisfying this criteria is the "diffusing continuum" theory proposed by Bedford and Stern(3). Other models, for example, the "effective stiffness" theory and the "continuum mixture" theory were proposed by Sun, Achenbach and Herrman(2) and by Hegemier, Gurtman and Nayfeh(4), respectively. The dispersion relations for harmonic waves derived from these approximate models were compared with the exact theory and many arguments have been made on the accuracy of the assumption used in these theories.

An analytical study on the transient wave propagation parallel to the layers of a linearly elastic laminated composite was made first by Peck and Gurtman(7) by using the elastic theory, and then Scott(8) developed this approach to a three layered plate. In those studies, they derived the formal solutions by superposition of the infinite series of harmonic waves, and gave the far-field solutions by using the head-of-the-pulse approximation. Their results are valid for waves far from the loaded end where the shape of the wave front becomes stational. The approximate theories mentioned above were also used by Sve and Whittier(9), by Hegemier, Gurtman and Nayfeh(4) and by Hegemier and Bache(10) to solve the transient waye propagation problems in the laminated materials, but they also used the head-of-the-pulse approximation and presented the far-field solusions. Accordingly, the behaviors of the stress waves near the loaded end where the wave front changes its shape as it travels were not discussed in these analyses.

Real composite materials of interest in engineering exhibit marked viscoelastic property. Wave shape modification can also occur as a result of spatial attenuation owing to the viscosity of the material and so in the study on a wave propagation in a viscoelastic composite material, it is necessary to consider not only a geometric dispersion but also a spatial attenuation. Concerning the waves propagting parallel to the layers of the viscoelastic laminates, the dispersion relations for harmonic waves were investigated by Stern, Bedford and Yew(11) and by Tanaka and Imano(12). For the transient waves, however, no analytical method has been reported up to the present.

In this paper, transient wave propagation in an elstic and a viscoelastic laminated composite is treated. These laminated media are composed of an infinite periodic array of two alternating layers which differ in material properties and in thickness. Waves propagating parallel to the layers in semi-infinite laminates loaded by impulsive forces at the end face are analyzed and wave shape modifications near the loaded end are considered.

In order to describe the fundamental equations of motion for the transient wave problem of a laminated medium approximately, the diffusing continuum theory is employed in the present work which was proposed by Bedford and Stern(3) to determine dispersion relations for harmonic waves traveling parallel to the laminates. According to this theory, the propagation process can be described approximately by a set of two-coupled partial differential equations. This approximate theory is reasonably simple and, at the same time, is known to be able to predict fairly accurately the response of harmonic waves in the elastic laminates, and is also applicable to the waves in the viscoelastic laminates.

As an analytical technique to find solutions for transient waves in media, there has been an integral transform method such as Fourier-Laplace transform method. For example, using this method, Payton(13) investigated the dynamic bond stress in a composite structure subjected to a sudden pressure rise. In his analysis, however, the numerical calculations were used because of the complexity of the inverse transformation for the integral solution. Hence, it seems to be difficult to treat the transient waves in viscoelasic laminates by the method. In the present paper, a new approach to the analytical study on transient waves in laminated media is proposed which is based on the wave front expansion technique.

The wave front expansion technique was first introduced by Achenbach and Reddy(14) to find solutions for transient wave with single wave front propagating in a linearly viscoelastic rod. Using this technique, C.T.Sun(15) derived the solutions for a rod of linear Maxwell material or a standard viscoelastic solid, and compared the results with the solutions by the Laplace transfrom technique. In these papers, as an analytical tool, the theory of propagating surfaces of discontinuity was used, and the stress or prticle velocity at an arbitrary location was expressed in the form of a Taylor expansion of time t about the arrival time of the wave front. But the composite materials treated here are composed of two alternating layers which differ in material, and so, for transient wave propagation, two wave fronts will occur in each layer as a result of the differnt natural propagation speeds in the two matrials. Applying the wave front expansion technique to the present problems, the state quantities, such as stress, strain or particle velocity at an arbitrary location of each layer, should be expressed by a superposition of two discontinuous functions which are caused by two wave fronts, and the coupling relations of the discontinuities of two wave fronts of each layer should be determined.

In Chapter 2 and Chapter 3 of this paper, transient waves in the elastic laminated composite are treated, and behaviors of strain waves near the loaded end are discussed. In Chapter 2, analytical method to apply the wave front expansion technique to the problem of transient wave with multi-wave-fronts is presented. By expressing the equations of motion in terms of the strains of two kinds of constituents, stress waves in semi-infinite elastic laminates loaded by a surface pressure at the end are investigated. The solutions are compared with the solutions by conventional Fourier-Laplace transform method. In Chapter 3, analytical treatment derived in Chapter 2 is extended to apply to the velocity boundary condition. The equations of motion are expressed in terms of the particle velocities. As an example, an elastic laminated composite impacted by a rigid body is investigated.

Chapter 4 and Chapter 5 are concerned with the stress waves in viscoelastic laminated composites. In Chapter 4, the transient waves in the semi-infinite viscoelastic laminates loaded by a surface pressure are treated. In Chapter 5, impact problems of viscoelastic laminates and rigid bodies are investigated. The techniques introduced in Chapter 2 and Chapter 3 are extended to analyze the transient waves in such viscoelastic medium which were difficult to treat by the conventional method. Analytical formulations are obtained for the medium of viscoelastic layers that obey the general linear viscoelastic relation. The solutions show a geometric dispersion and a spatial attenuation caused by the viscosity of the material.

In Chapter 6, experimental work on the stress waves in a laminated composite is described. The composite structure used in this experiment is made of layers of aluminum and copper. The strain waves generated by impact are measured by strain gages at the differnt layers in the composite. The analytical calculations are compared with the experimental results. Applicability of the diffusing continuum theory for the transient waves is also discussed.

CHAPTER 2

Transient Waves in Elastic Laminates

2.1 Fundamental Equations

The problem treated is a transient wave propagation along the layers of an elastic laminated composite. The composite is supposed to be an infinite periodic array of two alternating elastic layers which differ in material properties and thickness as shown in Fig.2.1.

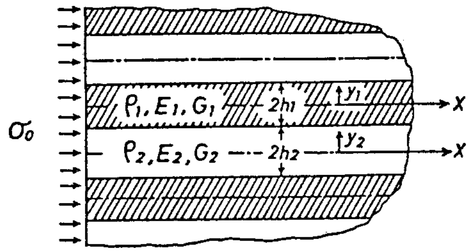


Fig.2.1 Laminated composite

A coordinate system is placed with the xz-plane as the center plane of each layer, the x-axis in the direction of propagation, and y_j-axis (j=1,2) perpendicular to the xz-plane. The composite is semi-infinite, bounded by the x=0 plane, and is suddenly loaded by a surface pressure on the x=0 plane. Assuming that no motion takes place in the z-direction, the problem is a two dimensional one.

It will suffice to examine only one pair of adjacent layers since deformations will be identical in each corresponding layer.

We assume that the equation of motion (x-direction) in each layer is given in the form of the following integral:

$$\rho_j \int_0^{h_j} \frac{\partial^2 U_j}{\partial t^2}\, dy_j = E_j \int_0^{h_j} \frac{\partial^2 U_j}{\partial x^2}\, dy_j + \int_0^{h_j} \frac{\partial \tau_{jxy}}{\partial y_j}\, dy_j , \quad j=1,2 \qquad (2.1)$$

where j=1,2 denotes the layers, and the j-th layer is characterised by density ρ_j, elastic modulus E_j, and layer thickness $2h_j$; $U_j(x,y_j,t)$ and $\tau_{jxy}(x,y_j,t)$ are the displacement in the x-direction and the shear stress, respectively, and t is time.

The average displacement is defined as follows:

$$\bar{U}_j(x,t) = \frac{1}{h_j} \int_0^{h_j} U_j(x, y_j, t)\, dy_j , \qquad j = 1, 2 \qquad (2.2)$$

By use of Eq.(2.2), Eq.(2.1) may be expressed as

$$\rho_j \frac{\partial^2 \bar{U}_j}{\partial t^2} = E_j \frac{\partial^2 \bar{U}_j}{\partial x^2} + \frac{1}{h_j} \Big| \tau_{jxy} \Big|_0^{h_j} , \qquad j = 1, 2 \qquad (2.3)$$

According to the diffusing continuum theory (see APPENDIX A), the second term of Eq.(2.3) can be expressed approximately by

$$\frac{1}{h_j} \Big| \tau_{jxy} \Big|_0^{h_j} \fallingdotseq \frac{B}{h_j} (\bar{U}_2 - \bar{U}_1) \qquad (2.4)$$

where

$$B = 3 G_1 G_2 / (h_1 G_2 + h_2 G_1)$$

Substitution of (2.4) into Eq.(2.3) yields the equations of motion which can be written as follows

$$
\left.\begin{aligned}
\frac{\partial^2 \bar{U}_1}{\partial t^2} &= c_1^2 \frac{\partial^2 \bar{U}_1}{\partial x^2} + \frac{B}{\rho_1 h_1}(\bar{U}_2 - \bar{U}_1) \\
\frac{\partial^2 \bar{U}_2}{\partial t^2} &= c_2^2 \frac{\partial^2 \bar{U}_2}{\partial x^2} + \frac{B}{\rho_2 h_2}(\bar{U}_1 - \bar{U}_2)
\end{aligned}\right\} \qquad (2.5)
$$

where $c_j = (E_j/\rho_j)^{1/2}$, j=1,2, are the elastic-wave speeds in each material. It is clear from Eq.(2.5) that each constituent motion is coupled with a momentum transfer term which depends on the relative displacement only. By introducing the following new variables

$$
u_1 = \bar{U}_1/h_1 , \quad u_2 = \bar{U}_2/h_1 , \quad \xi = x/h_1 , \quad \tau = c_1 t/h_1
$$

Eq.(2.5) can be written in non-dimensional from:

$$
\left.\begin{aligned}
\frac{\partial^2 u_1}{\partial \tau^2} &= \frac{\partial^2 u_1}{\partial \xi^2} + b(u_2 - u_1) \\
\frac{\partial^2 u_2}{\partial \tau^2} &= c^2 \frac{\partial^2 u_2}{\partial \xi^2} + b\zeta^2(u_1 - u_2)
\end{aligned}\right\} \qquad (2.6)
$$

where

$$
b = \frac{h_1 B}{\rho_1 c_1^2} , \quad c = \frac{c_2}{c_1} , \quad \zeta^2 = \frac{\rho_1 h_1}{\rho_2 h_2}
$$

2.2 Integral Solution

In this section, a solution by integral transformations is presented for a transient wave propagating along the layers of a laminated composite. The mathematical treatment described here closely follows that of Payton(13) who developed it to study the dynamic bond stress in a composite structure.

(a) The specific problem treated here is stress waves in an elastic laminated composite subjected to a uniform normal stress of a step-function of time t on the x=0 plane, as shown in Fig.2.1. Assume that the composite is initially undisturbed, then the initial conditions are

$$u_1 = u_2 = \frac{\partial u_1}{\partial \tau} = \frac{\partial u_2}{\partial \tau} = 0 , \qquad at \quad \tau = 0 \qquad (2.7)$$

and the boundary conditions are

$$\frac{\partial u_1}{\partial \xi} = E \frac{\partial u_2}{\partial \xi} = - \varepsilon_0 H(\tau) , \qquad at \quad \xi = 0 \qquad (2.8)$$

$$u_1 = u_2 = \frac{\partial u_1}{\partial \xi} = \frac{\partial u_2}{\partial \xi} = 0 , \qquad at \quad \xi = +\infty \qquad (2.9)$$

where $E=E_2/E_1$, $\varepsilon_0 = \sigma_0/E_1$ and $H(\tau)$ is a Heaviside unit-step function.

In order to obtain an integral solution to the problem, apply the Fourier cosine transform to Eq.(2.6) on a dimensionless distance ξ and Laplace transform on a dimensionless time τ. Denote the Fourier cosine and Laplace transform of $u_j(\xi,\tau)$ as

$u_j^*(k,\tau)$ and $\bar{u}_j(\xi,s)$ respectively. Eliminating the transformed displacement $\bar{u}_2^*(k,s)$ from the transformed equation (2.6), we obtain the transformed displacement $\bar{u}_1^*(k,s)$ in layer 1 as follows

$$\bar{u}_1^*(k,s) = \left(\frac{2}{\pi}\right)^{1/2}\varepsilon_0 \frac{s^2+c^2k^2+b\zeta^2(1+h)}{s\{s^4+[(1+c^2)k^2+(1+\zeta^2)b]s^2+[c^2k^4+bk^2(c^2+\zeta^2)]\}} \qquad (2.10)$$

where $h=h_2/h_1$. Equation (2.10) can be rewritten in the form

$$\bar{u}_1^*(k,s) = \left(\frac{2}{\pi}\right)^{1/2}\varepsilon_0 \frac{s^2+c^2k^2+b\zeta^2(1+h)}{s(s^2+\phi_1^2)(s^2+\phi_2^2)} \qquad (2.11)$$

where

$$\phi_1(k) = \left(\frac{1}{2}\right)^{1/2}[(1+c^2)k^2+(1+\zeta^2)b+\psi(k)]^{1/2} \qquad (2.12)$$

$$\phi_2(k) = \left(\frac{1}{2}\right)^{1/2}[(1+c^2)k^2+(1+\zeta^2)b-\psi(k)]^{1/2} \qquad (2.13)$$

$$\psi(k) = \{[(1-c^2)k^2+(1-\zeta^2)b]^2+4b\zeta^2\}^{1/2} \qquad (2.14)$$

Performing Laplace inversion results in (for $\tau>0$)

$$u_1^*(k,\tau) = \left(\frac{2}{\pi}\right)^{1/2}\varepsilon_0\left[\frac{c^2k^2+b\zeta^2(1+h)}{c^2k^4+(c^2+\zeta^2)bk^2} + \frac{-\phi_1^2+c^2k^2+b\zeta^2(1+h)}{\phi_1^2\,\psi}\cos\phi_1\tau \right.$$
$$\left. - \frac{-\phi_2^2+c^2k^2+b\zeta^2(1+h)}{\phi_2^2\psi}\cos\phi_2\tau\right] \qquad (2.15)$$

Aplication of Fourier cosine inversion formula gives (for $\xi\geq 0$)

$$u_1(\xi,\tau) = \frac{2}{\pi}\varepsilon_0\int_0^\infty \frac{c^2k^2+b\zeta^2(1+h)}{c^2k^4+(c^2+\zeta^2)bk^2}\cos k\xi\,dk + \frac{2}{\pi}\varepsilon_0\int_0^\infty Q(k,\tau)\cos k\xi\,dk \qquad (2.16)$$

where

$$Q(k,\tau) = \frac{-\phi_1^2+c^2k^2+b\zeta^2(1+h)}{\phi_1^2\psi}\cos\phi_1\tau - \frac{-\phi_2^2+c^2k^2+b\zeta^2(1+h)}{\phi_2^2\psi}\cos\phi_2\tau \qquad (2.17)$$

Differentiating (2.16) with respect to a dimensionless distance ξ, we can obtain the strain $\varepsilon_1(\xi,\tau)$ as follows:

$$\varepsilon_1(\xi,\tau)=-\frac{2}{\pi}\varepsilon_0\int_0^\infty\frac{c^2k^2+b\zeta^2(1+h)}{c^2k^4+(c^2+\zeta^2)bk^2}k\cdot\sin k\xi dk-\frac{2}{\pi}\varepsilon_0\int_0^\infty Q(k,\tau)k\cdot\sin k\xi\,dk \tag{2.18}$$

The first term of Eq.(2.18) can be evaluated by residue theory, but the second term is too complicated to evaluate exactly. In order to calculate the second term, we use a numerical approximation as follows: Integration range $(0,\infty)$ of the second term may be separated into two ranges $(0,K)$ and (K,∞) where K is a suitably chosen large number. The integral between $(0,K)$ can be evaluated by Simpson's rule, and the integral between (K,∞) can be estimated asymptoticaly for large K:

$$\begin{aligned}
\varepsilon_1(\xi,\tau)\approx&-\varepsilon_0\Big[\frac{1+h}{1+c^2/\zeta^2}+\Big(1-\frac{1+h}{1+c^2/\zeta^2}\Big)\exp\Big\{-\big[(1+\zeta^2/c^2)b\big]^{1/2}\cdot\xi\Big\}\Big]\\
&-\frac{2\varepsilon_0}{\pi}\int_0^K\Big[\frac{-\phi_1^2+c^2k^2+b\zeta^2(1+h)}{\phi_1^2\psi}\cos\phi_1\tau-\frac{-\phi_2^2+c^2k^2+b\zeta^2(1+h)}{\phi_2^2\psi}\cos\phi_2\tau\Big]k\cdot\sin k\xi\cdot dk\\
&+\frac{\varepsilon_0}{\pi}\Big[\frac{\pi}{2}-S_i[K(\xi+\tau)]+\frac{(\xi-\tau)}{|\xi-\tau|}\Big\{\frac{\pi}{2}-S_i[K\cdot|\xi-\tau|]\Big\}\Big]\\
&-\frac{(1+h)b\zeta^2\varepsilon_0}{2\pi K^2(1-c^2)}\Big\{2\cos K\tau\cdot\sin K\xi-\frac{2}{c^2}\cos cK\tau\cdot\sin K\xi+(\xi+\tau)K\cdot\cos K(\xi+\tau)\\
&+(\xi-\tau)K\cdot\cos K(\xi-\tau)-\frac{1}{c^2}(\xi+c\tau)K\cdot\cos K(\xi+c\tau)-\frac{1}{c^2}(\xi-c\tau)K\cdot\cos K(\xi-c\tau)\\
&-\frac{\pi}{2}\Big[(\xi+\tau)^2\cdot K^2+(\xi-\tau)\cdot|\xi-\tau|\cdot K^2-\frac{1}{c^2}(\xi+c\tau)^2K^2-\frac{1}{c^2}(\xi-c\tau)|\xi-c\tau|\cdot K^2\Big]\\
&+(\xi+\tau)^2K^2S_i[K(\xi+\tau)]+(\xi-\tau)|\xi-\tau|K^2S_i[K|\xi-\tau|]-\frac{1}{c^2}(\xi+c\tau)^2K^2S_i[K(\xi+c\tau)]\\
&-\frac{1}{c^2}(\xi-c\tau)|\xi-c\tau|K^2S_i[K|\xi-c\tau|]\Big\}
\end{aligned}\tag{2.19}$$

where

$$S_i(x) = \text{Sine integral} = \int_0^x \frac{\sin p}{p}\,dp$$

(b) If the stress of the loaded end is given as an arbitrary function of time, the solution can be obtained by superposing solutions of Eq.(2.19). Denote the strain of the loaded end as $f(\tau)$ then using the fundamental solution (2.19), the total strain $\varepsilon_1(\xi,\tau)$ in layer 1 can be written as follows

$$\varepsilon_1(\xi,\tau) = \int_0^\tau \frac{\partial f(\tau')}{\partial \tau'} e_1(\xi,\tau-\tau')\,d\tau' \tag{2.20}$$

where $e_1 = \varepsilon_1(\xi,\tau)/\varepsilon_0$.

In the numerical calculation, following equation expressed in the form of summation is employed

$$\varepsilon_1(\xi,n\Delta\tau) = \sum_{r=0}^{n} \Delta f(r\Delta\tau)\cdot e_1[\,\xi,(n-r)\Delta\tau\,] \tag{2.21}$$

where

$$n = \tau/\Delta\tau \quad , \qquad r = \tau'/\Delta\tau$$

2.3 Wave Front Expansion Analysis

When a transient wave propagates in an elastic laminated composite which is composed of two constituents, each layer supports elastic-wave disturbances traveling at two different speeds, and two wave fronts will occur in each layer, since each layer is forced by the motion of the other through a coupling action of the interface.

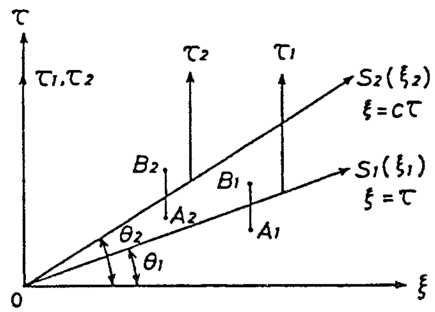


Fig.2.2 ξ-τ diagram

As shown in Fig.2.2, the positions of two wave fronts from the loaded end can be expressed as straight lines S_1 and S_2 in the ξ-τ diagram, and a discontinuity occurs across those lines. Now let us use two sets of skew coordinates S_ℓ-τ_ℓ instead of ξ-τ, and, as the measure variables along the lines S_1 and S_2, introduce ξ_1 and ξ_2 which are projective quantities of the S_1 and S_2 on the τ-axis, respectively. Let θ_ℓ be the angle between line S_ℓ $(\ell=1,2)$ and the ξ-axis. Then the relationships between the two coordinate systems ξ-τ and ξ_ℓ-τ_ℓ are

$$\xi_\ell = \xi \cdot \tan\theta_\ell \ , \qquad \tau_\ell = \tau - \xi \cdot \tan\theta_\ell \tag{2.22}$$

The relations between derivatives of any function $f(\xi,\tau)=f^{(l)}(\xi_l,\tau_l)$ defined in two coordinate systems are expressed as

$$\frac{\partial^n f}{\partial \xi^n}=\tan^n\theta_l\cdot\left(\frac{\partial}{\partial \xi_l}-\frac{\partial}{\partial \tau_l}\right)^n f^{(l)} \quad , \quad \frac{\partial^n f}{\partial \tau^n}=\frac{\partial^n f^{(l)}}{\partial \tau_l^n} \tag{2.23}$$

For the present problem treated here,

$$\tan\theta_1 = 1 \quad , \quad \tan\theta_2 = 1/c$$

Since S_l ($l=1,2$) is a characteristic curve for Eq.(2.6) and indicates the position of the wave front, τ_l is a dimensionless time measured from the arrival time of each wave front.

Equation (2.6) can be expressed in the form of strains by differentiation with respect to a non-dimensional distance ξ:

$$\left.\begin{aligned}\frac{\partial^2 \varepsilon_1}{\partial \tau^2}&=\frac{\partial^2 \varepsilon_1}{\partial \xi^2}+b(\varepsilon_2-\varepsilon_1)\\ \frac{\partial^2 \varepsilon_2}{\partial \tau^2}&=c^2\frac{\partial^2 \varepsilon_2}{\partial \xi^2}+b\zeta^2(\varepsilon_1-\varepsilon_2)\end{aligned}\right\} \tag{2.24}$$

By the use of (2.23), Eq.(2.24) can be expressed in terms of $\xi_l-\tau_l$ as follows

$$\left.\begin{aligned}&\frac{\partial^2 \varepsilon_1^{(1)}}{\partial \xi_1^2}-2\frac{\partial}{\partial \xi_1}\left(\frac{\partial \varepsilon_1^{(1)}}{\partial \tau_1}\right)+b(\varepsilon_2^{(1)}-\varepsilon_1^{(1)})=0\\ &\frac{\partial^2 \varepsilon_2^{(1)}}{\partial \xi_1^2}-2\frac{\partial}{\partial \xi_1}\left(\frac{\partial \varepsilon_2^{(1)}}{\partial \tau_1}\right)+\left(1-\frac{1}{c^2}\right)\left(\frac{\partial^2 \varepsilon_2^{(1)}}{\partial \tau_1^2}\right)+\frac{b\zeta^2}{c^2}(\varepsilon_1^{(1)}-\varepsilon_2^{(1)})=0\end{aligned}\right\} \tag{2.25}$$

or

$$\left.\begin{aligned}&\frac{\partial^2 \varepsilon_1^{(2)}}{\partial \xi_2^2}-2\frac{\partial}{\partial \xi_2}\left(\frac{\partial \varepsilon_1^{(2)}}{\partial \tau_2}\right)+(1-c^2)\left(\frac{\partial^2 \varepsilon_1^{(2)}}{\partial \tau_2^2}\right)+bc^2(\varepsilon_2^{(2)}-\varepsilon_1^{(2)})=0\\ &\frac{\partial^2 \varepsilon_2^{(2)}}{\partial \xi_2^2}-2\frac{\partial}{\partial \xi_2}\left(\frac{\partial \varepsilon_2^{(2)}}{\partial \tau_2}\right)+b\zeta^2(\varepsilon_1^{(2)}-\varepsilon_2^{(2)})=0\end{aligned}\right\} \tag{2.26}$$

To find solutions, we assume that at an arbitrary location of layer j, j=1,2, the strain $\varepsilon_j(\xi,\tau)$ can be expressed as the sum of two kinds of strain $\varepsilon_j^{(1)}(\xi_1,\tau_1)$ and $\varepsilon_j^{(2)}(\xi_2,\tau_2)$ which are caused by each wave front and are discontinuous across the line S_1 and S_2, respectively. Each strain $\varepsilon_j^{(\ell)}(\xi_\ell,\tau_\ell)$, j=1,2, is assumed to be in the form of MacLaurin series of τ_ℓ around the point just after the arrival time of each wave front:

$$\varepsilon_j(\xi,\tau) = \varepsilon_j^{(1)}(\xi_1,\tau_1) + \varepsilon_j^{(2)}(\xi_2,\tau_2) \quad , \quad j=1,2 \tag{2.27}$$

and

$$\varepsilon_j^{(\ell)}(\xi_\ell,\tau_\ell) = \sum_{n=0}^{\infty} A_{j,n}^{(\ell)}(\xi_\ell)\,\frac{\tau_\ell^n}{n!} \qquad \tau_\ell > 0 \;,\; \ell=1,2 \tag{2.28}$$

where $A_{j,n}^{(\ell)}$ is a discontinuos quantity of the n-th time derivative of the strain in layer j across the wave front S_ℓ, and is a continuous function of ξ,

$$A_{j,n}^{(\ell)}(\xi_\ell) = \left[\frac{\partial^n \varepsilon_j^{(\ell)}}{\partial \tau_\ell^n}\right]_{S_\ell} \tag{2.29}$$

where $[F]_{S_\ell}$ denotes the jump quantity of F across the wave line S_ℓ. The strains $\varepsilon_j^{(1)}$ and $\varepsilon_j^{(2)}$ satisfy respectively the system of Eq.(2.25) and (2.26). Substituting (2.28) into Eqs.(2.25) or (2.26) and comparing the coefficients of the same order of dimensionless time τ_ℓ, we obtain the following relations between discontinuities:

$$\left.\begin{aligned} \frac{dA_{1,n+1}^{(1)}}{d\xi_1} &= \alpha_{11}\left(A_{1,n}^{(1)} - A_{2,n}^{(1)}\right) + \alpha_{12}\frac{d^2A_{1,n}^{(1)}}{d\xi_1^2} \\ A_{2,n+2}^{(1)} &= \alpha_{13}\left(A_{1,n}^{(1)} - A_{2,n}^{(1)}\right) + \alpha_{14}\frac{d^2A_{2,n}^{(1)}}{d\xi_1^2} + \alpha_{15}\frac{dA_{2,n+1}^{(1)}}{d\xi_1} \end{aligned}\right\} \tag{2.30}$$

or

$$\left.\begin{aligned} A^{(2)}_{1,n+2} &= \alpha_{23}\,(A^{(2)}_{2,n} - A^{(2)}_{1,n}) + \alpha_{24}\frac{d^2A^{(2)}_{1,n}}{d\xi_2^2} + \alpha_{25}\frac{dA^{(2)}_{1,n+1}}{d\xi_2} \\ \frac{dA^{(2)}_{2,n+1}}{d\xi_2} &= \alpha_{21}\,(A^{(2)}_{2,n} - A^{(2)}_{1,n}) + \alpha_{22}\frac{d^2A^{(2)}_{2,n}}{d\xi_2^2} \end{aligned}\right\} \qquad (2.31)$$

where

$$\alpha_{11} = -b/2,\quad \alpha_{12} = 1/2,\quad \alpha_{13} = b\zeta^2/(1-c^2),\quad \alpha_{14} = c^2/(1-c^2),\quad \alpha_{15} = -2c^2/(1-c^2)$$
$$\alpha_{21} = -b\zeta^2/2,\quad \alpha_{22} = 1/2,\quad \alpha_{23} = -bc^2/(1-c^2).\quad \alpha_{24} = -1/(1-c^2),\quad \alpha_{25} = 2/(1-c^2)$$

Equations (2.30) and (2.31) show that the discontinuity $A^{(\ell)}_{j,n}(\xi_\ell)$ can be expressed in the form of an n-th order polynominal with respect to ξ_ℓ,

$$A^{(\ell)}_{j,n}(\xi_\ell) = \sum_{m=0}^{n} \gamma^{(\ell)}_{j,n,m}\frac{(\alpha_{\ell 1}\cdot\xi_\ell)^m}{m!}, \qquad \ell = 1,2 \qquad (2.32)$$

Substituting (2.32) into Eqs.(2.30) or (2.31), and comparing the terms of the same order of ξ_ℓ, we can derive the recurrent formulas for the coefficients $\gamma^{(\ell)}_{j,n,m}$,

$$\begin{aligned} \gamma^{(1)}_{1,n+1,m+1} &= (\gamma^{(1)}_{1,n,m} - \gamma^{(1)}_{2,n,m}) + \alpha_{11}\alpha_{12}\gamma^{(1)}_{1,n,m+2} \\ \gamma^{(1)}_{2,n+2,m} &= \alpha_{13}(\gamma^{(1)}_{1,n,m} - \gamma^{(1)}_{2,n,m}) + \alpha_{11}(\alpha_{15}\gamma^{(1)}_{2,n+1,m+1} + \alpha_{11}\alpha_{14}\gamma^{(1)}_{2,n,m+2}) \end{aligned} \qquad (2.33)$$

or

$$\begin{aligned} \gamma^{(2)}_{1,n+2,m} &= \alpha_{23}(\gamma^{(2)}_{2,n,m} - \gamma^{(2)}_{1,n,m}) + \alpha_{21}(\alpha_{25}\gamma^{(2)}_{1,n+1,m+1} + \alpha_{21}\alpha_{24}\gamma^{(2)}_{1,n,m+2}) \\ \gamma^{(2)}_{2,n+1,m+1} &= (\gamma^{(2)}_{2,n,m} - \gamma^{(2)}_{1,n,m}) + \alpha_{21}\alpha_{22}\gamma^{(2)}_{2,n,m+2} \end{aligned} \qquad (2.34)$$

From (2.32), the following relation holds for m>n.

$$\gamma^{(\ell)}_{j,n,m} = 0, \quad \text{as} \quad m > n, \quad \ell = 1,2$$

Equations (2.33) and (2.34) are connected with the boundary condition at the loaded end.

By the use of (2.28) and (2.32), strain (2.27) can be expressed as

$$\varepsilon_j(\xi,\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{\gamma_{j,n,m}^{(1)}\frac{(\alpha_{11}\xi_1)^m}{m!}\frac{\tau_1^n}{n!}+\gamma_{j,n,m}^{(2)}\frac{(\alpha_{21}\xi_2)^m}{m!}\frac{\tau_2^n}{n!}\right\} \quad (2.35)$$

At the loaded end $\xi=0$, $\xi_1=\xi_2=0$ and $\tau_1=\tau_2=\tau$, Eq.(2.35) now becomes

$$\varepsilon_j(0,\tau)=\sum_{n=0}^{\infty}(\gamma_{j,n,0}^{(1)}+\gamma_{j,n,0}^{(2)})\frac{\tau^n}{n!} \quad (2.36)$$

If the strain at $\xi=0$ is given in MacLaurin series of τ as follows

$$\varepsilon_j(0,\tau)=\sum_{n=0}^{\infty}a_{j,n}\frac{\tau^n}{n!} \quad (2.37)$$

then, comparing Eqs.(2.36) with (2.37), we obtain

$$a_{j,n}=\gamma_{j,n,0}^{(1)}+\gamma_{j,n,0}^{(2)}, \qquad j=1,2 \quad (2.38)$$

Thus, if the values of the coefficients $\gamma_{1,n,m}^{(2)}$ and $\gamma_{2,n,m}^{(1)}$ for n=0 and n=1, are known, all coefficients are determined successively by using Eqs.(2.33), (2.34) and (2.38). Those values for n=0 and n=1 are obtained as follows; The differential Eqs.(2.25) and (2.26) are integrated with respect to the corresponding time τ_ℓ, starting from a point A_ℓ below the wave line S_ℓ to a point B_ℓ above it, as shown in Fig.2.2. As A_ℓ and B_ℓ approach the wave line S_ℓ, the integral of $\varepsilon_j^{(\ell)}$ vanishes because $\varepsilon_j^{(\ell)}$ is finite, and the integral of $(\partial\varepsilon_j^{(\ell)}/\partial\tau_\ell)$ becomes

$$\lim_{A_\ell\to B_\ell}\int_{A_\ell}^{B_\ell}\frac{\partial\varepsilon_j^{(\ell)}}{\partial\tau_\ell}d\tau_\ell=\lim_{A_\ell\to B_\ell}\left\{\varepsilon_j^{(\ell)}(\xi_\ell,B_\ell)-\varepsilon_j^{(\ell)}(\xi_\ell,A_\ell)\right\}=[\varepsilon_j^{(\ell)}]_{S_\ell}$$

Thus, Eq.(2.25) yields after integration

$$\left.\begin{aligned} &\frac{\partial}{\partial \xi_1}\left[\varepsilon_1^{(1)}\right]_{s_1} = 0 \\ &-2\frac{\partial}{\partial \xi_1}\left[\varepsilon_2^{(1)}\right]_{s_1} + \left(1-\frac{1}{c^2}\right)\left[\frac{\partial \varepsilon_2^{(1)}}{\partial \tau_1}\right]_{s_1} = 0 \end{aligned}\right\} \qquad (2.39)$$

Repeating the same process to the second equation of (2.39), we have

$$\left[\varepsilon_2^{(1)}\right]_{s_1} = 0 \qquad (2.40)$$

and from Eqs.(2.39) and (2.40)

$$\left[\frac{\partial \varepsilon_2^{(1)}}{\partial \tau_1}\right]_{s_1} = 0 \qquad (2.41)$$

In the same way, the following relations are obtained from Eq.(2.26)

$$\frac{\partial}{\partial \xi_2}\left[\varepsilon_2^{(2)}\right]_{s_2} = 0 \quad , \quad \left[\varepsilon_1^{(2)}\right]_{s_2} = 0 \quad , \quad \left[\frac{\partial \varepsilon_1^{(2)}}{\partial \tau_2}\right]_{s_2} = 0 \qquad (2.42)$$

Equations (2.39)-(2.42) imply that the coefficients $\gamma_{j,n,m}^{(\ell)}$ for n=0 and n=1 should be

$$\begin{aligned} &\gamma_{1,0,0}^{(1)} = a_{1,0} \quad , \qquad \gamma_{2,0,0}^{(2)} = a_{2,0} \\ &\gamma_{1,0,0}^{(2)} = \gamma_{1,1,0}^{(2)} = \gamma_{1,1,1}^{(2)} = 0 \, , \qquad \gamma_{2,0,0}^{(1)} = \gamma_{2,1,0}^{(1)} = \gamma_{2,1,1}^{(1)} = 0 \end{aligned} \qquad (2.43)$$

Thus starting from the relations (2.43), the other coefficients are determined one by one.

2.4 Numerical Results

Example 2.1 We consider a semi-infinite elastic laminated composite subjected to a constant stress σ_0 at its end surface abruptly for $\tau>0$. From Eq.(2.36) and (2.37), we obtain the following relations for $\tau>0$:

$$\varepsilon_j(0,\tau) = \sum_{n=0}^{\infty}\left(\gamma_{j,n,0}^{(1)} + \gamma_{j,n,0}^{(2)}\right)\frac{\tau^n}{n!} = \frac{\sigma_0}{E_j}, \quad j = 1,2 \tag{2.44}$$

hence

$$\begin{aligned} &\gamma_{1,0,0}^{(1)} = \sigma_0/E_1, \qquad \gamma_{2,0,0}^{(2)} = \sigma_0/E_2 \\ &\gamma_{j,n,0}^{(1)} + \gamma_{j,n,0}^{(2)} = 0, \quad j = 1,2 \quad (n>0) \end{aligned} \tag{2.45}$$

Using Eqs.(2.33), (2.34) and (2.45), we can determine all coefficients successively.

In order to illustrate our results, let us consider a laminated composite consisting of two alternating layers of aluminum (layer 1) and copper (layer 2). Here, we regard two layers as elastic materials. The relevant material properties are as follows:

Aluminum (Layer 1)	Copper (Layer 2)
$\rho_1 = 0.2755\times10^{-9}$ kg.s^2/mm^4	$\rho_2 = 0.9133\times10^{-9}$ kg.s^2/mm^4
$E_1 = 7200.0$ kg/mm^2	$E_2 = 12000.0$ kg/mm^2
$G_1 = 2700.0$ kg/mm^2	$G_2 = 4200.0$ kg/mm^2

In Fig.2.3 (a),(b), the averaged strain of each layer is plotted aganist x for several values of t for two thickness ratios. One is the same thickness and the other is 1 to 5 for Al to Cu layers. The evaluation of solution (2.35) is performed by using finite terms. For comparison, an integral solution by Fourier and Laplace transformations (section 2.2) is also shown in the same figure. Broken lines in Fig.2.3(a) show the results obtained from the present method with 11-terms. The solution with 19-terms shows good agreement with the integral solution for the whole time range. In Fig.2.3(b), results by the present method with 25-terms are plotted in broken lines. They are in good agreement with integral solutions for time smaller than 40μsec. But for larger time t=50μsec, it is observed that the strain distribution apart from the wave front differs from the integral solution. To obtain better agreement for a lager time range, more terms in the series must be taken.

The results show that the strain distribution changes with time as a result of the energy exchange between two layers due to different natural propagation speeds in different layers and sometimes tensile strain region appears near the wave front. The wave modification in thin layer is more remarkable than in thick layer. It should be noted, however, that the amplitude of inherent wave front never changes in the individual layer.

Example 2.2 In this example we consider a laminated composite subjected to a stress linearly rising from 0 to σ_0 with rise time τ_0. In order to simplify mathematical treatment of the

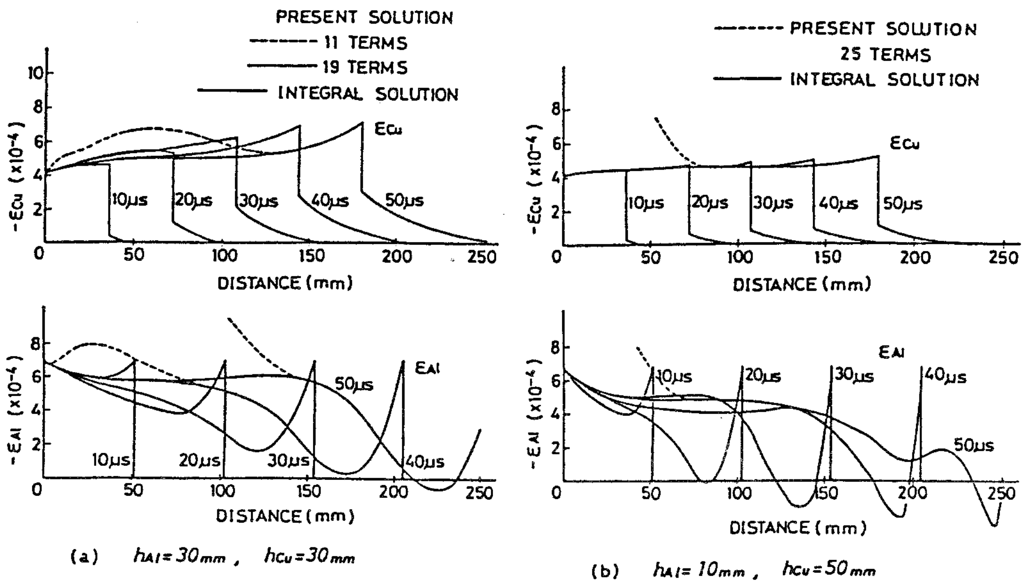


Fig.2.3 Strain distribution in an Al-Cu laminated composite subjected to a constant stress

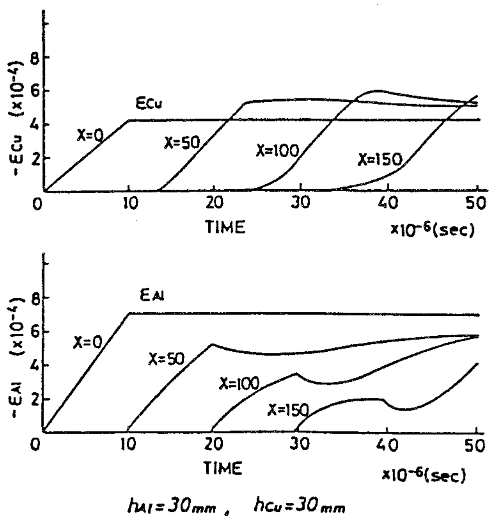


Fig.2.4 Strain distribution in an Al-Cu laminated composite subjected to a linearly rise stress

boundary condition, we consider a superposition of two boundary stresses linearly rising from $\tau=0$ and $\tau=\tau_0$ with equal stress rate of opposite sign. The superposition of the two stresses yields the boudary condition treated here.

For a linearly rising stress from $\tau=0$, Eqs.(2.36) and (2.37) can be written as follows:

$$\varepsilon_j(0,\tau) = \sum_{n=0}^{\infty}\left(\gamma_{j,n,0}^{(1)} + \gamma_{j,n,0}^{(2)}\right)\frac{\tau^n}{n!} = \frac{\sigma_0}{E_j}\frac{\tau}{\tau_0}\,,\quad j=1,2 \tag{2.46}$$

hence we can obtain the following relations

$$\begin{aligned} &\gamma_{j,0,0}^{(1)} + \gamma_{j,0,0}^{(2)} = 0\,, \qquad\qquad j=1,2 \\ &\gamma_{1,1,0}^{(1)} = \sigma_0/\tau_0 E_1\,, \qquad \gamma_{2,1,0}^{(2)} = \sigma_0/\tau_0 E_2 \\ &\gamma_{j,n+2,0}^{(1)} + \gamma_{j,n+2,0}^{(2)} = 0 \end{aligned} \tag{2.47}$$

As an example, the strains are calculated in the same composite as in example 2.1. In Fig.2.4, the strain of each layer is plotted against t for several values of x. The present solution with 19-terms is in good agreement with an integral solution for the whole time range.

From the numerical results, we can predict that the strain in each layer gradually approaches with time to the value $\varepsilon=0.52\times10^{-4}$ calculated by the effective modulus theory.

CHAPTER 3

Impact Problems of Elastic Laminates

3.1 Analysis for the Velocity Boundary Condition

Let us consider the transient wave propagation along layers in an elastic laminated composite which is semi-infinite, bounded by the x=0 plane, and is suddenly struck by a rigid body on the x=0 plane, as illustrated in Fig.3.1. In this case the boundary condition at the sruck end is given in terms of particle velocity.

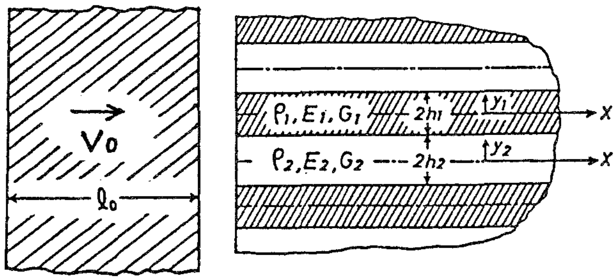


Fig.3.1 Geometry of problem

Equation of motion (2.6) is given in the form of the particle velocities after differentiation with respect to τ:

$$\left.\begin{aligned} \frac{\partial^2 v_1}{\partial \tau^2} &= \frac{\partial^2 v_1}{\partial \xi^2} + b(v_2 - v_1) \\ \frac{\partial^2 v_2}{\partial \tau^2} &= c^2 \frac{\partial^2 v_2}{\partial \xi^2} + b\zeta^2 (v_1 - v_2) \end{aligned}\right\} \tag{3.1}$$

where

$$b = h_1 B / \rho_1 C_1^2, \qquad c = C_2 / C_1, \qquad \zeta^2 = \rho_1 h_1 / \rho_2 h_2$$

By differentiating the Eq.(3.1) n times with respect to dimensionless time τ, and considering the finite discontinuities across the wave front, we obtain the following relations:

$$\left.\begin{aligned}\left[\frac{\partial^{n+2} v_1}{\partial \tau^{n+2}}\right] &= \left[\frac{\partial^{n+2} v_1}{\partial \xi^2 \partial \tau^n}\right] + b\left\{\left[\frac{\partial^n v_2}{\partial \tau^n}\right] - \left[\frac{\partial^n v_1}{\partial \tau^n}\right]\right\}\\ \left[\frac{\partial^{n+2} v_2}{\partial \tau^{n+2}}\right] &= c^2\left[\frac{\partial^{n+2} v_2}{\partial \xi^2 \partial \tau^n}\right] + b\zeta^2\left\{\left[\frac{\partial^n v_1}{\partial \tau^n}\right] - \left[\frac{\partial^n v_2}{\partial \tau^n}\right]\right\}\end{aligned}\right\} \tag{3.2}$$

By the use of (2.23), Eq.(3.2) can be expressed in terms of $\xi_\ell - \tau_\ell$:

$$\left.\begin{aligned}&\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n v_1^{(1)}}{\partial \tau_1^n}\right] - 2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1} v_1^{(1)}}{\partial \tau_1^{n+1}}\right] + b\left\{\left[\frac{\partial^n v_2^{(1)}}{\partial \tau_1^n}\right] - \left[\frac{\partial^n v_1^{(1)}}{\partial \tau_1^n}\right]\right\} = 0\\ &\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n v_2^{(1)}}{\partial \tau_1^n}\right] - 2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1} v_2^{(1)}}{\partial \tau_1^{n+1}}\right] + \left(1 - \frac{1}{c^2}\right)\left[\frac{\partial^{n+2} v_2^{(1)}}{\partial \tau_1^{n+2}}\right] + \frac{b\zeta^2}{c^2}\left\{\left[\frac{\partial^n v_1^{(1)}}{\partial \tau_1^n}\right] - \left[\frac{\partial^n v_2^{(1)}}{\partial \tau_1^n}\right]\right\} = 0\end{aligned}\right\} \tag{3.3}$$

and

$$\left.\begin{aligned}&\frac{\partial^2}{\partial \xi_2^2}\left[\frac{\partial^n v_1^{(2)}}{\partial \tau_2^n}\right] - 2\frac{\partial}{\partial \xi_2}\left[\frac{\partial^{n+1} v_1^{(2)}}{\partial \tau_2^{n+1}}\right] + (1 - c^2)\left[\frac{\partial^{n+2} v_1^{(2)}}{\partial \tau_2^{n+2}}\right] + bc^2\left\{\left[\frac{\partial^n v_2^{(2)}}{\partial \tau_2^n}\right] - \left[\frac{\partial^n v_1^{(2)}}{\partial \tau_2^n}\right]\right\} = 0\\ &\frac{\partial^2}{\partial \xi_2^2}\left[\frac{\partial^n v_2^{(2)}}{\partial \tau_2^n}\right] - 2\frac{\partial}{\partial \xi_2}\left[\frac{\partial^{n+1} v_2^{(2)}}{\partial \tau_2^{n+1}}\right] + b\zeta^2\left\{\left[\frac{\partial^n v_1^{(2)}}{\partial \tau_2^n}\right] - \left[\frac{\partial^n v_2^{(2)}}{\partial \tau_2^n}\right]\right\} = 0\end{aligned}\right\} \tag{3.4}$$

Following the procedure introduced in the previous chapter, the state quantities, such as strain or paticle velocity at an arbitrary location of each layer, can be expressed by a superposition of two discontinuous functions of non-dimensional time τ_ℓ which are caused by two wave fronts:

$$\varepsilon_j(\xi, \tau) = \sum_{n=0}^{\infty} \frac{\tau_1^n}{n!}\left[\frac{\partial^n \varepsilon_j^{(1)}}{\partial \tau_1^n}\right]_{\xi_1} + \sum_{n=0}^{\infty} \frac{\tau_2^n}{n!}\left[\frac{\partial^n \varepsilon_j^{(2)}}{\partial \tau_2^n}\right]_{\xi_2} \tag{3.5}$$

$$v_j(\xi,\tau)=\sum_{n=0}^{\infty}\frac{\tau_1^n}{n!}\left[\frac{\partial^n v_j^{(1)}}{\partial\tau_1^n}\right]_{\xi_1}+\sum_{n=0}^{\infty}\frac{\tau_2^n}{n!}\left[\frac{\partial^n v_j^{(2)}}{\partial\tau_2^n}\right]_{\xi_2} \tag{3.6}$$

Also, as shown in Chapter 2 , the jump quantities which are continuous functions along the wave line can be expressed in the form a polynomial of degree n with respect to ξ_ℓ :

$$\left[\frac{\partial^n v_j^{(\ell)}}{\partial\tau_\ell^n}\right]=\sum_{m=0}^{n}\varphi_{j,n,m}^{(\ell)}\frac{\xi_\ell^m}{m!}\ , \qquad j=1,2\ , \qquad \ell=1,2 \tag{3.7}$$

In order to obtain the strain ε_j as a function of the particle velocity v_j , we introduce the well-known relation given by

$$\frac{\partial\varepsilon_j}{\partial\tau}=\frac{\partial v_j}{\partial\xi} \tag{3.8}$$

Differentiating (3.8) n-1 times with respect to τ, and applying Eq.(2.23), we obtain the following relations:

$$\left[\frac{\partial^n\varepsilon_j^{(1)}}{\partial\tau_1^n}\right]=\frac{\partial}{\partial\xi_1}\left[\frac{\partial^{n-1}v_j^{(1)}}{\partial\tau_1^{n-1}}\right]-\left[\frac{\partial^n v_j^{(1)}}{\partial\tau_1^n}\right] \tag{3.9}$$

$$\left[\frac{\partial^n\varepsilon_j^{(2)}}{\partial\tau_2^n}\right]=\frac{1}{c}\left\{\frac{\partial}{\partial\xi_2}\left[\frac{\partial^{n-1}v_j^{(2)}}{\partial\tau_2^{n-1}}\right]-\left[\frac{\partial^n v_j^{(2)}}{\partial\tau_2^n}\right]\right\} \tag{3.10}$$

By substitution of (3.9) and (3.10) into (3.5), the strain can be expressed as

$$\varepsilon_j(\xi,\tau)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=1}^{2}\frac{1}{c^{(\ell)}}\left\{\varphi_{j,n-1,m+1}^{(\ell)}-\varphi_{j,n,m}^{(\ell)}\right\}\frac{\tau_\ell^n}{n!}\frac{\xi_\ell^m}{m!}\ , \tag{3.11}$$

where

$$c^{(\ell)}=\begin{cases}1 & \text{for } \ell=1\\ c & \text{for } \ell=2\end{cases}$$

Substituting (3.7) into Eqs.(3.3) and (3.4), and comparing the terms of the same order of ξ_ℓ; we can obtain the recurrent formulas for the coefficient $\varphi_{j,n,m}^{(\ell)}$:

$$\left.\begin{aligned} \varphi_{1,n+1,m+1}^{(1)} &= \frac{1}{2}\left\{ \varphi_{1,n,m+2}^{(1)} + b\left(\varphi_{2,n,m}^{(1)} - \varphi_{1,n,m}^{(1)} \right)\right\} \\ \varphi_{2,n+2,m}^{(1)} &= \lambda\left\{ \varphi_{2,n,m+2}^{(1)} - 2\varphi_{2,n+1,m+1}^{(1)} + \frac{b\zeta^2}{c^2}\left(\varphi_{1,n,m}^{(1)} - \varphi_{2,n,m}^{(1)} \right)\right\} \end{aligned}\right\} \tag{3.12}$$

and

$$\left.\begin{aligned} \varphi_{1,n+2,m}^{(2)} &= -\frac{\lambda}{c^2}\left\{ \varphi_{1,n,m+2}^{(2)} - 2\varphi_{1,n+1,m+1}^{(2)} + bc^2\left(\varphi_{2,n,m}^{(2)} - \varphi_{1,n,m}^{(2)} \right)\right\} \\ \varphi_{2,n+1,m+1}^{(2)} &= \frac{1}{2}\left\{ \varphi_{2,n,m+2}^{(2)} + b\zeta^2\left(\varphi_{1,n,m}^{(2)} - \varphi_{2,n,m}^{(2)} \right)\right\} \end{aligned}\right\} \tag{3.13}$$

where $\lambda = c^2/(1-c^2)$.

If the particle velocity at $\xi=0$ is given in a MacLaurin series of τ by

$$V_j(0,\tau) = \sum_{n=0}^{\infty} b_{j,n}\frac{\tau^n}{n!}, \qquad j=1,2 \tag{3.14}$$

then, we obtain the following relation from the boundary conditions (3.14)

$$\varphi_{j,n,0}^{(1)} + \varphi_{j,n,0}^{(2)} = b_{j,n}, \qquad j=1,2, \quad n=0,1,2,\cdots \tag{3.15}$$

In the same way mentioned in Eq.(2.43) the coefficients for n=0 and n=1 are given as

$$\begin{aligned} &\varphi_{1,0,0}^{(1)} = b_{1,0}, \qquad \varphi_{2,0,0}^{(2)} = b_{2,0} \\ &\varphi_{1,0,0}^{(2)} = \varphi_{1,1,0}^{(2)} = \varphi_{1,1,1}^{(2)} = 0, \qquad \varphi_{2,0,0}^{(1)} = \varphi_{2,1,0}^{(1)} = \varphi_{2,1,1}^{(1)} = 0 \end{aligned} \tag{3.16}$$

thus all coeffcents $\varphi_{j,n,m}^{(\ell)}$ are determined successively by using the recurence formulas (3.12) and (3.13) starting from the boundary conditions (3.15) and (3.16).

3.2 Numerical Results

The general solutions for impact problems obtained in the last section will be applied to some specific problems.

Example 3.1 Let us examine a transient wave in a semi-infinite elastic laminated composite struck by a rigid body of mass density ρ_0 and initial velocity v_0 on the x=0 plane. Denote the length of the rigid striker l_0 as shown in Fig.3.1, then its equation of motion is expressed as follows:

$$\frac{\partial v_j}{\partial \tau}(0,\tau) = \frac{\rho_1 h_1^2}{\rho_0 l_0 (h_1 + h_2)} \left\{ \varepsilon_1(0,\tau) + \frac{h_2 E_2}{h_1 E_1} \varepsilon_2(0,\tau) \right\} \tag{3.17}$$

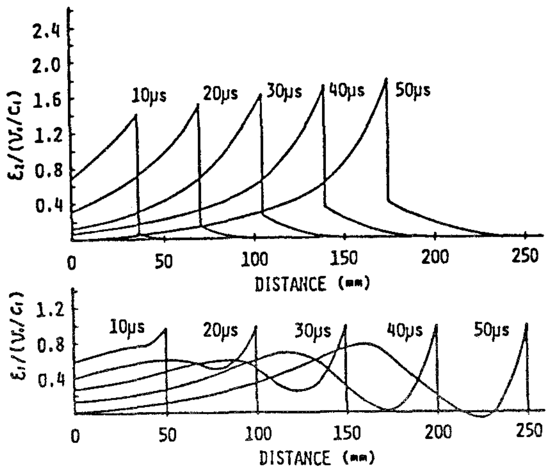
By the use of (3.11), the boundary condition can be expressed as:

$$\varphi^{(1)}_{j,n+1,0} + \varphi^{(2)}_{j,n+1,0} = \frac{\rho_1 h_1^2}{\rho_0 l_0 (h_1 + h_2)} \sum_{l=1}^{2} \frac{1}{C(l)} \left\{ \left(\varphi^{(l)}_{1,n-1,1} - \varphi^{(l)}_{1,n,0} \right) + \frac{h_2 E_2}{h_1 E_1} \left(\varphi^{(l)}_{2,n-1,1} - \varphi^{(l)}_{2,n,0} \right) \right\} \tag{3.18}$$

By employing Eqs.(3.12), (3.13) and (3.18), we can determine all coefficients successively.

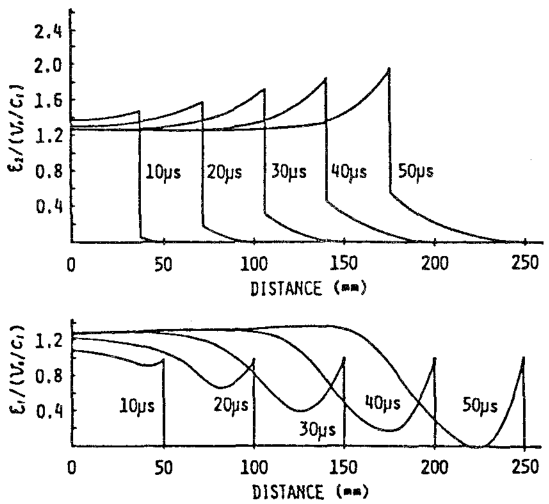
In Fig.3.2 the strain distribution in each layer at various times is illustrated. In the numerical calculation, the laminate properties are chosen as follows:

Layer 1	Layer 2
c_1=5000.0 m/sec	c_2=3500.0 m/sec
E_1=7200.0 kg/mm^2	E_2=12000.0 kg/mm^2
G_1=2700.0 kg/mm^2	G_2=4200.0 kg/mm^2



$h_1=30,\ h_2=30$

Fig.3.2 Strain distribution in an elastic laminated composite struck by a rigid body



$h_1=30,\ h_2=30$

Fig.3.3 Strain distribution in an elastic laminated composite subjected to a constant velocity impact

The calculations are performed by using the first 19 terms of the infinite series of Eq.(3.11). It is obvious that the strain distribution changes its shape with time, in the present case, the strain at the struck end jumps to the value of the wave front immediately after impact and then monotonically decreases to the value zero.

Example 3.2 The second problem concerns the propagation of a compressive wave generated by a constant velocity impact at the boundary of a half space at t=0. Mathematically, this problem can be represented by letting $\ell_0 \to \infty$ in Eq.(3.18), then the new boundary condition becomes

$$\varphi^{(1)}_{j,n+1,0} + \varphi^{(2)}_{j,n+1,0} = 0 \quad , \qquad j = 1,2 \tag{3.19}$$

In Fig.3.3 the strain distribution in the layer at various times is illustrated. The strains are calculated for the same composite as in example 3.1, and the solution (3.11) is approximated by 19 terms. The results show that the strain at the end of each layer, which is larger in layer 2 than in layer 1 immediately after impact, changes rapidly at first and subsequently continues to decrease in layer 2 or increase in layer 1, with diminishing rapidity, and ultimately approaches to the constant same value in both layers 1 and 2 calculated by the effective modulus theory.

CHAPTER 4

Transient Waves in Viscoelastic Laminates

4.1 Fundamental Equations

Consider an infinite periodic array of two alternating linearly viscoelastic layers, perfectly bonded at their interfaces. We will treat the transient wave propagation along the layers which are semi-infinite in length and are suddenly loaded by a surface pressure applied at its end x=0.

For the type of propagation problems being considered, a state of plane strain will be assumed to be in the z-direction. Applying the diffusing continuum theory (see APPENDIX B), the basic equations governing this composite can be written in the form:

(i) Equations of motion

$$\rho_1 \frac{\partial^2 \bar{u}_1}{\partial t^2} = \frac{\partial \bar{\sigma}_1}{\partial x} - \frac{P}{h_1} , \qquad \rho_2 \frac{\partial^2 \bar{u}_2}{\partial t^2} = \frac{\partial \bar{\sigma}_2}{\partial x} + \frac{P}{h_2} \tag{4.1}$$

(i) Constitutive relation

$$\frac{\partial \bar{u}_k}{\partial x} = J_{k0}\, \bar{\sigma}_k(x,t) + \int_{0^+}^{t} J_k^1(t-t')\cdot \bar{\sigma}_k(x,t')\,dt' , \quad k = 1, 2 \tag{4.2}$$

(i) Interaction term

$$P = B_0 \left\{ \bar{u}_1(x,t) - \bar{u}_2(x,t) \right\} + \int_{0^+}^{t} B^1(t-t') \left\{ \bar{u}_1(x,t') - \bar{u}_2(x,t') \right\} dt' \tag{4.3}$$

where

$$J_{k0} = J_k(0) \ , \quad J_k^i(t') = \frac{d^i J_k(t')}{dt'^i} \quad , \ B_0 = B(0) \ , \quad B^i(t') = \frac{d^i B(t')}{dt'^i}$$

In the foregoing the subscript k refers to the k-th laminate, and k-th layer is characterised by density ρ_k, creep function for the axial strain $J_k(t)$, relaxation modulus for the shear stress $G_k(t)$, and thickness $2h_k$. Also $\bar{\sigma}_k$ and $\bar{u}_k$ denote the averaged stress and displacement in x-direction, respectively. In Eq.(4.3), B(t) is a linear viscoelastic relaxation modulus which is a function of $G_k(t)$ and h_k (see APPENDIX B). It is clear from Eqs.(4.1)-(4.3) that the motions of each constituent are coupled by the interaction term which depends on the relative displacement only.

In order to make the equation of motion into non-dimentional form, we introduce the following new variables

$$\xi = x/h_1 \ , \quad \tau = c_1 t/h_1 \ , \quad \sigma_k = \bar{\sigma}_k/\sigma_0 \quad , \quad c = c_2/c_1 \ , \quad h = h_2/h_1$$

where $c_k = (1/\rho_k \cdot J_k)^{1/2}$, k=1,2, are the propagation velocities in each material.

Substituting (4.2) and (4.3) into Eq.(4.1), and expressing the results in terms of dimensionless stress, the equations of motion now becomes:

$$\left.\begin{aligned} \frac{\partial^2 \sigma_1}{\partial \tau^2} + 2\frac{\partial^2}{\partial \tau^2}\int_0^\tau \alpha_1^1(\tau-\tau')\cdot\sigma_1(\xi,\tau')\,d\tau' &= \frac{\partial^2 \sigma_1}{\partial \xi^2} - \psi \\ \frac{\partial^2 \sigma_2}{\partial \tau^2} + 2\frac{\partial^2}{\partial \tau^2}\int_0^\tau \alpha_2^1(\tau-\tau')\cdot\sigma_2(\xi,\tau')\,d\tau' &= c^2\frac{\partial^2 \sigma_2}{\partial \xi^2} + \zeta\cdot\psi \end{aligned}\right\} \qquad (4.4)$$

$$\frac{1}{K}\cdot\psi = \sigma_1(\xi,\tau) - J\cdot\sigma_2(\xi,\tau) + 2\int_0^{\tau}\left\{\alpha_1^1(\tau-\tau')\cdot\sigma_1(\xi,\tau') - J\cdot\alpha_2^1(\tau-\tau')\cdot\sigma_2(\xi,\tau')\right\}d\tau'$$

$$+ 2\int_0^{\tau}\beta^1(\tau-\tau')\cdot\left\{\sigma_1(\xi,\tau') - J\cdot\sigma_2(\xi,\tau')\right\}d\tau'$$

$$+ 4\int_0^{\tau}\beta^1(\tau-\tau')\cdot\int_0^{\tau'}\left\{\alpha_1^1(\tau-\tau'')\cdot\sigma_1(\xi,\tau'') - J\cdot\alpha_2^1(\tau-\tau'')\cdot\sigma_2(\xi,\tau'')\right\}d\tau''d\tau' \tag{4.5}$$

where

$$\psi = \frac{1}{\sigma_0}\frac{\partial P}{\partial \xi}\ ,\quad K = k_1\cdot J_{10}\cdot B_0\ ,\quad \zeta = \frac{c^2}{k}\ ,\quad \alpha_k^i(\tau) = \frac{J_k^i(\tau)}{2J_{k0}}\ ,\quad \beta^i(\tau) = \frac{B^i(\tau)}{2B_0}\ ,\ J = \frac{J_{20}}{J_{10}}$$

Differentiation of Eqs.(4.4) and (4.5) n times with respect to dimensionless time τ yields

$$\left.\begin{aligned}
&\frac{\partial^{n+2}\sigma_1}{\partial\tau^{n+2}} + 2\cdot\sum_{i=1}^{n+2}\alpha_{10}^i\frac{\partial^{n+2-i}\sigma_1}{\partial\tau^{n+2-i}} + 2\cdot\int_0^{\tau}\alpha_1^{n+3}(\tau-\tau')\cdot\sigma_1(\xi,\tau')\,d\tau' = \frac{\partial^{n+2}\sigma_1}{\partial\tau^n\partial\xi^2} - \frac{\partial^n\psi}{\partial\tau^n}\\
&\frac{\partial^{n+2}\sigma_2}{\partial\tau^{n+2}} + 2\cdot\sum_{i=1}^{n+2}\alpha_{20}^i\frac{\partial^{n+2-i}\sigma_2}{\partial\tau^{n+2-i}} + 2\cdot\int_0^{\tau}\alpha_2^{n+3}(\tau-\tau')\cdot\sigma_2(\xi,\tau')\,d\tau' = c^2\frac{\partial^{n+2}\sigma_2}{\partial\tau^n\partial\xi^2} + \zeta\frac{\partial^n\psi}{\partial\tau^n}
\end{aligned}\right\} \tag{4.6}$$

$$\frac{1}{K}\frac{\partial^n\psi}{\partial\tau^n} = \left\{\frac{\partial^n\sigma_1}{\partial\tau^n} - J\cdot\frac{\partial^n\sigma_2}{\partial\tau^n}\right\} + 2\cdot\sum_{i=1}^{n}\left\{\alpha_{10}^i\frac{\partial^{n-i}\sigma_1}{\partial\tau^{n-i}} - J\cdot\alpha_{20}^i\frac{\partial^{n-i}\sigma_2}{\partial\tau^{n-i}}\right\} + 2\cdot\int_0^{\tau}\left\{\alpha_1^{n+1}\sigma_1 - J\cdot\alpha_2^{n+1}\sigma_2\right\}d\tau'$$

$$+ 2\cdot\sum_{i=1}^{n}\beta_0^i\left\{\frac{\partial^{n-i}\sigma_1}{\partial\tau^{n-i}} - J\cdot\frac{\partial^{n-i}\sigma_2}{\partial\tau^{n-i}}\right\} + 2\cdot\int_0^{\tau}\beta^{n+1}\left\{\sigma_1 - J\cdot\sigma_2\right\}d\tau' + 4\cdot\sum_{i=0}^{n}\beta_0^i\cdot\sum_{j=1}^{n-i}\left\{\alpha_{10}^j\frac{\partial^{n-i-j}\sigma_1}{\partial\tau^{n-i-j}} - J\cdot\alpha_{20}^j\frac{\partial^{n-i-j}\sigma_2}{\partial\tau^{n-i-j}}\right\}$$

$$+ 4\cdot\sum_{i=1}^{n}\beta_0^i\cdot\int_0^{\tau}\left\{\alpha_1^{n+1-i}\cdot\sigma_1 - J\cdot\alpha_2^{n+1-i}\cdot\sigma_2\right\}d\tau' + 4\int_0^{\tau}\beta^{n+1}\int_0^{\tau'}\left\{\alpha_1^1\cdot\sigma_1 - J\cdot\alpha_2^1\cdot\sigma_2\right\}d\tau''d\tau' \tag{4.7}$$

Since the integral terms in each equations are continuous across any wave front, Eqs.(4.6) and (4.7) yield the following relations between finite discontinuities across the wave front:

$$\left.\begin{aligned}
&\left[\frac{\partial^{n+2}\sigma_1}{\partial\tau^{n+2}}\right] + 2\sum_{i=1}^{n+2}\alpha_{10}^i\left[\frac{\partial^{n+2-i}\sigma_1}{\partial\tau^{n+2-i}}\right] = \left[\frac{\partial^{n+2}\sigma_1}{\partial\tau^n\partial\xi^2}\right] - \left[\frac{\partial^n\psi}{\partial\tau^n}\right]\\
&\left[\frac{\partial^{n+2}\sigma_2}{\partial\tau^{n+2}}\right] + 2\sum_{i=1}^{n+2}\alpha_{20}^i\left[\frac{\partial^{n+2-i}\sigma_2}{\partial\tau^{n+2-i}}\right] = c^2\left[\frac{\partial^{n+2}\sigma_2}{\partial\tau^n\partial\xi^2}\right] + \zeta\left[\frac{\partial^n\psi}{\partial\tau^n}\right]
\end{aligned}\right\} \tag{4.8}$$

$$\frac{1}{K}\left[\frac{\partial^n \psi}{\partial \tau^n}\right] = \left[\frac{\partial^n \sigma_1}{\partial \tau^n}\right] - J\cdot\left[\frac{\partial^n \sigma_2}{\partial \tau^n}\right] + 2\cdot\sum_{i=1}^{n}\left\{(\alpha_{10}^i+\beta_0^i)\left[\frac{\partial^{n-i}\sigma_1}{\partial \tau^{n-i}}\right] - J\cdot(\alpha_{20}^i+\beta_0^i)\left[\frac{\partial^{n-i}\sigma_2}{\partial \tau^{n-i}}\right]\right\}$$
$$+ 4\cdot\sum_{i=1}^{n}\beta_0^i\sum_{j=1}^{n-i}\left\{\alpha_{10}^j\left[\frac{\partial^{n-i-j}\sigma_1}{\partial \tau^{n-i-j}}\right] - J\cdot\alpha_{20}^j\left[\frac{\partial^{n-i-j}\sigma_2}{\partial \tau^{n-i-j}}\right]\right\} \qquad (4.9)$$

where, finite jumps across the singular plane are denoted by square brackets following the usual convention.

4.2 Analysis

In analysis of transient wave propagation in a laminated medium, we apply the multi-wave-fronts expansion method which is presented in the foregoing chapter. In a laminated composite with two different viscoelastic materials, two wave fronts, i.e., two propagating discontinuities appear in each layer, too.

Let us consider the wave fronts generated by an impulsive surface pressure on the end of laminates. There are two wave fronts, i.e., $\xi=\tau$ and $\xi=c\tau$ in each layer, expressed as straight lines S_1 and S_2 in the $\xi-\tau$. One of them is an inherent wave front, and the other is a wave front produced by an influence of the other layer.

To simplify mathematical treatment of the propagating discontinuity, we introduce two sets of skew coordinates $S_\ell-\tau_\ell$, ($\ell=1,2$), instasd of $\xi-\tau$.

As shown in Chapter 2, the relations between derivatives of any function $f(\xi,\tau)=f^{(\ell)}(\xi_\ell,\tau_\ell)$ defined in two coordinate systems are expressed as follows:

$$\frac{\partial^n f}{\partial \xi^n}=\frac{1}{c^{(\ell)}}\cdot\left(\frac{\partial}{\partial \xi_\ell}-\frac{\partial}{\partial \tau_\ell}\right)^n f^{(\ell)}, \qquad \frac{\partial^n f}{\partial \tau^n}=\frac{\partial^n f^{(\ell)}}{\partial \tau_\ell^n} \tag{4.10}$$

For the problem treated here,

$$c^{(1)}=1 \quad \text{and} \quad c^{(2)}=c$$

To find solutions, we assume that at an arbitrary location of a layer k (k=1,2), the stress $\sigma_k(\xi,\tau)$ can be expressed as the sum of two kinds of stress $\sigma_k^{(1)}(\xi_1,\tau_1)$ and $\sigma_k^{(2)}(\xi_2,\tau_2)$, which are caused by each wave front and are discontinuous across the line S_1 and S_2, respectively.

Using (4.10), and considering that the stress $\sigma_k^{(\ell)}$ is continuous along the line S_ℓ, Eqs.(4.8) and (4.9) can be expressed in terms of $\xi_\ell-\tau_\ell$ as follows:

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n \sigma_1^{(1)}}{\partial \tau_1^n}\right]-2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1}\sigma_1^{(1)}}{\partial \tau_1^{n+1}}\right]-2\sum_{i=1}^{n+2}\alpha_{10}^{i}\left[\frac{\partial^{n+2-i}\sigma_1^{(1)}}{\partial \tau_1^{n+2-i}}\right]-\left[\frac{\partial^n \psi^{(1)}}{\partial \tau_1^n}\right]=0\\
&\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n \sigma_2^{(1)}}{\partial \tau_1^n}\right]-2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1}\sigma_2^{(1)}}{\partial \tau_1^{n+1}}\right]+\left(1-\frac{1}{c^2}\right)\left[\frac{\partial^{n+2}\sigma_2^{(1)}}{\partial \tau_1^{n+2}}\right]-\frac{2}{c^2}\sum_{i=1}^{n+2}\alpha_{20}^{i}\left[\frac{\partial^{n+2-i}\sigma_2^{(1)}}{\partial \tau_1^{n+2-i}}\right]+\frac{\zeta}{c^2}\cdot\left[\frac{\partial^n \psi^{(1)}}{\partial \tau_1^n}\right]=0
\end{aligned}\right\} \tag{4.11}$$

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial \xi_2^2}\left[\frac{\partial^n \sigma_1^{(2)}}{\partial \tau_2^n}\right]-2\frac{\partial}{\partial \xi_2}\left[\frac{\partial^{n+1}\sigma_1^{(2)}}{\partial \tau_2^{n+1}}\right]+(1-c^2)\left[\frac{\partial^{n+2}\sigma_1^{(2)}}{\partial \tau_2^{n+2}}\right]-2c^2\cdot\sum_{i=1}^{n+2}\alpha_{10}^{i}\left[\frac{\partial^{n+2-i}\sigma_1^{(2)}}{\partial \tau_2^{n+2-i}}\right]-c^2\cdot\left[\frac{\partial^n \psi^{(2)}}{\partial \tau_2^n}\right]=0\\
&\frac{\partial^2}{\partial \xi_2^2}\left[\frac{\partial^n \sigma_2^{(2)}}{\partial \tau_2^n}\right]-2\frac{\partial}{\partial \xi_2}\left[\frac{\partial^{n+1}\sigma_2^{(2)}}{\partial \tau_2^{n+1}}\right]-2\cdot\sum_{i=1}^{n+2}\alpha_{20}^{i}\left[\frac{\partial^{n+2-i}\sigma_2^{(2)}}{\partial \tau_2^{n+2-i}}\right]+\zeta\cdot\left[\frac{\partial^n \psi^{(2)}}{\partial \tau_2^n}\right]=0
\end{aligned}\right\} \tag{4.12}$$

$$\begin{aligned}
\frac{1}{K}\left[\frac{\partial^n \psi^{(\ell)}}{\partial \tau_\ell^n}\right]=&\left[\frac{\partial^n \sigma_1^{(\ell)}}{\partial \tau_\ell^n}\right]-J\cdot\left[\frac{\partial^n \sigma_2^{(\ell)}}{\partial \tau_\ell^n}\right]+2\cdot\sum_{i=1}^{n}\left\{(\alpha_{10}^{i}+\beta_0^{i})\left[\frac{\partial^{n-i}\sigma_1^{(\ell)}}{\partial \tau_\ell^{n-i}}\right]-J\cdot(\alpha_{20}^{i}+\beta_0^{i})\left[\frac{\partial^{n-i}\sigma_2^{(\ell)}}{\partial \tau_\ell^{n-i}}\right]\right\}\\
&+4\cdot\sum_{i=1}^{n}\beta_0^{i}\cdot\sum_{j=1}^{n-i}\left\{\alpha_{10}^{j}\left[\frac{\partial^{n-i-j}\sigma_1^{(\ell)}}{\partial \tau_\ell^{n-i-j}}\right]-J\cdot\alpha_{20}^{j}\left[\frac{\partial^{n-i-j}\sigma_2^{(\ell)}}{\partial \tau_\ell^{n-i-j}}\right]\right\}, \quad \ell=1,2
\end{aligned} \tag{4.13}$$

Each stress $\sigma_k^{(\ell)}(\xi_\ell,\tau_\ell)$ ($\ell=1,2$) is also assumed to be expressed in a MacLaurin series of τ_ℓ around a point just after the arrival time of each wave front:

$$\sigma_k(\xi,\tau) = \sigma_k^{(1)}(\xi_1,\tau_1) + \sigma_k^{(2)}(\xi_2,\tau_2)$$

$$= \sum_{n=1}^{\infty} \frac{\tau_1^n}{n!}\left[\frac{\partial^n \sigma_k^{(1)}}{\partial \tau_1^n}\right]_{\xi_1} + \sum_{n=0}^{\infty} \frac{\tau_2^n}{n!}\left[\frac{\partial^n \sigma_k^{(2)}}{\partial \tau_2^n}\right]_{\xi_2}, \qquad k=1,2 \tag{4.14}$$

The discontinuities can be expressed in an n-th order polynominal of ξ_ℓ with a spacial attenuation factor as follows,

$$\left[\frac{\partial^n \sigma_k^{(\ell)}}{\partial \tau_\ell^n}\right] = \sum_{m=0}^{n} \gamma_{k,n,m}^{(\ell)} \frac{\xi_\ell^m}{m!} e^{-\alpha_{\ell 0}^1 \cdot \xi_\ell}, \qquad k=1,2 \ . \ \ell=1,2 \tag{4.15}$$

Using Eqs.(4.14) and (4.15), we can write the solutions in the form

$$\sigma_k(\xi,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{k,n,m}^{(1)} \frac{\tau_1^n}{n!}\frac{\xi_1^m}{m!} e^{-\alpha_{10}^1 \xi_1} + \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{k,n,m}^{(2)} \frac{\tau_2^n}{n!}\frac{\xi_2^m}{m!} e^{-\alpha_{20}^1 \xi_2} \tag{4.16}$$

Substituting (4.15) into Eqs.(4.11)-(4.13), and comparing the terms of the same order of ξ_ℓ, we can obtain the recurrent formulas for the cofficient $\gamma_{k,n,m}^{(\ell)}$:

$$\left.\begin{aligned}
\gamma_{1,n+1,m+1}^{(1)} &= \frac{1}{2}\left\{ \Gamma_{n,m}^{(1)}(\gamma_1) - 2\cdot\sum_{l=1}^{n+2}\alpha_{10}^{l}\cdot\gamma_{1,n+2-l,m}^{(1)} - \gamma_{n,m}^{(1)}(\psi) \right\} \\
\gamma_{2,n+2,m}^{(1)} &= \lambda\left\{ \Gamma_{n,m}^{(1)}(\gamma_2) - 2\cdot\gamma_{2,n+1,m+1}^{(1)} - \frac{2}{c^2}\sum_{i=1}^{n+2}\alpha_{20}^{i}\cdot\gamma_{2,n+2-i,m}^{(1)} + \frac{\zeta}{c^2}\gamma_{n,m}^{(1)}(\psi) \right\}
\end{aligned}\right\} \tag{4.17}$$

$$\left.\begin{aligned}
\gamma_{1,n+2,m}^{(2)} &= -\frac{\lambda}{c^2}\left\{ \Gamma_{n,m}^{(2)}(\gamma_1) - 2\cdot\gamma_{1,n+1,m+1}^{(2)} - 2c^2\sum_{i=1}^{n+2}\alpha_{10}^{i}\cdot\gamma_{1,n+2-i,m}^{(2)} - c^2\gamma_{n,m}^{(2)}(\psi) \right\} \\
\gamma_{2,n+1,m+1}^{(2)} &= \frac{1}{2}\left\{ \Gamma_{n,m}^{(2)}(\gamma_2) - 2\cdot\sum_{i=1}^{n+2}\alpha_{20}^{i}\cdot\gamma_{2,n+2-i,m}^{(2)} + \zeta\cdot\gamma_{n,m}^{(2)}(\psi) \right\}
\end{aligned}\right\} \tag{4.18}$$

$$\Gamma^{(l)}_{n,m}(\gamma_k) = \gamma^{(l)}_{k,n,m+2} - 2\alpha^1_{l0}\gamma^{(l)}_{k,n,m+1} + (\alpha^1_{l0})^2\cdot\gamma^{(l)}_{k,n,m} + 2\alpha^1_{l0}\cdot\gamma^{(l)}_{k,n+1,m}$$

$$\frac{1}{K}\gamma^{(l)}_{n,m}(\psi) = \gamma^{(l)}_{1,n,m} - J\cdot\gamma^{(l)}_{2,n,m} + 2\cdot\sum_{i=1}^{n}\left\{(\alpha^i_{10}+\beta^l_0)\cdot\gamma^{(l)}_{1,n-i,m} - J\cdot(\alpha^i_{20}+\beta^l_0)\cdot\gamma^{(l)}_{2,n-i,m}\right\}$$

$$+4\sum_{i=1}^{n}\beta^l_0\cdot\sum_{j=1}^{n-i}\left\{\alpha^j_{10}\cdot\gamma^{(l)}_{1,n-i-j,m} - J\cdot\alpha^j_{20}\cdot\gamma^{(l)}_{2,n-i-j,m}\right\}, \quad k=1,2, \quad l=1,2 \tag{4.19}$$

where

$$\lambda = c^2/(1-c^2), \quad K = h_1\cdot J_{10}\cdot B_0$$

At the loaded end $\xi=0$, $\xi_1=\xi_2=0$ and $\tau_1=\tau_2=\tau$, and Eq.(4.16) now becomes

$$\sigma_k(0,\tau) = \sum_{n=0}^{\infty}\left\{\gamma^{(1)}_{k,n,0} + \gamma^{(2)}_{k,n,0}\right\}\frac{\tau^n}{n!}, \qquad k=1,2 \tag{4.20}$$

If the stress at $\xi=0$ is given in a MacLaurin series of τ by

$$\sigma_k(0,\tau) = \sum_{n=0}^{\infty} a_{k,n}\frac{\tau^n}{n!}, \qquad k=1,2 \tag{4.21}$$

then, comparing Eq.(4.20) with (4.21), we obtain

$$\gamma^{(1)}_{k,n,0} + \gamma^{(2)}_{k,n,0} = a_{k,n}, \qquad k=1,2 \tag{4.22}$$

Thus, if the values of the coefficents $\gamma^{(2)}_{1,n,m}$ and $\gamma^{(1)}_{2,n,m}$ for n=0 and n=1 are known, all the coefficients can be determined successively by using Eqs.(4.17)-(4.19) and (4.22). Those values for n=0 and n=1 are obtained as follows: Equation (4.4) expressed in terms of $\xi_1-\tau_1$ yields after integration across the wave line S_1:

$$\frac{\partial}{\partial\xi_1}[\sigma_1^{(1)}] + \alpha^1_{10}[\sigma_1^{(1)}] = 0, \qquad -2\frac{\partial}{\partial\xi_1}[\sigma_2^{(1)}] + \left(1-\frac{1}{c^2}\right)\left[\frac{\partial\sigma_2^{(1)}}{\partial\tau_1}\right] - 2\alpha^1_{20}[\sigma_2^{(1)}] = 0 \tag{4.23}$$

Repeating the same process to the second equation of (4.23), we obtain

$$[\sigma_2^{(1)}] = 0 \tag{4.24}$$

and substituting (4.24) into (4.23), we get

$$[\sigma_1^{(1)}] = \gamma_{1,0,0}^{(1)} e^{-\alpha_{10}^1 \xi_1}, \qquad [\sigma_2^{(1)}] = 0 \;, \qquad \left[\frac{\partial \sigma_2^{(1)}}{\partial \tau_1}\right] = 0 \tag{4.25}$$

Similarly, the following relations are obtained from Eq.(4.4) expressed in terms of $\xi_2 - \tau_2$:

$$[\sigma_2^{(2)}] = \gamma_{2,0,0}^{(2)} e^{-\alpha_{20}^1 \xi_2}, \qquad [\sigma_1^{(2)}] = 0 \;, \qquad \left[\frac{\partial \sigma_1^{(2)}}{\partial \tau_2}\right] = 0 \tag{4.26}$$

Equations (4.24)-(4.26) imply that the coefficients $\gamma_{k,n,m}^{(\ell)}$ for n=0 and n=1 should be

$$\gamma_{1,0,0}^{(1)} = a_{1,0} \;, \quad \gamma_{2,0,0}^{(2)} = a_{2,0} \;, \quad \gamma_{1,0,0}^{(2)} = \gamma_{1,1,0}^{(2)} = \gamma_{1,1,1}^{(2)} = 0 \;, \quad \gamma_{2,0,0}^{(1)} = \gamma_{2,1,0}^{(1)} = \gamma_{2,1,1}^{(1)} = 0 \tag{4.27}$$

Thus, starting from Eq.(4.27) and boundary condition (4.22), all coefficients are determined successively by using the recurrent formulas (4.17)-(4.19).

4.3 Viscoelastic Model

We consider a three-parameter solid for a constituent of linearly viscoelastic laminates. The corresponding axial creep function $J_k(t)$ and the relaxation modulus $G_k(t)$ for the shear stress are defined as follows (see APPENDIX C):

$$J_k(t) = \frac{1}{K_k + \frac{2}{3}G_{k0}}\left\{1 - \left(1 - \frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\right)\exp\left(-\frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\frac{G_{k0}}{2\eta_k}t\right)\right\} \qquad (4.28)$$

$$G_k(t) = \frac{G_{k0}}{2} + \frac{G_{k0}}{2}\exp\left(-\frac{G_{k0}}{2\eta_k}t\right) \qquad (4.29)$$

where K_k and η_k are the bulk modulus and the viscosity coefficient of the viscoelastic material, respectively, and $G_{k0}=G_k(0)$.

Differentiating Eq.(4.28) with respect to t, we obtain

$$J_k^i(0) = \frac{d^i J_k(0)}{dt^i} = -\left(1 - \frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\right)\left(-\frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\frac{G_{k0}}{2\eta_k}\right)^i \qquad (4.30)$$

$$\alpha_{k0}^i = \frac{J_k^i(0)}{2J_k(0)} = -\frac{G_{k0}}{3}\left(-\frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\frac{G_{k0}}{2\eta_k}\right)^i = -\frac{G_{k0}}{3}\left(-\frac{3\alpha_{k0}^1}{G_{k0}}\right)^i \qquad (4.31)$$

Applying the Laplace transformation to (4.29), we obtain

$$G_k^*(s) = \frac{1}{2}G_{k0}\left(\frac{1}{s} + \frac{1}{s+\mu_k}\right) = \frac{1}{2}G_{k0}\frac{2s+\mu_k}{s(s+\mu_k)} \qquad (4.32)$$

where

$$\mu_k = \frac{G_{k0}}{2\eta_k} = \frac{K_k + \frac{4}{3}G_{k0}}{K_k + \frac{2}{3}G_{k0}}\frac{3}{G_{k0}}\alpha_{k0}^1$$

Substituting (4.32) into Eq.(B9) (see APPENDIX C) we obtain the interaction coefficient

$$B^*(s)=\frac{3G_1^*(s)\cdot G_2^*(s)}{\hbar_2 G_1^*(s)+\hbar_1 G_2^*(s)}=B_0\,\frac{s^2+es+d}{s(s+\phi_1)(s+\phi_2)} \tag{4.33}$$

where

$$B_0=\frac{3G_{10}G_{20}}{\hbar_1 G_{20}+\hbar_2 G_{10}}, \qquad \phi_1=a+(a^2-b)^{1/2}, \qquad \phi_2=a-(a^2-b)^{1/2}$$

$$a=\frac{1}{4}\,\frac{2\hbar_1\mu_1 G_{20}+2\hbar_2\mu_2 G_{10}+\hbar_1\mu_2 G_{20}+\hbar_2\mu_1 G_{10}}{\hbar_1 G_{20}+\hbar_2 G_{10}},$$

$$b=\frac{1}{2}\mu_1\mu_2, \qquad e=\frac{1}{2}(\mu_1+\mu_2), \qquad d=\frac{1}{4}\mu_1\mu_2$$

Performing the Laplace inversion, we get from Eq.(4.33)

$$B(t)=B_0\left\{\frac{d}{\phi_1\phi_2}+\frac{\phi_1^2-e\phi_1+d}{\phi_1(\phi_1-\phi_2)}\exp(-\phi_1 t)+\frac{\phi_2^2-e\phi_2+d}{\phi_2(\phi_2-\phi_1)}\exp(-\phi_2 t)\right\} \tag{4.34}$$

and differentiating (4.34) with respect to t, we obtain

$$\beta_0^i=\frac{B^i(0)}{2B(0)}=K_1(-\phi_1)^i+K_2(-\phi_2)^i, \qquad i\geq 1 \tag{4.35}$$

where

$$K_1=\frac{1}{2}\,\frac{\phi_1^2-e\phi_1+d}{\phi_1(\phi_1-\phi_2)}, \qquad K_2=\frac{1}{2}\,\frac{\phi_2^2-e\phi_2+d}{\phi_2(\phi_2-\phi_1)}$$

4.4 Numerical Results

Example 4.1 Consider a semi-infinite viscoelastic laminated composite subjected to a constant stress σ_0 (dimensionless stress =1) at its end surface abruptly for $\tau > 0$. From Eqs.(4.20) and (4.21), we obtain the following relations for $\tau > 0$.

$$\sigma_k(0,\tau) = \sum_{n=0}^{\infty} \left\{ \gamma_{k,n,0}^{(1)} + \gamma_{k,n,0}^{(2)} \right\} \frac{\tau^n}{n!} = 1 \ , \qquad k = 1, 2 \tag{4.36}$$

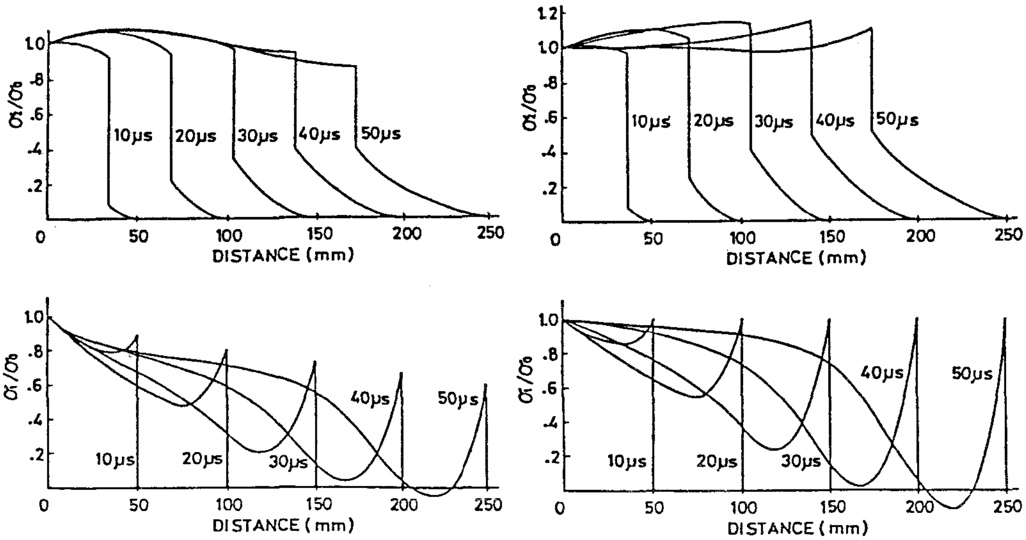
hence

$$\gamma_{1,0,0}^{(1)} = 1 \ , \quad \gamma_{2,0,0}^{(2)} = 1 \ , \quad \gamma_{k,n,0}^{(1)} + \gamma_{k,n,0}^{(2)} = 0 \ , \qquad k = 1, 2 \tag{4.37}$$

Using Eqs.(4.17)-(4.19) and (4.37), we can determine all coefficients successively.

The results are shown in Figs.4.1 (a),(b). The calculations are performed by using first 25 terms of the infinite series of Eq.(4.16). Fig.4.1(a) shows the stress waves in a layered medium composed of two sorts of viscoelastic materials, and in Fig.4.1(b), one layer is assumed to be an elastic. In both figures, the average stress in each layer is plotted against x for several values of t.

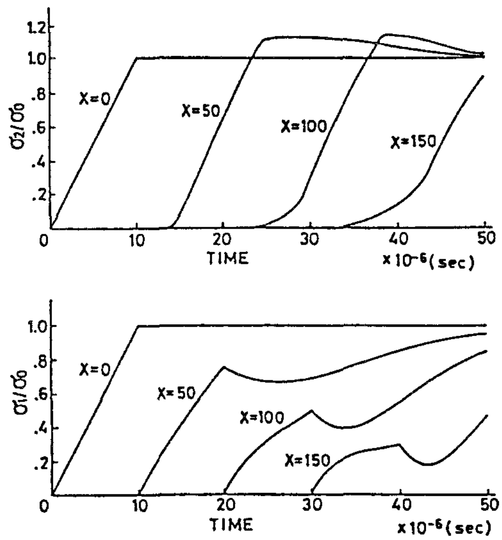
Both results show that the stress wave changes its shape with time. However it should be noted that the decrease of the amplitude of the wave front is owing only to the viscosity of the material, and is not affected by the coupling action through the interface. As shown in Fig.4.1(b) the wave front in elastic layer does not decrease at all.



$h_1=30mm$, $h_2=30mm$ $h_1=30mm$, $h_2=30mm$

(a) Viscoelastic-Viscoelastic (b) Elastic-Viscoelastic

Fig.4.1 Stress distribution in a viscoelastic laminated composite subjected to a constant stress



$h_1=30mm$, $h_2=30mm$

Fig.4.2 Stress distribution in an elastic-viscoelastic laminated composite subjected to a linealy rise stress

Example 4.2 In this example we consider a laminated composite subjected to a stress linearly rising from 0 to σ_0 with rise time τ_0.

In Fig.4.2, the stress of each layer is plotted against t for several values of x. Layer 1 and layer 2 are assumed to be an elastic and a viscoelastic material, respectively. Calculation is perfomed by using first 25 terms of solution (4.16). It is found that the stresses just behind the wave front decrease remarkably with time in the elastic layer with faster propagation velocity, and increase in the viscoelastic layer with slower propagation velocity.

CHAPTER 5

Impact Problems of Viscoelastic Laminates

5.1 Equations of Motion

The problem considered here is to determine the transient stress waves in viscoelastic laminates, caused by an impact of a rigid body. The analytical treatment derived in Chapter 4 is extended to apply to the velocity boundary condition.

For the wave motion along the layers, as shown in Chapter 4, the basic equations derived from the approximate continuum theory can be expressed by Eqs.(4.1)-(4.3). In the present chapter, instead of Eq.(4.2), the following constitutive relation is employed:

$$\bar{\sigma}_k(x,t) = E_{k0}\frac{\partial \bar{u}_k}{\partial x} + \int_{0^+}^{t} E_k^1(t-t')\frac{\partial \bar{u}_k}{\partial x}(x,t')\,dt' \quad , \qquad k = 1,2 \qquad (5.1)$$

where

$$E_{k0} = E_k(0), \qquad E_k^i(t') = \frac{d^i E_k(t')}{dt'^i}$$

Substitution of (5.1) into (4.1) and (4.3), and differentiation of them with respect to t, yield the following equations:

$$\left.\begin{aligned}
\frac{\partial^2 \bar{v}_1}{\partial t^2} &= c_1^2\frac{\partial^2 \bar{v}_1}{\partial x^2} + \frac{c_1^2}{E_{10}}\frac{\partial}{\partial t}\int_0^t E_1^1(t-t')\int_0^{t'}\frac{\partial^2 \bar{v}_1}{\partial x^2}(x,t'')\,dt''\cdot dt' - \frac{1}{\rho_1 h_1}\frac{\partial p}{\partial t}\\
\frac{\partial^2 \bar{v}_2}{\partial t^2} &= c_2^2\frac{\partial^2 \bar{v}_2}{\partial x^2} + \frac{c_2^2}{E_{20}}\frac{\partial}{\partial t}\int_0^t E_2^1(t-t')\int_0^{t'}\frac{\partial^2 \bar{v}_2}{\partial x^2}(x,t'')\,dt''\cdot dt' - \frac{1}{\rho_2 h_2}\frac{\partial p}{\partial t}
\end{aligned}\right\} \qquad (5.2)$$

$$\frac{\partial p}{\partial t} = B_{\circ}\left\{ \bar{v}_1(x,t) - \bar{v}_2(x,t) \right\} - \frac{\partial}{\partial t}\int_0^t B^1(t-t')\int_0^{t'}\left\{ \bar{v}_1(x,t'') - \bar{v}_2(x,t'') \right\} dt''\cdot dt' \quad (5.3)$$

where $v_k(x,t)$ is the particle velocity and $c_k = (E_k/\rho_k)^{1/2}$.

By introducing the following parameters

$$\xi = x/h_1 , \qquad \tau = c_1 t/h_1 , \quad v_k = \bar{v}_k/v_0 , \quad c = c_2/c_1 , \qquad h = h_2/h_1$$

Eqs.(5.2) and (5.3) can be rewritten in the non-dimensional form, where v_0 is an initial velocity of the rigid body. Differentiating them n times with respect to dimensionless time τ, we obtain

$$\left.\begin{aligned} \frac{\partial^{n+2} v_1}{\partial \tau^{n+2}} &= \frac{\partial^{n+2} v_1}{\partial \xi^2 \partial \tau^n} - 2\frac{\partial^{n+1}}{\partial \tau^{n+1}}\int_0^\tau e_1^1(\tau-\tau')\int_0^{\tau'}\frac{\partial^2 v_1}{\partial \xi^2}(\xi,\tau'')\,d\tau''\cdot d\tau' - \frac{\partial^n \psi}{\partial \tau^n} \\ \frac{\partial^{n+2} v_2}{\partial \tau^{n+2}} &= c^2\frac{\partial^{n+2} v_2}{\partial \xi^2 \partial \tau^n} - 2c^2\frac{\partial^{n+1}}{\partial \tau^{n+1}}\int_0^\tau e_2^1(\tau-\tau')\int_0^{\tau'}\frac{\partial^2 v_2}{\partial \xi^2}(\xi,\tau'')\,d\tau''\cdot d\tau' + \gamma\frac{\partial^n \psi}{\partial \tau^n} \end{aligned}\right\} \quad (5.4)$$

$$\frac{1}{b}\frac{\partial^n \psi}{\partial \tau^n} = \frac{\partial^n v_1}{\partial \tau^n} - \frac{\partial^n v_2}{\partial \tau^n} + 2\frac{\partial^{n+1}}{\partial \tau^{n+1}}\int_0^\tau \beta^1(\tau-\tau')\int_0^{\tau'}\left\{ v_1(\xi,\tau'') - v_1(\xi,\tau'') \right\} d\tau''\cdot d\tau' \quad (5.5)$$

where

$$\psi = \frac{1}{\rho_1 c_1 v_0}\frac{\partial p}{\partial \tau} , \qquad b = h_1 B_{\circ}/E_{10} , \qquad \gamma = \frac{\rho_1 h_1}{\rho_2 h_2} ,$$

$$e_k^i(\tau') = -\frac{E_k^i(\tau')}{2E_{k\circ}} , \qquad \beta^i(\tau') = \frac{B^i(\tau')}{2B_{\circ}}$$

5.2 Analysis

According to the analytical techniques proposed in Chapter 3, the state quantities, such as stress or particle velocity at an arbitrary location of each layer, can be expressed by:

$$\sigma_k(\xi,\tau) = \sum_{n=0}^{\infty} \frac{\tau_1^n}{n!}\left[\frac{\partial^n \sigma_k^{(1)}}{\partial \tau_1^n}\right]_{\xi_1} + \sum_{n=0}^{\infty} \frac{\tau_2^n}{n!}\left[\frac{\partial^n \sigma_k^{(2)}}{\partial \tau_2^n}\right]_{\xi_2} \tag{5.6}$$

$$v_k(\xi,\tau) = \sum_{n=0}^{\infty} \frac{\tau_1^n}{n!}\left[\frac{\partial^n v_k^{(1)}}{\partial \tau_1^n}\right]_{\xi_1} + \sum_{n=0}^{\infty} \frac{\tau_2^n}{n!}\left[\frac{\partial^n v_k^{(2)}}{\partial \tau_2^n}\right]_{\xi_2} \tag{5.7}$$

where

$$\sigma_k = \bar{\sigma}_k / \rho_1 c_1 v_0$$

Also, as shown in Chapter 4, the propagating discontitnuities in the viscoelastic materials can be expressed as follows:

$$\left[\frac{\partial^n v_k^{(l)}}{\partial \tau_l^n}\right] = \sum_{m=0}^{n} \varphi_{k,n,m}^{(l)} \cdot \frac{\xi_l^m}{m!} \exp(-e'_{l0}\xi_l), \quad k=1,2, \quad l=1,2 \tag{5.8}$$

In order to express the stress (5.6) as a function of the particle velocity v_k, we rewrite Eq.(5.1) as follows:

$$\frac{1}{\delta_k}\frac{\partial \sigma_k}{\partial \tau} = \frac{\partial v_k}{\partial \xi} - 2\frac{\partial}{\partial \tau}\int_0^{\tau} e'_k(\tau-\tau')\int_0^{\tau'} \frac{\partial v_k}{\partial \xi}(\xi,\tau'')\,d\tau''\,d\tau' \tag{5.9}$$

where

$$\delta_k = E_{k0} / E_{10}$$

After differentiating (5.9) n-1 times with respect to τ and considering the relation (4.10), we obtain the following relation:

$$\frac{c^{(\ell)}}{\delta_k}\frac{\partial^n \sigma_k^{(\ell)}}{\partial \tau_\ell^n} = \frac{\partial^n v_k^{(\ell)}}{\partial \xi_\ell \partial \tau^{n-1}} - \frac{\partial^n v_k^{(\ell)}}{\partial \tau_\ell^n} - 2\cdot\sum_{i=1}^{n-1} e_{k0}^i \frac{\partial^{n-i} v_k^{(\ell)}}{\partial \xi_\ell \partial \tau^{n-1-i}} - 2e_{k0}^n \int_{-\xi_\ell}^{\tau_\ell} \frac{\partial v_k^{(\ell)}}{\partial \xi_\ell} d\tau_\ell'$$

$$+2\sum_{i=1}^{n} e_{k0}^i \frac{\partial^{n-i} v_k^{(\ell)}}{\partial \tau_\ell^{n-i}} + 2\int_{-\xi_\ell}^{\tau_\ell} e_k^{n+1}(\tau_\ell - \tau_\ell') v_k^{(\ell)} d\tau_\ell' \qquad (5.10)$$

where

$$c^{(\ell)} = \begin{cases} 1 & \text{for } \ell = 1 \\ c & \text{for } \ell = 2 \end{cases}$$

Since the integral is continuous across the wave line, Eq.(5.10) yields the following relation between finite discontinuities:

$$\frac{c^{(\ell)}}{\delta_k}\left[\frac{\partial^n \sigma_k^{(\ell)}}{\partial \tau_\ell^n}\right] = \frac{\partial}{\partial \xi_\ell}\left[\frac{\partial^{n-1} v_k^{(\ell)}}{\partial \tau_\ell^{n-1}}\right] - \left[\frac{\partial^n v_k^{(\ell)}}{\partial \tau_\ell^n}\right] - 2\sum_{i=1}^{n-1} e_{k0}^i \frac{\partial}{\partial \xi_\ell}\left[\frac{\partial^{n-1-i} v_k^{(\ell)}}{\partial \tau_\ell^{n-1-i}}\right] + 2\sum_{i=1}^{n} e_{k0}^i \left[\frac{\partial^{n-i} v_k^{(\ell)}}{\partial \tau_\ell^{n-i}}\right] \qquad (5.11)$$

By employing (5.11) and (5.8), Eq.(5.6) becomes

$$\sigma_k(\xi,\tau) = \sum_{\ell=1}^{2} \frac{\delta_k}{c^{(\ell)}} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \Big\{ \varphi_{k,n-1,m+1}^{(\ell)} - e_{\ell 0}^1 \varphi_{k,n-1,m}^{(\ell)} - \varphi_{k,n,m}^{(\ell)}$$

$$-2\sum_{i=1}^{n-1} e_{k0}^i (\varphi_{k,n-1-i,m+1}^{(\ell)} - e_{\ell 0}^1 \varphi_{k,n-1-i,m}^{(\ell)})$$

$$+2\sum_{i=1}^{n} e_{k0}^i \cdot \varphi_{k,n-i,m}^{(\ell)} \Big\} \frac{\xi_\ell^m}{m!}\frac{\tau_\ell^n}{n!} \exp(-e_{\ell 0}^1 \xi_\ell) \qquad (5.12)$$

By the use of (4.10), and the consideration of discontinuities across the wave front, Eqs.(5.4) and (5.5) can be expressed in terms of $\xi_\ell - \tau_\ell$ as follows:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n v_1^{(1)}}{\partial \tau_1^n}\right] - 2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1} v_1^{(1)}}{\partial \tau_1^{n+1}}\right] = 2[\Phi_1^{(1)}] + \left[\frac{\partial^n \psi^{(1)}}{\partial \tau_1^n}\right] \\ &\frac{\partial^2}{\partial \xi_1^2}\left[\frac{\partial^n v_2^{(1)}}{\partial \tau_1^n}\right] - 2\frac{\partial}{\partial \xi_1}\left[\frac{\partial^{n+1} v_2^{(1)}}{\partial \tau_1^{n+1}}\right] + \left(1-\frac{1}{c^2}\right)\left[\frac{\partial^{n+2} v_2^{(1)}}{\partial \tau_1^{n+2}}\right] = 2[\Phi_2^{(1)}] - \frac{g}{c^2}\left[\frac{\partial^n \psi^{(1)}}{\partial \tau_1^n}\right] \end{aligned}\right\} \qquad (5.13)$$

$$\left.\begin{aligned}&\frac{\partial^2}{\partial\xi_2^2}\left[\frac{\partial^n U_1^{(2)}}{\partial\tau_2^{\,n}}\right]-2\frac{\partial}{\partial\xi_2}\left[\frac{\partial^{n+1}U_1^{(2)}}{\partial\tau_2^{\,n+1}}\right]-(1-c^2)\left[\frac{\partial^{n+2}U_1^{(2)}}{\partial\tau_2^{\,n+2}}\right]=2\left[\Phi_1^{(2)}\right]+c^2\left[\frac{\partial^n\psi^{(2)}}{\partial\tau_2^{\,n}}\right]\\&\frac{\partial^2}{\partial\xi_2^2}\left[\frac{\partial^n U_2^{(2)}}{\partial\tau_2^{\,n}}\right]-2\frac{\partial}{\partial\xi_2}\left[\frac{\partial^{n+1}U_2^{(2)}}{\partial\tau_2^{\,n+1}}\right]=2\left[\Phi_2^{(2)}\right]-g\left[\frac{\partial^n\psi^{(2)}}{\partial\tau_2^{\,n}}\right]\end{aligned}\right\}\tag{5.14}$$

$$\left[\Phi_k^{(\ell)}\right]=\sum_{i=1}^{n}e_{k0}^{i}\,\frac{\partial^2}{\partial\xi_\ell^2}\left[\frac{\partial^{n-i}U_k^{(\ell)}}{\partial\tau_\ell^{\,n-i}}\right]-2\sum_{i=1}^{n+1}e_{k0}^{i}\,\frac{\partial}{\partial\xi_\ell}\left[\frac{\partial^{n+1-i}U_k^{(\ell)}}{\partial\tau_\ell^{\,n+1-i}}\right]-\sum_{i=1}^{n+2}e_{k0}^{i}\left[\frac{\partial^{n+2-i}U_k^{(\ell)}}{\partial\tau_\ell^{\,n+2-i}}\right]$$

$$\frac{1}{b}\left[\frac{\partial^n\psi^{(\ell)}}{\partial\tau_\ell^{\,n}}\right]=\left[\frac{\partial^n U_1^{(\ell)}}{\partial\tau_\ell^{\,n}}\right]-\left[\frac{\partial^n U_2^{(\ell)}}{\partial\tau_\ell^{\,n}}\right]+2\sum_{i=1}^{n}\beta_0^{i}\left\{\left[\frac{\partial^{n-i}U_1^{(\ell)}}{\partial\tau_\ell^{\,n-i}}\right]-\left[\frac{\partial^{n-i}U_2^{(\ell)}}{\partial\tau_\ell^{\,n-i}}\right]\right\}\tag{5.15}$$

Substitution of (5.8) into Eqs.(5.13)-(5.15) yields the recurrent formulas for the coefficient $\varphi_{k,n,m}^{(\ell)}$ as follows:

$$\left.\begin{aligned}&\varphi_{1,n+1,m+1}^{(1)}=\frac{1}{2}\left\{\phi_{n,m}^{(1)}(\varphi_1)-2\cdot\phi_{n,m}^{(1)}(\Phi_1)-\phi_{n,m}^{(1)}(\psi)\right\}\\&\varphi_{2,n+2,m}^{(1)}=\lambda\left\{\phi_{n,m}^{(1)}(\varphi_2)-2\cdot\varphi_{2,n+1,m+1}^{(1)}-2\cdot\phi_{n,m}^{(1)}(\Phi_2)+\frac{g}{c^2}\phi_{n,m}^{(1)}(\psi)\right\}\end{aligned}\right\}\tag{5.16}$$

$$\left.\begin{aligned}&\varphi_{1,n+2,m}^{(2)}=-\frac{\lambda}{c^2}\left\{\phi_{n,m}^{(2)}(\varphi_1)-2\varphi_{1,n+1,m+1}^{(2)}-2\,\phi_{n,m}^{(2)}(\Phi_1)-c^2\phi_{n,m}^{(2)}(\psi)\right\}\\&\varphi_{2,n+1,m+1}^{(2)}=\frac{1}{2}\left\{\phi_{n,m}^{(2)}(\varphi_2)-2\cdot\phi_{n,m}^{(2)}(\Phi_2)+g\,\phi_{n,m}^{(2)}(\psi)\right\}\end{aligned}\right\}\tag{5.17}$$

$$\phi_{n,m}^{(\ell)}(\varphi_k)=\varphi_{k,n,m+2}^{(\ell)}-2\,e_{\ell0}^{1}\cdot\varphi_{k,n,m+1}^{(\ell)}+(e_{\ell0}^{1})^2\cdot\varphi_{k,n,m}^{(\ell)}+2\,e_{\ell0}^{1}\,\varphi_{k,n+1,m}^{(\ell)}$$

$$\begin{aligned}\phi_{n,m}^{(\ell)}(\Phi_k)=&\sum_{i=1}^{n}e_{k0}^{i}\left\{\varphi_{k,n-i,m+2}^{(\ell)}-2\cdot e_{\ell0}^{1}\cdot\varphi_{k,n-i,m+1}^{(\ell)}+(e_{\ell0}^{1})^2\cdot\varphi_{k,n-i,m}^{(\ell)}\right\}\\&-2\cdot\sum_{i=1}^{n+1}e_{k0}^{i}\left\{\varphi_{k,n+1-i,m+1}^{(\ell)}-e_{\ell0}^{1}\cdot\varphi_{k,n+1-i,m}^{(\ell)}\right\}+\sum_{i=1}^{n+2}e_{k0}^{i}\cdot\varphi_{k,n+2-i,m}^{(\ell)}\end{aligned}$$

$$\frac{1}{b}\cdot\phi_{n,m}^{(\ell)}(\psi)=\varphi_{1,n,m}^{(\ell)}-\varphi_{2,n,m}^{(\ell)}+2\cdot\sum_{i=1}^{n}\beta_0^{i}\left\{\varphi_{1,n-i,m}^{(\ell)}-\varphi_{2,n-i,m}^{(\ell)}\right\}\tag{5.18}$$

where

$$\lambda = c^2/(1-c^2)\ , \qquad b = h_1 \cdot B_0 / E_{10}$$

If the particle velocity at $\xi=0$ is given by

$$V_k(0,\tau) = \sum_{n=0}^{\infty} b_{k,n} \frac{\tau^n}{n!}\ , \qquad k=1,2 \tag{5.19}$$

then, we obtain as boundary conditions as follows

$$\varphi^{(1)}_{k,n,0} + \varphi^{(2)}_{k,n,0} = b_{k,n}\ , \qquad k=1,2 \tag{5.20}$$

Applying the same procedure introduced in section 4.2, we can determine the lowest coefficients as follows:

$$\begin{aligned} &\varphi^{(1)}_{1,0,0} = b_{1,0}\ , \qquad \varphi^{(2)}_{2,0,0} = b_{2,0} \\ &\varphi^{(2)}_{1,0,0} = \varphi^{(2)}_{1,1,0} = \varphi^{(2)}_{1,1,1} = 0\ , \qquad \varphi^{(1)}_{2,0,0} = \varphi^{(1)}_{2,1,0} = \varphi^{(1)}_{2,1,1} = 0 \end{aligned} \tag{5.21}$$

Thus, starting from Eqs.(5.20) and (5.21), all coefficients are obtained successively one by one.

5.3 Numerical Results

In the present section, specific problems of longitudinal impulsive motion of viscoelastic laminates subjected to an impact are treated by the methods previously developed.

Example 5.1 Let us consider the longitudinal compressive waves in a laminated medium struck on the contact plane x=0 by a rigid body with finite length l_0 and initial velocity v_0. In the analysis, the perfectly plane contact is assumed. Then, the equation of motion of the rigid body whose mass density is designated by ρ_0 is expressed as follows:

$$\frac{\partial U_k}{\partial \tau}(0,\tau) = \frac{\rho_1 h_1^2}{\rho_0 l_0 (h_1 + h_2)} \left\{ \sigma_1(0,\tau) + \frac{h_2}{h_1} \sigma_2(0,\tau) \right\} \tag{5.22}$$

By the use of Eq.(5.12), the boundary condition (5.22) can be expressed as

$$\varphi^{(1)}_{k,n+1,0} + \varphi^{(2)}_{k,n+1,0} = \frac{\rho_1 h_1^2}{\rho_0 l_0 (h_1 + h_2)} \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\delta_k h^{(k)}}{C^{(l)}} \Big\{ \varphi^{(l)}_{k,n-1,1} - e^1_{l0} \varphi^{(l)}_{k,n-1,0}$$

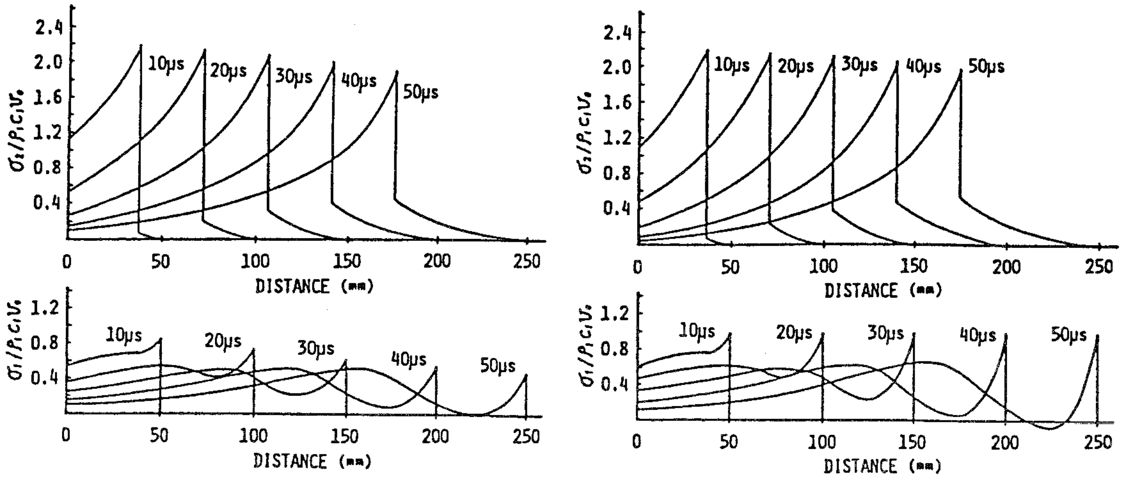
$$- \varphi^{(l)}_{k,n,0} - 2 \sum_{i=1}^{n-1} e^i_{k0} (\varphi^{(l)}_{k,n-1-i,1} - e^1_{l0} \cdot \varphi^{(l)}_{k,n-1-i,0}) + 2 \sum_{i=1}^{n} e^i_{k0} \varphi^{(l)}_{k,n-i,0} \Big\} \tag{5.23}$$

where

$$\delta_k = E_{k0}/E_{10}, \qquad h^{(k)} = h_k/h_1, \qquad C^{(l)} = C_l/C_1, \qquad k=1,2, \; l=1,2$$

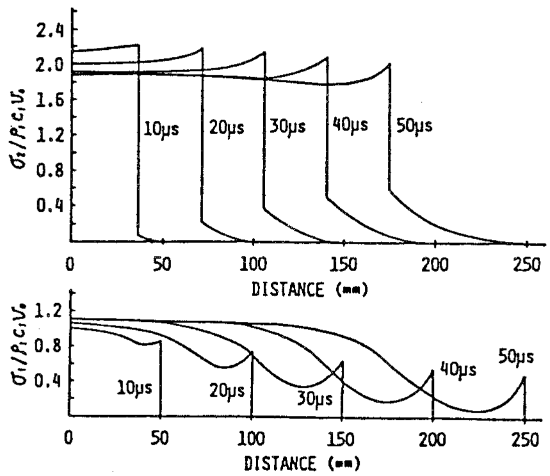
Using Eq.(5.23) and (5.16)-(5.18), all coefficients are obtained successively.

In Figs.5.1 (a),(b), the average stress in each layer of



(a) Viscoelastic-Viscoelastic $h_1=30$, $h_2=30$ (b) Elastic-Viscoelastic

Fig.5.1 Stress distribution in a viscoelastic laminated composite struck by a rigid body



$h_1=30$, $h_2=30$

Fig.5.2 Stress distribution in a viscoelastic laminated compostie subjected to a constant velocity impact.

laminated composite struck by a rigid body is plotted aganist x for several values of t. Fig.(a) is the case of two alternating viscoelastic layers, and Fig.(b) is the case of an elastic (layer 1) and viscoelastic material (layer 2). The calculation is performed by using first 25-terms of the infinite series (5.12).

Example 5.2 We now turn our attention to the stress waves in semi-infinite viscoelastic laminates due to a constant velocity impact, i.e., the velocity of the impact at the end of the laminates is kept constant. In this case, the corresponding boundary condition is obtained by letting $\ell_0 \to \infty$ in Eq.(5.23)

$$\varphi^{(1)}_{k,n+1,0} + \varphi^{(2)}_{k,n+1,0} = 0 \qquad (5.24)$$

The stress waves due to a constant velocity impact are plotted in Fig.5.2 aganist x for several values of t. Here again the composite used in Fig.5.1(a) is employed. Calculations are performed by using first 25-terms of infinite series (5.12).

The results show that, in the viscoelastic layers considered here, the wave front propagates with constant velocity and the magnitude of wave front decreases exponentially with time t. It is found that, in the viscoelastic laminates, wave shape modification occurs as the results of a geometric dispersion and a spacial attenuation.

CHAPTER 6

Experiments of Transitent Waves

In this chapter, experimental studies on a transient wave propagation in an elastic laminated structure are presented, and the strain wave modification along the layers observed experimentally is compared with the theoretical calculation mentioned in Chapter 2.

6.1. Experiments

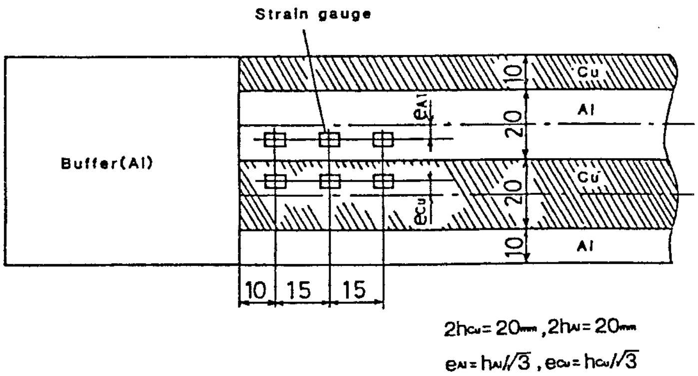


Fig.6.1 Specimen

The composite structure used in this experiments is composed of 4 plates of aluminum and copper. Each plate was pressure bonded to each other by using epoxy resin. The dimensions of specimen are shown in Fig.6.1. In order to realize the assumption of the infinite periodic array of layers approximately, the outer layer with half thickness of the corresponding inner layer is bonded on both sides of the specimen.

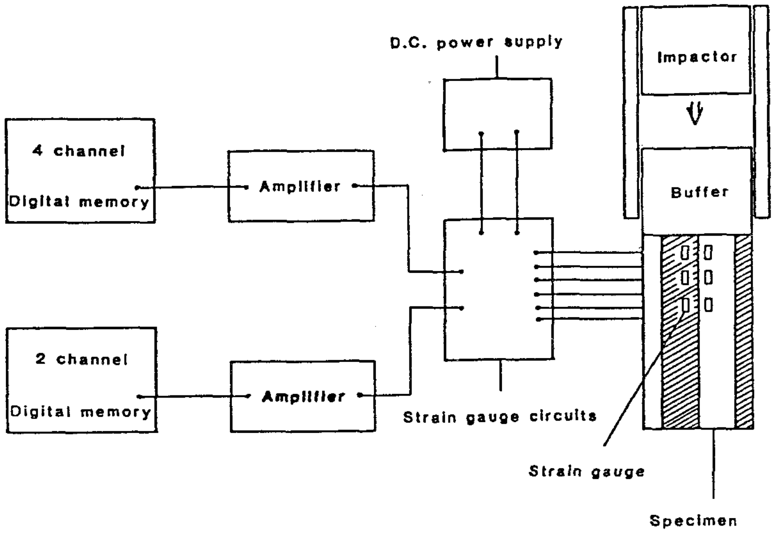


Fig.6.2 Experimental apparatus

The apparatus used to generate and observe the propagating strain pulse is shown in Fig.6.2. In this experiments, the aluminum buffer is attached at the top of the specimen as shown in Fig.6.1, and the strain pulse is generated by the impact of the aluminum striker on the buffer. Through the buffer with 120mm length, the shape of the transmitted wave front is corrected to be uniform and the plane wave enters the specimen. The strain waves in each layer of the composite are measured by strain gauges. In the present experiments, the strain gauges were bonded at the position of $h_j/\sqrt{3}$ ($2h_j$= thickness of layer) from the center line of each layer as shown in Fig.6.1. It is because that, according to the diffusing continuum theory (see APPENDIX A), strain at this point is equal to the average strain in the cross section of the layer.

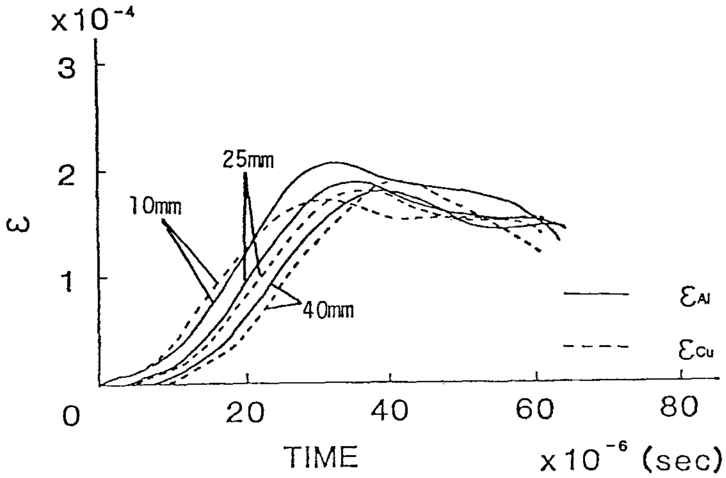
The strain measurement is planned to be finished before the release wave from the lateral surface of the specimen returens to the gauges position. The experiments are performed at relatively low stress levels, and so, the theoretical calculations are performed assuming all of the materials as linearly elastic. Material constants used in calculation are listed in Table 6.1.

Table 6.1

Aluminum (Layer 1)	Copper (Layer 2)
$\rho_1 = 0.2755 \times 10^{-9}$ kg.s²/mm⁴	$\rho_2 = 0.9133 \times 10^{-9}$ kg.s²/mm⁴
$E_1 = 7200.0$ kg/mm²	$E_2 = 12000.0$ kg/mm²
$G_1 = 2700.0$ kg/mm²	$G_2 = 4200.0$ kg/mm²

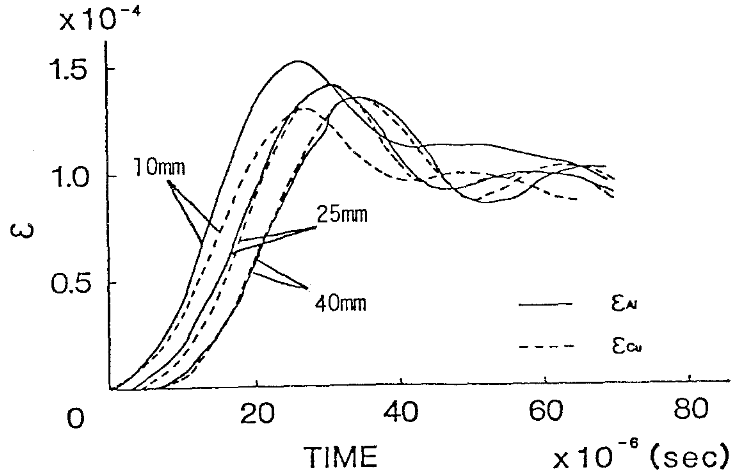
6.2. Experimental Results and Comparison with Theory

The comparisons of the wave forms propagating along the alunminum and copper layer are shown in Figs.6.3 and 6.4. Solid and dotted lines in these figures mean the strain variation with time in the aluminum and copper layer respectively, observed at the positions 10mm, 25mm and 40mm far from the loaded end. The results show that the strain wave changes its shape as it travels. The strain in the aluminum layer decreases its amplitude but on the other hand, in the copper layer, the strain wave increases its amplitude as it travels; and then the two strain levels in both layers gradually approach to the same value.



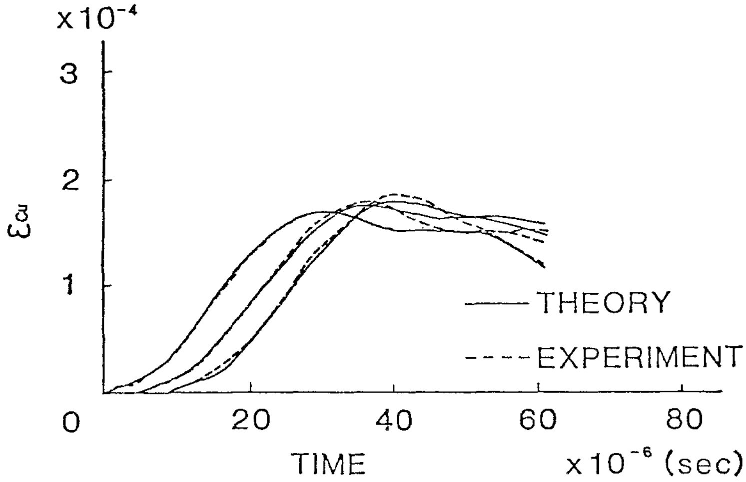
Strain variation observed experimentally

Fig.6.3 Experiment I

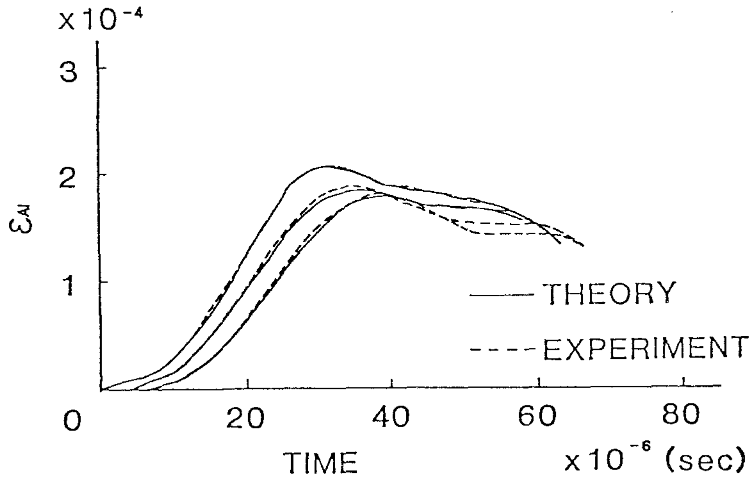


Strain variation observed experimentally

Fig.6.4 Experiment II

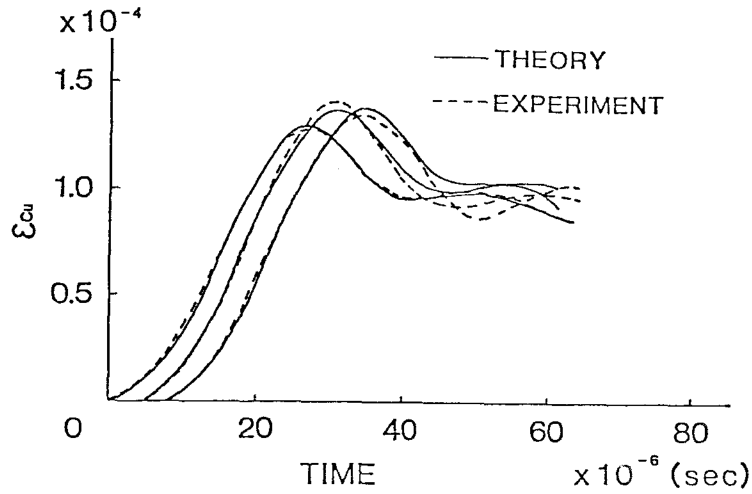


(b) Strains in the copper layer

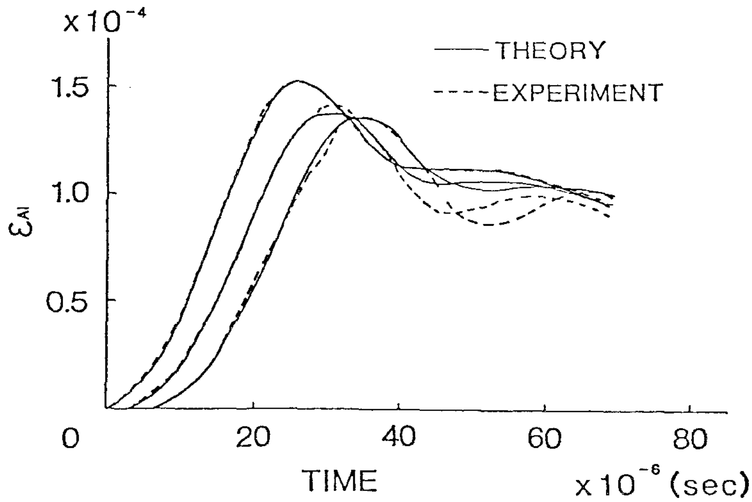


(a) Strains in the aluminum layer

Fig.6.5 Comparison of experiment I with theory



(b) Strains in the copper layer



(a) Strains in the aluminum layer

Fig.6.6 Comparison of experiment II with theory

The comparisons between experimental and theoretical results are shown in Figs.6.5(a),(b) and 6.6(a),(b). Figures (a) and (b) show the strains in the alminum and copper layer, respectively. Solid lines in Figs.6.5 and 6.6 are the strain variation observed in the experiments and are the same one as shown in Figs.6.3 and 6.4, respectively. Dotted lines are obtained by the theoretical calculations. Calculations are performed employing the strain variation measured near the loaded end (10mm far from the loaded end) as the boundary condition. Using a least-square criterion, the input strains are expressed by the 24-th order polynomials, and the calculations are performed by using the first 80-terms of the infinite series solution (2.35).

The agreement between experimental and theoretical results is good; especially for the arrival time of the wave front and the wave shape of rising part. Those results show that the diffusing continuum theory can predict all the important qualitative features of the transient wave, and also the quantitative agreement is fairly good.

CHAPTER 7

Conclusion

The transient waves in laminated composites have been treated, and the wave shape modification has been investigated by the diffusing continuum theory. Concerning the stress waves in a composite materials, usually the head-of-the-pulse approximation was used to obtain the far-field solutions. In the present work, we have been concerned with the near field where the wave front changes its shape as it travels.

In Chapter 2 and Chapter 3 of this paper, transient waves in the elastic laminated composite have been treated, and behaviors of strain waves near the loaded end have been discussed.

In Chapter 2, transient waves in semi-infinite elastic composite submitted to the surface pressure at the end of laminates have been investigated. In order to obtain an analytical solution near the loaded end, the multi-wave-fronts expansion method has been proposed which is useful to treat the transient waves with two discontinuous planes like as the wave in a laminated medium. Analytical results by the present method have been compared with those obtained by the integral solution.

In the theoretical analysis, two sets of skew coordinates along the characteristic curves have been employed. The use of these coordinates has facilitated the mathematical treatment of the propagating discontiuities in laminated medium. It has been shown, for the elastic laminates, that the discontinuity $[\partial^n \varepsilon_j^{(\ell)}/\partial \tau_\ell^n]$ at the wave fronts can be expressed by the n-th order polynomial of the spatial coordinate ξ_ℓ . Furthermore, It has also been shown that the coefficients of the each term of the discontiunity are determined by the recurrent formulas derived from the equations of motion for the elastic laminates; and infinite series solution obtained from the present analysis can be determined step by step from the lowest term. This approach is mathematically more simple than Achenbach and Reddy's one in which they obtained each term of the infinite series by integrating the differential equations one by one. And so, while in their approach, only 7-terms were used for the viscoelasitc single rod, in our approach, the first 25-terms solution has been obtained easily for the elastic laminates. It has been found that the present method is applicable to any boundary condition and the calculation time is reduced remarkably compared with the conventional integral solution. The numerical results have shown that the transient wave modification in the elasitc laminates occurs mainly as a result of the energy exchange between two layers due to different natural propagation speeds in different layers.

In Chaper 3, the analytical treatment introduced in Chapter 2 has been extended to apply to the velocity boundary condition. In this chapter, the equations of motion have been expressed in terms of the particle velocities, and the relations of the discontinuities between particle velocities and strains have been obtained. As the numerical calculations, two examples have been considered. The semi-infinite elastic laminates impacted by a rigid body has been first treated. The boundary condition for this problem has been obtained from the equilibrium of the motion for the rigid striker. The second example is concerned with the composite subjected to a constant velocity impact. For this case, the boundary condition has been easily obtained from the limiting case of Eq.(3.18) as $\ell_0 \to \infty$, and it has been shown that, at the impact end, the strains of two layers change with time and ultimately approach to the constant value.

Chapter 4 and Chapter 5 are concerned with the stress waves in viscoelastic laminated composites. The multi-wave-fronts expansion method introduced in previous Chapters has been extended to analyze the transient waves in such viscoelastic medium which were difficult to treat by the conventional method. In Chapter 4, the transient waves in the semi-infinite viscoelastic laminates have been investigated for the case of a surface pressure end loading. Impact problems of viscoelastic laminates with rigid bodies have been studied in Chapter 5. The diffusing continuum theory for the viscoelastic laminates has been employed. Analytical formulations have been constructed for the medium of viscoelastic layers that obey the general linear viscoelastic relation. It has been shown that, for the viscoelasic laminates, a propagating discontinuity $[\partial^n \sigma_k^{(\ell)} / \partial \tau_\ell^n]$ is expressed in the form of an n-th order polynominal with respect to ξ_1 accompanied with a spatial attenuation factor caused by the viscosity of the material, and the coefficients of the discontiunity are determined by the recurrent formulas obtained from the equations of motion for the viscoelastic laminates. Calculations have been performed for two kinds of composite materials composed of elastic and viscoelastic layers and two sorts of viscoelastic layers. From the numerical results for the viscoelastic laminate, it has been found that wave shape modification occurs as the results of a geometric dispersion and a spatial attenuation.

In Chapter 6, the transient wave propagation has been investigated experimentally by using an elastic laminated structure, and the results have been compared with theoretical ones. The agreement between experiment and calculation has been good enough, and it has been found that the theoretical treatment used in the present work can predict the important aspects of the dynamical behavior of laminated composite well.

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APPENDIX A

Interaction Term for Elastic Laminates

The interaction between two constituents in the laminated composite is caused by the shear stress acting on the interface of the layers. Therefore, in order to determine the interaction coefficient B(t), we should obtain the shear stress at the interface.

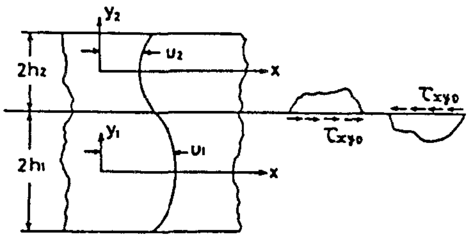

Fig.A1 Typical displacement profile in a laminated composite

The geometry of the layers and the profile of displacements are shown in Fig.A1, where U_1 and U_2 represent the displacement in the layer 1 and layer 2, respectively. The displacement and shear stress are continuous across the interface.

Now, let us assume that the displacement in the y_j-direction which is relatively small compared to the displacement in the x-direction may be neglected and the gradient of the shear stress is independent on y_j in each layer:

$$\frac{\partial \tau_{jxy}}{\partial y_j} = G_j \frac{\partial^2 U_j}{\partial y_j^2} = f_j \tag{A1}$$

Then the profile of a displacement in any cross-section of layer can be expressed as follows

$$U_j = \frac{f_j}{2 \cdot G_j} (y_j^2 - h_j^2) + K \quad , \qquad j = 1, 2 \tag{A2}$$

where K is an arbitrary function of x and t.

From Eq.(A1), the shear stress τ_{xy0} at the interface is given by

$$\tau_{xy0} = f_1 \cdot h_1 = - f_2 \cdot h_2 \tag{A3}$$

Now we define the average displacement of each layer by the following integral:

$$\bar{U}_j = \frac{1}{h_j} \int_0^{h_j} U_j \cdot dy_j = - \frac{f_j}{3 \cdot G_j} h_j^2 + K \tag{A4}$$

Then we may write

$$f_j \cdot h_j = - \frac{3 \cdot G_j}{h_j} \{ \bar{U}_j - K \} \quad , \qquad j = 1, 2 \tag{A5}$$

Substituting Eq.(A5) into Eq.(A3), we can obtain the shear stress at the interface

$$\tau_{xy0} = B \{ \bar{U}_1 - \bar{U}_2 \} \tag{A6}$$

where

$$B = \frac{3 G_1 \cdot G_2}{h_1 G_2 + h_2 G_1} \tag{A7}$$

APPENDIX B

Interaction Term for Viscoelastic Laminates

In the present appendix, the interaction term p and coefficient B(t) for the viscoelastic laminates will be determined by the methods developed in APPENDIX A.

By employing the shear relaxation modulus $G_k(t)$, (k=1,2), the relation between shear strain $\gamma_k(t)$ and shear stress $\tau_k(t)$ can be written in the form

$$\tau_k(t) = \gamma_{k0} \cdot G_k(t) + \int_0^{\tau} G_k(t-t') \frac{d\gamma_k(t')}{dt'} dt' \quad , \qquad k = 1, 2 \tag{B1}$$

where $\gamma_{k0} = \gamma_k(0)$. Assuming that the shear strain is expressed as $\partial U_k^*/\partial y_k$, it becomes, after Laplace transformation,

$$\tau_k^*(s) = \gamma_{k0} \cdot G_k^*(s) + G_k^*(s)\{ s\gamma_k^*(s) - \gamma_{k0} \} = s \cdot G_k^*(s) \frac{\partial U_k^*(s)}{\partial y_k} \quad , \quad k = 1, 2 \tag{B2}$$

Introducing the following assumptions,

$$\frac{\partial \tau_k^*(s)}{\partial y_k} = s \cdot G_k^*(s) \frac{\partial^2 U_k^*(s)}{\partial y_k^2} = f_k^*(s) \quad , \qquad k = 1, 2 \tag{B3}$$

the shear stress and the displacement in any cross-section of the layer become

$$\tau_k^*(s) = f_k^*(s) \cdot y_k \quad , \quad k = 1, 2 \tag{B4}$$

$$U_k^*(s) = \frac{f_k^*(s)}{2 \cdot s \cdot G_k^*(s)} (y_k^2 - h_k^2) + K^* \quad , \quad k = 1, 2 \tag{B5}$$

where K^* is a function of s.

From (B3), the shear stress τ_0^* at the interface is

$$\tau_0^*(s) = f_1^*(s) \cdot h_1 = - f_2^*(s) \cdot h_2 \tag{B6}$$

The average displacement of each layer becomes as follows

$$\bar{U}_k^*(s) = \frac{1}{h_k} \int_0^{h_k} U_k^*(s)\, dy_k = - \frac{f_k^*(s)}{3 \cdot S \cdot G_k^*(s)} h_k^2 + K^* , \quad k = 1, 2 \tag{B7}$$

Substituting (B7) into Eq.(B6), we can write the shear stress at the interface in terms of the average displacements

$$\tau_0^*(s) = S \cdot B^*(s) \left\{ \bar{U}_1^*(s) - \bar{U}_2^*(s) \right\} \tag{B8}$$

where

$$B^*(s) = \frac{3 G_1^*(s) \cdot G_2^*(s)}{G_2^*(s) \cdot h_1 + G_1^*(s) \cdot h_2} \tag{B9}$$

Applying the Laplace inversion to Eq.(B8), we obtain

$$\tau_0(t) = B_0 \left\{ \bar{U}_1(t) - \bar{U}_2(t) \right\} + \int_0^t B'(t - t') \left\{ \bar{U}_1(t') - \bar{U}_2(t') \right\} dt' \tag{B10}$$

where B(t) is the Laplace inversion of $B^*(s)$, and

$$B_0 = B(0) , \qquad B'(t') = \frac{dB(t')}{dt'} \tag{B11}$$

$\tau_0(t)$ in Eq.(B10) is equivalent to the interaction term p in Eq.(4.3).

APPENDIX C

Constitutive Relations for Viscoelastic Laminates

In this appendix, we will determine the constitutive relations for a viscoelastic material, applying the correspondence principle.

For the wave motion being considered, we assume that the strain in the y_k-direction is relatively small compared to the strain in the x-direction and can be neglected. Then Hooke's law for elastic material can be written,

$$\sigma_{kx} = (K_k + \tfrac{4}{3}G_k)\,\varepsilon_{kx} , \qquad \tau_{kxy} = G_k \cdot \gamma_{kxy} \tag{C1}$$

where K_k is the bulk modulus and G_k the shear modulus. We now introduce the correspondence principle represented by the following substitutions

$$3K_k \longleftrightarrow \frac{Q_k''}{P_k''} , \qquad 2G_k \longleftrightarrow \frac{Q_k'}{P_k'} \tag{C2}$$

and, using Eq.(C2), Eq.(C1) yields the corresponding relation for a viscoelastic material as follows

$$P_k' P_k'' \sigma_{kx} = \tfrac{1}{3}(\,P_k' Q_k'' + 2P_k'' Q_k'\,)\cdot \varepsilon_{kx} , \qquad P_k' \tau_{kxy} = \tfrac{1}{2} Q_k' \gamma_{kxy} \tag{C3}$$

where P", Q", P' and Q' are the operators which describe the viscoelasic behavior of the material.

We now apply this relation to specific material which is elastic in dilatation and is viscoelastic of the three-parameter solid in distortion:

$$P'_k = 1 + p'_{k1}\frac{d}{dt}, \qquad Q'_k = q'_{k0} + q'_{k1}\frac{d}{dt}$$
$$P''_k = 1, \qquad Q''_k = 3K_k \tag{C4}$$

Then Eq.(C3) becomes

$$\sigma_{kx} + p'_1\dot{\sigma}_{kx} = (K_k + \tfrac{2}{3}q'_{k0})\varepsilon_{kx} + (K_k p'_{k1} + \tfrac{2}{3}q'_{k1})\dot{\varepsilon}_{kx} \tag{C5}$$

$$\tau_{kxy} + p'_{k1}\dot{\tau}_{kxy} = \frac{1}{2}q'_{k0}\gamma_{kxy} + \frac{1}{2}q'_{k1}\dot{\gamma}_{kxy} \tag{C6}$$

From (C5) and (C6), the axial creep function, the axial and the shear relaxation modulus are obtained as follows:

$$J_k(t) = \frac{1}{K_k + \frac{2}{3}q'_{k0}}\left\{1 - \left(1 - \frac{K_k + \frac{2}{3}q'_{k0}}{K_k + \frac{4}{3}q'_{k1}/p'_{k1}}\right)\exp\left(-\frac{K_k + \frac{2}{3}q'_{k0}}{K_k + \frac{2}{3}q'_{k1}/p'_{k1}}\frac{t}{p'_{k1}}\right)\right\} \tag{C7}$$

$$E_k(t) = (K_k + \tfrac{2}{3}q'_{k0}) + \tfrac{2}{3}(q'_{k1}/p'_{k1} - q'_{k0})\exp(-t/p'_{k1}) \tag{C8}$$

$$G_k(t) = \frac{1}{2}\left\{q'_{k0} + \left(\frac{q'_{k1}}{p'_{k1}} - q'_{k0}\right)\exp(-t/p'_{k1})\right\} \tag{C9}$$

Substituting

$$p'_{k1} \to 2\eta_k/G_{k0}, \quad q'_{k0} \to G_{k0}, \quad q'_{k1} \to 4\eta_k \tag{C10}$$

we can rewrite Eqs.(C7)-(C9) as follows:

$$J_k(t) = \frac{1}{K_k + \frac{2}{3}G_{k0}}\left\{1 - \left(1 - \frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\right)\exp\left(-\frac{K_k + \frac{2}{3}G_{k0}}{K_k + \frac{4}{3}G_{k0}}\frac{G_{k0}}{2\eta_k}t\right)\right\} \tag{C11}$$

$$E_k(t) = (K_k + \tfrac{2}{3}G_{k0}) + \tfrac{2}{3}G_{k0}\exp\left(-\frac{G_{k0}}{2\eta_k}t\right) \tag{C12}$$

$$G_k(t) = \frac{G_{k0}}{2} + \frac{G_{k0}}{2}\exp\left(-\frac{G_{k0}}{2\eta_k}t\right) \tag{C13}$$