Studies on Integrability for Nonlinear Dynamical Systems and its Applications

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CHAPTER 1

Introduction

In this thesis, we study integrability for nonlinear dynamical systems including differential equations and discrete equations based on the soliton theory. Furthermore, we study applications of the soliton theory to numerical algorithms.

1. History of soliton theory

The notion of *soliton* means the solitary wave that travels stably and preserves its shape after interactions. The first literature about the soliton equations was presented in 1895 by Korteweg and de Vries. They presented the differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

which describes the propagation of a shallow water wave. The dispersion term $\partial^3 u/\partial x^3$ causes the wave to be scattered to many waves that have different phase velocities. The nonlinear term $u\,\partial u/\partial x$ varies the velocity of the wave according to the amplitude of the wave, then the wave stands erect and soon collapses. From those reasons, it was believed that there did not exist stable solitary wave for nonlinear evolution equations, until Korteweg and de Vries succeeded to derive the equation that had the exact solution of solitary wave. From the balance of dispersion and nonlinearity, the solution was obtained. The equation they presented is nowadays called *the KdV equation.*

Although the KdV equation was discovered at early year, the next development of it had not appeared until the research **[89]** by Zabusky and Kruskal in 1965. Using computers, they simulated the KdV equation numerically. They set the initial condition as the superposition of two pulses, both of which were exact solutions of solitary wave of the KdV equation. They computed a time evolution of the waves with periodic boundary condition. Two pulses moved to same direction by different velocities, because they had different amplitudes. The higher pulse traveled faster than the lower one. Zabusky and Kruskal observed the behaviors of interactions of pulses. From the results of the experiment, they discovered that each pulse preserved its shape and its velocity after the interactions. Moreover they discovered that positions of pulses were shifted at the interactions. That phenomenon is called a phase shift. Solitary waves behaved like particles. Then they named such solitary wave as the 'soliton' (a suffix '-on' stands for a particle). Their numerical experiments found a new phenomenon for nonlinear evolution equations. This discovery was also important as an example of contributions of computers to developments of mathematics.

Next epoch-making discovery was *the inverse scattering transform* (IST) [**23**], which was presented by Gardner, Greene, Kruskal, and Miura in 1967. By the IST, we transform a given evolution equation to a certain linear integral equation. Then we can solve initial value problem in principle. Another method for solving soliton equation was developed by Hirota in 1970s (cf. [**29**]–[**38**], [**43**], [**45**]). It is called *Hirota's direct method.* By the direct method, we can solve soliton equation directly not via the IST. The direct method firstly transform a given equation to so-called *Hirota's bilinear form.* Then we exactly obtain exact N-soliton solution by calculating a perturbation of the bilinear form. That solution is also expressed as a determinant. Such determinantal solution is called *the τ-function solution.* And the bilinear form is reduced to a certain identity of determinants.

The invention of the direct method also brought to us the techniques to discretize soliton equations (cf. [**39**]–[**42**], [**44**]). Preserving the structure of the τ-function, we do discretize the evolution equation, the independent variable transformation, the bilinear form, and the solution, simultaneously. Such discretization is sometimes called *an integrable discretization.* For example, the discrete KdV equation [**39**] is given by

$$u_n^{t+1} - u_{n+1}^{t-1} = \frac{1}{u_{n+1}^t} - \frac{1}{u_n^t}. \tag{1.2}$$

Many discrete soliton equations are now presented.

In early 1980s, Sato discovered that the τ-function of the Kadomtsev-Petviashvili (KP) equation is closely related to algebraic identities such as determinant identities. Moreover, he found that the totality of solutions for the KP equation and its higher order equations constitute an infinite dimensional Grassmann manifold.

2. Integrability conditions

The notion of integrability is rigidly defined for Hamilton systems. If a Hamilton system of N degree of freedom has N independent and mutually involutive integrals, then the system of ordinary differential equations (ODEs) is integrable in the sense in which the system can be linearized in terms of successive canonical transformations. This is the main result in the Liouville-Arnold theory. For partial differential equations (PDEs), there is no rigid definition determined yet. However there are candidates for integrability conditions of those systems. From studies on soliton equations, the following properties are now accepted as definitions of integrability for PDEs.

(1) Solvability by IST.

(2) Existence of N-soliton solution.

(3) Existence of infinite number of conserved quantities or symmetries.

(4) Existence of Lax pair **[53]**.

(5) Existence of bilinear form.

Generally it is not easy to obtain explicit solutions and conserved quantities for a given nonlinear equation. So we want to detect whether an equation is integrable or not beforehand. Thus the following integrability criteria have been proposed:

(a) The Painlevé test for ODE.

(b) The Weiss-Tabor-Carnevale (WTC) method for PDE.

(c) The singularity confinement test for discrete equation.

(d) The algebraic entropy test for discrete equation.

Those criteria are also used for deciding the values of parameters of an equation that has a possibility of integrability. We shall briefly introduce them.

We first consider ODE. The singularities of a linear ODE all depend on coefficients of the equation. However the singularities of a nonlinear equation often depend on initial values. We here consider a simple example

$$\frac{dy}{dx} + y^2 = 0\,. \tag{1.3}$$

The general solution of this equation is given by

$$y(x) = \frac{1}{x - C}\,. \tag{1.4}$$

The singularity of $y(x)$ occurs at $x = C$. Since the constant C is determined by $C = -1/y(0)$, the singular point is moved according to the initial value. Such singular point is called *a movable singular point.* If any movable singular point of an equation is not critical point, namely all movable singular points are poles, then it is called that the equation has *the Painlevé property.* The Painlevé property is used for a criterion of integrability of ODE. We shall briefly review the history of applications of the Painlevé property.

In 1889, Kowalevskya presented a new integrable case of the rigid body about fixed point. The equation of motion of the rigid body is sixth order ODE with six parameters. People at that time knew that only two cases of the equations are integrable when the parameters are specialized as some values. Those equations are called Euler's top and Lagrange's top respectively. In order to solve the equation, Kowalevskya restricted the solution to no movable singular point except for movable poles. Under that condition, she specified the parameters and succeeded to integrate the equation. The equation she presented is now called Kowalevskya's top.

In 1900s, Painlevé and co-workers presented so-called *the Painlevé equations.* They investigated nonautonomous second order ODEs, and enumerated all equations that had no movable critical point. They classified the equations and showed that the equations are essentially reduced to six types of new equations and known ones. Solutions of those six equations are called the Painlevé transcendents.

We here show how to check the Painlevé property of a given ODE. Let a movable singularity of $y(x)$ occur at $x = C$. Then we expand $y(x)$ around the point $x = C$ by the Laurent series

$$y(x) = (x - C)^a \sum_{j=0}^{\infty} y_j (x - C)^j . \tag{1.5}$$

We first check whether the singularity is a pole. It needs that the leading order a is a finite negative integer. If the leading order was a rational integer or an infinite integer, then the singularity became a branch point or an essential singularity. Next we check that the Laurent coefficients y_j have enough ambiguity. It needs that the number of arbitrary constants of y_j and the initial constant C is the same as the time of differentiations of the equation. If a and y_j satisfy those conditions and the expansion has no inconsistency, then it is said that the equation passes the Painlevé test.

We next consider PDE case. A conjecture about integrability for PDE was proposed by Ablowitz, Ramani, and Segur [**1, 2, 3**]. They stated that:

> Every nonlinear ODE obtained by an exact reduction of a nonlinear PDE that is solvable by IST has the Painlevé property.

Many soliton solutions are known to have this property. The KdV equation is actually reduced to an equation of elliptic function by a reduction of traveling wave solution. The modified KdV equation is reduced to the Painlevé equation of type II by a reduction using similarity solution.

However, it is impossible to check the Painlevé property of all ODEs obtained by all reduction of a given PDE. Thus Weiss, Tabor, and Carnevale proposed a method to check the Painlevé property of PDE directly not via reductions. This method is called *the WTC method* [**84**]. We briefly show the procedure of the WTC method. Let singularities of solution $u(x,t)$ for a nonlinear PDE occur on a manifold $\phi(x,t) = 0$. We assume that the function $\phi(x,t)$ is an arbitrary function, and that the solution is expressed as a formal Laurent series

$$u(x,t) = \phi(x,t)^a \sum_{j=0}^{\infty} u_j(x,t)\, \phi(x,t)^j . \tag{1.6}$$

We check that the leading order a is a finite negative integer, and that the number of arbitrary functions of u_j and ϕ is the same as the order of the differential equation. If a, u_j and ϕ satisfy those conditions and the expansion has no inconsistency, then it is said that the PDE has the

Painlevé property. If it is necessary to restrict u_j and ϕ to some conditions, then it is said that the equation has the conditional Painlevé property. An evolution equation that has a conditional Painlevé property is considered as a near-integrable system. In this thesis, we consider stability of such an equation.

Next we consider discrete equation. A criterion for discrete systems was first proposed by Grammaticos, Ramani, and Papageorgiou [25]. Their criterion is based on the property of *the singularity confinement* (SC). The SC property means that:

> The singularities of a discrete system are movable, i.e., they depend on initial conditions. And the memory of the initial conditions survives past the singularity by a few steps.

The property of the SC is accepted as a discrete version of the Painlevé property. The discrete Painlevé equations and many discrete soliton equation pass the SC test.

The SC test has been a useful criterion. However, Hietarinta and Viallet presented an equation that passes the SC test but has numerically chaotic property [28]. Then they proposed a more sensitive criterion. Their criterion is based on *the algebraic entropy* that is defined by the logarithmic average of a growth of degrees of iterations. The algebraic entropy test and the SC test are similar to each.

The SC type criteria are effective in reversible discrete systems such as soliton equations. However they are ineffective in irreversible discrete systems. For example, the arithmetic-harmonic mean algorithm [62],

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \tag{1.7}$$

has the explicit solution, however does not pass the SC test. We consider in the thesis integrability of such equations.

3. Integrable systems and numerical algorithms

The soliton theory has been developed in mathematics, physics and engineering. The optical soliton communication [26] is a famous example of application of the soliton theory to communication engineering. There are also applications to mathematical engineering. A close relationship between soliton equations and numerical algorithms has been pointed out. We enumerate those numerical algorithms and related integrable systems as follows.

- Matrix eigenvalue algorithms
  - 1-step of the QR algorithm is equivalent to time 1 evolution of the ordinary Toda equation [75] (see [73]).

  - The LR algorithm is equivalent to the discrete Toda equation [**40**] (see [**46**]).
  - The power method with the optimal shift is derived from an integrable discretization of the Rayleigh quotient gradient system (see [**60**]).
- Convergence acceleration algorithms
  - The recurrence relation of the ε-algorithm [**85**] (cf. the Shanks transform [**70**]) is equivalent to the discrete potential KdV equation (see [**68**]).
  - The ρ-algorithm [**86**] is equivalent to the discrete cylindrical KdV equation (see [**68**]).
  - The η-algorithm is equivalent to the discrete KdV equation (see [**56**]).
  - The n-th term of the E-algorithm is equivalent to the solution of the discrete hungry Lotka-Volterra equation (see [**76**]).
- Continued fraction algorithms (Padé approximations)
  - The recurrence relation of the qd algorithm for calculating continued fraction is equivalent to the discrete Toda equation.
  - The ordinary Toda equation gives a method for calculating Laplace transforms via the continued fraction (see [**61**]).
  - A new Padé approximation algorithm is formulated by using the discrete Schur flow (see [**55**]).
- Decoding algorithms
  - A BCH-Goppa decoding algorithm is designed by the Toda equation over finite fields (see [**59**]).
- Iteration methods having higher order convergence rate
  - The recurrence relation of the arithmetic-geometric mean algorithm has the solution of theta function (see [**18**]).
  - The recurrence relation of the arithmetic-harmonic mean algorithm has the solution of hyperbolic function (see [**62**]).

From these results, one may conjecture that a good numerical algorithm is regarded as an integrable dynamical system. Indeed, eigenvalue algorithms and acceleration algorithms, which are essentially linear convergent algorithms, pass the SC test of integrability criterion (cf. [**68**]). Moreover, they are proved to be equivalent to discrete soliton equations via Hirota's bilinear forms. However, some algorithms having higher order convergence rate do not pass this integrability criterion, as we mentioned in the previous section. It needs more discussions about integrability for such equations. We consider integrability of algorithms in the thesis. Furthermore, we develop numerical algorithms using the techniques in the soliton theory.

4. Outline of the thesis

The thesis is organized as follows.

In Chapter 2, we consider a generalized derivative nonlinear Schrödinger (GDNLS) equation. The equation is derived by adding two dispersion terms to the nonlinear Schrödinger (NLS) equation **[51, 26]**, which describes a propagation of pulses in optical fibers. The GDNLS equation has two parameters. We first construct a traveling wave solution for arbitrary values of parameters. We next investigate integrability of the GDNLS equation by the WTC method of the Painlevé test. We show that the equation has the Painlevé property and a conditional Painlevé property for some conditions of parameters. By numerical experiments, we examine stability of the traveling wave solutions in interactions.

In Chapter 3, we consider an extension of the Steffensen method **[72]**. The Steffensen method is an iteration method for finding a root of nonlinear equations. Its iteration function is constructed without any derivative function, and it has the second order convergence rate. The point to devise our extended method is that the iteration function is defined by using the k-th Shanks transform which is a sequence convergence acceleration algorithm. The convergence rate is shown to be of order $k + 1$. The use of the ε-algorithm avoids the direct calculation of Hankel determinants, which appear in the Shanks transform, and then diminishes the computational complexity. For a special case of the Kepler equation, it is shown that the numbers of mappings are actually decreased by the use of the extended Steffensen iteration.

In Chapter 4, we give new determinantal solutions for irreversible discrete equations. The equations considered are solvable chaotic systems and the discrete systems which are derived from iteration methods having higher order convergence rates. We deal with the hierarchy of the Newton type iterations (the Newton method and Nourein method **[64]**), that of the Steffensen type iterations (the Steffensen method and the extended Steffensen method in Chapter 3), and that of the Ulam-von Neumann system **[77]**. We obtain determinantal solutions for those systems including solvable chaotic systems in terms of addition formulas derived from some linear systems.

In Chapter 5, we finally state some remarks and further problems.

CHAPTER 2

Solution and Integrability of a Generalized Derivative Nonlinear Shrödinger Equation

1. Introduction

In this chapter, we consider the following equation,

$$\mathrm{i}\,U_t + \frac{1}{2}\,U_{xx} + |U|^2U + \mathrm{i}\alpha\,|U|^2U_x + \mathrm{i}\beta\,U^2U_x^* = 0\,, \tag{2.1}$$

where $U = U(x,t)$ is a complex variable and $*$ denotes a complex conjugate. Moreover, α and β are real parameters. Eq. (2.1) is reduced to the well-known nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\,U_t + \frac{1}{2}\,U_{xx} + |U|^2U = 0 \tag{2.2}$$

for $\alpha = \beta = 0$. Moreover, Eq. (2.1) yields two types of derivative nonlinear Schrödinger equations which are known to be integrable, namely the case of $\alpha : \beta = 1 : 0$ **[58]**

$$\mathrm{i}\,U_t + \frac{1}{2}\,U_{xx} + |U|^2U + \mathrm{i}\,|U|^2U_x = 0\,, \tag{2.3}$$

and the case of $\alpha : \beta = 2 : 1$ **[83]**

$$\mathrm{i}\,U_t + \frac{1}{2}\,U_{xx} + |U|^2U + 2\mathrm{i}\,|U|^2U_x + \mathrm{i}\,U^2U_x^* = 0\,. \tag{2.4}$$

Hereafter we call Eq. (2.1) a generalized derivative nonlinear Schrödinger (GDNLS) equation. We note that the GDNLS equation (2.1) can be regarded as a special case of the higher order nonlinear Schrödinger equation proposed by Kodama and Hasegawa **[51]**

$$\mathrm{i}\,U_t + \frac{1}{2}\,U_{xx} + |U|^2U + \mathrm{i}\alpha\,|U|^2U_x + \mathrm{i}\beta\,U^2U_x^* + \mathrm{i}\gamma\,U_{xxx} = 0 \tag{2.5}$$

which describes the pulses in optical fibers.

It is remarked that the term $|U|^2U$ can be eliminated by a gauge transformation **[49]**. Eqs. (2.3) and (2.4) without this term are known as the Chen-Lee-Liu (CLL) equation **[15]** and the Kaup-Newell (KN) equation **[50]**, respectively. The CLL equation was discussed by using the bilinear formalism by Nakamura and Chen **[58]**. Hirota **[47]** bilinearized the KN equation and showed that the CLL equation and the KN equation have the same bilinear forms. A class of solutions for the CLL, KN equations and their integrable generalization by Kundu **[52]**

$$\mathrm{i}\, U_t + \frac{1}{2}\, U_{xx} + 2\mathrm{i}\gamma\, |U|^2 U_x + 2\mathrm{i}(\gamma-1)\, U^2 U_x^* + (\gamma-1)(\gamma-2)\, |U|^4 U = 0\,, \tag{2.6}$$

where γ is a real parameter, has been constructed explicitly through the bilinear formalism by Kakei et al. **[49]**.

We first construct a traveling wave solution of the GDNLS equation (2.1) in Section 2. Motivated by a concrete form of the solution, we investigate the integrability of the GDNLS equation by using the Painlevé test in Section 3. Finally we examine a behavior of the traveling wave solution numerically in Section 4. In Section 5, we mention several remarks of this chapter.

2. Traveling wave solution

In this section, we construct a traveling wave solution for the GDNLS equation. Here we remark that the values of parameters α, β in Eq. (2.1) are taken to be arbitrary by the scale change except for the ratio β/α, and hence β/α can be regarded as a characteristic parameter of the equation.

Eq. (2.1) is invariant under the following transformation

$$U(x,t) = \frac{1}{\sqrt{k}}\tilde{U}(X,T)\, \mathrm{e}^{-\mathrm{i}V(X-\frac{V}{2}T)}\,, \tag{2.7}$$

where

$$x = k(X - VT)\,, \qquad t = k^2 T\,, \qquad k = 1 - V\alpha + V\beta \tag{2.8}$$

and V is an arbitrary constant. Taking this invariance into account, we first construct a stationary solution. We put

$$U(x,t) := r(x)\exp(\mathrm{i}\,\theta(x))\exp(\mathrm{i}\,\omega\, t)\,, \tag{2.9}$$

where $r(x)$ and $\theta(x)$ are real functions in x, and ω is a real constant. Substituting (2.9) into Eq. (2.1), we get

$$r_{xx} = 2\omega r - 2r^3 + r\theta_x^2 + 2(\alpha-\beta)r^3\theta_x \tag{2.10}$$

from the real part and

$$\theta_{xx} = -2\frac{r_x\theta_x}{r} - 2(\alpha+\beta)rr_x \tag{2.11}$$

from the imaginary part, respectively. The following ansatz is crucial for our construction of solution

$$\theta_x = \kappa r, \tag{2.12}$$

where κ is a constant. We obtain from Eq. (2.11)

$$(2\kappa + \alpha + \beta) r r_x = 0, \tag{2.13}$$

from which we have

$$\kappa = -\frac{\alpha + \beta}{2}. \tag{2.14}$$

Then Eq. (2.10) becomes

$$r_{xx} = 2\omega r - 2r^3 - \frac{1}{4}(\alpha + \beta)(3\alpha - 5\beta) r^5. \tag{2.15}$$

Integrating Eq. (2.15), we obtain

$$r^2 = \frac{8\omega \, e^{\pm 2\sqrt{2\omega}x}}{1 + 2\, e^{\pm 2\sqrt{2\omega}x} + \left(1 + \frac{2}{3}\omega(\alpha + \beta)(3\alpha - 5\beta)\right) e^{\pm 4\sqrt{2\omega}x}}. \tag{2.16}$$

Moreover, we get from Eqs. (2.12), (2.14) and (2.16)

$$\theta = -\sqrt{\frac{3(\alpha + \beta)}{3\alpha - 5\beta}} \tan^{-1}\left(\frac{1 + \left(1 + \frac{2}{3}\omega(\alpha + \beta)(3\alpha - 5\beta)\right) e^{\pm 2\sqrt{2\omega}x}}{\sqrt{1 + \frac{2}{3}\omega(\alpha + \beta)(3\alpha - 5\beta)}} \right), \tag{2.17}$$

where the following conditions should be satisfied that

$$\omega \geq 0, \qquad 1 + \frac{2}{3}\omega(\alpha + \beta)(3\alpha - 5\beta) \geq 0 \tag{2.18}$$

for the reality of r and θ. Substituting Eqs. (2.16) and (2.17) into Eq. (2.9), we have the stationary wave solution. Then, applying the transformation (2.7)–(2.8), we obtain the traveling wave solution. The result is expressed as

$$U(x,t) = \frac{p + p^*}{e^{-\phi} + Q\, e^{\phi^*}} \left(\frac{e^{-\phi} + Q\, e^{\phi^*}}{e^{-\phi} + P\, e^{\phi^*}} \right)^{\frac{N+1}{2}}. \tag{2.19}$$

Here we define function $\phi(x,t)$ as

$$\phi = px + \frac{1}{2} i p^2 t + \phi^{(0)}. \tag{2.20}$$

And we define parameters p, Ω, P, Q, and N as

$$p = (1 - V\alpha + V\beta)\Omega + iV, \tag{2.21}$$

$$\Omega = \pm\sqrt{2\omega}, \tag{2.22}$$

$$P = (1 - V\alpha + V\beta)\left(1 + \Omega\sqrt{-\frac{1}{3}(\alpha+\beta)(3\alpha - 5\beta)}\right), \tag{2.23}$$

$$Q = (1 - V\alpha + V\beta)\left(1 - \Omega\sqrt{-\frac{1}{3}(\alpha+\beta)(3\alpha - 5\beta)}\right), \tag{2.24}$$

$$N = \sqrt{\frac{3(\alpha+\beta)}{3\alpha - 5\beta}} \tag{2.25}$$

and $\phi^{(0)}$ and V are arbitrary constants. The condition (2.18) is also necessary here. This solution is characterized by the parameters ω and V for fixed α and β. The shape of the solution varies by the value of

$$D = PQ = (1 - V\alpha + V\beta)^2\left\{1 + \frac{2}{3}\omega(\alpha+\beta)(3\alpha - 5\beta)\right\}. \tag{2.26}$$

In fact, $|U|$ is given by

$$|U| = \sqrt{\frac{(p + p^*)^2\, e^{\phi+\phi^*}}{1 + 2\, e^{\phi+\phi^*} + D\, e^{2(\phi+\phi^*)}}}. \tag{2.27}$$

If D is sufficiently large, the solution has the soliton-like shape. For $D \sim 0$, it becomes

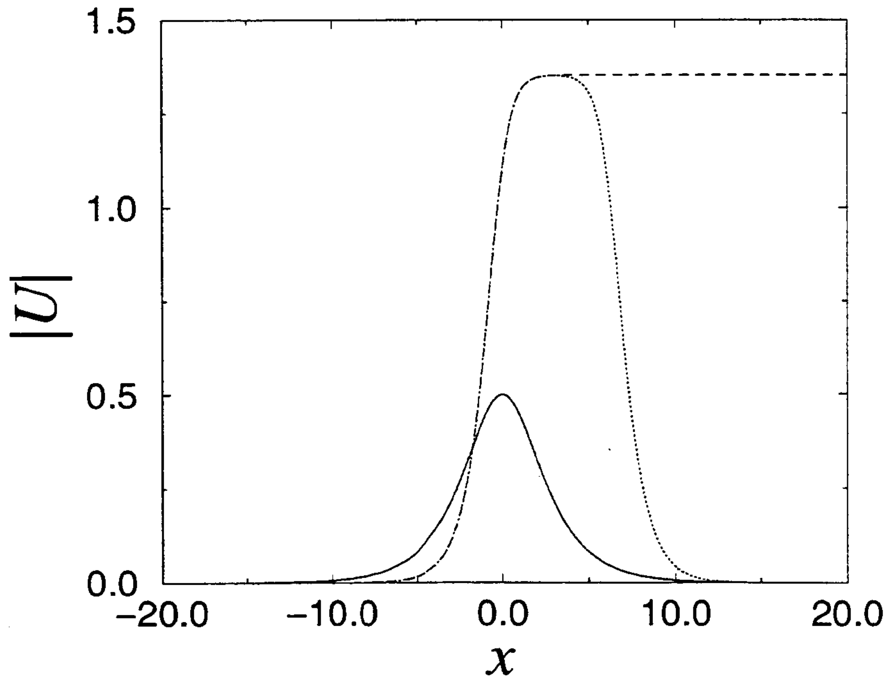

FIGURE 2.1. Shape of traveling wave solution for $\alpha = 1$, $\omega = 1/2$ and $V = 1/2$. Solid line: $\beta = 0$, $D = 0.5$. Dotted line: $\beta = 0.91355$, $D = 0.00000976$. Dashed line: $\beta = 0.9135528\cdots = (-1 + \sqrt{31})/5$, $D = 0$.

trapezoidal shape and for $D \to 0$, it has the kink-like shape as illustrated in Figure 2.1. Hence the traveling wave solution (2.19) may behave as a solitary wave.

Here we remark that if we take the limit $\alpha, \beta \to 0$, this solution is reduced to the 1-soliton solution of the NLS equation. Similarly, in the cases of $\beta/\alpha = 0$ and $\beta/\alpha = 1/2$, it gives the 1-soliton solution of Eqs. (2.3) and (2.4), respectively.

It should be emphasized that if N, which depends only on the ratio of α and β, is an odd integer, the traveling wave solution (2.19) is rational in exponential functions which is the common feature of soliton solutions. Thus it might be expected that in such cases, solitary waves of the GDNLS equation has good properties like that of integrable cases. The ratio of α and β in such cases are given by

$$\frac{\beta}{\alpha} = \frac{3m(m+1)}{5m(m+1)+2}, \qquad m = 0, 1, 2, \dots . \tag{2.28}$$

The cases $m = 0$ and $m = 1$ correspond to Eqs. (2.3) and (2.4), respectively, and they are known to be integrable as mentioned in the introduction.

Moreover, it should be noted that the 1-soliton solution for Eq. (2.6) obtained by Kakei et al. **[49]** has a quite similar form to Eq. (2.19). Indeed, we can check that 1-soliton solution of equation (2.6) for $\gamma = 0, 1$ equivalent to (2.19) for $N = 1, 3$, respectively, and not for other cases.

From the observation above, it may be natural to ask whether the cases of $m > 1$ in Eq. (2.28) are integrable or not. As for the integrability in a strict sense, the answer is no. In fact, Clarkson and Cosgrove **[17]** investigated the Painlevé property to the following equation,

$$\mathrm{i}\, u_t + u_{xx} + \mathrm{i}\alpha\, u u^* u_x + \mathrm{i}\beta\, u^2 u_x^* + \gamma\, u^3 u^{*2} + \delta\, u^2 u^* = 0\,, \tag{2.29}$$

where α, β, γ, and δ are real parameters, and shown that it is integrable only the case when it is equivalent to Eq. (2.6). However, we may expect some information from integrability test which distinguish the cases of Eq. (2.28) from other cases. We consider the integrability of the GDNLS equation (2.1) in the next section.

3. Painlevé test

In this section, we investigate the integrability of the GDNLS equation by using so-called the Painlevé test proposed by Weiss et al. **[84]**, and show that the GDNLS equation possesses "conditional Painlevé property" for the cases of Eq. (2.28).

Following to the procedure of the test, we regard that $u = U$ and $v = U^*$ are independent, and consider the GDNLS equation as a coupled system

$$\mathrm{i}\, u_t + \frac{1}{2}\, u_{xx} + u^2 v + \mathrm{i}\alpha\, u u_x v + \mathrm{i}\beta\, u^2 v_x = 0\,, \tag{2.30}$$

$$-\mathrm{i}\, v_t + \frac{1}{2}\, v_{xx} + v^2 u - \mathrm{i}\alpha\, v v_x u - \mathrm{i}\beta\, v^2 u_x = 0\,. \tag{2.31}$$

We assume the formal Laurent expansion around the zero points of some analytic function $\phi(x, t)$ for the solution of Eqs. (2.30) and (2.31)

$$u = \phi^a \sum_{j=0}^{\infty} u_j \phi^j\,, \qquad\qquad v = \phi^b \sum_{j=0}^{\infty} v_j \phi^j\,. \tag{2.32}$$

In this method, if

(1) there is no movable critical points, namely, the leading orders a and b are finite integers,

(2) the expansion (2.32) has sufficient number of arbitrary functions u_j and v_j,

(3) there is no incompatibilities in the expansion,

then it is regarded that the equation passes the Painlevé test, or it is said that the equation possesses the Painlevé property. In such case, it is usually believed that the equation is integrable. We show the concrete analysis in the following.

3.1. Leading order analysis. To get the leading power a and b, we substitute $u \sim u_0\,\phi^x$ and $v \sim v_0\,\phi^b$ into Eqs. (2.30) and (2.31). We obtain the relation

$$a + b = -1\,, \tag{2.33}$$

to adjust the leading order, and find

$$a = \frac{1}{2}\left(-1 \pm \sqrt{\frac{3(\alpha+\beta)}{3\alpha - 5\beta}}\right), \tag{2.34}$$

$$u_0 v_0 = \pm \mathrm{i}\, \phi_x \sqrt{\frac{3}{(\alpha+\beta)(3\alpha - 5\beta)}}\,. \tag{2.35}$$

Since a and b should be integers, we get the condition

$$\frac{\beta}{\alpha} = \frac{3m(m+1)}{5m(m+1)+2}\,, \qquad\qquad m = 0, 1, 2, \ldots \tag{2.36}$$

which is exactly the same as the condition (2.28).

3.2. Resonance analysis. The degree j is called *resonance* when u_j or v_j becomes an arbitrary function. The recurrence relation for u_j and v_j is given by

$$\begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \end{pmatrix}, \qquad j = 0, 1, 2, 3, \ldots . \tag{2.37}$$

Here we define elements $A_{11}^{(j)}, A_{12}^{(j)}, A_{21}^{(j)}$, and $A_{22}^{(j)}$ as

$$A_{11}^{(j)} = \frac{1}{2}(j + a - 1)(j + a)\phi_x^2 + \mathrm{i}\{\alpha(j + 2a) + 2\beta b\}u_0 v_0 \phi_x , \tag{2.38}$$

$$A_{12}^{(j)} = \mathrm{i}\{\alpha a + \beta(j + b)\}u_0^2 \phi_x , \tag{2.39}$$

$$A_{21}^{(j)} = -\mathrm{i}\{\alpha b + \beta(j + a)\}v_0^2 \phi_x , \tag{2.40}$$

$$A_{22}^{(j)} = \frac{1}{2}(j + b - 1)(j + b)\phi_x^2 - \mathrm{i}\{\alpha(j + 2b) + 2\beta a\}u_0 v_0 \phi_x . \tag{2.41}$$

And we define F_j and G_j as some polynomials of u_j, v_j and ϕ such that

$$F_j = F_j(u_0, \ldots, u_{j-1}, v_0, \ldots, v_{j-1}, \phi) , \tag{2.42}$$

$$G_j = G_j(u_0, \ldots, u_{j-1}, v_0, \ldots, v_{j-1}, \phi) . \tag{2.43}$$

Moreover we define $u_j = v_j = 0$ for $j < 0$.

We shall obtain the resonances. Coefficient u_j or v_j can be an arbitrary function when the condition

$$\det \begin{pmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{pmatrix} = \frac{1}{4}\phi_x^4 (j + 1)j(j - 2)(j - 3) = 0 \tag{2.44}$$

is satisfied. Hence we find that the resonances are

$$j = -1, 0, 2, 3 . \tag{2.45}$$

3.3. Compatibility condition. If the degree j is a resonance, the recurrence relation (2.37) should satisfy the compatibility condition

$$A_{11}^{(j)} : A_{21}^{(j)} = A_{12}^{(j)} : A_{22}^{(j)} = F_j : G_j \tag{2.46}$$

or

$$F_j = 0 , \quad G_j = 0 . \tag{2.47}$$

We shall check the compatibility for each resonance. Resonance $j = -1$ corresponds to the arbitrariness of ϕ. The compatibility condition is not necessary for $j = -1$. When $j = 0$, we have $F_0 = G_0 = 0$. When $j = 2$, we next obtain the relation

$$\frac{A_{11}^{(2)}}{A_{21}^{(2)}} = \frac{A_{12}^{(2)}}{A_{22}^{(2)}} = \frac{F_2}{G_2} = \pm\frac{2(2m+1)\alpha {u_0}^2}{(5m^2+5m+2)\mathrm{i}\phi_x}\,. \tag{2.48}$$

Thus we have checked the compatibility for the resonances $j = 0, 2$. The resonance $j = 0$ corresponds to the arbitrariness of u_0 or v_0, and $j = 2$ to that of u_2 or v_2. For $j = 3$, if the condition

$$\frac{v_0(m+1)(m-1)}{\phi_x^4(2m+1)}(2\phi_{tx}\phi_t\phi_x - \phi_{tt}\phi_x^2 - \phi_t^2\phi_{xx}) = 0\,, \tag{2.49}$$

or

$$\frac{u_0(m+2)m}{\phi_x^4(2m+1)}(2\phi_{tx}\phi_t\phi_x - \phi_{tt}\phi_x^2 - \phi_t^2\phi_{xx}) = 0\,, \tag{2.50}$$

is satisfied, then it is shown that the expansion is compatible. Therefore, for $m = 0$ and 1, the compatibility conditions are automatically satisfied. However, for $m = 2, 3, 4, \ldots$, the function $\phi(x, t)$ should satisfy

$$2\phi_{tx}\phi_t\phi_x - \phi_{tt}\phi_x^2 - \phi_t^2\phi_{xx} = 0 \tag{2.51}$$

to pass the test.

From this result, we may conclude that the GDNLS equation (2.1) possesses the Painlevé property for the cases of $m = 0$ and 1 in Eq. (2.28) which are known to be integrable. For $m > 1$, it does not pass the test in strict sense, but possesses "conditional Painlevé property" **[87, 88]**. For other cases, it does not pass the test.

It may be interesting to remark here that the condition (2.51) yields the dispersionless KdV equation

$$f_t - ff_x = 0 \tag{2.52}$$

by the dependent variable transformation

$$f = \frac{\phi_t}{\phi_x}\,. \tag{2.53}$$

We also note that exactly the same condition has appeared in the analysis of some system which describes the interaction of long and short water waves **[87, 88]**. In **[87, 88]**, Yoshinaga conjectured that the equation which passes the Painlevé test with the condition (2.51) has "finite-time integrability", since the solution of Eq. (2.52) loses analyticity in finite time as is well-known, and thus the assumption of the Painlevé test breaks.

4. Numerical experiments

4.1. Purpose. From the result of the Painlevé test, the GDNLS equation is not integrable in strict sense except for the cases $m = 0$ and 1 in Eq. (2.28). However, from the structure of the traveling wave solution, one may expect that the solitary waves behave like solitons even if the equation itself is not integrable. Motivated by this, we numerically solve the initial value problem for the GDNLS equation to check the following points:

(1) Stability of solitary waves in interactions.
(2) Existence of phase shift.
(3) Quantity of ripple which is generated by interactions.
(4) Any phenomenon which implies "finite-time integrability."

If (1) and (2) are observed, then it can be said that the solitary waves behave like solitons. We investigate (3) from the following reason: Suppose we observe the interaction of two different solitary waves. If the equation has a 2-soliton solution, it must approximate the initial state well at some t with some values of parameters. Then we may expect that the ripple which emerges through the interaction is quite small. Conversely, if the ripple which is observed for some values of α and β is small compared to other cases, then we may expect the existence of 2-soliton solution, or at least, it may be worth in further analysis. Moreover, it might be interesting to check whether the behavior of solutions differs or not by the cases that the GDNLS equation has the Painlevé property, the conditional Painlevé property and the other cases. From theoretical point of view, $\beta/\alpha = 0.6$ might be a critical point, since if the GDNLS equation possesses the conditional Painlevé property, then β/α should satisfy $0 \leq \beta/\alpha < 0.6$ from Eq. (2.36).

4.2. Method of numerical experiments. We adopt the spectral method for space, and the Runge-Kutta method for time integration. Range in space is from -50 to 50 and the number of mesh is $2^9 = 512$ points. Time interval is taken to be 0.01. We take superposition of two different traveling wave solutions as the initial value and calculate their time evolution. These two solitary waves are put with sufficient distance at $t = 0$. Then we fix the value of α as 1, and examine the time evolution with different values of β. The values of characteristic parameters of the traveling wave solutions are given by $\omega = 0.55$ and $V = 0.1$ for one wave, $\omega = 0.0075$ and $V = -2.0$ for another wave, respectively. Hereafter we call the former solitary wave pulse-1 and the latter pulse-2.

4.3. Results. Calculations have been performed until the solitary waves interact 10 times. We have checked the conserved quantity $\sigma = \int |U|^2 dx$ during the calculation as a measure of reliability. We see that σ is kept with sufficient accuracy. In fact, fluctuation of σ during the calculation is at most $\Delta\sigma/\sigma \sim 10^{-8}$, as shown in Table 2.1.

TABLE 2.1. Fluctuation of the conserved quantity $\Delta\sigma/\sigma$.

t	$\beta = 0$	$\beta = \frac{9}{16} = 0.5625$	$\beta = 0.8$
42	0	0	$1.6755916945128 \times 10^{-8}$
89	$9.3196098838920 \times 10^{-10}$	$1.9700283767298 \times 10^{-9}$	$1.9287796510011 \times 10^{-8}$
136	$1.9166390124162 \times 10^{-9}$	$2.7629271109169 \times 10^{-9}$	$2.0984692295199 \times 10^{-8}$
230	$3.9552707787706 \times 10^{-9}$	$4.2005398496527 \times 10^{-9}$	$2.3408647116396 \times 10^{-8}$
466	$8.8452799018246 \times 10^{-9}$	$7.6076397313715 \times 10^{-9}$	$2.7791032609452 \times 10^{-8}$

Figures 2.2 and 2.3 shows the behavior of solitary waves for the integrable case $\beta = 0$, and the case of $\beta = 0.5625 = 9/16$ ($m = 1$ in Eq. (2.28)), respectively. For small β, the solitary waves are stable in interaction. As β becomes larger, the change of the shape of solitary waves becomes large, which is illustrated in Figure 2.4.

Changes of heights of peaks and velocities for solitary waves after 10 times interactions for different β are shown in Figures 2.5 and 2.6, respectively.

Phase shifts in interaction are also observed for any β as shown in Figure 2.7. We note that the amounts of phase shift are measured by average of 10 times interactions.

Figure 2.8 shows the quantity of ripple after 10 times interactions of solitary waves. Here, it is measured by the ratio of integrated values of ripple to the conserved quantity σ. We see that the ripple is quite small for the integrable cases ($\beta = 0, 0.5$), as was expected. But it looks that it does not differ by the cases that the equation has the conditional Painlevé property (filled circles in Figure 2.8), and other cases (circles in Figure 2.8).

We have mentioned that $\beta = 0.6$ might be a critical point, but it looks that there is no drastic change in behavior of solitary waves at $\beta = 0.6$.

From these results, we may conclude that solitary waves are stable and behave like solitons at least for small β. Difference of behavior between the cases that the GDNLS equation has the conditional Painlevé property and the other cases was not observed in our calculations. In other words, we may conclude that soliton-like behavior of the solitary waves is the common property of the GDNLS equations regardless of the parameter β/α, as far as it is small.

As for the "finite-time integrability," we could not observe any such phenomenon that implies "finite-time integrability," e.g., break down of solitary waves in our numerical calculations.

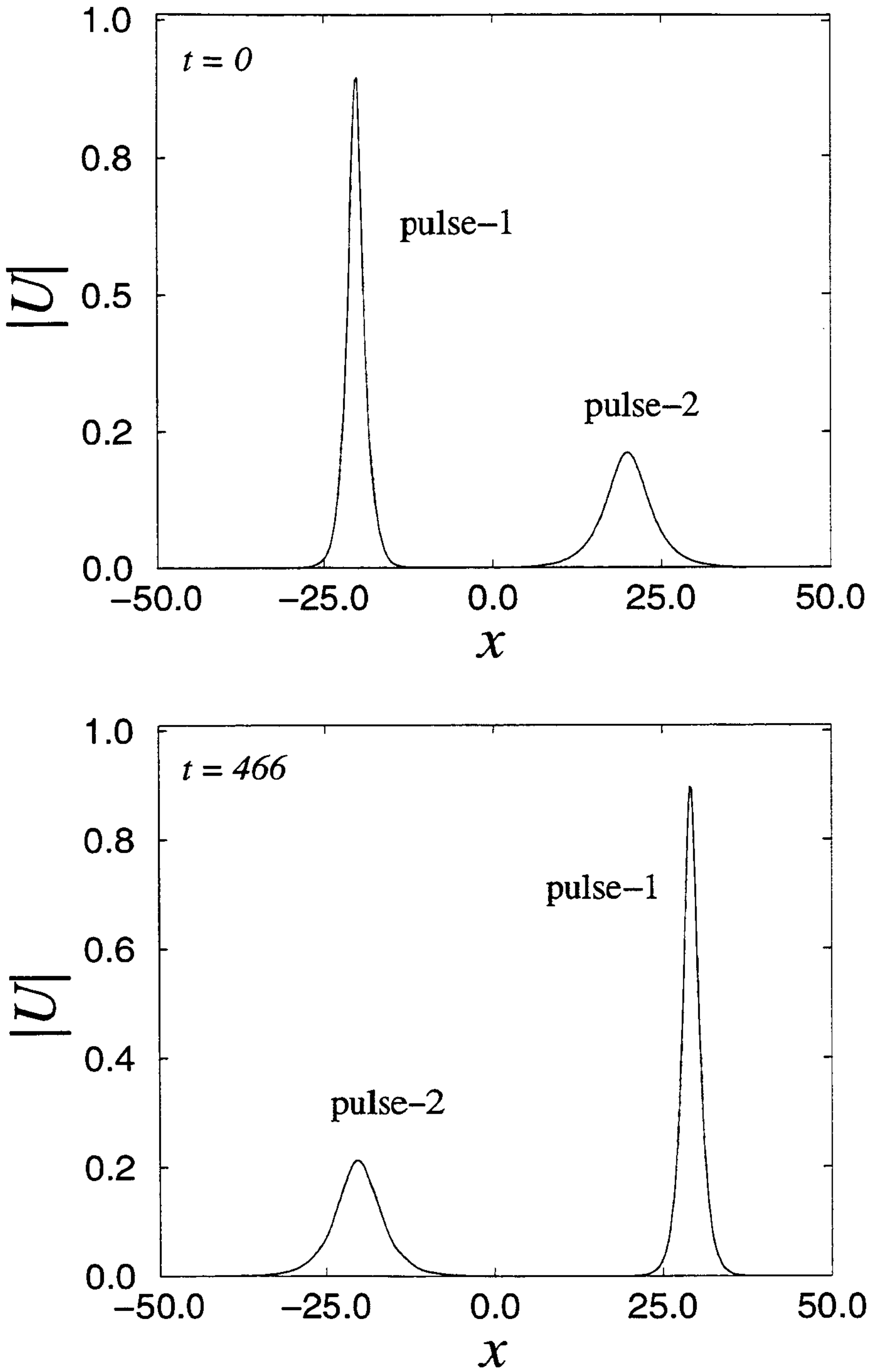

FIGURE 2.2. Behavior of solitary waves for $\alpha = 1, \beta = 0$.

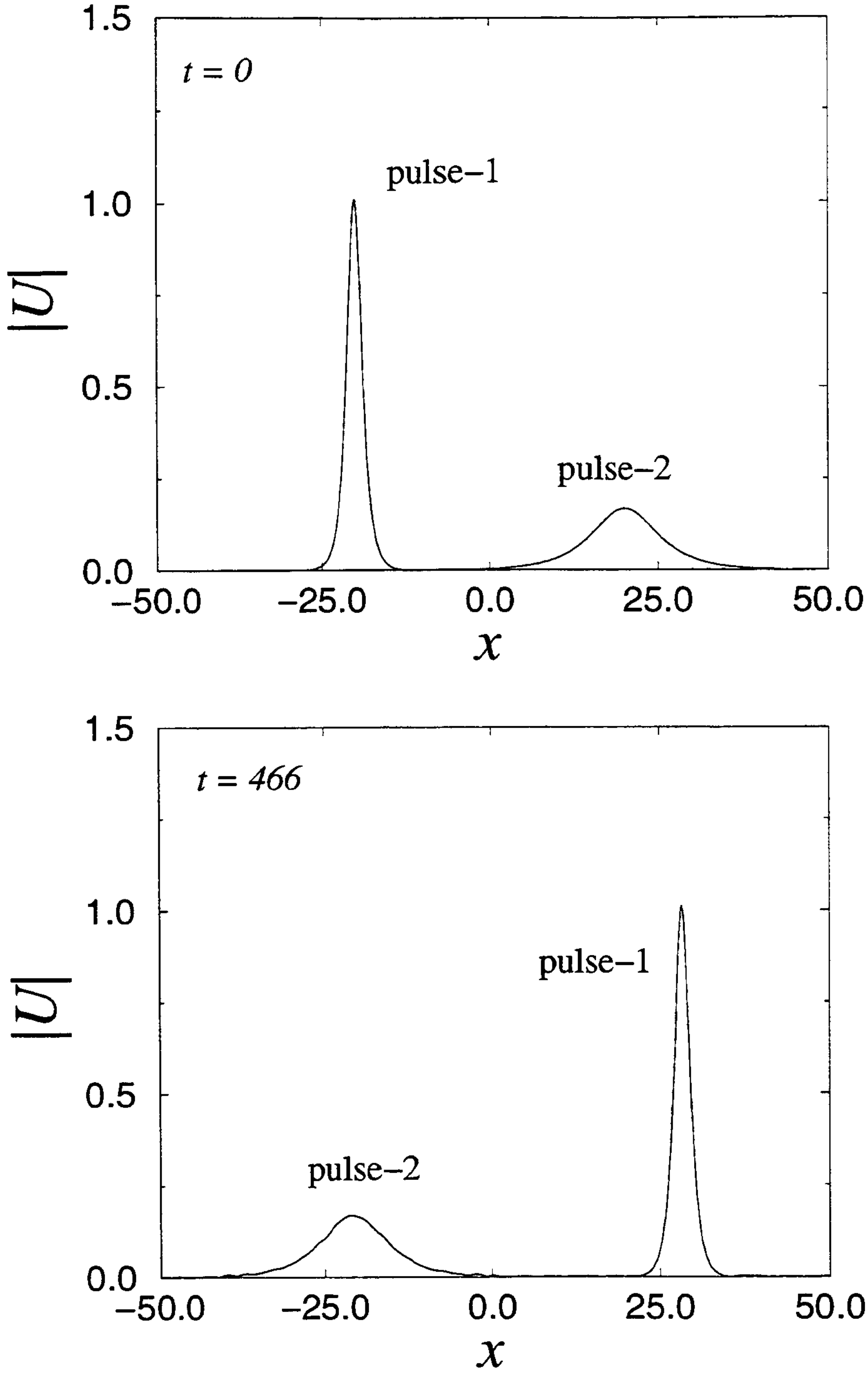

FIGURE 2.3. Behavior of solitary waves for $\alpha = 1, \beta = 0.5625$.

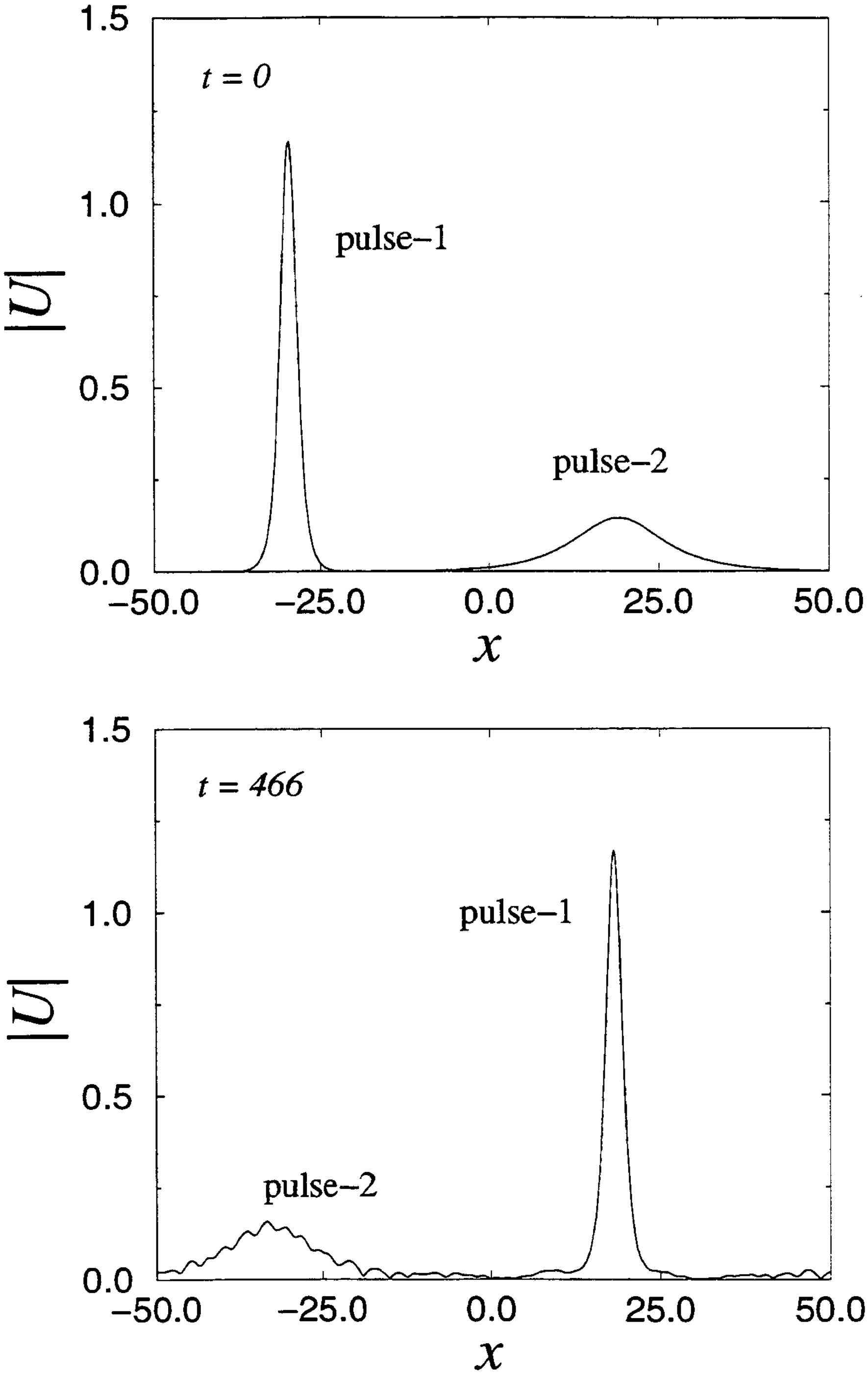

FIGURE 2.4. Behavior of solitary waves for $\alpha = 1, \beta = 0.8$.

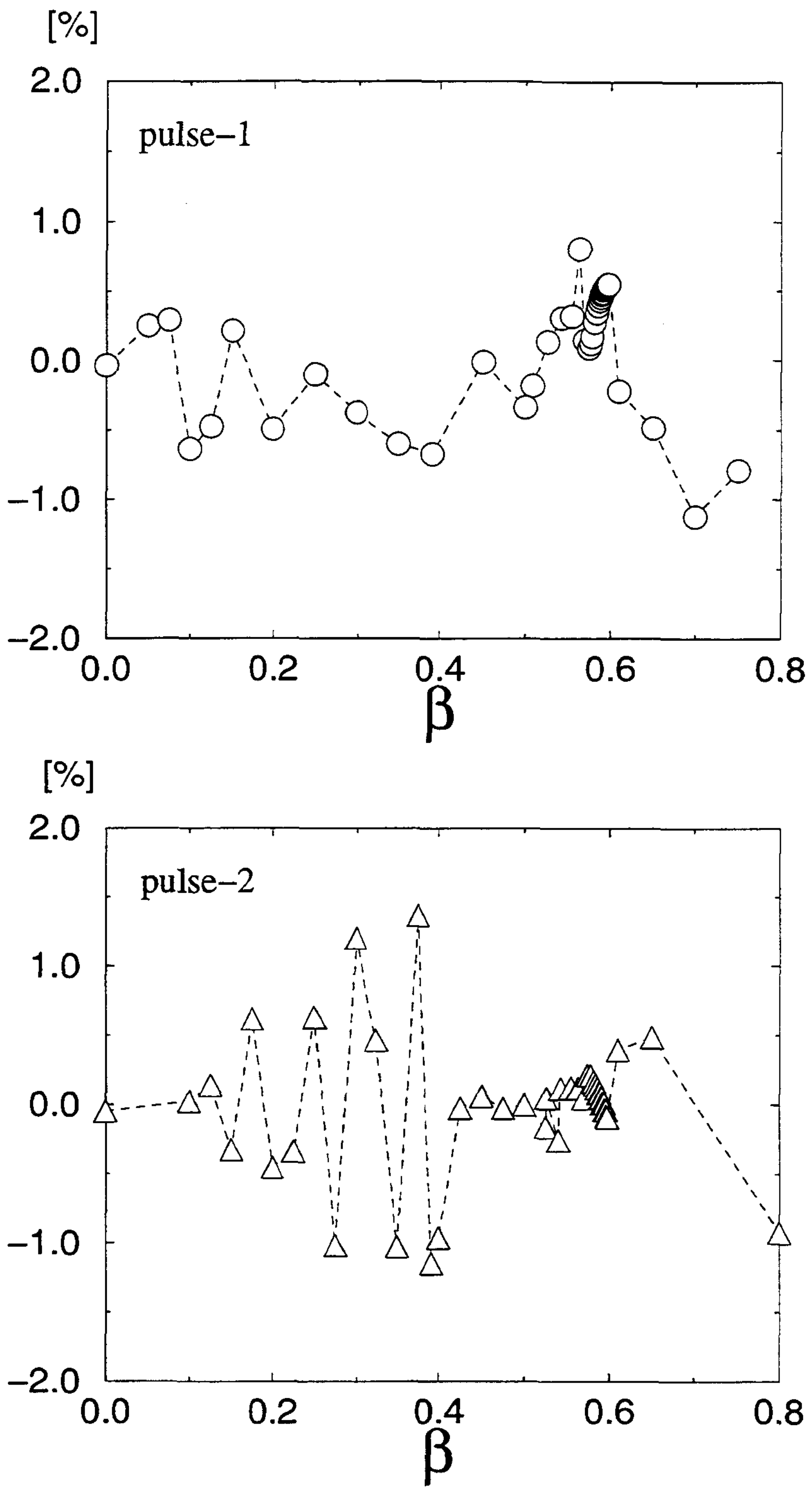

FIGURE 2.5. Changes of the peaks of pulse-1 and pulse-2 after 10 times interactions.

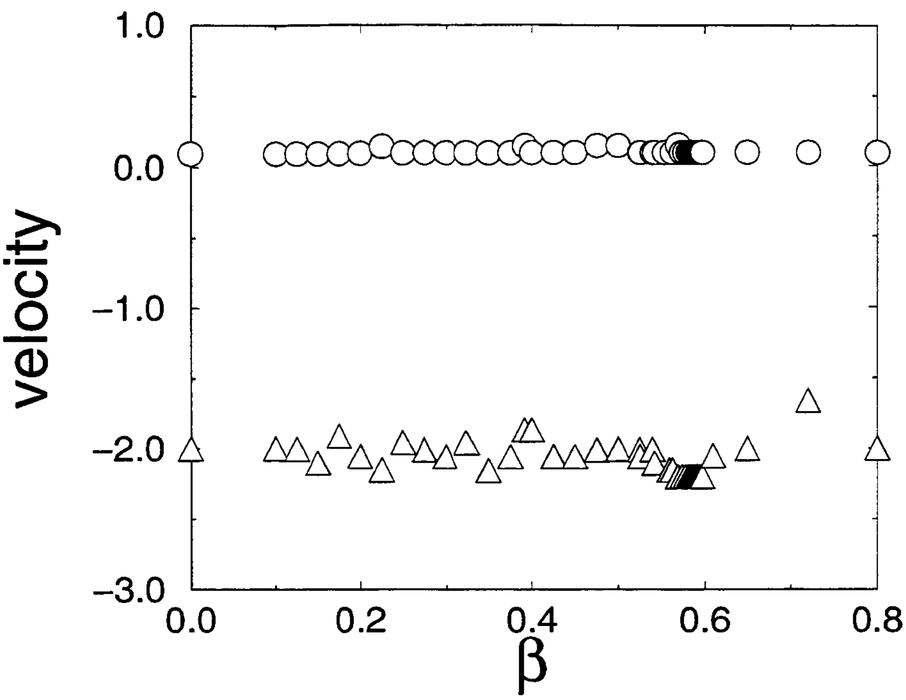

FIGURE 2.6. Velocities of solitary waves after 10 times interactions. Circle: pulse-1, triangle: pulse-2. Initial velocities are 0.1 and −2.0, respectively.

5. Concluding remarks

In this chapter, we have considered the GDNLS equation (2.1), and constructed a traveling wave solution (2.19) which is valid for any values of parameters. Motivated by the explicit form of the solution, we have applied the Painlevé test to the GDNLS equation, and shown that it possesses the Painlevé property in strict sense only for the known integrable cases, and the conditional Painlevé property for the cases of Eq. (2.36).

Numerical results imply that the traveling wave solution is stable in the interaction and behaves like a soliton for small β, regardless of the possession of the Painlevé property. Remarkable difference in the behavior of solitary waves between integrable and non-integrable cases was not observed, except that quantity of ripple generated by the interaction of solitary waves was small for integrable cases, as was expected.

As for the behavior of solitary waves for larger β, we could observe the change of shapes of solitary waves by the interaction. However, it looks that it is still insufficient to conclude that the solitary waves are not stable. Further theoretical analysis on stability may be necessary.

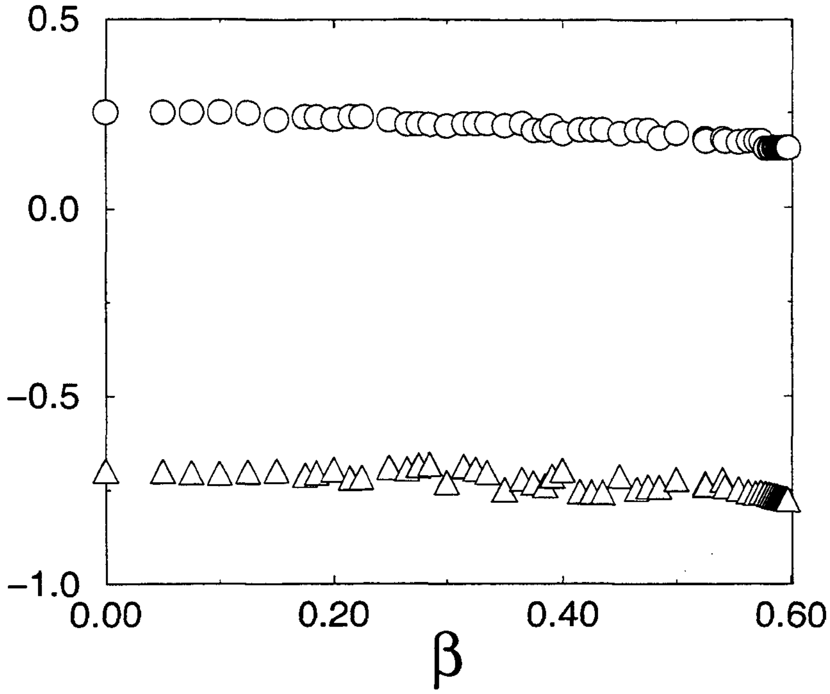

FIGURE 2.7. Quantity of position shift (phase shift) per one interaction. Circle: pulse-1, triangle: pulse-2.

In conclusion, it is expect that the soliton-like behavior of solitary waves for the GDNLS equation may be a "robust" property. Such behavior may be observed regardless of the value of parameter β/α, at least, as far as it is comparably small.

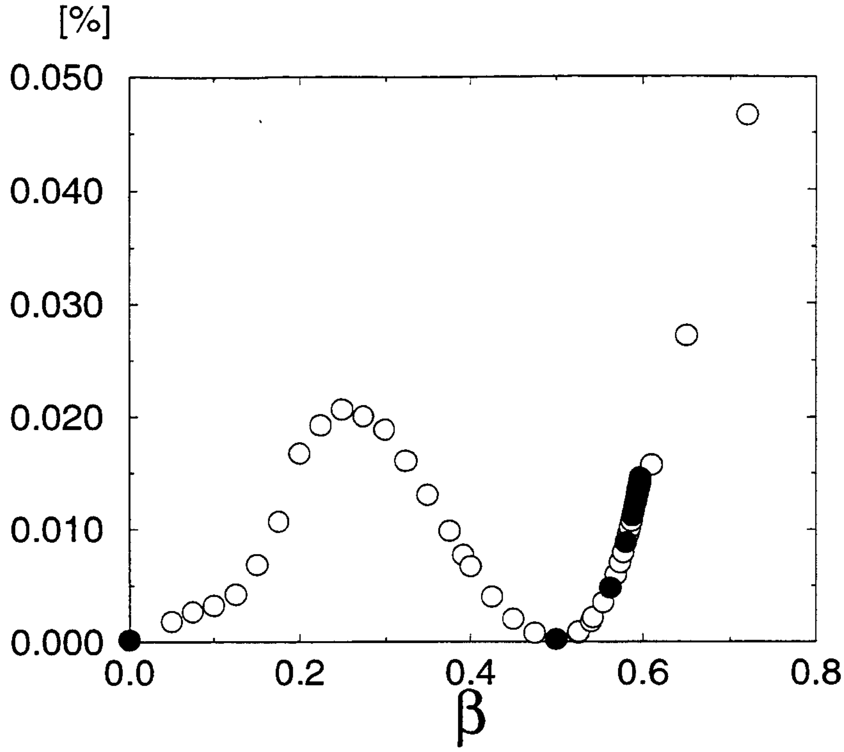

FIGURE 2.8. Quantity of ripple generated after 10 times interactions. β v.s. ratio of integrated values of ripple to the conserved quantity. Filled circle: the cases of Eq. (2.28), circle: other cases.

CHAPTER 3

An Extension of the Steffensen Iteration and Its Computational Complexity

1. Introduction

In this chapter, we consider iteration methods for finding a root of a single nonlinear equation $f(x) = 0$.

The Newton method is based on a first order approximation of the function $f(x)$. The sequence given by it generically converges locally and quadratically to a root α of $f(x)$. There have been many attempts to accelerate the Newton method. For example, some methods are designed based on a higher order approximation (cf. **[20]**), on a composition of the Newton iteration **[66]**, on a Padé approximation **[64, 16]**, on a modification of $f(x)$ in such a way that the convergence rate is increased **[22, 24]**, and so on.

The Steffensen method **[72]** is an iteration method which is applied to a nonlinear equation of the form $x = \phi(x)$. It also has the second order convergence rate, and its iteration function $\Phi(x)$ has no derivative of $\phi(x)$. The Steffensen method can be regarded as a discrete version of the Newton method. There are so many extensions for the Newton method, however, a few extension for the Steffensen method. The aim of this chapter is to develop a new iteration method of the Steffensen type having a higher order convergence rate.

In Section 2, we consider a relationship of the Newton method and the Steffensen method. In Section 3, we note that the Steffensen iteration function $\Phi(x)$ is congruent with the Aitken transform **[5]**. In Section 4, we introduce the k-th Shanks transform **[70]** which is a natural extension of the Aitken transform. When $k = 1$, the Shanks transform is reduced to the Aitken transform. In Section 5, we propose an extension of the Steffensen method in terms of the k-th Shanks transform. In Section 6, it is proved that the extension has the $(k+1)$-th order convergence rate provided that $\phi'(\alpha) \neq 0, \pm 1$. When $\phi(\alpha) = 0$, the iterated sequence has the $(k+2)2^{k-1}$-th order convergence rate. In Section 7, some numerical examples are given which demonstrate the efficacy of the extended Steffensen iteration. For a special case of the Kepler equation, it is shown that the numbers of mappings are actually decreased by the extended Steffensen iteration. In Section 8, we state the remarks of this chapter.

2. The Newton method and the Steffensen method

Let us consider the Newton iteration for the equation $f(x) = 0$. The Newton iteration is given by

$$x_{n+1} = N(x_n) := x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \ldots, \tag{3.1}$$

where the initial approximation x_0 is sufficiently close to a root α. The function $f(x)$ should be in C^2-class on an interval I such that $\alpha \in I$. If $f'(\alpha) \neq 0$ and $\max |N'(x)| < 1$ on I, then the sequence $x_0, x_1, x_2, \ldots$ converges to α quadratically.

To introduce the Steffensen iteration [**72**], we consider the equation $x = \phi(x)$ by setting

$$\phi(x) := x + f(x). \tag{3.2}$$

We prepare the sequence $\{y_j\}$ generated by the simple iteration

$$y_{j+1} = \phi(y_j), \qquad j = 0, 1, \ldots. \tag{3.3}$$

If the sequence $\{y_j\}$ converges to a number α, then it follows from $\alpha = \phi(\alpha)$ that $f(\alpha) = 0$. The contraction principle guarantees the convergence provided that $\max |\phi'(x)| < 1$. Furthermore, the convergence rate of the sequence $\{y_j\}$ is linear if $\phi'(\alpha) \neq 0$. Let us call such $\phi(x)$ the simple iteration function.

The Steffensen iteration is an iteration method for finding a root of the nonlinear equation of the form $x = \phi(x)$. There is *no derivative* in the Steffensen iteration function. Let us define the recurrence formula

$$x_{n+1} = \Phi(x_n) := x_n - \frac{\left(\phi(x_n) - x_n\right)^2}{\phi(\phi(x_n)) - 2\phi(x_n) + x_n}, \qquad n = 0, 1, \ldots, \tag{3.4}$$

where $\phi(x)$ is defined by (3.2). Here $\Phi(x)$ is the iteration function of the Steffensen iteration which generates the sequence $x_0, x_1, x_2, \ldots$. If $x_n \to \alpha$ as $n \to \infty$, then α is a root of $x = \phi(x)$. Even if the sequence $\{y_j\}$ given by the simple iteration (3.3) diverges, the Steffensen iteration (3.4) may converge to α more faster than does linear order method provided that $\phi(x)$ is in C^1-class, $x_0 \in I$ and $\phi'(\alpha) \neq 1$. Especially, if $\phi(x)$ is in C^2-class, the rate is *quadratic*, or equivalently, of the second order. The condition $\max |\phi'(x)| < 1$ is not necessary in this case [**66**, pp. 241–246]. Furthermore, a global convergence theorem is given in [**27**, pp. 90–95]. See for an abstract form of the Steffensen iteration [**65**]. An extension of the Steffensen iteration for systems of nonlinear equations is proposed in [**27**, p. 116] and a local convergence theorem is shown in [**63**].

The Steffensen iteration has its origin in a linear interpolation formula of $f(x)$. Let us briefly review this geometrical feature. A root α of $f(x) = 0$ is the intersection point of the curve $y = f(x)$ and the x-axis in xy-plain (see Figure 3.1). We consider the line through the two points $(a_0, f(a_0))$ and $(a_1, f(a_1))$ on the curve. Here a_1 is defined by

$$a_1 := \phi(a_0). \tag{3.5}$$

The intersection point $\bar{\alpha}$ of the line and the x-axis gives an approximation of α. It follows from $a_1 - a_0 = f(a_0)$ that

$$\bar{\alpha} := a_0 - \frac{f(a_0)}{\dfrac{f(a_1) - f(a_0)}{a_1 - a_0}} = a_0 - \frac{\left(\phi(a_0) - a_0\right)^2}{\phi(\phi(a_0)) - 2\phi(a_0) + a_0}. \tag{3.6}$$

Thus this approximation formula gives rise to the Steffensen iteration function (3.4). Let us set $h := a_1 - a_0$. Taking the limit that the line approaches to the tangential line at $(a_0, f(a_0))$, i.e., $a_1 \to a_0$, we derive

$$\bar{\alpha} = a_0 - \frac{f(a_0)}{\dfrac{f(a_0 + h) - f(a_0)}{h}} \quad \to \quad a_0 - \frac{f(a_0)}{f'(a_0)} \quad \text{as} \quad h \to 0. \tag{3.7}$$

In this limit, $\bar{\alpha}$ goes to the estimation of α by the Newton method (3.1). Thus we can regard the Steffensen iteration as *a discrete version of the Newton method.* This leads us to believe that an acceleration of the Steffensen iteration is a meaningful problem.

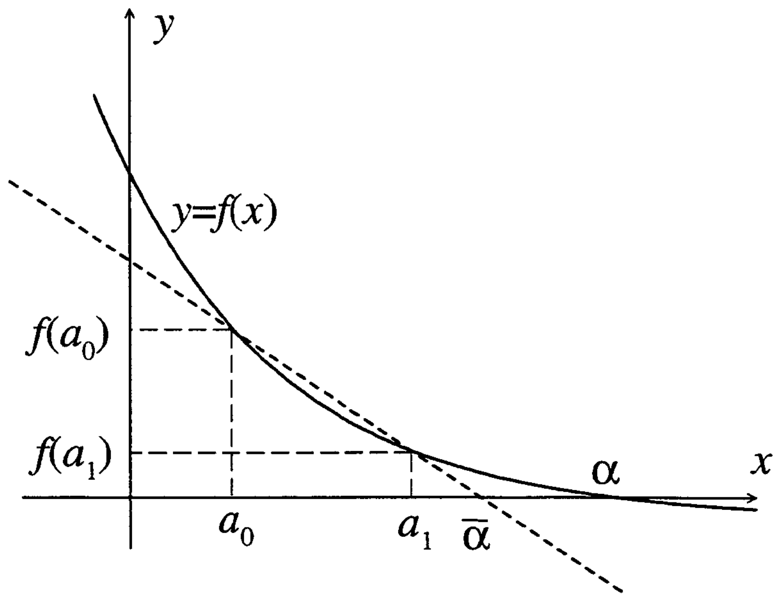

FIGURE 3.1. Graphical explanation of the Steffensen iteration

3. The Steffensen method and the Aitken transform

Let us introduce the Aitken transform [**5**]. It is a sequence transform to accelerate the convergence of a given sequence $\{y_j\}$. The Aitken transform is given by

$$\bar{y}_j = y_j - \frac{(y_{j+1} - y_j)^2}{y_{j+2} - 2y_{j+1} + y_j}, \qquad j = 0, 1, 2, \ldots . \tag{3.8}$$

If the sequence $\{y_j\}$ converges to a finite limit y_∞, then the sequence $\{\bar{y}_j\}$ converges to the same limit y_∞ faster than $\{y_j\}$. In general (cf. [**10**, pp. 1–2]), we consider some sequences $\{S_j\}$, $\{T_j\}$, and a sequence transform such that $A : S_j \to T_j$. If the sequences $\{S_j\}$ and $\{T_j\}$ converge to the same limit α and satisfy the condition

$$\lim_{j\to\infty} \frac{T_j - \alpha}{S_j - \alpha} = 0, \tag{3.9}$$

then the sequence transform A is called *sequence convergence accelerator.*

The Steffensen iteration function $\Phi(x_n)$ is equivalent to the Aitken transform of the three numbers x_n, $\phi(x_n)$ and $\phi(\phi(x_n))$. Namely, we have

$$\Phi(x_n) = \bar{y}_0 := y_0 - \frac{(y_1 - y_0)^2}{y_2 - 2y_1 + y_0}, \qquad y_0 = x_n, \qquad y_1 = \phi(x_n), \qquad y_2 = \phi(\phi(x_n)) \tag{3.10}$$

for each $n = 0, 1, \ldots$. It should be noted that the sequence $\{\bar{y}_j\}$ accelerated by the Aitken transform is different from the sequence $\{x_n\}$ generated by the Steffensen iteration (3.4). We can find that $x_{n+1} = \bar{y}_0$ and $x_{n+2} \neq \bar{y}_1$ in general, even if $x_n = y_0$. In order to use the Aitken acceleration, we must prepare the whole sequence $\{y_j\}$. Moreover, if the convergence rate of $\{y_j\}$ is linear, then the convergence rate of $\{\bar{y}_j\}$ is so (cf. [**6**]). The Aitken acceleration only guarantees that the sequence $\{\bar{y}_j\}$ converges faster than $\{y_j\}$ does in general. This property is in sharp contrast to the Steffensen iteration.

4. The Shanks transform and the ε-algorithm

The k-th Shanks transform [**70**] is a natural extension of the Aitken transform. It is defined by a ratio of Hankel determinants of $2k + 1$ numbers $y_j, \ldots, y_{j+2k}$ by

$$e_k(y_j) := \frac{A_k^{(j)}}{B_k^{(j)}}, \qquad j = 0, 1, 2, \ldots . \tag{3.11}$$

Here we define the numerator $A_k^{(j)}$ as a Hankel determinant of $y_j, \ldots, y_{j+2k}$ by

$$A_k^{(j)} := \overbrace{\begin{vmatrix} y_j & y_{j+1} & \cdots & y_{j+k} \\ y_{j+1} & y_{j+2} & \cdots & y_{j+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j+k} & y_{j+k+1} & \cdots & y_{j+2k} \end{vmatrix}}^{k+1}, \tag{3.12}$$

and the denominator $B_k^{(j)}$ as a Hankel determinant of $\Delta^2 y_j, \ldots, \Delta^2 y_{j+2k-2}$ by

$$B_k^{(j)} := \overbrace{\begin{vmatrix} \Delta^2 y_j & \Delta^2 y_{j+1} & \cdots & \Delta^2 y_{j+k-1} \\ \Delta^2 y_{j+1} & \Delta^2 y_{j+2} & \cdots & \Delta^2 y_{j+k} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^2 y_{j+k-1} & \Delta^2 y_{j+k} & \cdots & \Delta^2 y_{j+2k-2} \end{vmatrix}}^{k}, \tag{3.13}$$

where Δ is the forward difference operator such that

$$\Delta y_j := y_{j+1} - y_j, \qquad \Delta^2 y_j := y_{j+2} - 2y_{j+1} + y_j. \tag{3.14}$$

When $k = 1$, the Shanks transform is reduced to the Aitken transformation (3.8). Computation of determinants usually needs a plenty of multiplications and additions. In order to decrease the amount of the computations and to avoid the cancellation in the calculation of the Hankel determinants, we make use of *the ε-algorithm* [85], [9, pp. 40–51]. The sequence $\{e_k(y_j) | j = 0, 1, \ldots\}$ of the Shanks transform is determined directly by the recurrence relation

$$\varepsilon_{-1}^{(j)} = 0, \qquad \varepsilon_0^{(j)} = y_j, \qquad j = 0, 1, 2, \ldots, \tag{3.15}$$

$$\varepsilon_{i+1}^{(j)} = \varepsilon_{i-1}^{(j+1)} + \frac{1}{\varepsilon_i^{(j+1)} - \varepsilon_i^{(j)}}, \qquad i = 0, 1, 2, \ldots, \qquad j = 0, 1, 2, \ldots, \tag{3.16}$$

through

$$e_k(y_j) = \varepsilon_{2k}^{(j)}, \qquad j = 0, 1, \ldots. \tag{3.17}$$

The amount of computations (3.16) to get $e_k(y_j)$ is only $k(2k + 2n + 1)$. It should be remarked that the ε-algorithm has a numerical *stability*.

5. An extension of the Steffensen iteration

The Shanks transform is originally a sequence convergence accelerator for a given sequence. We apply the Shanks transform to define an iteration function, where the sequence $\{y_j\}$ is replaced by that of the simple iterations (3.3). Let x_0 be an initial approximation of a root α of a nonlinear equation $x = \phi(x)$. For a fixed natural number k, we introduce the following iteration function

$$x_{n+1} = \Phi_k(x_n) := \frac{A_k(x_n)}{B_k(x_n)}, \qquad n = 0, 1, 2, \ldots. \tag{3.18}$$

Here we define $A_k(x)$ and $B_k(x)$ as

$$A_k(x) := \begin{vmatrix} \phi_0(x) & \phi_1(x) & \cdots & \phi_k(x) \\ \phi_1(x) & \phi_2(x) & \cdots & \phi_{k+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_k(x) & \phi_{k+1}(x) & \cdots & \phi_{2k}(x) \end{vmatrix}, \tag{3.19}$$

$$B_k(x) := \begin{vmatrix} \delta^2\phi_0(x) & \delta^2\phi_1(x) & \cdots & \delta^2\phi_{k-1}(x) \\ \delta^2\phi_1(x) & \delta^2\phi_2(x) & \cdots & \delta^2\phi_k(x) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^2\phi_{k-1}(x) & \delta^2\phi_k(x) & \cdots & \delta^2\phi_{2k-2}(x) \end{vmatrix}. \tag{3.20}$$

The number x_{n+1} becomes a new starting value for the next iteration. Here $\phi_j(x)$ and $\delta^2\phi_j(x)$ are compositions of the simple iteration function $\phi(x)$ and their linear combinations defined by

$$\phi_0(x) := x, \quad \phi_j(x) := \overbrace{\phi(\phi(\cdots\phi}^{j}(x)\cdots)), \qquad j = 1, 2, 3, \ldots, 2k, \tag{3.21}$$

$$\delta^2\phi_j(x) := \phi_{j+2}(x) - 2\phi_{j+1}(x) + \phi_j(x), \qquad j = 0, 1, \ldots, 2k-2, \tag{3.22}$$

respectively. If a denominator in the formula (3.18) happens to be zero, we set $x_{n+1} = x_n$. Especially, $\Phi_1(x)$ is just the Steffensen iteration function (3.4). Let us call (3.18) *the extended Steffensen iteration.*

6. Convergence rate of the extended Steffensen iteration

We now consider the convergence rate of the extended Steffensen iteration (3.18). The main results in this chapter are as follows.

THEOREM 3.1. *If $\phi(x)$ is in C^{k+1}-class and $\phi'(\alpha) \neq 0, \pm 1$, then the extended Steffensen iteration has the $(k+1)$-th order convergence rate. Namely, $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^{k+1}$ for some constant C.*

Proof. Without loss of generality we can assume $\alpha = 0$ where α is a root of $x = \phi(x)$. We shall compute the leading term of the Taylor expansion of the iteration function $\Phi_k(x)$ around $x = 0$.

Let us perform the following operations to the determinants $A_k(x)$, $B_k(x)$. Setting

$$c_n := \frac{d^n\phi(0)}{dx^n}, \qquad n = 1, 2, \dots, \tag{3.23}$$

we first subtract the i-th row multiplied by c_1 from the $(i+1)$-th row for $i = 1, 2, \dots$. On the next step, we subtract the i-th row multiplied by $c_1{}^2$ from the $(i+1)$-th row for $i = 2, 3, \dots$. We do the similar operations recursively. Then we can express the Hankel determinants (3.19) and (3.20) as

$$A_k(x) = \begin{vmatrix} a_{1,0}(x) & a_{1,1}(x) & \cdots & a_{1,k}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,k+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,k}(x) & a_{k+1,k+1}(x) & \cdots & a_{k+1,2k}(x) \end{vmatrix}, \tag{3.24}$$

$$B_k(x) = \begin{vmatrix} b_{1,0}(x) & b_{1,1}(x) & \cdots & b_{1,k-1}(x) \\ b_{2,1}(x) & b_{2,2}(x) & \cdots & b_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,k-1}(x) & b_{k,k}(x) & \cdots & b_{k,2k-2}(x) \end{vmatrix}. \tag{3.25}$$

Here we define $a_{m,j}(x)$ and $b_{m,j}(x)$ as

$$a_{1,j}(x) := \phi_j(x), \qquad j = 0, 1, \dots, 2k, \tag{3.26}$$

$$a_{m+1,j}(x) := a_{m,j}(x) - c_1{}^m\, a_{m,j-1}(x), \quad m = 1, 2, \dots, k, \quad j = m, m+1, \dots, 2k, \tag{3.27}$$

$$b_{1,j}(x) := \delta^2\phi_j(x), \qquad j = 0, 1, \dots, 2k-2, \tag{3.28}$$

$$b_{m+1,j}(x) := b_{m,j}(x) - c_1{}^m\, b_{m,j-1}(x), \quad m = 1, 2, \dots, k-1, \quad j = m, m+1, \dots, 2k-2. \tag{3.29}$$

First we consider the Steffensen case where $k = 1$. By the Taylor expansion of $A_1(x)$ we see

$$a_{1,0}(x) = x, \qquad a_{1,1}(x) = c_1\, x + \frac{1}{2}c_2\, x^2 + \cdots, \tag{3.30}$$

$$a_{2,1}(x) = \frac{1}{2}c_2 x^2 + \cdots, \qquad a_{2,2}(x) = \frac{1}{2}c_1{}^2 c_2\, x^2 + \cdots. \tag{3.31}$$

Obviously, we have

$$A_1(x) = \frac{1}{2}c_1(c_1 - 1)c_2\, x^3 + \cdots, \tag{3.32}$$

$$B_1(x) = (c_1 - 1)^2\, x + \frac{1}{2}(c_1{}^2 + c_1 - 2)c_2\, x^2 + \cdots. \tag{3.33}$$

It follows from the condition $c_1 \neq 0, 1$ that $\Phi_1(x) = \mathrm{O}(x^2)$ as $x \to 0$. This proves the quadratic convergence.

Next we show that $\Phi_k(x) = O(x^{k+1})$ for any natural number k. The functions $a_{m,j}(x)$ take the form

$$a_{m,j}(x) = \phi_j(x) + \sum_{i=1}^{m-1} \beta_i^{(m)} \phi_{j-i}(x)\,, \qquad \beta_i^{(m)} := (-1)^i \sum_{0<p_1<\cdots<p_i<m} c_1{}^{p_1+p_2+\cdots+p_i}\,. \tag{3.34}$$

This can be checked by using the recurrence relation (3.27). We consider the n-th order derivative of the composition $\phi_j(x) = \phi_{j-1}(\phi(x))$, which is expressed as

$$\frac{d^n\phi_j(x)}{dx^n} = \sum_{r=1}^{n} \left(\frac{d^r\phi_{j-1}(\phi)}{d\phi^r} \sum_{\substack{q_1+\cdots+q_r=n \\ q_1\geq\cdots\geq q_r>0}} C(q_1,\ldots,q_r) \frac{d^{q_1}\phi(x)}{dx^{q_1}} \cdots \frac{d^{q_r}\phi(x)}{dx^{q_r}} \right) \tag{3.35}$$

for $n = 1, 2, \ldots$. Here $C(q_1,\ldots,q_r)$ are unique constants. We define the constants $C(q_1, q_2, \ldots)$ for $q_i \in \{0, 1, 2, \ldots\}$, $i = 1, 2, \ldots$, as follows:

(i) $C(q_1,\ldots,q_j,0) = C(q_1,\ldots,q_j)$,

(ii) $C(\ldots,q_i,\ldots,q_j,\ldots) = 0$ if $q_i < q_j$,

(iii) $C(1) = 1$,

$$C(q_1,\ldots,q_r) = \sum_{i=1}^{r} \kappa\, C(q_1,\ldots,q_{i-1},q_i-1,q_{i+1},\ldots,q_r) \text{ if } q_r > 0,$$

where κ is the number of the non-negative integers having the same values as $q_i - 1$ in the set $\{q_1,\ldots,q_{i-1},q_i-1,q_{i+1},\ldots,q_r\}$, namely,

$$\kappa := \#\{n = q_i - 1 \mid n \in \{q_1,\ldots,q_{i-1},q_i-1,q_{i+1},\ldots,q_r\}\}. \tag{3.36}$$

By use of (3.35) and $C(1,\ldots,1) = 1$, we write $\phi_j^{(n)}(0) := d^n\phi_j(0)/dx^n$ as

$$\phi_j^{(n)}(0) = c_1{}^n \phi_{j-1}^{(n)}(0) + \sum_{r=1}^{n-1} \gamma_r^{(n)} \phi_{j-1}^{(r)}(0)\,, \qquad \gamma_r^{(n)} := \sum_{\substack{q_1+\cdots+q_r=n \\ q_1\geq\cdots\geq q_r>0}} C(q_1,\ldots,q_r) c_{q_1} c_{q_2} \cdots c_{q_r}\,. \tag{3.37}$$

Using (3.34) and (3.37), we see for $a_{m,j}^{(n)}(0) := d^n a_{m,j}(0)/dx^n$ as follows:

$$\begin{aligned}
a_{m,j}^{(n)}(0) &= \phi_j^{(n)}(0) + \sum_{i=1}^{m-1} \beta_i^{(m)} \phi_{j-i}^{(n)}(0) \\
&= c_1{}^n \phi_{j-1}^{(n)}(0) + \sum_{r=1}^{n-1} \gamma_r^{(n)} \phi_{j-1}^{(r)}(0) + \sum_{i=1}^{m-1} \beta_i^{(m)} \left(c_1{}^n \phi_{j-i-1}^{(n)}(0) + \sum_{r=1}^{n-1} \gamma_r^{(n)} \phi_{j-i-1}^{(r)}(0) \right) \\
&= c_1{}^n \left(\phi_{j-1}^{(n)}(0) + \sum_{i=1}^{m-1} \beta_i^{(m)} \phi_{j-1-i}^{(n)}(0) \right) + \sum_{r=1}^{n-1} \gamma_r^{(n)} \left(\phi_{j-1}^{(r)}(0) + \sum_{i=1}^{m-1} \beta_i^{(m)} \phi_{j-1-i}^{(r)}(0) \right) \\
&= c_1{}^n a_{m,j-1}^{(n)}(0) + \sum_{r=1}^{n-1} \gamma_r^{(n)} a_{m,j-1}^{(r)}(0)\,.
\end{aligned} \tag{3.38}$$

We insert (3.38) into the n-th order derivative of (3.27) to derive

$$a^{(n)}_{m+1,j}(0) = ({c_1}^n - {c_1}^m)a^{(n)}_{m,j-1}(0) + \sum_{r=1}^{n-1} \gamma^{(n)}_r a^{(r)}_{m,j-1}(0) . \tag{3.39}$$

Assume that $a_{m,j}(x) = \mathrm{O}(x^m)$, namely,

$$a^{(n)}_{m,j}(0) = 0 , \qquad \text{for} \quad n < m, \tag{3.40}$$

$$a^{(n)}_{m,j}(0) \neq 0 , \qquad \text{for} \quad n = m. \tag{3.41}$$

The right hand side of (3.39) is equal to 0 for $n \leq m$. While $a^{(m+1)}_{m+1,j}(0)$ is not equal to 0 when $c_1 \neq 0, 1$. Then it follows that

$$a^{(n)}_{m+1,j}(0) = 0 , \qquad \text{for} \quad n < m+1, \tag{3.42}$$

$$a^{(n)}_{m+1,j}(0) \neq 0 , \qquad \text{for} \quad n = m+1. \tag{3.43}$$

This implies that $a_{m+1,j}(x) = \mathrm{O}(x^{m+1})$. By induction we find that $a_{m,j}(x) = O(x^m)$ for any natural number m. Therefore, the Taylor expansion of $a_{m,j}(x)$ is given by

$$a_{m,j}(x) = a^{(m)}_{m,j}(0)\, x^m + \cdots . \tag{3.44}$$

On the other hand, we can easily find that

$$b_{m,j} = a_{m,j+2} - 2\, a_{m,j+1} + a_{m,j} , \qquad m = 1, 2, \ldots, k, \qquad j = m-1, m, \ldots, 2k-2 \tag{3.45}$$

from the definition (3.29). Then we obtain

$$b^{(n)}_{m,j}(0) = 0 , \qquad \text{for} \quad n < m, \tag{3.46}$$

$$b^{(n)}_{m,j}(0) = ({c_1}^m - 1)^2 a^{(m)}_{m,j}(0) \neq 0 , \qquad \text{for} \quad n = m \tag{3.47}$$

from (3.38), (3.44), (3.45) and the condition $c_1 \neq \pm 1$. Hence we have

$$b_{m,j}(x) = b^{(m)}_{m,j}(0)\, x^m + \cdots \tag{3.48}$$

for any natural number m.

Finally we consider the determinants $A_k(x)$ and $B_k(x)$. Let S_n be the set of permutations $\sigma = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ i_0 & i_1 & \cdots & i_{n-1} \end{pmatrix}$ of n-items. By virtue of (3.24), (3.25), (3.44) and (3.48), we see

$$A_k(x) = \sum_{\sigma \in S_{k+1}} \operatorname{sgn} \sigma \cdot a_{1,i_0} a_{2,1+i_1} \cdots a_{k+1,k+i_k} = L\, x^{(k+1)(k+2)/2} + \cdots , \tag{3.49}$$

$$B_k(x) = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot b_{1,i_0} b_{2,1+i_1} \cdots b_{k,k-1+i_{k-1}} = M\, x^{k(k+1)/2} + \cdots . \tag{3.50}$$

Here we define constant L as

$$L = \begin{vmatrix} a^{(1)}_{1,0}(0) & a^{(1)}_{1,1}(0) & \cdots & a^{(1)}_{1,k}(0) \\ a^{(2)}_{2,1}(0) & a^{(2)}_{2,2}(0) & \cdots & a^{(2)}_{2,k+1}(0) \\ \vdots & \vdots & \ddots & \vdots \\ a^{(k+1)}_{k+1,k}(0) & a^{(k+1)}_{k+1,k+1}(0) & \cdots & a^{(k+1)}_{k+1,2k}(0) \end{vmatrix}, \tag{3.51}$$

and M as

$$M = \begin{vmatrix} b^{(1)}_{1,0}(0) & b^{(1)}_{1,1}(0) & \cdots & b^{(1)}_{1,k-1}(0) \\ b^{(2)}_{2,1}(0) & b^{(2)}_{2,2}(0) & \cdots & b^{(2)}_{2,k}(0) \\ \vdots & \vdots & \ddots & \vdots \\ b^{(k)}_{k,k-1}(0) & b^{(k)}_{k,k}(0) & \cdots & b^{(k)}_{k,2k-2}(0) \end{vmatrix}. \tag{3.52}$$

This means that $\Phi_k(x) = \mathrm{O}(x^{k+1})$. The extended Steffensen iteration defined by the k-th Shanks transform has the $(k+1)$-th order convergence rate. □

In Theorem 3.1, we use the sequence generated by the simple iteration (3.3) with the iteration function (3.2). In the remaining part of this section, we replace the iteration function (3.2) by the Newton iteration function (3.1). To this end, let us set the function $\phi(x)$ in (3.18) as $\phi(x) := N(x) = x - f(x)/f'(x)$. If $f(x)$ is in C^2-class on the interval I and satisfy $f'(\alpha) \neq 0$ and $f''(\alpha) \neq 0$, then the function $\phi(x)$ satisfies $\phi'(\alpha) = 0$ and $\phi''(\alpha) \neq 0$ and the Newton iteration $\{y_{j+1} = \phi(y_j)\}$ locally converges to α quadratically. We have

Theorem 3.2. *If $\phi(x)$ is in $\mathrm{C}^{(k+2)2^{k-1}}$-class and $\phi'(\alpha) = 0$, $\phi''(\alpha) \neq 0$, then the extended Steffensen iteration has the $(k+2)2^{k-1}$-th order convergence rate. Namely, $|x_{n+1} - \alpha| \leq C|x_n - \alpha|^{(k+2)2^{k-1}}$ for some constant C.*

Proof. We restrict ourselves to the case where $\alpha = 0$, for simplicity. Along the line which is similar to Theorem 3.1, we shall compute the Taylor coefficients $\phi_j^{(n)}(0)$ of $\phi_j(x)$. From (3.37) and the conditions $c_1 = 0, c_2 \neq 0$, it is turned out that

$$\phi_j^{(n)}(0) = 0, \qquad \text{for} \quad n < 2^j - 1, \tag{3.53}$$

$$\phi_j^{(n)}(0) \neq 0, \qquad \text{for} \quad n = 2^j. \tag{3.54}$$

Then we find $\phi_j(x) = \mathrm{O}(x^{2^j})$ and $\delta^2\phi_j(x) = \mathrm{O}(x^{2^j})$. We consider the Hankel determinant

$$A_k(x) = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}\,\sigma \cdot \phi_{i_0}\phi_{1+i_1}\phi_{2+i_2}\cdots\phi_{k+i_k}. \tag{3.55}$$

The leading term of $A_k(x)$ is given by the term $\phi_{i_0}\phi_{1+i_1}\cdots\phi_{k+i_k} = \mathrm{O}(x^{2^{i_0}+\cdots+2^{k+i_k}})$ which has the minimal degree in x. The degree becomes minimal when $i_0 = k$, $i_1 = k-1$, $i_2 = k-2, \ldots,$

$i_k = 0$. It follows that $A_k(x) = O(x^{(k+1)2^k})$. Similarly, $B_k(x) = O(x^{k2^{k-1}})$. Consequently, we see $\Phi_k(x) = O(x^{(k+2)2^{k-1}})$ which completes the proof of Theorem 3.2. □

In the book of Ostrowski [**66**, p. 252] a composition of the Newton iterations is formulated which has third-order convergence. The iteration in Theorem 3.2 with $k = 1$ provides third-order convergence. The extended Steffensen iteration in this case is also a composition of the Newton iterations, however, it is rather different from that in [**66**].

7. Numerical examples and computational complexity

In this section we present explicit examples to demonstrate how the extended Steffensen iteration acts. The computational complexity is also discussed.

All results of the numerical experiments are computed on the Intel Pentium Pro Processor 200 MHz. In Example 1 and 2, we examine the new iteration methods by use of the Mathematica version 3 (Wolfram Research, Inc.). In Example 3, we program them by the GNU C compiler version 2.7.2.

Example 1. The nonlinear equation to be solved is

$$f(x) = \exp(-x) - x = 0, \tag{3.56}$$

which has the unique solution $\alpha = 0.567143290409781 04129\cdots$. In order to apply Theorem 3.1, we set the iteration function $\phi(x)$ as

$$\phi(x) = \exp(-x). \tag{3.57}$$

It should be noted that $\phi'(\alpha) \neq 0, \pm 1$ and $\phi(x)$ satisfies the condition of Theorem 3.1. We compare several iteration methods. They are the simple iteration (3.3), the Steffensen iteration (3.4), and the extended Steffensen iteration (3.18) with $k = 2, 3, 4$. We choose the initial approximation as $x_0 = 0$, and generate the sequence $\{x_n\}$ until the condition

$$|f(x_{n^*})| < 10^{-r}, \qquad r = 1000 \tag{3.58}$$

is satisfied. Then x_{n^*} gives an approximation of the solution α. We compute the sequences in the multi precision arithmetic. In Figure 3.2, the quantity $\log_{10}|f(x_n)|$ is illustrated to estimate the error. In Table 3.1, we give the number n^* of iterations and an estimation of the convergence rate,

$$\frac{\log_{10}\left|\dfrac{x_{n^*-1} - x_{n^*}}{x_{n^*-2} - x_{n^*}}\right|}{\log_{10}\left|\dfrac{x_{n^*-2} - x_{n^*}}{x_{n^*-3} - x_{n^*}}\right|}, \tag{3.59}$$

by using four numbers $x_{n^*-3}, x_{n^*-2}, x_{n^*-1}$ and x_{n^*}.

It is shown that the iteration numbers n^* crucially depend on the iteration methods. On the convergence rate in Table 3.1, the estimated values are very close to the theoretical values for all iterations.

Example 2. Let us consider the same equation $f(x) = \exp(-x) - x = 0$ as in Example 1. We here replace the iteration function $\phi(x)$ by the Newton iteration function

$$\phi(x) = x + \frac{\exp(-x) - x}{\exp(-x) + 1}. \tag{3.60}$$

Obviously, $\phi'(\alpha) = 0$, $\phi''(\alpha) \neq 0$. Namely, $\phi(x)$ holds the condition in Theorem 3.2. Set the initial approximation as $x_0 = 0$. The sequences are computed in the multi precision arithmetic. In Figure 3.3 and Table 3.2, the Newton method (3.1) and the extended Steffensen iteration (3.18) with $k = 1, 2, 3, 4$ are illustrated. The estimated convergence rates seem to be good approximations of the theoretical rates.

TABLE 3.1. Number of iterations and convergence rate. (Example 1)

	number n^* of iterations	convergence rate	
		numerical	theoretical
simple iteration	4059	1.00†	1
Steffensen iteration	10	2.00	2
extended Steffensen iteration, $k = 2$	7	3.00	3
extended Steffensen iteration, $k = 3$	5	4.00	4
extended Steffensen iteration, $k = 4$	4	5.00	5

† This value is obtained by x_i,x_{i+1},x_{i+2} and x_{n^*} for $i = 10, 11, \ldots, 4045$. For $i > 4045$, the estimation of the convergence rate is quite different from 1.00.

TABLE 3.2. Number of iterations and convergence rate. (Example 2)

	number n^* of iterations	convergence rate	
		numerical	theoretical
Newton method	11	2.00	2
extended Steffensen iteration, $k = 1$	7	3.00	3
extended Steffensen iteration, $k = 2$	4	8.00	8
extended Steffensen iteration, $k = 3$	3	20.04	20
extended Steffensen iteration, $k = 4$	2	—‡	48

‡ Since x_{n^*-3} dose not exist, it is impossible to estimate the convergence rate.

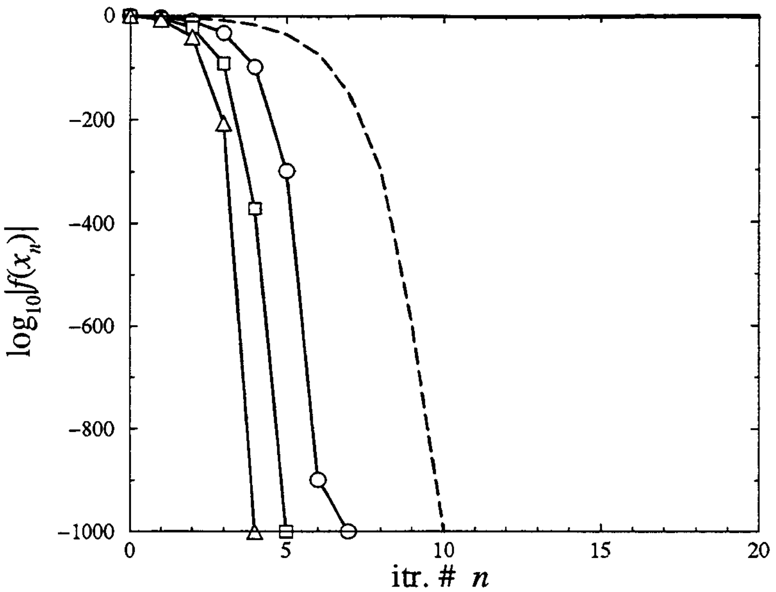

FIGURE 3.2. A comparison of $\log_{10}|f(x_n)|$ of several iteration methods when $\phi'(\alpha) \neq 0, \pm 1$. (Example 1) Solid line: simple iteration. Dashed line: Steffensen iteration. Circles, squares and triangles denote the extended Steffensen iteration for $k = 2, 3$, and 4, respectively.

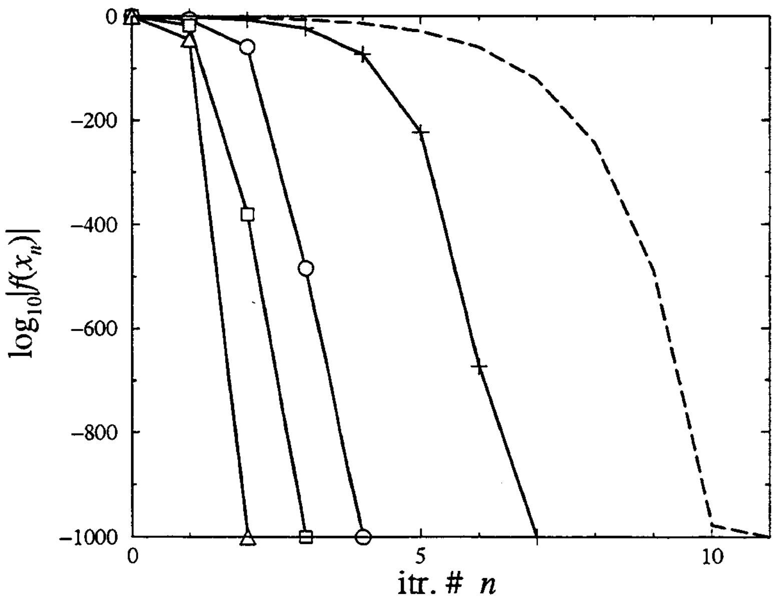

FIGURE 3.3. A comparison of $\log_{10}|f(x_n)|$ of several iteration methods when $\phi'(\alpha) = 0$, $\phi''(\alpha) \neq 0$. (Example 2) Dashed line: Newton method. Pluses, circles, squares and triangles denote the extended Steffensen iteration for $k = 1, 2, 3$, and 4, respectively.

Example 3. To discuss the computational complexity and the convergence property we solve the Kepler equation

$$f(x) := x - l - e\sin(x) = 0 \tag{3.61}$$

for various l and e, by using the simple iteration, the Newton method, the Steffensen iteration and the extended Steffensen iteration with $k = 2$. The Kepler equation appears in orbit determination in celestial mechanics and x, l and e are the eccentric anomaly, the mean anomaly and the

TABLE 3.3. Number of iterations and total numbers of mappings. (Example 3)

	number n^* of iterations			total numbers of mappings		
	average	maximal	$l = \frac{18\pi}{180}$ $e = 0.95$	average	maximal	$l = \frac{18\pi}{180}$ $e = 0.95$
simple iteration	45.94	2903	33	45.94	2903	33
Newton method	10.18	886	30	20.36	1772	60
Steffensen iteration	3.88	30	30	7.76	60	60
extended Steffensen iteration, $k = 2$	3.87	632	7	15.48	2528	28

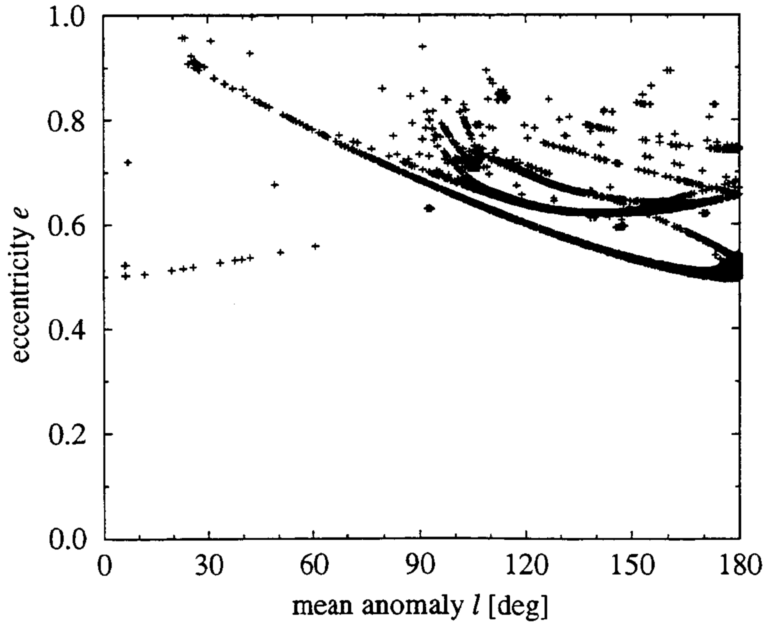

FIGURE 3.4. The parameters (l, e) for which the Newton iterations do not converge. (Example 3)

eccentricity, respectively. We solve the Kepler equation for x, where the remaining parameters l and e are fixed such that $0 \leq l \leq \pi$, $0 < e \leq 1$. Let $x_0 = l$ be the initial value. Let us set $\phi(x) := l + e\sin(x)$ and insert $\phi(x)$ into the iteration functions of the Steffensen and the extended

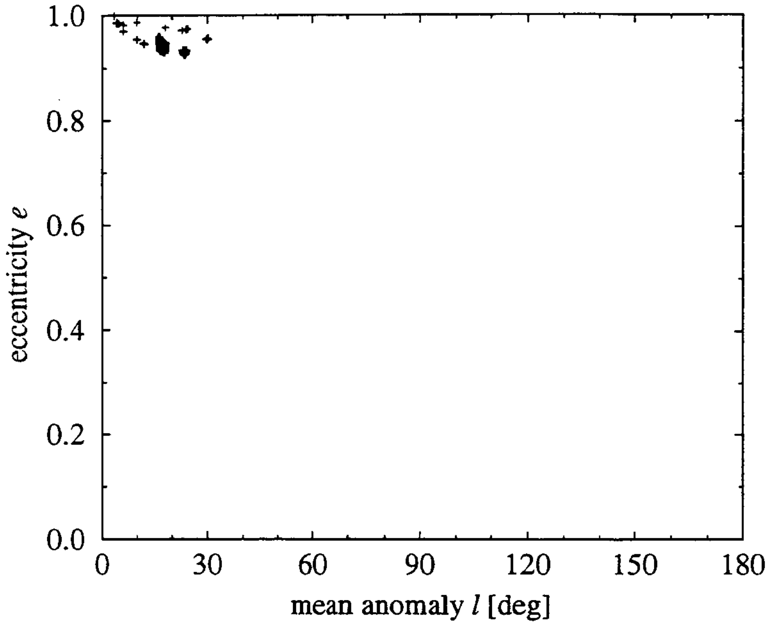

FIGURE 3.5. The parameters (l, e) for which the Steffensen iterations do not converge. (Example 3)

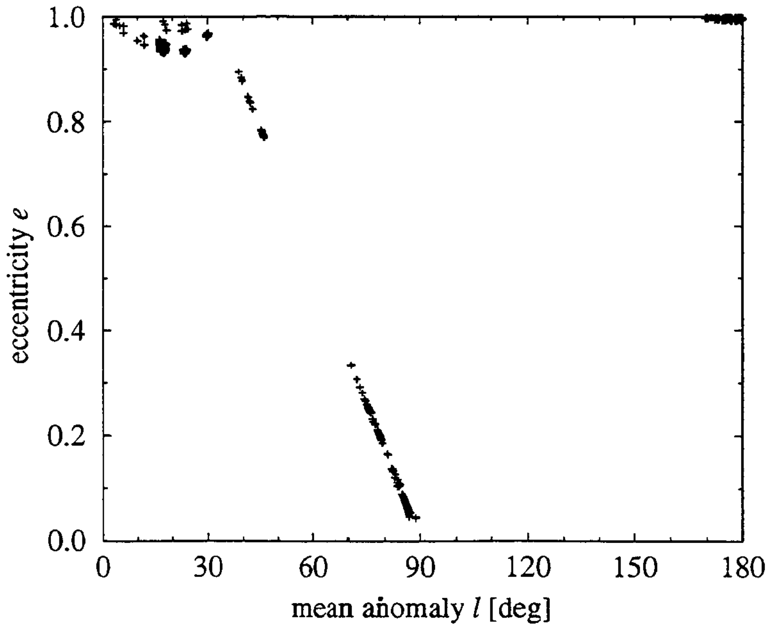

FIGURE 3.6. The parameters (l, e) for which the extended Steffensen iterations for $k = 2$ do not converge. (Example 3)

Steffensen iterations. We use $|f(x_{n*})| < 10^{-13}$ as the stopping criterion in the double precision arithmetic.

We first show the convergence property of the iterations. The simple iteration always converges for any pair of l and e. The marks in Figures 3.4, 3.5, and 3.6, indicate the pairs (l, e) for which the iterations do not converge. The mesh sizes of l and e in the figures are $0.01\pi/180$ and 0.001, respectively. We see that the Steffensen type iterations converges in more cases than the Newton method. There are some parameters for which the Steffensen iteration converge but the extended Steffensen iteration does not. The ratios of the number of all grid points to that of the marks in Figures 3.4, 3.5, and 3.6 are 0.06400% (Newton method), 0.02732% (Steffensen iteration) and 0.03536% (the extended Steffensen iterations), respectively.

Next, we illustrate the computational complexity with Table 3.3. We solve the Kepler equation for all parameters (l, e) such that $l = i\pi/180$, $i = 0, 1, \ldots, 180$ and $e = 0.01\,j$, $j = 1, 2, \ldots, 100$. The maximal and averaged numbers of iterations of each iteration method are shown in Table 3.3. The amount of computations of the ε-algorithm in the extended Steffensen iteration is negligible as compared with that of the mapping ϕ. Thus *the total numbers of mappings* are essential as well as the numbers of iterations in order to estimate the computational complexity. The simple iteration, the Newton method, the Steffensen iteration and the extended Steffensen iteration ($k = 2$), respectively, needs 1, 2, 2 and 4 mappings in one iteration. The total numbers of mappings are also shown in Table 3.3. The averaged and maximal total numbers of mappings of the Steffensen iteration is less than those of any other methods. However, the Steffensen iteration is the worst when $l = 18\pi/180$, $e = 0.95$. While the extended Steffensen iteration works well. For these special parameters, the extended Steffensen iteration is superior than other iterations.

8. Concluding remarks

In this chapter, we consider an extension of the Steffensen iteration in terms of the Shanks transform. The resulting iteration method does not need any derivatives and has a higher order convergence rate. If $\{\phi(y_j)\}$ converges linearly, then the sequence $\{\Phi_k(x_n)\}$ defined by using the k-th Shanks transform has the $(k+1)$-th order convergence rate (see Theorem 3.1). Here $\Phi_1(x)$ is just the Steffensen iteration function. On the other hand, if $\{\phi(y_j)\}$ converges quadratically, like the Newton sequence, then the iterated sequence $\{\Phi_k(x_n)\}$ has remarkably the $(k+2)2^{k-1}$-th order convergence rate (see Theorem 3.2). These theoretical convergence rates can be found in numerical examples (Examples 1, 2).

For the implementation of the extended Steffensen iteration, the stable ε-algorithm is especially useful to decrease the amount of computations in the calculation of Hankel determinants.

Consequently, the numbers of mappings take a major part of the computational complexity. It is shown (Example 3) that the extended Steffensen iteration with $k = 2$ has the minimal numbers of mappings in a special case of the Kepler equation. Moreover, the extended Steffensen iteration converges for more cases of parameters than the Newton method.

After the completion of this research the authors are told the references **[10]**, **[48]** by Professor N. Osada, which considers a generalized Steffensen iteration without any discussion on computational complexity. The idea in **[48]** is essentially the same as that in this thesis, however, there is no explicit numerical examples and no comparison to other iteration methods.

CHAPTER 4

Determinantal Solutions for Solvable Chaotic Systems and Iteration Methods Having Higher Order Convergence Rates

1. Introduction

The singularity confinement (SC) is a useful integrability criterion for discrete nonlinear dynamical systems [**25**]. The discrete Painlevé equations and many discrete soliton equations pass the SC test. However the SC test is not sufficient to identify integrability. In the literature [**28**], Hietarinta and Viallet presented a discrete dynamical system which passes the SC test but possesses a numerically chaotic property. Then they proposed a more sensitive integrability test [**28, 8**] using the algebraic entropy. The algebraic entropy is defined by the logarithmic average of a growth of degrees of iterations. Both test are similar to each, and the algebraic entropy test is a more precise criterion than the SC test.

Many of good numerical algorithms are deeply connected to the nonlinear integrable systems. For example, the recurrence relation of the qd-algorithm, which is used for calculating a continued fraction, is equivalent to the discrete time Toda equation. And the recurrence relation of the ε-algorithm [**85**], which is a sequence convergence accelerator, is equivalent to the discrete potential KdV equation. From these results, one may conjecture that good numerical algorithms can be regared as integrable dynamical systems. Indeed, many of linearly convergent algorithms such as eigenvalue algorithms and sequence accelerators pass the SC type criteria (cf. [**68**]), and they are proved to be equivalent to soliton equations. However, the algorithms having higher order convergence rates, which give irreversible dynamical systems, do not pass the SC type criteria. The techniques in the nonlinear integrable systems cannot be directly adapted to them.

The arithmetic-harmonic mean (AHM) algorithm [**62**] is an irreversible system having an explicit solution, however does not pass the SC type criteria. According to the setting of initial conditions, it behaves as an algorithm having the second order convergence rate, or as a solvable chaotic system. In this chapter, we investigate such discrete dynamical systems and obtain their determinantal solutions. We deal with the Ulam-von Neumann (UvN) system [**77**] which is a solvable chaotic system, and with the discrete dynamical systems derived from the Newton method, an extension of the Newton method, the Steffensen method [**72**], and the extended

Steffensen method proposed in Chapter 3, which are iteration methods having higher order convergence rates.

In Section 2, we show the trigonometric solutions for the AHM algorithm and the UvN system in terms of addition formulas. Moreover we show the hierarchy of the UvN system. The AHM algorithm is equivalent to the Newton method for a quadratic equation. In Section 3, we introduce the Newton method and the Nourein method **[64, 16]** which is an extension of the Newton method. Applying these methods to a quadratic equation, we present the hierarchy of the Newton type iterations. In Section 4, we give addition formulas of the determinants of certain tridiagonal matrices. In Section 5, we show determinantal solutions for the discrete Riccati equation. In Section 6, we obtain determinantal solutions for the hierarchy of the Newton type iterations. In Section 7, determinantal solutions for the hierarchy of the UvN system are derived. In Section 8, we obtain determinantal solutions for the hierarchy of the Steffensen type iterations. In Section 9, we give some remarks.

2. Trigonometric solutions for solvable chaos systems

In this section, we introduce solvable chaotic systems which have trigonometric solutions. We shall show that these solutions are obtained in terms of some addition formulas.

Firstly, we consider the iteration

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \frac{2\,a_n b_n}{a_n + b_n}, \qquad n = 0, 1, 2, \ldots, \tag{4.1}$$

which is called the arithmetic-harmonic mean (AHM) algorithm **[62]**. The AHM algorithm has the following solutions. For the case $a_0 > b_0 > 0$, we have

$$a_n = N_1 \coth(2^n \sigma_1), \qquad b_n = N_1 \tanh(2^n \sigma_1). \tag{4.2}$$

For the case $a_0 > 0$, $b_0 < 0$, we have

$$a_n = N_2 \cot(2^n \sigma_2), \qquad b_n = -N_2 \tan(2^n \sigma_2). \tag{4.3}$$

Here the positive constants N_1, N_2, σ_1 and σ_2 are uniquely determined by the initial values a_0 and b_0. The solutions (4.2) and (4.3) are derived from the double angle formulas of $\coth(x)$ and $\cot(x)$,

$$\coth(2x) = \frac{\coth(x) + \tanh(x)}{2}, \qquad \tanh(2x) = \frac{2\,\coth(x)\tanh(x)}{\coth(x) + \tanh(x)}, \tag{4.4}$$

$$\cot(2x) = \frac{\cot(x) - \tan(x)}{2}, \qquad \tan(2x) = \frac{2\,\cot(x)\tan(x)}{\cot(x) - \tan(x)}, \tag{4.5}$$

respectively. The AHM algorithm has the conserved quantity $I = a_n b_n$, which can be easily checked by (4.1). Thus $I = a_0 b_0$. Using the conserved quantity I, we introduce the variable u_n such that $u_n = a_n = I/b_n$. Then we have the discrete dynamical system

$$u_{n+1} = \frac{1}{2}\left(u_n + \frac{I}{u_n}\right). \tag{4.6}$$

The system (4.6) can be also derived by applying the Newton method to the quadratic equation $f(z) = z^2 - I = 0$. The behaviors of u_n are illustrated in Figures 4.1 and 4.2. When the case $I = a_0 b_0 > 0$, the sequence u_n quadratically converges to the positive root of I (see Figure

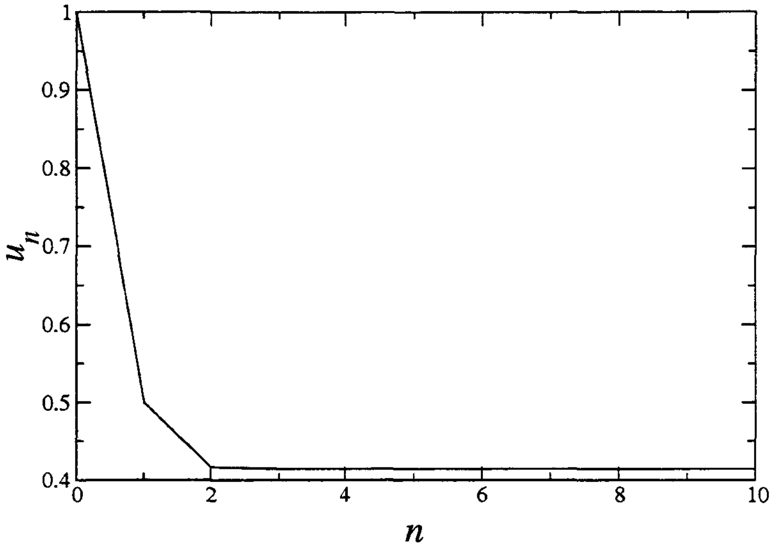

FIGURE 4.1. Behavior of the Newton method (4.6) for the case $I = a_0 b_0 > 0$.

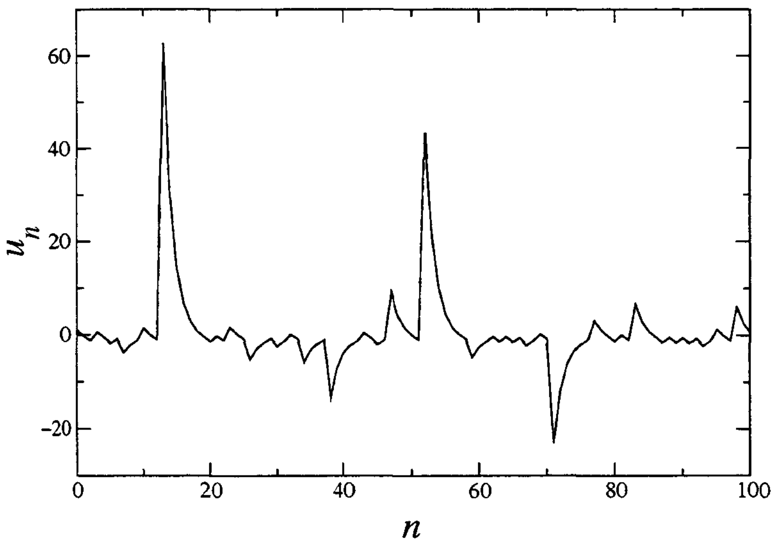

FIGURE 4.2. Behavior of the Newton method (4.6) for the case $I = a_0 b_0 < 0$.

4.1). When the case $I = a_0 b_0 < 0$, it behaves as a solvable chaotic system (see Figure 4.2). Its invariant measure is $\mu(dx) = dx/(\pi(1 + x^2))$, and its Lyapunov exponent is $\log 2$ (cf. **[79]**).

Next, we consider the solvable logistic map, or the Ulam-von Neumann (UvN) system **[77]**,

$$0 < u_0 < 1\,, \qquad u_{n+1} = 4u_n(1 - u_n)\,, \qquad n = 0, 1, 2, \dots\,. \tag{4.7}$$

A solution for (4.7) is obtained by

$$u_n = \sin^2(2^n \sigma_3)\,, \tag{4.8}$$

which is derived from the double angle formula of $\sin^2(x)$,

$$\sin^2(2x) = 4\sin^2(x)(1 - \sin^2(x))\,. \tag{4.9}$$

Here the constant σ_3 is determined by the initial value u_0. The invariant measure of the UvN system is $\mu(dx) = dx/(\pi\sqrt{x(1-x)})$, and the Lyapunov exponent of it is $\log 2$ (cf. **[79]**). By virtue of the n-tuple angle formulas of trigonometric functions, the higher order systems of the UvN system are given by

$$u_{n+1}^{(2)} = 4\,u_n^{(2)}(1 - u_n^{(2)})\,, \tag{4.10}$$

$$u_{n+1}^{(3)} = u_n^{(3)}(3 - 4\,u_n^{(3)})^2\,, \tag{4.11}$$

$$u_{n+1}^{(4)} = 16\,u_n^{(4)}(1 - u_n^{(4)})(1 - 2\,u_n^{(4)})^2\,, \tag{4.12}$$

$$u_{n+1}^{(5)} = u_n^{(5)}(5 - 4\,u_n^{(5)}(5 - 4\,u_n^{(5)}))^2\,, \tag{4.13}$$

and so on (cf. **[80]**). The superscripts m of $u_n^{(m)}$ denote the order of the hierarchy. Their invariant measures are all $\mu(dx) = dx/(\pi\sqrt{x(1-x)})$, and their Lyapunov exponents are respectively $\log m$ for $m = 3, 4, 5, \dots$. Another generalization of the UvN system having Jacobi or Weierstrass elliptic function solution is discussed in **[78]**.

An aim of this chapter is to obtain determinantal solutions for the discrete dynamical systems (4.6), (4.7) and their hierarchies. The hierarchy of (4.6) is introduced in Section 3, and the hierarchy of (4.7) already appear above.

3. The Newton method and the Nourein method

In this section, we introduce the Newton method (cf. **[13]**) and an extension of the Newton method for finding a root of an equation $f(z) = 0$. Furthermore we present a hierarchy of discrete dynamical systems given by the Newton type iterations.

The Newton method is given by

$$u_{n+1} = N(u_n)\,, \qquad n = 0, 1, 2, \dots\,, \tag{4.14}$$

$$N(z) = z - \frac{f(z)}{f'(z)}\,. \tag{4.15}$$

Here the prime denotes $f'(z) = df(x)/dz$. The Nourein method **[64, 16]**, which is an extension of the Newton method based on the Padé approximation, is given by

$$u_{n+1} = N_p(u_n)\,, \qquad n = 0, 1, 2, \dots\,, \tag{4.16}$$

$$N_p(z) = z - f(z)\frac{H_p(z)}{H_{p+1}(z)}\,, \tag{4.17}$$

where $H_p(z)$ are defined by

$$H_0(z) = 1\,, \qquad H_p(z) = \begin{vmatrix} c_1 & c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{p-1} & c_{p-2} & c_{p-3} & c_{p-4} & \cdots & c_0 \\ c_p & c_{p-1} & c_{p-2} & c_{p-3} & \cdots & c_1 \end{vmatrix}\,, \qquad p = 1, 2, 3, \dots\,, \tag{4.18}$$

and $c_j(z)$ denote

$$c_j(z) = \frac{1}{j!}\frac{d^j f(z)}{dz^j}\,, \qquad j = 0, 1, 2, \dots\,. \tag{4.19}$$

The convergence rate of the Nourein method is of order $p + 2$. When $p = 0$, 1, and 2, then the Nourein method (4.16) is reduced to the Newton method, the Halley method (cf. [**19**, pp. 220–221], [**64**]), and the Kiss method (cf. [**64**]), respectively.

Applying the Newton method (4.14) and the Nourein method (4.16) to the quadratic equation

$$f(z) := z^2 + 2b\,z + c = 0\,, \tag{4.20}$$

we obtain the following discrete dynamical systems for $p = 0, 1, 2, \dots$,

$$u^{(2)}_{n+1} = \frac{(u^{(2)}_n)^2 - c}{2\,u^{(2)}_n + 2b}\,, \tag{4.21}$$

$$u^{(3)}_{n+1} = \frac{(u^{(3)}_n)^3 - 3c\,u^{(3)}_n - 2bc}{3\,(u^{(3)}_n)^2 + 6b\,u^{(3)}_n + (4b^2 - c)}\,, \tag{4.22}$$

$$u^{(4)}_{n+1} = \frac{(u^{(4)}_n)^4 - 6c\,(u^{(4)}_n)^2 - 8bc\,u^{(4)}_n - (4b^2 - c)c}{4\,(u^{(4)}_n)^3 + 12b\,(u^{(4)}_n)^2 + 4(4b^2 - c)\,u^{(4)}_n + 4b(2b^2 - c)}\,, \tag{4.23}$$

and so on. The superscripts $m := p + 2$ of $u_n^{(m)}$ denote the order of the hierarchy. In Section 6, we shall obtain determinantal solutions for the hierarchy of the discrete systems (4.21)–(4.23).

4. Addition formula for tridiagonal determinant

In order to get solutions for the discrete dynamical systems corresponding to iteration methods and solvable chaotic systems, we derive an addition formula for tridiagonal determinants, which is an extension of addition formula for trigonometric function.

In this section, we present four lemmas for determinants. Let us consider the sequence of determinants of tridiagonal matrices,

$$\tau_n := \overbrace{\begin{vmatrix} \alpha & \beta & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & 1 & \alpha \end{vmatrix}}^{n}, \qquad n = 1, 2, 3, \dots, \tag{4.24}$$

where α and β are arbitrary complex constants. We set $\tau_{-1} := 0$ and $\tau_0 := 1$. It should be noted that τ_n is a monic polynomial of α of degree n. We can prove the following elementary lemmas.

LEMMA 4.1 (Three-term recurrence relation).

$$\tau_{-1} = 0, \qquad \tau_0 = 1, \qquad \tau_{n+1} = \alpha \tau_n - \beta \tau_{n-1}, \qquad n = 0, 1, 2, \dots. \tag{4.25}$$

Proof. In terms of the expansion of the determinant τ_{n+1} with respect to the last row, we derive (4.25). □

Here let us assume that β is a real positive constant. Setting

$$x := \frac{\alpha}{2\sqrt{\beta}}, \qquad T_0(x) := 1, \qquad T_n(x) := \frac{\tau_n}{2\sqrt{\beta^n}} - \frac{\tau_{n-2}}{2\sqrt{\beta^{n-2}}}, \qquad n = 1, 2, \dots, \tag{4.26}$$

we obtain the recurrence relation

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \qquad n = 1, 2, \dots, \tag{4.27}$$

from (4.25). The functions $T_n(x)$ are the Chebyshev polynomials of the first kind. The Chebyshev polynomials can be also expressed as

$$T_n(x) := \cos(n \arccos(x)). \tag{4.28}$$

Thus the determinants τ_n can be related to the trigonometric functions.

LEMMA 4.2 (Addition formula).

$$\tau_{n+m} = \tau_n \tau_m - \beta \tau_{n-1} \tau_{m-1}, \qquad n, m = 0, 1, \dots. \tag{4.29}$$

Proof. Formula (4.29) is a consequence of the Laplace expansion (cf. [**71**]) for the determinant τ_{n+m} with respect to the first n rows. We give an alternative proof here. Let us assume that $\tau_n \neq 0$. From a determinant partitioning formula for block matrices, it follows that

$$\tau_{n+m} = \tau_n \left| \begin{pmatrix} \alpha & \beta & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & 1 & \alpha \end{pmatrix} - \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} \alpha & \beta & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & 1 & \alpha \end{pmatrix}^{-1} \begin{pmatrix} & & & \\ & & & \\ & & & \\ \beta & & & \end{pmatrix} \right| \Bigg\} m \,. \tag{4.30}$$

We then have

$$\tau_{n+m} = \tau_n \begin{vmatrix} \alpha - \beta\tau_{n-1}/\tau_n & \beta & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & 1 & \alpha \end{vmatrix} . \tag{4.31}$$

Expanding the first row, we obtain

$$\tau_{n+m} = \tau_n(\alpha\tau_{m-1} - \beta\tau_{m-2}) - \beta\tau_{n-1}\tau_{m-1} \,. \tag{4.32}$$

Using Lemma 4.1, we derive (4.29). We have proved Lemma 4.2. □

Lemma 4.3 (Linear-bilinear identity).

$$\begin{aligned} &2\tau_m\tau_n - \tau_{m-1}\tau_{n+1} - \tau_{m+1}\tau_{n-1} \\ &\qquad = \beta^n(2\tau_{m-n} - \alpha\,\tau_{m-n-1}) \,, \qquad m \geq n \,, \qquad n = 0, 1, 2, \ldots \,. \end{aligned} \tag{4.33}$$

Proof. From Lemma 4.2, it follows that

$$2\tau_m\tau_n - \tau_{m-1}\tau_{n+1} - \tau_{m+1}\tau_{n-1} = \beta(2\tau_{m-1}\tau_{n-1} - \tau_{m-2}\tau_n - \tau_m\tau_{n-2}) \,. \tag{4.34}$$

Note that the indices of the right hand side of this relation are decreased by 1 rather than those of the left hand side. Calculating this relation recursively, we have

$$2\tau_m\tau_n - \tau_{m-1}\tau_{n+1} - \tau_{m+1}\tau_{n-1} = \beta^n(2\tau_{m-n}\tau_0 - \tau_{m-n-1}\tau_1 - \tau_{m-n+1}\tau_{-1}) \,. \tag{4.35}$$

From $\tau_{-1} = 0$, $\tau_0 = 0$ and $\tau_1 = \alpha$, this relation becomes to (4.33). We have proved Lemma 4.3. □

Lemma 4.4 (Differential relation). *If β is independent of α, then*

$$\frac{\partial\tau_{n+2}}{\partial\alpha} - \beta\frac{\partial\tau_n}{\partial\alpha} = (n+2)\tau_{n+1} \,, \qquad n = 0, 1, \ldots \,. \tag{4.36}$$

If α and β depend on the same parameter t then

$$\frac{\partial \tau_{n+2}}{\partial t} - \beta \frac{\partial \tau_n}{\partial t} = (n+2)\frac{\partial \alpha}{\partial t}\tau_{n+1} - (n+1)\frac{\partial \beta}{\partial t}\tau_n, \qquad n = 0, 1, \ldots. \tag{4.37}$$

Proof. A partial differentiation with respect α leads to

$$\frac{\partial \tau_{n+2}}{\partial \alpha} = \sum_{j=0}^{n+1} \tau_j \tau_{n+1-j}. \tag{4.38}$$

From Lemma 4.2, it follows that

$$\frac{\partial \tau_{n+2}}{\partial \alpha} = (n+2)\tau_{n+1} + \beta \sum_{j=1}^{n} \tau_{j-1}\tau_{n-j} = (n+2)\tau_{n+1} + \beta \sum_{j=0}^{n-1} \tau_j \tau_{n-1-j}. \tag{4.39}$$

From (4.38), we obtain

$$\frac{\partial \tau_{n+2}}{\partial \alpha} = (n+2)\tau_{n+1} + \beta \frac{\partial \tau_n}{\partial \alpha}. \tag{4.40}$$

Thus we have proved (4.36). The relation (4.37) can be proved by a similar line of thought. □

5. Determinantal solution for the discrete Riccati equation

In order to get solutions for the discrete dynamical systems corresponding to the Newton type iterations and the Steffensen type iterations, we give the determinantal solution for the discrete Riccati equation.

Let us consider the discrete Riccati equation

$$x_{n+1} = \frac{a\, x_n + b}{c\, x_n + d}, \qquad n = 0, 1, 2, \ldots, \tag{4.41}$$

where x_n is a complex variable, and a, b, c and d are complex constants. When we set the parameters as

$$x_n = X(t), \quad t = n\delta, \quad a = 1 + B\delta, \quad b = C\delta, \quad c = -A\delta, \quad d = 1 - B\delta, \tag{4.42}$$

and take the limit as $\delta \to 0$, then we have the differential Riccati equation

$$\frac{dX(t)}{dt} = A\, X(t)^2 + 2B\, X(t) + C \tag{4.43}$$

with the constant coefficients A, B and C.

A determinantal solution for Eq. (4.41) is obtained by

$$x_n = \frac{x_0\, \tau_n - (x_0\, d - b)\, \tau_{n-1}}{\tau_n - (a - x_0\, c)\, \tau_{n-1}}, \qquad n = 0, 1, 2, \ldots. \tag{4.44}$$

Here τ_n is the determinant of the degree n defined by

$$\tau_{-1} = 0, \qquad \tau_0 = 1, \qquad \tau_n = \overbrace{\begin{vmatrix} \alpha & \beta & & & \\ 1 & \alpha & \beta & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \alpha & \beta \\ & & & 1 & \alpha \end{vmatrix}}^{n}, \qquad n = 1, 2, 3, \dots, \tag{4.45}$$

where α and β denote $\alpha = a + d, \beta = ad - bc$.

From Lemma 4.1, the determinants τ_n satisfy the linear difference equation

$$\tau_{-1} = 0, \qquad \tau_0 = 1, \qquad \tau_{n+1} - (a+d)\tau_n + (ad - bc)\tau_{n-1} = 0, \qquad n = 0, 1, 2, \dots. \tag{4.46}$$

Substituting (4.44) into (4.41) using (4.46), we check that (4.44) gives a solution.

6. Determinantal solutions for hierarchy of the Newton iteration

In this section, we obtain determinantal solutions for the hierarchy of the Newton type iterations (4.21)–(4.23). The hierarchy is derived by applying the Newton type methods (4.14), (4.16) to the quadratic equation $f(z) = z^2 + 2bz + c$.

6.1. Determinantal solutions. We begin to consider the following discrete Riccati equation

$$v_{n+1} = \frac{v_0 v_n - c}{v_n + (v_0 + 2b)}, \qquad n = 0, 1, 2, \dots, \tag{4.47}$$

where b, c and v_0 are arbitrary complex values. Note that the initial value v_0 is included in the coefficients of the recurrence relation. From the determinantal solution for the discrete Riccati equation in Section 5, we obtain a solution for (4.47)

$$v_n = v_0 - B\frac{F_{n-1}}{F_n}, \qquad n = 0, 1, 2, \dots. \tag{4.48}$$

Here A and B denote $A = f'(v_0)$, $B = f(v_0)$, and F_n are the determinants defined by

$$F_{-1} = 0, \qquad F_0 = 1, \qquad F_n = \overbrace{\begin{vmatrix} A & B & & & \\ 1 & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A & B \\ & & & 1 & A \end{vmatrix}}^{n}, \qquad n = 1, 2, 3, \dots. \tag{4.49}$$

Next, we consider addition formulas for v_n, which are resulted from the following theorem.

THEOREM 4.1 (Addition formula). *The solution v_n (4.48) for Eq. (4.47) satisfies the relation*

$$v_{(m+1)n-1} = \frac{v_{mn-1}v_{n-1} - c}{v_{mn-1} + v_{n-1} + 2b}, \qquad m, n = 1, 2, 3, \ldots . \tag{4.50}$$

This relation gives the m-tuple addition formulas

$$v_{mn-1} = N_{m-2}(v_{n-1}), \qquad m = 2, 3, 4, \ldots, \qquad n = 0, 1, 2, \ldots, \tag{4.51}$$

where we define functions $N_p(z)$ by

$$N_p(z) := z - f(z)\frac{H_p(z)}{H_{p+1}(z)}, \qquad p = 0, 1, 2, \ldots, \tag{4.52}$$

$$H_0(z) := 1, \quad H_p(z) := \overbrace{\begin{vmatrix} f'(z) & f(z) & & & \\ 1 & f'(z) & f(z) & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & f'(z) & f(z) \\ & & & 1 & f'(z) \end{vmatrix}}^{p}. \tag{4.53}$$

Proof. First we shall prove the relation (4.50). From Lemmas 4.1 and 4.2, the determinants $F_{(m+1)n-2}$ and $F_{(m+1)n-1}$ are given by

$$F_{(m+1)n-2} = F_{mn-1}F_{n-1} - B\,F_{mn-2}F_{n-2}, \tag{4.54}$$

$$F_{(m+1)n-1} = A\,F_{mn-1}F_{n-1} - B\,F_{mn-1}F_{n-2} - B\,F_{mn-2}F_{n-1}. \tag{4.55}$$

Inserting (4.54) and (4.55) into

$$v_{(m+1)n-1} = v_0 - B\frac{F_{(m+1)n-2}}{F_{(m+1)n-1}}, \tag{4.56}$$

we have

$$v_{(m+1)n-1} = v_0 - B\frac{F_{mn-1}F_{n-1} - BF_{mn-2}F_{n-2}}{AF_{mn-1}F_{n-1} - BF_{mn-2}F_{n-1} - BF_{mn-1}F_{n-2}}. \tag{4.57}$$

Rearranging (4.57), we obtain

$$v_{(m+1)n-1} = \frac{\left(v_0 - B\dfrac{F_{mn-2}}{F_{mn-1}}\right)\left(v_0 - B\dfrac{F_{n-2}}{F_{n-1}}\right) - c}{\left(v_0 - B\dfrac{F_{mn-2}}{F_{mn-1}}\right) + \left(v_0 - B\dfrac{F_{n-2}}{F_{n-1}}\right) + 2b}. \tag{4.58}$$

The relation (4.58) leads to the proof of (4.50).

Next we shall prove the addition formulas (4.51). When $m = 2$, the formula (4.51) can be easily shown by using (4.50). Let us assume that v_{mn-1} satisfy (4.51) for a certain $m \geq 2$. Then we shall check that $v_{(m+1)n-1}$ satisfy the relation (4.51). From the assumption, we rewrite (4.50) as

$$v_{(m+1)n-1} = \frac{N_{m-2}(v_{n-1})\, v_{n-1} - c}{N_{m-2}(v_{n-1}) + v_{n-1} + 2b}\,. \tag{4.59}$$

From (4.52) it follows that

$$v_{(m+1)n-1} = v_{n-1} - \frac{f(v_{n-1})H_{m-1}(v_{n-1})}{(2v_{n-1} + 2b)H_{m-1}(v_{n-1}) - f(v_{n-1})H_{m-2}(v_{n-1})}\,. \tag{4.60}$$

Since $H_p(z)$ satisfy

$$H_p(z) = f'(z)\, H_{p-1}(z) - f(z)\, H_{p-2}(z)\,, \tag{4.61}$$

we have

$$v_{(m+1)n-1} = v_{n-1} - f(v_{n-1})\frac{H_{m-1}(v_{n-1})}{H_m(v_{n-1})} = N_{m-1}(v_{n-1})\,. \tag{4.62}$$

By induction, the addition formulas (4.51) are proved. □

Finally, we introduce the variables $u_n^{(m)}$ defined by

$$u_n^{(m)} := v_{m^n-1}\,, \qquad m = 2, 3, 4, \ldots, \qquad n = 0, 1, 2, \ldots. \tag{4.63}$$

Thus let us consider the map

$$u_0^{(m)} = v_0 \;\mapsto\; u_1^{(m)} = v_{m-1} \;\mapsto\; u_2^{(m)} = v_{m^2-1} \;\mapsto\; u_3^{(m)} = v_{m^3-1} \;\mapsto\; \cdots. \tag{4.64}$$

By virtue of the m-tuple addition formulas (4.51), we thus obtain the hierarchy of the discrete dynamical systems

$$u_{n+1}^{(2)} = \frac{(u_n^{(2)})^2 - c}{2\, u_n^{(2)} + 2b}\,, \tag{4.65}$$

$$u_{n+1}^{(3)} = \frac{(u_n^{(3)})^3 - 3c\, u_n^{(3)} - 2bc}{3(u_n^{(3)})^2 + 6b\, u_n^{(3)} + (4b^2 - c)}\,, \tag{4.66}$$

$$u_{n+1}^{(4)} = \frac{(u_n^{(4)})^4 - 6c(u_n^{(4)})^2 - 8bc\, u_n^{(4)} - (4b^2 - c)c}{4(u_n^{(4)})^3 + 12b\,(u_n^{(4)})^2 + 4(4b^2 - c)u_n^{(4)} + 4b^2(2b^2 - c)}\,, \tag{4.67}$$

and so on. These discrete systems are the same as the Newton type iterations (4.21)–(4.23). Therefore we obtain the determinantal solutions for the hierarchy of the Newton iterations by

$$u_n^{(m)} = u_0^{(m)} - B\frac{F_{m^n-2}}{F_{m^n-1}}\,, \qquad m = 2, 3, 4, \ldots, \qquad n = 0, 1, 2, \ldots, \tag{4.68}$$

from (4.48), (4.49), (4.63) and $A = f'(u_0^{(m)})$, $B = f(u_0^{(m)})$.

It is to be remarked that the determinantal solution (4.68) is also expressed as the continued fraction

$$u_n^{(m)} = u_0^{(m)} - \underbrace{\frac{B|}{|A} - \frac{B|}{|A} \cdots - \frac{B|}{|A}}_{m^n - 1} . \tag{4.69}$$

6.2. Other solutions. In the previous subsection, we have constructed the determinantal solutions (4.68) in terms of only four arithmetic operations. Here we ease this restriction. Let us allow to use the operation of square root. Then solutions of other type are obtained as follows,

$$u_n^{(m)} = u_0^{(m)} - \frac{r_2\, {r_1}^{m^n} - r_1\, {r_2}^{m^n}}{{r_1}^{m^n} - {r_2}^{m^n}} , \qquad n = 0, 1, 2, \ldots . \tag{4.70}$$

Here r_1 and r_2 are the roots of the characteristic equation

$$x^2 - A\,x + B = 0 , \qquad A = f'(u_0^{(m)}) , \qquad B = f(u_0^{(m)}) , \tag{4.71}$$

which is given by the three-term recurrence relation of F_n. When we use the roots λ_1, λ_2 of $f(z) = 0$, then we have

$$u_n^{(m)} = \frac{\lambda_2\,(u_0^{(m)} - \lambda_1)^{m^n} - \lambda_1\,(u_0^{(m)} - \lambda_2)^{m^n}}{(u_0^{(m)} - \lambda_1)^{m^n} - (u_0^{(m)} - \lambda_2)^{m^n}} , \qquad n = 0, 1, 2, \ldots . \tag{4.72}$$

The solution (4.72) is also expressed as

$$u_n^{(m)} = (\psi^{-1} \circ {R_m}^n \circ \psi)(u_0^{(m)}) , \tag{4.73}$$

where we define the functions $R(z)$, $\psi(z)$ as

$$R_m(z) := z^m , \qquad \psi(z) := \frac{z - \lambda_1}{z - \lambda_2} . \tag{4.74}$$

This result implies that the map N_{m-2} is conjugate with the map R_m, namely

$$N_{m-2} = \psi^{-1} \circ R_m \circ \psi . \tag{4.75}$$

The relation (4.75) yields the Julia set of the map N_{m-2} by

$$J(N_{m-2}) = \{\, w \,|\, w = \psi^{-1}(z), |z|^2 = 1, z \in \mathbb{C} \} . \tag{4.76}$$

The relation (4.75) with $m = 2$ was originally found by Cayley in 1879 (cf. [**67**]) for the Newton method.

7. Determinantal solutions for hierarchy of the Ulam-von Neumann system

7.1. Determinantal solutions. We begin to consider the following linear difference equation

$$v_{-1} = 2, \qquad v_0 = A, \qquad v_{n+1} - A\,v_n + B\,v_{n-1} = 0, \qquad n = 0, 1, 2, \dots, \tag{4.77}$$

where A, B are arbitrary complex constants. A determinantal solution for Eq. (4.77) is obtained by

$$v_n = F_{n+1} - B\,F_{n-1}, \qquad n = 0, 1, 2, \dots, \tag{4.78}$$

where F_n is the determinant of the degree n defined by

$$F_{-1} = 0, \qquad F_0 = 1, \qquad F_n = \overbrace{\begin{vmatrix} A & B & & & \\ 1 & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A & B \\ & & & 1 & A \end{vmatrix}}^{n}, \qquad n = 1, 2, 3, \dots. \tag{4.79}$$

Next we consider addition formulas for v_n, which are given by the following theorem.

THEOREM 4.2 (Addition formula). *The solution v_n for Eq.* (4.77) *satisfies the relation*

$$v_{(m+1)n-1} = v_{mn-1}v_{n-1} - B^n\,v_{(m-1)n-1}, \qquad m = 1, 2, 3, \dots, \qquad n = 1, 2, 3, \dots. \tag{4.80}$$

This relation gives the m-tuple addition formulas

$$v_{mn-1} = G_m(v_{n-1}) - B^n\,G_{m-2}(v_{n-1}), \qquad m = 2, 3, 4, \dots, \qquad n = 1, 2, 3, \dots, \tag{4.81}$$

where $G_m(z)$ are defined by

$$G_0(z) = 1, \qquad G_m(z) = \overbrace{\begin{vmatrix} z & B^n & & & \\ 1 & z & B^n & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & z & B^n \\ & & & 1 & z \end{vmatrix}}^{m}, \qquad m = 1, 2, 3, \dots. \tag{4.82}$$

Proof. First we shall prove the relation (4.80). From (4.78) and Lemma 4.2, it follows that

$$\begin{aligned} v_{(m+1)n-1} &= F_{mn+n} - B F_{(mn-1)+(n-1)} \\ &= (F_{mn}F_n - B F_{mn-1}F_{n-1}) - B(F_{mn-1}F_{n-1} - B F_{mn-2}F_{n-2}) \\ &= (F_{mn} - B F_{mn-2})(F_n - B F_{n-2}) - B(2F_{mn-1}F_{n-1} - F_{mn-2}F_n - F_{mn}F_{n-2}) . \end{aligned} \tag{4.83}$$

From (4.78) and Lemma 4.3, we have

$$v_{(m+1)n-1} = v_{mn-1}v_{n-1} - B^n(2F_{mn-n} - A F_{mn-n-1}) . \tag{4.84}$$

From (4.78) and Lemma 4.1, we obtain

$$v_{(m+1)n-1} = v_{mn-1}v_{n-1} - B^n v_{(m-1)n-1} . \tag{4.85}$$

Thus we have proved (4.80).

Next we shall prove (4.81). When $m = 2$, we can easily check (4.81) by (4.80). We assume that v_{mn-1} satisfy (4.81) for a certain $m \geq 2$. Then we shall show that $v_{(m+1)n-1}$ satisfy (4.81). From (4.80) and the assumption, we have

$$\begin{aligned} v_{(m+1)n-1} &= v_{mn-1}v_{n-1} - B^n v_{(m-1)n-1} \\ &= v_{n-1}(G_m(v_{n-1}) - B^nG_{m-2}(v_{n-1})) - B^n(v_{n-1}G_{m-1}(v_{n-1}) - B^nG_{m-3}(v_{n-1})) \\ &= (v_{n-1}G_m(v_{n-1}) - B^nG_{m-1}(v_{n-1})) - B^n(v_{n-1}G_{m-2}(v_{n-1}) - B^nG_{m-3}(v_{n-1})) . \end{aligned} \tag{4.86}$$

Since $G_m(z)$ satisfy

$$G_{m+1}(z) = zG_m(z) - B^n G_{m-1}(z) , \tag{4.87}$$

we obtain

$$v_{(m+1)n-1} = G_{m+1}(v_{n-1}) - B^n G_{m-1}(v_{n-1}) . \tag{4.88}$$

By induction, we have proved (4.81). □

In this paragraph, we finally derive the UvN hierarchy. We introduce the variables $u_n^{(m)}$ such that

$$u_n^{(m)} := v_{m^n-1} , \qquad m = 2,3,4,\dots , \qquad n = 0,1,2,\dots . \tag{4.89}$$

Namely, we consider the map

$$u_0^{(m)} = v_0 \mapsto u_1^{(m)} = v_{m-1} \mapsto u_2^{(m)} = v_{m^2-1} \mapsto u_3^{(m)} = v_{m^3-1} \mapsto \cdots . \tag{4.90}$$

By virtue of the m-tuple addition formulas (4.81), we derive a hierarchy of nonautonomous discrete dynamical systems

$$u_{n+1}^{(2)} = (u_n^{(2)})^2 - 2B^n \,, \tag{4.91}$$

$$u_{n+1}^{(3)} = (u_n^{(3)})^3 - 3B^n u_n^{(3)} \,, \tag{4.92}$$

$$u_{n+1}^{(4)} = (u_n^{(4)})^4 - 4B^n (u_n^{(4)})^2 + 2B^{2n} \,, \tag{4.93}$$

$$u_{n+1}^{(5)} = (u_n^{(5)})^5 - 5B^n (u_n^{(5)})^3 + 5B^{2n} u_n^{(5)} \,, \tag{4.94}$$

and so on. We remark that determinantal solutions for systems (4.91)–(4.94) can be obtained from (4.78) and (4.89). When we set $B = 0$ and replace the variable $u_n^{(2)}$ such that

$$u_n^{(2)} \to 1 - 2u_n \,, \tag{4.95}$$

we derive a solvable logistic map

$$u_{n+1} = 2\, u_n(1 - u_n) \,, \qquad n = 0, 1, 2, \dots \,. \tag{4.96}$$

The system (4.96) is not chaotic system for initial value $0 < u_0 < 1$, and it converges to $1/2$ exponentially. Next we set $B = 1$ and replace the variables $u_n^{(m)}$ such that

$$u_n^{(m)} \to 2(1 - 2u_n^{(m)}) \,. \tag{4.97}$$

Then we obtain the UvN hierarchy from (4.91)–(4.94) by

$$u_{n+1}^{(2)} = 4\, u_n^{(2)}(1 - u_n^{(2)}) \,, \tag{4.98}$$

$$u_{n+1}^{(3)} = u_n^{(3)}(3 - 4u_n^{(3)})^2 \,, \tag{4.99}$$

$$u_{n+1}^{(4)} = 16\, u_n^{(4)}(1 - u_n^{(4)})(1 - 2\, u_n^{(4)})^2 \,, \tag{4.100}$$

$$u_{n+1}^{(5)} = u_n^{(5)}(5 - 4\, u_n^{(5)}(5 - 4\, u_n^{(5)}))^2 \,, \tag{4.101}$$

and so on. Furthermore we obtain the determinantal solutions for the UvN hierarchy by

$$u_n^{(m)} = \frac{1}{2} - \frac{1}{4}\,(F_{m^n} - F_{m^n-2}) \,, \qquad m = 2, 3, 4, \dots \,, \qquad n = 0, 1, 2, \dots \,, \tag{4.102}$$

where $A = 2(1 - 2u_0^{(m)})$, $B = 1$ and F_n are defined by (4.79).

Relationship to the known determinantal solution **[11]** and the analytic solution **[69]** of the logistic map is not clear. The determinants which appear in **[11]** look rather different from F_{2^n}. Indeed, the value of the parameter μ of the logistic map $u_{n+1} = \mu\, u_n(1 - u_n)$ is not specified in **[11]** and **[69]**. Recently the quadratic map (4.7) in real and complex domains is reviewed in **[54]**.

7.2. Lyapunov exponents. Let us restrict the initial value $u_0^{(m)}$ to real values such that $0 < u_0^{(m)} < 1$. We shall compute the Lyapunov exponents of the UvN hierarchy without use of explicit invariant measures. Let us write $u_{n+1}^{(m)} = \Psi(u_n^{(m)})$ for $n = 0, 1, 2, \ldots$. The Lyapunov exponents are expressed as

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left| \Psi'(u_j^{(m)}) \right| . \tag{4.103}$$

Here we consider the partial differentiation of $\Psi(u_j^{(m)})$ with respect to $u_0^{(m)}$, which are given by

$$\frac{\partial \Psi(u_j^{(m)})}{\partial u_0^{(m)}} = \Psi'(u_j^{(m)}) \frac{\partial u_j^{(m)}}{\partial u_0^{(m)}} . \tag{4.104}$$

From (4.104) and $u_j^{(m)} = \Psi(u_{j-1}^{(m)})$, it follows that

$$\Psi'(u_j^{(m)}) = \frac{\dfrac{\partial \Psi(u_j^{(m)})}{\partial u_0^{(m)}}}{\dfrac{\partial \Psi(u_{j-1}^{(m)})}{\partial u_0^{(m)}}} , \qquad j = 0, 1, 2, \ldots, n-1 . \tag{4.105}$$

From (4.105) and $u_n^{(m)} = \Psi(u_{n-1}^{(m)})$, the Lyapunov exponents (4.103) are written as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial u_n^{(m)}}{\partial u_0^{(m)}} \right| . \tag{4.106}$$

Inserting the solution (4.102) into (4.106), we obtain

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \left(F_{m^n} - F_{m^n-2} \right)}{\partial u_0^{(m)}} \right| . \tag{4.107}$$

Using Lemma 4.4, we have

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log |m^n F_{m^n-1}| = \log m + \lim_{n\to\infty} \frac{1}{n} \log |F_{m^n-1}| . \tag{4.108}$$

Since F_n satisfy the second order linear difference equation

$$F_{n+1} - 2(1 - u_0^{(m)}) F_n + F_{n-1} = 0 , \tag{4.109}$$

the determinants F_n can be also expressed as

$$F_n = c_1 r_1{}^n + c_2 r_2{}^n . \tag{4.110}$$

Here r_1, r_2 are the roots of the characteristic equation

$$x^2 - 2(1 - u_0^{(m)}) x + 1 = 0 , \tag{4.111}$$

and c_1, c_2 are determined by the initial condition. From the condition $0 < u_0^{(m)} < 1$, it follows that $|r_1| = |r_2| = 1$. Thus there exists a positive constant M such that $0 \leq |F_n| < M$. Moreover we remove the zeros of $F_{m^n-1}(u_0^{(m)}) = 0$ from the initial region $0 < u_0^{(m)} < 1$, then we have $0 < |F_n| < M$. Therefore we obtain $L_0 < \log|F_{m^n-1}| < L_1$, where L_0 and L_1 are certain positive constants. It is concluded that

$$\lim_{n\to\infty} \frac{1}{n} \log|F_{m^n-1}| = 0\,. \tag{4.112}$$

We can state:

THEOREM 4.3. *Let us restrict the initial values* $u_0^{(m)}$ *to real values such that* $0 < u_0^{(m)} < 1$. *Then the Lyapunov exponents of the UvN hierarchy are* $\log m$.

8. Determinantal solutions for hierarchy of the Steffensen iteration

In this section, we give determinantal solutions for the discrete dynamical systems corresponding to the extended Steffensen method which is proposed in Chapter 3.

Let us consider the quadratic equation

$$f(z) := z^2 + 2bz + c = 0\,, \tag{4.113}$$

where z is a complex variable, and b, c are some complex constants. Rearranging Eq. (4.113), we have the equation,

$$z = \frac{(a-b)\,z - c}{z + (a+b)} =: \phi(z)\,, \tag{4.114}$$

where a is an auxiliary and arbitrary constant. We write the right hand side of Eq. (4.114) as $\phi(z)$. We consider the extended Steffensen method (3.18) for Eq. (4.113) with the simple iteration function $\phi(z)$. The hierarchy of the Steffensen type iterations are given by

$$u_{n+1}^{(m)} = \Phi_{m-1}(u_n^{(m)})\,, \qquad m = 2, 3, 4, \ldots\,, \qquad n = 0, 1, 2, \ldots\,, \tag{4.115}$$

from (3.18). In **[6, 7]**, Arai, Okamoto, and Kametaka find a new addition formula for $\cot(x)$ in terms of addition formulas for a three parameter family of functions. The aim of this section is to obtain determinantal solutions for the hierarchy (4.115) by using a theorem in **[6, 7]**.

To find solutions, we begin to consider the simple iteration

$$v_{n+1} = \phi(v_n) = \frac{(a-b)\,v_n - c}{v_n + (a+b)}\,, \qquad j = 0, 1, 2, \ldots\,. \tag{4.116}$$

We present addition formulas of v_n, which are resulted from the following theorem.

THEOREM 4.4 (Addition formula). *Let the auxiliary parameter a be set by $a = b + v_0$. The sequence v_n generated by the iteration* (4.116) *satisfies the m-tuple addition formulas*

$$v_{m(n+m-1)-1} = \overbrace{\begin{vmatrix} v_{n-1} & v_n & \cdots & v_{n+m-2} \\ v_n & v_{n+1} & \cdots & v_{n+m-1} \\ \vdots & \vdots & & \vdots \\ v_{n+m-2} & v_{n+m-1} & \cdots & v_{n+2m-3} \end{vmatrix}}^{m} \Bigg/ \overbrace{\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & v_{n-1} & v_n & \cdots & v_{n+m-2} \\ 1 & v_n & v_{n+1} & \cdots & v_{n+m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & v_{n+m-2} & v_{n+m-1} & \cdots & v_{n+2m-3} \end{vmatrix}}^{m+1}, \tag{4.117}$$

for $m = 2, 3, 4, \dots$ and $n = 1, 2, 3, \dots$.

Proof. In order to prove it, we first introduce the theorem in the literature [**6, 7**]. Let $p(z)$ be the three parameter family of functions defined by

$$p(z) := p(\alpha, \beta, \gamma; z) = \frac{\alpha\gamma^z - \beta}{\gamma^z - 1} \tag{4.118}$$

with $\alpha - \beta \neq 0, \gamma \neq 0$. Then functions $p(z)$ satisfies the addition formula

$$p(x_1 + x_2 + \cdots + x_m + y_1 + y_2 + \cdots + y_m) =$$

$$\begin{vmatrix} p(x_1+y_1) & p(x_1+y_2) & \cdots & p(x_1+y_m) \\ p(x_2+y_1) & p(x_2+y_2) & \cdots & p(x_2+y_m) \\ \vdots & \vdots & & \vdots \\ p(x_m+y_1) & p(x_m+y_2) & \cdots & p(x_m+y_m) \end{vmatrix} \Bigg/ \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & p(x_1+y_1) & p(x_1+y_2) & \cdots & p(x_1+y_m) \\ 1 & p(x_2+y_1) & p(x_2+y_2) & \cdots & p(x_2+y_m) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & p(x_m+y_1) & p(x_m+y_2) & \cdots & p(x_m+y_m) \end{vmatrix}, \tag{4.119}$$

provided that $\gamma^{x_i+y_j} \neq 1$.

We remark that $p(z)$ satisfies the relations

$$p(z+1) = \frac{(\alpha\gamma - \beta)\, p(z) - \alpha\beta(\gamma - 1)}{(\gamma - 1)\, p(z) + (\alpha - \beta\gamma)}, \tag{4.120}$$

$$p(x+y) = \frac{p(x)p(y) - \alpha\beta}{p(x) + p(y) - (\alpha + \beta)}. \tag{4.121}$$

In order to adapt the addition formula (4.119) to v_n, we compare the recurrence relations (4.116) with (4.120). From the comparison, we set the parameters as

$$\alpha + \beta = -2b, \qquad \alpha\beta = c, \qquad \gamma = \frac{2a + \alpha - \beta}{2a - \alpha + \beta}. \tag{4.122}$$

Thus we obtain $v_n = p(n + n_0)$, where the integer n_0 is determined by an initial condition. Here we choose $n_0 = 1$, namely, $v_0 = p(1)$. Then we should restrict the auxiliary parameter a as

$$a := b + v_0. \tag{4.123}$$

Inserting $x_i = n+i-1$, $y_j = j-1$ into (4.119), we therefore obtain the addition formulas (4.117).

□

Next we assume that there exists a natural number l for which $u_n^{(m)} = v_{l-1}$ holds for each step n. From Theorem 4.4 and the assumption, the iteration (4.115) is rewritten as

$$u_{n+1}^{(m)} = \Phi_{m-1}(u_n^{(m)}) = \Phi_{m-1}(v_{l-1}) = v_{m(l+m-1)-1} \tag{4.124}$$

for each step n. We shall determine a natural number l for each step n. Starting $u_0^{(m)} = v_0$ and $l = 1$, and computing the relation (4.124) recursively, we have

$$u_1^{(m)} = \Phi_{m-1}(u_0^{(m)}) = \Phi_{m-1}(v_0) = v_{m^2-1}\,, \qquad \text{if } l = 1\,, \tag{4.125}$$

$$u_2^{(m)} = \Phi_{m-1}(u_1^{(m)}) = \Phi_{m-1}(v_{m^2-1}) = v_{m^3+m^2-m-1}\,, \qquad \text{if } l = m^2\,, \tag{4.126}$$

$$u_3^{(m)} = \Phi_{m-1}(u_2^{(m)}) = \Phi_{m-1}(v_{m^3+m^2-m-1}) = v_{m^4+m^3-m-1}\,, \qquad \text{if } l = m^3+m^2-m\,, \tag{4.127}$$

and so on. By induction, we obtain

$$u_n^{(m)} = v_{m^{n+1}+m^n-m-1} = v_{(m+1)(m^n-1)}\,, \qquad n = 0, 1, 2, \ldots. \tag{4.128}$$

The relation (4.128) yields the map

$$u_0^{(m)} = v_0 \quad \mapsto \quad u_1^{(m)} = v_{(m+1)(m-1)} \quad \mapsto \quad u_2^{(m)} = v_{(m+1)(m^2-1)} \quad \mapsto \quad \cdots \tag{4.129}$$

which is the extended Steffensen iteration.

We finally obtain determinantal solution. Since the recurrence relation (4.116) is a discrete Riccati equation (4.41), the determinantal solution for v_n can be obtained from (4.44) and (4.45). By virtue of (4.128), we finally obtain the determinantal solution for (4.115) by

$$u_n^{(m)} = u_0^{(m)} - B\frac{F_{(m+1)(m^n-1)-1}}{F_{(m+1)(m^n-1)}}\,, \qquad n = 0, 1, 2, \ldots\,, \tag{4.130}$$

where $A = f'(u_0^{(m)})$, $B = f(u_0^{(m)})$ and

$$F_{-1} = 0\,, \qquad F_0 = 1\,, \qquad F_n = \overbrace{\begin{vmatrix} A & B & & & \\ 1 & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & A & B \\ & & & 1 & A \end{vmatrix}}^{n}\,, \qquad n = 1, 2, 3, \ldots. \tag{4.131}$$

It should be noted that the determinantal solution (4.130)–(4.131) is also expressed as the continued fraction

$$u_n^{(m)} = u_0^{(m)} - \underbrace{\frac{B|}{|A} - \frac{B|}{|A} \cdots - \frac{B|}{|A}}_{(m+1)(m^n-1)}. \tag{4.132}$$

We have constructed the determinantal solution (4.130)–(4.131) by only using four arithmetic operations. Here we ease this restriction. Let us allow to use the operation of square root. Another type solution for (4.115) is obtained by

$$u_n^{(m)} = p\left(\lambda_1,\ \lambda_2,\ \frac{u_0^{(m)} - \lambda_2}{u_0^{(m)} - \lambda_1};\ m^{n+1} + m^n - m\right) \tag{4.133}$$

from (4.118), (4.122), (4.128) and $v_n = p(n+1)$. Here λ_1 and λ_2 denote the roots of the equation $f(z) = 0$.

9. Concluding remarks

In this chapter, we have obtained the determinantal solutions for irreversible discrete equations. We have dealt with the hierarchy of the UvN system, and the hierarchies of discrete dynamical systems which are derived by applying the Newton type iterations and the Steffensen type iterations to a quadratic equation. According to the setting of parameters and initial conditions, these systems give rise to algorithms having higher order convergence rates, or solvable chaotic systems. For all cases, we have constructed the explicit solutions in a unified way.

Firstly, we have obtained the determinantal solutions v_n for the second order linear difference equation and the discrete Riccati equation. We have derived the addition formulas for the solutions v_n (Theorems 4.1, 4.2, 4.4). At the next step, we have focused only on the values v_{m^n} for integers $m \geq 2$. Then we have introduced the new variables $u_n^{(m)} = v_{m^n}$ for each m. Finally, we have showed that the addition formulas yield the irreversible dynamical systems of $u_n^{(m)}$. As a result, we have derived the hierarchies of new solvable irreversible dynamical systems and have obtained their determinantal solutions simultaneously.

From the determinantal solutions for the UvN hierarchy, we have obtained the Lyapunov exponents of them without explicit use of invariant measures (Theorem 4.3).

CHAPTER 5

Concluding Remarks

In this thesis, we have studied integrability of a continuous evolution equation and some discrete equations. As an application of the soliton theory, we have proposed a numerical algorithm based on the techniques in the nonlinear integrable systems.

In Chapter 2, we have considered the GDNLS equation. We first have constructed the traveling wave solution which is valid for any real values of parameters. We have applied the Painlevé test to the GDNLS equation for detecting integrability. We have shown that the equation possesses the Painlevé property in a strict sense only for the known integrable cases of parameters. Therefore we have shown that it possesses a conditional Painlevé property for an infinite number of cases of conditions for parameters, which is the same condition as that of the single-valued property of the traveling wave solution. When the GDNLS equation has the conditional Painlevé property, it is necessary for the function $\phi(x, t)$ to satisfy an equation which is transformed to the dispersionless KdV equation. We remark the interesting fact that the same condition for $\phi(x, t)$ appeared at the Painlevé analysis of the long and short wave interaction equation by Yoshinaga **[87, 88]**. Next we have examined stability of the solitary wave by the numerical simulation. Remarkable difference between integrable case and non-integrable case has not been observed, except for the quantities of ripples generated by interactions. The traveling wave solution is stable in interactions and behaves like a soliton. In conclusion, the GDNLS equation is a near-integrable system which has a conditional Painlevé property and a stable soliton-like traveling wave solution. Further theoretical analysis on stability may be necessary.

In Chapter 3, we have proposed an extension of the Steffensen iteration for finding a root α of the nonlinear equation $x = \phi(x)$. We have developed the extended Steffensen method in terms of the k-th Shanks transform which is a sequence convergence acceleration algorithm. The resulting iteration method does not need any derivative. And it has a higher order convergence rate, although the Shanks transform is originally a linearly convergent algorithm. If the equation satisfies $\phi'(\alpha) \neq 0, \pm 1$, then the sequence generated by the extended Steffensen method has the $(k + 1)$-th order convergence rate (Theorem 3.1). On the other hand, if the equation satisfies $\phi'(\alpha) = 0$, then the extended Steffensen iteration has remarkably the $(k+2)2^{k-1}$-th order convergence rate (Theorem 3.2). These theoretical convergence rates have been verified in numerical examples (Examples 1, 2). For the implementation of the extended Steffensen iteration, the ε-algorithm is especially useful to decrease the amount of computations in the calculation of Hankel determinants. This algorithm is stable for errors and equivalent to the discrete potential KdV equation. Consequently, computation due to the numbers of mappings takes a major part of the computational complexity. We have shown that the extended Steffensen iteration with $k = 2$ has the minimal numbers of mappings in a special case of the Kepler equation (Example 3). Moreover, the extended Steffensen iteration converges for more cases of parameters than the Newton method.

In Chapter 4, we have obtained the determinantal solutions for irreversible discrete equations. We have dealt with the hierarchy of the UvN system, and the hierarchies of discrete dynamical systems which are derived by applying the Newton type iterations and the Steffensen type iterations to a quadratic equation. According to the setting of parameters and initial conditions, these systems give rise to algorithms having higher order convergence rates, or solvable chaotic systems. For all cases, we have constructed the solutions in a unified way. Firstly, we have obtained the determinantal solutions v_n for some linear systems. We have derived the addition formulas for the solutions v_n. At the next step, we have focused only on the values v_{m^n} for integers $m \geq 2$. Then we have introduced the new variables $u_n^{(m)} = v_{m^n}$ for each m. Finally, we have showed that the addition formulas yield the irreversible dynamical systems of $u_n^{(m)}$. As a result, we have obtained the hierarchies of new solvable irreversible dynamical systems and their determinantal solutions simultaneously.

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Bibliography

[1] M. J. Ablowitz, A. Ramani, and H. Segur, Nonlinear evolution equations and ordinary differential equations of Painlevé type, Lett. Nuovo Cimento **23** (1978), 333–338.

[2] M. J. Ablowitz, A. Ramani, and H. Segur, A connection between nonlinear evolution and ordinary differential equations of P-type I, J. Math. Phys. **21** (1980), 715–721.

[3] M. J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, 1981.

[4] R. L. Adler and T. J. Rivlin, Ergodic and mixing properties of Chebyshev polynomials, Proc. Amer. Math. Soc. **15** (1964), 794–796.

[5] A. C. Aitken, On Bernoulli's numerical solution of algebraic equations, Proc. Roy. Soc. Edinburgh **46** (1926), 289–305.

[6] M. Arai, K. Okamoto, and Y. Kametaka, Aitken-Steffensen acceleration and a new additional formula for Fibonacci numbers, Proc. Japan Acad. A **62** (1986), 5–7.

[7] M. Arai, K. Okamoto, and Y. Kametaka, Aitken acceleration and Fibonacci numbers, Japan J. Appl. Math. **5** (1988), 145–152.

[8] M. P. Bellon, C.-M. Viallet, Algebraic entropy, Commun. Math. Phys. **204** (1999), 425–437.

[9] C. Brezinski, Accélération de la convergence en analyse numérique, Lecture Notes in Math. **584**, Springer-Verlag, Berlin, 1977.

[10] C. Brezinski, and R. Zagria, Extrapolation Methods: Theory and Practice, Elsevier, Amsterdam, 1991.

[11] M. Bruschi, Determinantal solution of the logistic map, J. Phys. A **31** (1998), L153–L155.

[12] F. L. Bauer, Nonlinear sequence transformations, Approximation of functions, Garbedian ed., Elsevier, New York, 1965, 134–151.

[13] F. Cajari, Historical note on the Newton-Raphson method of approximation, Amer. Math. Monthly **18** (1911), 29–33.

[14] F. Cariello and M. Tabor, Painlevé expansions for nonintegrable evolution equations, Physica D **39** (1989), 77–94.

[15] H. H. Chen, Y. C. Lee, and C. S. Liu, Integrability of nonlinear Hamilton systems by inverse scattering method, Physica Scripta **20** (1979), 490–500.

[16] G. Claessens, G. Loizou, and L. Wuytack, Comments on a root finding method using Padé approximation, BIT **17** (1977), 360–361.

[17] P. A. Clarkson and C. M. Cosgrove, Painlevé analysis of the non-linear Schrödinger family of equations, J. Phys. A **20** (1987), 2003–2024.

[18] D. A. Cox, The arithmetic-geometric mean of Gauss, L'Enseignement Mathématique **30** (1984), 275–330.

[19] A. Cuypt and L. Wuytack, Nonlinear Methods in Numerical Analysis, North-Holland, 1987.

[20] J. E. Dennis and R. B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice Hall, Englewood Cliffs, 1983.

[21] R. L. Devaney, An Introduction to Chaotic Systems, Benjamin/Cummings, New York, 1986.

[22] W. F. Ford and J. A. Pennline, Accelerated convergence in Newton's method, SIAM Rev. **38** (1996), 658–659.

[23] C. S. Gardner, J. M. Greene, M D. Kruskal, and R. M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. **19** (1967), 1095–1097.

[24] J. Gerlach, Accelerated convergence in Newton's method, SIAM Rev. **36** (1994), 272–276.

[25] B. Grammaticos, A. Ramani, and V. Papageorgiou, Do integrable mappings have the Painlevé property?, Phys. Rev. Lett. **67** (1991), 1825–1828.

[26] A. Hasegawa and Y. Kodama, Solitons in Optical Communications, Clarendon Press, Oxford, 1995.

[27] P. Henrici, Elements of Numerical Analysis, John Willey & Sons, New York, 1964.

[28] J. Hietarinta and C. Viallet, Singularity confinement and chaos in discrete systems, Phys. Rev. Lett. **81** (1998), 325–328.

[29] R. Hirota and K. Suzuki, Studies on lattice solitons by using electrical networks, J. Phys. Soc. Japan **28** (1970), 1366–1367.

[30] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. **27** (1971), 1192–1194.

[31] R. Hirota, Exact three-soliton solution of the two-dimensional sine-Gordon equation, J. Phys. Soc. Japan **35** (1973), 1566.

[32] R. Hirota, Exact envelope-soliton solution of a nonlinear wave equation, J. Math. Phys. **14** (1973), 805–809.

[33] R. Hirota, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, J. Math. Phys. **14** (1973), 810–814.

[34] R. Hirota, Exact N-soliton solution of a nonlinear lumped network equation, J. Phys. Soc. Japan **35** (1973), 286–288.

[35] R. Hirota, Exact N-soliton solution of nonlinear lumped self-dual network equations, J. Phys. Soc. Japan **35** (1973), 289–294.

[36] R. Hirota, A new form of Bäcklund transformations and its relation to the inverse scattering problem, Prog. Theor. Phys. **52** (1974), 1498–1512.

[37] R. Hirota and J. Satsuma, N-soliton solutions of nonlinear network equations describing a Volterra system, J. Phys. Soc. Japan **40** (1976), 891–900.

[38] R. Hirota and J. Satsuma, A variety of nonlinear network equations generated from the Bäcklund transformation for the Toda lattice, Supp. Prog. Theor. Phys. **59** (1976), 64–100.

[39] R. Hirota, Nonlinear partial difference equations. I. A difference analogue of the Korteweg-de Vries equation, J. Phys. Soc. Japan **43** (1977), 1424–1433.

[40] R. Hirota, Nonlinear partial difference equations. II. Discrete-time Toda equation, J. Phys. Soc. Japan **43** (1977), 2074–2078.

[41] R. Hirota, Nonlinear partial difference equations. III. Discrete sine-Gordon equation, J. Phys. Soc. Japan **43** (1977), 2079–2086.

[42] R. Hirota, Nonlinear partial difference equations. IV. Bäcklund transformation for the discrete-time Toda equation, J. Phys. Soc. Japan **45** (1978), 2079–2086.

[43] R. Hirota and J. Satsuma, A simple structure of superposition formula of the Bäcklund transformation, J. Phys. Soc. Japan **45** (1978), 1741–1750.

[44] R. Hirota, Nonlinear partial difference equations. V. Nonlinear equations reducible to linear equations, J. Phys. Soc. Japan **46** (1979), 312–319.

[45] R. Hirota and M. Wadati, A functional integral representation of the soliton solution, J. Phys. Soc. Japan **47** (1979), 1385–1386.

[46] R. Hirota, S. Tsujimoto, and T. Imai, Difference scheme of soliton equations, Future Directions of Nonlinear Dynamics in Physical and Biological Systems, P. L. Christiansen Ed., Plenum Press, 1993, 7–15.

[47] R. Hirota, unpublished.

[48] A. Iserles, Complex dynamics and convergence acceleration, IMA J. Numer. Anal. **11** (1991), 205–240.

[49] S. Kakei, N. Sasa, and J. Satsuma, Bilinearization of a generalized derivative nonlinear Schrödinger equation, J. Phys. Soc. Japan **64** (1995), 1519–1523.

[50] D. J. Kaup and A. C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. **19** (1978) 798–801.

[51] Y. Kodama and A. Hasegawa, Nonlinear pulse propagation in a monomode dielectric guide, IEEE J. QE-**23** (1987), 510–524.

[52] A. Kundu, Landou-Lifshitz and higher-oder nonlinear systems gauge generated from nonlinear Schrödinger type equations, J. Math. Phys. **25** (1984) 3433–3438.

[53] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. **21** (1968), 467–490.

[54] M. Lyubich, The quadratic family as a qualitatively solvable model of chaos, Notices Amer. Math. Soc. **47** (2000), 1042–1052.

[55] A. Mukaihira and Y. Nakamura, Schur flow for orthogonal polynomials on the unit circle and its integrable discretization, J. Comput. Appl. Math., to appear.

[56] A. Nagai and J. Satsuma, Discrete soliton equations and convergence acceleration algorithms, Phys. Lett. A **209** (1995), 305–312.

[57] A. Nagai, T. Tokihiro and J. Satsuma, The Toda molecule equation and the ε-algorithm, Math. Comput. **67** (1998), 1565–1575.

[58] A. Nakamura and H. H. Chen, Multi-soliton of a derivative Schrödinger equation, J. Phys. Soc. Japan **49** (1980), 813–816.

[59] Y. Nakamura, The BCH-Goppa decoding as a moment problem and a tau-function over finite fields, Phys. Lett. A **223** (1996), 75–81.

[60] Y. Nakamura, K. Kajiwara, and H. Shiotani, On an integrable discretization of the Rayleigh quotient gradient system and the power method with a shift, J. Comput. Appl. Math. **96** (1998), 77–90.

[61] Y. Nakamura, Calculating Laplace transformations in terms of the Toda molecule, SIAM J. Sci. Comput. **20** (1998), 306–317.

[62] Y. Nakamura, Algorithms associated with arithmetic, geometric and harmonic means and integrable systems, J. Comput. Appl. Math., to appear.

[63] T. Noda, The Aitken-Steffensen formula for systems of nonlinear equations V, Proc. Japan Acad. A **68** (1992), 37–40.

[64] A.-W. M. Nourein, Root determination by use of Padé approximants, BIT **16** (1976), 291–297.

[65] N. Osada, Extrapolation method for some singular fixed point sequences, Numer. Algorithms **3** (1992), 335–344.

[66] A. M. Ostrowski, Solution of Equations and Systems of Equations, Second Edition, Academic Press, New York, 1966.

[67] H.-O. Peitgen and P. R. Richter, The Beauty of Fractals, Springer-Verlag, 1986.

[68] V. Papageorgiou, B. Grammaticos, and A. Ramani, Integrable lattices and convergence acceleration algorithms, Phys. Lett. A **179** (1993), 111–115.

[69] S. Rabinovich, G. Berkolaiko, S. Buldyrev, A. Shehter and S. Havlin, "Logistic map": an analytical solution, Physica A **218** (1995), 457–460.

[70] D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, J. Math. and Phys. **34** (1955), 1–42.

[71] G. E. Shilov, Linear Algebra, Dover, New York, 1977.

[72] J. F. Steffensen, Remarks on iteration, Skand. Aktuar Tidskr. **16** (1933), 64–72.

[73] W. W. Symes, The QR algorithm and scattering for the finite nonperiodic Toda lattice, Physica D **4** (1982), 275–280.

[74] G. Szegö, Orthogonal Polynomials, Colloquium Publications Vol. 23, Amer. Math. Soc., Providence, 1939.

[75] M. Toda, Vibration of a chain with nonlinear interaction, J. Phys. Soc. Japan **22** (1967), 431–436.

[76] S. Tsujimoto, Studies on Discrete Nonlinear Integrable Systems, Doctor Thesis, Waseda Univ., 1997.

[77] S. M. Ulam and J. von Neumann, On combination of stochastic and deterministic processes, the 53rd summer meeting of AMS, Bull. Amer. Math. Soc. **53** (1947), 1120.

[78] K. Umeno, Method of constructing exactly solvable chaos, Phys. Rev. E **55** (1997), 5280–5284.

[79] K. Umeno, Superposition of chaotic processes with convergence to Lévy's stable law, Phys. Rev. E **58** (1998), 2644–2647.

[80] K. Umeno, Exactly solvable chaos and addition theorems of elliptic functions, RIMS Kokyuroku **1098** (1999), 104–117.

[81] R. Vein and P. Dale, Determinants and Their Applications in Mathematical Physics, Applied Mathematical Sciences **134**, Springer-Verlag, New York, 1999.

[82] A. P. Veselov, What is an integrable mapping?, in: What Is Integrability?, V. E. Zakharov ed., Springer-Verlag, Berlin, 1991, 251–272.

[83] M. Wadati, K. Konno, and Y. H. Ichikawa, A generalization of inverse scattering method, J. Phys. Soc. Japan **46** (1979), 1965–1966.

[84] J. Weiss, M. Tabor, and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. **24** (1983), 522–526.

[85] P. Wynn, On a device for computing the $e_m(S_m)$ transformation, Math. Tables Aids Comput. **10** (1956), 91–96.

[86] P. Wynn, On a procrustean technique for the numerical transformation of slowly convergent sequences and series, Proc. Camb. Phil. Soc. **52** (1956), 663–671.

[87] T. Yoshinaga and T. Kakutani, Solitary and e-shock waves in a resonant system between long and short waves, J. Phys. Soc. Japan **63** (1994), 445–549.

[88] T. Yoshinaga, Painlevé analysis of the long and short wave interaction equation, J. Phys. Soc. Japan **64** (1995), 3193–3200.

[89] N. J. Zabusky and M. D. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. **15** (1965), 240–243.

List of Authors Papers Cited in the Thesis and Related Works

Original Papers

[1] Koichi Kondo, Kenji Kajiwara, and Kai Matsui, Solution and integrability of a generalized derivative nonlinear Schrödinger equation, J. Phy. Soc. Japan **66** (1997), 60–66. (Chapter 2)

[2] Koichi Kondo and Yoshimasa Nakamura, An extension of the Steffensen iteration and its computational complexity, Interdisciplinary Information Sciences **4** (1998), 129–138. (Chapter 3)

[3] Koichi Kondo and Yoshimasa Nakamura, Determinantal solutions of solvable chaotic systems, preprint. (Chapter 4)

Proceedings Papers

[1] Koichi Kondo and Yoshimasa Nakamura, Newton-Steffensen-Shanks, RIMS Kokyuroku, Kyoto Univ.
近藤弘一, 中村佳正, ニュートン・ステファンセン・シャンクス, 京都大学数理解析研究所講究録 **1020** (1997), 28–38.

[2] Koichi Kondo and Yoshimasa Nakamura, A new iteration method of the Steffensen type having higher order convergence rate, RIMS Kokyuroku, Kyoto Univ.
近藤弘一, 中村佳正, 高次収束する Steffensen 型反復解法, 京都大学数理解析研究所講究録 **1040** (1998), 228–236.

[3] Koichi Kondo and Yoshimasa Nakamura, Determinantal solutions for the Newton method and solvable chaotic systems, RIAM Kokyuroku, Kyushu Univ.
近藤弘一, 中村佳正, ニュートン法と可解カオスの行列式解, 九州大学応用力学研究所研究集会報告 **11ME-S4** (2000), 53–58.

[4] Satoshi Tsujimoto and Koichi Kondo, Molecule solutions of discrete soliton equations and orthogonal polynomials, RIMS Kokyuroku, Kyoto Univ.
辻本諭, 近藤弘一, 離散方程式の分子解と直交多項式, 京都大学数理解析研究所講究録 **1170** (2000), 1–8.