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THEORETICAL STUDIES OF REACTOR NOISE ANALYSIS
BASED ON STATISTICAL PHYSICS

Doctor Thesis

by

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CHAPTER 1

Introduction

§1.1 Introduction

The term "reactor noise" is now settled to be a technical one for fluctuations in the output signal from a neutron or other sensor in a nuclear reactor. The fluctuations representing an excess or a deficiency from the average, result from many stochastic or random elemental processes such as neutron-scattering, absorption, fission, energy transport and coolant boiling etc. In other words, a nuclear reactor has fluctuations or "reactor noise" included in its state variables such as neutron number and temperature etc., which have macroscopically constant values in a steady state.

Since the fluctuations are random functions of time, we can obtain useful information concerning the dynamic behavior of the system by suitably processing them. We have a possibility of gaining not only the static but also the dynamic information of the system even in a steady state, by making use of "reactor noise". Such information is usually utilized for determining reactor time constants and dynamic characteristics, for assisting correct operation and for giving an early warning on unusual phenomena in the reactor system without perturbing the system. Therefore, the reactor noise analysis shows various potentiality of refinement on technical, economical and safety aspects, and is closely related to the diagnosis technique of reactor and the control theory etc.

Physicists have also been attracted by the problem, since the reactor noise phenomena include nuclear, thermal and hydrodynamic processes occurring in a non-equilibrium state. Consequently, one of the purposes of this thesis is to stress that the reactor noise phenomena is a good

example in the non-linear non-equilibrium statistical physics.

§1.2 Reactor Noise Analysis

The reactor noise analysis has already a history extending over a quarter of a century. The early theoretical work in this field was carried out by Courant and Wallance (1947)⁽¹⁾. This was followed with various techniques and theories developed by many authors, e.g., Orndoff (1957)⁽²⁾, Pál (1958)⁽³⁾, Moore (1959)⁽⁴⁾, Cohn (1960)⁽⁵⁾ and others. Until now, a large number of papers have been published. Various kinds of monographs and review papers have been also published by Thie (1963)⁽⁶⁾, Uhrig (1970)⁽⁷⁾, Seifritz and Stegemann (1971)⁽⁸⁾, Kosaly (1973)⁽⁹⁾, Saito (1970, 1974)⁽¹⁰⁾⁽¹¹⁾ and Williams (1974)⁽¹²⁾. When we analyze the nuclear reactor noise, it is appropriate to classify reactor noise phenomena into two categories; (1) zero power reactor noise (2) power reactor noise.

§1.2.1 Survey of Zero Power Reactor Noise Analysis

Since the thermal, mechanical and hydraulic effects are absent in a zero power reactor, we can describe it by state variables only of neutron and precursor. Namely, the fluctuations arise entirely from nuclear events such as fission, scattering, decay and absorption.

There are many established experimental methods for zero power reactor noise. One is the method in the time domain — the Rossi- α experiment, Feynmann- α experiment, waiting time analysis, auto- and cross-correlation function measurements or polarity correlation analysis etc. The other is the method in the frequency domain — the auto- and cross-power spectral density methods or the polarity correlation analysis etc.

By using these methods, we can estimate many reactor constants (prompt neutron-decay constant, time constant of precursor, neutron lifetime etc.) and can determine many reactor kinetic parameters (reactivity, reactor power, transfer function etc.).

The theory of zero power reactor noise has almost been established through various kinds of approaches or formulations made in many countries. Here, we will discuss the Kolmogorov and the Langevin methods as examples.

(a) The Kolmogorov Formalism

Pal⁽³⁾ and Bell⁽¹³⁾ calculated the probability distribution of neutrons, including delayed neutron effects, by means of the first collision probability method. This method is related to the backward equation within the framework of the Kolmogorov formalism for branching processes.

However, we discuss the forward Kolmogorov equation that was early presented by Courant and Wallance⁽¹⁾, since this is a simple description in order to understand the neutron population dynamics. For simplicity we neglect delayed neutrons. In the point model of zero power reactor the system is assumed to develop as a Markovian process and is described by a probability $P(N,t)$ that N neutrons are present in the system at time

t. Then, we can set up a balance equation for $P(N,t)$ as follows:

$$\begin{aligned}
 P(N,t+\Delta t) &= P(N-1,t) \times S\Delta t \\
 &+ \sum_{n=0}^{\infty} P(N-n+1,t) \times p(n) \times \frac{1}{\ell} \Delta t (N-n+1) \\
 &+ P(N,t) \times (1-S\Delta t - \Delta t \frac{N}{\ell}). \quad (1.1)
 \end{aligned}$$

Probability that there are N neutrons at time $t+\Delta t$
 Probability that there are (N-1) neutrons at time t
 Probability that the source emits one neutron during Δt

Probability that there are (N-n+1) neutrons at time t
 Probability that one neutron causes a absorption during Δt , given that there are (N-n+1) neutrons in the system

Probability that a neutron loss gives rise to n new neutrons

Probability that there are N neutrons at time t
 Probability that no neutrons are produced during Δt

When $\Delta t \rightarrow 0$, this equation reduces to:

$$\begin{aligned}
 \frac{d}{dt}P(N,t) &= \{P(N-1,t) - P(N,t)\}S \\
 &+ \sum_{n=0}^{\infty} P(N-n+1,t) \frac{N-n+1}{\ell} p(n) - \frac{N}{\ell} P(N,t). \quad (1.2)
 \end{aligned}$$

We may also obtain the moment equations instead of directly solving the above probability distribution equation (1.2).

(b) The Langevin Formalism

This method was suggested by Cohn⁽⁵⁾ in the field of neutron physics, and consists of a stochastic differential equation (the point reactor kinetic equation) with a random source term. This term accounts for a correlated neutron term resulting from the branching processes. The magnitude of the random source is determined by the Schottky formula. This is a rather heuristic formulation

and, at the same time, has an advantage of being applicable to power reactor noise. Generally, the Langevin equation is used as one of fundamental and important equations in the irreversible process of the statistical physics in order to describe not only the Brownian motion of particle but also many body problems.

The fluctuations $\delta x(t)$ satisfy the following equation

$$\frac{d}{dt}\delta x(t) = K \delta x(t) + R(t), \quad (1.3)$$

where K is the regression constant and $R(t)$ is a random term. The power spectrum of $R(t)$ is white [$\langle R(t_1)R(t_2) \rangle \propto \delta(t_1 - t_2)$]. The Langevin equation is a phenomenological stochastic equation to determine the process $\delta x(t)$ from the known process $R(t)$.

Usually, the behavior of the system is enough specified by estimating only the lower moments of $\delta x(t)$. In fact, we need only the mean values and correlation functions to specify the system, if the random force in Eq. (1.3) is Gaussian. Although the process of $R(t)$ in a zero power reactor is not Gaussian, the higher moments of $R(t)$ is not necessary in evaluating them, since the Langevin equation is linear. Namely, the evolution equation of moments do not have the hierarchical structure. On the other hand, the Gaussian distribution of neutron number has been usually observed experimentally. This ambiguity will be made clear in the later part of this thesis [cf. Chap. 2].

Although many varieties of formalism are present in zero power reactor noise theories, we may conclude that the main reason of its success comes from the following four hypotheses.

- (1) Markovian process
- (2) Linearity
- (3) Stationary process
- (4) Ergode hypothesis

The assumptions (3) and (4) are necessary for a practical

data processing or for an analysis of fluctuations.

§1.2.2 Power Reactor Noise Analysis

While the knowledge of zero power reactor noise stimulates basic scientific interest, the information obtained from the analysis of reactor noise should be applied to practical engineering purpose. In other words, it must be useful for diagnostics of existing faults of power reactors. In the case of zero power reactor noise, we have constructed simple models, since the elemental processes involved are clearly separable by the differences of their time scales. On the contrary, a power reactor has very different situations in noise from a zero power reactor, as it is usually a very complicated system involving a wide variety of phenomena such as nuclear, thermal, hydrodynamical and mechanical processes etc.

We will list below many processes and the ranges of frequencies over which they are important.

- 1) Aging processes ($<10^{-4}$ Hz)
- 2) Xe and Sm poisoning (10^{-4} - 10^{-2} Hz)
- 3) Control rod movements (10^{-3} - 10^{-1} Hz)
- 4) Coolant temperature (10^{-3} - 10^0 Hz)
- 5) Delayed neutron effects (10^{-2} - 10^0 Hz)
- 6) Fuel temperature (10^{-1} - 10^1 Hz)
- 7) Reactivity feedback of pressure (10^{-1} - 10^1 Hz)
- 8) Reactivity feedback of voids . (10^{-1} - 10^1 Hz)
- 9) Prompt neutron decay constant . (10^1 - 10^4 Hz)
- 10) Prompt neutron life time . . . (10^4 - 10^7 Hz)

Moreover, the patterns of fluctuations depend on not only the specific type of reactor but also conditions of its operation.

From these points, the noise analysis of power reactors is still in its infancy due mainly to a lack both of general theory and of knowledge about the complicated

noise mechanisms involved. Of course, there exist several formulations to attack such a complicated power reactor noise; the Langevin method, probability distribution method and input-output method of control theory etc. On the basis of these formulations, it is necessary to establish a general and transparent theory from the fundamental standpoint. Thereupon, the theory must be extended with greater applicability from zero power reactors to practical power reactors by an appropriate generalization of the mentioned frameworks. The following directions of generalization are considered as main themes of this thesis:

- (1) Non-linearity [Chaps. 2 & 3]
- (2) Non-stationary process [Chaps. 2 & 3]
- (3) Non-Markovian process [Chap. 5].

§1.3 Physical Basis of Reactor Noise and Main Subject of This Thesis

Before generalization of frameworks it is essential to study reactor noise phenomena from the statistical physics point of view. The number of neutron density in a reactor is an example of macrovariable defined as an extensive quantity. Though the magnitude of fluctuations of macrovariable is usually small in a macroscopic level, they are composed of a great number of interacting or noninteracting elemental units. For example, the fluctuations observed in a neutron detector result from the sum of statistical processes of many neutrons in a reactor. From this, it is one of the fundamental problems not only

in the reactor noise theory but also in the statistical mechanics to find how macroscopic reactions are explained from a number of microscopic processes.

It should be noted that a reactor noise is one of the phenomena in open systems since it has a neutron source and accompanies leakage of neutron and energy release etc. Therefore, it is important to attack the reactor noise within the framework of open systems. When a reactor is at power state, the non-linear effects of temperature feedback mechanisms become important. Then the power reactor is one of non-linear systems and is at times even in a non-stationary state. Furthermore, there remain flow mixing and turbulence in a coolant region as indefinite physical phenomena. Therefore, it is also important to treat the reactor noise phenomena as a non-linear non-equilibrium system. The generalization of reactor noise theory in this direction corresponds to the above-mentioned point (1) and (2) in §1.2.2.

In a non-linear system, the non-linearity brings variety of phenomena. Hence, the unified theory applicable to it must possess the possibility of sorting the information contained in it and of classifying its states in order to describe the various phenomena covering wide areas in the non-linear system. In the general framework of non-linear non-equilibrium systems, it is necessary to study what universal patterns, if there are, may appear in reactor noise phenomena by understanding the essential features without being puzzled its complicated aspects. The idea of the universal patterns suggests us the necessity of making use of the analogies between the reactor noise phenomena and the ones⁽¹⁴⁾⁻⁽¹⁶⁾ that are found in general non-linear non-equilibrium systems. Especially, the view of generalized phase transition phenomena will be useful in this sense.

Based on these considerations, we will use the system size expansion method⁽¹⁷⁾ recently developed in the

non-linear non-equilibrium statistical physics and apply it to the power reactor noise phenomena. Within this method, we will provide a new parameter in the reactor noise analysis. Hence, we will treat the general theory so as to be useful not only to the nuclear fission reactor noise but also to the nuclear fusion reactor noise.

Finally, it is most important to discuss the stability of power reactor from the practical viewpoint of safety. For diagnostic techniques based on the reactor noise analysis, it is an essential problem to discern the information particularly concerning the stability, by processing data of fluctuations. Before the system approaches an unstable region due to a certain change of its mechanisms, the fluctuations will show earlier symptom for instabilization of system than the mean values. In this case, the fluctuations may be well used as "forerunner phenomena", and then contain the useful information for instability. By catching this unusual phenomena beforehand, the reactor noise analysis serves to detection of anomalies or stability monitoring. Then, the method of system recognition developed in this line will be applicable not only to the reactor noise analysis but also to many other fields.

The quality of information is also transformed. A typical example is an operation of projecting information by observation. Namely, the equation describing fluctuations becomes a non-Markovian type as in the contraction of information in the statistical physics. Therefore, the generalization of reactor noise theory in this direction corresponds to the above-mentioned point (3) in §1.2.2. Furthermore, the stochastic model, e.g., the autoregressive model, which has been actively applied to the reactor noise for the system identification, and that was recently developed by Akaike⁽¹⁸⁾ in the time series analysis and control on the base of experimental data, possesses the non-Markovian property in its equation.

In the considerations as mentioned above, we will

study the reactor noise in this thesis. Let us state the contents of each chapter.

In Chap. 2, a new approach to the reactor noise theory is developed on the basis of recent studies on the theory of non-linear non-equilibrium statistical physics. Within the approach a basic equation is derived in a quite general form, which yields a solution of asymptotic character for large system. In the lowest-order approximation, that is, in the normal case, the formalism yields the conventional equations. This leads to a clear description of the relation between the Langevin and the Kolmogorov methods, and substantiates the assumption of Gaussian distribution of the number of neutrons. A rigorous analysis of the newly derived equation is made with the aid of flow patterns representing the Hamilton-Jacobi equation.

In Chap. 3, "irreversible circulation of fluctuation", α , is introduced as a new variable for the analysis of reactor noise in the normal case, which further develops our formulation based on the system size expansion method in Chap. 2. It is shown that α — considered in conjunction with the variance σ — provides useful information about reactor noise, apart from the data we usually obtain on power spectral density. The relations holding between α and the conventionally used variables are given for the case of steady state. The present formalism is applied to a non-linear system with three degrees of freedom (total neutron number, fuel energy and coolant energy), to examine numerically the behavior of the fluctuations. It is shown that the so-called soft- and hard-mode instabilities can be distinguished by observing α . It is also demonstrated that appropriate processing of such quantities as α and σ will provide advanced information on instabilities in power reactors.

In Chap. 4, the irreversible circulation α is proposed as a new statistical variable for a multi-dimensional time series analysis. It is a concept indicating the circula-

tion in a space of random state variables, while the variance σ is the concept corresponding to the spread of distribution of fluctuations. The pair of quantities σ and α are minimum necessary intergral indices to characterize any power spectral density and will be convenient for data retrieval. It is discussed that α is also less sensitive to filter effect in the low frequency region than σ .

In Chap. 5, a non-Markovian Langevin equation for the fluctuations of known state variables is proposed. The non-Markovian effect is due to "hidden" state variables. The random force does not necessarily meet the requirement of the causality condition. Then, a new relation between the fluctuation and the dissipation is also discussed.

Chapter 6 will be devoted to summarize the results obtained in each chapter and discuss the future problems of reactor noise.

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CHAPTER 2

System Size Expansion and Reactor Noise

§2.1 Introduction

The theory of zero-power reactor noise is now well established, and has provided us with useful methods of practical application to reactor technology. This success may be attributed to the physical consistency of the linear Langevin equation. It has led to general acceptance of the idea that fluctuations in power reactors can be described by simply adding a Langevin term or a random force to the rate equation. This phenomenological prescription is, however, not well substantiated by physical evidence. While the linear Langevin equation would appear intuitively to be plausible, some question remains as to the validity of extending it to non-linear and/or non-stationary cases.

Thus, it has become a problem of general concern in reactor noise theory to find a suitable formalism for non-linear and/or non-stationary cases. Such a formalism must be based on the theory of general non-linear non-equilibrium statistical physics.

The purpose of this chapter is to describe the neutron density fluctuations by means of the system size expansion theory developed by van Kampen⁽¹⁾, Kubo et al.⁽²⁾ and Tomita et al.⁽³⁾

The main point of the theory is an expansion in powers of the system size Ω to derive a new master equation for the probability $P(\mathbf{x}, t)$ of finding the system in a state \mathbf{x} at time t . (The state \mathbf{x} is represented by a set of macrovariables.) In general it is difficult to obtain a transparent solution of the master equation when it embodies non-linearity in a non-equilibrium situation. Using the space independence of the transitions between

states in a macrosystem, we can obtain an asymptotic solution of the master equation in the form

$$P(x, t) \propto e^{\Omega \phi_0(x, t)}$$

We shall show that this solution is quite useful for discussing non-linear non-stationary reactor noise. It also gives us a deeper insight into the conventional noise theory for zero-power reactors.

In the above system size expansion, it is logical to distinguish the intensive quantities from the extensive. For example, it stands to reason to use as variable the neutron density (the intensive variable) instead of the total number of neutrons (the extensive variable). In equilibrium state the magnitude of the density fluctuations around the mean value is of order $\Omega^{-1/2}$. This also holds in the usual non-stationary case, as it has been pointed out by van Kampen. We call such a case "normal". In the conventional reactor noise theory, no attention is paid to this dependence on system size, and this leaves an ambiguity, which is instanced in the theoretical result that branching processes generate a non-Gaussian distribution of neutrons, whereas a Gaussian distribution is actually observed in zero-power reactors.

On the basis of the theory of system size expansion, we shall derive a fundamental equation for the density fluctuations without assuming Gaussian random force. In the normal case, our formalism yields what corresponds to equations derived from the current Kolmogorov and Langevin formalisms. We shall prove that our equation can be replaced by an equivalent Hamilton-Jacobi expression in order to obtain a solution, for zero-power reactors, in the form of flow pattern. It is easy to extend this treatment to bring to light the characteristic patterns of non-linear effects occurring in power reactors. Our proposed equation is a kind of generalization of the Langevin equation that extends the consideration from a

stationary to a non-stationary case. And we shall show that this system size expansion method leads to a very useful and appropriate formalism for considering various reactor noise phenomena.

In this chapter we intend to develop a theory of power reactor noise in non-linear and/or non-stationary cases⁽⁴⁾⁻⁽⁶⁾.

We shall outline in §2.2 the method of system size expansion on the basis of the Markov assumption. In §2.3, we shall examine the range of applicability of the normal case. And the formalism presented in §2.2 will be used in §2.4 to clarify the ambiguity existing in the relation between the Langevin and the Kolmogorov methods, and also to extend the theory to the non-stationary case. In §2.5, we shall show flow patterns using the Hamilton-Jacobi method. This will be followed in §2.6 with the discussion of a system with randomly fluctuating parameters. The final §2.7 will give summary conclusions.

§2.2 System Size Expansion⁽²⁾

Let $X = \{X_i\}$ ($i=1,2,\dots,N$) be a set of N extensive macrovariables of the whole system. We regard them as stochastic variables including their fluctuation. The system is characterized by the Markovian process and is described by the master equation

$$\frac{\partial}{\partial t} P(X, t) = \int d\Delta X W(X - \Delta X, \Delta X, t) P(X - \Delta X, t) - \int d\Delta X W(X, \Delta X, t) P(X, t), \quad (2.1)$$

where $P(X, t)$ is the probability distribution function of

\mathcal{X} at time t , and $W(\mathcal{X}, \Delta\mathcal{X}, t)$ is the transition probability per unit time from \mathcal{X} to $\mathcal{X} + \Delta\mathcal{X}$. The integral in Eq. (2.1) is to be replaced by a summation if the X_i s are discrete. The right-hand side of this equation is expanded into a series in the form known as the Kramers-Moyal expansion:

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathcal{X}, t) &= - \int d\Delta\mathcal{X} (1 - e^{-\Delta\mathcal{X} \frac{\partial}{\partial \mathcal{X}}}) W(\mathcal{X}, \Delta\mathcal{X}, t) P(\mathcal{X}, t) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial \mathcal{X}}\right)^n \cdot C_n(\mathcal{X}, t) P(\mathcal{X}, t), \end{aligned} \quad (2.2)$$

where

$$C_n(\mathcal{X}, t) = \int d\Delta\mathcal{X} (\Delta\mathcal{X})^n W(\mathcal{X}, \Delta\mathcal{X}, t) \quad (2.3)$$

is the n -th moment of the transition probability.

Under the usual conditions of a macroscopic system, each event of a transition is localized. Hence the individual event can be described by intensive variables, and the transition probability W is proportional to its size Ω . This local nature is explicitly described by

$$W(\mathcal{X}, \Delta\mathcal{X}, t) = \Omega w(\mathbf{x}, \Delta\mathcal{X}, t), \quad (2.4)$$

where Ω is the system size, and \mathbf{x} the set of intensive macrovariables normalized per unit volume corresponding to \mathcal{X} , i.e.,

$$\mathbf{x} = \frac{\mathcal{X}}{\Omega} = \varepsilon \mathcal{X}, \quad \varepsilon = \Omega^{-1}. \quad (2.5)$$

Using a corresponding scale change

$$C_n(\mathbf{x}, t) = \int d\Delta\mathcal{X} (\Delta\mathcal{X})^n w(\mathbf{x}, \Delta\mathcal{X}, t) = \varepsilon C_n(\mathcal{X}, t)$$

and

$$\Omega^n P(\mathcal{X}, t) = P(\mathbf{x}, t), \quad (2.6)$$

we obtain the basic equation;

$$\begin{aligned} \frac{\partial}{\partial t} P(\mathbf{x}, t) &= - \frac{1}{\varepsilon} \int d\Delta\mathcal{X} (1 - e^{-\varepsilon \Delta\mathcal{X} \frac{\partial}{\partial \mathbf{x}}}) w(\mathbf{x}, \Delta\mathcal{X}, t) P(\mathbf{x}, t) \\ &= \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1}}{n!} \left(-\frac{\partial}{\partial \mathbf{x}}\right)^n \cdot C_n(\mathbf{x}, t) P(\mathbf{x}, t). \end{aligned} \quad (2.7)$$

For simplicity, let us limit to one macrovariable case ($N=1$) in the remaining part of this chapter. The fundamental solution of Eq. (2.7) with the initial condition

$$\lim_{t \rightarrow t_0} P(x, t | x_0, t_0) = \delta(x - x_0)$$

is given in the form

$$P(x, t | x_0, t_0) = T \left[e^{-\frac{1}{\varepsilon} \int_{t_0}^t ds H(x, \varepsilon \frac{\partial}{\partial x}, s)} \right] \delta(x - x_0), \quad (2.8)$$

where T is the time ordering operator and

$$H(x, \varepsilon \frac{\partial}{\partial x}, s) = \int d\Delta X \{ 1 - e^{-\varepsilon \Delta X (\frac{\partial}{\partial x})} \} w(x, \Delta X, s). \quad (2.9)$$

It is convenient to rewrite Eq. (2.8) in x representation;

$$\begin{aligned} \delta(x - x_0) &= \int \frac{d\beta}{\varepsilon} e^{i \frac{\beta}{\varepsilon} (x - x_0)} \\ P(x, t | x_0, t_0) &= \langle x | T e^{-\frac{1}{\varepsilon} \int_{t_0}^t ds H(x, \varepsilon \frac{\partial}{\partial x}, s)} | x_0 \rangle. \end{aligned} \quad (2.10)$$

Using the property of the Markov process, and noting the incommutability of x and $\frac{\partial}{\partial x}$, i.e.,

$$\begin{aligned} &\langle x' | e^{-\frac{\Delta t}{\varepsilon} H(x', \varepsilon \frac{\partial}{\partial x'}, t)} | x'' \rangle \\ &= \int_{-\infty}^{\infty} \frac{d\beta}{\varepsilon} e^{\left\{ -\frac{\Delta t}{\varepsilon} [H(x', i\beta, t) - i\beta \frac{x' - x''}{\Delta t}] + O(\Delta t^2) \right\}}, \end{aligned}$$

we express $P(x, t | x_0, t_0)$ by the path integral,

$$P(x, t | x_0, t_0) = \int d\mathcal{D}(x, \beta) e^{-\frac{1}{\varepsilon} \int_{t_0}^t ds [H(x, i\beta, s) - i\beta \dot{x}]}. \quad (2.11)$$

For the asymptotic evolution of the path integral about q , the method of steepest descent should prove useful. If there exists a suitable path of the macrosystem evolution, we can choose the col at $q=p$ determined by

$$\frac{\partial}{\partial p} H(x, p, s) = \dot{x},$$

and write

$$P(x, t | x_0, t_0) = \int d\mathcal{D}(x) e^{\frac{1}{\varepsilon} \int_{t_0}^t ds L(x, \dot{x}, s)}, \quad (2.12)$$

in terms of the Lagrangian L defined by

$$L(x, \dot{x}, t) = -H(x, p, t) + p \dot{x} . \quad (2.13)$$

Choosing the path that maximizes the action integral

$$\phi_0(x, t | x_0, t_0) = \int_{t_0}^t ds L(x, \dot{x}, s),$$

we have

$$P(x, t | x_0, t_0) = C' e^{\frac{1}{\epsilon} [\phi_0(x, t | x_0, t_0) + O(\epsilon)]} . \quad (2.14)$$

Thus, if the system has the initial condition

$$P(x_0, t_0) = C'' e^{\frac{1}{\epsilon} \phi_0(x_0, t_0)} ,$$

the probability distribution function of the macrovariable as determined by Eq. (2.7) has the asymptotic form

$$P(x, t) = C e^{\frac{1}{\epsilon} [\phi_0(x, t) + O(\epsilon)]} , \quad (2.15)$$

unless the system changes through a critical point (See Appendix). Moreover, this solution is not necessarily unique. If the probability has the form of expression (2.15), we say that the system has an "extensive property".

The solution (2.15) provides us with a convenient method for dealing with non-linear systems. This approach will be discussed later in association with the Hamilton-Jacobi method.

The basic equation (2.7) contains an expansion in terms of a small parameter ϵ , and one may terminate the expansion at a finite order. If the expansion is terminated at the second order, one obtains what is in form a Fokker-Planck type equation which has generally non-linear coefficients about x . On the other hand, van Kampen has pointed out that, in a case where the coefficients c_1 and c_2 do not explicitly depend on time, the above termination of expansion leads to inconsistent results when one takes account of the fact that the relative magnitude of fluctuations around the most probable path depends on the system size. In other words, in the normal case the

fluctuation is expected to be small quantity of order $\varepsilon^{1/2}$ compared with the mean value. We hence let⁽¹⁾

$$x = y(t) + \varepsilon^{1/2} \xi, \quad (2.16)*$$

and introduce a probability distribution function of ξ , defined by

$$P(\xi, t) = \varepsilon^{1/2} P(y(t) + \varepsilon^{1/2} \xi, t). \quad (2.17)$$

Then the basic equation (2.7) is transformed into the equation,

$$\frac{\partial}{\partial t} P(\xi, t) - \varepsilon^{1/2} \dot{y}(t) \frac{\partial}{\partial \xi} P(\xi, t) = \sum_{n=1}^{\infty} \frac{\varepsilon^{n/2}}{n!} \left(-\frac{\partial}{\partial \xi}\right)^n C_n(y(t) + \varepsilon^{1/2} \xi) P(\xi, t). \quad (2.18)$$

Comparing the term of $\varepsilon^{-1/2}$ on both sides, we find the phenomenological law for $y(t)$:

$$\frac{d}{dt} y(t) = C_1(y(t)). \quad (2.19)$$

Inserting this result into Eq. (2.18), we obtain an equation for the probability distribution function $P(\xi, t)$ of the fluctuations around the macroscopic value $y(t)$:

$$\begin{aligned} \frac{\partial}{\partial t} P(\xi, t) = & -\varepsilon^{-1/2} \frac{\partial}{\partial \xi} [C_1(y + \varepsilon^{1/2} \xi) - C_1(y)] P(\xi, t) \\ & + \sum_{n=2}^{\infty} \frac{\varepsilon^{n/2}}{n!} \left(-\frac{\partial}{\partial \xi}\right)^n C_n(y + \varepsilon^{1/2} \xi) P(\xi, t). \end{aligned} \quad (2.20)$$

Retaining only the lowest order terms on the right-hand side of Eq. (2.20), we have

$$\frac{\partial}{\partial t} P(\xi, t) = -\frac{\partial}{\partial \xi} \frac{\partial C_1(y)}{\partial y} \xi P(\xi, t) + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} C_2(y) P(\xi, t). \quad (2.21)$$

Equation (2.21) possesses the same form as the linear Fokker-Planck equation with the time-dependent coefficients determined by the most probable path $y(t)$. The solution of Eq. (2.21) has a Gaussian distribution for the fluctuation ξ :

$$P(\xi, t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\xi^2}{2\sigma}}, \quad (2.22)$$

* The applicability of this relation will be discussed in §2.3.

where

$$\sigma = \int \xi^2 P(\xi, t) d\xi, \quad (2.23)$$

and σ satisfies the following equation,

$$\frac{d}{dt} \sigma = 2 C_1(\gamma) \sigma + C_2(\gamma). \quad (2.24)$$

The basic equation (2.7), the concept of extensive property (2.15) and the scaling relation (2.16) are the bases of our new approach to the reactor noise theory. A summary of the present formulation is illustrated in Fig. 2.1. In the following sections, we apply this formalism to noise problems to demonstrate its effectiveness.

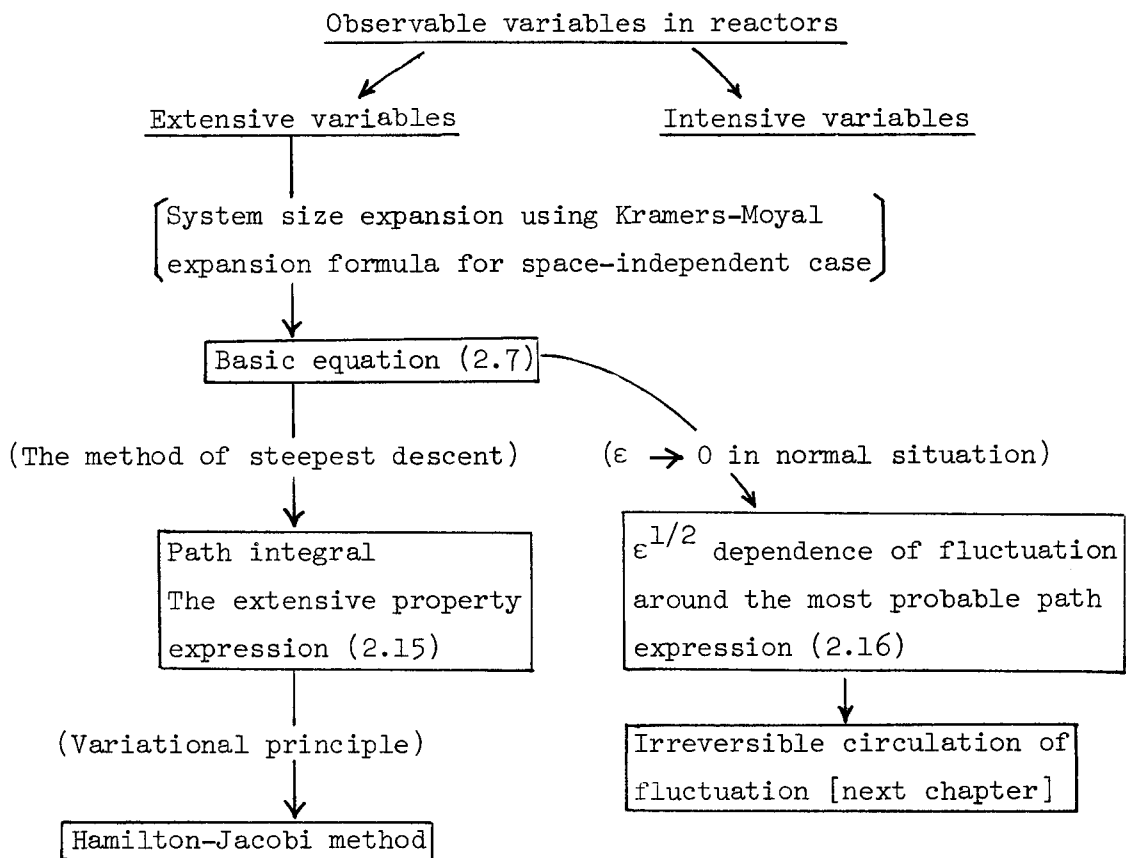


Fig. 2.1 Conceptual flow chart of system size expansion method

§2.3 Validity of the Normal Case ⁽⁶⁾

For the investigation of the validity of the normal case in a steady state system, we examine the two lowest order terms in the ϵ expansion of the master equation by using the perturbation theory. The ϵ^0 and $\epsilon^{1/2}$ terms in Eq. (2.20) give

$$\frac{\partial}{\partial t} P(\xi, t) = (H_0 + \epsilon^{1/2} H_1) P(\xi, t), \quad (2.25)$$

where

$$H_0 = -\frac{\partial}{\partial \xi} c_1'(y) \xi + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} c_2(y)$$

$$H_1 = -\frac{1}{2} \frac{\partial}{\partial \xi} c_1''(y) \xi^2 + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} c_2'(y) \xi - \frac{1}{6} \frac{\partial^3}{\partial \xi^3} c_3(y).$$

The probability distribution function $P(\xi, t) = P_0(\xi, t) + \epsilon^{1/2} P_1(\xi, t)$ is substituted into Eq. (2.25) to obtain the Fokker-Planck equation of order ϵ^0

$$\frac{\partial}{\partial t} P_0(\xi, t) = H_0 P_0(\xi, t), \quad [\text{cf. (2.21)}] \quad (2.26)$$

and of the next order $\epsilon^{1/2}$

$$\frac{\partial}{\partial t} P_1(\xi, t) = H_0 P_1(\xi, t) + H_1 P_0(\xi, t). \quad (2.27)$$

In the case of a steady state system, the solution of Eq. (2.26) is

$$P_0^s(\xi) = \frac{1}{\sqrt{2\pi\sigma_0^s}} e^{-\frac{\xi^2}{2\sigma_0^s}}, \quad (2.28)$$

so that

$$H_1 P_0^s \propto -\left[c_1''(y^s) + \frac{3c_2'(y^s)}{2\sigma_0^s} + \frac{c_3(y^s)}{2(\sigma_0^s)^2} \right] \xi e^{-\frac{\xi^2}{2\sigma_0^s}}$$

$$+ \left[\frac{c_1''(y^s)}{2\sigma_0^s} + \frac{c_2'(y^s)}{2(\sigma_0^s)^2} + \frac{c_3(y^s)}{6(\sigma_0^s)^3} \right] \xi^3 e^{-\frac{\xi^2}{2\sigma_0^s}}. \quad (2.29)$$

It is seen that in the order $\epsilon^{1/2}$ the effects on $P_0(\xi, t)$ coming from the terms c_1'' , c_2' and c_3 are qualitatively similar. Thus, we will adopt only the non-linear term $c_1(y^s)$ as a representative in order to study the non-linear effect in a steady state.

Then, we will be able to put the non-linear Langevin equation

$$\frac{d}{dt} x = C_1(x) + F(t), \quad (2.30)$$

where $F(t)$ is Gaussian random force and

$$\begin{cases} \langle F(t) \rangle = 0 \\ \langle F(t) F(t') \rangle = \varepsilon D \delta(t-t') \end{cases} \quad (2.31)$$

Using the scaling relations $x = y^S + \varepsilon^{1/2} \xi$ ($c_1(y^S) = 0$) and $F(t) = \varepsilon^{1/2} f(t)$, we obtain

$$\frac{d}{dt} \xi = C'_1(y^S) \xi + \varepsilon^{1/2} \frac{C''_1(y^S)}{2} \xi^2 + \dots + f(t). \quad (2.32)$$

We represent Eq. (2.32) in the form of Fokker-Planck equation

$$\frac{\partial}{\partial t} P(\xi, t) = -\frac{\partial}{\partial \xi} \left\{ C'_1(y^S) \xi + \varepsilon^{1/2} \frac{C''_1(y^S)}{2} \xi^2 + \dots \right\} P(\xi, t) + \frac{1}{2} \frac{\partial^2}{\partial \xi^2} D P(\xi, t). \quad (2.33)$$

For a steady state solution of Eq. (2.33), we obtain

$$\begin{aligned} P^S(\xi) &\propto e^{-\int \xi d\xi' \frac{2}{D} \left\{ C'_1(y^S) \xi' + \varepsilon^{1/2} \frac{C''_1(y^S)}{2} \xi'^2 + \dots \right\}} \\ &\approx e^{-\frac{1}{D} \left\{ C'_1(y^S) \xi^2 + \varepsilon^{1/2} \frac{C''_1(y^S)}{3} \xi^3 + \dots \right\}} \end{aligned} \quad (2.34)$$

From Eq. (2.26),

$$\frac{d}{dt} \sigma_0^S = 2 C'_1(y^S) \sigma_0^S + D = 0; \quad \sigma_0^S = -\frac{D}{2 C'_1(y^S)}. \quad (2.35)$$

Thus

$$P^S(\xi) \propto e^{-\frac{1}{2\sigma_0^S} \xi^2 + \varepsilon^{1/2} \frac{C''_1(y^S)}{3D} \xi^3 + \dots} \quad (2.36)$$

The system becomes unstable when the system parameter approaches a marginal point ($c'_1(y^S) = 0$) (See Appendix) and also the variance σ_0^S becomes divergent. Instead of $[\sigma_0^S]^{-1}$, we consider the smallness parameter $\tilde{\varepsilon} = (\gamma - \gamma_C) / \gamma_C$ ($[\sigma_0^S]^{-1} \rightarrow 0$ for $\tilde{\varepsilon} \rightarrow 0$). We must take into consideration the non-Gaussian terms in Eq. (2.36) when $[\sigma_0^S]^{-1} \xi^2$ and $\varepsilon^{1/2} \xi^3$ are comparable near an instability point. Namely, we cannot neglect the non-linear effect in this case.

Hence we can draw the conclusion that the lowest-order approximation formalism in the normal case is accurate enough so long as the system is in a stable domain. On the other hand, near a marginal point, the accuracy of our description should be determined by the relation between $\epsilon^{1/2}$ and $\tilde{\epsilon}$ values. Namely, we have to treat the effect of non-linearity of system near unstable states where the validity of the normal case does not hold since the perturbation treatment is not suitable for this situation. A further detailed discussion will be given by Kanemoto et al. ⁽⁷⁾

§2.4 Theory of Zero Power Reactor and Gaussian Assumption

Two methods — those of Langevin and Kolmogorov — have mainly been utilized hitherto in theoretical considerations of zero-power reactor noise ⁽⁸⁾⁻⁽¹⁰⁾. These two methods, however, possess some ambiguity in their mutual relations. We will clarify this ambiguity in what follows, using our formalism presented in §2.2, and we will further ascertain the validity of the Gaussian distribution of the number of neutrons.

We will begin by outlining these methods to bring the ambiguity into relief. For simplicity, we will limit our problem to the zero-power reactor model with no delayed neutrons.

The master equation of the Kolmogorov method is

$$\frac{\partial}{\partial t} P(X, t) = \int d\Delta X W(X - \Delta X, \Delta X) P(X - \Delta X, t) - \int d\Delta X W(X, \Delta X) P(X, t), \quad (2.37)$$

and we let

$$W(X, \Delta X) = S_e \delta_{\Delta X, 1} + \sum_{m=0}^{\infty} \frac{X}{l} p(m) \delta_{\Delta X, m-1}, \quad (2.38)$$

where X is the total number of neutrons in the reactor, S_e the rate of emission from neutron source "in the whole system", which is proportional to the system size (i.e., $S_e = s\Omega$)*, l is the mean neutron life time, and $p(m)$ the probability that m neutrons will be reproduced when a neutron is absorbed. In this case, $W(X, \Delta X)$ is time-independent and the integrals in Eq. (2.37) are to be replaced by summations. Namely,

$$\begin{aligned} \frac{\partial}{\partial t} P(X, t) = & S_e P(X-1, t) - S_e P(X, t) \\ & - \frac{X}{l} P(X, t) + \sum_{m=0}^{\infty} \frac{X-m+1}{l} p(m) P(X-m+1, t). \end{aligned} \quad \begin{array}{l} \text{[cf. (1.2)]} \\ (2.39) \end{array}$$

The right-hand side of Eq. (2.39) is expanded in the same manner as in §2.2.

$$\frac{\partial}{\partial t} P(X, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial X}\right)^n C_n(X) P(X, t), \quad (2.40)$$

where

$$C_n(X) = \overline{(m-1)^n} X + S_e,$$

and

$$\overline{m^n} = \sum_{m=0}^{\infty} m^n p(m).$$

Here, \bar{m} and $\overline{m^2}$ correspond respectively to k and k_2 in current nomenclature.

The Langevin method, on the other hand, is based on the equation

$$\frac{d}{dt} X(t) = \frac{\rho}{l} X(t) + S_e + F(t), \quad (2.41)$$

where $\rho = k-1$, and $F(t)$ is the random force. The following assumptions are made about this random force:

* Here we neglect multiple emission of neutrons from the source.

- (1) The process of $F(t)$ is stationary and Gaussian*.
 (2) The power spectrum of $F(t)$ is white:

$$\langle F(t_1) F(t_2) \rangle = \sigma_{eF} \delta(t_1 - t_2).$$

We represent Eq. (2.41) rigorously in the form of a Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left(\frac{p}{l} x + S_e \right) P(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma_{eF} P(x, t), \quad (2.42)$$

in order to compare Eq. (2.41) with Eq. (2.40). The comparison would indicate that the Gaussian assumption in the Langevin method seems to correspond to the termination of the Kramers-Moyal expansion. But if we simply cut off the terms for $n > 2$ in Eq. (2.40), we have

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} C_1(x) P(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} C_2(x) P(x, t), \quad (2.43)$$

which is different from Eq. (2.42). This ambiguity will be systematically clarified in the following.

We will here restrict our attention to the subcritical state which may be regarded as the normal case. Then, as seen from the preceding sections, we can adopt the scaling relation (2.16) i.e. $x=y(t)+\epsilon^{1/2}\xi$, and Eq. (2.21) yields a Gaussian distribution of fluctuations ξ around the most probable path. This means that our formalism is self-consistent. Thus, in our case we obtain the Fokker-Planck equation (2.21), with $c_1(y)$ and $c_2(y)$ determined from Eq. (2.38). Converting ξ to x , we obtain

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left(\frac{p}{l} x + S \right) P(x, t) + \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2} \left(\frac{k_2 - 2k + 1}{l} y + S \right) P(x, t), \quad (2.44)$$

and

$$\frac{d}{dt} y(t) = \frac{p}{l} y(t) + S. \quad (2.45)$$

 * For simplicity we use this assumption. The applicability of the Langevin method is not necessarily restricted to the stationary case and it will be shown later that the conventional theory can apply only to the "normal" case.

Equations (2.42), (2.43) and (2.44) differ from one another despite their similarity in form. It is important to note here that Eqs. (2.44) and (2.45) hold in a non-stationary situation. In fact, multiplying Eq. (2.44) by $(x-y(t))^2$, summing over all possible values of x , and using the notation $\sigma(t)$, we have

$$\langle (x-y(t))^2 \rangle = \varepsilon \langle \xi^2 \rangle = \varepsilon \sigma(t),$$

which leads to

$$\frac{d}{dt} \sigma(t) = 2 \frac{\rho}{l} \sigma(t) + \frac{k_2 - 2k + 1}{l} y(t) + S, \quad (2.46)$$

a relation that can also be derived directly from Eq. (2.24). Equations (2.45) and (2.46) are none other than the mean and variance equations given by the conventional Kolmogorov method. The solution of Eq. (2.44) with the initial condition

$$\lim_{t \rightarrow t_0} P(x, t | x_0, t_0) = \delta(x - x_0)$$

is

$$P(x, t | x_0, t_0) = \frac{1}{\sqrt{2\pi \varepsilon \sigma(t)}} e^{-\frac{(x-y(t))^2}{2\varepsilon \sigma(t)}}, \quad (2.47)$$

where

$$y(t) = x_0 e^{\frac{\rho}{l}(t-t_0)} - \frac{lS}{\rho} \left\{ 1 - e^{\frac{\rho}{l}(t-t_0)} \right\}, \quad (2.48)$$

and

$$\sigma(t) = \left\{ c_1(y(t)) \right\}^2 \int_{t_0}^t \frac{c_2(y(t'))}{\left\{ c_1(y(t')) \right\}^2} dt'. \quad (2.49)$$

Taking the limit $t \rightarrow \infty$, we have the stationary solution

$$P(x, \infty) = \frac{1}{\sqrt{2\pi \varepsilon \sigma_\infty}} e^{-\frac{(x-y_\infty)^2}{2\varepsilon \sigma_\infty}}, \quad (2.50)$$

where

$$y_\infty = \frac{lS}{1-k},$$

and

$$\sigma_\infty = \frac{lS}{2(1-k)^2} \{k_2 - 3k + 2\}.$$

The Gaussian distribution has also been observed experimentally^{(11) (12)}. This substantiates the validity of our formalism within the order ϵ . Hence we obtain for the correlation function in the range $t > \tau$,

$$\begin{aligned}
 \langle \Delta x_t \Delta x_\tau \rangle &= \langle x_t x_\tau \rangle - \langle x_t \rangle \langle x_\tau \rangle \\
 &= \int dx_\tau y(t|x_\tau) x_\tau P(x_\tau, \tau | x_0, t_0) \\
 &\quad - \langle x_\tau \rangle \int dx_\tau \int dx_t P(x_t, t | x_\tau, \tau) P(x_\tau, \tau | x_0, t_0) x_\tau \\
 &= \int dx_\tau y(t|x_\tau) \{ x_\tau - \langle x_\tau \rangle \} P(x_\tau, \tau | x_0, t_0) \\
 &= \epsilon \sigma(t) e^{-\frac{\rho}{\ell}(t-\tau)}, \tag{2.51}
 \end{aligned}$$

where $y(t|x_\tau)$ is given by the expression (2.48) with $x_0 \rightarrow x_\tau$ and $t \rightarrow \tau$, while $\sigma(\tau)$ is the variance given by Eq. (2.46). If we let $t \rightarrow \infty$, $\tau \rightarrow \infty$ and $t-\tau=u$, we obtain the same result via the conventional Langevin method.

To compare our formalism with the Langevin method, we rewrite the Fokker-Planck type equation (2.42) in terms of the intensive variable:

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left(\frac{\rho}{\ell} x + S \right) P(x, t) + \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2} \sigma_F P(x, t), \tag{2.52}$$

where

$$S = \frac{S_e}{\Omega}, \quad \sigma_F = \frac{\sigma_{eF}}{\Omega}.$$

The solution of this equation corresponds to the stationary solution of Eq. (2.44). Hence we have the Schottky formula

$$\sigma_F = \frac{k_2 - k}{\ell} y_s - \frac{\rho}{\ell} y_s + S; \quad y_s = -\frac{\ell S}{\rho}. \tag{2.53}$$

We have seen that system size expansion using the relation (2.16) affords us a systematic method which includes the conventional Langevin method for stationary states and which can be extended to a non-stationary case.

In other words, appropriate truncation of our expansion

sion covers the conventional theory as the normal case: Eqs. (2.45), (2.46) and correlation function derived in the normal case are none other than the equations obtained with the Kolmogorov and Langevin methods. If we require a more accurate approximation, such as for deriving information on $\overline{\frac{3}{m}}$, we should truncate Eq. (2.20) at a higher order. Such procedure, however, is tedious, and in the next section, we will propose another means of deriving the same results more directly with advantage taken of the extensive property.

§2.5 Phase Flow Pattern

Equations (2.39) and (2.44) yield identical results for the mean value and variance in the non-stationary case. But if we take higher-order terms of ϵ into consideration, some difference must necessarily appear between the two results, since a branching process is non-Gaussian.

In order to examine this difference we utilize the extensive property mentioned in §2.2 that is applicable to a subcritical reactor. Inserting the expression (2.15) into the basic equation (2.7), we obtain

$$\frac{\partial}{\partial t} \phi_0(x, t) = - \int d\Delta\lambda w(x, \Delta\lambda, t) \left[1 - e^{-\Delta\lambda \left(\frac{\partial \phi_0}{\partial x} \right)} \right]. \quad (2.54)$$

Then, by using the WKB approximation of quantum mechanics (where $\hbar = (\text{Planck constant})/2\pi$ corresponds to ϵ), we can find the Hamilton-Jacobi equation in the following manner: If we rewrite the extensive property (2.15) in the form

$$P(x,t) = C e^{\frac{1}{\epsilon} \phi_0(x,t)} = C e^{\frac{1}{\epsilon} \int dt' \frac{d\phi_0}{dt'}} = C e^{\frac{1}{\epsilon} \int dt' L(x, \dot{x}, t')}, \quad (2.55)$$

and compare this with Eq. (2.12), we find as the Lagrangian

$$L = \frac{d\phi_0}{dt} = \frac{\partial \phi_0}{\partial t} + \dot{x} \frac{\partial \phi_0}{\partial x}. \quad (2.56)$$

Then, defining the generalized momentum $p = \frac{\partial \phi_0}{\partial x}$, we have the Hamiltonian

$$H = \dot{x} p - L = -\frac{\partial \phi_0}{\partial t}. \quad (2.57)$$

This is the Hamilton-Jacobi partial differential equation,

$$\frac{\partial \phi_0}{\partial t} + H(x, p, t) = 0, \quad \text{for } x \text{ and } p. \quad (2.58)$$

Instead of solving the partial differential equation (2.58), we can solve the canonical equations

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \end{cases} \quad (2.59)$$

While, in general, Eq. (2.59) cannot be solved analytically, we can still visualize the characteristics possessed by the solution with the aid of flow patterns. This method suggests itself as a useful approach for the case of a non-linear system, and it can be considered to be a kind of pattern recognition.

In a simple example of subcritical zero-power reactor, the exact treatment of the Kolmogorov method gives

$$H(x, p) = s(1 - e^{-p}) + \frac{\kappa}{\ell} (1 - G(p) e^p), \quad (2.60)$$

where

$$G(p) = \sum_{m=0}^{\infty} p(m) e^{-mp}.$$

If $s(t)$ is constant, $H(x, p)$ does not depend explicitly on time, and the flow induced by Eq. (2.59) is determined by $H(x, p) = E$. Figure 2.2 shows this flow pattern.

Next, we will show the flow pattern of the approximate

treatment derived in §2.2 which corresponds to the Gaussian approximation of the fluctuation around the most probable path. Converting ξ to x in Eq. (2.21),

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \{C_1'(y(t))[x-y(t)] + C_1(y(t))\} P(x, t) + \frac{\epsilon}{2} \frac{\partial^2}{\partial x^2} C_2(y(t)) P(x, t), \quad (2.61)$$

and using Eq. (2.22), i.e.

$$P(\xi, t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\xi^2}{2\sigma}} = \frac{\epsilon^{\frac{1}{2}}}{\sqrt{2\pi\epsilon\sigma}} e^{-\frac{(x-y(t))^2}{2\epsilon\sigma}} = \epsilon^{\frac{1}{2}} P(x, t), \quad (2.62)$$

we obtain the time-dependent Hamiltonian applicable to the present case:

$$H(x, p, t) = \{C_1'(y(t))[x-y(t)] + C_1(y(t))\} p - \frac{C_2(y(t))}{2} p^2. \quad (2.63)$$

Then the characteristic equations of the zero-power reactor are

$$\begin{cases} \frac{dx_t}{dt} = \frac{\partial H}{\partial p_t} = \frac{p}{l} x_t + S - \left\{ \frac{k_2 - 2k + 1}{l} y(t) + S \right\} p_t \\ \frac{dp_t}{dt} = -\frac{\partial H}{\partial x_t} = -\frac{p}{l} p_t \\ \frac{dy(t)}{dt} = \frac{p}{l} y(t) + S. \end{cases} \quad (2.64)$$

We can easily solve these equations under the initial conditions

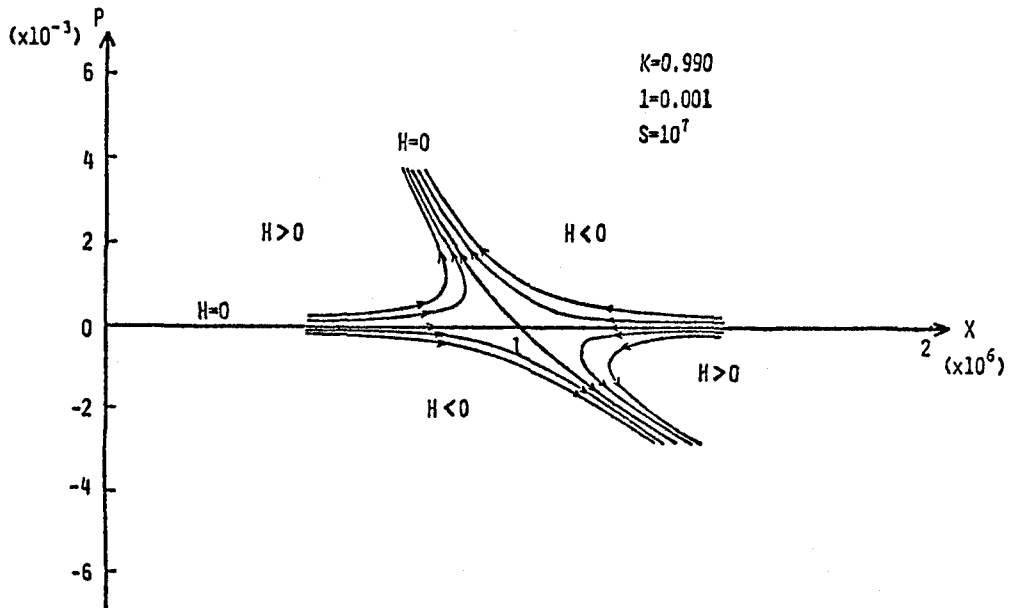
$$\begin{cases} x(t_0) = x_{t_0} \\ p(t_0) = -\frac{1}{\sigma_0} (x_{t_0} - \eta) \\ y(t_0) = \eta, \end{cases}$$

corresponding to an initial Gaussian distribution:

$$P(x_0, t_0) = e^{\frac{1}{\epsilon} \phi_0(x_{t_0}, t_0)}; \quad \phi_0(x_{t_0}, t_0) = -\frac{1}{2\sigma_0} (x_{t_0} - \eta)^2.$$

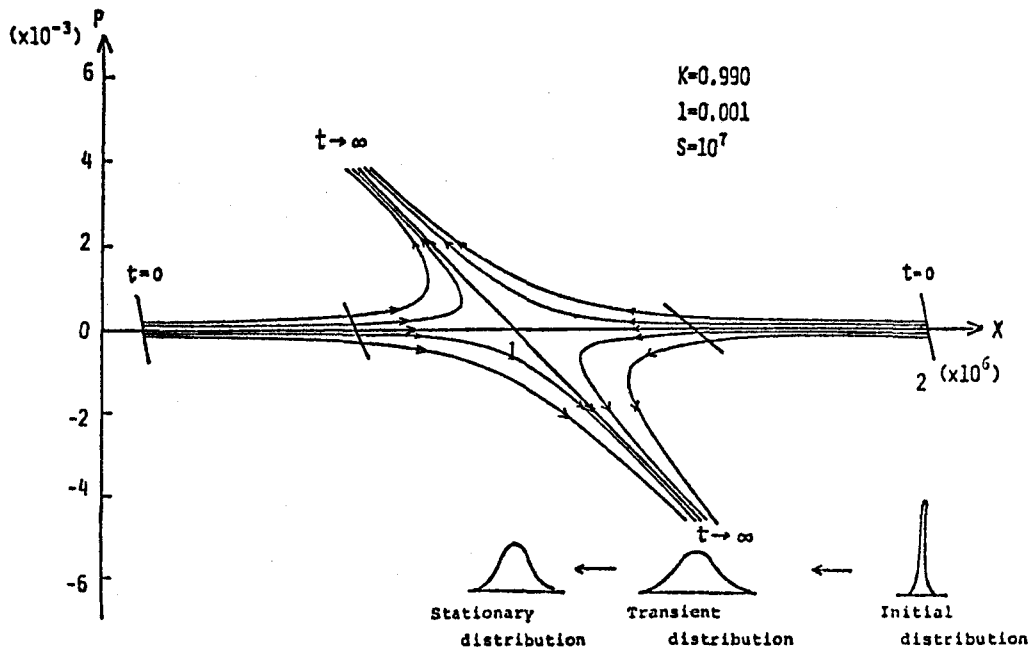
The flow pattern for this is illustrated in Fig. 2.3. In this figure the straight lines represent the time propagation of the Gaussian distribution. The slope of the line is inversely proportional to the variance.

We can see from Eq. (2.7) that each c_n is accompanied



Exact treatment of Kolmogorov method Eq. (2.60)

Fig. 2.2 Flow pattern of zero-power reactor



Approximate treatment in the normal case Eq. (2.63)

Fig. 2.3 Flow pattern of zero-power reactor

by a higher-order term of ϵ depending on the index n . Now, in the above two treatments the Hamiltonian (2.63) comprises only c_1 , c_1' and c_2 , while the Hamiltonian (2.60) contains all the higher-order terms c_n . The effect of these higher-order terms appears in Fig. 2.2 with the slight bending seen in the line of the stationary distribution $P(x, t=\infty)$, which is absent from the corresponding line in Fig. 2.3. Thus we may conclude that the effect of the higher-order terms is not significant for zero-power reactor noise analysis, on account of the large volume of the reactors in the normal case. As a first approximation, therefore, we can obtain an amply accurate result by assuming "normal".

The foregoing finding provides us with a useful and convenient method for studying non-linear, non-equilibrium noise problems. This characteristic permits us to introduce a new variable α , that is very suitable for analyzing non-stationary reactor noise, as it will be shown in the next chapter. In a case where an evident non-Gaussian distribution of fluctuations around the most probable path is found in contrast to the above normal case, we should take account this effect by taking up the higher-order terms of c_n to clarify the non-Gaussian mechanism.

§2.6 Application to a Non-linear Reactor System
with Random Parameters

The non-linear behavior of reactors with random parameters has not so far been treated by taking into consideration the difference between the extensive and intensive variables. In the case of a point reactor, the stochastic differential equation with random parameters has been expressed by Williams⁽¹³⁾⁻⁽¹⁵⁾ in the form

$$\frac{d}{dt} X(t) = \frac{\rho(t)}{\ell} X(t) + S_e(t), \quad (2.65)$$

where the delayed neutrons have been ignored, while $X(t)$ is the number of neutrons, ℓ the mean lifetime of neutrons, $\rho(t)$ the reactivity and $S_e(t)$ an independent source term. Williams assumed that $\rho(t)$ and $S_e(t)$ are stationary Gaussian random variables representing parameters and source excitations, respectively. In what follows, we assume that their noise are white. In a source-free case, the inconsistency of Williams' treatment is obvious if we compare the two equations — (a) obtained by integrating the Fokker-Planck equation (23) of Williams' reference (13) multiplied by X over all possible states of X , namely

$$\frac{d}{dt} \langle X(t) \rangle = \frac{\rho_0}{\ell} \langle X(t) \rangle, \quad (2.66)$$

and (b) obtained by differentiating Eq. (3.13) of Williams' reference (14) for $p=1$ with respect to t , namely

$$\frac{d}{dt} \langle X(t) \rangle = \left(\frac{\rho_0}{\ell} + \frac{\sigma}{2\ell^2} \right) \langle X(t) \rangle. \quad (2.67)$$

Here we have also used Eq. (3.6) of Williams' reference (14), i.e.

$$\sigma_x^2 = \int_0^t dt' \int_0^t dt'' R_\Delta(t'-t'') = \int_0^t dt' \int_0^t dt'' \sigma \delta(t'-t'') = \sigma t. \quad (2.68)$$

Where does this difference between Eqs. (2.66) and (2.67) come from? * In the latter treatment Williams also derived

* This is also discussed in references (15), (17) and (18).

a non-Gaussian distribution function of X by Gaussian random parametric excitation. While his treatment is mathematically self-consistent, its physical meaning is not clear, since he used the Gaussian assumption in a non-normal situation in which the Ω -dependence of the mean value and that of the fluctuation are of the same order. Such a non-normal situation may occur in particular cases. The Gaussian assumption should be applied more appropriately to the normal case. We will further clarify the inconsistency in more concrete terms.

If the macroscopic system has the most probable path in a stable situation (See Appendix), it stands to reason that the distribution of the fluctuations around the most probable path should become Gaussian. Since we assume that the system lies in the normal case (or, in other words, the system is subcritical), the processes of $\rho(t)$ and $S_e(t)$ are Gaussian. Hence, in accordance with the method of van Kampen, they must depend on the system size Ω by the relations

$$\left\{ \begin{array}{l} \rho(t) = \rho_0 + \varepsilon^{\frac{1}{2}} \Delta(t) \quad \langle \Delta(t) \rangle = 0 \\ \varepsilon S_e(t) = S_0 + \varepsilon^{\frac{1}{2}} S(t) \quad \langle S(t) \rangle = 0 \\ \langle \Delta(t_1) \Delta(t_2) \rangle = \sigma \delta(t_1 - t_2) \\ \langle S(t_1) S(t_2) \rangle = \sigma_s \delta(t_1 - t_2) \\ \langle \Delta S \rangle = \langle S \Delta \rangle = 0 \end{array} \right. \quad (2.69)$$

Under these assumptions, we can obtain from Eq. (2.65) the Fokker-Planck equation in terms of intensive variables.

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} C_1(x) P(x, t) + \frac{\varepsilon}{2} \frac{\partial^2}{\partial x^2} C_2(x) P(x, t), \quad (2.70)$$

where

$$x = \frac{X}{\Omega} \quad ; \quad s = \frac{S_e}{\Omega} \quad ; \quad c_i(x) = \frac{C_i(X)}{\Omega} \quad (i=1, 2),$$

and

$$\begin{cases} C_1(x) = \frac{\rho_0}{l} x + S + \varepsilon \frac{\sigma}{2l^2} x \\ C_2(x) = \frac{\sigma}{l^2} x^2 + \sigma_s . \end{cases} \quad (2.71)$$

As mentioned earlier, we can let

$$x = y(t) + \varepsilon^{\frac{1}{2}} \xi .$$

Substituting this into Eq. (2.70) and comparing the terms of order $\varepsilon^{-1/2}$, we obtain

$$\frac{d}{dt} y(t) = \frac{\rho_0}{l} y(t) + S . \quad (2.72)$$

In a source free case, this result coincides with the first equation of Williams. From the terms of order ε^0 , we obtain

$$\frac{\partial}{\partial t} P(\xi, t) = -\frac{\rho_0}{l} \frac{\partial}{\partial \xi} \xi P(\xi, t) + \frac{1}{2} \left(\frac{\sigma}{l^2} y^2(t) + \sigma_s \right) \frac{\partial^2}{\partial \xi^2} P(\xi, t) . \quad (2.73)$$

This equation is the linear Fokker-Planck equation with time-dependent coefficients, and leads to a solution of Gaussian distribution. From these results, we conclude that in so far as the random parametric excitation $\rho(t)$ has the Ω -dependence given by Eq. (2.69), the system has a Gaussian distribution of fluctuations around the most probable path.

When Williams calculated the moments of the Fokker-Planck equation from the Langevin equation, he used the "discrete" Langevin equation (Ito's method) for zero power reactors, and the "continuous" Langevin equation (Stratonovich's method) for power reactors that are affected by such factors as bubble formation and temperature fluctuations. Now, the difference between Eqs. (2.66) and (2.67) arises from this discrepant treatment in the two cases of power reactors. It should also be noted that this difference is present in parametric random excitation, while the difference is absent in source random excitation on account of the linearity

of the Langevin equation. We have proved earlier that, even in a power reactor, continuous treatment should yield the same equations as those obtained from discrete treatment on account of the dependence on the system size in the normal case. This is why the discrepancy between the two treatments disappears in the normal case. Thus the necessity becomes felt for distinguishing the non-normal from the normal reactor state. Furthermore we see that the non-Gaussian effect pointed out by Williams plays an important role when $\rho(t)$ has a non-normal Ω -dependence caused by non-linear and other effects in a particular or abnormal situation. Even in such a case, however, we can estimate these effects by the method developed in this chapter, unless the transition probability of states is of non-local nature.

§2.7 Conclusion and Discussion

We have explained the system size expansion method in §2.2. This was followed in §2.3 with an examination of the validity of the "normal" assumption in a steady state. In §§2.4 and 2.5, it was demonstrated that the results obtained on the basis of conventional reactor noise theory are equivalent to those of the normal case in our formalism within an order of ϵ . For the zero-power reactor, we have presented the flow patterns of its evolution, showing that for such a reactor we can adopt the Gaussian assumption without incurring error beyond the order of ϵ . This indicates that our approach has a possibility of shedding

a fresh light on the established approaches based on the linear stationary ergodic Markovian processes. Hence, the reactor noise, particularly in-core noise, can be or should be treated with the framework of statistical physics for non-linear non-equilibrium open systems. Finally, in §2.6, a non-linear reactor with random parameters was discussed as an example of application of our theory.

Though the point model of reactor has been treated in this chapter, it is straightforward to extend the applicability of our formalism to space-dependent phenomena if the configuration space can be divided into cubic cells. In this case, instead of the system size expansion, the cell size expansion as in chemical reactions⁽¹⁹⁾ may be adopted. The length l of a cell is chosen larger than the micro-characteristic length l_{\min} (for example, mean free path), so that the mean value taken in a cell is meaningful. The l must be also smaller than the macro-characteristic length l_{\max} , so that the variation of mean value $y(t)$ between neighboring cells should be small and hence the spatial difference has the meaning of derivative in the spatially continuous approximation. It is important to take variables describing the spatial tendency of the system in which all the transitions are limited to occur only in a relevant scale, in order to make use of $1/\Omega$ -expansion.

When we treat a power reactor, its state requires to be classified in more detail than currently practiced, on account of the significant differences brought by such effects as non-linearity and feedback. When Williams analyzed a reactor mechanism with consideration given to fluctuations, he distinguished the discrete from the continuous treatment. This is not sufficient, and we have additionally distinguished the non-normal from normal fluctuations, based on the statistical physics considerations. As we have pointed out §2.6, it is important to study whether or not it is valid to assume Gaussian

distribution of the fluctuations around the most probable path. In the case of a power reactor where the non-linearity of the phenomenological law plays an important role, the dependence of the fluctuations on the system size presents aspects different from the normal case discussed here. If a power reactor is subject to non-Gaussian effects, the higher-order terms $c_n(x)$ should serve as clue to studying their physical mechanism. The order of system size dependence should also be useful as an index to aid in classifying the reactor state.

When we use the Markovian assumption in a power reactor, we should use a state vector instead of one variable as in §2.2. The stability condition of a power reactor in reactor dynamics can be classified into domains corresponding to stable, unstable, semi-stable states and so on. Within one state space, the state condition of a power reactor can be specified by means of certain parameters. When the system lies in a stable or unstable domain or in the case of transition from one domain to another, it is useful to draw flow patterns such as shown in this chapter for the analysis of the stability with account taken of fluctuations. It should further be noted that the power spectral density is not well-defined in a non-stationary state. This makes it indispensable in a non-stationary case to seek variables that can provide information corresponding to that available from power spectral density in a stationary case. For example, in the 'phase' of limit cycle, an eligible variable would be the irreversible circulation of fluctuation α to be defined by further examination of the present formalism. This will be the subject of the next chapter.

APPENDIX

We define the terminology necessary for representing the various states of the macrosystem. In general, the phenomenological equation (2.19) may be assumed to be expanded in reference to the stationary or equilibrium state in a form

$$\frac{d}{dt}y(t) = C_1(y) = \nu_1 y + \nu_2 y^2 + \nu_3 y^3 + \dots$$

The different kinds of states shall be classified and defined as follows:

- a) Stable state: $\nu_1 < 0$
- b) Unstable state: $\nu_1 > 0$
- c) Marginal state: $\nu_1 = 0, \nu_2 \neq 0$
- d) Critical state: $\nu_1 = 0, \nu_2 = 0, \nu_3 \neq 0$.

This classification is based on current practice in statistical physics. The critical state does not necessarily correspond to a critical reactor. The marginal point, in particular, will play an important role in the next chapter in relation to change in behavior of the reactor from stable to limit cycle region in a non-linear system.

The subcritical reactor state corresponds to the condition (a), since $c_1(y)$ is linear and $\nu_1 = \rho/\lambda < 0$. A supercritical reactor corresponds to the condition (b). When a reactor becomes just critical, i.e., $\rho/\lambda = 0$, the usual point reactor model no longer includes non-linear terms. This means that we must in this case take account of ν_2, ν_3, \dots which, in comparison with ν_1 , could be neglected in supercritical and subcritical reactors. Hence there is need for refinement of the conventional point reactor model to cover properly the just-critical state.

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CHAPTER 3

Irreversible Circulation of Fluctuation in Reactor Noise

§3.1 Introduction

A power reactor is usually a complicated system involving non-linear and/or feedback effects due to nuclear, thermal, hydrodynamical, mechanical and various other phenomena⁽¹⁾⁽²⁾. Power reactors have intrinsically fluctuating components. Furthermore, they are also operated in a non-equilibrium or non-stationary state. When the system is in a non-stationary state, the power spectral density is not well-defined. This makes it necessary to find new variables that include the requisite information to account for the non-stationary state, through extension of the stationary power spectral density. It is major subject of power reactor noise theory how to formulate equations for these complicated phenomena⁽³⁾⁻⁽⁵⁾.

The first step in this direction was made in the previous chapter⁽³⁾, which took up as basis the system size expansion method of non-linear non-equilibrium statistical physics⁽⁶⁾⁻⁽¹⁰⁾. In Chap. 2, it was seen that in the normal case the non-Gaussian effect on a macrovariable (total neutron number) appears only within the order $\varepsilon (= \Omega^{-1})$, owing to the macroscopic nature of the system size Ω , despite the fact that a non-Gaussian process — a branching process — exists in zero-power reactors.

It is the purpose of this chapter to find out to what extent the non-linear non-equilibrium power reactor can be described with use made of the system size expansion method of Chap. 2, under the assumption that the system belongs to the "normal" case.

To describe reactor noise, we shall propose a new parameter — the "irreversible circulation of fluctuation"

α —, which was first introduced by Tomita⁽⁸⁾ to define a cyclic balance in an off-equilibrium situation. Much useful information can be expected to accrue from examination of α in conjunction with the currently considered variance σ and the power spectral density, even for steady states. Examples of such information would include discerning of forerunner phenomena preceding various instabilities of power reactors, through appropriate processing of data from α and σ . Thus, it will be seen that α and σ are very useful in extending the theory to non-stationary cases, when the system is in the normal case.

In §3.2, we shall incorporate the irreversible circulation of fluctuation α in a time-dependent formalism, based on the system size expansion method. This will be followed in §3.3 by the consideration of a stationary state, in order to relate the new variable α to conventional variables. §3.4 will serve to demonstrate the effectiveness of the theory, by treating a specific type of reactor representing widely current forms of such reactor. Conclusion and discussion are presented in §3.5.

§3.2 System Size Expansion and Irreversible Circulation of Fluctuation

Treatment of the non-linear Langevin equation⁽¹¹⁾ leaves some ambiguity both in the mathematical and physical aspects, as mentioned in §2.6. This has induced us to base our approach on the probability distribution function and

the Markov assumption. Then the system under consideration is described by the master equation (2.1):

$$\frac{\partial}{\partial t} P(X, t) = \int \{W(X-\Delta X, \Delta X, t) P(X-\Delta X, t) - W(X, \Delta X, t) P(X, t)\} d\Delta X. \quad (3.1)$$

By using

$$x = \frac{X}{\Omega} = \varepsilon X \quad (\varepsilon = \Omega^{-1}), \quad W(X, \Delta X, t) = \Omega w(x, \Delta X, t), \quad P(x, t) = \Omega^N P(X, t),$$

Eq. (3.1) can be rewritten in a power series of ε , known as the Kramers-Moyal expansion (2.7):

$$\frac{\partial}{\partial t} P(x, t) = \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1}}{n!} \left(-\frac{\partial}{\partial x}\right)^n \cdot c_n(x, t) P(x, t), \quad (3.2)$$

where

$$c_n(x, t) \equiv \int d\Delta X (\Delta X)^n w(x, \Delta X, t). \quad (3.3)$$

As mentioned in §§2.2, 2.3 and 2.5, it is possible to disregard detailed information concerning the far wing distribution in the normal case, owing to the macroscopic nature of the system. This permits us to adopt in this case the scaling relation

$$x = \mathcal{Y}(t) + \varepsilon^{\frac{1}{2}} \mathcal{E}, \quad [\mathcal{X} = \{x_i\} \quad (i=1, 2, \dots, N)] \quad (3.4)$$

where $\mathcal{Y}(t)$ and \mathcal{E} are respectively the sets of mean values and the fluctuations around them. The normal character of the system is thus represented by the fluctuations of order $\varepsilon^{1/2}$. Then, Eq. (3.2) can be further replaced by an equation written in terms of the variables \mathcal{E} instead of \mathcal{X} :

$$\frac{\partial}{\partial t} P(\mathcal{E}, t) - \varepsilon^{-\frac{1}{2}} \dot{\mathcal{Y}}(t) \frac{\partial P(\mathcal{E}, t)}{\partial \mathcal{E}} = \sum_{n=1}^{\infty} \frac{\varepsilon^{\frac{n-2}{2}}}{n!} \left(-\frac{\partial}{\partial \mathcal{E}}\right)^n \cdot c_n(\mathcal{Y}(t) + \varepsilon^{\frac{1}{2}} \mathcal{E}, t) P(\mathcal{E}, t). \quad (3.5)$$

Within the order $\varepsilon^{-1/2}$

$$\frac{d}{dt} \mathcal{Y}(t) = C_1(\mathcal{Y}(t), t), \quad (3.6)$$

and within the next order ε^0

$$\frac{\partial}{\partial t} P(\mathcal{E}, t) = -\frac{\partial}{\partial \mathcal{E}} [K(\mathcal{Y}(t), t) \cdot \mathcal{E} P(\mathcal{E}, t)] + \frac{1}{2} \frac{\partial^2}{\partial \mathcal{E}^2} [D(\mathcal{Y}(t), t) P(\mathcal{E}, t)], \quad (3.7)$$

where $K(\mathcal{Y}(t), t) = \partial C_1(\mathcal{Y}(t), t) / \partial \mathcal{Y}(t)$ is the regression matrix, and $D(\mathcal{Y}(t), t) = c_2(\mathcal{Y}(t), t)$ the diffusion matrix.

Equation (3.7) is the linear Fokker-Planck equation with the coefficients depending on time, through the time-dependent most-probable path $\mathcal{Y}(t)$ even in the case of time-independent w . Equation (3.7) may be written as a conservation relation of probability:

$$\frac{\partial}{\partial t} P(\xi, t) = - \frac{\partial}{\partial \xi} G(\xi, t), \quad (3.8)$$

where

$$G(\xi, t) = K(\mathcal{Y}, t) \xi P(\xi, t) - \frac{1}{2} \frac{\partial}{\partial \xi} D(\mathcal{Y}, t) P(\xi, t)$$

stands for the probability flux. From the Eq. (3.6) we obtain the evolution equation for a small deviation $\delta\mathcal{Y}$:

$$\frac{d}{dt} \delta\mathcal{Y}(t) = K(\mathcal{Y}, t) \delta\mathcal{Y}(t), \quad (3.9)$$

and from Eq. (3.7) we have

$$\frac{d}{dt} \sigma(t) = K(\mathcal{Y}(t), t) \sigma(t) + \widetilde{K(\mathcal{Y}(t), t)} \sigma(t) + D(\mathcal{Y}(t), t), \quad (3.10)$$

where σ is the matrix of variance, defined by

$$\sigma(t) = \langle \xi(t) \xi(t) \rangle = \int \xi \xi P(\xi, t) d\xi, \quad (3.11)$$

and the tilde \sim denotes the transposed matrix.

It is easy to show that Eq. (3.7) is satisfied by the Gaussian distribution

$$P(\xi, t) = \sqrt{\frac{\det(\mathcal{G}(t))}{(2\pi)^N}} e^{-\frac{1}{2} \xi \cdot \mathcal{G}(t) \cdot \xi}, \quad (3.12)$$

where

$$\sigma(t) \mathcal{G}(t) = \mathcal{G}(t) \sigma(t) = \mathbf{1}.$$

Taking advantage of this property, we can transform Eq. (3.7) into the more convenient form

$$\begin{aligned} \frac{\partial}{\partial t} P(\xi, t) &= \frac{\partial}{\partial \xi} (K(t) \sigma(t) + \frac{1}{2} D(t)) \frac{\partial}{\partial \xi} P(\xi, t) \\ &= \frac{\partial}{\partial \xi} \mathbf{E} \frac{\partial}{\partial \xi} P(\xi, t), \end{aligned} \quad (3.13)$$

where $K(t)$ and $D(t)$ are abbreviations of $K(\mathcal{Y}(t), t)$ and $D(\mathcal{Y}(t), t)$. Dividing \mathbf{E} into (a) symmetric and (b) anti-

symmetric parts, and noting the relation (3.10), we obtain

$$\begin{aligned}
 E &= K(t)\sigma(t) + \frac{1}{2}D(t) \\
 &= \frac{1}{2}\left\{K(t)\sigma(t) + \overline{K(t)\sigma(t)} + D(t)\right\} + \frac{1}{2}\left\{K(t)\sigma(t) - \overline{K(t)\sigma(t)}\right\} \\
 &= \frac{1}{2}\dot{\sigma}(t) - \alpha(t),
 \end{aligned} \tag{3.14}$$

where we have introduced a new variable defined by the relation

$$\alpha(t) \equiv \frac{1}{2}\left\{\overline{K(t)\sigma(t)} - K(t)\sigma(t)\right\}. \tag{3.15}$$

From the mathematical property of antisymmetric matrices,

$$\frac{\partial}{\partial \xi} \alpha(t) \frac{\partial}{\partial \xi} P(\xi, t) = 0. \tag{3.16}$$

This means that $\alpha(t)$ does not contribute to the time evolution of $P(\xi, t)$. Then, in a system approaching steady state, the symmetric part $\dot{\sigma}$ disappears, leaving the anti-symmetric part α still remaining. The physical meaning of $\alpha(t)$ is closely related to the cyclic balance in the non-equilibrium situation of an open system. Further details of the irreversible circulation of fluctuation $\alpha(t)$ have been presented by Tomita et al. (8)-(10)

It should be noted here that $\alpha(t)$ defined by the expression (3.15) would not be directly observable quantity, since while $\sigma(t)$ is observable, the regression matrix $K(t)$ is not. We shall therefore derive another definition of $\alpha(t)$ that involves only observable variables. For this purpose, noting the property of $\alpha(t)$ related to the two-time gate probability⁽⁹⁾ manifested in Eq. (3.14), we derive the equation of motion for ξ . Then the conditional probability over an arbitrary path from (ξ, t) to (ξ', τ) can be written

$$P(\xi', \tau | \xi, t) \propto e^{-\frac{1}{2}(\xi' - \delta y(\tau)) \cdot g(t) \cdot (\xi - \delta y(t))}, \quad (\tau > t) \tag{3.17}$$

with the initial conditions $\sigma(t) = 0$ and $\delta y(t) = \xi$.

Letting $\tau-t=\Delta t$ and $\xi'-\xi=\Delta\xi$ and using Eqs. (3.9) and (3.10) for an infinitesimal time interval, we can rewrite the conditional probability in the form of a path integral covering a finite time interval:

$$P(\xi_2, t_2 | \xi_1, t_1) \propto \int d\mathcal{O}(\xi) e^{-\int_{t_1}^{t_2} \mathcal{O}(\xi, \dot{\xi}) dt}, \quad (3.18)$$

where

$$\mathcal{O}(\xi, \dot{\xi}) = \frac{1}{4} (\dot{\xi} - K(t)\xi) \cdot R(t) \cdot (\dot{\xi} - K(t)\xi), \quad R(t) = 2D(t)^{-1}. \quad (3.19)$$

The equation of motion for ξ can be expressed on the basis of the variational principle by the Euler-Lagrange equation⁽⁹⁾

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{O}}{\partial \dot{\xi}} \right) - \left(\frac{\partial \mathcal{O}}{\partial \xi} \right) \\ = R(t) \ddot{\xi} - \left\{ R(t)K(t) - \widetilde{K}(t)R(t) - \dot{R}(t) \right\} \dot{\xi} \\ - \left\{ \dot{R}(t)K(t) + R(t)\dot{K}(t) + \widetilde{K}(t)R(t)K(t) \right\} \xi = 0. \end{aligned} \quad (3.20)$$

The equation of motion is decomposed into the independent modes of

$$(a) \quad \dot{\xi}_+ = K(t)\xi \quad (\text{forward evolution}) \quad (3.21)$$

$$(b) \quad \dot{\xi}_- = (K(t) + D(t)g(t))\xi \quad (\text{backward evolution}). \quad (3.22)$$

This leads us to another definition of $\alpha(t)$:

$$\begin{aligned} \alpha(t) &= \frac{1}{2} \left(\widetilde{K(t)\sigma(t)} - K(t)\sigma(t) \right) \\ &= \frac{1}{2} \left(\langle \xi(t) \dot{\xi}_+(t) \rangle - \langle \dot{\xi}_+(t) \xi(t) \rangle \right). \end{aligned} \quad (3.23)$$

This definition of $\alpha(t)$ shows that $\alpha(t)$ is an observable variable. In a three-dimensional case, $\alpha(t)$ is also expressed in vector representation by

$$\alpha(t) = \frac{1}{2} \langle [\xi(t) \times \dot{\xi}(t)] \rangle. \quad (3.24)$$

This expression would attribute to $\alpha(t)$ the meaning of mean angular momentum of fluctuations.

Thus far, time-dependence has been considered throughout in the formulations, since our aim is to study the non-linear non-stationary behavior of systems; but in the next section we shall consider a steady state, for the purpose of examining the relation between $\alpha(t)$ and the other observable variables. We shall find a third expression of $\alpha(t)$, applicable to steady state, which is convenient for such an examination.

§3.3 Relation of α to the Conventional Variables^{(4) (5)}

Since for ordinary power reactors Eq. (3.6) is non-linear, on account of feedback and/or non-linear phenomena, it is difficult to examine analytically global behavior of its solution over the whole region. The behavior in time of fluctuations is complicated even in the normal case, on account of the regression matrix K and the diffusion matrix D in Eq. (3.7), which depend on time through the mean value $\bar{y}(t)$. To obviate this complication, we shall consider a steady state, where $\bar{y}(t)$ is constant. All necessary information on fluctuations is then contained in K_s and D_s , where the subscript s means steady state. This implies that both $\sigma(t)$ and $\alpha(t)$ are expressed solely in terms of K_s and D_s . In the same way, the relation between the conventional variables and $\alpha(t)$ can be found in this special case, as it will be shown in this section.

First of all, the conventional variables are summarized below. The correlation matrix in a steady state is defined by

$$\begin{aligned}
C(t) &= \langle \xi(0) \xi(t) \rangle \\
&= \int d\xi_0 \int d\xi \xi_0 \xi P(\xi, t | \xi_0) P^s(\xi_0), \quad (t \geq 0). \quad (3.25)
\end{aligned}$$

Then, the time derivative of Eq. (3.25) can be transformed by partial integration after inserting Eq. (3.7):

$$\begin{aligned}
\frac{d}{dt} C(t) &= \langle \xi(0) \dot{\xi}(t) \rangle \\
&= \int d\xi_0 \int d\xi \xi_0 \xi \frac{\partial}{\partial t} P(\xi, t | \xi_0) P^s(\xi_0) \\
&= \langle \xi(0) (K_s \xi(t)) \rangle \\
&= C(t) \widetilde{K}_s. \quad (3.26)
\end{aligned}$$

Introducing the Laplace-Fourier transform

$$S(\omega) = \int_0^\infty dt e^{i\omega t} C(t),$$

we obtain from Eq. (3.26)

$$S(\omega) = -\sigma_s [i\omega + \widetilde{K}_s]^{-1}, \quad (3.27)$$

where σ_s is given by the relation

$$K_s \sigma_s + \widetilde{K}_s \sigma_s + D_s = 0, \quad (3.28)$$

The power spectral density matrix for a normal case is written

$$\begin{aligned}
P(\omega) &= \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \langle (x(0) - \mathcal{Y}^s)(x(t) - \mathcal{Y}^s) \rangle e^{i\omega t} dt \\
&= \int_{-\infty}^{\infty} \langle \xi(0) \xi(t) \rangle e^{i\omega t} dt \\
&= S(\omega) + S^\dagger(\omega), \quad (3.29)
\end{aligned}$$

where $S^\dagger(\omega)$ denotes the Hermite conjugate of $S(\omega)$. In the last equality in Eq. (3.29) we have used the obvious relation

$$\langle \xi(t) \xi(0) \rangle = \langle \xi(0) \xi(-t) \rangle,$$

for steady state. Substitution of Eq. (3.27) into

Eq. (3.29) gives

$$P(\omega) = -\sigma_s \frac{1}{i\omega + K_s} - \frac{1}{-i\omega + K_s} \sigma_s, \quad (3.30)$$

which can also be written

$$P(\omega) = G_T(i\omega) D_s \overline{G_T(-i\omega)}, \quad G_T(i\omega) = [i\omega - K_s]^{-1}, \quad (3.31)$$

with use made of the relation (3.28).

Here, we also should pay attention to the definitions of power spectral density since the system size expansion method provides different Ω -dependent power spectral densities.

$$P_X(\omega) = \Omega^2 P_x(\omega) = \Omega P(\omega)$$

$$P_X(\omega) = \int_{-\infty}^{\infty} \langle (X(0) - Y^s)(X(t) - Y^s) \rangle e^{i\omega t} dt$$

$$P_x(\omega) = \int_{-\infty}^{\infty} \langle (x(0) - y^s)(x(t) - y^s) \rangle e^{i\omega t} dt,$$

where $P_X(\omega)$ is used in the conventional reactor noise theory of the point model.

The above considerations confirm that full information about the fluctuations of steady systems can be embodied — within the order ε — in the self- and cross-power spectral densities. For example, the variance σ_s of the fluctuations is determined through the power spectral density by

$$\sigma_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Re } P(\omega) d\omega. \quad (3.32)$$

We shall now consider the relation of σ_s to the power spectral density. From Eq. (3.29)

$$\frac{d}{dt} \langle \xi(0) \xi(t) \rangle = \langle \xi(0) \dot{\xi}(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega) P(\omega) e^{-i\omega t} d\omega, \quad (3.33)$$

and

$$\langle \dot{\xi}(t) \xi(0) \rangle = \overline{\langle \xi(0) \dot{\xi}(t) \rangle} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega) \overline{P(\omega)} e^{-i\omega t} d\omega, \quad (3.34)$$

where we have replaced $\overline{P(\omega)}$ by $\overline{P(\omega)}$, the complex conjugate of $P(\omega)$, since the power spectral density is

Hermitian. Taking the limit $t \rightarrow 0$ in Eq. (3.34),

$$\begin{aligned}\alpha_s &= \frac{1}{2} (\langle \xi \dot{\xi} \rangle - \langle \dot{\xi} \xi \rangle) \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} (-i\omega) [P(\omega) - \overline{P(\omega)}] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega \operatorname{Im} P(\omega) d\omega.\end{aligned}\quad (3.35)$$

The probable reason why α has not been considered in the past, either in theoretical or experimental studies on reactor noise is that, in so far as steady states are concerned, the power spectral density suffices to obtain all the necessary information. However, it has been shown that both α and σ are useful as concise integral measures of the information contained in the power spectral density in steady state.

Next, a relation between α and correlation matrix will be shown. The correlation matrix is given by the evolution equation (3.26):

$$C(t) = \sigma_s e^{\widetilde{K}_s t}, \quad (t \geq 0). \quad (3.36)$$

In a similar way, we obtain the another correlation matrix

$$\widetilde{C}(t) = \langle \xi(t) \xi(0) \rangle = e^{K_s t} \sigma_s, \quad (t \geq 0). \quad (3.37)$$

Employing Eqs. (3.36) and (3.37), we easily obtain the same relation as Eq. (3.23) in a steady state without using the variational principle:

$$\begin{aligned}\alpha_s &\equiv \frac{1}{2} (\widetilde{K}_s \sigma_s - K_s \sigma_s) \\ &= \frac{1}{2} \{ \dot{C}(+0) - \dot{\widetilde{C}}(+0) \} \\ &= \frac{1}{2} \{ \langle \xi(0) \dot{\xi}(+0) \rangle - \langle \dot{\xi}(+0) \xi(0) \rangle \}.\end{aligned}\quad (3.38)$$

We extend, in a steady state, the time domain of the correlation matrix to negative region, i.e.,

$$\widetilde{C}(t) = \langle \xi(t) \xi(0) \rangle = \langle \xi(0) \xi(-t) \rangle = C(-t), \quad (t \geq 0).$$

The correlation matrix is

$$C(t) = \langle \xi(0) \xi(t) \rangle = \begin{cases} \sigma_s e^{\tilde{K}_s t} & (t \geq 0) \\ e^{K_s |t|} \sigma_s & (t \leq 0). \end{cases} \quad (3.39)$$

In general, the regression matrix K is not necessarily symmetric in a non-equilibrium situation. This means that cross-correlation functions are asymmetric with respect to the time inversion. However, we can see that self-correlation functions involved in $C(t)$ are always symmetric. Then, we shall connect α with the cross-correlation functions. To do this, we note that Eq. (3.28) becomes in a steady state,

$$\begin{aligned} K_s &= -\sigma_s (\tilde{K}_s + \mathcal{G}_s D_s) \mathcal{G}_s \\ &= -\sigma_s \overline{(K_s + D_s \mathcal{G}_s)} \mathcal{G}_s, \end{aligned} \quad (3.40)$$

which yields

$$K_s^n = (-1)^n \sigma_s \overline{(K_s + D_s \mathcal{G}_s)^n} \mathcal{G}_s,$$

where \mathcal{G}_s is the inverse of σ_s ($\sigma_s \mathcal{G}_s = \mathcal{G}_s \sigma_s = 1$). Then we can obtain

$$\begin{aligned} C(t) &= e^{K_s |t|} \sigma_s \quad (t \leq 0) \\ &= \sigma_s + \sum_{n=1}^{\infty} \frac{1}{n!} (K_s |t|)^n \sigma_s \\ &= \sigma_s + \sum_{n=1}^{\infty} \frac{1}{n!} \sigma_s \overline{[-(K_s + D_s \mathcal{G}_s) |t|]^n} \\ &= \sigma_s e^{-\overline{K_s + D_s \mathcal{G}_s} |t|}. \end{aligned} \quad (3.41)$$

From the definition (3.38) and the relation (3.28),

$$K_s + D_s \mathcal{G}_s = -\sigma_s \tilde{K}_s \mathcal{G}_s = -(K_s + 2\alpha_s \mathcal{G}_s).$$

Finally we obtain

$$C(t) = \sigma_s e^{\overline{K_s + 2\alpha_s \mathcal{G}_s} |t|}, \quad (t \leq 0). \quad (3.42)$$

Comparing Eq. (3.36) with Eq.(3.42), we can easily understand that α_s represents the irreversibility in time in a state far from equilibrium.

Hence, the existence of α shows the asymmetry of cross-correlation functions with respect to the origin of time. The asymmetry of cross-correlation functions are familiar in experiments of reactor noise, while auto-correlation functions are always symmetric.

Having clarified the relations between α and conventional variables, we can proceed further to research how α is related to a joint probability distribution function for two time points. We have a Gaussian type solution from Eq. (3.7). Since the existence of α shows the rotational freedom of the Gaussian distribution, we fail to notice the existence of α without the concept of time series. So we must treat the two-time gate probability in order to investigate this freedom even in a steady state. From Eq. (3.7) in a steady state, we obtain the following conditional probability

$$P(\xi', t+\tau | \xi, \tau) \propto e^{-\frac{1}{2}[\xi' - \delta y(t+\tau)] g(t+\tau) [\xi' - \delta y(t+\tau)]}, (t > 0), (3.43)$$

with the initial conditions $\sigma(\tau)=0$ and $\delta y(\tau)=\xi$ where

$$\sigma(t+\tau) g(t+\tau) = 1, \quad \delta y(t+\tau) = e^{K_s t} \xi.$$

Then, the joint probability in the Markov process is

$$W_2^s(\xi', t+\tau; \xi, \tau) = P(\xi', t+\tau | \xi, \tau) P^s(\xi). \quad (3.44)$$

From Eqs. (3.43) and (3.44) we get the following equation:

$$\begin{aligned} & -2 \ln W_2^s(\xi', t+\tau; \xi, \tau) \\ & \simeq (\xi', \xi) \begin{pmatrix} g(t+\tau) & , & -g(t+\tau) e^{K_s t} \\ -e^{\tilde{K}_s t} g(t+\tau) & , & e^{\tilde{K}_s t} g(t+\tau) e^{K_s t} + g_s \end{pmatrix} \begin{pmatrix} \xi' \\ \xi \end{pmatrix} \\ & = (\xi', \xi) A^{-1} \begin{pmatrix} \xi' \\ \xi \end{pmatrix}, \end{aligned}$$

where the 2 by 2 matrix A , whose elements are N by N matrices, is evaluated by noting that the matrices do not commute in general, so that

$$A = \begin{pmatrix} e^{K_s t} \sigma_s e^{\tilde{K}_s t} + \int_0^t dt' e^{K_s t'} D_s e^{\tilde{K}_s t'}, & e^{K_s t} \sigma_s \\ \sigma_s e^{\tilde{K}_s t} & \sigma_s \end{pmatrix}. \quad (3.45)$$

From Eq. (3.10) in a steady state with the initial condition $\sigma(\tau) = \sigma_s$, we can obtain the following solution:

$$\begin{aligned} \sigma(t+\tau) &= e^{K_s t} \sigma_s e^{\tilde{K}_s t} + \int_t^{t+\tau} dt' e^{K_s(t+\tau-t')} D_s e^{\tilde{K}_s(t+\tau-t')} \\ &= e^{K_s t} \sigma_s e^{\tilde{K}_s t} + \int_0^t dt' e^{K_s t'} D_s e^{\tilde{K}_s t'}. \end{aligned} \quad (3.46)$$

Then, from Eqs. (3.45) and (3.46), we have

$$A = \begin{pmatrix} \sigma(t+\tau) & e^{K_s t} \sigma_s \\ \sigma_s e^{\tilde{K}_s t} & \sigma_s \end{pmatrix} = \begin{pmatrix} \sigma_s & e^{K_s t} \sigma_s \\ \sigma_s e^{\tilde{K}_s t} & \sigma_s \end{pmatrix}. \quad (3.47)$$

since $\sigma(t+\tau) = \sigma(t) = \sigma_s$ in a steady state. Finally we obtain, from Eq. (3.42)

$$W_2^s(\xi', t+\tau; \xi, \tau) \approx e^{-\frac{1}{2}(\xi', \xi) A^{-1} \begin{pmatrix} \xi' \\ \xi \end{pmatrix}}, \quad (3.48)$$

where

$$A = \begin{pmatrix} \sigma_s & C(t) \\ C(t) & \sigma_s \end{pmatrix} = \begin{pmatrix} \sigma_s & \sigma_s e^{-\tilde{K}_s + 2\alpha_s \mathcal{Q}_s(-t)} \\ \sigma_s e^{\tilde{K}_s t} & \sigma_s \end{pmatrix}.$$

In a steady state of the system, the one-gate probability has a fixed Gaussian shape and the joint probability represents the time evolution which is determined by K_s and α_s . It is concluded that α is a one of fundamental observable variables. The examination of α must be important in order to obtain the dynamical information of this rotational freedom in a reactor system.

In the next section we shall show that, even in a steady state, suitable processing of α and σ can serve to

provide a deeper insight into the various states of power reactors. We are led to a new viewpoint, based on which the system is conveniently analyzed through mathematical and physical examination of α , and we can deal with information on complicated power reactors by evaluation of α .

§3.4 Application to a Concrete Model of Reactor^{(4) (5)}

It has already been mentioned in §3.1 that a power reactor in operation is a very complicated system, involving a wide variety of phenomena. In the theoretical analysis, therefore, a number of assumptions and a simple model would have to be adopted, which must not, however, obscure the essential features. This section will present our formalism concerning a simple point (homogeneous) model of power reactor, assuming a reactor in the "normal" case.

The reactor of interest is of KUR-type, a water-moderated and -cooled reactor, discussed elsewhere in detail by Morishima⁽¹²⁾⁻⁽¹⁴⁾. For simplicity, we neglect the delayed neutrons and coolant velocity fluctuations which have been taken into account in his model. Accordingly, the state vector $X = \{X_i\}$ ($i=1,2,3$) consists of

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} N \\ \frac{C_v^f P_f V_f}{\rho} \theta_f \\ \frac{C_v^c P_c V_c}{\rho} \theta_c \end{pmatrix}, \quad (3.49)$$

where X_1 is the total number of neutrons in the reactor, X_2 the total energy content in the fuel region divided by the fission energy q , and X_3 the corresponding quantity for the coolant region. The other notations are summarized at end of this chapter. The transition probability is given by

$$\begin{aligned}
 W(X, \Delta X, t) = & S \delta_{\Delta X_1, 1} \delta_{\Delta X_2, 0} \delta_{\Delta X_3, 0} + \Lambda_c X_1 \delta_{\Delta X_1, -1} \delta_{\Delta X_2, 0} \delta_{\Delta X_3, 0} \\
 & + \sum_{\nu=0}^{\infty} \Lambda_f X_1 p_f(\nu) \delta_{\Delta X_1, \nu-1} \delta_{\Delta X_2, 1} \delta_{\Delta X_3, 0} \\
 & + h_t (X_2 - \eta X_3) \delta_{\Delta X_1, 0} \delta_{\Delta X_2, -1} \delta_{\Delta X_3, 1} \\
 & + \lambda_r (X_3 - Y_3^{in}) \delta_{\Delta X_1, 0} \delta_{\Delta X_2, 0} \delta_{\Delta X_3, -1}, \quad (3.50)
 \end{aligned}$$

where all the quantities representing the probabilities of elementary process, and the net changes of state quantities are identical with those adopted by Morishima (see Table 1).

Table 1 Elementary events in transition probability (3.50)

Elementary event	Its rate per unit time	Net change per event
Neutron source	S	+1 neutron
Neutron removal	$\Lambda_c X_1$	-1 neutron
Fission	$\Lambda_f X_1 p_f(\nu)$	$\left\{ \begin{array}{l} (\nu-1) \text{ neutrons} \\ +1 \text{ energy in fuel} \end{array} \right.$
Heat transfer	$h_t (X_2 - \eta X_3)$	$\left\{ \begin{array}{l} -1 \text{ energy in fuel} \\ +1 \text{ energy in coolant} \end{array} \right.$
Heat removal	$\lambda_r (X_3 - Y_3^{in})$	-1 energy in coolant

In our formulation, particular attention must be paid to the difference between the extensive and intensive quantities. The reaction rates — Λ_f for fission and Λ_c for capture — are volume-averaged over the reactor:

$$\begin{cases} \Lambda_f = \frac{V_f}{\Omega} \Lambda_f^{(f)} \\ \Lambda_c = \frac{V_f}{\Omega} \Lambda_c^{(f)} + \frac{V_c}{\Omega} \Lambda_c^{(c)} \end{cases} \quad (3.51)$$

All the reaction rates contained in the right-hand side of the expression (3.51) are assumed to be intensive quantities depending linearly on temperatures:

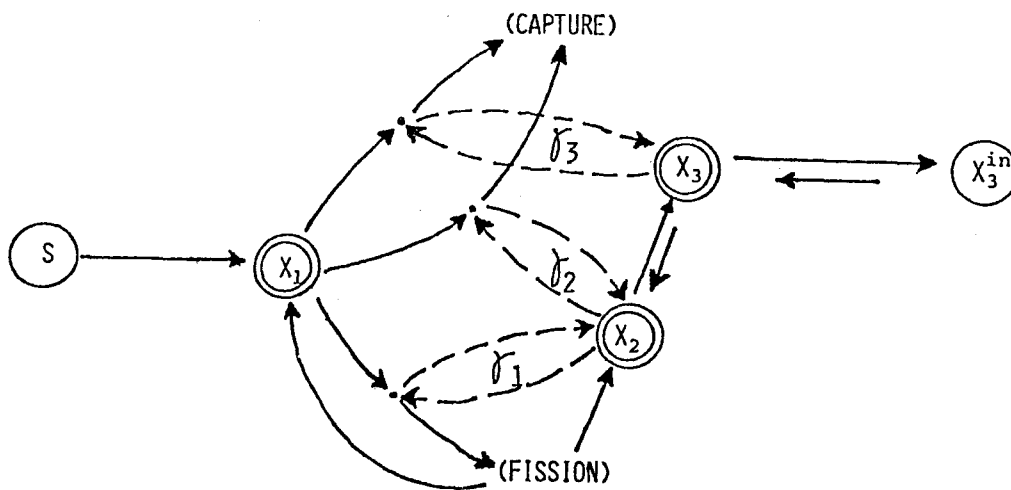
$$\begin{cases} \Lambda_f^{(f)} = \Lambda_{f0}^{(f)} \{ 1 - \gamma_1 (\theta_f - \theta_f^0) \} \\ \Lambda_c^{(f)} = \Lambda_{c0}^{(f)} \{ 1 - \gamma_2 (\theta_f - \theta_f^0) \} \\ \Lambda_c^{(c)} = \Lambda_{c0}^{(c)} \{ 1 - \gamma_3 (\theta_c - \theta_c^0) \} \end{cases} , \quad (3.52)$$

where the reaction rates $\Lambda_{f0}^{(f)}$, $\Lambda_{c0}^{(f)}$ and $\Lambda_{c0}^{(c)}$ are standard values and θ_f^0 and θ_c^0 are chosen to be the steady-state values defined later. The numerical values of the parameters are given in Table 2. Some of these values have been replaced by corresponding effective quantities, as we have simplified Morishima's model.

Table 2 Model parameters

\bar{v} -----	2.473	$\Lambda_{c0}^{(f)} / \Lambda_{c0}^{(c)}$ -----	30
\bar{v}^2 -----	7.337	H (cm) -----	64.7
q (MeV) -	200	ρ_f (g/cm ³) --	10.0
S (sec ⁻¹)	10 ⁹	ρ_c (g/cm ³) --	1.0
l_s (sec) -	10 ⁻⁴	θ_{in} (°C) -----	30
Ω (cm ³) -	1.053x10 ⁵	C_v^f (cal/g°C)	0.05
V_f (cm ³) -	3.74x10 ⁴	C_v^c (cal/g°C)	1.0
V_c (cm ³) -	6.79x10 ⁴	γ_2 (°C) -----	2x10 ⁻⁵
P (kW) --	1000		

The network of various processes in the model is symbolically represented in Fig. 3.1. The nomenclature used in this model is also listed at the end of this chapter.



The parameters γ_1 , γ_2 and γ_3 are the linear temperature feedback coefficients of expression (3.52)

Fig. 3.1 Reaction network of power reactor presented by expression (3.50)

In short, this model is characterized by its being an open system subject to linear feedback mechanism and in a non-equilibrium state.

The system size expansion method applied to the right-hand side of the mean evolution equation (3.6) yields

$$C_1(\mathcal{Y}) = \begin{pmatrix} \frac{1}{l_s} \{ \rho - a_1(\mathcal{Y}_2 - \mathcal{Y}_2^s) - a_2(\mathcal{Y}_3 - \mathcal{Y}_3^s) \} \mathcal{Y}_1 + S \\ \frac{1}{l_s \bar{\nu}} \{ 1 - b(\mathcal{Y}_2 - \mathcal{Y}_2^s) \} \mathcal{Y}_1 - k_t(\mathcal{Y}_2 - \eta \mathcal{Y}_3) \\ k_t(\mathcal{Y}_2 - \eta \mathcal{Y}_3) - \lambda_r(\mathcal{Y}_3 - \mathcal{Y}_3^m) \end{pmatrix}, \quad (3.53)$$

where

$$l_s = \frac{\Delta \Omega}{\bar{\nu} V_f \Lambda_{40}^{(+)}}, \quad \rho = \frac{\bar{\nu} - 1}{\bar{\nu}} - \frac{l_s}{\Delta \Omega} (V_f \Lambda_{co}^{(+)} + V_c \Lambda_{co}^{(c)})$$

$$S = \frac{S}{\Delta \Omega}, \quad a_1 = \frac{\beta \Delta \Omega}{c_f^+ \rho_f V_f} \left(\frac{\bar{\nu} - 1}{\bar{\nu}} r_1 - \frac{V_f l_s}{\Delta \Omega} \Lambda_{co}^{(+)} r_2 \right)$$

$$b = \frac{\beta \Delta \Omega}{c_f^+ \rho_f V_f} \gamma_1, \quad a_2 = \frac{\beta \Delta \Omega}{c_f^+ \rho_c V_c} \left(\frac{V_f l_s}{\Delta \Omega} \Lambda_{co}^{(c)} r_3 \right).$$

In the evolution equation (3.10) for the variance, $\mathbf{K}(t)$ and $\mathbf{D}(t)$ are expressed by

$$K_{ij}(\mathcal{Y}) = \frac{\partial}{\partial y_j} C_i^{(i)}(\mathcal{Y}), \quad (3.54)$$

$$\left\{ \begin{array}{l} D^{(11)}(\mathcal{Y}) = \frac{\nu(\nu-1)}{l_s \bar{\nu}} \{1 - b(y_2 - y_2^s)\} y_1 - \frac{1}{l_s} \{P - a_1(y_2 - y_2^s) - a_2(y_3 - y_3^s)\} y_1 + S \\ D^{(12)}(\mathcal{Y}) = \frac{\nu-1}{l_s \bar{\nu}} \{1 - b(y_2 - y_2^s)\} y_1 \\ D^{(13)}(\mathcal{Y}) = 0 \\ D^{(22)}(\mathcal{Y}) = \frac{1}{l_s \bar{\nu}} \{1 - b(y_2 - y_2^s)\} y_1 + k_t (y_2 - \eta y_3) \\ D^{(23)}(\mathcal{Y}) = -k_t (y_2 - \eta y_3) \\ D^{(33)}(\mathcal{Y}) = k_t (y_2 - \eta y_3) + \lambda_r (y_3 - y_3^{in}). \end{array} \right. \quad (3.55)$$

We shall first examine the stability condition of a steady state. The fact that we are considering a steady state assures satisfaction of the condition

$$C_i(\mathcal{Y}^s) = 0.$$

$$\mathcal{Y}^s = \begin{pmatrix} y_1^s \\ y_2^s \\ y_3^s \end{pmatrix} = \begin{pmatrix} -\frac{l_s S}{P} \\ \eta y_3^s + \frac{1}{k_t l_s \bar{\nu}} y_1^s \\ y_3^{in} + \frac{1}{\lambda_r l_s \bar{\nu}} y_1^s \end{pmatrix}. \quad (3.56)$$

Hence, the expression (3.54) and (3.55) become

$$K_s = \begin{pmatrix} \frac{P}{l_s} & , & -\frac{a_1}{l_s} y_1^s & , & -\frac{a_2}{l_s} y_1^s \\ \frac{1}{l_s \bar{\nu}} & , & -\frac{b}{l_s \bar{\nu}} y_1^s - k_t & , & k_t \eta \\ 0 & , & k_t & , & -k_t \eta - \lambda_r \end{pmatrix}, \quad (3.57)$$

and

$$D_s = \begin{pmatrix} \frac{\nu(\nu-1)}{l_s \bar{\nu}} y_1^s + 2S, & \frac{\nu-1}{l_s \bar{\nu}} y_1^s, & 0 \\ \frac{\nu-1}{l_s \bar{\nu}} y_1^s, & \frac{2}{l_s \bar{\nu}} y_1^s, & -\frac{1}{l_s \bar{\nu}} y_1^s \\ 0, & -\frac{1}{l_s \bar{\nu}} y_1^s, & \frac{2}{l_s \bar{\nu}} y_1^s \end{pmatrix}. \quad (3.58)$$

The steady state can be classified according to the eigenvalues of the regression matrix. A linear stable region, for example, is determined by the Routh-Hurwitz criterion. From Eq. (3.9) in a steady state, i.e., $\frac{d}{dt} \delta y(t) = K_s \delta y(t)$, we have the characteristic equation

$$Z^3 + H_1 Z^2 + H_2 Z + H_3 = 0, \quad (3.59)$$

where

$$\begin{cases} H_1 = -\text{Tr } K_s \\ H_2 = \det K_s \cdot \text{Tr } K_s^{-1} \\ H_3 = -\det K_s, \end{cases}$$

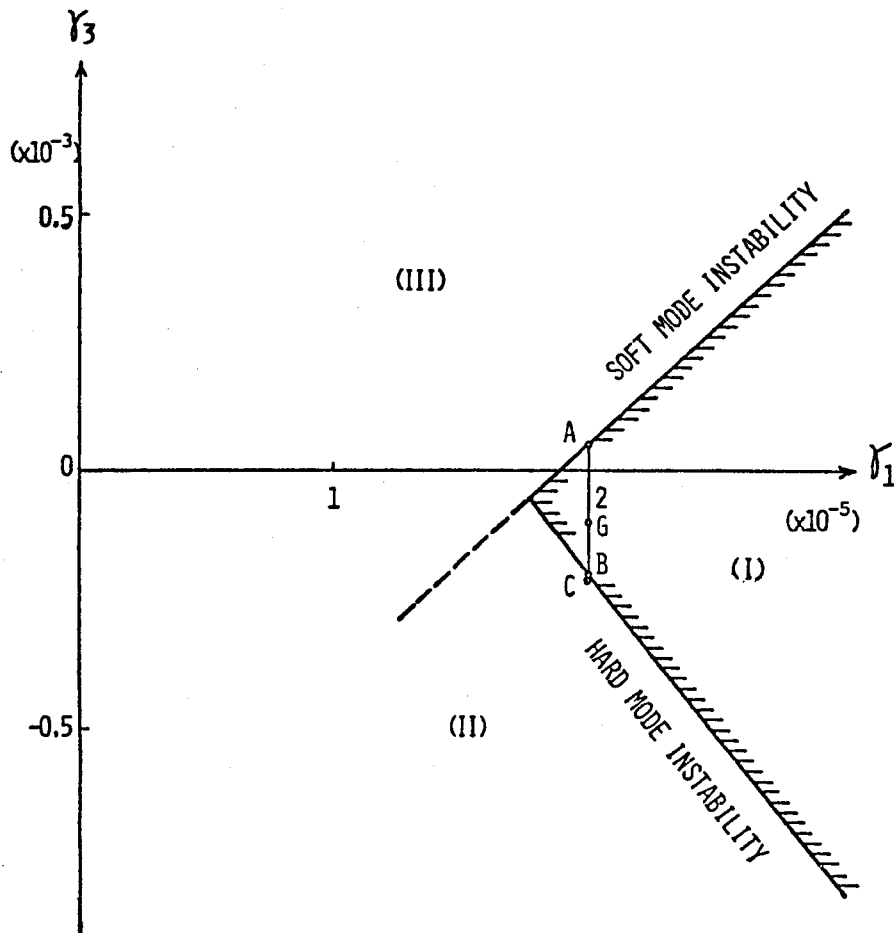
from which we have for the condition of stability

$$H_1 > 0, H_2 > 0, H_3 > 0 \quad \text{and} \quad H_1 H_2 > H_3. \quad (3.60)$$

Figure 3.2 shows the linear stable region in the parameter space where γ_2 is fixed. The values of the parameters used in this calculation are also given in Table 2.

It should be noted that the stable domain for δy coincides with for δ , both depending on the eigenvalues of K_s in the normal case.

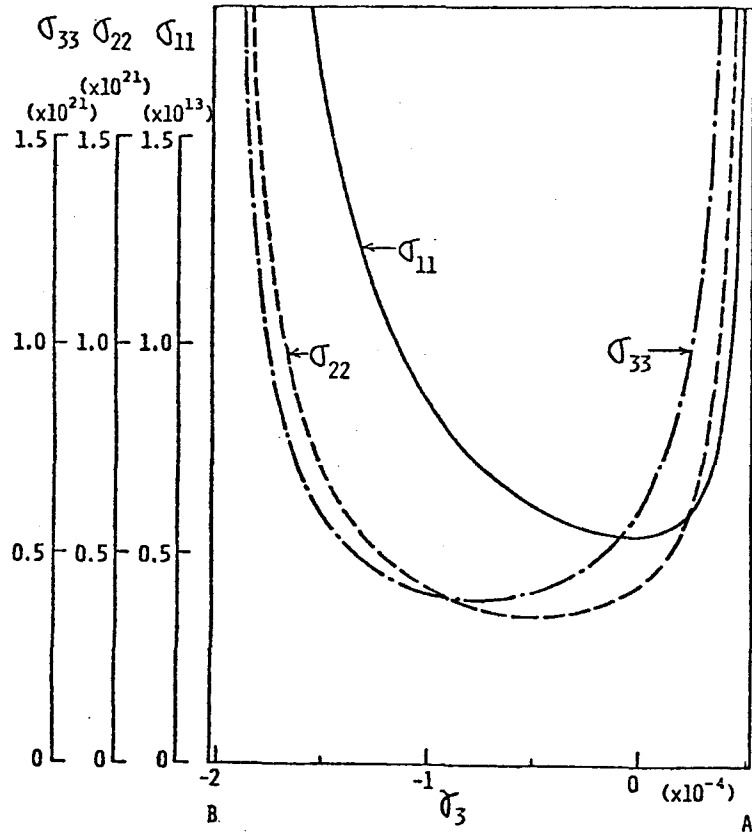
When the parameters change under a certain condition, an instability may be produced by feedback or non-linear effects. The three degrees of freedom possessed by the system give rise to different types of instability. For instance, the system becomes unstable when the real part of any root of Eq. (3.59) approaches zero. Two typical cases may occur. One is that in which one real root approaches zero. This instability is called the "soft-mode instability". Another case of instability is that



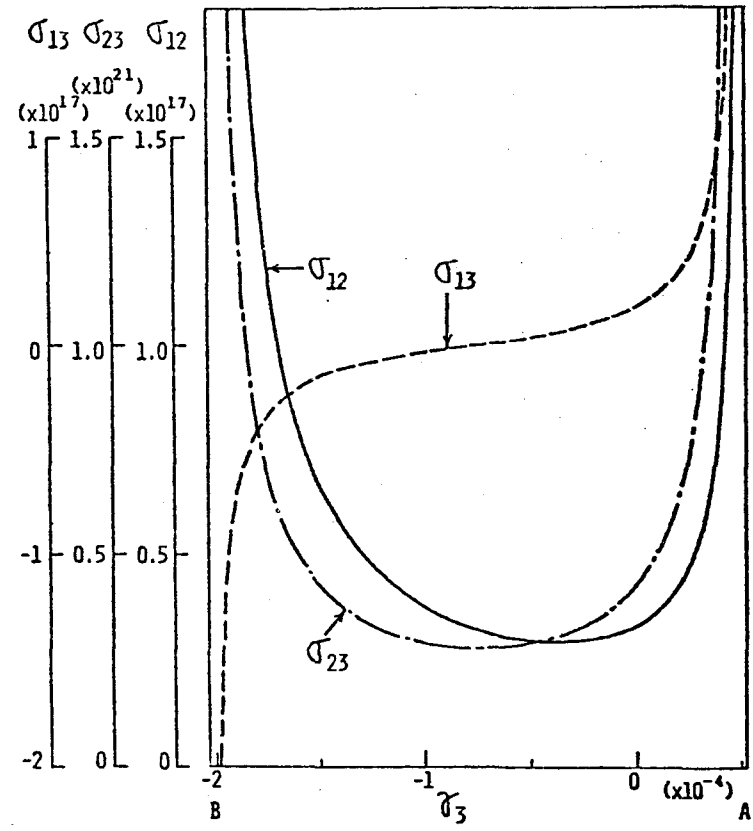
The domains (I), (II) and (III) are stable, unstable focus and col regions, respectively. In particular, there is a limit cycle region in a part of (II). The boundaries of the stable region are marginal states. Points A, B, C and G are on a straight line ($\gamma_1 = 2 \times 10^{-5}$, $\gamma_2 = 2 \times 10^{-5}$), with $\gamma_3 = 5.22598 \times 10^{-5}$ ($= \gamma_3^s$), -2.04815×10^{-4} ($= \gamma_3^h$), -2.0686×10^{-4} and -1.0×10^{-4} , respectively.

Fig. 3.2 Classification of reactor states in parameter space based on Routh-Hurwitz criterion

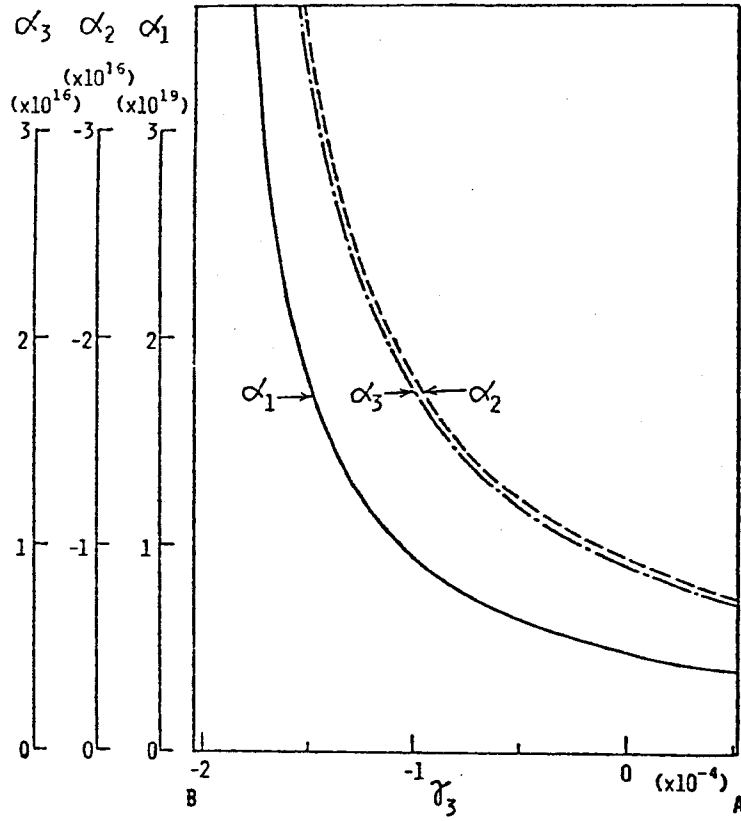
in which the real part of a pair of conjugate roots approaches zero. This is called the "hard-mode instability". In fact, the steady state becomes unstable at the boundary of the stable region in Fig. 3.2. The parameter γ_3 is regarded as a slow-varying quantity. If the



(a)



(b)



(C)

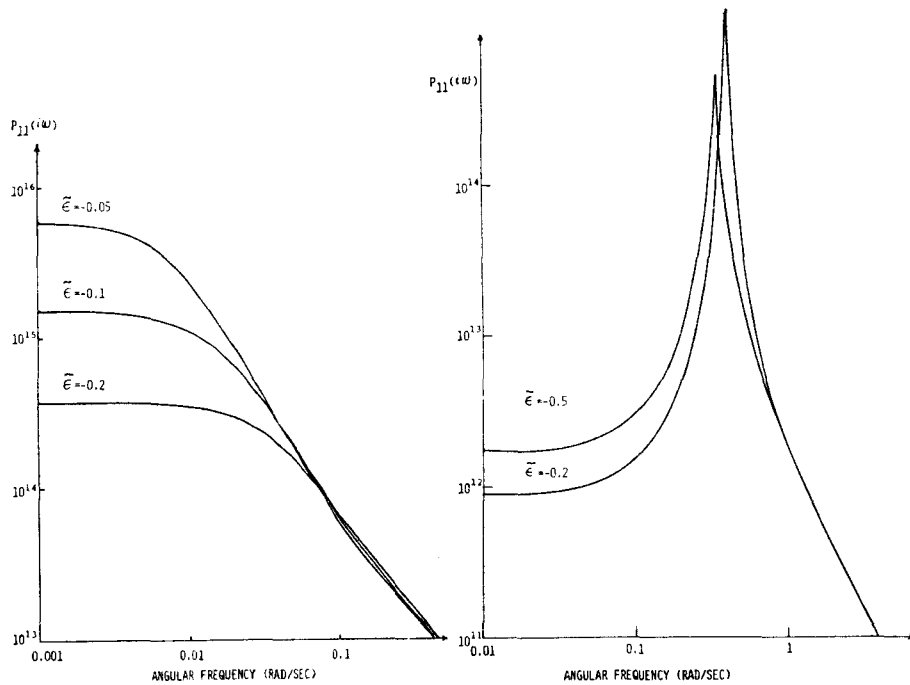
- (a) Variance σ_{ii}
- (b) Covariance σ_{ij}
- (c) Irreversible circulation of fluctuation α

$\alpha_i = \alpha_{ik}$, $\alpha_{ij} = -\alpha_{ji}$ where (i,j,k) is a cyclic permutation of 1, 2, 3.

Fig. 3.3 Steady state values of σ and α on the line AB in stable region of Fig. 3.2

parameter γ_3 changes, the variance σ diverges at the two points corresponding to A and B on the abscissa of Fig. 3.3. This situation occurs when the coolant temperature feedback changes on account of any event occurring in the coolant, such as turbulence and boiling.

These instabilities are seen in the (1,1) component of the power spectral density presented in Fig. 3.4.



(a) Near soft mode

(b) Near hard mode

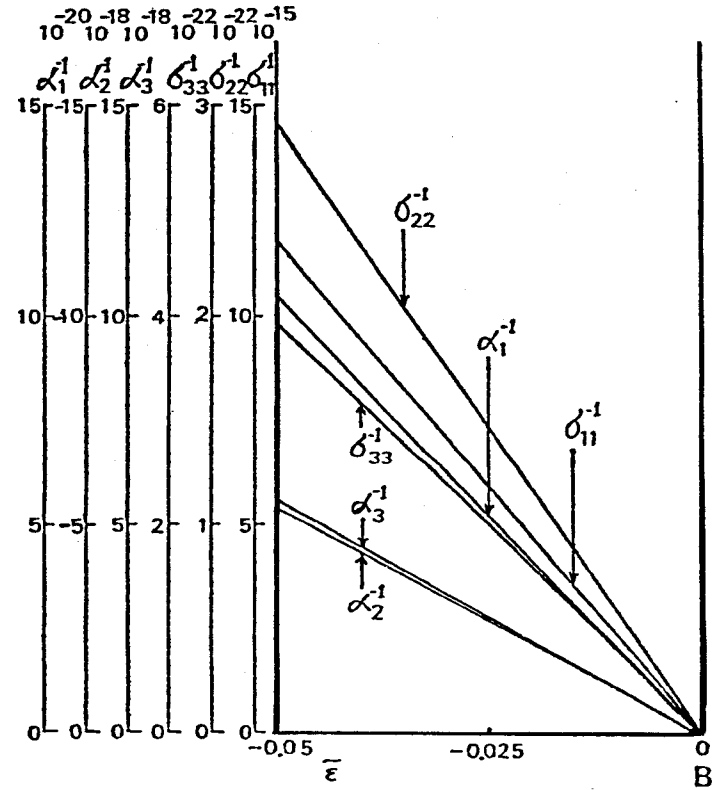
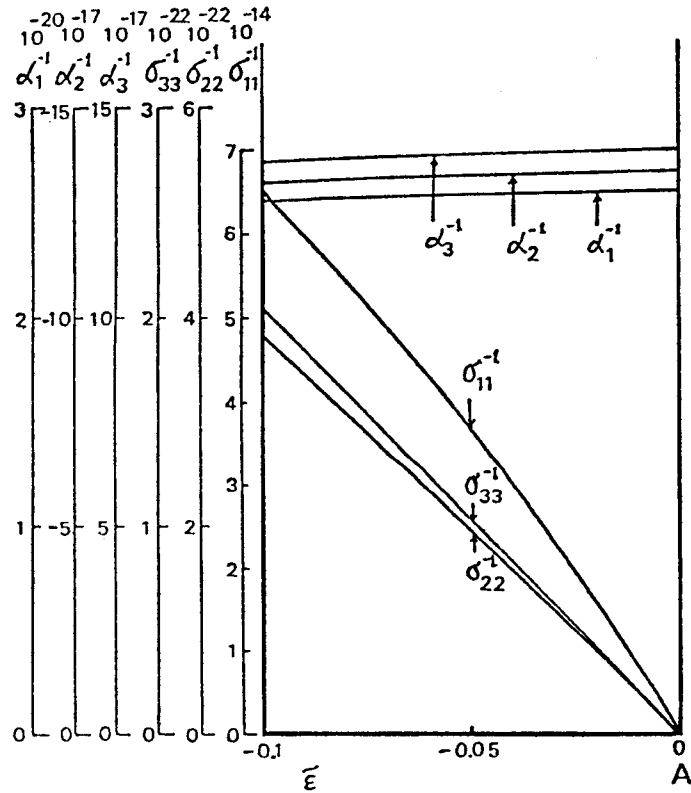
instability $(\gamma_3^c = \gamma_3^s)$

instability $(\gamma_3^c = \gamma_3^h)$

Fig. 3.4 (1,1) component of power spectral density

$$\tilde{\epsilon} = (\gamma_3 - \gamma_3^c) / \gamma_3^c$$

As γ_3 approaches the point A, the zero frequency component of the single mode increases divergently. This is the case of the soft-mode instability. The hard-mode instability appears when some finite frequency component of a pair of modes diverges at point B. These instabilities are also shown by σ and α near the transition points A and B in Fig. 3.5. Thus, the analogy with generalized phase transitions in far from equilibrium serves to



(a) Soft mode instability

(b) Hard mode instability

(α is finite near $\gamma_3^c = \gamma_3^s$)

(α is infinite near $\gamma_3^c = \gamma_3^h$)

Fig. 3.5 Steady State values of σ_{11}^{-1} , σ_{22}^{-1} , σ_{33}^{-1} , α_1^{-1} , α_2^{-1} and α_3^{-1} $\bar{\epsilon} = (\gamma_3 - \gamma_3^c) / \gamma_3^c$

$\alpha_i = \alpha_{jk}$, $\alpha_{ij} = -\alpha_{ji}$ where (i, j, k) is a cyclic permutation of 1, 2, 3.

describe reactor noise phenomena.

Information only concerning σ will not enable us to predict which type of instability will occur. This will be possible by taking α into consideration as well as σ , as it is suggested by the fact that σ is related to the symmetric part of the expression (3.14), and α to the antisymmetric. This is evident also from an examination of Fig. 3.3-c or 3.5, where α remains finite in the case of soft mode instability, while a corresponding instability of the hard mode makes the same variable diverge. This can be analytically proved in the following manner: The quantity α is related to the imaginary part of the power spectral density, as seen from Eq. (3.35). This imaginary part is, by definition, an odd function of ω :

$$\begin{aligned} \text{Im } P(\omega) &= \frac{1}{2i} (P(\omega) - \overline{P(\omega)}) \\ &= \int_{-\infty}^{\infty} C(t) \frac{e^{i\omega t} - e^{-i\omega t}}{2i} dt \\ &= \int_{-\infty}^{\infty} C(t) \sin \omega t dt. \end{aligned} \quad (3.61)$$

Since ω has appeared in Eq. (3.31) in combination $i\omega$, the imaginary part must be proportional to ω , i.e.,

$$\text{Im } P(\omega) \propto \omega \cdot f_e(\omega), \quad (3.62)$$

where $f_e(\omega)$ is a certain given even-function of ω . In the case of soft mode instability, the diverging zero-frequency component of $P(\omega)$ is canceled out in the integral $\omega \cdot \text{Im } P(\omega)$ of α defined by Eq. (3.35), and α remains finite in contrast to σ .

When the value of the parameter γ_3 declines beyond the boundary point B, instability appears in the system, which thereupon becomes subject to a limit cycle. As an example, the limit cycle at point C in Fig. 3.2 has been analyzed numerically, with the result shown in Fig. 3.6. This may be interpreted as the circulation inherent to fluctuations making its appearance and taking the form of

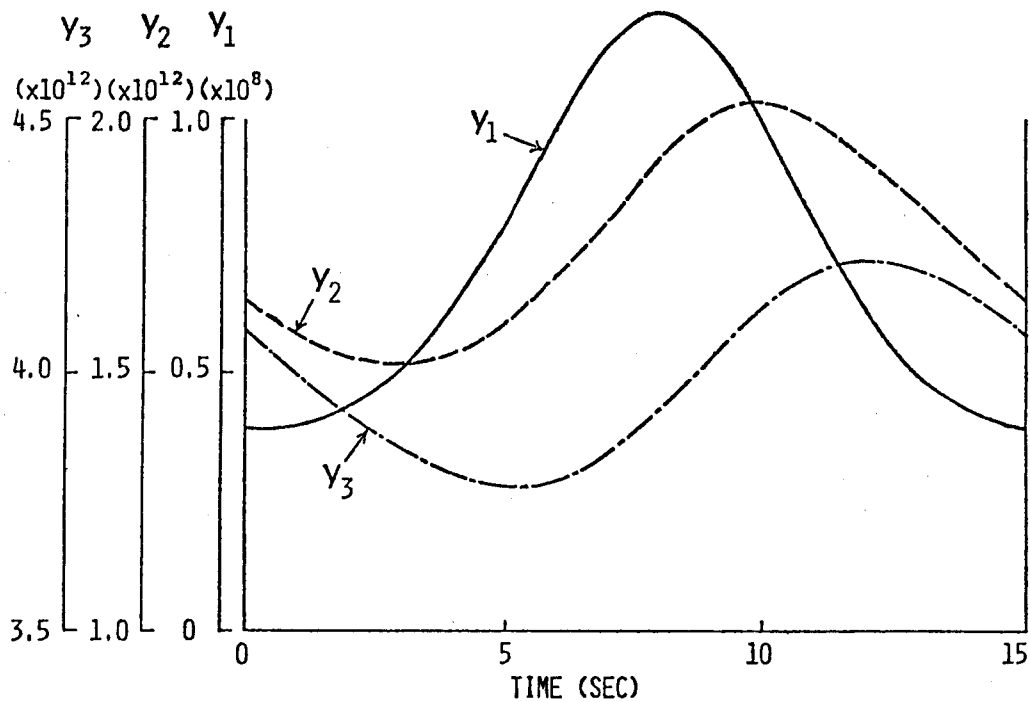
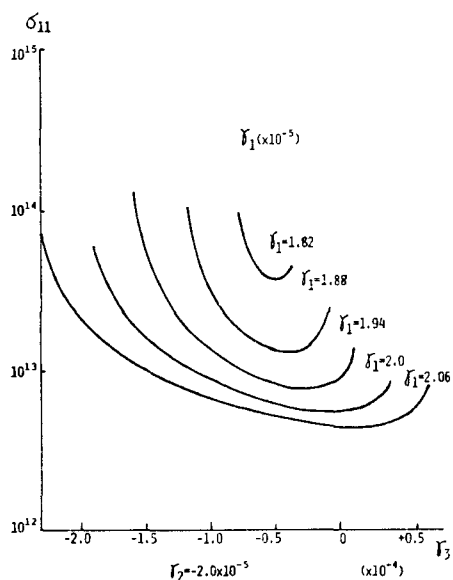


Fig. 3.6 Limit cycle at C point in Fig. 3.2

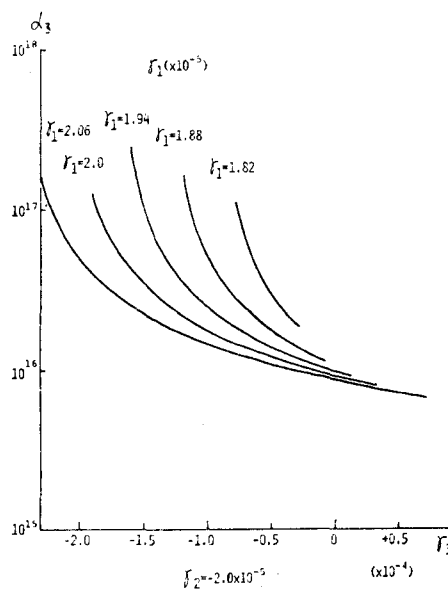
limit cycle in the mean evolution. In this situation, the quantities α and σ provide information on "forerunner phenomena" of unstable states, while the mean value remains constant until γ_3 reaches A or B. In addition, we are able to observe, as in Fig. 3.7 the changes brought up on α_3 and σ_{11} under stable steady state by variations in the parameter γ_1 and γ_3 .

The stability diagram of the system being specified by parameter γ_3 (coolant temperature feedback coefficient) is also shown in Fig. 3.8 for a change of power of a reactor according to the linear stability criterion of Eq. (3.60). [cf. Fig. 3.2]

Next, we shall illustrate how α is related to the asymmetry of correlation functions in this model. Typical patterns of correlation functions at point H ($\gamma_3 = 1 \times 10^{-5}$) in Fig. 3.8 are given from Eqs. (3.36) and (3.42), and shown in Fig. 3.9.

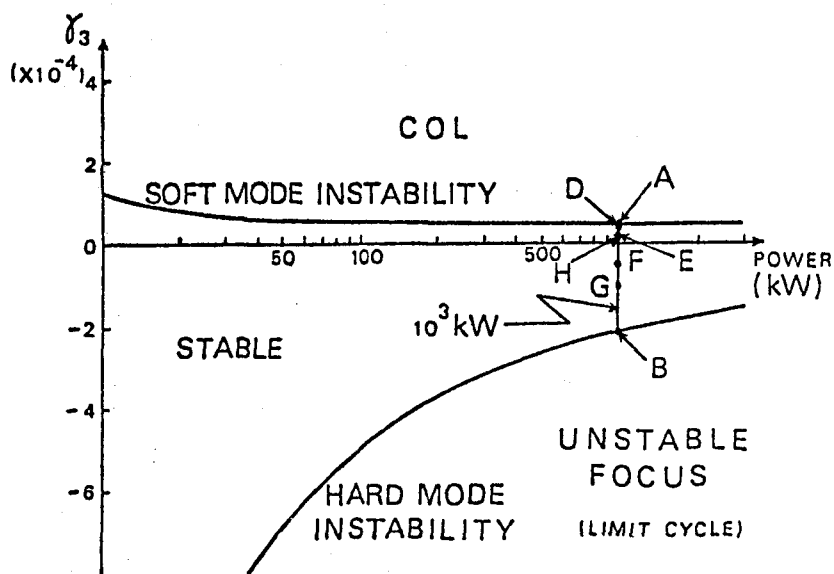


(a) Variance σ_{11}



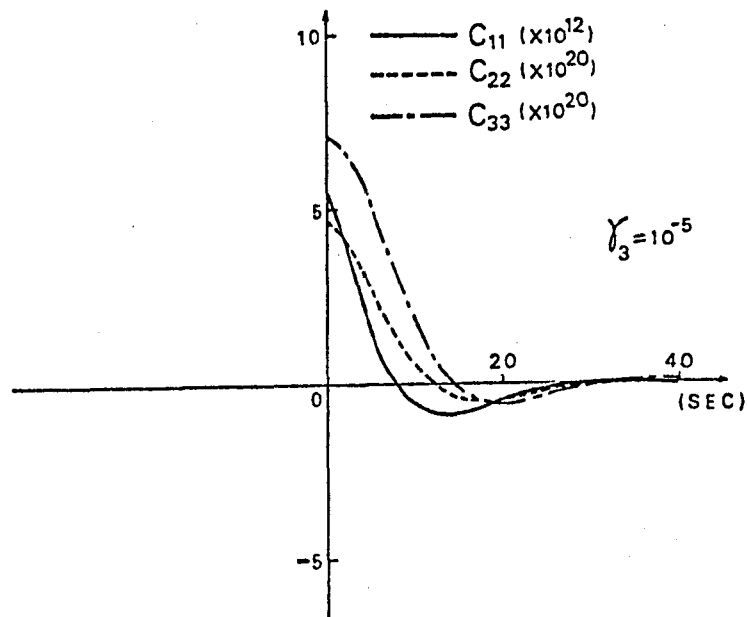
(b) Irreversible circulation of fluctuation α_3

Fig. 3.7 Steady state values of σ_{11} and α_3 in stable region by variations in γ_1 and γ_3

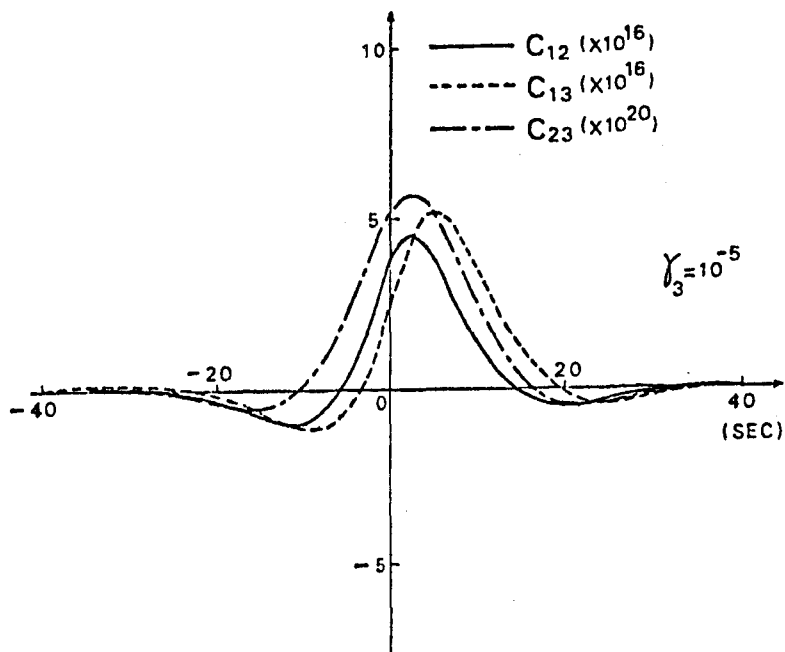


Points A, B, D, E, F, G and H are on a straight line (power=1000kW), with $\gamma_3 = 5.22598 \times 10^{-5}$ ($=\gamma_3^s$), -2.04815×10^{-4} ($=\gamma_3^h$), 4.0×10^{-5} , 2.0×10^{-5} , -5.0×10^{-5} , -10.0×10^{-5} and 1.0×10^{-5} , respectively.

Fig. 3.8 Diagrammatical classification of stability of system



(a) Auto-correlation functions



(b) Cross-correlation functions

Fig. 3.9 Correlation functions at point H in Fig. 3.8

Figure 3.9-b represents asymmetry of cross-correlation functions due to the existence of the irreversible circulation of fluctuation α as mentioned in §3.3. In each figure, C_{ii} 's or C_{ij} 's ($i, j=1, 2, 3; i \neq j$) have similar tendency of variation with time. This allows us to adopt C_{11} and C_{12} for the representatives of auto- and cross-correlation functions. The characteristics of reactor have significant effects on these patterns of correlation functions. Figure 3.10 shows that these patterns change when the parameter γ_3 changes on the line AB in the stable region of Fig. 3.2 or 3.8.

Thus, it is important to find the way how to process information of reactor noise in the cases of where changes of these patterns occur one after another. It is also necessary to systematize various observable variables in complicated reactor noise phenomena and to know mutual correspondence among them. Then, let us illustrate this by the above-mentioned reactor model. Figure 3.11 shows mutual correspondence of observable variables.

Hence, we can estimate a position of operating reactor state in the stability diagram by taking advantage of α which is an index for types of instability (soft or hard). We should make use of reactor noise not only estimating reactor dynamics or kinetics parameters but also treating reactor stability. Moreover, with the increase of reliability of reactor noise experiments, it is useful to adopt various possible representation of an observable variable, corresponding experimental situations. In other words, reactor noise experiments are, in general, rather difficult and we should not draw any conclusion on the reactor state until we ascertain the agreement of results derived from various different measurements.

Finally, we summarize essential equations or relations in the steady stable state of the normal case in Fig. 3.12.

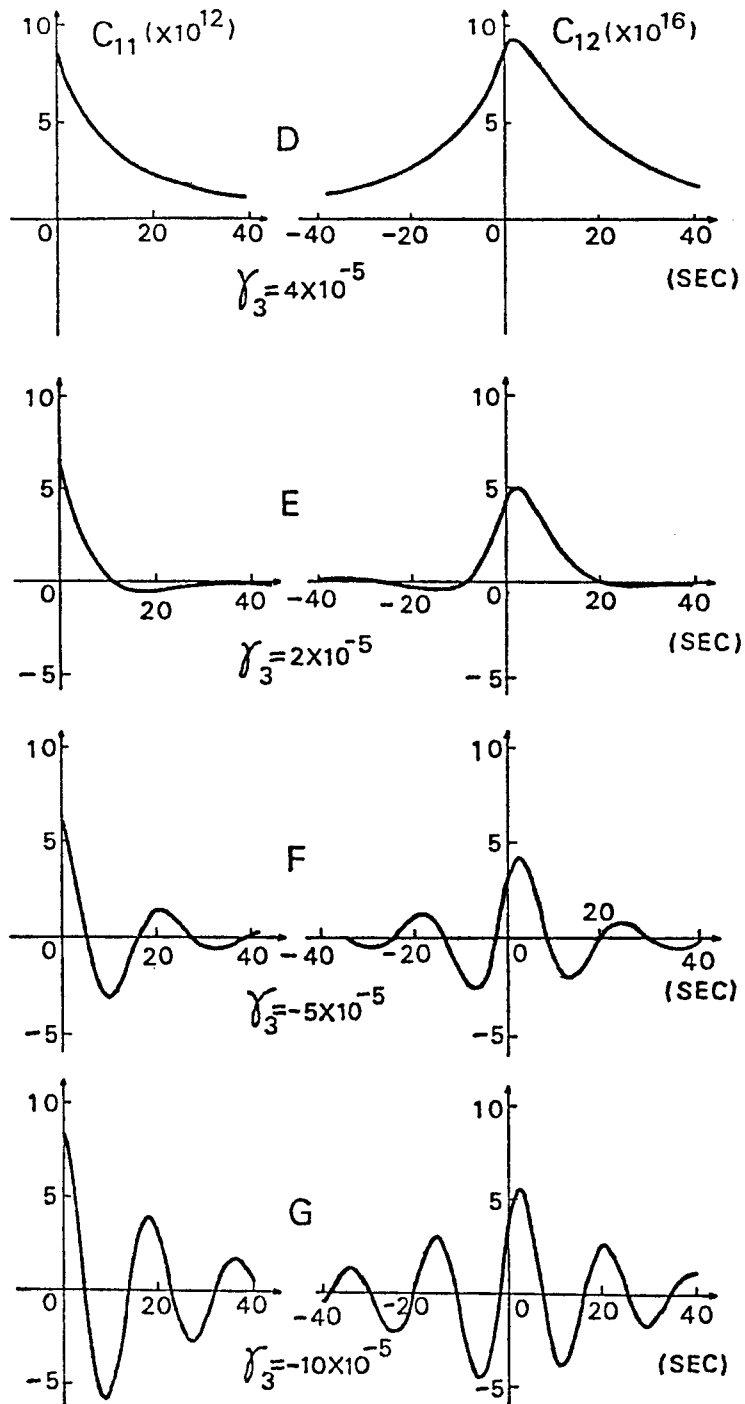


Fig. 3.10 Patterns of correlation functions at points D, E, F and G in Fig. 3.8

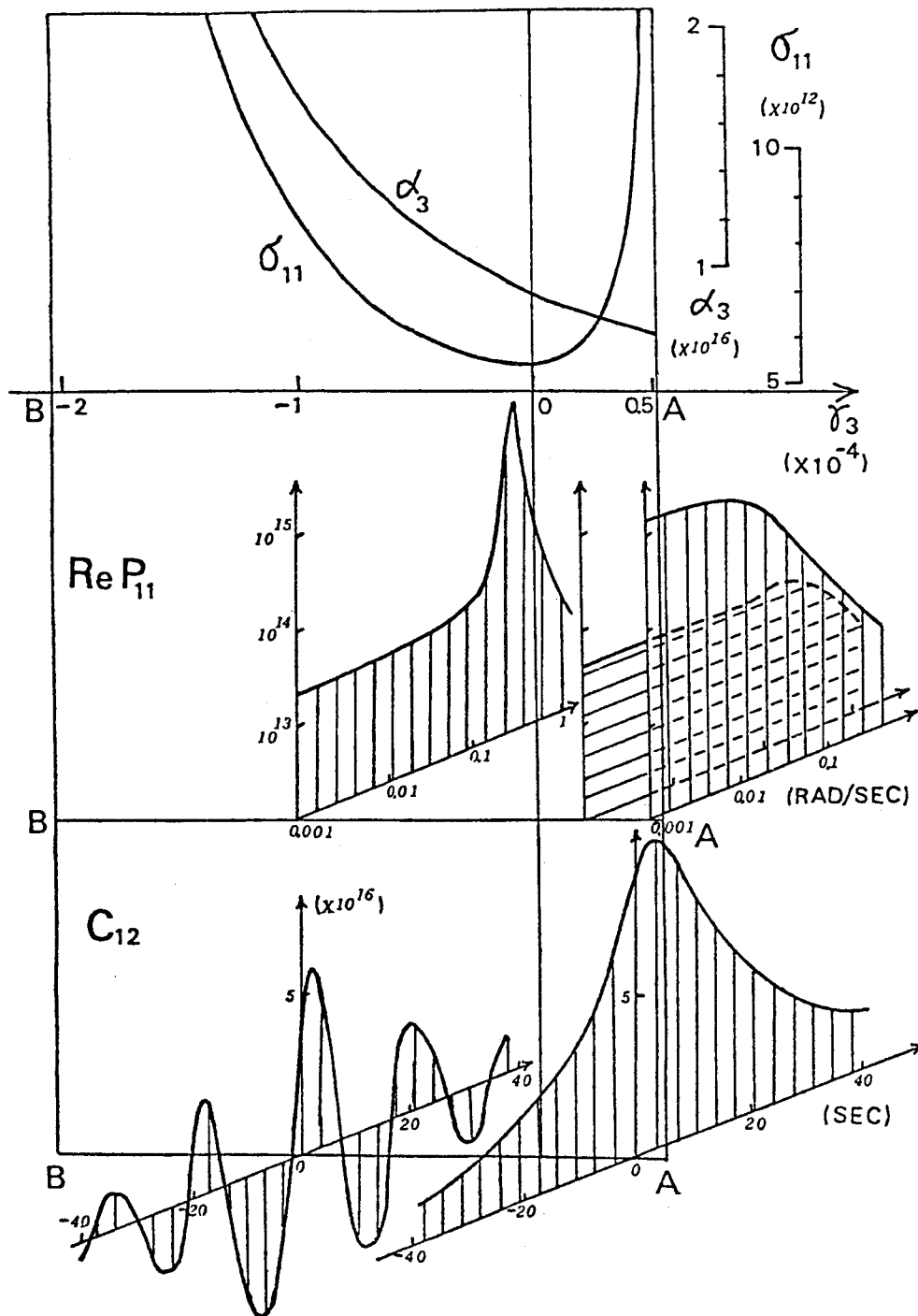


Fig. 3.11 Example of processing information in reactor noise

< Normal case > $x = y^s + \varepsilon^{\frac{1}{2}} \xi$

Evolution equations

{	Mean value	$y = y^s + \delta y$ ($C_1(y^s) = 0$)
	Small deviation	$\frac{d}{dt} \delta y = K_s \delta y$ ↓ [in §3.4] <div style="border: 1px solid black; padding: 2px; display: inline-block;">soft mode instability hard mode instability</div>
	Fluctuations	$\frac{\partial}{\partial t} P(\xi, t) = -\frac{\partial}{\partial \xi} K_s \xi P(\xi, t) + \frac{1}{2} \frac{\partial}{\partial \xi} D_s \frac{\partial}{\partial \xi} P(\xi, t)$
		$\frac{d}{dt} \xi = K_s \xi + F$ [in Chap.4] <div style="display: inline-block; vertical-align: middle;"> $\left\{ \begin{array}{l} \langle F \rangle = 0 \\ \langle F_t F_{t'} \rangle = D_s \delta(t-t') \end{array} \right.$ </div>

Relations among statistical variables

{	Correlation function	$C(t) = \langle \xi(0) \xi(t) \rangle = \begin{cases} \sigma_s e^{K_s t} & (t \geq 0) \\ e^{K_s t} \sigma_s & (t \leq 0) \end{cases}$ $= \frac{1}{2\pi} \int P(\omega) e^{-i\omega t} d\omega$
	Power spectral density	$P(\omega) = \int C(t) e^{i\omega t} dt$ $= [i\omega - K_s]^{-1} D_s [i\omega + \bar{K}_s]^{-1}$
	Variance	$\sigma_s = \langle \xi \xi \rangle = C(0) = \frac{1}{2\pi} \int P(\omega) d\omega$ ($K_s \sigma_s + \bar{K}_s \sigma_s + D_s = 0$)
	Irreversible circulation of fluctuation	$\alpha_s = \frac{1}{2} (\bar{K}_s \sigma_s - K_s \sigma_s) = \frac{1}{2} \{ \dot{C}(+0) - \dot{C}(-0) \}$ $= \frac{1}{2\pi} \int \omega \text{Im} P(\omega) d\omega$

Fig. 3.12 Essential equations or relations in the steady stable state of the normal case

§3.5 Conclusion and Discussion

The irreversible circulation of fluctuation α has been introduced as a new variable for representing the amount of reactor noise, and the usefulness of α in reactor noise theory has been demonstrated for non-linear and/or non-stationary cases.

We have found for steady state the relations of α to the variables treated in conventional reactor noise theory. We have also applied our formalism to a simple model in order to analyze its behavior numerically. The analysis of the experimental data is under way and we also hope other authors may find occasion to compare these numerical results with actually measured reactor noise.

The section 3.4 gave the condition for power reactor stability, represented by domains marked out in a diagram in some parameter space. Transition from a stable region to one of limit cycles has been shown in Fig. 3.2. This would suggest that reactor noise theory might possibly be formulated from the viewpoint of generalized phase transition phenomena — "generalized" meaning the occurrence of such phase transition in a non-equilibrium open system.

While the mean value of ψ is constant in a stable region, its fluctuations can be utilized as a "forerunner phenomenon" to foretell undesirable instability, through appropriate information processing. From the numerical results obtained with our model, we have found that the divergence of both α and σ indicates hard mode instability, while soft mode instability is characterized by infinite σ and finite α . Thus, we see that α can serve effectively as an index for instabilities. Beyond the threshold of hard mode instability a limit cycle appears. While in the normal case the amplitude of this limit cycle is small in comparison with the mean value, and limit cycle is considered to be stable, it should

nonetheless be carefully examined whether the reactor is safe or not in the limit cycle state. Such a decision will be made by processing the information on α and σ .

It should be noted in passing that it is important to check the present results against those from other independent methods of information processing, in order to increase the reliability of the ensuing judgement on reactor safety. And the values obtained by such parallel procedures must coincide with the above results, so long as the conditions do not differ.

It is to be noted that the definitions of α and σ given in § 3.2 hold also in the case of non-stationary states. As an example, as the parameter γ_3 changes from the point G in the stable region of Fig. 3.2 to point C in the limit cycle region, $y_1(t)$, $\sigma_{11}(t)$ and $\alpha_3(t)$ vary in time as shown in Fig. 3.13.

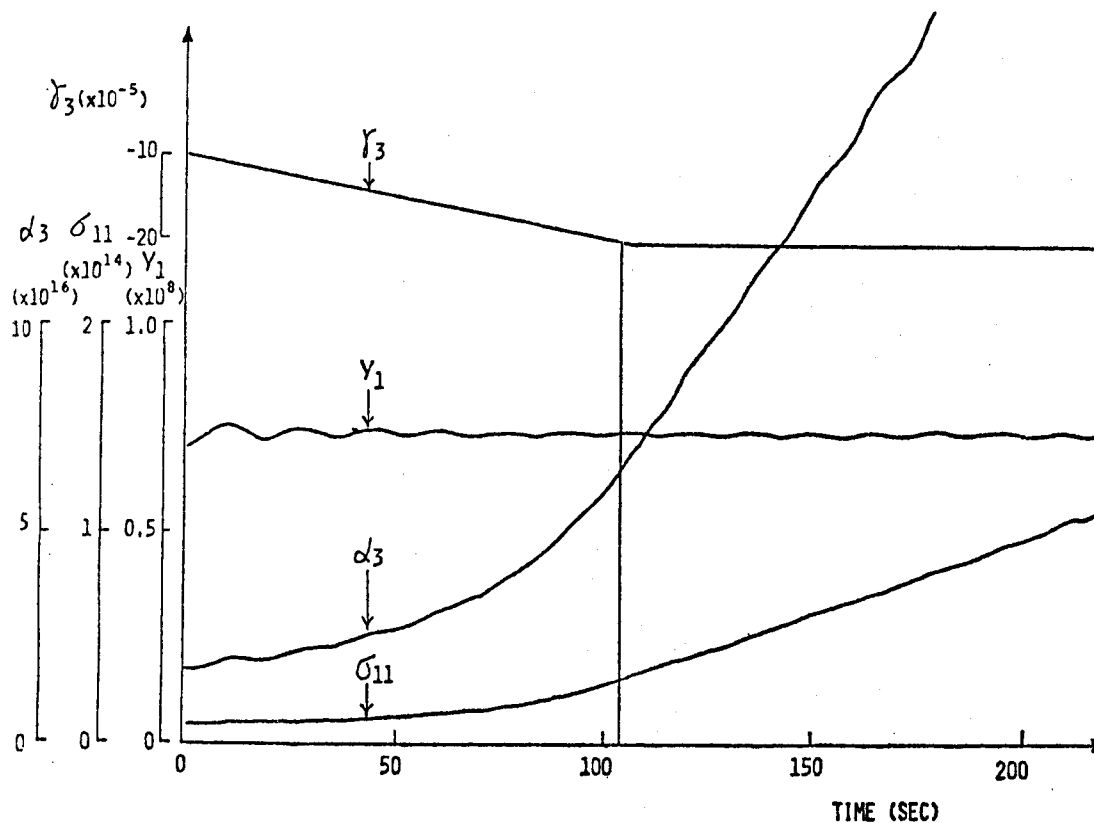


Fig. 3.13 Temporal behavior of $y_1(t)$, $\sigma_{11}(t)$ and $\alpha_3(t)$. γ_3 changes from G point to C point in Fig. 3.2

The mean value $y_1(t)$ oscillates and slowly approaches the limit cycle. The change of $\sigma_{11}(t)$ is faster than that of $y_1(t)$. It is of particular interest to see that the irreversible circulation of fluctuation $\alpha_3(t)$ changes most rapidly, on account of its embodying the effect of the time derivative in the expression (3.23).

In the usual operation of a power reactor, there are not only intrinsic fluctuations due to various causes but also extrinsic fluctuations generated by external random phenomena. It is thus indispensable to classify power reactor states according to the statistical nature of the fluctuations. A transparent method is also necessary in the study of reactor noise phenomena. Based on these considerations, we have treated the power reactor stochastically, using the system size expansion method, with the aim of obtaining clear and useful results both in mathematical and physical aspects. Our particular intention has been to gain a detailed insight into power reactor noise phenomena by examining the information contained in α (rotational information on fluctuations) for the "normal" case.

Further study on the characteristic behavior of power reactors is called for, to be undertaken from the viewpoint not only of reactor noise but also of reactor safety and diagnosis. The present study has been based on a number of assumptions and simplifications which have resulted in neglect of many factors that may be significant in the very complicated behavior of actual reactors. A number of corrections to our formalism should become necessary from these considerations. This will be the subject of subsequent studies.

Nomenclature

$X_1 = \Omega x_1$: Total number of neutrons
$X_2 = \Omega x_2$: Total energy content in fuel region
$X_3 = \Omega x_3$: Total energy content in coolant region
θ_f	: Fuel temperature ($^{\circ}\text{C}$)
θ_c	: Coolant temperature ($^{\circ}\text{C}$)
$S = \Omega s$: Neutron source (sec^{-1})
Λ_f	: Average fission reaction rate (sec^{-1})
Λ_c	: Average capture reaction rate (sec^{-1})
$\Lambda_f^{(f)}$: Fission reaction rate in fuel region (sec^{-1})
$\Lambda_c^{(f)}$: Capture reaction rate in fuel region (sec^{-1})
$\Lambda_c^{(c)}$: Capture reaction rate in coolant region (sec^{-1})
$p_f(\nu)$: Probability of ν neutrons to be born in a fission
q	: Fission energy (MeV)
$P = Y_1^S q / \bar{\nu} \ell_s$: Average power level (kW)
$\gamma_1, \gamma_2, \gamma_3$: Temperature coefficients defined in Eq. (3.52) ($^{\circ}\text{C}^{-1}$)
ℓ_s	: Mean neutron lifetime (sec)
ρ	: Reactivity
$\Omega = V_f + V_c$: Volume of reactor (cm^3)
V_f	: Volume of fuel region (cm^3)
V_c	: Volume of coolant region (cm^3)
C_v^f	: Heat Capacity of coolant at constant volume (kcal/g $^{\circ}\text{C}$)

C_v^c : Heat capacity of coolant at constant volume
(kcal/g°C)

ρ_f : Mass density of fuel (g/cm³)

ρ_c : Mass density of coolant (g/cm³)

$h_t = h_0 / C_v^f \rho_f V_f$: Heat transfer coefficient (sec⁻¹)

$h_0 = 11.7 (\theta_f^0 - \theta_c^0)^{0.33}$: Heat transfer coefficient (kW/°C)

$\theta_f^0 - \theta_c^0 = P / h_0$: Temperature difference of fuel and coolant
(°C)

$\lambda_r = 2v/H$: Heat removal coefficient of coolant (sec⁻¹)

$v = 1.16P^{0.25}$: Coolant velocity (cm/sec)

H : Channel height (cm)

$$\eta = C_v^f \rho_f V_f / C_v^c \rho_c V_c$$

θ_3^{in} : Inlet temperature of coolant (°C)

$$[Y_3^{in} = C_v^c \rho_c V_c \theta_3^{in} / q]$$

$$\overline{v^n} = \sum_{v=0}^{\infty} v^n p_f(v)$$

$$\theta_f^0 = Y_2^s q / C_v^f \rho_f V_f$$

$$\theta_c^0 = Y_3^s q / C_v^c \rho_c V_c$$

APPENDIX

The irreversible circulation of fluctuation α exists even in a linear system and is nonvanishing also in a steady state. The α is characteristic in the system having more than two degrees of freedom. Thus, the α is a useful concept even in a simple zero power reactor model with a single delayed neutron group. The transition probability is

$$W(N, C, \Delta N, \Delta C) = S \delta_{\Delta N, 1} \delta_{\Delta C, 0} + \lambda C \delta_{\Delta N, 1} \delta_{\Delta C, -1} + \sum_{\nu_0=0}^{\infty} \sum_{\nu_1=0}^{\infty} \frac{N}{\ell} p(\nu_0, \nu_1) \delta_{\Delta N, \nu_0-1} \delta_{\Delta C, \nu_1}, \quad (A.1)$$

with the notation:

- N : number of neutrons [=Ωn]
 C : number of precursors [=Ωc]
 S : neutron source [=Ωs]
 λ : decay constant of precursor
 p(ν₀, ν₁) : probability of ν₀ neutrons and ν₁ precursors emitted in an absorption
 ℓ : neutron lifetime.

As the equation is linear in this case,

$$\frac{\partial c_i(\psi)}{\partial \psi} = K_B \quad \text{or} \quad c_i(\psi) = K_B \psi.$$

From ε-expansion, we obtain for the mean and the variance equations

$$\frac{d}{dt} \begin{pmatrix} n \\ c \end{pmatrix} = K_B \begin{pmatrix} n \\ c \end{pmatrix} + \begin{pmatrix} S \\ 0 \end{pmatrix} \quad (A.2)$$

$$\frac{d}{dt} \sigma(t) = K_B \sigma(t) + \overline{K_B \sigma(t)} + D \quad (A.3)$$

respectively, where

$$K_B = \begin{pmatrix} -\frac{1-k(1-\beta)}{\ell}, & \lambda \\ \frac{\beta k}{\ell}, & -\lambda \end{pmatrix}, \quad D = \begin{pmatrix} \frac{n}{\ell}(k_{2p}-2k_p+1) + \lambda C + S, & \frac{n}{\ell}(k_{pd}-k_d) - \lambda C \\ \frac{n}{\ell}(k_{pd}-k_d) - \lambda C, & \frac{n}{\ell}k_{2d} + \lambda C \end{pmatrix},$$

with the notation:

$$\begin{aligned}
 k &= \langle v \rangle = \langle v_0 \rangle + \langle v_1 \rangle & k_d &= \langle v_1 \rangle = \beta \langle v \rangle & k_p &= \langle v_0 \rangle \\
 k_{2p} &= \langle v_0^2 \rangle & k_{pd} &= \langle v_0 v_1 \rangle & k_{2d} &= \langle v_1^2 \rangle.
 \end{aligned}$$

The steady state solutions of mean and variance equations are

$$\begin{aligned}
 \bar{n}^s &= \frac{S\ell}{1-k}, & C^s &= \frac{S\beta k}{(1-k)\lambda} \\
 \sigma_{11}^s &= \left[1 + \frac{(1-k+\lambda\ell)(k_2-k) - (1-k)C'}{2(1-k)(1-k_p+\lambda\ell)} \right] \bar{n}^s \\
 \sigma_{12}^s &= \sigma_{21}^s = \frac{k_d(k_2-k) + (1-k)C'}{2(1-k)(1-k_p+\lambda\ell)} \bar{n}^s \\
 \sigma_{22}^s &= \left[\frac{k_{2d}+k_d}{2} + \frac{k_d^2(k_2-k) + (1-k)k_d C'}{2(1-k)(1-k_p+\lambda\ell)} \right] \frac{\bar{n}^s}{\lambda\ell},
 \end{aligned}$$

where $c' = (k_2 - k) - (k_{2p} - k_p)$ and $k_2 = k_{2p} + 2k_{pd} + k_{2d}$. After straightforward calculation, we obtain

$$\alpha^s = \frac{\bar{n}^s}{2\ell(1-k_p+\lambda\ell)} \left[(k_2-k)(k_d-\lambda\ell) + (k_{2p}-k_p)\lambda\ell \right] + k_{pd} \frac{\bar{n}^s}{\ell}. \quad (\text{A.4})$$

Next, we examine the stability of this system, using Eq. (3.9), namely,

$$\frac{d}{dt} \begin{pmatrix} \delta n \\ \delta C \end{pmatrix} = K_B \begin{pmatrix} \delta n \\ \delta C \end{pmatrix},$$

and the characteristic equation is

$$z^2 + \left(\lambda + \frac{1-k(1-\beta)}{\ell} \right) z + \lambda \frac{1-k}{\ell} = 0.$$

When the reactor approaches critical state, $k \rightarrow 1$, only soft mode instability occurs in this system. This result also corresponds to the limiting case of three component system mentioned in this chapter when the power decreases to zero as in Fig. 3.8.

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CHAPTER 4

Irreversible Circulation as a New Parameter for Time Series Analysis

§4.1 Introduction

We have treated α as a new variable in the system size expansion method in Chap. 3⁽¹⁾⁽²⁾. However, the usefulness of α is not restricted to the normal case of this formalism. Here, we will propose to adopt the irreversible circulation of fluctuation α as a new variable in cases of general fields⁽²⁾⁽³⁾. Therefore, it is the purpose of this chapter to find out to what extent we can discuss general properties of α directly derived from noise experimental data as independently of modeling and assumption about the process as possible.

When we analyze a stationary noise-time-series along this line⁽⁴⁾⁽⁵⁾, the correlation matrix $C_{ij}(t)$ and/or the power spectral density matrix $P_{ij}(\omega)$ are the usual quantities invoked for the statistical characterization of the process. However, in practice, these quantities do not necessarily lead to a direct understanding or a concise specification of the physical situation. In this case lower order moments will often be rather convenient in characterizing the process than the whole spectrum. The variance matrix is a typical example. In this chapter we will propose a new statistical quantity "circulation" α as a necessary complement to the variance σ in characterizing any multi-dimensional time series. We will also discuss that the property of α is less sensitive to filter effect in the low frequency region than σ . Finally, the property of α that we have mentioned in this thesis will be summarized at the end of this chapter.

§4.2 Integral Index for "Noise" (3)

Suppose the system under consideration is described by a set of variables $\mathcal{X}=\{x_i\}$ ($i=1,2,\dots,N$), then the statistical properties of a stationary process in the space of \mathcal{X} is characterized by the correlation matrix which is defined by

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N x_i(k) x_j(k+n), \quad (\text{discrete sampling}) \quad (4.1)$$

or by

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt_0 \langle x_i(t_0) x_j(t_0+t) \rangle \equiv C_{ij}(t), \quad (4.2)$$

or its Fourier transform, i.e. the power spectral density

$$C_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} P_{ij}(\omega), \quad (4.3)$$

or

$$P_{ij}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} C_{ij}(t). \quad (4.4)$$

Let us split the correlation function into two parts

$$C_{ij}(t) = C_{ij}^{(s)}(t) + C_{ij}^{(a)}(t), \quad (4.5)$$

where

$$C_{ij}^{(s)}(t) \equiv \frac{1}{2} \left\{ \langle x_i(t_0) x_j(t_0+t) \rangle + \langle x_i(t_0) x_j(t_0-t) \rangle \right\} \quad (\text{symmetric part})$$

and

$$C_{ij}^{(a)}(t) \equiv \frac{1}{2} \left\{ \langle x_i(t_0) x_j(t_0+t) \rangle - \langle x_i(t_0) x_j(t_0-t) \rangle \right\}. \quad (\text{anti-symmetric part}).$$

In terms of the power spectral density

$$P_{ij}(\omega) = P_{ji}^*(\omega) = P'_{ij}(\omega) + i P''_{ij}(\omega), \quad (4.6)$$

where

$$P'_{ij}(\omega) = P'_{ij}(-\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} C_{ij}^{(s)}(t), \quad (\text{even function of } \omega)$$

$$P''_{ij}(\omega) = -P''_{ij}(-\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} C_{ij}^{(a)}(t), \quad (\text{odd function of } \omega),$$

and * indicates the complex conjugate. Conversely,

$$C_{ij}^{(s)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} P_{ij}'(\omega) \quad (4.7)$$

$$C_{ij}^{(a)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} P_{ij}''(\omega). \quad (4.8)$$

To obtain an integral index one may put $t=0$ in these relations. It is well known that the relation (4.7) yields the variance σ , i.e.

$$\sigma_{ij} = C_{ij}^{(s)}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega P_{ij}'(\omega) = \sigma_{ji}. \quad (4.9)$$

The corresponding procedure on the relation (4.8) leads to

$$C_{ij}^{(a)}(0) = 0,$$

however, a non-vanishing index is obtained when we put $t=0$ after differentiation, i.e.

$$\alpha_{ij} = \dot{C}_{ij}^{(a)}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \omega P_{ij}''(\omega) = -\alpha_{ji}. \quad (4.10)$$

This is the quantity by which we propose to complement the information given by σ in characterizing the random process in a concise way. For the mathematical property of α , we can also obtain the following short time expansion of correlation matrix

$$\begin{aligned} C(\Delta t) &= \langle \chi(0) \chi(\Delta t) \rangle \\ &= C^{(s)}(\Delta t) + C^{(a)}(\Delta t) \\ &= \sigma + o(\Delta t)^2 + \alpha \Delta t + o(\Delta t)^3. \end{aligned} \quad (4.11)$$

Hence, the α is the first differential coefficient of correlation matrix. It should be pointed out that the variance is not sufficient to characterize a multi-dimensional random process. This is because a distribution may have any amount of internal circulation even if the spread is uniquely characterized by the variance.

The quantity α may be written as

$$\alpha = \frac{1}{2} \langle [\chi, \dot{\chi}] \rangle = \frac{1}{2} (\langle \chi \dot{\chi} \rangle - \langle \dot{\chi} \chi \rangle), \quad (4.12)$$

and it stands for the average angular momentum or the areal velocity of the distribution. Therefore it is clear that the quantity α is necessary in characterizing a distribution together with the variance σ . The pair of quantities σ and α may serve as minimum necessary topological indices in the classification of time series and will be essential in data retrieval. In relation to α_{ij} a generalized angular velocity $\dot{\theta}_{ij}$ may also be defined, e.g. by

$$\dot{\theta}_{ij} = \frac{\alpha_{ij}}{\sqrt{\det \sigma_2}}, \quad (4.13)$$

where

$$\sigma_2 = \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}.$$

For the particular case in which the situation may be approximated by a two dimensional system, $\dot{\theta}_{ij}$ acquires a concrete meaning.

In view of the relation (4.10) the quantity α is expected less sensitive to low frequency modification of the bare spectrum either by a weak non-stationarity or by a low frequency cut filter effect. In the next section, we will discuss this in detail by describing a concrete example.

§4.3 An Example

We study a nuclear reactor noise phenomenon as an example of observable time series⁽⁶⁾⁽⁷⁾. As the direct analysis of the experimental data using the new concept is still under way, we here demonstrate the use of our

proposal on a theoretical model adopted by Morishima⁽⁸⁾⁻⁽¹⁰⁾ in order to describe the data obtained by Utsuro et al.⁽¹¹⁾ for the Kyoto University Reactor (KUR). [cf. We have already mentioned the simplified Morishima's model of this reactor in Chap. 3.] The reactor which has a linear feedback mechanism is described by five variables (N, C, F, M, Q), i.e. the total number of neutron N, that of precursors C, the fuel energy F, the coolant energy M and the coolant flow rate Q. The network of the reaction is shown in Fig. 4.1.

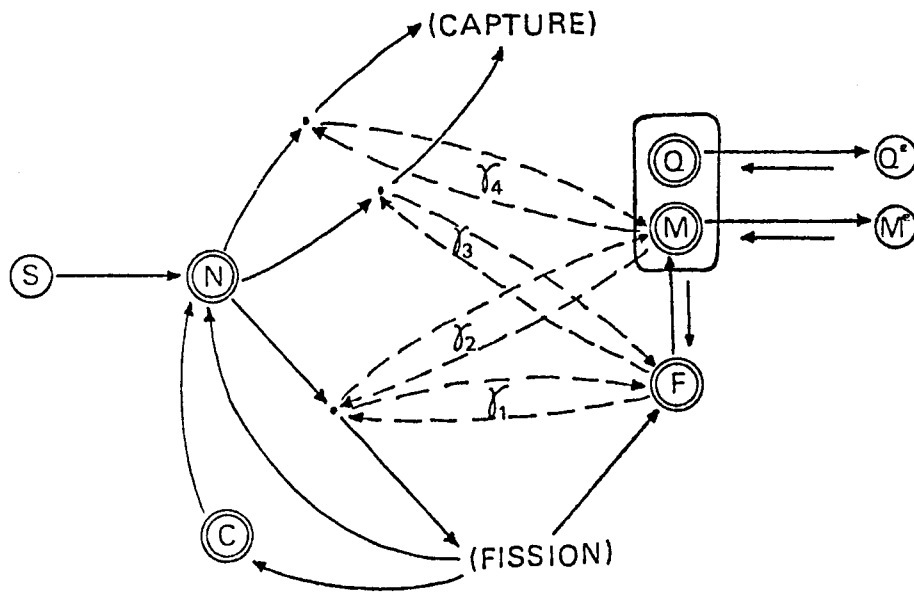


Fig. 4.1 Reactor network with five components
N, C, F, M and Q

When the reactor approaches a critical state, $P_{NN}(\omega)$ in the low frequency region rises up in proportion to its power as seen in Fig. 4.2. This event results from energy or temperature fluctuations in fuel and coolant regions. A reactor also has the property of the integral action or the reset action. We calculate the imaginary part of $P_{NM}(\omega)$ in Fig. 4.3 since N and M are easily measured among five components.

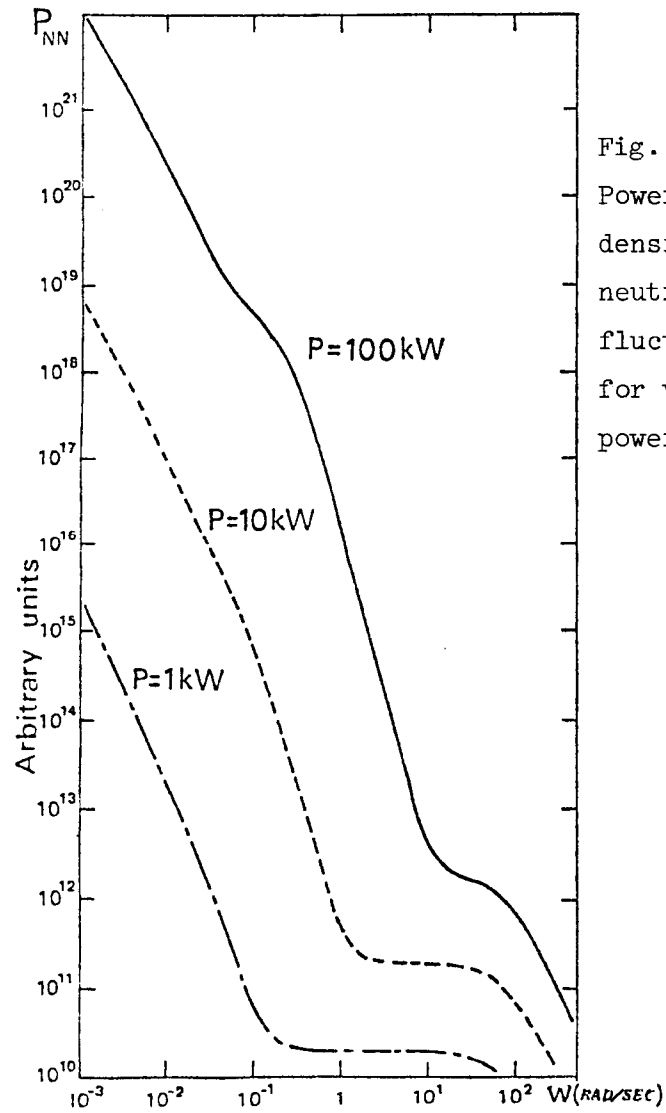


Fig. 4.2
Power spectral density of neutron fluctuations for values of power levels

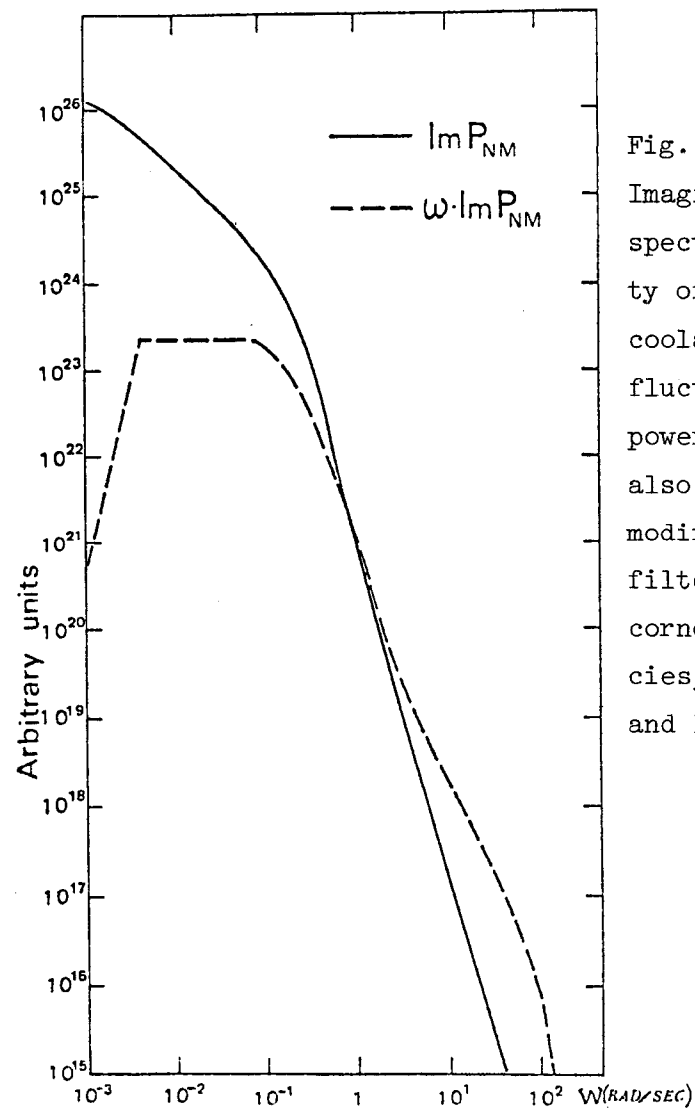


Fig. 4.3
Imaginary power spectral density of neutron-coolant energy fluctuations at power=100kW and also $\omega \text{Im}P_{NM}(\omega)$ modified by filter having corner frequencies, 4.8828×10^{-3} and 10^2 rad/sec

Sample data obtained from a system are usually filtered prior to data processing. Here we used a filter having a 12 db per octave rolloff outside the ranges from 4.8828×10^{-3} to 10^2 rad/sec in order to get the theoretical estimate. The low cut filter has not a little influence on P_{NN} or σ_{NN} . On the other hand, we can conclude that α is less sensitive to low cut filter effects since $\text{Im} P(\omega)$ is an odd function of ω and α is composed of $\text{Im} P(\omega)$ multiplied by ω in Eq. (4.10). This is shown by the broken line of the filtered $\omega \text{Im} P_{NM}(\omega)$ in Fig. 4.3.

§4.4 Discussion

The power spectral density and/or the correlation function are the usual statistical quantities observed when we treat "noise" or time series. In extracting information from the power spectral density, the coherence function $\gamma_{ij}^2(\omega) = |P_{ij}(\omega)|^2 / [P_{ii}(\omega) \cdot P_{jj}(\omega)]$, and the phase function $\theta_{ij}(\omega) = \tan^{-1} [P''_{ij}(\omega) / P'_{ij}(\omega)]$ are useful quantities for a fixed frequency. In contrast, the variance and the irreversible circulation of fluctuation are the integral indices of power spectral density. The latter may also be obtained from finite time observation. In this sense, these three kinds of quantity are complementary to one another in the noise analysis.

We have discussed the general properties of α without referring to a mechanism in §4.2. On the other hand, if the system is described by a linear Langevin equation, we

can find out another eminent property of α in its framework. Namely, we will discuss that some of the integral indices derived from power spectral density in this system will have no meaning since they are divergent except σ and α : Such moments are meaningless in the Markovian process as will be proved below⁽²⁾.

Suppose a multi-dimensional stochastic differential equation with random forces is given in the form⁽¹²⁾

$$\frac{d}{dt} \mathbf{x} = \mathbf{K} \mathbf{x} + \mathbf{F}, \quad (4.14)$$

where

$$\langle \mathbf{x}_0 \mathbf{F}_t \rangle = 0, \quad \langle \mathbf{F}_t \mathbf{F}_{t'} \rangle = \mathbf{D} \delta(t-t') \quad \text{and} \quad \langle \mathbf{F}_t \rangle = 0.$$

This linear approximation is not too restricted to the applicability of the formalism, since the amplitude of fluctuations is usually small as mentioned in Chaps. 2 and 3.

The power spectral density is given by the Fourier transform⁽¹³⁾, i.e.,

$$\begin{aligned} \mathbf{P}(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \mathbf{x}(0) \mathbf{x}(t) \rangle \quad [\text{cf. (3.31)}] \\ &= \mathbf{G}(\omega) \mathbf{D} \widetilde{\mathbf{G}}(-i\omega), \quad \mathbf{G}(\omega) = [i\omega - \mathbf{K}]^{-1} \quad (4.15) \end{aligned}$$

where the tilder \sim denotes the transposed matrix. In a stable system, $\mathbf{P}(\omega)$ has no singularity as seen from Eq. (4.15). The possibility of divergence of the indices occurs in the process of integration. Hence, let us examine the ω -dependence of $\mathbf{P}(\omega)$. In Eq. (4.15),

$$[i\omega - \mathbf{K}]^{-1} = \frac{1}{\det(i\omega - \mathbf{K})}. \quad [\text{cofactor matrix}]$$

The element of cofactor matrix is of the following type:

$$(i\omega)^{N-1} + A_{N-2} (i\omega)^{N-2} + \dots, \quad (A_i: \text{constant})$$

Namely,

$$[i\omega - \mathbf{K}]_{ij}^{-1} \sim \frac{(i\omega)^{N-1} + \dots}{(i\omega)^N + \dots}.$$

Then, we obtain

$$\begin{aligned} \operatorname{Re} P(\omega) &\sim \frac{\omega^{2N-2} + \dots}{\omega^{2N} + \dots} \\ \operatorname{Im} P(\omega) &\sim \frac{A_{2N-3} \omega^{2N-3} + \dots}{\omega^{2N} + \dots} \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \sigma &\propto \int_{-\infty}^{\infty} \operatorname{Re} P(\omega) d\omega \xrightarrow{\text{(for large } \omega)} \int \frac{1}{\omega} d\omega = [\text{finite}] \\ \alpha &\propto \int_{-\infty}^{\infty} \omega \operatorname{Im} P(\omega) d\omega \xrightarrow{\text{(for large } \omega)} \int \frac{1}{\omega^2} d\omega = [\text{finite}]. \end{aligned} \quad (4.17)$$

The other integral indices are zero or divergent. For example,

$$\begin{aligned} \int_{-\infty}^{\infty} \omega^2 \operatorname{Re} P(\omega) d\omega &\xrightarrow{\text{(for large } \omega)} \int d\omega \propto [\text{divergent}] \\ \int_{-\infty}^{\infty} \omega^3 \operatorname{Im} P(\omega) d\omega &\xrightarrow{\text{(for large } \omega)} \int d\omega \propto [\text{divergent}]. \end{aligned} \quad (4.18)$$

No integral index except σ and α exists in a stable macroscopic system. Consequently, we see that σ and α are only useful indices for patterns of power spectral density in a stable state.

If the random force $F(t)$ of Eq. (4.14) is Gaussian along the spirit of the central limit theorem, the Langevin equation (4.14) is reduced to the linear Fokker-Planck equation (3.7) of a steady state in Chap. 3. Hence, the new variable α is applicable to the Langevin technique in the time series analysis.

Finally, let us summarize the usefulness of the irreversible circulation of fluctuation α which has been clarified in this thesis.

[1] α is an indispensable variable, as a kind of generalized angular momentum, for analyzing main features of non-deterministic systems just as the variance is for a measure of the width of probability distribution⁽³⁾.

[2] α is useful as an index for the hard or soft mode instability, especially in a system having many degrees of freedom⁽¹⁾. It is also related to the cyclic balance

in a non-equilibrium situation of an open system⁽¹⁴⁾.

[3] α is related to the asymmetry of cross-correlation functions in a steady state⁽²⁾.

[4] α and σ are the integral indices of power spectral density in a stable system⁽¹⁾⁽²⁾. Namely, they are useful for indices of classification of patterns in the experimental data and serve as reactor state indices in reactor diagnosis when the information of reactor stability is needed⁽³⁾.

[5] α and σ are applicable in a case of non-stationary states under the Markovian assumption⁽¹⁾.

[6] Possible ways of representation of an observable as Fig. 3.12 of Chap. 3 are useful for processing experimental data with more confidence⁽²⁾.

We can conclude that α serves as an integral index for classification of time series not only in the reactor noise analysis but also in the time series analysis in general.

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CHAPTER 5

Hidden State Variables and a Non-Markovian Formulation of Reactor Noise

§5.1 Introduction

A number of articles have recently been devoted to a better understanding of complicated fluctuation phenomena in power reactor⁽¹⁾⁻⁽⁴⁾. A unified and rational theory can be provided if one finds, and manipulates, a sufficient number of state variables so that our system is described by a set of coupled Markovian Langevin equations. It is, however, difficult in some situation to find all the state variables necessary for the Markovian description; one is usually confronted with the occurrence of some hidden (or unknown) variables. This may occur in cases for power reactors, and makes it necessary to introduce phenomenological quantities such as an external noise source to understand the actually observed experimental results. Searches for hidden state variables and the physical interpretation of the phenomenological external noise source are important tasks in this approach.

As an alternative approach, one may use non-Markovian Langevin equations only for the known variables. This is based on the well-known fact that a contraction of information, i.e., a reduction of the number of the state variables, introduces the non-Markovian character into the new description⁽⁵⁾. Attempts have been made by Saito⁽⁶⁾⁽⁷⁾⁽¹⁾ and recently by Morishima⁽⁸⁾. In accordance with the theory of Brownian motion⁽⁹⁾⁽¹⁰⁾, Saito employed the following equation:

$$\frac{d}{dt} \mathbf{f}(t) = - \int_{t_0}^t \mathbf{B}(t-t') \mathbf{f}(t') dt' + \mathbf{Q}(t) + \mathbf{R}(t), \quad (5.1)$$

for a set of the fluctuations of state variables, $\mathbf{f} = \text{col}(f_1, \dots, f_n)$. Here $\mathbf{R}(t)$ denotes random stochastic

forces and $\mathbf{Q}(t)$ represents deterministic forces applied externally. The vector \mathbf{f} represents the fluctuations of the physical quantities, so that $\langle \mathbf{f} \rangle = 0$, where $\langle \dots \rangle$ denotes the ensemble average. Since the magnitudes of the fluctuations are usually small, we assume that \mathbf{f} obeys a linear equation like Eq. (5.1). Saito examined that Eq. (5.1) leads to an incorrect result if one assumes the causality condition^{(7) (1)}

$$\langle \mathbf{R}(t_0 + t) \mathbf{f}(t_0) \rangle = 0, \quad (t > 0).$$

Morishima introduced a course-graining operator \mathcal{P} to reduce the number of state variables⁽⁸⁾. The resulting equation for $\mathcal{P}\mathbf{f}(t)$ is, however, identical with Eq. (5.1). Contrary to Saito's conclusion, his formulation needs the causality condition.

The purpose of this chapter is to formulate the non-Markovian effect due to some hidden variables $\mathcal{g} = \text{col}(g_1, \dots, g_m)$, with the hope of resolving the above-mentioned confusion.

In §§5.2, 5.3 and 5.4, a non-Markovian Langevin equation is derived, a formula for the power spectral density is obtained, and a new fluctuation-dissipation theorem for $\mathbf{R}(t)$ and $\mathbf{B}(t)$ is found. In §5.5, a simple example is treated. Finally, some concluding remarks are included in §5.6.

§ 5.2 Non-Markovian Langevin Equation ⁽¹¹⁾

Let us begin by assuming that a sufficient number of physical quantities $\mathbf{f}_c = \text{col}(f_1, \dots, f_n, g_1, \dots, g_m)$ can be chosen so that any of the correlation times between the $n+m$ random forces in Eq. (5.1) is short compared with the time resolution of our experimental observation. The fluctuation-dissipation theorem ⁽⁹⁾ then gives

$$\mathbf{B}(t-t') = \delta(t-t') \mathbf{B}, \quad (5.2)$$

and in the case of no external force Eq. (5.1) leads to

$$\frac{d}{dt} \mathbf{f}_c(t) = -\mathbf{B} \mathbf{f}_c(t) + \mathbf{R}_c(t). \quad (5.3)$$

Such a set of physical quantities is referred to as a complete set of state variables ⁽⁵⁾. In this chapter, we shall take no account of the non-Markovian effect described by Eq. (5.1). The non-Markovian effect to be discussed here is the one due to hidden variables, i.e., due to incompleteness in our knowledge of all the state variables.

In the light of Eq. (5.3) we may say that the problem has been solved already; the non-Markovian formulation is unnecessary in principle. However, the complete set \mathbf{f}_c consists of many physical quantities in general, and may involve some unknown variables, particularly in the case of power reactors. It is therefore desirable the Markovian process of the complete set of state variables in terms of a less number of known variables only.

There is no essential difficulty in such a formulation; one has only to eliminate the unknown variables \mathbf{g} from the coupled equations (5.3), which we now write

$$\frac{d}{dt} \begin{pmatrix} \mathbf{f}(t) \\ \mathbf{g}(t) \end{pmatrix} = - \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}(t) \\ \mathbf{g}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{pmatrix}, \quad (5.4)$$

where \mathbf{B}_{11} , \mathbf{B}_{12} , \mathbf{B}_{21} and \mathbf{B}_{22} are $n \times n$, $n \times m$, $m \times n$ and $m \times m$ matrices, respectively, and $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ consist of

random forces for f and g , respectively. We want to discuss the time-correlation matrix $\langle f(t_0+t)f(t_0) \rangle$ between the known variables. At the initial time t_0 of the correlation, the unknown physical quantities $g(t_0)$ cannot be considered vanishing, because these quantities, if they are appreciable, must have lifetimes greater than the time resolution of our experimental observation. With this initial condition, one eliminates $g(t)$ from Eq. (5.4) to find the following equation:

$$\frac{d}{dt} f(t) = -\int_{t_0}^t B(t-t') f(t') dt' + Z(t-t_0) g(t_0) + F(t), \quad (5.5)$$

where

$$B(t) = \delta(t) B_{11} + Z(t) B_{21} \quad (5.6)$$

$$Z(t) = -B_{12} e^{-t B_{22}} \quad (5.7)$$

$$F(t) = X(t) + \int_{t_0}^t Z(t-t') Y(t') dt'. \quad (5.8)$$

Equation (5.5) is what we were looking for, and one of the main results of this chapter. It differs from the well known equation (5.1) by the additional stochastic term $Z(t-t_0) g(t_0)$, which describes the effect on $f(t)$ of the decays of the hidden physical quantities. The new terms contains the stochastic quantities $g(t_0)$, and cannot be regarded as the deterministic term $Q(t)$ in Eq. (5.1). Equation (5.5) is, therefore, substantially new in spite of its similarity in form to Eq. (5.1)

The causality condition

$$\langle F(t_0+t) f(t_0) \rangle = 0, \quad (t > 0) \quad (5.9)$$

obviously holds true. Equation (5.5) is identical in form with Eq. (5.1), if one puts in Eq. (5.5)

$$R(t) = Z(t-t_0) g(t_0) + F(t). \quad (5.10)$$

The random force thus defined, however, violates the causality condition because

$$\langle \mathcal{G}(t_0) \mathbf{f}(t_0) \rangle \neq 0$$

in general*. When the variables $\mathcal{G}(t)$ does not couple with $\mathbf{f}(t)$, i.e., when $\mathbf{B}_{12}=0$, the set $\mathbf{f}(t)$ is complete. Then, $\mathbf{Z}(t)=0$, and Eq. (5.5) reduces to Eq. (5.3) for the known variables $\mathbf{f}(t)$. In this sense, Eq. (5.5) is a generalization of Eq. (5.3) to the case where there are some hidden variables. It may also be said that Eq. (5.5) is equivalent to Eq. (5.3) for the known and hidden variables $\mathbf{f}_c(t)$, as is obvious from the derivation of Eq. (5.5).

It is well known in statistical mechanics that hidden variables give rise to the non-Markovian effect on the motion of known variables⁽⁵⁾. Equation (5.5) exhibits an example of such a general rule.

* In the special case of $\langle \mathcal{G}(t_0) \mathbf{f}(t_0) \rangle = 0$, the causality condition $\langle \mathbf{R}(t_0+t) \mathbf{f}(t_0) \rangle = 0$ is satisfied. It is therefore seen that the usual non-Markovian formalism based on Eq. (5.1) does involve the assumption of $\langle \mathcal{G}(t_0) \mathbf{f}(t_0) \rangle = 0$, i. e., the unknown variables \mathcal{G} are orthogonal to \mathbf{f} . These unknown variables \mathcal{G} can be made orthogonal to \mathbf{f} by the transformation $\mathcal{G}' = \mathcal{G} - \langle \mathcal{G} \mathbf{f} \rangle \langle \mathbf{f} \mathbf{f} \rangle^{-1} \mathbf{f}$. The usual non-Markovian formalism is then applicable. This method is, however, inconvenient for finding hidden physical quantities \mathcal{G} .

§5.3 Time Correlations and Their Power Spectra

By virtue of the causality condition (5.9), one readily finds from Eq. (5.5) the following kinetic equation:

$$\frac{d}{dt} \langle \mathbf{f}(t_0+t) \mathbf{f}(t_0) \rangle = - \int_0^t \mathbf{B}(t-t') \langle \mathbf{f}(t_0+t') \mathbf{f}(t_0) \rangle dt' + \mathbf{Z}(t) \langle \mathcal{G}(t_0) \mathbf{f}(t_0) \rangle, \quad (5.11)$$

for the time-correlation matrix $\langle \mathbf{f}(t_0+t) \mathbf{f}(t_0) \rangle$. Hereafter we assume that our system is in a stationary state. The Fourier-Laplace transform of Eq. (5.11) leads to the power spectral density matrix

$$\begin{aligned} \Phi(\omega) &= \int_0^\infty e^{-i\omega t} \langle \mathbf{f}(t_0+t) \mathbf{f}(t_0) \rangle dt + \text{h.c.} \\ &= [i\omega + \mathbf{B}(i\omega)]^{-1} \{ \langle \mathbf{f} \mathbf{f} \rangle + \mathbf{Z}(i\omega) \langle \mathcal{G} \mathbf{f} \rangle \} + \text{h.c.}, \quad (5.12) \end{aligned}$$

where $\langle \mathbf{f} \mathbf{f} \rangle = \langle \mathbf{f}(t_0) \mathbf{f}(t_0) \rangle$, etc., $\mathbf{B}(i\omega)$ and $\mathbf{Z}(i\omega)$ are the Fourier-Laplace transform of $\mathbf{B}(t)$ and $\mathbf{Z}(t)$, respectively, and h.c. stands for the Hermitian conjugate of the preceding matrix. When the hidden variables are discarded, $\mathbf{Z}(i\omega) = 0$ and $\mathbf{B}(i\omega) = \mathbf{B}_{11}$. The power spectral density (5.12) then reduces to the well-known one in the Markovian formalism

$$\Phi(\omega) = [i\omega + \mathbf{B}_{11}]^{-1} \Phi_F [-i\omega + \widetilde{\mathbf{B}}_{11}]^{-1}, \quad (5.13)$$

where $\widetilde{\mathbf{B}}_{11}$ is the transpose of \mathbf{B}_{11} , and

$$\Phi_F = \mathbf{B}_{11} \langle \mathbf{f} \mathbf{f} \rangle + \langle \mathbf{f} \mathbf{f} \rangle \widetilde{\mathbf{B}}_{11}. \quad (5.14)$$

§5.4 Fluctuation-Dissipation Theorem

In the Markovian equation (5.3), the friction \mathbf{B} and the random force \mathbf{R}_c satisfy a general relation, called the second fluctuation-dissipation theorem⁽⁹⁾. Since our non-Markovian equation (5.5) is equivalent to Eq. (5.3), one expects that there is also a definite relationship between $\mathbf{B}(t)$ and $\mathbf{F}(t)$ involved in Eq. (5.5). For convenience, we consider instead of $\mathbf{F}(t)$ the random force $\mathbf{R}(t)$ defined by Eq. (5.10), and as seen in the Appendix, the following fluctuation-dissipation theorem is obtained:

$$\begin{aligned}\Phi_{\mathbf{R}}(i\omega) &= \int_0^{\infty} e^{-i\omega t} \langle \mathbf{R}(t_0+t) \mathbf{R}(t_0) \rangle dt \\ &= \mathbf{B}(i\omega) \langle \mathbf{f} \mathbf{f} \rangle + \mathbf{Z}(i\omega) \langle \mathbf{g} \mathbf{f} \rangle [-i\omega + \tilde{\mathbf{B}}_{11}]^{-1} - \langle \mathbf{f} \dot{\mathbf{f}} \rangle.\end{aligned}\quad (5.15)$$

Comparing this with the fluctuation-dissipation theorem for Eq. (5.1)

$$\int_0^{\infty} e^{-i\omega t} \langle \mathbf{R}(t_0+t) \mathbf{R}(t_0) \rangle dt = \mathbf{B}(i\omega) \langle \mathbf{f} \mathbf{f} \rangle - \langle \mathbf{f} \dot{\mathbf{f}} \rangle, \quad (5.16)$$

or that for Eq. (5.3),

$$\int_0^{\infty} e^{-i\omega t} \langle \mathbf{R}_c(t_0+t) \mathbf{R}_c(t_0) \rangle dt = \mathbf{B} \langle \mathbf{f}_c \mathbf{f}_c \rangle - \langle \mathbf{f}_c \dot{\mathbf{f}}_c \rangle, \quad (5.17)$$

one sees that the second term on the right-hand side of Eq. (5.15) is characteristic to the present formalism. The appearance of this term reflects the violation of the causality condition, $\langle \mathbf{R}(t_0+t) \mathbf{f}(t_0) \rangle \neq 0$.

In the following argument, it is useful to express the power spectral density (5.12) in terms of the power spectrum of the random force $\mathbf{R}(t)$,

$$\begin{aligned}\Phi_{\mathbf{R}}(\omega) &= \Phi_{\mathbf{R}}(i\omega) + \text{h. c.} \\ &= \left\{ \mathbf{B}(i\omega) \langle \mathbf{f} \mathbf{f} \rangle + \mathbf{Z}(i\omega) \langle \mathbf{g} \mathbf{f} \rangle (-i\omega + \tilde{\mathbf{B}}_{11})^{-1} \right\} + \text{h. c.}\end{aligned}\quad (5.18)$$

After a little manipulation, Eq. (5.12) is rewritten as

$$\Phi(\omega) = [i\omega + \mathbf{B}(i\omega)]^{-1} \mathbf{A}(\omega) [-i\omega + \tilde{\mathbf{B}}(-i\omega)]^{-1}; \quad (5.19)$$

$$A(\omega) = \Phi_R(\omega) + Z(i\omega) \langle \mathcal{R} f \rangle \widetilde{B}_{21} \widetilde{Z}(-i\omega) + \text{c. c.} \quad (5.20)$$

Equation (5.19) is the most general expression of the power spectral density, and a useful version of the result

$$\Phi_c(\omega) = [i\omega + B]^{-1} \langle R_c R_c \rangle [-i\omega + \widetilde{B}]^{-1}$$

based on Eq. (5.3). If the hidden variables are neglected, Eqs. (5.19) and (5.20) reduce to Eqs. (5.13) and (5.14), respectively.

§5.5 Application to a Zero Power Reactor with Delayed Neutrons

As a simple illustration of the present formalism, let us consider the effect of delayed neutrons. The density fluctuations of total neutrons, $\delta N(t) = N(t) - \langle N(t) \rangle$, and that of the precursors, $\delta C(t) = C(t) - \langle C(t) \rangle$, are assumed to constitute a complete set of the state variables [cf. Appendix of Chap. 3]. Let us treat $\delta C(t)$ as a hidden variable. Equations (5.6) and (5.7) are written as

$$B(t) = \alpha_p \delta(t) - \frac{\beta k}{l} Z(t), \quad (5.21)$$

and

$$Z(t) = \lambda e^{-\lambda t}, \quad (5.22)$$

where $\alpha_p = \frac{1}{l} \{1 - k(1 - \beta)\}$ is the decay constant of the prompt neutrons, λ that of the precursors, l the neutron lifetime, k the multiplication factor and β the mean fraction

of the delayed neutrons. Equation (5.5) thus becomes

$$\begin{aligned} \frac{d}{dt} \delta N(t) = & -\alpha_p \delta N(t) + \frac{\lambda \beta k}{l} \int_{t_0}^t e^{-\lambda(t-t')} \delta N(t') dt' \\ & + \lambda e^{-\lambda(t-t_0)} \delta C(t_0) + F(t). \end{aligned} \quad (5.23)$$

The term $\lambda \exp[-\lambda(t-t_0)] \delta C(t_0)$ describes the decay rate of the initial precursors. The power spectral density of the neutrons is given from Eq. (5.19) by

$$\Phi(\omega) = \frac{A(\omega)}{|i\omega + B(i\omega)|^2}, \quad (5.24)$$

where

$$\begin{aligned} B(i\omega) &= \alpha_p - \frac{\lambda \beta k}{l(i\omega + \lambda)} \\ A(\omega) &= \Phi_R(\omega) - \frac{2\lambda^2 \beta k \langle \delta C \delta N \rangle}{l(i\omega + \lambda)(-i\omega + \lambda)}, \end{aligned}$$

with

$$\Phi_R(\omega) = \left\{ \left(\alpha_p - \frac{\lambda \beta k}{l(i\omega + \lambda)} \right) \langle \delta N^2 \rangle + \frac{\lambda}{i\omega + \lambda} \langle \delta C \delta N \rangle (-i\omega + \alpha_p) \right\} + C. C.$$

This result, of course, coincides with that⁽⁷⁾ of the Markovian formalism with the state variables $\mathbf{f} = \text{col}(\delta N, \delta C)$, i.e., the (1,1)'th element of

$$[i\omega + \mathbf{B}]^{-1} \mathbf{D} [-i\omega + \tilde{\mathbf{B}}]^{-1},$$

where

$$\mathbf{D} = \mathbf{B} \langle \mathbf{f} \mathbf{f} \rangle + \langle \mathbf{f} \mathbf{f} \rangle \tilde{\mathbf{B}},$$

and

$$\mathbf{B} = \begin{pmatrix} \alpha_p & -\lambda \\ -\frac{\beta k}{l} & \lambda \end{pmatrix}.$$

The non-Markovian treatment of this problem has been examined by Saito⁽⁷⁾, and he pointed out that Eq. (5.1) leads to an incorrect result because of the violation of the causality. The present formalism is free from this difficulty, since it correctly takes care of the non-causality.

§5.6 Concluding Remarks

On the basis of the assumption that there exists a complete set of the state variables, we have derived the non-Markovian Langevin equation (5.5) for an incomplete set of state variables. It has been shown that the causality condition and the fluctuation-dissipation theorem do not hold in the usual sense of the non-Markovian stochastic theory. Equation (5.19) has been given for the power spectral density.

If we know all the physical quantities that constitute a complete set of state variables, the present formalism reduces to the usual approach based on the Markovian equation (5.3). Our formalism will, therefore, be most useful when we want to find out unknown state variables from experimental results.

It should be mentioned that the present formalism is a specialization of Morishima's attempt⁽⁸⁾. If his formalism is applied to the present case, his operator P is given by the projection operator

$$P = \begin{pmatrix} U, 0 \\ 0, 0 \end{pmatrix},$$

where U is the $n \times n$ unit matrix so that

$$P \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}.$$

The term $Z(t-t_0) \mathbf{g}(t_0)$ in Eq. (5.5), however, is missing in his formalism.

Although Eq. (5.5) has been derived from the phenomenological Markovian equation (5.3), the most rational basis will be given by applying Mori's formalism⁽¹⁰⁾ to the case with some hidden variables. This approach will clarify the assumption underlying Eq. (5.3), i.e., the validity of the Markovian description with a full set of state variables.

The applicability of the present formalism is not restricted to the analysis of the power reactor noises.

The fluctuation phenomena are one of the important subjects in physics and engineering. For example, the time correlation of the atomic density fluctuations plays a fundamental role in slow neutron scattering theory. On the basis of Mori's Langevin type equation⁽¹⁰⁾, the density fluctuation of classical liquids has been analyzed by Akcasu et al.⁽¹²⁾ It seems quite likely that in liquids there are some hidden quasi-particles associated with collective motions of the atoms. Hence, the new term $Z(t-t_0) \mathcal{G}(t_0)$ in our Langevin equation (5.5) seems to give an important contribution to the cross sections of liquids.

APPENDIX

— Derivation of Eq. (5.15) —

To find the relation between $\mathbf{B}(t)$ and $\mathbf{R}(t)$, we proceed in a way quite similar to that in the proof of the fluctuation-dissipation theorem (21) of Ref. (9). Equation (5.5) gives

$$\mathbf{R}(t_0+t) = \dot{\mathbf{f}}(t_0+t) + \mathbf{B}_{11} \mathbf{f}(t_0+t) + \int_0^t \mathbf{Z}(t-t') \mathbf{B}_{21} \mathbf{f}(t_0+t') dt'. \quad (\text{B.1})$$

With the aid of the causality condition (5.9), one gets

$$\begin{aligned} \langle \mathbf{R}(t_0+t) \mathbf{R}(t_0) \rangle &= \langle \dot{\mathbf{f}}(t_0+t) \dot{\mathbf{f}}(t_0) \rangle \\ &+ \int_0^t \mathbf{B}(t-t') \langle \mathbf{f}(t_0+t') \dot{\mathbf{f}}(t_0) \rangle dt' + \mathbf{Z}(t) \langle \mathcal{G}(t_0) \mathbf{f}(t_0) \rangle \widetilde{\mathbf{B}}_{11}. \end{aligned} \quad (\text{B.2})$$

The Fourier-Laplace transform of Eq. (B.2) is carried out by making use of Eq. (5.12), the fact that $\langle \mathbf{f}(t_0+t) \dot{\mathbf{f}}(t_0) \rangle \rightarrow \langle \mathbf{f}(t_0+t) \rangle \langle \dot{\mathbf{f}}(t_0) \rangle = 0$ ($t \rightarrow \infty$), and the stationary condition $\langle \mathbf{f}(t_0+t) \dot{\mathbf{f}}(t_0) \rangle = -\langle \dot{\mathbf{f}}(t_0+t) \mathbf{f}(t_0) \rangle$. One thus arrives at the fluctuation-dissipation theorem (5.15).

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CHAPTER 6

Summary and Conclusions

In this thesis, mainly using the system size expansion method, we have developed the reactor noise theory from the viewpoint that the reactor noise is an example of non-linear non-equilibrium statistical physics.

We have reviewed in Chap. 1 the conventional analyses of reactor noise and also outlined the physical basis to study reactor noise phenomena.

In Chap. 2, we have substantiated the assumption of Gaussian distribution of the number of neutrons in the zero power reactor, owing to the macroscopic nature of the system size. We have also applied our method to the non-linear Langevin formalism which was obscurely used in the reactor noise, and pointed out the importance of assumption of the normal case in order to classify states of system in detail.

In the normal case of our formalism, we have introduced the irreversible circulation of fluctuation α as a new variable for the analysis of reactor noise as in Chaps. 3 and 4. The α has the information of the rotational freedom of fluctuations, and also serves to be the integral index for power spectral density in conjunction with the variance. We have emphasized that large effects of α are expected not only in the reactor noise analysis but also in the time series analysis. To show the effectiveness of α , we have studied the theoretical model of the KUR type of reactor described by the three components. In this analysis, we have pointed out that the α is useful as an index to specify the type of instability (soft-mode or hard-mode). In many physically interesting examples as well as the reactor noise, there appears a stable limit cycle beyond the hard mode instability. In this case, the limit cycle may be interpreted

as a macroscopic manifestation of the circulation which is inherent in the fluctuations around the steady state below threshold. Namely, we have shown that in this case the fluctuations can be utilized as "forerunner phenomena" to foretell instability through appropriate information processing. We have also suggested that the reactor noise theory should be formulated from the viewpoint of generalized phase transition phenomena, in order to make active use of analogies with theories or methods developed in many other related fields and to get unified point of view.

In Chap. 5, we have made clear that the contraction of information by a reduction of hidden variables introduces the non-Markovian character into the description of fluctuations. This is only the first step of the trial to study the information contained in fluctuations and the quality change of information through a general contraction of information. Therefore, it is necessary to study the reactor noise from this "interscience" point of view. The author hopes that the thesis stands as a bridge between the reactor noise analysis and the other related fields, especially statistical physics, time series analysis and information processing etc.

The present study in this thesis has been based on a number of assumptions and simplifications, which have resulted in neglect of many factors that may be important in the complicated phenomena of actual power reactors. Further study on the reactor noise analysis is called for, in proportion to scale up of reactor power plant, as follows; for example,

- (1) generalization to space-dependent problems
- (2) development of system analysis such as multivariate analysis to treat a complex system
- (3) advancement of pattern recognition technique for diagnosis of anomalies by using actual data.

These are still remained as future problems of reactor noise analysis.

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