The University of Osaka
Institutional Knowledge Archive

Title	INVARIANT STATES ON C*-ALGEBRAS
Author(s)	楠田, 雅治
Citation	大阪大学, 1987, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/2061
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

INVARIANT STATES ON C^*-ALGEBRAS

MASAHARU KUSUDA

FACULTY OF ENGINEERING SCIENCE

OSAKA UNIVERSITY

1987

To my son Kazuya

IN MEMORY OF HIS BIRTH

Acknowledgements

I would like to take this opportunity to thank Professor Charles J. K. Batty for his fruitful discussions through the joint work.

Special thanks go to Professor Akitaka Kishimoto and Professor Shôichiro Sakai for their helpful suggestions and stimulating discussions.

I would like to thank Professor Yoshiomi Nakagami and Professor Niro Yanagihara for their warm encouragements and valuable suggestions.

My deepest thanks go to Professor Osamu Takenouchi, my adviser, for his guidance, constant encouragements and instructive suggestions throughout the course of this research.

A few words can not adequately express the thanks I owe to my wife, Chieko Kusuda. While I was engrossed in studying and writing, she created a warm home for me and our son Kazuya, and through her constant support and friendship provided an enriched human context for my life.

Contents

Introduction.

This thesis is devoted to the study of some problems concerning invariant states on C*-algebras with group actions. Though we have various objects to be studied in C*-dynamical systems, the most important one among them is presumably invariant states. The use of C*-algebras themselves in physics has been promoted by Segal, Haag and others. The physical interpretation of the C*-algebra is that it is the algebra generated by "the observables". It is quite common that, in many physical problems, there is a naturally defined group acting on the system in question which keeps invariant the system as a whole. In the corresponding C*-version, we shall admit that a group acts as a group of automorphisms on the C*-algebra. Then, it may be natural to restrict our attention to the states which are invariant under the group action, i.e., invariant states.

It is not too much to say that when we make the research of invariant states, substantial discussions are centered around the existence of invariant states such as KMS states, or ground states, the extensions of invariant states on an invariant C*-subalgebra to the whole algebra preserving the specific properties, or the equivalence between classes of invariant states. We pay our attention mainly on these subjects in this thesis.

This thesis consists of three chapters. We study the invariant states on C*-algebras with one-parameter automorphism groups in chapters I and II, and those on C*-algebras with actions by locally compact groups in chapter III. Now we explain briefly the contents of each chapter.

In chapter I, we discuss how the ideal structure of C*-crossed product is related to the existence of ground states.

Let (A, G, α) be a C*-dynamical system. Recently, the ideal structure of the crossed product $A\times_{\alpha}G$ has attracted considerable attention, particularly in the cases when G is abelian; for when G is abelian, spectral theory may be applied with good effect (cf. [24, or 25]). In the case when G is R, A is a UHF C*-algebra, and α is of product type, Bratteli[6] proved that $A\times_{\alpha}G$ is not simple since A admits a ground state for α; i.e., a state φ such that for all x in A and y in A^a, the space of entire analytic elements in A, the function $f(t) = \varphi(x\alpha_t(y))$ extends to be analytic in the upper half-plane and bounded there by $\| x \| \| y \|$. Subsequently, Pedersen and Takai[26] extended this result to the case when A and α are arbitrary (but $G = R$). Our primary objective in this chapter is to prove that the existence of a ground state is equivalent to the existence of a proper ideal of the crossed product which is monotonely increasing up to the whole algebra under the dual action.

In the last section, we shall generalize our results to cover a certain C*-dynamical system (A, G, α) where G is an abelian, connected and compact group.

In chapter II, we discuss the equivalence between the notions of passive states and spectrally passive states on UHF C*-algebras.

Let (A, R, α) be a C*-dynamical system. The notion of "passive" states had been introduced by Pusz and Woronowicz[31]. It was derived from the second law of thermodynamics. A convenient mathematical formulation is that a state φ of a unital C*-algebra A is *passive* if

$$-i\,\varphi(u^*\delta(u)) \geqq 0$$

for all unitary elements u which belong to both the domain $D(\delta)$ of the infinitesimal generator δ of α and the principal connected component of the unitary group of A. KMS states with some positive inverse temperature and ground states are passive. The converse is not true in general. In fact, though any mixture of passive states is passive, a non-trivial mixture of KMS states with different temperatures is neither a KMS state nor a ground state.

Recently, De Cannière[9] defined an α-invariant state φ of A to be *spectrally passive* if

$$\varphi(x^*x) \leqq \varphi(xx^*)$$

for all x in $A^{\alpha}(-\infty, 0)$. Here $A^{\alpha}(-\infty, 0)$ denotes the spectral subspace of A corresponding to the open interval

$(-\infty, 0)$, which is defined to be the closed linear span of all the elements of the form $\alpha_f(x) = \int f(t)\alpha_t(x)\,dt$, where x is in A, and f is a function in $L^1(R)$ whose inverse Fourier transform has compact support in $(-\infty, 0)$. Moreover, he showed that φ is spectrally passive if and only if

$$-i\,\varphi(x\delta(x)) \geqq 0$$

for any self-adjoint element x in $D(\delta)$. If φ is passive, taking $u = e^{itx}$ and differentiating twice with respect to t, it immediately follows that the above inequality holds. Thus all passive states are spectrally passive. De Cannière then asked whether all spectrally passive states were passive. Later, Batty[3] gave a partial answer that for a group action G commuting with α, any G-central spectrally passive state is passive. Here we remark that even in a full matrix algebra there are spectrally passive states which are not G-central for any G commuting with α. However when A is a full matrix algebra, passivity and spectral passivity are equivalent, which is seen from a result of Lenard[21], and this was pointed out by De Cannière.

Now we have a question whether passivity and spectral passivity are equivalent on UHF C*-algebras. In this chapter, as a step toward this problem, we consider the case where a UHF C*-algebra has a one-parameter automorphism group generated by the closure of a commutative normal *-derivation of finite type (see Powers and Sakai[29], Sakai[35, 36, 37] for the details).

In section 1, we show that every passive state is spectrally passive for any one-parameter automorphism group on finite dimensional C*-algebras.

In section 2, we show that if δ is a closed *-derivation densely defined on a unital C*-algebra A, then each unitary element in both the domain of δ and the connected component of the identity in the unitary group of A has the form

$$e^{ia_1}e^{ia_2}\ldots\ldots\ldots\ldots e^{ia_m}$$

for some self-adjoint elements $a_1, a_2, \ldots\ldots\ldots, a_m$ in the domain of δ. This result is used to show the main theorem in the next section.

In section 3, we show the main theorem in this chapter. That is, we show that if a UHF C*-algebra has a one-parameter automorphism group generated by the closure of a commutative normal *-derivation of finite type, then passivity and spectral passivity are equivalent for such a C*-dynamical system. This result is applicable to all one-dimensional lattice systems with finite range interaction and one-dimensional Ising model at arbitrary temparature.

In chapter III, we discuss the extensions of certain invariant states, and observe how these extensions characterize the C*-dynamical systems.

Attempts to extend a factor state φ on a C*-algebra B to a factor state on a larger C*-algebra A were first partially accomplished by the use of the notion of weak expectations for the GNS representation π_φ, that is, linear contractions P of A into $\pi_\varphi(B)''$ such that $P|_B = \pi_\varphi$. The eventual solutions of the attempts[22, 27] were variants of this method.

In the case when there is an action α of an amenable group G on A leaving B invariant, an analogous problem is to consider an α-invariant state φ of B which is centrally ergodic in the sense that

$$\pi_\varphi(B)'' \cap \pi_\varphi(B)' \cap u_G^{\varphi}{}' = \mathbb{C}\cdot 1,$$

where $(\pi_\varphi, u^\varphi, H_\varphi)$ is the associated covariant representation of (B, G, α), and to try to find an extension to a centrally ergodic state of A. It was shown in [4] that this can be done by the method of [1] if B is (semi)nuclear. The von Neumann algebra theory developed in [22, 27] is not sufficient to provide a general solution. A corollary from the positive answer to this problem is that if A is separable and G-central (and B is nuclear), then B is also G-central.

The purpose of this chapter is to clarify the covariant situation. In section 1, we consider the problem lifted to the C*-crossed products. Let (A, G, α) be a C*-dynamical system, and let B be an α-invariant C*-subalgebra of A. For a covariant representation (π, u, H) of B, the existence of a weak expectation $\hat{Q}$ of $A\times_\alpha G$ for the representation $\pi\times u$ (with respect to the subalgebra $L^1(B, G)$) is shown to be equivalent to the existence of a (covariant) completely positive contraction Q of A into $(\pi(B) \cup u_G)''$ such that $Q|_B = \pi$. Such a contraction Q will be called to be a covariant weak expectation.

In section 2, we show that, for an α-invariant state φ on B, there are bijective correpondences between covariant weak expectations Q of A into $(\pi_\varphi(B) \cup u_G^\varphi)''$, weak expectations $\hat{Q}$ of $A \times_\alpha G$ into $(\pi_\varphi(B) \cup u_G^\varphi)''$, certain α-invariant extensions of φ to A, and certain $(\alpha \otimes 1)$-invariant states of $A \otimes_{max} (\pi_\varphi(B) \cup u_G^\varphi)'$.

In section 3, we show that if there is a covariant weak expectation of A into $\pi_\varphi(B)''$ for a centrally ergodic state φ, φ can extend to a centrally ergodic state on A.

In section 4, it is observed that, if A is G-central, then Q and $\hat{Q}$ always exist (for each α-invariant state).

In section 5, we discuss G-abelianness of C*-dynamical system (A, G, α) with a compact group G. We show the equivalence of G-abelianness of A, commutativity of the fixed point algebra of A, and ergodicity of certain class of invariant states.

Preliminaries and notations.

We shall summarize definitions and notations about some objects in C*-dynamical systems to be used in this thesis.

0.1. A *C*-dynamical system* is a triple (A, G, α) consisting of a C*-algebra A, a locally compact group G, a continuous homomorphism α of G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A.

0.2. Let (A, G, α) be a C*-dynamical system. Then a state φ on A is *α-invariant* if $\varphi(\alpha_t(x)) = \varphi(x)$ for all x in A and all t in G.

0.3. Let (A, G, α) be a C*-dynamical system. Then the *C*-crossed product* $A \times_\alpha G$ for (A, G, α) is defined as the enveloping C*-algebra of $L^1(A, G)$, the set of all Bochner integrable A-valued functions on G equipped with the following Banach *-algebra structure:

$$(xy)(t) = \int_G x(s)\alpha_s(y(s^{-1}t))\, ds,$$

$$x^*(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$

$$\| x \|_1 = \int_G \| x(s) \| \, ds,$$

where ds is the left Haar measure of G and $\Delta(t)$ is the associated modular function on G.

We consider the actons of A and G on $A\times_\alpha G$ given by

$$(ax)(t) = ax(t) \qquad (a \in A),$$

$$(\lambda_s x)(t) = x(s^{-1}t) \qquad (s \in G),$$

for all x in $L^1(A, G)$. We may embed A and G (or λ_G) into the multiplier algebra $M(A\times_\alpha G)$ of $A\times_\alpha G$ under these actions.

We assume that G is a locally compact abelian group. The *dual action* $\hat{\alpha}$ of α is defined on $A\times_\alpha G$ by the formula

$$\hat{\alpha}_\gamma(x)(t) = \overline{< t, \gamma >}x(t),$$

where γ in $\hat{G}$ and x in $L^1(A, G)$, and where $< t, \gamma >$ denotes the value of γ at t.

0.4. A *covariant representation* for (A, G, α) is a triple (π, u, H), where π is a (non-degenerate) representation of A on a Hilbert space H, and u is a strongly continuous unitary representation of G on H such that

$$\pi(\alpha_t(x)) = u_t\pi(x)u_t^*$$

for all x in A and all t in G.

Let φ be a state on A. Then there exists the cyclic representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ of A with a cyclic vector ξ_φ such that $(\pi_\varphi(x)\xi_\varphi|\xi_\varphi) = \varphi(x)$ for all x in A. If φ is α-invariant, a unitary representation u^φ is defined by

$$u_t^{\varphi}\pi_{\varphi}(x)\xi_{\varphi} = \pi_{\varphi}(\alpha_t(x))\xi_{\varphi}$$

for all x in A and all t in G. Then we obtain a covariant representation $(\pi_{\varphi}, u^{\varphi}, H_{\varphi}, \xi_{\varphi})$ for (A, G, α).

If (π, u, H) is a covariant representation of (A, G, α), there is a non-degenerate representation $(\pi \times u, H)$ of $A \times_{\alpha} G$ such that

$$(\pi \times u)(x) = \int_G \pi(x(t))u_t \, dt$$

for all x in $L^1(A, G)$. Moreover, the correspondence $(\pi, u, H) \to (\pi \times u, H)$ gives a bijection onto the non-degenerate representations of $A \times_{\alpha} G$.

0.5. We denote by $\mathbb{C}$, R, and Z the set of complex numbers, the set of real numbers, and the set of integers respectively.

Chapter I. C*-dynamical systems with ground states.

Let (A, G, α) be a C*-dynamical system. Recently, the ideal structure of the crossed product $A\times_\alpha G$ has attracted considerable attention, particularly in the cases when G is abelian; for, when G is abelian, spectral theory may be applied with good effect (cf. [24, or 25]). In the case when G is R, A is a UHF C*-algebra, and α is of product type, Bratteli[6] proved that $A\times_\alpha G$ is not simple since A admits a ground state for α; i.e., a state φ such that for all x in A and y in A^a, the space of entire analytic elements in A, the function $f(t) = \varphi(x\alpha_t(y))$ extends to be analytic in the upper half-plane and bounded there by $\|x\| \|y\|$. Subsequently, Pedersen and Takai[26] extended this result to the case when A and α are arbitrary (but $G = R$). Our primary objective in this chapter is to prove that the existence of a ground state is equivalent to the existence of a proper ideal of the crossed product which is monotonely increasing up to the whole algebra under the dual action.

In last section, we shall generalize our results to cover certain C*-dynamical system (A, G, α) where G is an abelian, connected and compact group.

I.1. Ideal structure of C*-crossed products.

Let (A, R, α) be a C*-dynamical system. If A admits a ground state, the crossed product is not simple. In this section, we elaborate on this point by showing the existence of some proper ideal of the crossed product which is monotonely increasing up to the whole algebra under the dual action.

First of all, we review some conditions equivalent to the condition in the definition of ground states mentioned in the introduction.

Proposition 1.1. Let (A, R, α) be a C*-dynamical system, δ the generator of α, and φ a state of A. Then the following conditions are equivalent:

(i) φ is a ground state.

(ii) φ is α-invariant, and if $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ is the cyclic covariant representation associated with φ, then $Sp(u^\varphi) \subset R_+$, where $Sp(u^\varphi)$ denotes the spectrum of u^φ.

(iii) $\varphi(x^*x) = 0$ for all x in $A^\alpha(-\infty, 0)$, where $A^\alpha(-\infty, 0)$ denotes the spectral subspace of A corresponding to the open interval $(-\infty, 0)$ in R.

(iv) $-i\,\varphi(x^*\delta(x)) \geqq 0$ for all x in the domain of δ.

(v) There is a positive (not necessarily bounded) operaror h on H_φ with $h\xi_\varphi = 0$ such that

$$e^{ith}\pi_\varphi(x)e^{-ith} = \pi_\varphi(\alpha_t(x))$$

for all t in R and x in A.

We are referred to [8, or 24] for the proof. Note that (ii), (iii), and (iv) are used in this section, in the next section, and in chapter II respectively.

Proposition 1.2. *Let* (A, R, α) *be a* C^**-dynamical system, and let* $\hat{\alpha}$ *denote the dual action of* R *on* $A\times_\alpha R$*. If* A *has a ground state, then there exists a proper closed ideal* I *in* $A\times_\alpha R$ *such that if* $\lambda_1 < \lambda_2$*, then* $\hat{\alpha}_{\lambda_1}(I) \subsetneqq \hat{\alpha}_{\lambda_2}(I)$ *and for any* $\lambda > 0$, $A\times_\alpha R = \overline{\bigcup_{Z_+\ni n} \hat{\alpha}_{n\lambda}(I)}$*, where* Z_+ *denotes the set of non-negative integers.*

Proof. Let φ be a ground state. Let $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ be the cyclic covariant representation associated with φ. Then u_t^φ has the spectral decomposition

$$u_t^\varphi = \int_R e^{its}\, dE(s),$$

and the support of E $(= Sp(u^\varphi))$ is contained in R_+, where R_+ is the set of all non-negative real numbers (see [24, 8.12.5]). For each $k \geqq 0$, $R \ni t \to e^{ikt} u_t^\varphi$ is a unitary representation on H_φ. Put $v_k(t) = e^{ikt} u_t^\varphi$. Then $(\pi_\varphi, v_k, H_\varphi)$ is a covariant representation. Hence, we can consider the corresponding representation $(\pi_\varphi \times v_k, H_\varphi)$ of the crossed product. We denote the direct sum of $\{\pi_\varphi \times v_k\}_{k \geqq 0}$ by $\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)$. Let

$$I = \ker\Big(\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)\Big)$$

be the kernel of $\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)$. Now fix a positive number λ

arbitrarily. Then we have

$$\hat{\alpha}_\lambda(I) = \hat{\alpha}_\lambda(\ker(\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)))$$

$$= \ker((\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)) \circ \hat{\alpha}_{-\lambda})$$

$$= \ker(\bigoplus_{k \geqq 0} (\pi_\varphi \times v_{k+\lambda}))$$

$$= \ker(\bigoplus_{k \geqq \lambda} (\pi_\varphi \times v_k)) \supset \ker(\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)) = I.$$

Take any f in $L^1(R)$. By an easy consequence of Plancherel's theorem, there is a sequence $\{f_n\}$ in $L^1(R)$ such that $\mathrm{supp}(\hat{f}_n)$ is compact for each n and $\| f_n - f \|_{L^1} \to 0$, where $\hat{f}_n$ denotes the inverse Fourier transform of f_n. Since $\mathrm{supp}(\hat{f}_n)$ is compact, we can choose a natural number $m(n)$ such that $\mathrm{supp}(\hat{f}_n(\cdot + m(n)\lambda))$ is contained in R_- for each f_n, where R_- is the set of all negative numbers. In particular, $\mathrm{supp}(\hat{f}_n(\cdot + m(n)\lambda + k))$ is contained in R_- for any $k \geqq 0$. Put $g_n(t) = e^{im(n)\lambda t} f_n(t)$ and $(x \otimes g)(t) = g(t)x$ for any g in $L^1(R)$ and x in A. Then we have

$$(*) \qquad \hat{\alpha}_{m(n)\lambda}(x \otimes g_n)(t) = (x \otimes f_n)(t).$$

For any $k \geqq 0$,

$$(\pi_\varphi \times v_k)(x \otimes g_n) = \int_R \pi_\varphi(g_n(t)x) v_k(t)\, dt$$

$$= \pi_\varphi(x) \int_R g_n(t) e^{ikt} u_t^\varphi \, dt$$

$$= \pi_\varphi(x) \int_R \int_R g_n(t) e^{ikt} e^{its} dt \, dE(s)$$

$$= \pi_\varphi(x) \int_R \hat{f}_n(s + m(n)\lambda + k)\, dE(s) = 0,$$

since the support of E is contained in R_-. Therefore,

$$\Big(\bigoplus_{k \geqq 0} (\pi_\varphi \times v_k)\Big)(x \otimes g_n) = 0$$

for any n. Since $x \otimes g_n$ belongs to I, we have

$$\{x \otimes f_n\} \subset \bigcup_{Z_+ \ni m} \hat{\alpha}_{m\lambda}(I)$$

from (*). Since

$$\| x \otimes f_n - x \otimes f \| \leqq \| x \otimes f_n - x \otimes f \|_{L^1}$$
$$= \| x \| \, \| f_n - f \|_{L^1} \to 0,$$

We have $x \otimes f \in \overline{\bigcup_{Z_+ \ni n} \hat{\alpha}_{n\lambda}(I)}$. Now we easily see that the ideal I is strictly contained in $\hat{\alpha}_\lambda(I)$. Thus, if $\lambda_1 < \lambda_2$, then we have $\hat{\alpha}_{\lambda_1}(I) \subsetneqq \hat{\alpha}_{\lambda_2}(I)$ and $L^1(A, R) \subset \overline{\bigcup_{Z_+ \ni n} \hat{\alpha}_{n\lambda}(I)}$. Therefore, we obtain the desired result. Q.E.D.

We recall that for any proper ideal, there is a primitive ideal containing it. Hence, we obtain the following.

Corollary 1.3. *Let (A, R, α) be a C*-dynamical system. If A has a ground state, then $A \times_\alpha R$ contains a primitive ideal ζ such that for any $\lambda > 0$, $A \times_\alpha R = \overline{\bigcup_{Z_+ \ni n} \hat{\alpha}_{n\lambda}(\zeta)}$.*

Assume that a C*-algebra A is unital. Then Powers and Sakai[29, 2.3] proved that there exists a ground state if α is approximately inner.

Corollary 1.4. *Let* (A, R, α) *be a C*-dynamical system, where* A *is unital. If* α *is approximately inner, the statement of Corollary* 1.3 *holds.*

The above corollary is an extension of the result of Bratteli[6, 3.2], in which he assumed that A is a UHF C*-algebra with a product type action of R. Moreover, under the same assumptions, he showed the same result for an abelian, connected and compact group such that the cardinality of the dual group does not exceed the power of the continuum, instead of R. As for this case, we shall discuss in section 3.

I.2. Existence of ground states.

In this section, we consider the converse of Proposition 1.2 and show how the ideal structure of the crossed product is related to the existence of ground states.

Theorem 2.1. *Let* (A, R, α) *be a C*-dynamical system, where* A *is a unital C*-algebra. Then the following conditions are equivalent:*

(i) *There is a ground state of* A *for* α.

(ii) *There is a proper closed ideal* I *of* $A\times_\alpha R$ *such that* $I \subset \hat{\alpha}_\lambda(I)$ *for any* $\lambda \geqq 0$ *and the union of* $\hat{\alpha}_\lambda(I)$ *with all* λ *in* R *is dense in* $A\times_\alpha R$.

Proof. (i) $\Rightarrow$ (ii). This follows from Proposition 1.2.

(ii) $\Rightarrow$ (i). Let B be an α-invariant hereditary C^*-subalgebra of A. Put

$$J_B = \{ f \in L^1(R) \mid x \otimes f \in I \text{ for any } x \in B \}.$$

Then, J_B is a closed ideal. Set

$$Z(J_B) = \{ t \in R \mid \hat{f}(t) = 0 \text{ for any } f \in J_B \}.$$

Since $I \subset \hat{\alpha}_\lambda(I)$ for any $\lambda \geqq 0$, we have $J_B \subset \hat{\alpha}_\lambda(J_B)$. Hence, when $Z(J_B)$ is not empty, we have $Z(J_B) \supset Z(J_B) + \lambda$ for any $\lambda \geqq 0$. Thus, if $Z(J_B)$ is neither empty nor R, it is a half-line (i.e., it is of the form $[r, \infty)$, where r is a real number). We denote the greatest lower bound (including ∞) of $Z(J_B)$ by $m(B)$, where we set $m(B) = \infty$ (resp. $-\infty$) in the case that $Z(J_B)$ is empty (resp. R). Let n be a positive integer and let H_n be the set of α-invariant hereditary C^*-subalgebras B of A with $m(B) \geqq -n$. When B_n denotes the α-invariant hereditary C^*-subalgebra generated by those in H_n, we have $B_n \in H_n$. Let $\widetilde{B}_n$ be the linear span of elements of the form axb with $a \in B' \in H_n$, $b \in B'' \in H_n$, $x \in A$. Since $\widetilde{B}_n$ is dense in B_n, it follows that $y \otimes g \in I$ for all $y \in B_n$ and $g \in L^1(R)$ with $\mathrm{supp}(\hat{g}) \subset (-\infty, -n)$

We assert that $\bigcup_n B_n$ is dense in A. Otherwise, denoting the closure of $\bigcup_n B_n$ by B_∞, we have an α-invariant state φ of A such that $\varphi = 0$ on B_∞ since B_∞ is α-invariant. Through the representation $\pi_\varphi \times u^\varphi$ of $A \times_\alpha R$, define a state $\overline{\varphi}$ of $A \times_\alpha R$ by the cyclic vector ξ_φ, where

$(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ is the cyclic covariant representation of A associated with φ. By Kishimoto's theorem[14], the ideal I is densely spanned by elements of the form $x \otimes f$ with $x \in A$, $f \in L^1(R)$. Therefore, there are $x \otimes f$ in I and $\lambda \geqq 0$ such that

$$\overline{\varphi}(\hat{\alpha}_\lambda(x \otimes f)) = 0 \quad (\text{i.e.,} \quad \varphi(x) \neq 0).$$

Let $H(x^*Ax)$ be the α-invariant hereditary C*-subalgebra generated by x^*Ax. Then $H(x^*Ax) \in H_n$ for some $n > 0$, which implies $\varphi(x^*x) = 0$ (i.e., $\varphi(x) = 0$). Thus, we have reached a contradiction. Therefore, $\bigcup_n B_n$ is dense in A.

Suppose $B_n \neq A$ for any n. Since B_n is the hereditary C*-subalgebra of A for each n, $\| x - 1 \| \geqq 1$ for any $x \in B_n$, which implies $1 \notin B_\infty = \overline{\bigcup_n B_n}$. Hence, $B_n = A$ for some $n > 0$. Therefore if $\operatorname{supp}(\hat{f}) \subset (-\infty, -n)$, we have $1 \otimes f \in I$.

Now we take a covariant representation (π, u, H) of A such that $\ker(\pi \times u) \supset I$. Then, $Sp(u) \subset [-n, \infty)$. Put

$$(*) \qquad \delta = \inf(Sp(u)).$$

We take a sequence $\{\xi_k\}$ in H such that

$$Sp_u(\xi_k) \subset [\delta, \delta + k^{-1}]$$

and $\| \xi_k \| = 1$ for each k. Putting $\varphi_k(x) = (\pi(x)\xi_k | \xi_k)$, then we have a weak* limit point φ of $\{\varphi_k\}$ in the state space of A. We take an arbitrary positive number ε. If $x \in A$, with $Sp_\alpha(x) \subset (-\infty, -\varepsilon)$, then

$$Sp_u(\pi(x)\xi_k) \subset Sp_\alpha(x) + Sp_u(\xi_k)$$

(see [24, 8.2.4]), and so it follows easily from (*) that $\varphi_k(x^*x) = 0$ for $k^{-1} < \varepsilon$. (See[7, or 24] for the spectral theory.) Hence, we have $\varphi(x^*x) = 0$ for all $x \in A$ with $Sp_\alpha(x) \subset (-\infty, 0)$. Q.E.D.

I.3. Ground states for compact abelian groups.

Let G be a locally compact abelian group. Suppose that P is a semigroup in the dual group $\hat{G}$ which is closed and has two additional properties;

$$P \cap (-P) = \{0\}, \qquad P \cup (-P) = \hat{G}$$

(P is said to be a positive cone). Under these conditions, P induces an order in $\hat{G}$, i.e., $\gamma \geqq \gamma'$ if $\gamma - \gamma' \in P$ (see [32]).

Let (A, G, α) be a C*-dynamical system. Then we call an α-invariant state φ a ground state for (G, P) if $Sp(u^\varphi) \subset P$.

Now let G be an abelian, connected and compact group. Assume that the cardinality of $\hat{G}$ does not exceed the power of the continuum. Then, $\hat{G}$ has a positive cone P which induces an archimedean order in $\hat{G}$ (see [32, 8.1.2]). Now we have the following:

Proposition. *If A has a ground state for (G, P) where P induces an archimedean order in $\hat{G}$, then $A\times_\alpha G$ contains a proper ideal I such that $\hat{\alpha}_{\lambda_1}(I) \subsetneq \hat{\alpha}_{\lambda_2}(I)$ for $\lambda_1 < \lambda_2$ in $\hat{G}$ and if $\lambda > 0$, then $A\times_\alpha G = \overline{\bigcup_{Z\ni n} \hat{\alpha}_{n\lambda}(I)}$.*

We sketch the proof. By [32, 8.1.2], there is an order preserving isomorphism of $\hat{G}$ onto a subgroup of R. Hence, it is easily seen that there exists τ in P such that $\bigcap_n (n\tau + P)$ is empty, i.e., $\bigcup_n (n\tau + P^c) = \hat{G}$, where P^c is the complement of P in $\hat{G}$. If $\gamma \geqq \gamma' \geqq 0$, $\gamma + P^c \supset \gamma' + P^c$. Hence, $\{n\tau + P^c\}_n$ is an increasing sequence. For any f in $L^1(G)$, there exists a sequence $\{f_n\}$ in $L^1(G)$ such that $\mathrm{supp}(\hat{f}_n)$ is compact for each n and $\| f_n - f \|_{L^1} \to 0$, where $\hat{f}_n$ is the inverse Fourier transform of f_n. Then, for each n, there exists a natural number $m(n)$ such that $\mathrm{supp}(\hat{f}_n) \subset m(n)\tau + P^c$. Hence, we have $\mathrm{supp}(\hat{f}_n(\cdot + m(n)\tau + \gamma)) \subset P^c$ for any γ in P. Thus, we can obtain the desired result by the same method used in section 1.

Chapter II. Passive states on UHF C*-algebras.

Let (A, R, α) be a C*-dynamical system, where R is a locally compact group of the real numbers. The notion of "passive" states had been introduced by Pusz and Woronowicz[31]. It was derived from the second law of thermodynamics.
A convenient mathematical formulation is that a state φ of a unital C*-algebra A is *passive* if

$$-i\varphi(u^*\delta(u)) \geqq 0$$

for all unitary elements u which belong to both the domain $D(\delta)$ of the infinitesimal generator δ of α and the principal connected component of the unitary group of A.
KMS states with some positive inverse temperature and ground states are passive. The converse is not true in general.
In fact, though any mixture of passive states is passive, a non-trivial mixture of KMS states with different temperatures is neither a KMS state nor a ground state.

Recently, De Cannière[9] defined an α-invariant state φ of A to be *spectrally passive* if

$$\varphi(x^*x) \leqq \varphi(xx^*)$$

for all x in $A^\alpha(-\infty, 0)$. Here $A^\alpha(-\infty, 0)$ denotes the spectral subspace of A corresponding to the open interval $(-\infty, 0)$, which is defined to be the closed linear span of all the elements of the form $\alpha_f(x) = \int f(t)\alpha_t(x)\, dt$, where x is in A, and f is a function in $L^1(R)$ whose inverse Fourier transform has compact support in $(-\infty, 0)$. Moreover, he showed that φ is spectrally passive if and only if

$$-i\,\varphi(x\delta(x)) \geqq 0$$

for any self-adjoint element x in $D(\delta)$. If φ is passive, taking $u = e^{itx}$ and differentiating twice with respect to t, it immediately follows that the above inequality holds. Thus all passive states are spectrally passive. De Cannière then asked whether all spectrally passive states were passive. Later, Batty[3] gave a partial answer that for a group action G commuting with α, any G-central spectrally passive state is passive. Here we remark that even in a full matrix algebra there are spectrally passive states which are not G-central for any G commuting with α. However when A is a full matrix algebra, passivity and spectral passivity are equivalent, which is seen from a result of Lenard[21], and this was pointed out by De Cannière.

Now we have a question whether passivity and spectral passivity are equivalent on UHF C*-algebras. In this chapter, as a step toward this problem, we consider the case where a UHF C*-algebra has a one-parameter automorphism group generated by the closure of a commutative normal *-derivation of finite type (see Powers and Sakai[29], Sakai[35, 36, 37] for the details).

In section 1, we show that every passive state is spectrally passive for any one-parameter automorphism group on finite dimensional C*-algebras.

In section 2, we show that if δ is a closed *-derivation densely defined on a unital C*-algebra A, then each unitary element in both the domain of δ and the connected component of the identity in the unitary group of A has the form

$$e^{ia_1}e^{ia_2}\cdots\cdots\cdots e^{ia_m}$$

for some self-adjoint elements $a_1, a_2, \ldots\ldots\ldots, a_m$ in the domain of δ. This result is used to show the main theorem in the next section.

In section 3, we show the main theorem in this chapter. That is, we show that if a UHF C*-algebra has a one-parameter automorphism group generated by the closure of a commutative normal *-derivation of finite type, then passivity and spectral passivity are equivalent for such a C*-dynamical system. This result is applicable to all one-dimensional lattice systems with finite range interaction and one-dimensional Ising model at arbitrary temparature.

II.1. Passivity on finite dimensional C*-algebras.

In this section, we show that every passive state is spectrally passive on finite dimensional C*-algebras for any one-parameter automorphism group. First of all, we prove that passivity and spectral passivity are equivalent in a full matrix algebra. This fact is already well-known (cf. [9, 21]), and it is a consequence of the discussions in [9, 21] based on spectral analysis. But we report here its more elementary proof.

Consider a C*-dynamical system (A, R, α), where A is an $n \times n$-matrix algebra. Let τ be the tracial state on A. Any state φ of A is given by a density matrix ρ with $\rho \geqq 0$, $\tau(\rho) = 1$, and $\varphi(x) = \tau(\rho x)$ for all x in A. Now α can be written as the form $\alpha_t(x) = e^{ith}xe^{-ith}$ with some self-adjoint matrix h in A. If φ is α-invariant, ρ and h commute.

Sublemma([21]). *Under the above notations, we suppose that ρ and h commute. If the eigenvalues ρ_i of ρ and h_i of h satisfy $(\rho_i - \rho_j)(h_j - h_i) \geqq 0$ for all i and j, then we have $\tau(\rho u^*hu) \geqq \tau(\rho h)$ for all unitary matrices u in A.*

Proof. Let $u = (u_{ij})$ be a unitary matrix. Then we have

$$\tau(\rho u^*hu) = \sum_{i,j} \rho_i h_j |u_{ij}|^2$$

and

$$\tau(\rho h) = \sum_i \rho_i h_i.$$

Since the matrix $(|u_{ij}|^2)$ is doubly stochastic, it is given by a convex combination of the permutation matrices from Birkhoff-von Neumann's theorem. Hence we have

$$\tau(\rho u^* h u) = \sum_\sigma \lambda_\sigma \sum_i \rho_i h_{\sigma(i)}$$

for $\lambda_\sigma \geqq 0$ and $\sum_\sigma \lambda_\sigma = 1$, where σ runs over permutations of $1, 2, \ldots\ldots\ldots\ldots, n$. If $(\rho_i - \rho_j)(h_j - h_i) \geqq 0$, then we have

$$\sum_i \rho\ h_{\sigma(i)} \geqq \sum_i \rho_i h_i$$

(e.g., [13, Theorem 368]), which implies the desired result.

Q.E.D.

Now suppose that a state φ of a full matrix algebra A is spectrally passive. Then ρ and h commute. Therefore we may suppose that $h = \sum_i h_i e_{ii}$, where h_i $(1 \leqq i \leqq n)$ are real numbers, and that $\rho = \sum_i \rho_i e_{ii}$, $\rho_i \geqq 0$, $\sum_i \rho_i = 1$, where we denote by $\{e_{ij}\}$ the matrix units of A.

Take a self-adjoint element $x = e_{ij} + e_{ji}$. Then we have

$$\begin{aligned} -i\,\varphi(x\delta(x)) &= -i\tau(\rho x\delta(x)) = \tau(\rho x[h, x]) \\ &= \tau(\rho x h x) - \tau(\rho x^2 h) \end{aligned}$$

$$= (\rho_i h_j + \rho_j h_i) - (\rho_i h_i + \rho_j h_j)$$

$$= (\rho_i - \rho_j)(h_j - h_i) \geqq 0.$$

(Note that $-i\varphi(x\delta(x)) = \sum_{i,j} \rho_i(h_j - h_i)|x_{ij}|^2$ for $x = x^* = \sum_{i,j} x_{ij}e_{ij}$.) Since $-i\varphi(u^*\delta(u)) = \tau(\rho u^*hu) - \tau(\rho h)$ for all unitary matrices u in A, it follows from Sublemma that φ is passive. Thus we have the following.

Lemma 1.1. *Any spectrally passive state of a full matrix algebra is passive.*

Proposition 1.2. *Let (A, R, α) be a C^*-dynamical system, where A is a finite dimensional C^*-algebra. Then any spectrally passive state of A is passive.*

Proof. Since α is uniformly continuous, α is inner, i.e., $\alpha_t(x) = e^{ith}xe^{-ith}$ for all x in A, with some self-adjoint element h in A. Now let $A = \sum_j \oplus Ap_j$ be the central decomposition, where $\{p_j\}$ is the family of orthogonal minimal central projections in A. For each j, Ap_j is a full matrix algebra. Since $\alpha_t(p_j) = p_j$, Ap_j is α-invariant. Suppose that a state φ is spectrally passive. Then, φ is spectrally passive on Ap_j. By Lemma 1.1, φ is passive on Ap_j. Take any unitary element u in A. Since up_j is a unitary element in Ap_j,

$$-i\,\varphi((up_j)^*\delta(up_j)) \geqq 0$$

for all j. Since $\{p_j\}$ is the family of orthogonal central projections with $\delta(p_j) = 0$, a straightforward computation shows

$$- i\,\varphi(u^*\delta(u)) = -i\,\varphi\Big(\sum_j u^*p_j\delta(\sum_k up_k)\Big)$$

$$= \sum_j -i\,\varphi((up_j)^*\delta(up_j)) \geqq 0.$$

This completes the proof. Q.E.D.

II.2. Lemmas for closed *-derivations.

In this section, we prepare some lemmas to show our main theorem in the next section.

Lemma 2.1. *Let δ be a closed *-derivation densely defined in a unital C*-algebra. Let u be a unitary element in $D(\delta)$, the domain of δ, with $\| u - 1 \| < 1$. Then there exists a self-adjoint element a in $D(\delta)$ such that $u = e^{ia}$.*

Proof. Since $\| u - 1 \| < 1$, the spectrum of u is contained in the domain of the principal logarithm . Hence, we have

$$\mathrm{Log}\, u = - \sum_{n=1}^{\infty} \frac{1}{n}(1 - u)^n.$$

Moreover, it is easily seen that

$$\delta\left(\frac{1}{p}(1-u)^p\right) = \sum_{k=0}^{p-1} \frac{1}{p}(1-u)^k \delta(1-u)(1-u)^{p-k-1}.$$

Put $a_n = -\sum_{p=1}^{n} \frac{1}{p}(1-u)^p$. Then for $m \geqq n$, we have

$$\| \delta(a_m) - \delta(a_n) \|$$

$$= \| \sum_{p=1}^{m} \sum_{k=0}^{p-1} \frac{1}{p}(1-u)^k \delta(1-u)(1-u)^{p-k-1}$$

$$- \sum_{p=1}^{n} \sum_{k=0}^{p-1} \frac{1}{p}(1-u)^k \delta(1-u)(1-u)^{p-k-1} \|$$

$$\leqq \sum_{p=n+1}^{m} \sum_{k=0}^{p-1} \frac{1}{p} \| (1-u) \|^k \| \delta(1-u) \| \, \| (1-u) \|^{p-k-1}$$

$$= \sum_{p=n+1}^{m} \sum_{k=0}^{p-1} \frac{1}{p} \| (1-u) \|^{p-1} \| \delta(1-u) \|$$

$$= \sum_{p=n+1}^{m} \| (1-u) \|^{p-1} \| \delta(1-u) \| .$$

As $m, n \to \infty$, we have

$$\| \delta(a_m) - \delta(a_n) \| \to 0.$$

Since δ is closed, we conclude that $\mathrm{Log}\, u$ belongs to $D(\delta)$. Taking $a = -i\mathrm{Log}\, u$, we obtain $u = e^{ia}$ with $a = a^*$ in $D(\delta)$.

Q.E.D.

Remark 2.2. Since $D(\delta)$ is a Banach algebra with the graph norm, we can obtain the lemma by using the holomorphic functional calculas.

Lemma 2.3. *Let δ be a closed *-derivation densely defined on a unital C^*-algebra A. Take any unitary element u in $D(\delta) \cap U_0$, where U_0 is the connected component of the identity in the unitary group of A. Then there exist self-adjoint elements $a_1, a_2, \ldots\ldots, a_m$ in $D(\delta)$ such that*

$$u = e^{ia_1} e^{ia_2} \ldots\ldots e^{ia_m}.$$

Proof. Since u belongs to U_0, we can choose self-adjoint elemnts $h_1, h_2, \ldots\ldots, h_{m-1}$ in A such that

$$u = e^{ih_1} e^{ih_2} \ldots\ldots e^{ih_{m-1}}.$$

Since $D(\delta)$ is dense in A, there exist self-adjoint elements $a_1, a_2, \ldots\ldots, a_{m-1}$ in $D(\delta)$ such that

$$\| u - e^{ia_1} e^{ia_2} \ldots\ldots e^{ia_{m-1}} \| < 1.$$

Since $e^{ia_1}, e^{ia_2}, \ldots\ldots e^{ia_{m-1}}$ belong to $D(\delta)$ by [28], it follows from Lemma 2.1 that

$$e^{-ia_{m-1}} \ldots\ldots e^{-ia_2} e^{-ia_1} u = e^{ia_m}$$

for some self-adjoint element a_m in $D(\delta)$.
This completes the proof. Q.E.D.

II.3. Passive states for C*-dynamical systems associated with commutative normal *-derivations of finite type.

In this section we establish our main theorem. Sakai [35, 36, 37] introduced the notion of commutative normal *-derivations on UHF C*-algebras. A commutative normal *-derivation is defined as follows.

Let A be a UHF C*-algebra. Then a *-derivation δ_0 is said to be a *commutative normal *-derivation* if there is an increasing sequence $\{A_n\}$ of finite type I subfactors (containing the identity) in A such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A and the domain $D(\delta_0)$ of δ_0 is $\bigcup_{n=1}^{\infty} A$; moreover, there is a sequence of mutually commuting self-adjoint elements $\{h_n\}$ in A such that $\delta_0(a) = i[h_n, a]$ for all a in A_n $(n = 1, 2, \ldots\ldots\ldots)$. By Sakai[35], δ_0 has a canonical extension δ such that δ is a generator and

$$e^{t\delta}(a) = \lim_{n\to\infty} e^{t\delta_n}(a)$$

for all a in A, where $\delta_n(\cdot) = [ih_n, \cdot]$. Under this setting, we consider the C*-dynamical system (A, R, α), where $\alpha_t = e^{t\delta}$ for all t in R.

A (commutative) normal *-derivation δ_0 is said to be of finite type if we can choose h_n from the domain of δ_0 for all n. This is equivalent to $\delta_0(D(\delta_0)) \subset D(\delta_0)$.

Now we show the main theorem.

Theorem. *Under the above notation, suppose that* δ_0 *is of finite type. Then the following statements* (i) *and* (ii) *are equivalent.*

(i) *A state* φ *of* A *is spectrally passive for* α.

(ii) *A state* φ *of* A *is passive for* α.

Proof. We have only to prove that (i) implies (ii).

Let B_n be the C*-subalgebra of A generated by A_n and h_n. Then we have $[h_m, h_n] = 0$ and $[ih_m, A_n] \subset B_n$ for all $m \geqq n$. Hence B_n is invariant under δ_m. Since we have $\delta_m = \delta_n$ on B_n, B_n is α-invariant. Thus, we can consider the C*-dynamical systems (B_n, R, α) for $n = 1, 2, \ldots..$ Since $h_n \in \bigcup_m A_m$ for all n, B_n is a finite dimensional C*-algebra.

Now suppose that a state φ of A is spectrally passive for α. Then we can consider φ as a spectrally passive state over the C*-dynamical systems (B_n, R, α) for all n. Here, we remark that B_n contains the identity of A. It follows from Proposition 1.2 that φ is passive over (B_n, R, α) for all n.

It is well-known that a unitary group of a UHF C*-algebra is connected. Let u be a unitary element in $D(\delta)$. Then it follows from Lemma 2.3 that u is of the form

$$e^{ia_1}e^{ia_2}\ldots\ldots e^{ia_j}\ldots\ldots e^{ia_m}$$

for some self-adjoint elements $a_1, a_2, \ldots\ldots\ldots, a_m$ in $D(\delta)$. Since $\bigcup_{n=1}^{\infty} A_n$ is a core for δ, for each j $(1 \leqq j \leqq m)$, we can choose a sequence of self-adjoint elements $\{a_{j(n)}\}$

$(n = 1, 2, \ldots\ldots)$ from $\bigcup_{n=1}^{\infty} A_n$ such that

$$\| a_{j(n)} - a_j \| \to 0$$

and

$$\| \delta(a_{j(n)}) - \delta(a_j) \| \to 0$$

as $n \to \infty$. For each n, we may suppose that $a_{j(n)} \in A_n$ for all j $(1 \leqq j \leqq m)$. On the other hand, it follows from [28] that

$$e^{ia_{1(n)}}, e^{ia_{2(n)}}, \ldots\ldots\ldots\ldots, e^{ia_{m(n)}} \in D(\delta)$$

and

$$\delta(e^{ia_{j(n)}}) = i\int_0^1 e^{ita_{j(n)}} \delta(a_{j(n)}) e^{i(1-t)a_{j(n)}}\, dt$$

for all n. Hence, we have that

$$\lim_{n\to\infty} \delta(e^{ia_{j(n)}}) = \delta(e^{ia_j}) = i\int_0^1 e^{ita_j} \delta(a_j) e^{i(1-t)a_j}\, dt$$

Putting $u_n = e^{ia_{1(n)}} e^{ia_{2(n)}} \ldots\ldots\ldots\ldots\ldots e^{ia_{m(n)}}$, we see that u_n is contained in B_n and

$$\| u_n - u \| \to 0$$

as $n \to \infty$. Moreover, since

$$\delta(u_n) = \sum_{j=1}^{m} e^{ia_{1(n)}} e^{ia_{2(n)}} \ldots \delta(e^{ia_{j(n)}}) \ldots\ldots e^{ia_{m(n)}}$$

and

$$\delta(u) = \sum_{j=1}^{m} e^{ia_1} e^{ia_2} \ldots \delta(e^{ia_j}) \ldots\ldots\ldots e^{ia_m},$$

we have

$$\| \delta(u_n) - \delta(u) \| \to 0.$$

Since the identity of A is contained in B_n for all n,

u_n is a unitary element in B_n. Passivity of φ for (B_n, R, α) shows that

$$-i\,\varphi(u_n^*\delta(u_n)) \geqq 0.$$

Since we have $\varphi(u_n^*\delta(u_n)) \to \varphi(u^*\delta(u))$, we have

$$-i\,\varphi(u^*\delta(u)) \geqq 0.$$

This completes the proof. Q.E.D.

Chapter III. Extensions of invariant states.

In this chapter, we discuss the extensions of certain invariant states, and observe how these extensions characterize the C*-dynamical systems.

Attempts to extend a factor state φ on a C*-algebra B to a factor state on a larger C*-algebra A were first partially accomplished by the use of the notion of weak expectations for the GNS representation π_φ, that is, linear contractions P of A into $\pi_\varphi(B)''$ such that $P|_B = \pi_\varphi$. The eventual solutions of the attempts[22, 27] were variants of this method.

In the case when there is an action α of an amenable group G on A leaving B invariant, an analogous problem is to consider an α-invariant state φ of B which is centrally ergodic in the sense that

$$\pi_\varphi(B)'' \cap \pi_\varphi(B)' \cap u_G^{\varphi}{}' = \mathbb{C}\cdot 1,$$

where $(\pi_\varphi, u^\varphi, H_\varphi)$ is the associated covariant representation of (B, G, α), and to try to find an extension to a centrally ergodic state of A. It was shown in [4] that this can be done by the method of [1] if B is (semi)nuclear. The von Neumann algebra theory developed in [22, 27] is not sufficient to provide a general solution. A corollary from the positive answer to this problem is that if A is separable and G-central (and B is nuclear), then B is also G-central.

The purpose of this chapter is to clarify the covariant situation. In section 1, we consider the problem lifted to the C*-crossed products. Let (A, G, α) be a C*-dynamical system, and let B be an α-invariant C*-subalgebra of A. For a covariant representation (π, u, H) of B, the existence of a weak expectation $\hat{Q}$ of $A\times_{\alpha}G$ for the representation $\pi\times u$ (with respect to the subalgebra $L^1(B, G)$) is shown to be equivalent to the existence of a (covariant) completely positive contraction Q of A into $(\pi(B)\cup u_G)''$ such that $Q|_B = \pi$. Such a contraction Q will be called to be a covariant weak expectation.

In section 2, we show that, for an α-invariant state φ on B, there are bijective correpondences between covariant weak expectations Q of A into $(\pi_\varphi(B)\cup u^\varphi_G)''$, weak expectations $\hat{Q}$ of $A\times_\alpha G$ into $(\pi_\varphi(B)\cup u^\varphi_G)''$, certain α-invariant extensions of φ to A, and certain $(\alpha\otimes 1)$-invariant states of $A\otimes_{max}(\pi_\varphi(B)\cup u^\varphi_G)'$.

In section 3, we show that if there is a covariant weak expectation of A into $\pi_\varphi(B)''$ for a centrally ergodic state φ, φ can extend to a centrally ergodic state on A.

In section 4, it is observed that, if A is G-central, then Q and $\hat{Q}$ always exist (for each α-invariant state).

In section 5, we discuss G-abelianness of C*-dynamical system (A, G, α) with a compact group G. We show the equivalence of G-abelianness of A, commutativity of the fixed point algebra of A, and ergodicity of certain class of invariant states.

III.1. Covariant weak expectations.

Let (A, G, α) be a C*-dynamical system, and B be an α-invariant C*-subalgebra of A. Let (π, u, H) be a covariant representation of (B, G, α) and put

$$\mathcal{M} = (\pi(B) \cup u_G)''.$$

Definition 1.1. A *covariant weak expectation* for (π, u, H) is a completely positive contraction $Q : A \to \mathcal{M}$ such that $Q|_B = \pi$ and $Q(\alpha_t(a)) = u_t Q(a) u_t^*$ $(a \in A,\ t \in G)$.

In the above definition , if the action α is trivial, Q is a linear contraction from A into $\pi(B)''$ such that $Q|_B = \pi$. Such a linear contraction is called to be a *weak expectation* for (π, H) of a C*-subalgebra B (see [1]).

Definition 1.2. A *weak expectation* for $(\pi \times u, H)$ is a linear contraction $\hat{Q} : A \times_\alpha G \to \mathcal{M}$ such that

$$\hat{Q}(y) = (\pi \times u)(y)$$

for all y in $L^1(B, G)$.

Note that this definition is not quite covered by the definition of weak expectations in [1], since there is no reason, a priori, why it is automatically possible to embed $B \times_\alpha G$ in $A \times_\alpha G$, or to factor $\pi \times u$ through B_G. Here we denote by B_G a C*-subalgebra of $A \times_\alpha G$ generated by $L^1(B, G)$. In general, B_G is a quotient algebra of $B \times_\alpha G$; the algebras coincide if G is amenable.

Remark 1.3. If Q is a conditional expectation from A onto B with $Q \circ \alpha_t = \alpha_t \circ Q$ for all t in G, then $B \times_\alpha G$ is automatically a C*-subalgebra of $A \times_\alpha G$. Indeed, let Φ be a positive definite function on G into B^* (cf. [24, 7.6.7]). Then define a positive definite function Ψ on G into A^* by

$$\Psi(t)(a) = \Phi(t)(Q(a))$$

for all a in A. Since we can check the condition (ii) of Theorem 1.1 in [18], $B \times_\alpha G$ is a C*-subalgebra of $A \times_\alpha G$.

Roughly speaking, we can say that any covariant weak expectation is a weak expectation commuting with the group action. In the case where there exists a covariant weak expectation for some covariant representation (π, u, H) of an α-invariant C*-subalgebra B of A, the above remark might suggest that $B \times_\alpha G$ should be a C*-subalgebra of $A \times_\alpha G$. But we have the following example.

Example 1.4. Let (π, u, H) be a covariant representation of a C*-subalgebra B of A. Let (A_0, G, β) be a C*-dynamical system and B_0 be a β-invariant C*-subalgebra such that $B_0 \times_\beta G$ can not be embedded in $A_0 \times_\beta G$. Then we consider a C*-dynamical system $(A \oplus A_0, G, \alpha \oplus \beta)$, a C*-subalgebra $B \oplus B_0$, and a covariant representation (ρ, u, H) defined by $\rho(b \oplus b_0) = \pi(b)$ for all $b \oplus b_0$ in $B \oplus B_0$.

Theorem 1.5. *Let (A, G, α) be a C^*-dynamical system, and let B be an α-invariant C^*-subalgebra of A. Suppose that (π, u, H) is a covariant representation of B, and put $\mathcal{M} = (\pi(B) \cup u_G)''$. Then there is a bijective correspondece between covariant weak expectations $Q : A \to \mathcal{M}$ for (π, u, H) and weak expectations $\hat{Q} : A \times_\alpha G \to \mathcal{M}$ for $(\pi \times u, H)$. The correspondece is given by*

$$\hat{Q}(x) = \int_G Q(x(t))u_t \, dt$$

for all x in $L^1(A, G)$.

Proof. Suppose that $Q : A \to \mathcal{M}$ is a covariant weak expectation for (π, u, H). Define $\hat{Q} : L^1(A, G) \to \mathcal{M}$ by the above formula. Then we have

$$\begin{aligned} \hat{Q}(x^*) &= \int_G \Delta(t)^{-1} Q(\alpha_t(x(t^{-1})^*))u_t \, dt \\ &= \int_G \Delta(t)^{-1} u_t Q(x(t^{-1}))^* \, dt \\ &= \int_G u_t^* \, Q(x(t))^* \, dt \\ &= \hat{Q}(x)^*. \end{aligned}$$

Moreover, for y in $L^1(B, G)$, we have

$$\begin{aligned} \hat{Q}(y) &= \int_G Q(y(t))u_t \, dt \\ &= \int_G \pi(y(t))u_t \, dt \end{aligned}$$

$$= (\pi \times u)(y).$$

Let ξ be a unit vector in H. Consider the map $\Psi : G \to A^*$ defined by

$$\Psi(t)(a) = (Q(a)u_t\xi | \xi).$$

For t_i in G and a_i in A, we have

$$\sum_{i,j=1}^{n} \Psi(t_i^{-1}t_j)(\alpha_{t_i^{-1}}(a_i^* a_j))$$
$$= \sum_{i,j=1}^{n} (u^*_{t_i} Q(a_i^* a_j)u_{t_i} u^*_{t_i} u_{t_j}\xi | \xi) \geqq 0$$

by [40, IV.3.4]. Thus Ψ is positive definite. Since we have $\Psi(e)(a) = (Q(a)\xi|\xi)$, $\Psi(e)$ is a state of A. By [24, 7.6.8], there is a state ω_ξ of $A\times_\alpha G$ such that

$$\Psi(t)(a) = \omega_\xi(a\lambda_t)$$

where the same symbols are used to denote the canonical extension of ω_ξ to the multiplier algebra $M(A\times_\alpha G)$, A is embedded in $M(A\times_\alpha G)$, and λ is the unitary representation of G in $M(A\times_\alpha G)$. For $x = x^*$ in $L^1(A, G)$, we have

$$\omega_\xi(x) = \int_G \omega_\xi(x(t)\lambda_t)\, dt$$

$$= \int_G (Q(x(t))u_t\xi | \xi)\, dt$$

$$= (\hat{Q}(x)\xi | \xi).$$

Thus we obtain that

$$|(\hat{Q}(x)\xi | \xi)| \leqq \|x\|_{A\times_\alpha G}.$$

Since $\hat{Q}(x)^* = \hat{Q}(x^*) = \hat{Q}(x)$, we see that $\|\hat{Q}(x)\| \leqq \|x\|_{A\times_\alpha G}$. Hence $\hat{Q}$ extends by continuity to a bounded self-adjoint linear map, also denoted by $\hat{Q}$, of $A\times_\alpha G$ into $\mathcal{M}$ which is a contraction on the self-adjoint part. Then $\hat{Q}$ extends an ultraweakly continuous linear map, also denoted by $\hat{Q}$, of $(A\times_\alpha G)^{**}$ into $\mathcal{M}$ which is a contraction between the self-adjoint parts. Furthermore, we have $\pi\times u = \hat{Q}\circ\Phi$ where $\Phi : B\times_\alpha G \to B_G$ is the canonical $*$-homomorphism, so this identity remains valid for the ultraweakly continuous extensions. Since $\pi\times u$ is non-degenerate, $\hat{Q}(p) = I_H$, where p is the identity of B_G^{**}. So p is a projection in $(A\times_\alpha G)^{**}$. Now, if I is the identity of $(A\times_\alpha G)^{**}$, then we have

$$\begin{aligned}&\|I_H \pm \hat{Q}(I - p)\|\\ &= \|\hat{Q}(p \pm (I - p)\|\\ &\leqq \|p \pm (I - p)\| = 1.\end{aligned}$$

Hence we obtain that $\hat{Q}(I - p) = 0$. So we see that $\hat{Q}(I) = I_H$. For x in $(A\times_\alpha G)^{**}$ with $0 \leqq x \leqq I$, we have

$$\|I_H - \hat{Q}(x)\| \leqq \|I - x\| \leqq 1.$$

Since $\hat{Q}(x)$ is self-adjoint, $\hat{Q}(x) \geqq 0$. Thus $\hat{Q}$ is positive. Since $\hat{Q}(I) = I_H$, $\hat{Q}$ is a contraction on $(A\times_\alpha G)^{**}$ and hence on $A\times_\alpha G$ (see [7, 3.2.6]).

Let $\{f_i\}$ be an approximate identity for $L^1(G)$. For a in A, put $(a\otimes f_i)(t) = f_i(t)a$, so $a\otimes f_i \in L^1(A, G)$ and $a\otimes f_i \to a$ ultraweakly in $(A\times_\alpha G)^{**}$. Then we have

$$Q(a) = \lim \left(\int_G f_i(t)u_t\, dt\right)Q(a) = \lim \hat{Q}(a\otimes f_i) = \hat{Q}(a),$$

where the limit is taken in the ultraweak topology.

Conversely, let $\hat{Q} : A\times_\alpha G \to \mathcal{M}$ be a weak expectation for $(\pi\times u, H)$. Then $\hat{Q}$ extends to an ultraweakly continuous mapping, also denoted by $\hat{Q}$, of $(A\times_\alpha G)^{**}$ into $\mathcal{M}$. Furthermore, the kernel of Φ is contained in the kernel of $\pi \times u$, so there is a representation ρ of B_G such that $\pi \times u = \rho \circ \Phi$ and $\hat{Q}$ is a weak expectation for ρ in the sense of [1]. By [1, 2.1], $\hat{Q}$ is completely positive, and satisfies the module property:

$$\hat{Q}(y_1 x y_2) = \rho(y_1)\hat{Q}(x)\rho(y_2)$$

for y_1, y_2 in B_G^{**} and x in $(A\times_\alpha G)^{**}$. Identifying A with its image in $M(A\times_\alpha G)$, put $Q = \hat{Q}|_A$. Then Q is a completely positive contraction of A into $\mathcal{M}$. Moreover, we have

$$Q(b) = \hat{Q}(b) = \rho(b) = \pi(b)$$

for b in B and

$$Q(\alpha_t(a)) = \hat{Q}(\lambda_t a \lambda_t^*) = \rho(\lambda_t)\hat{Q}(a)\rho(\lambda_t^*) = u_t Q(a) u_t^*$$

for a in A. Thus Q is a covariant weak expectation. For x in $L^1(A, G)$, since we have $x = \int_G x(t)\lambda_t \, dt$ where the integral is ultraweakly convergent in $(A\times_\alpha G)^{**}$, we have

$$\begin{aligned} \hat{Q}(x) &= \int_G \hat{Q}(x(t)\lambda_t)\, dt \\ &= \int_G \hat{Q}(x(t))\rho(\lambda_t)\, dt \\ &= \int_G Q(x(t))u_t\, dt. \end{aligned}$$

This establishes the bijective correspondence. We complete the proof. Q.E.D.

Remark 1.6. From the proof of Theorem 1.5, we see that a covariant weak expectation Q satisfies the module property

$$Q(b_1 a b_2) = \pi(b_1)Q(a)\pi(b_2)$$

for $a \in A$ and $b_1, b_2 \in B$. This may also be deduced from Stinespring's Theorem [23, or 38] for any completely positive mapping $Q : A \to \mathcal{M}$ such that $Q|_B = \pi$.

Remark 1.7. There is a standard argument to show that any linear contraction $Q : A \to \mathcal{M}$, such that $Q|_B = \pi$, is positive. Indeed, let p be the identity of B^{**}, so that p is a projection in A^{**}. Now Q extends to an ultraweakly continuous linear contraction, also denoted by Q, of A^{**} into $\mathcal{M}$, whose restriction to B^{**} is the normal extension of π. Since π is non-degenerate, we see that $Q(p) = I_H$. Let ω be any normal state of $\mathcal{M}$. Then we have $\|\omega \circ Q\| = 1$. Hence, we have $(\omega \circ Q)(p) = 1 = \|p\|$. Thus $\omega \circ Q$ is positive by [34, 1.5.2]. Hence Q is positive.

Moreover, Q is completely positive if it satisfies any one of the following additional properties:

(i) Q is a complete contraction,

(ii) Q maps A into $\pi(B)''$ (see [1, 2.1]),

(iii) Q is covariant, and for t_i in G and a_i in A,

$$\sum_{i,j=1}^{n} u^*_{t_i} Q(a^*_i a_j) u_{t_j} \geq 0$$

(see the proof of Theorem 1.5).

Example 1.8. In general, Q may not be completely positive, even if it is covariant. For example, let A be the C*-algebra M_2 of 2×2 complex matrices, B be the subalgebra of diagonal matrices in 2×2 complex matrices, $G = \{0, 1\}$, $\alpha_1 = \mathrm{Ad}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, π be the identity representation of B on $\mathbb{C}^2$, $u_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and Q be the transpose map.

Example 1.9. A covariant weak expectation Q may fail to map A into $\pi(B)''$. For example, let $A = M_2 \otimes M_2$, $B = M_2 \otimes I_2$, $G = U(2)$, $\alpha_t = \mathrm{Ad}\,(t \otimes \bar{t})$, $H = \mathbb{C}^2 \otimes \mathbb{C}^2$, $\pi(b \otimes I_2) = b \otimes I_2$ $(b \in M_2)$, $u_t = t \otimes \bar{t}$. Then (π, u, H) is a covariant representation of (B, G, α) with u-invariant cyclic vector $2^{-\frac{1}{2}}((1,0) \otimes (1,0) + (0,1) \otimes (0,1))$, and $\pi(B)'' = \pi(B) = M_2 \otimes I_2$, $\mathcal{M} = M_2 \otimes M_2$. The identity representation $Q = \rho$ of A is a covariant weak expectation , mapping A onto $\mathcal{M}$. Here $\hat{Q}$ is just $\rho \times u$.

Remark 1.10. Suppose that G is amenable, and let m be an invariant mean on $L^\infty(G)$. Suppose that there is a completely positive contraction $P ; A \to \mathcal{M}$ such that $P|_B = \pi$. Then there is a covariant weak expectation $Q : A \to \mathcal{M}$ given by

$$(Q(a)\xi |\ \eta) = m(t \to (u_t^* P(\alpha_t(a))u_t \xi |\ \eta))$$

for ξ and η in H. In particular, if there is an injective von Neumann algebra $\mathcal{N}$ such that $\pi(B)'' \subset \mathcal{N} \subset \mathcal{M}$, then there is a weak expectation $\hat{Q} : A \times_\alpha G \to \mathcal{M}$. If B is nuclear, one may take $\mathcal{N} = \pi(B)''$ or $\mathcal{N} = \mathcal{M}$ since $B \times_\alpha G$ is nuclear (see [12, Proposition 14]). If B is seminuclear (cf. [20]), there is a weak expectation $P : A \to \pi(B)''$ and hence a covariant weak expectation $Q : A \to \mathcal{M}$.

III. 2. Applications to invariant states.

Let (A, G, α) be a C*-dynamical system, and let B be an α-invariant C*-subalgebra of A. For an α-invariant state φ, we denote by $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ the GNS representation of φ. Put $\mathcal{M}_\varphi = (\pi_\varphi \times u^\varphi)(B \times_\alpha G)''$. In this section, we establish bijective correspondences between covariant weak expectations of A into $\mathcal{M}_\varphi$, weak expectations of $A \times_\alpha G$ into $\mathcal{M}_\varphi$, certain α-invariant extensions of φ to A, and certain $(\alpha \otimes 1)$-invariant states of $A \otimes_{max} \mathcal{M}_\varphi'$.

Recall that there is an affine homeomorphism between α-invariant states φ of B and states $\tilde{\varphi}$ of $B \times_\alpha G$ with $\tilde{\varphi}(\lambda_t) = 1$ for all t in G, given by

$$\tilde{\varphi}(y) = \int_G \varphi(y(t))\, dt$$

for y in $L^1(B, G)$ (see, for example, [2, 4.1]). The GNS representation of $\tilde{\varphi}$ is $(\pi_\varphi \times u^\varphi, H_\varphi)$.

Theorem 2.1. *Let* (A, G, α) *be a* C^**-dynamical system, and let* φ *be an* α*-invariant state of an* α*-invariant* C^**-subalgebra* B *with associated covariant representation* $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ *of* (B, G, α). *We denote by* $\mathcal{M}_\varphi$ *the von Neumann algebra generated by* $\pi_\varphi(B) \cup u^\varphi_G$. *Then There are bijective correspondences between:*

(i) $(\alpha \otimes 1)$*-invariant states* ω *of* $A \otimes_{\max} \mathcal{M}'_\varphi$ *such that*

$$(*) \qquad \omega(b \otimes d) = (\pi_\varphi(b) d\xi_\varphi \mid \xi_\varphi)$$

for b *in* B *and* d *in* $\mathcal{M}'_\varphi$,

(ii) *covariant weak expectations* $Q : A \to \mathcal{M}_\varphi$ *for* $(\pi_\varphi, u^\varphi, H_\varphi)$

(iii) α*-invariant states* ψ *of* A *such that* $\psi|_B = \varphi$ *and* $E_\psi \pi_\psi(A) E_\psi \subset \mathcal{M}_\varphi$, *where* E_ψ *is the projection of* H_ψ *onto* H_φ,

(iv) *weak expectations* $\hat{Q} : A \times_\alpha G \to \mathcal{M}_\varphi$ *for* $(\pi_\varphi \times u^\varphi, H_\varphi)$,

(v) *states* $\tilde{\omega}$ *of* $(A \times_\alpha G) \otimes_{\max} \mathcal{M}'_\varphi$ *such that*

$$(**) \qquad \tilde{\omega}(x \otimes d) = \int_G (\pi_\varphi(x(t)) d\xi_\varphi \mid \xi_\varphi)\, dt$$

for x *in* $L^1(B, G)$,

(vi) *states* $\tilde{\psi}$ *of* $A \times_\alpha G$ *such that* $\tilde{\psi} \circ \Phi = \tilde{\varphi}$ *and* $E_{\tilde{\psi}} \pi_{\tilde{\psi}}(A \times_\alpha G) E_{\tilde{\psi}} \subset \mathcal{M}_\varphi$, *where* Φ *is the *-homomorphism of* $B \times_\alpha G$ *onto* B_G.

Proof. The proof of [1, 2.3] shows that there is a correspondence between states ω of $A \otimes_{max} \mathcal{M}'_\varphi$ satisfying (*) and completely positive contraction $Q : A \to \mathcal{M}_\varphi$ such that $Q|_B = \pi_\varphi$. (The proof in [1] did not use the assumption that the C*-subalgebra D is ultraweakly dense in $\pi_\varphi(B)'$ except to show that $Q(A) \subset \pi_\varphi(B)'$ $(=D'')$. Now taking $D = D''$ $= \mathcal{M}'_\varphi$, the same proof gives the present result.) Furthermore, Q is covariant

$$\Longleftrightarrow \quad (Q(\alpha_t(a))\pi_\varphi(b_1)d\xi_\varphi|\pi_\varphi(b_2)\xi_\varphi) = (u_tQ(a)u_t{}^*\pi_\varphi(b_1)d\xi_\varphi|\ \pi_\varphi(b_2)\xi_\varphi)$$
$$(a \in A;\ \ b_1,\ b_2 \in B;\ t \in G;\ \ d \in \mathcal{M}'_\varphi)$$

$$\Longleftrightarrow \quad (Q(b_2^*\alpha_t(a)b_1)d\xi_\varphi|\xi_\varphi) = (Q(\alpha_{t^{-1}}(b_2^*)a\alpha_{t^{-1}}(b_1))u_t{}^*du_t\,\xi_\varphi|\xi_\varphi)$$
$$(a \in A;\ b_1,\ b_2 \in B;\ t \in G;\ d \in \mathcal{M}'_\varphi)$$

$$\Longleftrightarrow \quad \omega(b_2^*\alpha_t(a)b_1 \otimes d) = \omega(\alpha_{t^{-1}}(b_2^*)a\alpha_{t^{-1}}(b_1) \otimes d)$$
$$(a \in A;\ b_1,\ b_2 \in B;\ t \in G;\ d \in \mathcal{M}'_\varphi)$$

$$\Longleftrightarrow \quad \omega(\alpha_t(a) \otimes d) = \omega(a \otimes d) \quad (a \in A;\ t \in G;\ d \in \mathcal{M}'_\varphi)$$

$$\Longleftrightarrow \quad \omega \text{ is } (\alpha \otimes 1)\text{-invariant.}$$

This establishes the correspondence between (i) and (ii).

It was also shown in [1, 2.3] that the restriction map of the state space of $A \otimes_{max} \mathcal{M}'_\varphi$ into the state space of A gives an affine homeomorphism between states ω satisfying (*) and states ψ of A with $\psi|_B = \varphi$ and $E_\psi\pi_\psi(A)E_\psi \subset \mathcal{M}_\varphi$. Clearly, if ω is $(\alpha \otimes 1)$-invariant, then ψ is α-invariant. On the other hand, if ψ is α-invariant, then it follows, for example by the uniqueness of ω, that ω is $(\alpha \otimes 1)$-invariant.

This establishes the correspondence between (i) and (iii).

The correspondence between (ii) and (iv) is immediate from Theorem 1.5, while the correspondences between (iv), (v) and (vi) again follow from [1]. One merely has to observe that the condition (**) is equivalent to the requirement that

$$\tilde{\omega}(x \otimes d) = ((\pi_\varphi \times u^\varphi)(x)d\xi_\varphi \mid \xi_\varphi),$$

and that if $\tilde{\omega}$ exists, then we have $\tilde{\omega}(y \otimes 1) = \tilde{\varphi}(y)$ for y in $L^1(B, G)$. Thus $\tilde{\varphi}$ factors through B_G. Hence $\pi_\varphi \times u^\varphi$ induces a representation ρ_φ of B_G and the weak expectations $\hat{Q}$ for $(\pi_\varphi \times u^\varphi, H_\varphi)$ correspond to the weak expectations for the representation $(\rho_\varphi, H_\varphi)$ of the C*-subalgebra B_G. Q.E.D.

Remark 2.2. The correspondences of Theorem 2.1 are all affine homeomorphisms in the weak* and point-ultraweak topologies.

Remark 2.3. The correspondence between (iii) and (vi) is the canonical correspondence between α-invariant states ψ of A and states $\tilde{\psi}$ of $A \times_\alpha G$ with $\tilde{\psi}(\lambda_t) = 1$ for t in G.

III.3. Extensions of centrally ergodic states.

It is useful to apply the notion of weak expectations to the extension problem of a state on a C*-subalgebra. But for a given C*-dynamical system (A, G, α), the usual weak expectation is not necessarily useful when we attempt to extend an α-invariant state on an α-invariant C*-subalgebra to A preserving the α-invariance. Fortunately this is possible by applying the covariant weak expectations. In this section, we discuss an extension of a centrally ergodic state on an α-invariant C*-subalgebra assuming the existence of a covariant weak expectation.

Here, recall that an α-invariant state φ on A is called to be centrally ergodic if

$$\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap u_G^{\varphi\,\prime} = \mathbb{C}\cdot 1,$$

where $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ denotes the cyclic covariant representation associated with φ.

Let (A, G, α) be a C*-dynamical system, and let φ be a centrally ergodic state. When A is not unital, we denote by $\tilde{A}$ the unital C*-algebra obtained by the adjunction of 1 to A. Moreover, we denote by $\tilde{\alpha}$ and $\tilde{\varphi}$ the canonical extensions of α and φ to $\tilde{A}$ respectively. Then we remark that φ is centrally ergodic for α if and only if $\tilde{\varphi}$ is centrally ergodic for $\tilde{\alpha}$.

Theorem 3.1. *Let (A, G, α) be a C^*-dynamical system. Let φ be a centrally ergodic state on an α-invariant C^*-subalgebra B of A. If there exists a covariant weak expectation Q from A into $\pi_\varphi(B)''$, then φ is extended to a centrally ergodic state on A.*

Proof. By the above remark, we may assume that A and B have the common identity.

Now put $\varphi_Q(x) = (Q(x)\xi_\varphi | \xi_\varphi)$ for x in A. Then φ_Q is an α-invariant state of A and $\varphi_Q(x) = \varphi(x)$ for all x in B. We denote by W the set of all covariant weak expectations from A into $\pi_\varphi(B)''$ and put $S = \{ \varphi_P \mid P \in W \}$. Then since W is non-empty and compact convex in the point-ultra weak topology, S is weak* compact convex in A^*. Let ψ be an extremal point of S. Then there exists an element Q in W corresponding to ψ with $\psi = \varphi_Q$. We show that φ_Q is centrally ergodic on A. Let (π_Q, u^Q, H_Q, ξ_Q) be the GNS representation of φ_Q. Since $\varphi_Q|_B = \varphi$, we can consider H_φ as a closed subspace of H_Q and we denote by E the projection of H_Q onto H_φ. For $x \in A$, $y, z \in B$, we have

$$((E\pi_Q(x)E)\pi_Q(y)\xi_Q | \pi_Q(z)\xi_Q)$$

$$= (\pi_Q(x)E\pi_Q(y)\xi_Q | E\pi_Q(z)\xi_Q)$$

$$= (\pi_Q(x)\pi_Q(y)\xi_Q | \pi_Q(z)\xi_Q)$$

$$= \varphi_Q(z^*xy)$$

$$= (Q(z^*xy)\xi_Q | \xi_Q)$$

$$= (Q(x)\pi_\varphi(y)\xi_\varphi | \pi_\varphi(z)\xi_\varphi)$$

$$= (Q(x)\pi_Q(y)\xi_Q | \pi_Q(z)\xi_Q) .$$

This means that

$$(*) \qquad E\pi_Q(x)E = Q(x)$$

for all x in A.

Suppose that p is a projection in $\pi_Q(A)'' \cap \pi_Q(A)' \cap u_G^{Q}{}'$. Then we have $EpE \in \pi_\varphi(B)''$ by $(*)$. Now we must show that

$$EpE \in \pi_\varphi(B)'' \cap \pi_\varphi(B)' \cap u_G^{\varphi}{}' = \mathbb{C}\cdot 1$$

Since B is α-invariant, u_t^Q leaves $\pi_Q(B)\xi_Q$ invariant. For x and y in B, we have

$$((Eu_t^Q pE)\pi_Q(x)\xi_Q | \pi_Q(y)\xi_Q)$$

$$= (pE\pi_Q(x)\xi_Q | u_t^{Q*}\pi_Q(y)\xi_Q)$$

$$= (pE\pi_Q(x)\xi_Q | Eu_t^{Q*}\pi_Q(y)\xi_Q)$$

$$= (Eu_t^Q EpE\pi_Q(x)\xi_Q | \pi_Q(y)\xi_Q).$$

Hence we obtain

$$(**) \qquad Eu_t^Q pE = Eu_t^Q EpE.$$

For x in B, since we have

$$u_t^Q \pi_Q(x)\xi_Q = \pi_Q(\alpha_t(x))\xi_Q$$

$$= \pi_\varphi(\alpha_t(x))\xi_\varphi$$

$$= u_t^\varphi \pi_\varphi(x)\xi_\varphi$$

$$= u_t^{\varphi}\pi_Q(x)\xi_Q ,$$

we see that

(***) $$u_t^Q E = E u_t^{\varphi} E = u_t^{\varphi} E.$$

Thus we have by (**) and (***)

$$\begin{aligned}(EpE)u_t^{\varphi} &= Ep(Eu_t^{\varphi}E) = Epu_t^Q E = Eu_t^Q pE \\ &= Eu_t^Q(EpE) = Eu_t^Q E(EpE) = (u_t^{\varphi}E)(EpE) \\ &= u_t^{\varphi}(EpE).\end{aligned}$$

Therefore, we see $EpE \in u_G^{\varphi}{}'$. For x in B, we have

$$\begin{aligned}(EpE)(E\pi_Q(x)E) &= Ep\pi_Q(x)E = E\pi_Q(x)pE \\ &= (E\pi_Q(x)E)(EpE).\end{aligned}$$

Since $E\pi_Q(x)E = \pi_{\varphi}(x)$ for all x in B, we see that EpE belongs to $\pi_{\varphi}(B)'' \cap \pi_{\varphi}(B)'$. Thus we have

$$EpE \in \pi_{\varphi}(B)'' \cap \pi_{\varphi}(B)' \cap u_G^{\varphi}{}' .$$

Let $EpE = \lambda\cdot 1$ for $0 \leqq \lambda \leqq 1$. Define a positive linear functional ω on A by

$$\omega(x) = (p\pi_Q(x)\xi_Q|\xi_Q).$$

Since $\omega(1) = (p\xi_Q|\xi_Q) = (EpE\xi_Q|\xi_Q) = \lambda$ and φ_Q majorizes ω, $\lambda = 0$ or $\lambda = 1$ implies $\omega = 0$ or $\varphi_Q = \omega$, that is, $p = 0$ or $p = 1$.

Suppose that λ is neither 0 nor 1. Define positive linear maps Q_1, Q_2 from A into $\pi_{\varphi}(B)''$ by

$$Q_1(x) = \frac{1}{\lambda}\, Ep\pi_Q(x)E$$

and

$$Q_2(x) = \frac{1}{1-\lambda} E(1-p)\pi_Q(x)E$$

for x in A. Then , for x in B, we have

$$Q_1(x) = \frac{1}{\lambda} Ep\pi_Q(x)E = E\pi_Q(x)E = Q(x).$$

Moreover, since u_t^φ leaves $\pi_Q(B)\xi_Q$ invariant, we have

$$((Eu_t^Q p\pi_Q(x)u_t^{Q*}E)\pi_Q(y)\xi_Q | \pi_Q(z)\xi_Q)$$

$$= (p\pi_Q(x)u_t^{Q*}E\pi_Q(y)\xi_Q | u_t^{Q*}E\pi_Q(z)\xi_Q)$$

$$= (p\pi_Q(x)u_t^{Q*}E\pi_Q(y)\xi_Q | Eu_t^{Q*}E\pi_Q(z)\xi_Q)$$

$$= ((Eu_t^Q Ep\pi_Q(x)u_t^{Q*}E)\pi_Q(y)\xi_Q | \pi_Q(z)\xi_Q).$$

Thus we obtain

$$Eu_t^Q p\pi_Q(x)u_t^{Q*}E = Eu_t^Q Ep\pi_Q(x)u_t^{Q*}E$$

for all x in A. Hence, we have by using (***)

$$\begin{aligned} Q_1(\alpha_t(x)) &= \frac{1}{\lambda} Ep\pi_Q(\alpha_t(x))E \\ &= \frac{1}{\lambda} Epu_t^Q\pi_Q(x)u_t^{Q*}E \\ &= \frac{1}{\lambda} Eu_t^Q p\pi_Q(x)u_t^{Q*}E \\ &= \frac{1}{\lambda} Eu_t^Q Ep\pi_Q(x)u_t^{Q*}E \\ &= \frac{1}{\lambda} u_t^\varphi Ep\pi_Q(x)Eu_t^{\varphi*} \\ &= u_t^\varphi Q_1(x)u_t^{\varphi*}. \end{aligned}$$

Therefore we know $Q_1 \in W$. Since a simple observation shows $E(1 - p)E = (1 - \lambda)\cdot 1$, similarly we see $Q_2 \in W$. It is easy to check that $\varphi_Q = \lambda \varphi_{Q_1} + (1 - \lambda)\varphi_{Q_2}$. Since λ is neither 0 or 1, the extremality of φ_Q in S shows $\varphi_Q = \varphi_{Q_1} = \varphi_{Q_2}$. Since $\varphi_{Q_1} = \frac{1}{\lambda}\omega$, we have $p = \lambda\cdot 1$. Therefore we have $\lambda = 1$ or $\lambda = 0$, which is a contradiction. Thus we obtain $p = 1$ or $p = 0$.

Q.E.D.

Corollary 3.2. *Let (A, G, α) be a C^*-dynamical system, where G is amenable. If an α-invariant C^*-subalgebra B of A is nuclear, then any centrally ergodic state on B is extended to a centrally ergodic state on A.*

III.4. G-centrality of C*-dynamical systems.

Let (A, G, α) be a C*-dynamical system, and let B be an α-invariant C*-subalgebra of A. In this section, we apply Theorem 2.1 to the question whether B is G-central, assuming that A is G-central and G is amenable.

Recall that an α-invariant state ψ of A is said to be *G-abelian* if, for each a, b in A and u^{ψ}-invariant vector η in H_{ψ},

$$\inf |(\pi_{\psi}(a'b - ba')\eta|\eta)| = 0$$

where the infimum is taken over all a' in the convex hull of $\{\alpha_t(a) | t \in G\}$. Moreover, A is said to be *G-abelian* if every α-invariant state ψ is G-abelian; equivalently, for each ψ, $\mathcal{M}_{\psi}'$ $(= \pi_{\psi}(A)' \cap u_G^{\psi}{}')$ is abelian; equivalently, the α-invariant states of A form a Choquet simplex (cf. [7, 4.3.11]).

Proposition 4.1. *Suppose that G is amenable, and A is G-abelian. For each α-invariant state φ of B, there is a covariant weak expectation for $(\pi_{\varphi}, u^{\varphi}, H_{\varphi})$.*

Proof. The first step is to note that B is G-abelian. This is well known, but for completeness we give the proof. We have to show that for each α-invariant state φ, and a, b in B,

$$(*) \quad \inf_{a'} | \varphi(a'b - ba')| = 0.$$

Since G is amenable, there is an α-invariant state ψ of A extending φ, and then (*) follows from the G-abelianness of ψ.

Now $\mathcal{M}_{\varphi}'$ $(= \pi\ (B)' \cap u_G')$ is abelian, so $\mathcal{M}_{\varphi}$ is injective by [26, 10.15]. Hence the existence of a weak expectation $Q : A\times_{\alpha}G \to \mathcal{M}_{\varphi}$ follows, since B_G is isomorphic to $B\times_{\alpha}G$. Q.E.D.

Recall also that an α-invariant state ψ of A is said to be *G-central* if each a, b in A, u^{ψ}-invariant vector η in H_{ψ}, and x in $\pi_{\psi}(A)'$,

$$\inf |(\pi_{\psi}(a'b - ba')x\eta \mid \eta)| = 0$$

where the infmum is taken over all a' in the convex hull of $\{\alpha_t(a) \mid t \in G\}$. Moreover, A is said to be *G-central* if each α-invariant state ψ is G-central; equivalently, A is G-central if $\pi_{\psi}(A)' \cap u_G^{\psi\,\prime} \subset \pi_{\psi}(A)''$ for each α-invariant state ψ; equivalently, the α-invariant states form a Choquet simplex whose boundary measures are subcentral (e.g., [7, 4.3.14]).

In [4], attention was given to the question whether B is G-central, assuming that A is G-central and G is amenable. In separable cases, it is enough to show that every centrally ergodic state φ of B is compressible in A (that is, there is a weak expectation $P : A \to \pi_{\varphi}(B)''$ for π_{φ}). Proposition 4.1 shows that there exist covariant weak expectations $Q : A \to \mathcal{M}_{\varphi}$, but in general there is no reason to suppose that φ is compressible.

One non-amenable instance when the existence of Q implies the existence of P is described in the following result.

Proposition 4.2. *Let G be the unitary group of the C^*-subalgebra $\tilde{B}$ spanned by B and the identity of A (adjointed to A if necessary), and let α be the inner action of G on A. Let φ be an α-invariant trace state of B. Then any covariant weak expectation $Q : A \to \mathcal{M}_{\varphi}$ maps A into $\pi_{\varphi}(B)''$. Conversely, any weak expectation $P : A \to \pi_{\varphi}(B)''$ is covariant.*

Proof. It is possible to prove the first statement directly, but we give an alternative proof using the correspondences developed above. Let ψ be the α-invariant state of A corresponding to Q given by Theorem 2.1. The α-invariance means that ψ is B-central ($\psi(ab) = \psi(ba)$ for a in A and b in B), and by [1, 3.1], ψ corresponds to a weak expectation $P : A \to \pi_{\varphi}(B)''$. Since the correspondences are the same and one-to-one, we conclude that $P = Q$.

Conversely, the covariance of P follows from the identity:

$$P(\alpha_v(a)) = P(vav^*) = \pi_{\varphi}(v)P(a)\pi_{\varphi}(v^*) = u_v^{\varphi}P(a)u_v^{\varphi}{}^*$$

for a in A and any unitary element v in $\tilde{B}$. Q.E.D.

III.5. G-abelianness of C*-dynamical systems by compact actions.

In this section, we characterize G-abelian C*-dynamical systems by compact actions in terms of extensions of certain class of invariant states.

Proposition 5.1. *Let (A, G, α) be a C*-dynamical system. Let φ be an α-invariant state of an α-invariant C*-subalgebra B of A such that*

$$(\pi_\varphi(B) \cup u_G^\varphi)'' \cap (\pi_\varphi(B) \cup u_G^\varphi)' = \mathbb{C}\cdot 1.$$

Suppose that there is a covariant weak expectation P for for $(\pi_\varphi, u^\varphi, H_\varphi)$. Then there exists an α-invariant state ψ of A such that

$$(\pi_\psi(A) \cup u_G^\psi)'' \cap (\pi_\psi(A) \cup u_G^\psi)' = \mathbb{C}\cdot 1.$$

and $\psi|_B = \varphi$.

Proof. Put

$$\tilde{\varphi}(x) = ((\pi_\varphi \times u^\varphi)(x)\,\xi_\varphi \mid \xi_\varphi)$$

for x in $L^1(B, G)$. Then $\tilde{\varphi}$ is a factor state of B_G. On the other hand, P induces a weak expectation $\hat{P}$ from $A\times_\alpha G$ into $(\pi_\varphi(B) \cup u_G^\varphi)''$. Therefore, there is a weak expectation $\hat{Q}$ from $A\times_\alpha G$ into $(\pi_\varphi(B) \cup u_G^\varphi)''$ such that $\tilde{\psi}(x) = (\hat{Q}(x)\xi_\varphi \mid \xi_\varphi)$ is a factor state of $A\times_\alpha G$ (see [1, or 41]). Then there exists a covariant weak expectation Q from A into $(\pi_\varphi(B) \cup u_G^\varphi)''$ corresponding to $\hat{Q}$ by Theorem 1.5. Put

$$\psi(a) = (Q(a)\xi_\varphi \mid \xi_\varphi)$$

for a in A. Then ψ is α-invariant and $\psi|_B = \varphi$. Moreover, for x in $L^1(A, G)$, we have

$$
\begin{aligned}
&((\pi_\psi \times u^\psi)(x)\xi_\psi | \xi_\psi) \\
&= \int_G (\pi_\psi(x(t))u_t^\psi \xi_\psi | \xi_\psi)\, dt \\
&= \int_G (\pi_\psi(x(t))\xi_\psi | \xi_\psi)\, dt \\
&= \int_G (Q(x(t))\xi_\varphi | \xi_\varphi)\, dt \\
&= \int_G (Q(x(t))u_t^\varphi \xi_\varphi | \xi_\varphi)\, dt \\
&= (\widehat{Q}(x)\xi_\varphi | \xi_\varphi) \\
&= \widetilde{\psi}(x).
\end{aligned}
$$

Therefore, $\pi_\psi \times u^\psi$ is a factor representation of $A \times_\alpha G$. This means $(\pi_\psi(A) \cup u_G^\psi)'' \cap (\pi_\psi(A) \cup u_G^\psi)' = \mathbb{C}\cdot 1$. Q.E.D.

Theorem 5.2. *Let (A, G, α) be a C*-dynamical system with a compact group G. Suppose that the fixed point algebra of A is non-zero. Then the following statements are equivalent.*

(i) *The fixed point algebra A^α is commutative.*

(ii) *A is G-abelian.*

(iii) *Every α-invariant state φ of A which yields $(\pi_\varphi(A) \cup u_G^\varphi)'' \cap (\pi_\varphi(A) \cup u_G^\varphi)' = \mathbb{C}\cdot 1$ is ergodic.*

Proof. (i) $\Rightarrow$ (ii). This follows from $\pi_\varphi(A)'' \cap u_G^{\varphi}{}'$ $= \pi_\varphi(A^\alpha)''$ for any α-invariant state φ on A.

(ii) $\Rightarrow$ (iii). Since A is G-abelian, $(\pi_\varphi(A) \cup u_G^\varphi)'$ is commutative. Hence, we have

$$(\pi_\varphi(A) \cup u_G^\varphi)' \subset (\pi_\varphi(A) \cup u_G^\varphi)'' \cap (\pi_\varphi(A) \cup u_G^\varphi)' = \mathbb{C}\cdot 1,$$

which implies ergodicity of φ.

(iii) $\Rightarrow$ (i). Let E be a conditinal expectation from A onto A^α given by

$$E = \int_G \alpha_t \, dt.$$

Take any factor state ψ of A^α. Then by Proposition 5.1, ψ extends to an α-invariant state φ of A such that

$$(\pi_\varphi(A) \cup u_G^\varphi)'' \cap (\pi_\varphi(A) \cup u_G^\varphi)' = \mathbb{C}\cdot 1.$$

By assumption, φ is ergodic. Since E gives a bijective correspondence between α-invariant states of A and states of A^α, $\psi = \varphi|_B$ is pure. In general, a C*-algebra is commutative if and only if every factor state is pure (cf. [42]). So, A^α is commutative. Q.E.D.

Remark 5.3. In general, we can not replace the statement (iii) by the statement that every centrally ergodic state is ergodic.

References

[1] R. J. Archbold and C. J. K. Batty; Extensions of factorial states of C*-algebras, J. Funct. Anal. 63 (1986), 86-100.

[2] C. J. K. Batty; Simplexes of states of C*-algebras, J. Operator Theory 4 (1980), 3-23.

[3] C. J. K. Batty; The KMS condition and passive states, J. Funct. Anal. 46 (1982), 246-257.

[4] C. J. K. Batty; G-central subalgebras, and extensions of KMS states, J. Funct. Anal. 66 (1986), 11-20.

[5] C. J. K. Batty and M. Kusuda; Weak expectations in C*-dynamical systems.

[6] O. Bratteli; Crossed products of UHF algebras by product type actions, Duke Math. J. 46 (1979), 1-23.

[7] O. Bratteli and D. W. Robinson; Operator algebras and quantum statistical mechanics I, Springer-Verlag, Berlin, Heidelberg, and New York, 1979.

[8] O. Bratteli and D. W. Robinson; Operator algebras and quantum statistical mechanics II, Springer-Verlag, Berlin, Heidelberg, and New York, 1981.

[9] J. De Cannière; A spectral characterization of KMS states, Comm. Math. Phys. 84 (1982), 187-206.

[10] J. Dixmier; C*-algebras, North-Holland, Amsterdam, New York, Oxford, 1982.

[11] S. Doplicher, D. Kastler, and D. W. Robinson; Covariance algebras in field theory and statistical mechanics, Comm. Math. Phys. 3 (1966), 1-28.

[12] P. Green; The local structure of twisted covariance algebras, Acta Math. 140 (1978), 191-250.

[13] G. H. Hardy, J. E. Littlewood and G. Polya; Inequalities, Cambridge University Press, Cambridge, 1934.

[14] A. Kishimoto; Ideals of C*-crossed products by locally compact abelian groups, Proc. Sympos. Pure Math. vol. 38, Part 1, Amer. Math. Soc., Providence, R. I., 1982, pp. 365-368.

[15] A. Kishimoto; C*-crossed products by R, Yokohama Math. J. 30 (1982), 151-164.

[16] M. Kusuda; Crossed products of C*-dynamical systems with ground states, Proc. Amer. Math. Soc. 89 (1983), 273-278.

[17] M. Kusuda; Passive states for C*-dynamical systems associated with commutative normal *-derivations of finite type, Math. Japon. 30, No. 6 (1985), 913-921.

[18] M. Kusuda; Hereditary C*-subalgebras of C*-crossed products.

[19] C. Lance; On nuclear C*-algebras, J. Funct. Anal. 12 (1973), 157-176.

[20] C. Lance; Tensor products and nuclear C*-algebras, Proc. Sympos. Pure Math. vol. 38, Part 1, Amer. Math. Soc., Providence, R. I., 1982, pp 379-399.

[21] A. Lenard; Thermodynamical proof of the Gibbs formula for elementary quantum systems, J. Stat. Phys. 19 (1978), 576-586.

[22] R, Longo; Solution of the factorial Stone-Weierstrass, conjecture, An application of standard W*-inclusions, Invent. Math. 76 (1984), 145-155.

[23] V. I. Paulsen; Completely bounded maps and dilations, Pitman Research Notes in Math. 146, Longman Scientific and Technical, England, 1986.

[24] G. K. Pedersen; C*-algebras and their automorphism groups, Academic Press, London, New York, 1979.

[25] G. K. Pedersen; Dynamical systems and crossed products, Proc. Sympos. Pure Math. vol. 38, Part 1, Amer. Math. Soc. , Providence, R. I., 1982, pp. 271-283.

[26] G. K. Pedersen and H. Takai; Crossed products of C*-algebras by approximately uniformly continuous actions, Math. Scand. 45 (1979), 282-288.

[27] S. Popa; Constructing semiregular maximal abelian subalgebras in factors, Invent. Math. 76 (1984), 157-161.

[28] R. T. Powers; A remark on the domain of an unbounded derivation of a C*-algebra, J. Funct. Anal. 18 (1975), 85-95.

[29] R. T. Powers and S. Sakai; Unbounded derivations in operator algebras, J. Funct. Anal. 19 (1975), 81-95.

[30] R. T. Powers and S. Sakai; Existence of ground states and KMS states for approximately inner dynamics, Comm. Math. Phys. 39 (1975), 273-288.

[31] W. Pusz and S. L. Woronowicz; Passive states and KMS states for general quantum systems, Comm. Math. Phys. 58 (1978), 273-290.

[32] W. Rudin; Fourier analysis on groups, Interscience, New York, 1962.

[33] S. Sakai; On a characterization of type I C*-algebras, Bull. Amer. Math. Soc. 72 (1966), 508-512.

[34] S. Sakai; C*-algebras and W*-algebras, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

[35] S. Sakai; On commutative normal *-derivations, Comm. Math. Phys. 43 (1975), 39-40.

[36] S. Sakai; On commutative normal *-derivations II, J. Funct. Anal. 21 (1976), 203-208.

[37] S. Sakai; On commutative normal *-derivations III, Tôhoku Math. J. 28 (1976), 583-590.

[38] W. F. Stinespring; Positive functions on C*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216.

[39] S. Strătilă, Modular theory in operator algebras, Abacus Press, England, 1981.

[40] M. Takesaki; Theory of operator algebras I, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[41] S.-K. Tsui; Factor state extension on nuclear C*-algebras, Yokohama Math. J. 29 (1981), 157-160.

[42] S. Wright; On factorial states, Rocky Mountain J. 12 (1982), 569-579.

Masaharu KUSUDA
Department of Applied Mathematics
Faculty of Engineering Science
Osaka University
Toyonaka, Osaka 560
Japan