On the large time behavior of solutions for some degenerate quasilinear parabolic systems

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Dedicated to Professor Hiroki Tanabe on his sixthieth birthday

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1. Introduction

We consider the large time behavior of the solutions for the following Cauchy problem :

$$
(1.1) \qquad \begin{aligned} u_t &= (u^m)_{xx} - v^n u^n \quad \text{in } \mathbf{R} \times (0, \infty) \\ v_t &= (v^m)_{xx} - u^n v^n \quad \text{in } \mathbf{R} \times (0, \infty) \end{aligned}
$$

with initial conditions

$$
(1.2) \qquad u(\cdot, 0) = u_0 \text{ and } v(\cdot, 0) = v_0 \text{ on } \mathbf{R}.
$$

Here, $m > 1$ and $n \geq 1$ are real numbers. Throughout this paper, we assume that $m > 1$ and $n \geq 1$.

By [10], the following properties are shown :

When the reaction arises among some reactions, for each reactant the equation for reaction-diffusion takes the form

$$
\frac{\partial C}{\partial t} = \text{ div } D \text{ grad } C + q',
$$

where C is the concentration, D is the diffusion coefficient and q' is the amount of material formed through chemical reactions per unit volume per unit time. When a reaction arises among n molecules of a substance A and n molecules of a substance B and does not reverse, that is to say, when the reaction is written as

$$
nA + nB \underset{\not\leftarrow}{\rightarrow} \text{product},
$$

then $q^{'}$ of both equations for A and B are proportional to $-C_A^n C_B^n$, where C_A and C_B are the concentrations of the substances A and B, respectively. That is to say, the concentrations C_A and C_B satisfy the equation

$$(1.3) \qquad \begin{aligned} \frac{\partial C_A}{\partial t} &= \text{div } D_A \text{ grad } C_A - kC_A^n C_B^n \\ \frac{\partial C_B}{\partial t} &= \text{div } D_B \text{ grad } C_B - kC_A^n C_B^n, \end{aligned}$$

where k is a positive constant. Here we omit the equation of the concentration of the product, since the concentration does not need to study C_A and C_B in our situation.

In this paper we consider (1.3) in case of $D_A = C_A^{m-1}$ and $D_B = C_B^{m-1}$. Then equations (1.1) are equivalent to the equations (1.3).
We make the following assumptions (A. I.) on the initial data u_0 and v_0 :

(A. I.) (1) The functions u_0 and v_0 are nonnegative and continuous on $\mathbf{R}$,

(2) they have compact support and are not identically zero on $\mathbf{R}$.

Moreover, in this paper, we assume that every function is bounded and nonnegative.

If $u_0 \equiv v_0$ on $\mathbf{R}$, the solutions u and v of (1.1) and (1.2) would coinside in $\mathbf{R} \times [0, \infty)$ and satisfy the following Cauchy problem with $p = 2n$:

$$(1.4) \qquad u_t = (u^m)_{xx} - u^p \ \text{ in } \mathbf{R} \times (0, \infty)$$

$$(1.5) \qquad u(\cdot, 0) = u_0 \ \text{ on } \mathbf{R}.$$

As for the study of the large time behavior of solution for (1.4) and (1.5), it is important to investigate the large time behavior of supports and L^∞-norms of the solutions. Therefore many authors have studied on supports and L^∞-norms of the solutions (See [1], [5], [7] - [9], [11] - [13] etc.).

The support and L^∞-norm of the solution u of (1.4) and (1.5) have the following properties :

If $1 \leq p < m$, then $\bigcup_{t \geq 0}$ supp $u(\cdot,t)$ is bounded in $\mathbf{R}$.

$-\inf\{\text{supp } u(\cdot,t)\}$, $\sup\{\text{supp } u(\cdot,t)\} \sim \log t$ if $1 < p = m$.

$-\inf\{\text{supp } u(\cdot,t)\}$, $\sup\{\text{supp } u(\cdot,t)\} \sim t^{(p-m)/(2p-2)}$

if $\max(1, p-2) < m < p$.

$-\inf\{\text{supp } u(\cdot,t)\}$, $\sup\{\text{supp } u(\cdot,t)\} \sim t^{1/(m+1)}$ if $1 < m < p-2$.

$\log(| u(\cdot,t) |_{\infty,\mathbf{R}}) \sim -t$ if $1 = p < m$.

$| u(\cdot,t) |_{\infty,\mathbf{R}} \sim t^{-1/(p-1)}$ if $\max(1,p-2) < m$ and $1 < p$.

$| u(\cdot,t) |_{\infty,\mathbf{R}} \sim t^{-1/(m+1)}$ if $1 < m < p-2$.

Here $a(t) \sim b(t)$ means that there exist two positive constants c_1 and c_2 satisfying

$$c_1 a(t) \leq b(t) \leq c_2 a(t) \quad \text{for any sufficiently large } t.$$

In this paper, for the initial data u_0 and v_0 satisfying the following assumption, we consider the solutions of (1.1) and (1.2).

$$\text{(A. II.)} \qquad u_0 \not\equiv v_0 \text{ on } \mathbf{R} \text{ and } 0 \leq u_0 \leq v_0 \text{ on } \mathbf{R}.$$

The purposes of this paper is to investigate whether the large time behavior of the solutions for the system (1.1) differences from the behavior of the solutions for the equation (1.4).

The following is our main theorem.

Theorem 1.1. Let $m > 1$ and $n \geq 1$ and suppose that u_0 and v_0 satisfy (A.I.) and (A.II.).

The initial problem (1.1) - (1.2) has a unique pair of solutions u and v

which are nonnegative. Then, the support and L^∞-norm of u and v have the following properties :

$$\cup_{t\geq 0}\text{supp } u(\cdot,t) \text{ is bounded in } \mathbf{R}, \text{if } 2n-1<m.$$

$$-\inf\{\text{supp } u(\cdot,t)\}, \sup\{\text{supp } u(\cdot,t)\} \sim \log t \text{ if } 2n-1=m.$$

$$-\inf\{\text{supp } u(\cdot,t)\},\ \sup\{\text{supp } u(\cdot,t)\} \sim t^{(1/2)\{1-(\frac{m-1}{n-1})(1-\frac{n}{m+1})\}}$$

$$\text{if } 2n-2<m<2n-1.$$

$$-\inf\{\text{supp } u(\cdot,t)\},\ \sup\{\text{supp } u(\cdot,t)\} \sim t^{1/(m+1)} \text{ if } m<2n-2.$$

$$\log |\, u(\cdot,t)\, |_{\infty,\mathbf{R}} \sim -t^{m/(m+1)} \qquad \text{if } n=1.$$

$$|\, u(\cdot,t)\, |_{\infty,\mathbf{R}} \sim t^{-\{1/(n-1)\}\{1-n/(m+1)\}} \quad \text{if } 2n-2<m \text{ and } 1<n.$$

$$|\, u(\cdot,t)\, |_{\infty,\mathbf{R}} \sim t^{-1/(m+1)} \qquad \text{if } m<2n-2.$$

In all of the above cases, the solution v satisfies

$$-\inf\{\text{supp } v(\cdot,t)\},\ \sup\{\text{supp } v(\cdot,t)\} \sim t^{1/(m+1)}$$

and

$$|\, v(\cdot,t)\, |_{\infty,\mathbf{R}} \sim t^{-1/(m+1)}.$$

Therefore, the behavior of v is independent of the behavior of u.

By Theorem 1.1 we see that the large time behaviors of u and v in our case are different from the behaviors in case of $u_0 \equiv v_0$. That is to say, the behavior of solutions for the system (1.1) is essentially different from one for the equation (1.3).
And we remark that the solutions v and u is similar to the solutions of (2.2) in Section 2 and (5.1) in Section 5, respectively (See Lemma 5.2.).

In particular, we have :

Corollary 1.2. Under the assumptions of Theorem 1.1, the supports of u and v of the system has the following properties :

If $2n-1 < m$, then $\bigcup_{0\leq t} \text{supp} \ \ u(\cdot,t)$ is bounded in $\mathbf{R}$.

If $1 < m \leq 2n-1$, then $\bigcup_{0\leq t} \text{supp} \ \ u(\cdot,t) = \mathbf{R}$.

And, for all of the above cases, $\bigcup_{0\leq t} \text{supp} \ \ v(\cdot,t) = \mathbf{R}$.

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2. Notations and definitions

Throughout this paper, we use the following notations and definitions. For any measurable subsets E of $\mathbf{R}$ or $\mathbf{R} \times [0,\infty)$, the usual norms of the spaces $L^q(E)$ for $1 \le q \le \infty$ are denoted by $| \cdot |_{q,E}$ and $C_0(E)$ is the space whose elements are compactly supported continuous functions in E. As for the other function spaces, we use the notations and the definitions in [14].

Next we shall define the solutions of (1.4) and (1.5). For $p \ge 1$ and $\mu \ge 0$, the operator $B^{(\mu)}$ is defined with the domain

$$D(B^{(\mu)}) = \{w \in L^1(\mathbf{R}) \ : \ (| u |^{m-1} u)_{xx} \in L^1(\mathbf{R})\}$$

by

$$B^{(\mu)}w = (| w |^{m-1} w)_{xx} - \mu | w |^{p-1} w \quad \text{for } w \in D(B^{(\mu)}).$$

By [2], it is shown that $B^{(\mu)}$ is m-dissipative in $L^1(\mathbf{R})$. Therefore, by [6], it is shown that a contraction semigroup $T^{(\mu)}(t)$ on $L^1(\mathbf{R})$ is defined by

$$T^{(\mu)}(t)w = \lim_{\lambda \searrow 0}(1 - \lambda B^{(\mu)})^{-[t/\lambda]}w \quad \text{for } t \ge 0 \text{ and } w \in D(B^{(\mu)}),$$

where $[\cdot]$ is the Gauss function. Then, for $w_0 \in L^1(\mathbf{R})$ we define the solutions w of (1.4) with $w(\cdot,0) = w_0$ by

$$w(\cdot,t) = T^{(1)}(t)w_0. \tag{2.1}$$

We also consider the equation :

$$z_t = (z^m)_{xx} \quad \text{in } \mathbf{R} \times (0,\infty), \tag{2.2}$$

with initial condition

$$z(\cdot,0) = z_0 \quad \text{on } \mathbf{R}. \tag{2.3}$$

For $z_0 \in L^1(\mathbf{R})$, we define the solution of (2.3) and (2.4) by

$$z(\cdot,t) = T^{(0)}(t)z_0. \tag{2.4}$$

For a positive constant M we see

$$\bar{z}(x,t;M) \equiv t^{-1/(m+1)} \left(a^2 - b_m t^{-2/(m+1)} x^2\right)_+^{1/(m-1)} \tag{2.5}$$

$$\text{for} \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

where $a = a_m M^{(m-1)/(m+1)}$ with a certain constant a_m and $b_m = (m-1)/(2m(m+1))$. We call the function (2.5) the self-similar solution or the explicit solution (of (2.2)). We can observe that the self-similar solution satisfies (2.2) in $\mathbf{R} \times (0,\infty)$ and that the corresponding initial condition is $M\delta_0$, where δ_0 is the delta function.

The Banach space X denotes $(L^1(\mathbf{R}))^2$ with the norm

$$|\,(u,v)\,|_X = |\,u\,|_{1,\mathbf{R}} + |\,v\,|_{1,\mathbf{R}} \quad \text{for} \quad (u,v) \in X.$$

We shall define an operator A with the domain

$$D(A) \;=\; \{(u,v) \in X \;:\; (|\,u\,|^{m-1}\,u)_{xx},\; (|\,v\,|^{m-1}\,v)_{xx} \in L^1(\mathbf{R})\},$$

by

$$A(u,v) = \left((|\,u\,|^{m-1}\,u)_{xx} - |\,v\,|^n |\,u\,|^{n-1}\,u, (|\,v\,|^{m-1}\,v)_{xx} - |\,u\,|^n |\,v\,|^{n-1}\,v\right)$$

By Lemma 3.3 in [16] it is shown that A is m-dissipative in X and that a contraction semigroup $S(t)$, $t \geq 0$, on X is defined by

$$S(t)V = \lim_{\lambda \to 0} (1 - \lambda A)^{-[t/\lambda]} V \quad \text{for} \quad t \geq 0 \quad \text{and} \quad V \in X.$$

Then, we define the solutions u and v of (1.1) - (1.2) by

$$(u(\cdot,t), v(\cdot,t)) = S(t)(u_0, v_0). \tag{2.6}$$

3. The existence and the uniqueness of generalized solution

In this section we shall consider the following Cauchy problem :

(3.1) $$w_t = (w^m)_{xx} - Pw^q \quad \text{in} \quad \mathbf{R} \times [0, \infty)$$

(3.2) $$w(\cdot, t) = w_0 \quad \text{on} \quad \mathbf{R}$$

where $q \geq 1$, P is a function on $\mathbf{R} \times [0, \infty)$ and w_0 is a continuous function on $\mathbf{R}$.

Throughout this section, we assume $q \geq 1$.

Definition 3.1. We say that w is a generalized solution of (3.1) if w belongs to $C([0, \infty)\ ;\ L^1(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0, \infty))$, and for t_0, t_1, a.e. x_0 and a.e. x_1 such that $0 \leq t_0 < t_1$, $x_0 < x_1$, the following integral identity holds :

(3.3) $$I(u, f, E) \equiv \int_{t_0}^{t_1} \int_{x_0}^{x_1} \{w^m f_{xx} + w f_t - P w^q f\}\, dx dt - \left[\int_{x_0}^{x_1} wf\ dx\right]_{t_0}^{t_1} - \left[\int_{t_0}^{t_1} w^m f_x\ dt\right]_{x_0}^{x_1} = 0,$$

for $f \in C^{2,1}(E)$ satisfying

$$f(x_0, t) = f(x_1, t) = 0 \quad \text{for} \quad t \in [t_0, t_1],$$

where we set $E = [x_0, x_1] \times [t_0, t_1]$.

Definition 3.1 is slightly different from ones in [9] and [11] - [13]. That is to say, they have assumed that the solutions are continous in $\mathbf{R} \times [0, \infty)$, while we do not assume such a continuity.

Remark 3.2. Under (A.I), the solutions of (1.1) and (1.2) defined in Section 2 are generalized solutions. Since this is shown by the following lemma 3.3 - 3.5 and the standard argument, we omit the proof.

Lemma 3.3. Let $P \in C^{2,1}(\mathbf{R} \times [0,\infty)) \cap L^\infty(\mathbf{R} \times [0,\infty))$ and $w_0 \in C_0(\mathbf{R})$. Then there exists a unique generalized solution w of (3.1) - (3.2). Moreover, w is continous in $\mathbf{R} \times [0,\infty)$ satisfying

$$0 \le w(x,t) \le \bar{z}(x,t+1;M), \ | w_0 |_{\infty,\mathbf{R}} \ \text{in } \mathbf{R} \times [0,\infty),$$

where $\bar{z}(x,t;M)$ is the self-similar solution such that

$$w_0(\cdot) \le \bar{z}(\cdot,1;M) \quad \text{on } \mathbf{R}.$$

Proof. Let $K =| w_0 |_{\infty,\mathbf{R}} +1$. Then we can see that there exists a sequence of smooth functions w_{0n} satisfying the following properties :

(i) $1/n < w_{0n}(x) < K$ for $x \in (-n,n)$,

(ii) $w_{0n}(\pm n) = K$,

(iii) w_{0n} is strictly monotonically decreasing with respect to n and uniformly converges to w_0 in any finite intervals as $n \to \infty$.

We shall consider the following boundary value problem of the form

$$w_t = (w^m)_{xx} - Pw^q \quad \text{in } Q_n \equiv (-n,n) \times (0,n), \tag{3.4}$$

$$w(\pm n,t) = K \quad \text{on } [0,n), \tag{3.5}$$

$$w(\cdot,0) = w_{0n} \quad \text{on } [-n,n]. \tag{3.6}$$

Due to Theorem 4.4 in [14], we see that the problem (3.4)-(3.6) has a unique classical solution $w_n \in C(\bar{Q}_n) \cap H_{loc}^{2+\alpha,1+\alpha/2}(Q_n)$ $(0 < \alpha < 1)$ satisfying

$$0 < w_n(x,t) \le K \quad \text{for } (x,t) \in \bar{Q}_n. \tag{3.7}$$

By the comparison theorem and (3.7), it follows that the sequence of the solutions w_n is monotonically decreasing with respect to n. Therefore, for $(x,t) \in \mathbf{R} \times [0,\infty)$, there exists $\lim_{n\to\infty} w_n(x,t)$. Denote w the limit. For t_0, t_1, x_0, x_1 with $0 \le t_0 < t_1$, $x_0 < x_1$ and for $f \in C^{2,1}([x_0,x_1] \times [t_0,t_1])$ with $f(x_0,t) = f(x_1,t) = 0$, w_n satisfy the integral identity

$$I(w_n,\ f,\ [x_0,x_1] \times [t_0,t_1]) \ = \ 0,$$

and hence, w satisfies the integral identity

$$I(w,\ f,\ [x_0,x_1] \times [t_0,t_1]) \ = \ 0. \tag{3.8}$$

By a similar argument to the proofs of Theorem 6 and Theorem 8 in [12], we can prove that w belongs to $C(\mathbf{R} \times [0,\infty))$. By a similar argument to the proof of Theorem 3 in [12], w satisfies moreover,

$$0 \le w(x,t) \le \bar{z}(x,t+1;M) \text{ and } 0 \le w(x,t) \le \mid w_0 \mid_{\infty,\mathbf{R}} \text{ for } (x,t) \in \mathbf{R}\times[0,\infty)$$

where $\bar{z}$ is the self-similar solution such that

$$w_0(x) \le \bar{z}(x,1\ ;\ M) \quad \text{for } x \in \mathbf{R}.$$

Therefore, w is a required generalized solution.

To prove the uniqueness we let $\tilde{w}$ be another generalized solution of (3.1) and (3.2). Set (cf. Theorem 2 in [13])

$$A_n \ = \ A_n(x,t) \ = \ \int_0^1 m\{\theta w_n + (1-\theta)\tilde{w}\}^{m-1} d\theta$$

and

$$C_n \ = \ C_n(x,t) \ = \ \int_0^1 qP\{\theta w_n + (1-\theta)\tilde{w}\}^{q-1} d\theta.$$

Let $T \in (0,n)$ and let $r \in (0,n)$ be a point where

$$I(\tilde{w},\ f,\ [-r,r] \times [0,T]) = 0,$$

holds for $f \in C^{2,1}([-r,r] \times [0,T])$ with

$$f(\pm r,t) = 0 \quad \text{for } \ t \in [0,T].$$

Then w_n and $\tilde{w}$ satisfy

(3.9) $$\left[\int_{-r}^{r}\{w_n(x,t)-\tilde{w}(x,t)\}f(x,t)dx\right]_0^T$$
$$= -\left[\int_0^T\{(w_n(x,t))^m-(\tilde{w}(x,t))^m\}f_x(x,t)dx\right]_{-r}^{r}$$
$$+\int_0^T\int_{-r}^{r}\{A_n(x,t)f_{xx}+f_t-C_n(x,t)f\}\{w_n-\tilde{w}\}dxdt.$$

By (3.7), there exist two sequences of smooth positive functions $A_{nkr}(x,t)$ and $C_{nkr}(x,t)$ with the following properties :

$$\lim_{k\to\infty} A_{nkr}(x,t)=A_n(x,t) \quad \text{a.e. in } [-r,r]\times[0,T],$$
$$\frac{1}{2}\delta_n \le A_{nkr}(x,t) \le K_{1n}$$
$$\text{for } k\ge 1 \text{ and a.e. in } [-r,r]\times[0,T],$$

$$\lim_{k\to\infty} C_{nkr}(x,t)=C_n(x,t) \quad \text{a.e. in } [-r,r]\times[0,T],$$

and

$$0\le C_{nkr}(x,t)\le K_{2n}$$
$$\text{for } k\ge 1 \text{ and a.e. in } [-r,r]\times[0,T],$$

where

$$\delta_n=\{\min_{(x,t)\in\bar{Q}_n} w_n(x,t)\}^{m-1},$$
$$K_{1n}=2m\left[\max\{|\, w_n\,|_{\infty,\mathbf{R}\times[0,\infty)},\ |\,\tilde{w}\,|_{\infty,\mathbf{R}\times[0,\infty)}\}\right]^{m-1}$$

and

$$K_{2n}=2q\left[\max\{|\, w_n\,|_{\infty,\mathbf{R}\times[0,\infty)},\ |\,\tilde{w}\,|_{\infty,\mathbf{R}\times[0,\infty)}\}\right]^{q-1}|\, P\,|_{\infty,\mathbf{R}\times[0,\infty)}\,.$$

Then the first boundary value problem

$$\begin{cases} f_t+A_{nkr}f_{xx}-C_{nkr}f=0 & \text{in } [-r,r]\times[0,T], \\ f(\cdot,T)=f_0(\cdot) & \text{on } [-r,r], \\ f(\pm r,t)=0 & \text{on } [0,T] \end{cases}$$

has a unique classical solution $f=f^{nkr}$, here f_0 is an arbitary smooth function such that

$$\text{supp } f_0\subset(-r,r) \quad\text{and}\quad |\, f_0\,|_{\infty,\mathbf{R}}\le 1.$$

Substituting the function $f = f^{nkr}$ into (3.9), we observe that

$$\left[\int_{-r}^{r} \{w_n(x,t) - \tilde{w}(x,t)\} f(x,t) dx\right]_0^T$$
$$= - \left[\int_0^T \{(w_n(x,t))^m - (\tilde{w}(x,t))^m\} f_x(x,t) dt\right]_{-r}^{r}$$
$$+ \int_0^T \int_{-r}^{r} (A_n - A_{nrk})(w_n - \tilde{w}) f_{xx} dx dt$$
$$+ \int_0^T \int_{-r}^{r} (C_{nkr} - C_n)(w_n - \tilde{w}) f dx dt.$$

Taking the limit as $k \to \infty$, $n \to \infty$ and $r \to \infty$ in this order we get by Lemma 3.6 in [12],

$$\int_{\mathbf{R}} \{w(x,T) - \tilde{w}(x,T)\} f_0(x) dx = 0,$$

which simplies $w(\cdot,T) = \tilde{w}(\cdot,T)$ a.e. on $\mathbf{R}$.

Since T is arbitrary, we conclude $w = \tilde{w}$ a.e. in $\mathbf{R} \times [0,\infty)$.

Q.E.D.

Lemma 3.4. Let $w_{i,0} (i = 1,2)$ be functions on $\mathbf{R}$ with compact support and let $w_i (i = 1,2)$ be generalized solutions for

$$w_t = (w^m)_{xx} - P_i w^q \quad \text{in } \mathbf{R} \times [0,\infty)$$

with $w_i(\cdot,0) = w_{i,0}$, where $P_i (i = 1,2)$ are functions on $\mathbf{R} \times [0,\infty)$. Then, we have

$$| w_1(\cdot,t) - w_2(\cdot,t) |_{1,\mathbf{R}} \leq | w_{1,0} - w_{2,0} |_{1,\mathbf{R}}$$
$$+ \int_0^t | P_1(\cdot,s) w_1^q(\cdot,s) - P_2(\cdot,s) w_2^q(\cdot,s) |_{1,\mathbf{R}} \, ds$$
$$\text{for } \ t \in [0,\infty).$$

The proof is given in a quite similar way as in the proof of uniqueness part in Lemma 3.3 and omitted.

Combining Lemma 3.3 with Lemma 3.4 and using approximation procedure we can prove the following.

Lemma 3.5. Let P be a function on $\mathbf{R} \times [0, \infty)$ and let w_0 be a compactly supported function on $\mathbf{R}$. Then, there exists a unique generalized solution w of (3.1) and (3.2) and w satisfies

$$0 \leq w(x,t) \leq \bar{z}(x, t+1; M), \mid w_0 \mid_{\infty, \mathbf{R}},$$

$$\text{for } t \geq 0 \text{ and a.e. } x \in \mathbf{R},$$

where $\bar{z}$ is the self-similar solution such that

$$0 \leq w_0(\cdot) \leq \bar{z}(\cdot, 1; M) \quad \text{a.e. on } \mathbf{R}.$$

4. Comparison theorems

In this section, we shall define the generalized supersolutions and the generalized subsolutions and give some comparison results.

By the same argument as in the proof of Lemma 4.1, 4.2 in [16], we can show the next lemma, the proof being omitted.

Lemma 4.1. Let u_0, v_0, $\tilde{u}_0$ and $\tilde{v}_0$ belong to $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and satisfy

$$0 \leq u_0 \leq \tilde{u}_0 \leq \tilde{v}_0 \leq v_0 \quad \text{a.e. on } \mathbf{R}.$$

Let (u, v) and $(\tilde{u}, \tilde{v})$ be two pairs of solutions of (1.1) with initial data (u_0, v_0) and $(\tilde{u}_0, \tilde{v}_0)$, respectively. Then, the functions satisfy, $t \geq 0$,

$$0 \leq u(\cdot, t) \leq \tilde{u}(\cdot, t) \leq \tilde{v}(\cdot, t) \leq v(\cdot, t) \quad \text{a.e. on } \mathbf{R}.$$

Definition 4.2. Let G be a connected open subset of $\mathbf{R} \times (0, \infty)$. A function w belonging to $C([0, \infty)\ ;\ L^1(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0, \infty))$ is called a generalized super (sub) solution of (3.1) in G, if for t_0, t_1, a.e. x_0 and a.e. x_1 such that $0 \leq t_0 < t_1$, $x_0 < x_1$ and $[x_0, x_1] \times [t_0, t_1] \subset \bar{G}$, the following integral inequality holds (see (3.3)) :

$$\begin{aligned} I(w,\ f,\ E) \quad &\leq\ 0 \\ &(\geq\ 0) \end{aligned}$$

for $f \in C^{2,1}(G)$ with $f(x_0, t) = f(x_1, t) = 0$, $t_0 \leq t \leq t_1$, where we recall $E = [x_0, x_1] \times [t_0, t_1]$.

Lemma 4.3. Let P and w_0 be functions in $L^\infty(\mathbf{R} \times [0, \infty))$ and $C_0(\mathbf{R})$, respectively. Let w be a generalized solution of (3.1) with $w(\cdot, 0) = w_0$ and let $\tilde{w}$ be a generalized super (sub) solution of (3.1). Then, if

$$\begin{aligned} w_0 \quad &\leq\ \tilde{w}(\cdot, 0) \quad \text{a.e. on } \mathbf{R}, \\ &(\geq\ \tilde{w}(\cdot, 0)) \end{aligned}$$

we have, for $t \geq 0$,

$$w(\cdot,t) \;\leq\; \tilde{w}(\cdot,t) \qquad \text{a.e. on } \mathbf{R}.$$
$$(\geq\; \tilde{w}(\cdot,t))$$

Proof. By a quite similar argument obtaining uniqueness part in Lemma 3.3 we can prove

$$\int_{\mathbf{R}} (w(x,t) - \tilde{w}(x,t)) f_0(x)\, dx \;\leq\; 0 \tag{4.1}$$

for any function $f_0(x)$, which yields the desired result. The details are omitted.

Q.E.D.

For $T > 0$, let ℓ be a smooth function in $[T, \infty)$ such that

$$\ell(T) > 0 \quad \text{and} \quad \ell'(\cdot) \geq 0 \quad \text{in } [T, \infty).$$

and let

$$G = \{(x,t) \;:\; t > T \quad \text{and} \quad x \in (-\ell(t),\; \ell(t))\}.$$

S_ℓ and $S_{-\ell}$ denote the subsets $\{(\ell(t),\; t)\;;\; t \in [0,\infty)\}$ and $\{(-\ell(t),\; t)\;;\; t \in [T,\infty)\}$ of $\mathbf{R} \times [0,\infty)$ respectively.

Lemma 4.4. Let w be a generalized solution of (3.1) with $w(\cdot,0) = w_0 \in C_0(\mathbf{R})$ and let $\tilde{w}$ be a generalized super (sub) solution of (3.1) in G that belonging to $C(\bar{G})$.

P in (3.1) satisfies that

$$(S_\ell \cup S_{-\ell}) \;\cap\; \text{supp } P \;=\; \phi,$$

Then w is Hölder continuous in some neighborhood of $S_\ell \cup S_{-\ell}$. Moreover, if

$$w(\cdot,T) \;\leq\; \tilde{w}(\cdot,T) \qquad \text{a.e. on } [-\ell(T), \ell(T)]$$
$$(\geq\; \tilde{w}(\cdot,T))$$

and if

$$\begin{array}{llll} w(\ell(t),t) & < & \tilde{w}(\ell(t),t) & \text{for } t \geq T, \\ & (> & \tilde{w}(\ell(t),t)) & \\ w(-\ell(t),t) & < & \tilde{w}(-\ell(t),t) & \text{for } t \geq T, \\ & (> & \tilde{w}(-\ell(t),t)) & \end{array}$$

then we have

$$\begin{array}{llll} w(x,t) & \leq & \tilde{w}(x,t) & \text{for } t \geq T \text{ and a.e. } x \in [-\ell(t),\ell(t)]. \\ & (\geq & \tilde{w}(x,t)) & \end{array}$$

Proof. Let an arbitrary constant $T_1 > T$ be fixed. There exists a constant $\delta = \delta(T, T_1, P) \in (0, T)$ such that

$$\left(E_\delta^{(+)}(t) \cup E_\delta^{(-)}(t)\right) \cap \text{supp } P = \phi \quad \text{for } t \in [T, T_1],$$

where, for $\delta > 0$ and $t \geq T$, $E_\delta^{(+)}(t)$ and $E_\delta^{(-)}(t)$ denote $[\ell(t) - \delta,\ \ell(t) + \delta] \times [t - \delta,\ t + \delta]$ and $[-\ell(t) - \delta,\ -\ell(t) + \delta] \times [t - \delta,\ t + \delta]$ respectively.

Then, there exists a sequence of smooth functions $P_j \in L^\infty(\mathbf{R} \times [0, \infty))$ such that

$$| P_j |_{\infty, \mathbf{R}\times[0,\infty)} \leq | P |_{\infty, \mathbf{R}\times[0,\infty)} \quad \text{for } j \geq 1,$$

$$\lim_{j \to \infty} P_j(x,t) = P(x,t) \quad \text{for a.e. } (x,t) \in \mathbf{R} \times [0, \infty)$$

and

$$\left(E_{\frac{5}{6}\delta}^{(+)}(t) \cup E_{\frac{5}{6}\delta}^{(-)}(t)\right) \cap \text{ supp } P = \phi$$

$$\text{for } j \geq 1 \quad \text{and} \quad t \in [T, T_1].$$

Let w_{0n} satisfy the properties (i) - (iii) in the proof of Lemma 3.3. For $j \geq 1$, let w_{jn} be the classical solutions of

$$w_t = (w^m)_{xx} - P_j w^q \quad \text{in } Q_n$$

with (3.5) and (3.6), where Q_n denotes $(-n, n) \times (0, n)$.

Then, since w_{jn} are positive in $\overline{Q_n}$, w_{jn} are smooth in Q_n.

Now, we shall show the uniformly Hölder continuity of the solutions w_{jn} in $E^{(+)}_{\delta/2}(t)$ and $E^{(-)}_{\delta/2}(t)$.
We shall omit $(+)$ and $(-)$ from $E^{(+)}_{\cdot}(\cdot)$ and $E^{(-)}_{\cdot}(\cdot)$, respectively.

We fix $t_0 \in [T, T_1]$ and $(x_1, t_1) \in E_{2\delta/3}(t_0)$ arbitrarily.

Let ψ_0 be a smooth function such that

$$\psi_0(x)\begin{cases} = 1 & \text{on } [-1,1] \\ \in (0,1) & \text{on } (-2,-1)\cup(1,2) \\ = 0 & \text{on } (-\infty,-2)\cup(2,\infty), \end{cases}$$

and we set

$$\psi_\delta(x,t) = \frac{t-(t_0-\frac{5}{6}\delta)}{t_1-(t_0-\frac{5}{6}\delta)}\psi_0\left(\frac{x-x_1}{\delta/24}\right)$$
$$\text{in } \mathbf{R}\times\mathbf{R}.$$

We set also

$$\phi(y) = Ny(2-y) \quad \text{for } y \geq 0,$$

with $N = (4m/(m-1))(|\, w_0\, |_{\infty,\mathbf{R}} + 1)^{m-1}$.

Let an arbitrary $j \geq 1$ and an arbitrary n satisfying $E_{5\delta/6}(t_0) \subset Q_n$ be fixed. We shall omit j and n from w_{jn} and P_j. Setting $(m/(m-1))w^{m-1} = \phi(q)$, we see $0 \leq q \leq 1/4$ and

$$q_t = (m-1)\phi q_{xx} + \left[(m-1)\phi\frac{\phi''}{\phi'} + \phi'\right](q_x)^2 - \lambda\frac{P\phi^\beta}{\phi'}, \tag{4.2}$$
$$\text{in } (-n,n)\times[t_0-\frac{5}{6}\delta,\infty)$$

with $\beta = (m+n-2)/(m-1)$ and $\lambda = m((m-1)/m)^\beta$.

We differentiate (4.2) with respect to x, multiply by $q_x\psi_\delta^2$ and consider a point (x_2, t_2) of $E_{1\delta}$ where the function $z = (q_x\psi_\delta)^2$ attains a maximum in $[-n,n]\times[t_0-(5\delta/6),\infty)$. Since we may assume $t_2 > t_0-(5\delta/6)$ without loss of generality, we observe that

$$z_t(x_2,t_2)\geq 0,\;\; z_x(x_2,t_2)=0 \;\;\text{and}\;\; z_{xx}(x_2,t_2)\leq 0.$$

Then, at such a point we have the following inequality :

$$\begin{aligned}(4.3)\qquad & \left[-m\phi'' - (m-1)\phi\left(\frac{\phi''}{\phi'}\right)'\right]\psi_\delta^2(q_x)^4 \\ \leq\; & \left[\psi_\delta\psi_{\delta t} + 3(m-1)\phi(\psi_{\delta x})^2 - (m-1)\phi\psi_\delta\psi_{\delta xx}\right](q_x)^2 \\ & - \left[(m+1)\phi' + 2(m-1)\phi\frac{\phi''}{\phi'}\right]\psi_\delta\psi_{\delta x}(q_x)^3.\end{aligned}$$

Set

$$a_1 = |\,\psi_{\delta t}\,|_{\infty,E_{\frac{5}{6}\delta}(t_0)},\;\; a_2 = |\,\psi_{\delta x}\,|_{\infty,E_{\frac{5}{6}\delta}(t_0)} \;\;\text{and}\;\; a_3 = |\,\psi_{\delta xx}\,|_{\infty,E_{\frac{5}{6}\delta}(t_0)}\,.$$

Note that

$$0\leq q\leq 1/4 \;\;\text{in}\;\; [-n,n]\times[t_0-\frac{5}{6}\delta,\infty)$$

and

$$(4.4)\qquad 0<\frac{3}{2}N\leq\phi'(q)\leq 2N,\quad \phi''(q)=-2N$$
$$\text{and}\;\; \left|\,\frac{\phi''}{\phi'}\,\right|\leq\frac{4}{3}\quad\text{in}\;\;[-n,n]\times[t_0-\tfrac{5}{6}\delta,\infty).$$

By (4.3) and (4.4), we obtain

$$\psi_\delta^2(q_x)^4\leq C_1(q_x)^2 + C_2\psi_\delta\,|\,q_x\,|^3,$$

with

$$\begin{aligned} C_1 &= \frac{1}{2Nm}(a_1 + N(m-1)a_3 + 3N(m-1)a_2^2), \\ C_2 &= \frac{a_2}{3Nm}(7m-1). \end{aligned}$$

Therefore we have

$$z(x,t)\leq |\,z\,|_{\infty,E_{\frac{2}{3}\delta}(t_0)}\leq 2\left(C_1+\frac{C_2^2}{2}\right)\;\;\text{for}\;(x,t)\in E_{\frac{2}{3}\delta(t_0)},$$

and hence,

$$(4.5)\qquad \left| \frac{m}{m-1}(w^{m-1})_x(x,t) \right|^2 \leq 8N^2\left(C_1+\frac{C_2^2}{2}\right)$$

$$\text{for } (x,t) \in E_{\frac{2}{3}\delta}(t_0).$$

By (4.5) and Theorem 8 in [13], for $j \geq 1$ and n such that $\bigcup_{t_0\in[T,T_1]} E_{5\delta/6}(t_0) \subset Q_n$, the solutions w_{jn} satisfy

$$(4.6)\qquad \langle w_{jn}\rangle^{(\alpha)}_{x,E_{\delta/2}(t_0)} + \langle w_{jn}\rangle^{(\alpha/3)}_{t,E_{\delta/2}(t_0)} \leq C_\delta, \text{ for } t_0 \in [T,T_1],$$

where $\alpha = \min(1, 1/(m-1))$ and C_δ is a positive constant depending only on $| w_0 |_{\infty,\mathbf{R}}$, $| P |_{\infty,\mathbf{R}\times[0,\infty)}$ and δ.
Set $E^{(+)} = \bigcup_{t_0\in[T,T_1]} E^{(+)}_{\delta/2}(t_0)$ and $E^{(-)} = \bigcup_{t_0\in[T,T_1]} E^{(-)}_{\delta/2}(t_0)$. By (4.6) and Ascoli - Arzelà theorem, for each $j \geq 1$, a subsequence of the solutions w_{jn} uniformly converges to w_j on $E^{(+)} \cup E^{(-)}$ as $n \to \infty$. Moreover, we obtain

$$(4.7)\qquad \langle w_j\rangle^{(\alpha)}_{x,E^{(i)}} + \langle w_j\rangle^{(\alpha/3)}_{t,E^{(i)}} \leq C_\delta \quad \text{for } i = +,-.$$

By Lemma 3.3, Lemma 3.5, Lebegue's convergence theorem and Gronwall's inequality, we have

$$(4.8)\qquad \lim_{j\to\infty} \sup_{t\in[0,T_1+\delta]} | w_j(\cdot,t) - w(\cdot,t) |_{1,\mathbf{R}} = 0.$$

By (4.7), (4.8) and Ascoli - Arzelà theorem, there exists a subsequence of the solutions w_j which uniformly converges to w on $E^{(+)} \cup E^{(-)}$.

Therefore, the generalized solution w is Hölder continuous in $E^{(+)} \cup E^{(-)}$.

Let $\tilde{w} \in C(G)$ be a generalized supersolution in G.

There exists a positive constant η which has the following property ; For any sufficiently large n and j, w_{jn} satisfy that

$$(4.9)\qquad w_{jn}(x,t) < \tilde{w}(x,t)$$

$$\text{for } t \in [T,T_1] \text{ and } | x | \in [\ell(t)-\eta,\ \ell(t)]$$

For any integer $H \geq 1$ and $h = 0, 1, 2, \cdots, H-1$, we set $t_h^{(H)} = T + (T_1 - T)h/H$ and $G_h^{(H)} = [-\ell(t_h^{(H)}),\ \ell(t_h^{(H)})] \times [t_h^{(H)}, t_{h+1}^{(H)}]$.

We fix a large H such that

$$0 \leq \ell(t_{h+1}^{(H)}) - \ell((t_h^{(H)}) \leq \eta \ \text{ for } h = 0, 1, \cdots, H-1.$$

Repeating the same argument obtaining uniqueness part in Lemma 3.3 we can prove

$$\int_{-\ell(t_0)}^{\ell(t_0)} \{w(x, t_1) - \tilde{w}(x, t_1)\} f_0(x) dx \ \leq \ 0 \tag{4.10}$$

where $f \in C^{2,1}(G_0)$ be an arbitrary function such that $f(\pm\ell(t_0),\ t) = 0$ for $t \in [t_0, t_1]$ and consequently

$$\int_{-\ell(t_0)}^{\ell(t_0)} (w(x, t_1) - \tilde{w}(x, t_1))_+ \, dx \ \leq \ 0. \tag{4.11}$$

By (4.9) and (4.11) we have

$$w(\cdot, t_1) \ \leq \ \tilde{w}(\cdot, t_1) \quad \text{a.e. on } [-\ell(t_1),\ \ell(t_1)]. \tag{4.12}$$

From (4.12), the same argument yields

$$w(\cdot, t_2) \ \leq \ \tilde{w}(\cdot, t_2) \quad \text{a.e. on } [-\ell(t_2),\ \ell(t_2)].$$

Repeating this procedure we arrive at

$$w(\cdot, T_1) \ \leq \ \tilde{w}(\cdot, T_1) \quad \text{a.e. on } [-\ell(T_1),\ \ell(T_1)].$$

Since $T_1 > T$ is arbitrary, we conclude

$$w(\cdot, t) \ \leq \ \tilde{w}(\cdot, t) \ \text{ for } t \geq T \text{ and a.e. on } [-\ell(t),\ \ell(t)].$$

Q.E.D.

Lemma 4.5. Let w_0 in (3.2) belong to $C_0(\mathbf{R})$.

Let w be a generalized solution of (3.1) and (3.2). And let $\tilde{w}$ be a generalized supersolution of (3.1) in G and be continuous and positive in $\bar{G}$. Suppose that :

$$w(\cdot,T) \ \leq \ \tilde{w}(\cdot,T) \quad \text{a.e. on } [-\ell(T),\ \ell(T)],$$
$$\text{supp } w \ \cap \ (S_\ell \cup S_{-\ell}) = \phi.$$

Then, w and $\tilde{w}$ satisfy

$$w(x,t) \ \leq \ \tilde{w}(x,t) \ \text{ for } t \geq T \ \text{ and a.e. } x \in [-\ell(t),\ell(t)].$$

The proof is given in a quite similar way as in the one of Lemma 4.4 and omitted.

Finally, we state for following.

Lemma 4.6. Let $\ell(t) \equiv \ell_0$ on $[T,\infty)$.

Let P in (3.1) and w_0 in (3.2) belong to $L^\infty(\mathbf{R} \times [0,\infty)) \cap C^{2,1}(\mathbf{R} \times [0,\infty))$ and $C_0(\mathbf{R})$ respectively.

Let w be the generalized solution of (3.1) and (3.2). And let $\tilde{w}$ be a generalized subsolution of (3.1) in G and be continuous in $\bar{G}$.

Suppose that :

$$w(\cdot,T) \ \geq \ \tilde{w}(\cdot,T) \quad \text{on } [-\ell_0,\ell_0]$$
$$w(\ell_0,t) \ \geq \ \tilde{w}(\ell_0,t) \quad \text{and } \ w(-\ell_0,t) \geq \tilde{w}(-\ell_0,t) \ \text{ for } t \geq T.$$

Then, w and $\tilde{w}$ satisfy

$$w \ \geq \ \tilde{w} \ \text{ in } \ \bar{G}.$$

The proof is standard and omitted.

5. The large time behavior of the solutions for $w_t = (w^m)_{xx} - \lambda(t+1)^{-n/(m+1)} w^n$

In this section, we consider the large time behavior of solutions for the following equation :

$$(5.1) \qquad w_t = (w^m)_{xx} - \lambda(1+t)^{-n/(m+1)} w^n \quad \text{in} \quad \mathbf{R} \times (0,\infty)$$

with initial condition

$$(5.2) \qquad w(\cdot,0) = w_0 \quad \text{on } \mathbf{R},$$

where $\lambda > 0$ and w_0 satisfies (A.I.) in Introduction.

In order to investigate the large time behavior of the generalized solution for (5.1), we shall derive an estimate of $|\ (w^{m-1})_x(\cdot,t)\ |_{\infty,\mathbf{R}}$. The following is proved similarly as in the proof of Lemma 3.1 in [8].

Lemma 5.1. Let w be the generalized solution of (5.1) - (5.2). Then, we have

$$|\ (w^{m-1})_x(\cdot,t)\ |_{\infty,\mathbf{R}} \leq\ C\left(t^{-1}\ |\ w(\cdot,t/2)\ |_{\infty,\mathbf{R}}^{m-1}\right)^{1/2} \quad \text{for} \ \ t>0,$$

where C is a positive constant independent of t and w_0.

Our main result in this section is as follows.

Lemma 5.2. Let $p_* = nm/(m+1-n)$. Let w_0 satisfy the assumptions (A.I.) and w be the generalized solution of (5.1) and (5.2). Then, the support and L^∞-norm of w have the following properties :

$\bigcup_{t\geq 0} \mathrm{supp} w(\cdot,t)$ is bounded in $\mathbf{R}$, if $2n-1<m$.

$-\inf\{\mathrm{supp} w(\cdot,t)\},\ \ \sup\{\mathrm{supp} w(\cdot,t)\} \sim \log t \quad$ if $m = 2n-1$.

$-\inf\{\mathrm{supp} w(\cdot,t)\},\ \ \sup\{\mathrm{supp} w(\cdot,t)\} \sim t^{(p_*-m)/(2p_*-2)}$
if $2n-2 < m < 2n-1$.

$-\inf\{\mathrm{supp} w(\cdot,t)\},\ \ \sup\{\mathrm{supp} w(\cdot,t)\} \sim t^{1/(m+1)}$ if $m < 2n-2$.

$$\log(|\, w(\cdot,t)\,|_{\infty,\mathbf{R}}) \sim -t^{\frac{m}{m+1}} \quad \text{if } n=1.$$
$$|\, w(\cdot,t)\,|_{\infty,\mathbf{R}} \sim t^{-1/(p_*-1)} \quad \text{if } 2n-2<m \text{ and } n>1.$$
$$|\, w(\cdot,t)\,|_{\infty,\mathbf{R}} \sim t^{-1/(m+1)} \quad \text{if } 1<m<2n-2.$$

Proof. For simplicity we assume that $w_0(0)>0$.

(I) Case : $n=1$.

Let $\bar{w}$ be a generalized solution of (2.3) with initial condition $\bar{w}(\cdot,0)=w_0$. We set for $t\geq 0$

$$\rho(t)=\exp\left(-\lambda\int_0^t(1+s)^{\frac{-1}{m+1}}ds\right)$$

and

$$\nu(t)=\int_0^t \rho(s)^{m-1}ds.$$

Then, we can observe

$$w(x,t)=\rho(t)\bar{w}(x,\nu(t)) \quad \text{for} \quad (x,t)\in\mathbf{R}\times(0,\infty),$$

and the result follows from [17].

(II) Case : $2n-1<m$ and $n>1$.

Let w^* be a generalized solution of (1.4) with $p=p_*$ and with $w^*(\cdot,0)=w_0$. In Introduction we describe the estimate of supp $w^*(\cdot,t)$ and $|\, w^*(\cdot,t)\,|_{\infty,\mathbf{R}}$, which is the required one also for $w(t)$. Suppose that λ is so large to satisfy :

(5.3) $$w^*(x,t)^{\frac{n-1}{m+1-n}}\leq\lambda^{1/n}(1+t)^{\frac{-1}{m+1}} \quad \text{for} \quad (x,t)\in\mathbf{R}\times(0,\infty).$$

For such a constant λ, since w is a generalized subsolution of (3.1) with $P=w^{*(n-1)/(m+1-n)}$ and $q=n$ we obtain by Lemma 3.3, Lemma 4.3 and (5.3) that $w^*\geq w$ in $\mathbf{R}\times[0,\infty)$. Let a and b be positive constants satisfying $a^{m-1}b^2 = 1$ and set $\tilde{w}(x,t) = aw(bx,t)$ for $(x,t) \in \mathbf{R} \times (0,\infty)$. Then $\tilde{w}$ is a generalized solution of the following equation :

$$w_t = (w^m)_{xx} - \lambda a^{1-n}(1+t)^{-n/(m+1)}w^n \quad \text{in } \mathbf{R} \times (0,\infty).$$

Therefore, we obtain the upper estimates of $\text{supp} w(\cdot,t)$ and $| w(\cdot,t) |_{\infty,\mathbf{R}}$.

Let $\tilde{h}_0 \in (0,1)$ and let $\tilde{h}$ be the solution of the following Cauchy problem :

$$\begin{cases} (\tilde{h}^m)'' = \mu(\tilde{h}^n - \tilde{h}) \quad \text{on } (0,\infty) \\ \tilde{h}(0) = \tilde{h}_0, \quad \tilde{h}'(0) = 0 \text{ , where } \mu = \lambda^{\frac{n-2}{n-1}} \left(\frac{m+1}{(n-1)(m+1-n)}\right)^{\frac{1}{n-1}} \end{cases}$$

It is easy to observe that $\hat{h}$ has a zero point, and let δ be the first zero point of $\tilde{h}$. Then, there exists an nontrivial and nonnegative solution h for

$$\begin{cases} \mu(h^n - h) = (h^m)_{xx} \quad \text{on} \quad (-\delta,\ \delta), \\ h(\delta) = h(-\delta) = 0 \end{cases}$$

such that

$$w_0 \geq h \quad \text{on} \quad (-\delta, \delta). \tag{5.4}$$

Let τ be the solution of the following Cauchy problem :

$$\begin{cases} \tau'(t) = -\lambda(1+t)^{-n/(m+1)}\tau^n \quad \text{on} \quad (0,\infty), \\ \tau(0) = \tau_0, \end{cases} \tag{5.5}$$

where $\tau_0 = \min\{1, ((m+1)/(\lambda(n-1)(m-n+1)))^{-1/(n-1)}\}$
Then, we observe that

$$(\tau h)_t \leq ((\tau h)^m)_{xx} - \lambda(1+t)^{-n/(m+1)}(\tau h)^n \text{in} (-\delta,\delta) \times (0,\infty). \tag{5.6}$$

By (5.4), (5.6) and Lemma 4.6, we get that $h\tau \leq w$ in $(-\delta,\delta) \times (0,\infty)$. Moreover, the decay rate of τ in t is equal to the one which we want to show. Therefore, we have a lower estimate of $| w(\cdot,t) |_{\infty,\mathbf{R}}$.

(III) Case : $m = 2n - 1$.

Let $c > 0$ and set $\tilde{\lambda} = \lambda(m-1)^{(m+n-2)/(m-1)}$. We consider the Cauchy Problem :

$$(5.7) \quad \begin{cases} (q^m)'' + cq' + q - \tilde{\lambda} q^n = 0 \quad \text{on} \quad [0, \eta) \\ q(0) = \tilde{\lambda}^{-1/(n-1)}, \ 0 < q < \tilde{\lambda}^{-1/(n-1)}, \ q' < 0 \quad \text{on} \quad [0, \eta), \end{cases}$$

where η is a positive constant. This problem has a solution for some $\eta > 0$ and the behavior of it is known (See [1].). Using this we can construct desired generalized supersolution and subsolutions (See [5].).

(IV) Case : $2n - 2 < m < 2n - 1$.

Set

$$(5.8) \qquad w_*(x,t) = A(1+t)^{-a}(D - x^2(1+t)^{-b})_+^{1/(m-1)} \quad \text{for } (x,t) \in \mathbf{R} \times (0,\infty)$$

where $a = 1/(p_* - 1)$, $b = (p_* - m)/(p_* - 1)$ and A and D are positive constants.

Then we obtain that

$$\begin{aligned} L(w_*) &= w_{*t} - (w_*^m)_{xx} + \lambda(1+t)^{-n/(m+1)} w_*^n \\ &= -A(m-1)^{-1} x^2 (1+t)^{-am-2b} \psi^{(-m+2)/(m-1)} \times \\ &\quad \times \left(4(m-1)^{-1} m A^{m-1} - b(1+t)^{a(m-1)+b-1}\right) \\ &\quad -A(1+t)^{-a-1} \psi^{1/(m-1)} \Big(a - 2(m-1)^{-1} m A^{m-1} (1+t)^{-a(m-1)-b+1} \\ &\quad -\lambda A^{n-1} (1+t)^{-a(n-1)+1-n/(m+1)} \psi^{(n-1)/(m-1)}\Big) \\ &\qquad \text{for } (x,t) \in \mathbf{R} \times (0,\infty). \end{aligned}$$

Since $a(m-1) + b - 1 = 0$, we have

$$(5.9) \qquad L(w_*) \leq -A(m-1)^{-1} x^2 (1+t)^{-am-2b} \psi^{(-m+2)/(m-1)} \times$$

$$\times \left(4(m-1)^{-1}mA^{m-1}-b\right) - A(1+t)^{-a-1}\psi^{1/(m-1)} \times$$
$$\times \left(a-2(m-1)^{-1}mA^{m-1}-\lambda A^{n-1}\psi^{(n-1)/(m-1)}\right)$$
$$\text{for } (x,t) \in \mathbf{R} \times (0,\infty),$$

where $\psi = \psi(x,t) = \left(D - x^2(1+t)^{-b}\right)_+$. Since $2a > b$, there exists a positive constant A such that $2a > 4m(m-1)^{-1}A^{m-1} > b$. Let such an A be fixed. Moreover, there exists a sufficiently small constant $D > 0$ such that $a - 2(m-1)^{-1}mA^{m-1} - \lambda A^{n-1}D^{(n-1)/(m-1)} > 0$ and $w_0(x) \geq A(D-x^2)_+^{1/(m-1)}$ on $\mathbf{R}$ and we shall fix such a constant D. Then we have $L(w_*) \leq 0$ in $\mathbf{R} \times (0,\infty)$. Since w_{*t}, $(w_*^m)_{xx} \in L^1(\mathbf{R} \times [0,T])$ for $T > 0$, w_* is a generalized subsolution of (5.1). Therefore, we have by Lemma 3.3 and Lemma 4.3, that $w_* \leq w$ in $\mathbf{R} \times (0,\infty)$. Thus, we obtain the lower estimates of $\operatorname{supp} w(\cdot,t)$ and $|\ w(\cdot,t)\ |_{\infty,\mathbf{R}}$.

Let w^* be a solution of (5.5) with the initial condition $w^*(\cdot,0) = |\ w_0\ |_{\infty,\mathbf{R}}$. By Lemma 3.2, there exists a smooth function ℓ in $[0,\infty)$ such that

$$\ell(0) > 0,\ \ell' \geq 0 \text{ in } [0,\infty),$$
$$\operatorname{supp} w(\cdot,t) \subset (-\ell(t),\ell(t)) \text{ for } t \geq 0.$$

Set $G = \{(x,t)\ :\ t > 0 \text{ and } x \in (-\ell(t),\ell(t))\}$. Since w^* is a generalized supersolution of (5.1) in G, we find by Lemma 4.5 an upper estimate of $|\ w(\cdot,t)\ |_{\infty,\mathbf{R}}$. On the other hand, we see by Lemma 5.1

(5.10) $$|\ (w^{m-1})_x(\cdot,t)\ |_{\infty,\mathbf{R}} \leq Ct^{-(1/2)(1+(m-1)/(p_*-1))}.$$

By a similar argument to one which is used for the porous medium equation, we can show that $\operatorname{supp} w(\cdot,t)$ is an interval $(\zeta_1(t),\zeta_2(t))$ for large t and that

(5.11) $$\zeta_i'(t) = \frac{m}{m-1}(w^{m-1})_x(\zeta_i(t),t) \text{ for large } t, i = 1,2.$$

By (5.10) and (5.11), we see

(5.12) $$|\ \zeta_i'(t)\ | \leq Ct^{-(1/2)(1+(m-1)/(p_*-1))},$$

where C is a positive constant. Integrating (5.12) from 0 to t, we have upper the desired estimates of $|\zeta_1(t)|$ and $|\zeta_2(t)|$.

(V) Case : $m < 2n-2$.

We set again

$$w_*(x,t) = A(1+t)^{-a}(D - x^2(1+t)^{-b})_+^{1/(m-1)} \quad \text{for} \quad (x,t) \in \mathbf{R} \times (0,\infty).$$

We put for $\varepsilon > 0$

(5.13)

$$a = 1/(m+1)+\varepsilon,\ b = 2/(m+1) - \varepsilon(m-1),\ A = \left(\tfrac{m-1}{2m(m+1)}\right)^{1/(m-1)},$$

$$D = (\lambda^{-1}A^{1-n}\varepsilon)^{(m-1)/(n-1)} \quad \text{and} \quad E = AD^{1/(m-1)}.$$

Then, for any sufficiently small $\varepsilon > 0$, we get

$$w_0(x) \geq E\chi_{[-D^{1/2}, D^{1/2}]}(x) \quad \text{for} \quad x \in \mathbf{R}, \tag{5.14}$$

where χ is a characteristic function. We fix such an $\varepsilon > 0$. By (5.9), (5.13) and (5.14), we see that w_* is a generalized subsolution of (5.1), and $w \geq w_*$ in $\mathbf{R} \times [0,\infty)$. Set $T_1 = (\varepsilon\lambda^{-1}A^{1-n}2^{-1})^{-1/q} - 1$, which $q = (2n-2-m)/(m+1) > 0$. Then we see

$$w(x,T_1) \geq A(1+T_1)^{-a(1)}\left(D(1) - x^2(1+T_1)^{-b(1)}\right)_+^{1/(m-1)} \quad \text{on } \mathbf{R},$$

with $a(1) = 1/(m+1)+\varepsilon/2$, $b(1) = 2/(m+1) - \varepsilon(m-1)/2$ and $D(1) = D(1+T_1)^{-\varepsilon(m-1)/2}$. Setting

$$w_{(*1)}(x,t) = A(1+t)^{-a(1)}\left(D(1) - x^2(1+t)^{-b(1)}\right)_+^{1/(m-1)} \text{ in } \mathbf{R} \times (0,\infty).$$

We see

$$L(w_{(*1)}) \leq 0 \quad \text{in} \quad \mathbf{R} \times [T_1, \infty)$$

and

$$w_{(*1)t},\ (w_{(*1)}^m)_{xx} \in L^1(\mathbf{R} \times [0,T]) \quad \text{for } T > 0.$$

Thus, $w_{(*1)}(x,t)$ is a generalized subsolution of (5.1) in $\mathbf{R} \times [T_1, \infty)$, and we have by Lemma 4.3 $w_{(*1)} \leq w$ in $\mathbf{R} \times [T_1, \infty)$.

For any positive integer j, we put $a(j) = 1/(m+1) + \varepsilon 2^{-j}$, $b(j) = 2/(m+1) - \varepsilon(m-1)2^{-j}, T_j = (\varepsilon \lambda^{-1} A^{1-n} 2^{-j})^{-1/q} - 1$ and $D(j+1) = D(j)(1+T_j)^{-\varepsilon(m-1)2^{-j-1}}$. Setting

$$w_{(*1)}(x,t) = A(1+t)^{-a(j)} \left(D(j) - x^2(1+t)^{-b(j)}\right)_+^{1/(m-1)} \text{ in } \mathbf{R} \times (0, \infty).$$

We get similarly $w \geq w_{(*j)}$ in $\mathbf{R} \times [T_j, \infty)$. Therefore, we get for $t \in [T_j, T_{j+1}]$, $j \geq 1$,

$$(1+t)^{-1/(m+1)} \mid \inf\{\mathrm{supp} w(\cdot,t)\} \mid \geq D(j)^{1/2}(1+T_{j+1})^{-\varepsilon(m-1)2^{-j}},$$

$$(1+t)^{-1/(m+1)} \mid \sup\{\mathrm{supp} w(\cdot,t)\} \mid \geq D(j)^{1/2}(1+T_{j+1})^{-\varepsilon(m-1)2^{-j}},$$

and

$$(1+t)^{1/(m+1)} \mid w(\cdot,t) \mid_{\infty,\mathbf{R}} \geq A D(j)^{1/(m-1)}(1+T_{j+1})^{-\varepsilon 2^{-j}}.$$

Since $\lim_{j\to\infty} D(j) = D(\infty) > 0$ and $\lim_{j\to\infty}(1+T_{j+1})^{-\varepsilon 2^{-j}} = 1$, we obtain the lower estimates of $\mathrm{supp} w(\cdot,t)$ and $\mid w(\cdot,t) \mid_{\infty,\mathbf{R}}$.

Let w^* be a generalized solution of (2.1) with initial condition $w^*(\cdot,0) = w_0$. Then we can show that w^* is a generalized supersolution of (5.1). Therefore, we obtain the upper estimates of $\mathrm{supp} w(\cdot,t)$ and $\mid w(\cdot,t) \mid_{\infty,\mathbf{R}}$.

Q.E.D.

6. Regularity and semiconvexity of the solution for (1.4) and (1.5) in case of $\inf_{x\in\mathbf{R}} w_0(x) > 0$

In this section, we let the function w_0 belong to $C(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and satisfy

$$\inf_{x\in\mathbf{R}} w_0(x) \ > \ 0. \tag{6.1}$$

We shall consider the regularity and the semiconvexity of the solution for (1.4) and (1.5) under this assumption.

By Theorem 0 in [8], we know the following proposition.

Proposition 6.1. Let $w_0 \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$ satisfy (6.1). Then, there exists a unique classical solution of (1.4) and (1.5), and the solution is smooth in $\mathbf{R} \times (0, \infty)$.

Now we shall show main results of this section.

Lemma 6.2. Let $w_0 \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$ satisfy (6.1) and let w be the classical solution of (1.4) and (1.5). Then, w satisfies

$$| (w^{m-1})_x(\cdot, t) |_{\infty,\mathbf{R}} \leq D_{n,m} D_1 (t + D_2)^{-\frac{m-1}{2n-1}-1} \times$$
$$\times \{ D_2^{-\frac{m-1}{2n-1}} - (t + D_2)^{-\frac{m-1}{2n-1}} \}^{-1/2} \quad \text{for} \quad (x, t) \in \mathbf{R} \times (0, \infty),$$

where $D_{n,m}$ is a positive constant depending only on m and n,

$$D_1 = \left(\frac{| w_0 |_{\infty,\mathbf{R}}}{\inf_{x\in\mathbf{R}} w_0(x)} \right)^{\frac{2n+m-2}{2}} \times \left\{ 2(\inf_{x\in\mathbf{R}} w_0(x))^{-2n+1} \ - \ | w_0 |_{\infty,\mathbf{R}}^{-2n+1} \right\}^{1/2}$$

and

$$D_2 = \frac{2}{2n-1} (\inf_{x\in\mathbf{R}} w_0(x))^{-2n+1}.$$

Proof. Set $W = (m/(m-1))w^{m-1}$. Then, W satisfies the equation :

$$W_t \ = \ (m-1)WW_{xx} + \ | W_x |^2 \ - \lambda W^\beta \quad \text{in } \mathbf{R} \times (0, \infty),$$

with $\beta = (2n+m-2)/(m-1)$ and $\lambda = m^{\beta-1}(m-1)^{\beta}$.

By the comparison theorem, we have

$$(6.2)\quad \mu(t+C^*)^{-\frac{m-1}{2n-1}} \ \geq\ W(x,t)\ \geq\ \mu(t+C_*)^{-\frac{m-1}{2n-1}} \text{ in } \mathbf{R}\times[0,\infty),$$

with $\mu = \{\lambda(2n-1)/(m-1)\}^{-(m-1)/(2n-1)}$, $C_* = (2n-1)^{-1}(\inf_{x\in\mathbf{R}} w_0(x))^{-2n+1}$ and $C^* = (2n-1)^{-1} \mid w_0 \mid_{\infty,\mathbf{R}}^{-2n+1}$.

Set

$$\tilde{Q}(x,t) = (t+2C_*)^{\alpha}(W(x,t) - \mu(t+2C_*)^{-\frac{m-1}{2n-1}}) \text{ in } \mathbf{R}\times[0,\infty),$$

where $\alpha = (2n+m-2)/(2n-1)$.

Then, $\tilde{Q}$ is positive in $\mathbf{R}\times[0,\infty)$ and we obtain the following estimates :

$$(6.3)\qquad \tilde{Q}(x,t)\ \leq\ \mu\frac{m-1}{2n-1}\left(\frac{2C_*}{C^*}\right)^{\alpha}(2C_*-C^*) \quad \text{in } \mathbf{R}\times[0,\infty)$$

and

$$(6.4)\qquad \tilde{Q}(x,t)\ \geq\ \mu\frac{m-1}{2n-1}C_* \text{ in } \mathbf{R}\times[0,\infty).$$

Setting

$$r(s) = \left((2C_*)^{1-\alpha} - (\alpha-1)s\right)^{-\frac{1}{\alpha-1}} \quad \text{on } [0,S_*)$$

with $S_* = (2C_*)^{1-\alpha}(\alpha-1)^{-1}$, we find

$$\begin{cases} r' = r^{\alpha} & \text{on } (0,S_*) \\ r(0) = 2C_*. \end{cases}$$

Using $r(s)$ we set

$$Q(x,s) = \tilde{Q}(x, r(s)-2C_*) \text{ in } \mathbf{R}\times(0,S_*).$$

Then,

$$(6.5)\qquad Q_s \ =\ (m-1)(Q+\mu r)Q_{xx} + \mid Q_x\mid^2 + \lambda\mu^{\beta}r^{\alpha}$$
$$-\lambda\mu^{\beta}r^{\alpha}(1+\mu^{-1}r^{-1}Q)^{\beta} + \alpha r^{\alpha-1}Q$$
$$\text{in } \mathbf{R}\times(0,S_*).$$

Set $N = 8\mu(m-1)(2n-1)^{-1}(2C_*/C^*)^{\alpha}(2C_* - C^*)$ and $\phi(y) = Ny(1-y)$. By a quite similar argument obtaining (4.5), since we can prove

$$(6.6) \qquad | Q_x(x,s) |^2 \leq \frac{N}{ms} \quad \text{for } (x,s) \in \mathbf{R} \times (0, S_*).$$

Therefore we omit the prove of (6.6).

Since

$$s = \frac{1}{(\alpha - 1)}\{(2C_*)^{1-\alpha} - (t + 2C_*)^{1-\alpha}\},$$

it follows from (6.6) that

$$\begin{aligned} | W_x(x,t) | \ \leq \ & D_{n,m}\left[\left(\frac{C_*}{C^*}\right)^{\alpha}(2C_* - C^*)\right]^{1/2}\left\{(2C_*)^{1-\alpha} - (2C_* + t)^{1-\alpha}\right\}^{-\frac{1}{2}} \times \\ & \times\ (t + 2C_*)^{-\alpha} \text{ in } \mathbf{R} \times (0,\infty), \end{aligned}$$

for a certain $D_{n,m} > 0$.

Q.E.D.

Lemma 6.3. Let $m \geq 2n$. Let $w_0 \in C(\mathbf{R}) \cap L^{\infty}(\mathbf{R})$ satisfy (6.1) and let w be the classical solution of (1.4) and (1.5). Then, the solution w satisfies the following inequality :

$$\begin{aligned} (w^{m-1})_{xx}(x,t) \ \geq \ & -K(t + D_2)^{-\frac{m-1}{2n-1}-1} \times \\ & \times\{D_2^{-\frac{m-1}{2n-1}} - (t + D_2)^{-\frac{m-1}{2n-1}}\}^{-1} \\ & \qquad \text{in } \mathbf{R} \times (0,\infty), \end{aligned}$$

with $K = \max((m-1)/(m(m+1)),\ \hat{D}_{n,m}D_1^2)$, where D_1 and D_2 are the constants in Lemma 6.2 and $\hat{D}_{n,m}$ is a positive constant depending only on n and m.

Proof. We differntiate the equation in (6.5) twice with respect to x and set $P = Q_{xx}$ to get

$$(6.7) \qquad P_s \ = \ (m-1)(Q + \mu r)P_{xx} \ + \ 2mQ_xP_x \ + \ (m+1)P^2$$

$$-\alpha r^{\alpha-1}\{(1+\mu^{-1}r^{-1}Q)^{\beta-1}-1\}P$$

$$-(\beta-1)\alpha\mu^{-1}r^{\alpha-2}(1+\mu^{-1}r^{-1}Q)^{\beta-2}(Q_x)^2$$

$$\text{in } \mathbf{R}\times(0,S_*).$$

We shall consider the differential operator :

$$\text{(6.8)}\quad \tilde{L}(\theta) = \theta_s-(m-1)(Q+\mu r)\theta_{xx}-2mQ_x\theta_x-(m+1)\theta^2$$
$$+\alpha r^{\alpha-1}\{(1+\mu^{-1}r^{-1}Q)^{\beta-1}-1\}\theta \text{ in } \mathbf{R}\times(0,S_*).$$

Then from (6.4), (6.6) and (6.7) it follows that

$$\tilde{L}(P) \geq -\left(\mu\frac{m-1}{2n-1}C_*\right)^{-1}\frac{N}{ms}(\beta-1)\alpha r^{\alpha-1}(\mu^{-1}r^{-1}Q)(1+\mu^{-1}r^{-1}Q)^{\beta-2}$$
$$\text{in } \mathbf{R}\times(0,S_*).$$

Let $k>0$ and $0<\eta<S_*$. We substitute $\hat{P}=-k/(s-\eta)$ into (6.7) to get (note that $1<\beta\leq 2$)

$$\tilde{L}(\hat{P}) \leq \frac{k}{(s-\eta)^2}-\frac{(m+1)k^2}{(s-\eta)^2}-(\beta-1)\alpha r^{\alpha-1}(\mu^{-1}r^{-1}Q)(1+\mu^{-1}r^{-1}Q)^{\beta-2}\frac{k}{s-\eta}$$
$$\text{in } \mathbf{R}\times(\eta,S_*).$$

If the positive constant k satisfies

$$\left(\mu\frac{m-1}{2n-1}C_*\right)^{-1}\frac{N}{m} \leq k \leq (m+1)k^2,$$

we have

$$\text{(6.9)}\qquad \tilde{L}(P) \geq \tilde{L}(\hat{P}) \quad \text{in } \mathbf{R}\times(\eta,S_*).$$

By Theorem 5.1 in Chapter 7 of [14] P is bounded in $\mathbf{R}\times[\eta,(S_*+\eta)/2]$ and there exists a positive constant ε_0 such that

$$\text{(6.10)}\qquad P \geq \hat{P} \quad \text{in } \mathbf{R}\times(\eta,\eta+\varepsilon_0).$$

Thus, choosing

$$k=\max\left\{\frac{1}{m+1},\ \left(\mu\frac{m-1}{2n-1}C_*\right)^{-1}\frac{N}{m}\right\},$$

we have

$$P \geq \hat{P} = -\frac{k}{s-\eta} \quad \text{in } \mathbf{R} \times (\eta, S_*).$$

Since η is an arbitrary constant such that $0 < \eta < S_*$, we get

$$Q_{xx} \geq -\frac{k}{s} \quad \text{in } \mathbf{R} \times (0, S_*).$$

Therefore it follows that

$$\frac{m}{m-1}(w^{m-1})_{xx} \geq -(t+2C_*)^{-\alpha} k \left[(2C_*)^{1-\alpha} - (t+2C_*)^{1-\alpha}\right]^{-1}$$
$$\text{in } \mathbf{R} \times (0, \infty).$$

Q.E.D.

7. The estimates of v in $(\text{supp } u)^c$

In this section we shall consider the estimates of v in $(\text{supp } u)^c$, where u and v are the solutions for (1.1) and (1.2). Throughout this section, we assume $2n-2 < m$ and that u_0 and v_0 satisfy (A.I.) and (A.II.). Set $d_0 = (b_m)^{-1/2} a_m (| v_0 |_{1,\mathbf{R}} - | u_0 |_{1,\mathbf{R}})^{(m-1)/(m+1)}$, where a_m and b_m are the constants in (2.5).

For $d \in (0, d_0)$, we set $h = h(d) = b_m^{1/(m-1)} (d_0^2 - d^2)^{1/(m-1)}$.

Lemma 7.1. Let u and v be the solutions of (1.1) and (1.2).

For $d \in (0, d_0)$ and $\varepsilon \in (0, d_0 - d)$ such that $h(d+\varepsilon) \geq (3/4)h(d)$, there exists a positive constant $T_1 = T_1(d, \varepsilon)$ satisfying the property :

$$\text{supp } u(\cdot, t) \subset \left[-\frac{d}{3} t^{\frac{1}{m+1}}, \ \frac{d}{3} t^{\frac{1}{m+1}} \right] \quad \text{for } t \geq T_1.$$

Moreover, for $t \geq T_1$, there exist $x_1 = x_1(d, \varepsilon, t) \in [dt^{1/(m+1)}, \ (d+\varepsilon)t^{1/(m+1)}]$ and $x_2 = x_2(d, \varepsilon, t) \in [-(d+\varepsilon)t^{1/(m+1)}, \ -dt^{1/(m+1)}]$ such that

$$v(x_i, t) \ \geq \ \frac{2}{3} h(d) t^{-\frac{1}{m+1}} \quad \text{for } i = 1, 2.$$

Remark 7.2. For $\eta \in (0, \varepsilon)$, we put

$$G = \{(x,t) \ : \ t \geq T_1, \ x \in [-(d+\eta)t^{\frac{1}{m+1}}, \ (d+\eta)t^{\frac{1}{m+1}}]\}.$$

Applying Lemma 4.4 to the solution v, we see that v is continuous in $\{(x,t) \ : \ t \geq T_1, \ x \in [-(d+\varepsilon)t^{1/(m+1)}, \ -dt^{1/(m+1)}] \cup [dt^{1/(m+1)}, \ (d+\varepsilon)t^{1/(m+1)}]\}$

Proof of Lemma 7.1. For $\lambda > 0$ and $(f_0, g_0) \in X$ such that $f_0(x)$ and $g_0(x) \geq 0$ a.e. $x \in \mathbf{R}$, we put $(f_\lambda, g_\lambda) = (I - \lambda \mathcal{A})^{-1}(f_0, g_0)$, where $\mathcal{A}$ and X are the operator and the space in Section 2 respectively. Since $(| f_\lambda |^{m-1} f_\lambda)_{xx}$ and $(| f_\lambda |^{m-1} g_\lambda)_{xx}$ belong to $L^1(\mathbf{R})$, we have that

$$\int_{\mathbf{R}} g_\lambda(x) dx \ - \ \int_{\mathbf{R}} f_\lambda(x) dx \ = \ \int_{\mathbf{R}} g_0(x) dx \ - \ \int_{\mathbf{R}} f_0(x) dx.$$

Therefore, by the definition of the solutions u and v in Section 2, we get that

(7.1) $$| v(\cdot,t) |_{1,\mathbf{R}} - | u(\cdot,t) |_{1,\mathbf{R}} = M \quad \text{for } t \geq 0,$$

where $M = | v_0 |_{1,\mathbf{R}} - | u_0 |_{1,\mathbf{R}}$.

Let w be the generalized solution of (1.4) ($p = 2n$) with $w(\cdot,0) = u_0$. Then, since $2n-2 < m$, the solution w satisfies the following peroperties:

$$\lim_{t\to\infty} | w(\cdot,t) |_{1,\mathbf{R}} = 0$$

and there exists a positive constant T_{1*} such that

(7.2) $$\text{supp } w(\cdot,t) \subset [-\frac{d}{3}t^{\frac{1}{m+1}}, \frac{d}{3}t^{\frac{1}{m+1}}] \text{ for } t \geq T_{1*}.$$

Therefore, by Lemma 4.1, the solution u satisfies that

(7.3) $$\text{supp } u(\cdot,t) \subset \left[-\frac{d}{3}t^{\frac{1}{m+1}}, \frac{d}{3}t^{\frac{1}{m+1}}\right] \quad \text{for } t \geq T_{1*}$$

and

(7.4) $$\lim_{t\to\infty} | u(\cdot,t) |_{1,\mathbf{R}} = 0.$$

By (7.4), there exists $T_{2*} = T_{2*}(\varepsilon,d) \geq T_{1*}$ such that

(7.5) $$| u(\cdot,t) |_{1,\mathbf{R}} < \frac{1}{24}h(d)\varepsilon \text{ for } t \geq T_{2*}.$$

Let v^* be the generalized solution of (2.2) with $v^*(\cdot,T_{2*}) = v(\cdot,T_{2*})$ in $\mathbf{R} \times [T_{2*},\infty)$. From Lemma 4.1 it follows that

(7.6) $$v(x,t) \leq v^*(x,t) \text{ for } t \geq T_{2*} \text{ and a.e. } x \in \mathbf{R}.$$

By [7] and [15], the solution v^* satisfies that

(7.7) $$\sup\{\text{supp } v^*(\cdot,t)\}, \inf\{\text{supp } v^*(\cdot,t)\} \sim t^{\frac{1}{m+1}}$$

and

(7.8) $$\lim_{t\to\infty}\{t^{\frac{1}{m+1}} | v^*(\cdot,t) - \bar{v}(\cdot,t) |_{\infty,\mathbf{R}}\} = 0,$$

where $\bar{v}(x,t) = \bar{z}(x,t\ ;\ |\ v(\cdot, T_{2*})\ |_{1,\mathbf{R}})$ and $\bar{z}$ is the self - similar solution ((2.5)).

From (7.7) and (7.8), it follows that

$$(7.9) \qquad \lim_{t\to\infty} |\ v^*(\cdot,t)\ -\ \bar{v}(\cdot,t)\ |_{1,\mathbf{R}} = 0.$$

Now we shall show the existence of the point x_1.

By Remark 7.2 and (7.3), the negation of our conclusion is as follows :

There exist a constant $\varepsilon \in (0, d_0 - d)$ and a sequence of the points $t_n \in [T_{2*}, \infty)$ such that

$$h(d+\varepsilon) \geq \frac{3}{4}h(d), \ \lim_{n\to\infty} t_n = \infty$$

and

$$(7.10) \qquad v(x,t_n) < \frac{2}{3}h(d)t_n^{-\frac{1}{m+1}}$$

$$\text{for } n \geq 1 \text{ and } x \in [dt_n^{\frac{1}{m+1}},\ (d+\varepsilon)t_n^{\frac{1}{m+1}}].$$

From (7.6), (7.8) and (7.10), it follows that

$$(7.11) \ |\ v(\cdot,t_n)\ |_{1,\mathbf{R}} \ < \ \int_{\mathbf{R}\setminus I(t_n)} \bar{v}(x,t_n)dx + \ |\ v^*(\cdot,t_n) - \bar{v}(\cdot,t_n)\ |_{1,\mathbf{R}} + \frac{2}{3}h\varepsilon,$$

where

$$I(t) = [dt^{\frac{1}{m+1}},\ (d+\varepsilon)t^{\frac{1}{m+1}}] \ \text{ for } t > 0.$$

For any positive constant ε such that $h(d+\varepsilon) \geq (3/4)h(d)$, the function $\bar{v}$ satisfies that

$$(7.12) \qquad \bar{v}(x,t) \ \geq \ \frac{3}{4}h(d)t^{-\frac{1}{m+1}}$$

$$\text{for } t > 0 \ \text{ and } x \in I(t).$$

Since any generalized solutions z of (2.2) conserve the total mass :

$$| z(\cdot,t) |_{1,\mathbf{R}} = | z(\cdot,0) |_{1,\mathbf{R}},$$

it follows from (7.1) and (7.5) that

$$(7.13) \qquad | v(\cdot,t) |_{1,\mathbf{R}} + \frac{1}{24}h\varepsilon$$
$$> | v(\cdot,T_{2*}) |_{1,\mathbf{R}} = | v^*(\cdot,t) |_{1,\mathbf{R}} \quad \text{for any } t \geq T_{2*}.$$

By (7.12) and (7.13) we have

$$(7.14) | v(\cdot,t_n) |_{1,\mathbf{R}} > \int_{\mathbf{R}\setminus I(t_n)} \bar{v}(x,t_n)dx + \frac{3}{4}h\varepsilon$$
$$- | v^*(\cdot,t_n) - \bar{v}(\cdot,t_n) |_{1,\mathbf{R}} - \frac{1}{24}h\varepsilon \text{ for } n \geq 1.$$

From (7.11) and (7.14) it follows that

$$(7.15) \qquad | v^*(\cdot,t_n) - \bar{v}(\cdot,t_n) |_{1,\mathbf{R}} > \frac{1}{48}h\varepsilon \quad \text{for } n \geq 1.$$

The property (7.15) contradicts with (7.9). Thus, for any sufficiently large t, there exists a point x_1 having the required property.

By the similar argument, we can show the existence of the point x_2.

Q.E.D.

In the proof of the following lemma, since we shall use the scheme employed in the proof of Lemma 3.1 in [8], we omit the proof.

Lemma 7.3. For $d \in (0, d_0)$, there exist two positive constants $T_2 = T_2(d)$ and $C_2 = C_2(v_0, d)$ such that

$$| (v^{m-1})_x(x,t) | \leq C_2 t^{-\frac{m}{m+1}} \quad \text{for } t \geq T_2$$

$$\text{and a.e. } x \in (-\infty, -dt^{\frac{1}{m+1}}] \cup [dt^{\frac{1}{m+1}}, \infty).$$

By Lemma 7.1 and Lemma 7.3, we can show the following lemma.

Lemma 7.4. For $d \in (0, d_0)$, there exists a positive constant $T_3 = T_3(d, v_0)$ such that

$$v(dt^{\frac{1}{m+1}}, t)\ ,\ \ v(-dt^{\frac{1}{m+1}}, t) \ > \ \frac{1}{2}h(d)t^{\frac{-1}{m+1}}$$

$$\text{for } t \geq T_3.$$

Proof. Fix $d \in (0, d_0)$ and choose

$$\varepsilon \ = \ \frac{1}{2}\left\{\left(\frac{2}{3}h(d)\right)^{m-1} - \left(\frac{1}{2}h(d)\right)^{m-1}\right\} C_2^{-1}, \tag{7.16}$$

where C_2 is the constant in Lemma 7.3. Let $T_3 = \max(T_1, T_2)$.
Then, it follows from Lemma 7.1 and (7.16) that

$$\begin{aligned} v^{m-1}(dt^{\frac{1}{m+1}}, t) \ &\geq \ v^{m-1}(x_1, t) \ - \ C_2 t^{\frac{-m}{m+1}}(x_1 - dt^{-\frac{1}{m+1}}) \\ &\geq \ \left(\frac{2}{3}h(d)t^{-\frac{1}{m+1}}\right)^{m-1} \ - \ C_2 \varepsilon t^{-\frac{m-1}{m+1}} \\ &> \ \left(\frac{1}{2}h(d)t^{\frac{1}{m+1}}\right)^{m-1} \quad \text{for } \ t \geq T_3, \end{aligned}$$

where x_1 is the point in Lemma 7.1.
By a similar argument, we obtain

$$v^{m-1}\left(-dt^{\frac{1}{m+1}}, t\right) \ > \ \left(\frac{1}{2}h(d)t^{\frac{1}{m+1}}\right)^{m-1} \quad \text{for } \ t \geq T_3.$$

Q.E.D.

8. Proof of Theorem 1.1

Part I (Case $m < 2n-2$.).

Let w^* be the generalized solution of (2.2) with $w^*(\cdot,0)=v_0$. By [17], it is shown that the solution w^* satisfies

$$|\, w^*(\cdot,t)\,|_{\infty,\mathbf{R}} \sim t^{\frac{-1}{m+1}} \tag{8.1}$$

and

$$\sup(\mathrm{supp}\ w^*(\cdot,t)),\ -\inf(\mathrm{supp}\ w^*(\cdot,t)) \ \sim\ t^{\frac{1}{m+1}}. \tag{8.2}$$

Let w be the generalized solution of (1.4) ($p=2n$) with $w(\cdot,0)=v_0$. By the result stated in Introduction, the solution w satisfies

$$|\, w(\cdot,t)\,|_{\infty,\mathbf{R}} \sim t^{\frac{-1}{m+1}}, \tag{8.3}$$

and

$$\sup(\mathrm{supp}\ w(\cdot,t)),\ -\inf(\mathrm{supp}\ w(\cdot,t)) \ \sim\ t^{\frac{1}{m+1}}. \tag{8.4}$$

On the other hand, by Lemma 4.1 we have

$$w(x,t)\ \le\ v(x,t)\ \le w^*(x,t) \qquad \text{for } t\ge 0 \quad \text{and a.e. } x\in\mathbf{R}. \tag{8.5}$$

It follows from (8.1) - (8.5) that

$$|\, v(\cdot,t)\,|_{\infty,\mathbf{R}} \sim t^{\frac{-1}{m+1}} \tag{8.6}$$

and

$$\sup(\mathrm{supp}\ v(\cdot,t)),\ -\inf(\mathrm{supp}\ v(\cdot,t)) \ \sim\ t^{\frac{1}{m+1}}.$$

Therefore we have the estimates of $|\, v(\cdot,t)\,|_{\infty,\mathbf{R}}$ and $\mathrm{supp} v(\cdot,t)$.

By (8.6), there exists a positive constant λ such that

$$(v(x,t))^n\ \le\ \lambda(1+t)^{-\frac{n}{m+1}} \qquad \text{for } t\ge 0 \ \text{ and a.e. } x\in\mathbf{R}. \tag{8.7}$$

For such a λ, we let u_* be the generalized solution of (5.1) with $u_*(\cdot,0) = u_0$. By (8.7), the solution u_* is a subsolution of (3.1) with $P = v^n$ and $q = n$. Hence, by Lemma 4.1 and Lemma 4.3, we have

$$(8.8) \qquad u_*(x,t) \le u(x,t) \le w(x,t) \quad \text{in } \mathbf{R} \times [0,\infty).$$

By Lemma 5.2, the solution u_* satisfies

$$(8.9) \qquad | u_*(\cdot,t) |_{\infty,\mathbf{R}} \sim t^{-\frac{1}{m+1}}$$

and

$$(8.10) \qquad \sup(\text{supp } u_*(\cdot,t)), \ -\inf(\text{supp } u_*(\cdot,t)) \sim t^{\frac{1}{m+1}}.$$

From (8.3), (8.4) and (8.8) - (8.10), we obtain the required estimates of $| u(\cdot,t) |_{\infty,\mathbf{R}}$ and supp $u(\cdot,t)$.

Part II (Case of $2n-2 < m$.).

The following lemma gives some lower estimates of v in a subset of $(\text{ supp } u)^c$.

Lemma 8.1. Let

$$\zeta(t) = \begin{cases} \zeta_0 t^{(2n-m)_+/2(2n-1)} & \text{if } 2n \ne m, \\ \zeta_0 (\log(t))_+ & \text{if } 2n = m, \end{cases}$$

$$k(t) = A(t+K)^{\frac{1}{m+1}} + \zeta(t) \quad \text{for } t > 0$$

and let

$$v_*^{(1)}(x,t) = \left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} t^{-\frac{1}{m+1}} (A^2 - (x-k(t))^2 t^{-\frac{2}{m+1}})_+^{\frac{1}{m-1}} \text{ in } \mathbf{R}\times[0,\infty).$$

Then, u and v satisfy

$$\text{supp } u(\cdot,t) \subset [-\zeta(t), \ \zeta(t)] \text{ for } t \ge T_1$$

and

$$v(x,t),\ v(-x,t)\ \geq\ v_*^{(1)}(x,t)$$

$$\text{for } t \geq T_1 \text{ and a.e. } x \in (-\infty, At^{\frac{1}{m+1}}]$$

for certain positive constants k, ζ_0 and T_1.

Proof. Let w be the generalized solution of (1.4) with $w(\cdot,0) = u_0$. Then, as is noted in Introduction, it holds for sufficiently large ζ_0 and T_{1*} that

$$\operatorname{supp} w(\cdot,t)\ \subset\ [-\zeta(t),\ \zeta(t)] \text{ for } t \geq T_{1*}.$$

By Lemma 4.1, we have also

$$\operatorname{supp} u(\cdot,t)\ \subset\ [-\zeta(t),\ \zeta(t)] \text{ for } t \geq T_{1*}. \tag{8.11}$$

We set

$$A\ =\ (2^{m-1}+1)^{-1/2} d_0, \tag{8.12}$$

where d_0 is the constant in Section 7. By Lemma 7.4, (8.12) and $2n-2\ <\ m$, there exists a positive constant $T_{2*}\ =\ T_{2*}(A)\ \geq\ T_{1*}$ such that

$$v(At^{\frac{1}{m+1}},t),\ v(-At^{\frac{1}{m+1}},t)\ >\ \frac{1}{2}h(A)t^{-\frac{1}{m+1}} \quad \text{for } t \geq T_{2*} \tag{8.13}$$

and

$$\operatorname{supp} u(\cdot,t)\ \subset\ [-\frac{1}{2}At^{\frac{1}{m+1}},\ \frac{1}{2}At^{\frac{1}{m+1}}] \text{ for } t \geq T_{2*}, \tag{8.14}$$

where h is the function in Section 7.

Here, we take

$$K\ =\ 2^{m+1}T_{2*} \tag{8.15}$$

and set

$$G_{T_{2*}}\ =\ \{(x,t)\ ;\ t \geq T_{2*},\ x \in [-At^{\frac{1}{m+1}},\ At^{\frac{1}{m+1}}]\}.$$

We shall compare the function $v_*^{(1)}$ in this lemma and the solution v in $G_{T_{2*}}$.

By (8.15), the function $v_*^{(1)}$ satisfies

$$(8.16) \qquad v_*^{(1)}(x, T_{2*}) = 0 \quad \text{for} \quad x \in [-AT_{2*}^{\frac{1}{m+1}},\ AT_{2*}^{\frac{1}{m+1}}].$$

Since

$$k(t) \ > \ At^{\frac{1}{m+1}} \quad \text{for } t > 0$$

we have

$$(8.17) \qquad v_*^{(1)}(At^{\frac{1}{m+1}}, t) \ < \ \frac{1}{2}h(A)t^{-\frac{1}{m+1}} \quad \text{for} \quad t \geq T_{2*}$$

$$(8.18) \qquad v_*^{(1)}(-At^{\frac{1}{m+1}}, t) = 0 \quad \text{for } t \geq T_{2*}$$

and

$$(8.19) \qquad v_{*x}^{(1)}(x,t) \geq 0$$

$$\text{for } t \geq T_{2*} \ \text{ and a.e. } x \in [-At^{\frac{1}{m+1}},\ At^{\frac{1}{m+1}}].$$

By (8.11), the solution u satisfies

$$(8.20) \qquad (\text{ supp } u) \ \cap \ (\text{ supp } v_*^{(1)}) = \phi \quad \text{in} \quad G_{T_{2*}}.$$

Since the function $((m-1)/(2m(m+1)))^{1/(m-1)}t^{-1/(m+1)}(A^2-x^2t^{-2/(m+1)})_+^{1/(m-1)}$ satisfies (2.2) a.e. in $\mathbf{R} \times (0, \infty)$, we obtain by (8.19) and (8.20)

$$(8.21) \qquad v_{*\,t}^{(1)} \ - \ (v_*^{(1)m})_{xx} \ + \ u^n v_*^{(1)n}$$

$$= \ v_{*\,x}^{(1)}(-k'(t)) \ \leq \ 0 \quad \text{a.e. in } G_{T_{2*}},$$

That is, $v_*^{(1)}$ is a subsolution of (3.1) with $P = v^n$ and $q = n$ in $G_{T_{2*}}$. Thus, by (8.13), (8.14), (8.16) - (8.18) and Lemma 4.4, we conclude $v_*^{(1)} \leq v$ in $G_{T_{2*}}$.

Q.E.D.

Lemma 8.2. There exist a positive constant T_2 and a positive function ρ defined on $[T_2, \infty)$ such that

$$v(x,t) \ \geq \ \rho(t) \ > \ 0 \qquad \text{for } \ t \geq T_2 \ \text{ and a.e. } \ x \in [-At^{\frac{1}{m+1}}, \ At^{\frac{1}{m+1}}],$$

where A is the constant in Lemma 8.1.

Proof. Case : $2n-2 \ < \ m \ \leq \ 2n$.

Let w be the generalized solution of (1.4) with $w(\cdot, 0) = u_0$. Then, by [8], [9], [12] and [13], there exists a nonnegative constant T_{1*} such that for each $t \geq T_{1*}$, $\{x \in \mathbf{R} \ : \ w(x,t) > 0\}$ is an open interval in $\mathbf{R}$ containing $x = 0$. By Lemma 8.1, there exists a constant $T_{2*} \ \geq \ \max(T_{1*}, T_1)$ such that

$$v(x,t) \ \geq \ \varphi_0(x \ ; \ t) \qquad \text{for } \ t \geq T_{2*} \ \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, \ At^{\frac{1}{m+1}}], \tag{8.22}$$

where

$$\varphi_0(x \ ; \ t) = \left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} t^{-\frac{1}{m+1}} \left(\frac{A^2}{9} - (x - \frac{2}{3}At^{\frac{1}{m+1}})^2 t^{-\frac{2}{m+1}}\right)_+^{\frac{1}{m-1}}$$

and A and T_1 are the constants in Lemma 8.1.

For each $s \geq T_{2*}$, let $w(\cdot, \ \cdot \ ; \ s)$ be the generalized solution of (1.4) with $w(\cdot, \ s \ ; \ s) \ = \ \max\{w(\cdot, s), \ \varphi_0(\cdot \ ; \ s)\}$ in $\mathbf{R} \times [s, \infty)$ and let $\varphi(\cdot, \ \cdot \ ; \ s)$ be the generalized solution of (1.4) with $\varphi(\cdot, \ s \ ; \ s) \ = \ \varphi_0(\cdot \ ; \ s)$ in $\mathbf{R} \times [s, \infty)$. Then, by (8.21), Lemma 4.1 and Lemma 4.3, we have

$$\varphi(x, \ t \ ; \ s) \ \leq \ w(x, \ t \ ; \ s) \ \leq \ v(x,t) \qquad \text{for } s \geq T_{2*}, \ \text{any } t \geq s \ \text{ and a.e. } x \in \mathbf{R}. \tag{8.23}$$

We observe by [9], [12] and [13] that for each $t \geq s$, $\{x \in \mathbf{R} \ : \ \varphi(x, \ t \ ; \ s) \ > 0\}$ is an open interval in $\mathbf{R}$.

Since $m \leq 2n$, the result in Introduction implies that there exists a constant T_{3*} $(\geq T_{2*})$ such that

$$\{x \in \mathbf{R} \ : \ w(x,t) > 0\} \ \cap \ \{x \in \mathbf{R} \ : \ \varphi(x,\ t\ ; s) > 0\} \neq \phi \quad \text{for } t \geq T_{3*}.$$

Let $E_0(t) = \{x \in \mathbf{R} \ : \ w(x,t) > 0\}$, $E_s(t) = \{x \in \mathbf{R} \ : \ \varphi(x,t\ ;\ s) > 0\}$ and $E_R(t) = \{x \in \mathbf{R} \ : \ v_*^{(1)}(x,t) > 0\}$, where $v_*^{(1)}$ is the function in Lemma 8.1. Since $E_0(t)$, $E_R(t)$ and $E_s(t)$ extend as t increases we obtain by [9], [12] and [13] that

$$E_0(t) \ \cup \ E_R(t) \ \cup \ (\bigcup_{s \in [T_{2*},t]} E_s(t)) \ \supset \ [0, At^{\frac{1}{m+1}}] \quad \text{for } t \geq T_{3*}. \tag{8.24}$$

Since $[0, At^{1/(m+1)}]$ is compact there exists a finite sequence $\{s_j\}_{j=1}^{J} \subset [T_{2*}, t]$ such that

$$E_0(t) \ \cup \ E_R(t) \ \cup \ (\bigcup_{j=1}^{J} E_{s_j}(t)) \ \supset \ [0, At^{\frac{1}{m+1}}]. \tag{8.25}$$

By Lemma 3.3. w and $\varphi(\cdot,\ \cdot\ ;\ s)$ are continuous in $\mathbf{R} \times [0,\infty)$ and $\mathbf{R} \times [s,\infty)$ respectively. When we put

$$\rho_+(t) = \min\{\max(w(x,t),\ v_*^{(1)}(x,t),\ \varphi(x,\ t\ ;\ s_1), \cdots,$$
$$\varphi(x,\ t\ ;\ s_J))\ :\ x \in [0, At^{\frac{1}{m+1}}]\} \quad \text{for } t \geq T_{3*},$$

Lemma 4.1, (8.23) and (8.25) imply that the function ρ_+ is positive in $[T_{3*}, \infty)$ and satisfies

$$v(x,t) \ \geq \ \rho_+(t) \quad \text{for } t \geq T_{3*} \ \text{ and a.e. } \ x \in [0, At^{\frac{1}{m+1}}]. \tag{8.26}$$

By a similar argument, we find a positive constant T_{4*} and a positive function ρ_- defined on $[T_{4*}, \infty)$ such that

$$v(x,t) \geq \rho_-(t) \ \text{ for } t \geq T_{4*} \ \text{ and a.e. } \ x \in [-At^{\frac{1}{m+1}}, 0]. \tag{8.27}$$

Case : $m > 2n$.

By a result stated in Introduction and Lemma 4.1, there exists a positive constant b such that

(8.28) $$\text{supp } u(\cdot,t) \;\subset\; [-\frac{b}{2},\ \frac{b}{2}] \;\text{ for } t \geq 0.$$

By (8.28) and Lemma 4.4, v is continuous in $(\mathbf{R}\backslash(-2b/3,\ 2b/3)) \times [0,\infty)$. Hence, by Lemma 8.1, there exist two positive constants a and T_{5*} such that

(8.29) $$v(x,t) \;>\; at^{-\frac{m}{(m+1)}\cdot\frac{1}{m-1}}$$

$$\text{for } \; t \geq T_{5*} \text{ and } \; |\, x \,| \geq b.$$

For $T \geq 0$, we let $w(x,t;T)$ be the solution of

(8.30) $$w_t = (w^m)_{xx} - \lambda w \;\text{ in }\; \mathbf{R} \times (T,\infty)$$

with

$$w(\cdot,T;T) = v(\cdot,T) \;\text{ on }\; \mathbf{R},$$

where $\lambda = |\, v_0 \,|_{\infty,\mathbf{R}}^{2n-1}$.

By Remark 3.2, Lemma 3.5 and 4.1, we have

$$|\, u^n(\cdot,t)v^{n-1}(\cdot,t) \,|_{\infty,\mathbf{R}} \leq \; \lambda \quad \text{for } t \geq 0,$$

and hence, by the comparison theorem,

(8.31) $$w(\cdot,t;T) \leq v(\cdot,t) \text{ for } t \text{ and } T \text{ with } t \geq T \geq 0.$$

If we can show

(8.32) $$\bigcup_{T\geq 0}\bigcup_{t>T}(\text{ supp } w(\cdot,t;T))^{\circ} \supset [-b,b],$$

this together with (8.32) will give the desired result. In fact, for $x_0 \in [-b,b]$ there exist $t(x_0) > T(x_0)$ such that

$$w(x_0,t(x_0);T(x_0)) > 0.$$

Then, there exists an open interval $I(x_0)$ such that

$$x_0 \in I(x_0) \subset (\text{ supp } w(\cdot, t(x_0); T(x_0)))^\circ.$$

Since supp $w(\cdot, t; T(x_0))$ is monotonically non-decreasing with respect to t, we get

$$I(x_0) \subset (\text{ supp } w(\cdot, t; T(x_0)))^\circ$$

$$\text{for } t \geq t(x_0).$$

Since there exist finite points $\{x_j\}_{j=1}^J \subset [-b, b]$ such that $\bigcup_{j=1}^J I(x_j) \supset [-b, b]$, we obtain

$$v(x,t) \geq h(t) > 0$$

for any $x \in [-b, b]$ and any $t \geq t_*$, where $t^* = \max_j t(x_j)$ and $h(t) = \min_j \max_x w(x, t; T(x_j))$.

In order to show (8.32), we assume

$$\bigcup_{T \geq 0} \bigcup_{t > T} (\text{ supp } w(\cdot, t; T))^\circ \not\supset [-b, b].$$

Then there exists a point $x_* \in [-b, b]$ such that

$$w(x_*, t; T) = 0 \text{ for } t \text{ and } T \text{ with } t > T \geq 0.$$

Therefore, v_0 and u_0 satisfy $v_0(x_*) = u_0(x_*) = 0$. Note that

$$\text{(8.33)} \qquad \int_{x_*}^{\infty} v_0(y)dy > \int_{x_*}^{\infty} u_0(y)dy$$

or

$$\int_{-\infty}^{x_*} v_0(y)dy > \int_{-\infty}^{x_*} u_0(y)dy.$$

We assume (8.33) without loss of generality.

Let

$$v_{*0} = u_0 \chi_{(-\infty, x_*]} + v_0 \chi_{[x_*, \infty)},$$

$$u_{*0} = u_0$$

and let v_* and u_* be the solutions of (1.1) with $u_*(\cdot,0) = v_{*0}$ and $u_*(\cdot,0) = u_{*0}$. Then, by Lemma 4.1 we have for $t \geq 0$

$$v(x,t) \geq v_*(x,t) \geq u_*(x,t) \geq u(x,t) \geq 0 \tag{8.34}$$

$$\text{for a.e.} \quad x \in \mathbf{R}.$$

For $T \geq 0$, we let $w_*(\cdot,\cdot;T)$ be the solution of (8.30) with $w_*(\cdot,T;T) = v_*(\cdot,T)$. Then, since $T \geq 0$ $w(\cdot,\cdot;T) \geq w_*(\cdot,\cdot;T)$ in $\mathbf{R} \times (T,\infty)$, we get

$$w_*(x_*,t;T) = 0 \ \text{ for } t \text{ and } T \text{ with } t > T \geq 0.$$

For $T \geq 0$, we let $z_*(\cdot,\cdot;T)$ be the solution of (2.2) in $\mathbf{R} \times (T,\infty)$ with $z_*(\cdot,T;T) = v_*(\cdot,T)$.

Then, we have

$$w_*(x,t;T) = e^{\lambda(T-t)} z_*(x, \int_0^{t-T} e^{-(m-1)\lambda s} ds; T). \tag{8.35}$$

By the comparison theorem and (8.35), z_* satisfies

$$z_*(x,s+T;T) = 0 \text{ for } s \in \left(0, \frac{1}{(m-1)\lambda}\right) \tag{8.36}$$

and for $t \geq T$,

$$z_*(\cdot,t;T) \geq v_*(\cdot,t) \ \text{ a.e. on } \mathbf{R}. \tag{8.37}$$

For each $T \geq 0$, let

$$v_1(\cdot,\cdot;T) = z_*(\cdot,\cdot;T)\chi_{[x_*,\infty)}.$$

Then, we shall show that $v_1(\cdot,\cdot;T)$ is the solution of (2.2) in $\mathbf{R} \times (T, T + 1/(\lambda(m-1))$ with $v_1(\cdot,T;T) = v_*(\cdot,T)\chi_{[x_*,\infty)}$. For this we take $t_0, t_1 \in (T, T+1/(\lambda(m-1))]$ with $t_0 < t_1$ and $x_0, x_1 \in \mathbf{R}$ with $x_0 < x_1$.

For $\delta > 0$, we set

$$\rho_\delta(x) = \begin{cases} \exp(-(\delta^2 - (x-(x_*+\delta))^2)^{-1}) & \text{if } \mid x-(x_*+\delta) \mid < \delta, \\ 0 & \text{if } \mid x-(x_*+\delta) \mid \geq \delta \end{cases}$$

and

$$h_\delta(x) = \int_{-\infty}^{x} \rho_\delta(y)dy.$$

For $f \in C^{2,1}([t_0, t_1] \times [x_0, x_1])$ with $f(x_0, t) = f(x_1, t) = 0$, z_* satisfies

$$I(z_*, fh_\delta, [t_0, t_1] \times [x_0, x_1]) = 0.$$

Since there exists a constant $C = C(t_0, t_1)$ such that

$$| (z_*^{m-1})_x(\cdot, t; T) |_{\infty, \mathbf{R}} \leq C \quad \text{for } t \in [t_0, t_1],$$

we have

$$\begin{aligned} 0 &= I(z_*, fh_\delta, [t_0, t_1] \times [x_0, x_1]) \\ &= \int_{t_0}^{t_1} \int_{x_0}^{x_1} \{-(z_*^m)_x(f_x h_\delta + f\rho_\delta) + z_* f_t h_\delta\} dx dt \\ &\qquad -[\int_{x_0}^{x_1} z_* h_\delta f dx]_{t_1}^{t_1}, \end{aligned}$$

and, letting $\delta \to 0$,

$$I(z_* \chi_{[x_*, \infty)}, f, [t_0, t_1] \times [x_0, x_1]) = 0. \tag{8.38}$$

Therefore $v_1(\cdot, \cdot, ; T)$ is the solution of (2.2) in $\mathbf{R} \times (T, T + 1/(\lambda(m-1)))$.

Let w be the solution of (1.4) with $w(\cdot, 0) = u_0$ and let $v_2 = w\chi_{(-\infty, x_*]}$. By Lemma 4.1, we see

$$v_*(x, t) \geq w(x, t) \text{ for } t \geq 0 \text{ and a.e. } x \in \mathbf{R},$$

and hence, by (8.36) and (8.37),

$$w(x_*, t) = 0 \text{ for } t \geq 0.$$

By a similar argument we can show that v_2 is the solution of (1.4) ($p = 2n$) with $v_2(\cdot, 0) = u_0 \chi_{(-\infty, x_*]}$.

For $T \geq 0$, let $\tilde{v}(\cdot, \cdot; T) = v_1(\cdot, \cdot; T) + v_2$ and let $\tilde{u} = v_2$. Then $\tilde{v}(\cdot, \cdot; T)$ and $\tilde{u}$ are the solutions of (1.1) in $\mathbf{R} \times (T, T + 1/(\lambda(m-1)))$. First we take $T = 0$.

Since

$$\tilde{v}(\cdot, 0; 0) = v_{*0} \geq u_{*0} \geq \tilde{u}(\cdot, 0) \text{ a.e. on } \mathbf{R},$$

we have by Lemma 4.1 that for $t \in [0, 1/(\lambda(m-1))]$

$$\tilde{v}(\cdot, t; 0) \geq v_*(\cdot, t) \geq u_*(\cdot, t) \geq \tilde{u}(\cdot, t) \text{ a.e. on } \mathbf{R}. \tag{8.39}$$

Since $\tilde{u} = \tilde{v}$ if $x < x^*$ $v_* = u_*$ in $(-\infty, x_*] \times [0, 1/(\lambda(m-1))]$.

By induction, we conclude

$$v_* = u_* = w\chi_{(-\infty, x_*]} \text{ in } (-\infty, x_*] \times [0, \infty).$$

By a result stated in Introduction, $\bigcup_{t \geq 0} \text{supp } w(\cdot, t)$ is bounded in $\mathbf{R}$, while $\bigcup_{t \geq 0} \text{supp } v_*(\cdot, t)\chi_{(-\infty, x_*]}$ is not bounded in $\mathbf{R}$ by Lemma 7.1. This contradicts to the above equality, and (8.32) is now proved.

Q.E.D.

For $T, N > 0$, we shall consider the following boundary value problem:

$$w_t = (w^m)_{xx} - w^{2n} \text{ in } Q_T^N, \tag{8.40}$$

$$w(\pm N\zeta(t), t) = 4t^{-\frac{1}{2n-1}} \text{ on } [T, \infty), \tag{8.41}$$

where $Q_T^N = \{(x, t) \in \mathbf{R} \times [0, \infty) \ : \ t \geq T, \ x \in [-N\zeta(t), \ N\zeta(t)]\}$ and ζ is the function in Lemma 8.1.

Lemma 8.3. If $2n - 2 < m \leq 2n$, there exist two positive constants T_3, N and a positive classical solution w_b of (8.40) and (8.41) in $Q_{T_3}^N$ such that

$$u(x, t) \leq w_b(x, t) \leq v(x, t)$$

$$\text{for } t \geq T_3 \text{ and a.e. } x \in [-N\zeta(t), \ N\zeta(t)],$$

$$v(\pm N\zeta(t), t) > 5t^{-\frac{1}{2n-1}} \text{ in } [T_3, \infty),$$

$$\text{supp } w(\cdot,t) \subset \left[-\frac{N}{2}\zeta(t), \frac{N}{2}\zeta(t)\right] \quad \text{for } t \geq T_3,$$

where w is the generalized solution of (1.4) with $w(\cdot,0) = u_0$.

Proof. By a result in Introduction, there exist two positive constants $N \geq 2$ and T_{1*} such that

$$\text{(8.42)} \qquad \text{supp } w(\cdot,t) \subset [-\frac{N}{2}\zeta(t),\ \frac{N}{2}\zeta(t)] \quad \text{for } t \geq T_{1*}.$$

Hence, there exists a positive constant $T_{2*} \geq \max(T_{1*}, T_1)$ such that

$$\text{(8.43)} \qquad v_*^{(1)}(N\zeta(t),t) \ > \ 5t^{-\frac{1}{2n-1}} \quad \text{for } t \geq T_{2*},$$

where T_1 and $v_*^{(1)}$are the constant and the function in Lemma 8.1, respectively.

By (8.42), Lemma 4.1 and Lemma 4.4 v is continuous in $\mathbf{R} \times [T_{2*},\infty)\backslash Q_{T_{2*}}^{2N/3}$ and we have by (8.43) and Lemma 8.1 that

$$\text{(8.44)} \qquad v(N\zeta(t),t) \ > \ 5t^{-\frac{1}{2n-1}} \quad \text{for } t \geq T_{2*}.$$

Let $T_{3*} = \max(T_{2*}, T_2)$, where T_2 is the constant in Lemma 8.2. Then, by Lemma 4.1, Lemma 8.1 and Lemma 8.2, v satisfies

$$\text{(8.45)} \qquad v(x,T_{3*}) \ \geq \ \max\{v_*^{(1)}(x,T_{3*}),\ v_*^{(1)}(-x,T_{3*}),\ \rho(T_{3*}), \\ w(x,T_{3*})\} \quad \text{for a.e. } x \in [-N\zeta(T_{3*}), N\zeta(T_{3*})],$$

where A is the constant in Lemma 8.1 and ρ is the function in Lemma 8.2. By Theorem 0 in [8] w is smooth in $\{(x,t) \in \mathbf{R}\times(0,\infty) \ : \ w(x,t) > 0\}$ and by (8.42), (8.44), (8.45) there exists a positive function $w_{0b} \in H^{2+\beta}([-N\zeta(T_{3*}),\ N\zeta(T_{3*})])$, $0 < \beta < 1$, such that

$$\text{(8.46)} \qquad w(x,T_{3*}) \ \leq \ w_{0b}(x) \ \leq \ v(x,T_{3*})$$

$$\text{for a.e. } x \in [-N\zeta(T_{3*}),\ N\zeta(T_{3*})]$$

and that w_{0b} satisfies the compatibility condition of first order for (8.40) and (8.41) in $Q^N_{T_{3*}}$.

Since w_{0b} is positive on $[-N\zeta(T_{3*}),\ N\zeta(T_{3*})]$, we can show by Theorem 6.1 of Section 5 in [14] and the change of variables the existence of the positive solution $w_b \in H_{loc}^{2+\beta,1+\beta/2}(Q^N_{T_{3*}})$ of (8.40) and (8.41) with $w_b(\cdot, T_{3*}) = w_{0b}$ in $Q^N_{T_{3*}}$. Since w_b is a generalized solution of (1.4) in $Q^N_{T_{3*}}$, we obtain by (8.42), (8.46), Lemma 4.1 and Lemma 4.5 that

$$(8.47) \qquad u(x,t) \le w(x,t) \le w_b(x,t)$$

$$\text{for } t \ge T_{3*} \text{ and a.e. } x \in [-N\zeta(t), N\zeta(t)].$$

Thus, w_b is a subsolution of (3.1) with $P = u^n$ and $q = n$ in $Q^N_{T_{3*}}$ and by (8.43), (8.46), (8.47) and Lemma 4.4 we have

$$(8.48) \qquad u(x,t) \le w_b(x,t) \le v(x,t)$$

$$\text{for } t \ge T_{3*} \text{ and a.e. } x \in [-N\zeta(t), N\zeta(t)].$$

Q.E.D.

Lemma 8.4. If $2n < m$, there exists a positive constant b such that

$$\text{supp}\, u(\cdot, t) \subset [-\frac{b}{2}, \frac{b}{2}] \text{ for } t \ge 0.$$

Moreover, suppose that for such a constant b, there exist three functions $\bar{u}$, $\underline{v}$ and $w^* \in H_{loc}^{2+\beta,1+\beta/2}([-b,b] \times [T_4, \infty))$, $0 < \beta < 1$, $T_4 > 0$, satisfying the following properties :

$\bar{u}$ is a generalized supersolution for $w_t = (w^m)_{xx} - \underline{v}^n w^n$ in $[-b,b] \times [T_4, \infty)$, $\underline{v}$ is a generelized subsolution for $w_t = (w^m)_{xx} - \bar{u}^n w^n$ in $[-b,b] \times [T_4, \infty)$ and w^* is a generalized solution for $w_t = (w^m)_{xx} - w^{2n}$ in $[-b,b] \times [T_4, \infty)$,

$$\begin{aligned} \frac{1}{2} t^{-\frac{1}{2n-1}} \ &< \ \bar{u}(\pm b, t) \ \le \ w^*(\pm b, t) \ \le \ \underline{v}(\pm b, t) \\ &< \ 2t^{-\frac{1}{2n-1}} \ < \ v(\pm b, t) \ \text{ for } t \ge T_4, \end{aligned}$$

$$
\begin{aligned}
u(x,T_4) &\le \bar{u}(x,T_4) \le w^*(x,T_4) \le \underline{v}(x,T_4) \\
&\le v(x,T_4) \text{ for a.e. } x \in [-b,b], \\
& \quad u(x,t) \le w^*(x,t) \\
& \qquad \text{for } t \ge T_4 \text{ and a.e. } x \in [-b,b], \\
0 < \bar{u} &\le w^* \le \underline{v} \text{ in } [-b,b] \times [T_4,\infty).
\end{aligned}
$$

Then, those functions satisfy that

$$
\begin{aligned}
&u(x,t) \le \bar{u}(x,t) \le w^*(x,t) \le \underline{v}(x,t) \le v(x,t) \\
&\qquad \text{for } t \ge T_4 \text{ and a.e. } x \in [-b,b].
\end{aligned}
$$

Proof. We know already that there exists a positive constant b such that $\mathrm{supp} u(\cdot,t) \subset [-b/2, b/2]$ for $t \ge 0$. Let us fix such a constant b. We shall consider the following initial boundary value problems :

(8.49) $\qquad \mathcal{L}(w;q) \equiv w_t - (w^m)_{xx} + q^n w^n = 0 \text{ in } [-b,b] \times [T_4,\infty),$

(8.50) $\qquad w(b,t) = w(-b,t) = \dfrac{1}{2} t^{-\frac{1}{2n-1}} \text{ on } [T_4,\infty),$

(8.51) $\qquad w(x,T_4) = u_{0b} \text{ on } [-b,b]$

and

(8.52) $\qquad \mathcal{L}(w;q) = 0 \text{ in } [-b,b] \times [T_4,\infty),$

(8.53) $\qquad w(b,t) = w(-b,t) = 2t^{-\frac{1}{2n-1}} \text{ on } [T_4,\infty),$

(8.54) $\qquad w(x,T_4) = v_{0b} \text{ on } [-b,b]$

where q is an arbitrary function belonging to $L^\infty([-b,b] \times [T_4,\infty))$, $v_{0b} = 2T_4^{-1/(2n-1)}$ and u_{0b} is the convolution of $\max\{u(\cdot,T_4),\ (1/2)T_4^{-1/(2n-1)}\}$ and an appropriate mollifier. Therefore, u_{0b} and v_{0b} satisfy the compatibility condition of first order for (8.49) - (8.51) and (8.52) - (8.54), respectively.

Let $u_0 = v_0 = w^*$. Then there exist two sequences of the positive functions u_j and $v_j \in H_{loc}^{2+\beta,1+\beta/2}([-b,b]\times[T_4,\infty))$ satisfying the following properties :

For $j \geq 1$, v_j is the classical solution of $\mathcal{L}(w;u_{j-1}) = 0$ with (8.53) and (8.54) in $[-b,b]\times[T_4,\infty)$ and u_j is the classical solution of $\mathcal{L}(w;v_{j-1}) = 0$ with (8.50) and (8.51).

We can prove that there exist $\lim_{j\to\infty} v_j(x,t)$ and $\lim_{j\to\infty} u_j(x,t)$ in $[-b,b]\times[T_4,\infty)$. Setting $v_b(x,t) = \lim_{j\to\infty} v_j(x,t)$ and $u_b(x,t) = \lim_{j\to\infty} u_j(x,t)$, we can obtain

$$(8.55) \qquad u(x,t) \leq u_b(x,t) \leq w^*(x,t) \leq v_b(x,t) \leq v(x,t)$$

$$\text{for } t \geq T_4 \text{ and a.e } x \in [-b,b].$$

Let $\hat{v}_0 = \underline{v}$ and let $\hat{u}_0 = \bar{u}$. Then, for any $j \geq 1$, $\hat{u}_j$ is the classical solution for $\mathcal{L}(w;\hat{u}_{j-1}) = 0$ with (8.50) and (8.51), $\hat{v}_j$ is the classical solution for $\mathcal{L}(w;\hat{u}_{j-1}) = 0$ with (8.53) and (8.54). We can prove that there exist $\lim_{j\to\infty} \hat{v}_j(x,t)$ and $\lim_{j\to\infty} \hat{u}_j(x,t)$ in $[-b,b]\times[T_4,\infty)$. Setting $\hat{v}_b(x,t) = \lim_{j\to\infty} \hat{v}_j(x,t)$ and $\hat{u}_b(x,t) = \lim_{j\to\infty} \hat{u}_j(x,t)$, we obtain

$$(8.56) \quad \hat{u}_b \leq \bar{u} \leq \underline{v} \leq \hat{v}_b \leq |\, v_{0b} \,|_{\infty,[-b,b]} \quad \text{in } [-b,b]\times[T_4,\infty).$$

Therefore we can observe that

$$(8.57) \qquad \hat{u}_b(x,t) = u_b(x,t) \quad \text{for } t \geq T_4 \text{ and a.e. } x \in [-b,b],$$

$$(8.58) \qquad \hat{v}_b(x,t) = v_b(x,t) \quad \text{for } t \geq T_4 \text{ and a.e. } x \in [-b,b].$$

From (8.55) - (8.58) we conclude

$$u(x,t) \leq \bar{u}(x,t) \leq w^*(x,t) \leq \underline{v}(x,t) \leq v(x,t)$$

$$\text{for } t \geq T_4 \text{ and a.e. } x \in [-b,b].$$

Q.E.D.

Let $T > 0$ and let

$$G_T = \{(x,t) \in \mathbf{R}\times[T,\infty) \; : \; x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}]\},$$

where A is the constants in Lemma 8.1. Let us consider the following initial boundary value problem (I.B.) :

$$(I.B.)\begin{cases} v_t = (v^m)_{xx} - u^n v^n \text{ in } G_T, \\ u_t = (u^m)_{xx} - v^n u^n \text{ in } G_T, \\ v(\pm At^{\frac{1}{m+1}}, t) = \frac{8}{9}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}} t^{-\frac{1}{m+1}} \text{ on } [T,\infty), \\ u(\pm At^{\frac{1}{m+1}}, t) = t^{-\frac{1}{m+1}-2} \text{ on } [T,\infty), \\ v(\cdot, T) = v_{0b} \text{ on } [-AT^{\frac{1}{m+1}}, AT^{\frac{1}{m+1}}], \\ u(\cdot, T) = u_{0b} \text{ on } [-AT^{\frac{1}{m+1}}, AT^{\frac{1}{m+1}}]. \end{cases}$$

First, we shall construct some convenient initial functions.

Lemma 8.5. There exist a positive constant T_5 and two positive functions $\underline{v}$ and $\bar{u}$ such that

$$\underline{v}(\cdot;t),\ \bar{u}(\cdot;t) \in H^{2+\beta}([-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}])$$
$$\text{for } t \geq T_5,$$

$$v(x,t) \geq \underline{v}(x;t) \geq \bar{u}(x;t) \geq w(x,t) \geq u(x,t)$$
$$\text{for } t \geq T_5 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}],$$

$$\underline{v}(At^{\frac{1}{m+1}};t) = \underline{v}(-At^{\frac{1}{m+1}};t) = \frac{8}{9}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}} t^{-\frac{1}{m+1}}$$
$$\text{for } t \geq T_5,$$

$$\bar{u}(At^{\frac{1}{m+1}};t) = \bar{u}(-At^{\frac{1}{m+1}};t) = t^{-\frac{1}{m+1}-2} < \frac{1}{9}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}} t^{-\frac{1}{m+1}}$$
$$\text{for } t \geq T_5,$$

$$\min\{\underline{v}(x;t)\ ;\ x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}]\},\ \frac{1}{2}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}} t^{-\frac{1}{m+1}}$$
$$> \max\{\bar{u}(x;t)\ ;\ x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}]\}$$
$$\text{for } t \geq T_5$$

and for $T \geq T_5$ $\bar{u}(\cdot;T)$ and $\underline{v}(\cdot;T)$ satisfy the compatibility condition of first order for (I.B.), where A is the constant in Lemma 8.1 and w is the generalized solution of (1.4) with $w(\cdot,0) = u_0$.

Proof. Case : $2n-2 < m \leq 2n$.
Let α be the solution of the Cauchy problem

$$\begin{cases} \alpha' &= \alpha^{m+2n-1} - \alpha^{2n} \text{ in } (0,\infty) \\ \alpha(0) &= (1/2)^{\frac{1}{m-1}}. \end{cases}$$

Then, the solution α is monotone decreasing with respect to t and satisfies

$$\alpha(t) \sim t^{-\frac{1}{2n-1}}. \tag{8.59}$$

We set

$$\tilde{w}_*(x,t) = \alpha(t)\exp(x^2\alpha^{2n-1}(t)).$$

Then it follows that

(8.60)

$$\begin{aligned} \mathcal{L}(\tilde{w}_*) &= \tilde{w}_{*t} - (\tilde{w}_*^m)_{xx} + \tilde{w}_*^{2n} \\ &\leq -\exp(x^2\alpha^{2n-1})\{(2m-1) - (2n-1)\alpha^{2n-1}x^2\}\alpha^{m+2n-1} \\ &\quad +\alpha^{2n}\exp(x^2\alpha^{2n-1})\{\exp((2n-1)x^2\alpha^{2n-1}) - 1 - (2n-1)x^2\alpha^{2n-1}\} \\ &\qquad \text{in } \mathbf{R}\times(0,\infty). \end{aligned}$$

By (8.59) and (8.60), there exists a positive constant T_{1*} such that $\mathcal{L}(\tilde{w}_*) \leq 0$ for $t \geq T_{1*} + T_3$ and $x \in [-N\zeta(t), N\zeta(t)]$ and that

$$w_b(x,t) \geq \tilde{w}_*(x, t+T_{1*}) \tag{8.61}$$

$$\text{for } t \geq T_3 \text{ and } x \in [-N\zeta(t), N\zeta(t)],$$

where T_3, N and w_b are two constants and the function in Lemma 8.3 respectively and ζ is the function in Lemma 8.1.

Let us fix the positive constant T_{1*} satisfying (8.61) and let $w_*(x,t) = \tilde{w}_*(x, t + T_{1*})$.

By (8.61), Lemma 8.3 and the comparison theorem,
we obtain

$$w_*(x,t) \leq w_b(x,t) \leq v(x,t) \tag{8.62}$$

$$\text{for } t \geq T_3 \text{ and a.e. } x \in [-N\zeta(t), N\zeta(t)].$$

On the other hand we see by Lemma 4.5 and Lemma 8.3 that

$$w(x,t) \leq (2n-1)^{-\frac{1}{2n-1}}(t+d_1)^{-\frac{1}{2n-1}} \tag{8.63}$$

$$\text{for } (x,t) \in Q_{T_3}^N,$$

with $d_1 = (2n-1)^{-1}(|\, w(\cdot, T_3) \,|_{\infty,\mathbf{R}})^{-2n+1} - T_3$.
By (8.63), Lemma 4.1 and Lemma 8.3 we obtain

$$u(x,t) \leq (2n-1)^{-\frac{1}{2n-1}}(t+d_1)^{-\frac{1}{2n-1}} \tag{8.64}$$

$$\text{for } t \geq T_3 \text{ and a.e. } x \in \mathbf{R}.$$

By (8.62) and (8.63), there exists a positive constant $T_{2*} \geq T_3$ such that

$$\begin{aligned} u(x,t) &\leq (2n-1)^{-\frac{1}{2n-1}}(t+d_1)^{-\frac{1}{2n-1}} < \alpha(t+T_{1*}) \\ &\leq w_*(x,t) \leq w_b(x,t) \leq v(x,t) \end{aligned} \tag{8.65}$$

$$\text{for } t \geq T_{2*} \text{ and a.e. } x \in [-N\zeta(t), N\zeta(t)].$$

Here, let

$$\underline{v}_*(x;t) = \begin{cases} \frac{8}{9}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}} t^{-\frac{1}{m+1}} \\ \qquad \text{if } \mid x \mid \in \ [At^{\frac{1}{m+1}} - 1, \infty) \\ v_*^{(1)}(x,t) \text{ if } \mid x \mid \in \ (N\zeta(t), At^{\frac{1}{m+1}} - 1) \\ w_*(x,t) \text{ if } \mid x \mid \in \ [0, N\zeta(t)] \end{cases}$$

and let

$$\bar{u}^*(x;t) = \max\{w(x,t), t^{-2-\frac{1}{m+1}}\}.$$

Then, by Lemma 8.1, there exists a positive constant $T_{3*} \geq T_{2*}$ such that for $t \geq T_{3*}$ $\underline{v}_*(\cdot;t)$ and $\bar{u}^*(\cdot;t)$ satisfy the properties of this lemma except these reqularities.

Let $\underline{v}(\cdot;t)$ be the convolution of $v_*(\cdot;t)$ and an appropriate mollifier and let $\bar{u}(\cdot;t)$ be the convolution of $\bar{u}^*(\cdot;t)$ and an appropriate mollifier. Then, for $t \geq T_{3*}$, $\underline{v}(\cdot;t)$ and $\bar{u}(\cdot;t)$ satisfy the properties of this lemma.

In case of $2n < m$.

Let ρ be the function in Lemma 8.2.

Let T_{4*} be the positive constant such that

$$T_{4*} \geq T_1 \text{ and } T_2,$$

$$\frac{8}{9}\left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}}\, t^{-\frac{1}{m+1}} > (2n-1)^{-\frac{1}{2n-1}} t^{-\frac{1}{2n-1}}$$

$$\text{for any } t \geq T_{4*},$$

where T_1 and A are the constants in Lemma 8.1 and T_2 is the constant in Lemma 8.2.

We set

$$(8.66)\ w_0^*(x) = \max\{w(x,T_{4*}),\ \min(\rho(T_{4*}), (2n-1)^{-\frac{1}{2n-1}} T_{4*}^{-\frac{1}{2n-1}})\}$$

$$\text{for } x \in \mathbf{R}.$$

By Theorem 0 in [8], there exists a unique positive classical solution w^*

of (1.4) with $w^*(\cdot, T_{4*}) = w_0^*$ in $\mathbf{R} \times [T_{4*}, \infty)$ satisfying the following properties :

$$(8.67)\ (2n-1)^{-\frac{1}{2n-1}}(t+d_3)^{-\frac{1}{2n-1}} \leq w^*(x,t) \leq (2n-1)^{-\frac{1}{2n-1}}t^{-\frac{1}{2n-1}}$$

$$\text{for } (x,t) \in \mathbf{R} \times [T_{4*}, \infty),$$

where

$$d_3 = (2n-1)^{-1}\{\min(\rho(T_{4*}), (2n-1)^{-\frac{1}{2n-1}}T_{4*}^{-\frac{1}{2n-1}})\}^{-2n+1} - T_{4*}.$$

By Lemma 4.1 and the comparison theorem, the solution w^* satisfies

$$u(x,t) \leq w(x,t) \leq w^*(x,t) \quad \text{for } t \geq T_{4*}$$

$$\text{and a.e. } x \in \mathbf{R},$$

and w^* is a generalized subsolution of (3.1) with $P = u^n$ and $q = n$ in $G_{T_{4*}}$. Then, by (8.66), (8.67), Lemma 4.4 and the definitions of ρ and T_{4*} we have

$$(8.68) \qquad u(x,t) \leq w^*(x,t) \leq v(x,t)$$

$$\text{for } t \geq T_{4*} \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}].$$

Now, by a result in Introduction, Lemma 4.1 and Lemma 8.1 there exist two positive constants b and $T_{5*} \geq T_{4*}$ such that

$$\operatorname{supp} u(\cdot,t) \subset [-\frac{b}{2}, \frac{b}{2}] \quad \text{for } t \geq T_{5*},$$

$$AT_{5*}^{\frac{1}{m+1}} > b \geq 1,$$

$$v(x,t) > 5t^{-\frac{1}{2n-1}}$$

$$\text{for } t \geq T_{5*} \text{ and } x \in [-At^{\frac{1}{m+1}}, -b] \cup [b, At^{\frac{1}{m+1}}].$$

We set

$$\xi(t) = 2b\left(\frac{T+1}{t+1}\right)^{\frac{m-2n}{2n-1}} - \frac{b}{2},$$

$$\ell(t) = \frac{1}{S}(2b - \xi(t))^{-P},$$

with $P = 2Kb^2(T+1)^{(m-2n)/(2n-1)} + 2$, where $T > T_{5*}$. K and S are constants chosen later.

We set further

$$\begin{aligned} W(x,t) &= \frac{m}{m-1}(w^*(x,t))^{m-1}, \\ U(x,t) &= \frac{m}{m-1}(w^*(x,t))^{m-1}(1-\ell(x-\xi)_+^P) \end{aligned}$$

and

$$V(x,t) = \frac{m}{m-1}(w^*(x,t))^{m-1}(1+\ell(x-\xi)_+^P)\ .$$

Now, let us consider the differential operator :

$$\mathcal{M}(F;H) = F_t - (m-1)FF_{xx} - (F_x)^2 + \lambda H^{\frac{n}{m-1}} F^{\frac{n+m-2}{m-1}},$$

with $\lambda = m((m-1)/m)^{(2n+m-2)/(m-1)}$.

Then we observe that

(8.69) $$\mathcal{M}(W\ ;\ W) = 0 \ \text{ in } \ \mathbf{R} \times (T_{4*}, \infty).$$

We shall find T, K and S such that

(8.70) $$\mathcal{M}(U\ ;\ V) \geq 0 \ \text{ in } \ [-b,b] \times [T,\infty),$$

(8.71) $$\mathcal{M}(V\ ;\ U) \leq 0 \ \text{ in } \ [-b,b] \times [T,\infty).$$

First, we shall consider (8.70).

We see

(8.72)

$$\begin{aligned} \mathcal{M}(U;V) = & \Big[W_t(1-\ell(x-\xi)_+^P) - (m-1)WW_{xx}(1-\ell(x-\xi)_+^P)^2 \\ & - (W_x)^2(1-\ell(x-\xi)_+^P)^2 \\ & + \lambda W^{\frac{2n+m-2}{m-1}}(1+\ell(x-\xi)_+^P)^{\frac{n}{m-1}}(1-\ell(x-\xi)_+^P)^{\frac{n+m-2}{m-1}}\Big] \\ & + \Big[W\{-\ell'(x-\xi)_+^P + P\ell\xi'(x-\xi)_+^{P-1}\} - W^2P^2\ell^2(x-\xi)_+^{2P-2} \\ & +(m-1)P(P-1)W^2\ell(x-\xi)_+^{P-2}(1-\ell(x-\xi)_+^P)\Big] \end{aligned}$$

$$+\left[2mP\ell(x-\xi)_+^{P-1}WW_x(1-\ell(x-\xi)_+^P)\right]$$

$$= I + II + III \text{ in } [-b,b]\times[T,\infty).$$

Let $S > 1$. Then, we see for $y \in [0, 1/S]$

$$(8.73)\ (1-y)^{\frac{n-1}{m-1}}(1+y)^{\frac{n}{m-1}} - 1 \geq (1+\frac{1}{S})^{-\frac{m-2}{m-1}}(1-\frac{1}{S^2})^{\frac{n-1}{m-1}-1}\times$$

$$\times\left\{\frac{1}{m-1}(1 - \frac{2n-2}{S} - (2n-1)\frac{1}{S^2})\right\}y.$$

Let

$$C_{nm}(S) = (1+\frac{1}{S})^{-\frac{m-2}{m-1}}(1-\frac{1}{S^2})^{\frac{n-1}{m-1}-1}\times$$

$$\times\left\{\frac{1}{m-1}(1 - \frac{2n-2}{S} - (2n-1)\frac{1}{S^2})\right\}$$

and let S be a constant such that

$$(8.74)\qquad C_{nm}(S) \geq \frac{1}{2(m-1)}.$$

By (8.69), (8.73) and (8.74) we have

$$(8.75)\qquad I \geq \{\frac{\lambda}{2(m-1)}W^{\frac{2n+m-2}{m-1}} + (m-1)WW_{xx} + (W_x)^2\}\times$$

$$\times\ \ell(x-\xi)_+^P(1-\ell(x-\xi)_+^P) \text{ in } [-b,b]\times[T,\infty),$$

and hence, by (8.75) and Lemma 6.3, there exists a positive constant $T > T_{5*}$ such that

$$(8.76)\qquad I \geq 0 \text{ in } [-b,b]\times[T,\infty).$$

To treat II we observe

$$(8.77)\qquad \ell'(t) \leq 0.$$

Then, we have

(8.78)

$$II \geq (m-1)P(P-1)\ell(x-\xi)_+^{P-2}W\left\{1-(1+\frac{P}{(m-1)(P-1)})\ell(x-\xi)_+^P\right.$$

$$-\frac{1}{(m-1)(P-1)W}(x-\xi)_+ 2b(\frac{m-2n}{2n-1})(T+1)^{\frac{m-2n}{2n-1}}(t+1)^{-\frac{m-2n}{2n-1}-1}\Big\}$$

$$\text{in } [-b,b]\times[T,\infty).$$

Here, by choosing T larger, if neccessary, we have

$$W(x,t) \geq \left(\frac{2}{3}\right)^{m-1} \frac{m}{m-1}(2n-1)^{-\frac{m-1}{2n-1}} t^{-\frac{m-1}{2n-1}} \quad \text{for } t \geq T. \tag{8.79}$$

Now, we choose S, K such that

$$S \geq 3\left(1+\frac{2}{m-1}\right) \tag{8.80}$$

and

$$K \geq \left(\frac{3}{2}\right)^{m-1} \frac{3}{m}(2n-1)^{\frac{m+2n-2}{2n-1}}(m-1). \tag{8.81}$$

Then, by (8.78) - (8.81) we obtain

$$II \geq \frac{1}{3}(m-1)P(P-1)\ell(x-\xi)_+^{P-2}W^2 \quad \text{in } [-b,b]\times[T,\infty). \tag{8.82}$$

It follows from (8.80) - (8.82) that

(8.83)
$$II+III \geq (m-1)P(P-1)\ell(x-\xi)_+^{P-2}W^2\left\{\frac{1}{3} - \frac{m}{(m-1)K}\frac{|W_x|}{W}\right\}$$

$$\text{in } [-b,b]\times[T,\infty).$$

By choosing T larger, if neccessary, it follow from Lemma 6.2, (8.79) and (8.83) that

$$II + III \geq 0 \quad \text{in } [-b,b]\times[T,\infty). \tag{8.84}$$

Thus, from (8.76) and (8.84) we conclude (8.70).

If we choose S satisfying (8.74), (8.80) and

$$S > \left(1 - (\frac{3}{4})^{m-1}\right)^{-1},\ (2^{m-1}-1)^{-1}, \tag{8.85}$$

then by (8.79) we have that

$$(8.86)\qquad U(\pm b,t) > \frac{m}{m-1}\left(\frac{1}{2}\right)^{m-1}(2n-1)^{-\frac{m-1}{2n-1}}t^{-\frac{m-1}{2n-1}}$$

$$\text{in } [-b,b]\times[T,\infty).$$

By the definitions of the functions ℓ and ξ, we see

$$(8.87)\qquad V(x,T) = U(x,T) = W(x,T) \text{ on } [-b,b].$$

By a quite similar argument obtaining (8.70), since we can prove (8.71), we shall omit the proof of (8.71).

Further, (8.67) and (8.85) we have

$$(8.88)\qquad V(\pm b,t) < 2^{m-1}\frac{m}{m-1}(2n-1)^{-\frac{m-1}{2n-1}}t^{-\frac{m-1}{2n-1}}$$

$$\text{for } t \geq T.$$

Thus, (8.68), (8.70), (8.71), (8.86) - (8.88) and Lemma 8.4 we obtain

$$(8.89)\qquad u(x,t) \leq \left(\frac{m-1}{m}U(x,t)\right)^{\frac{1}{m-1}} \leq w^*(x,t)$$

$$\leq \left(\frac{m-1}{m}V(x,t)\right)^{\frac{1}{m-1}} \leq v(x,t)$$

$$\text{for } t \geq T \text{ and a.e. } x \in [-b,b].$$

Similarly, setting

$$\hat{U}(x,t) = W(x,t)(1-\ell(-x-\xi(t))_+^p),$$

and

$$\hat{V}(x,t) = W(x,t)(1+\ell(-x-\xi)_+^p),$$

these have the same estimates as U and V, respectively, and we obtain

$$(8.90)\qquad u(x,t) \leq \left(\frac{m-1}{m}\hat{U}(x,t)\right)^{\frac{1}{m-1}} \leq w^*(x,t)$$

$$\leq \left(\frac{m-1}{m}\hat{V}(x,t)\right)^{\frac{1}{m-1}} \leq v(x,t)$$

$$\text{for } t \geq T \text{ and a.e. } x \in [-b,b].$$

By the definitions of the functions ℓ and ξ, there exists a positive constant T_{6*} such that

$$(8.91)\qquad \ell(t)(x-\xi(t))_+ \;\geq\; \frac{1}{S}(\frac{5}{2}b)^{-P}(\frac{b}{4})^{P} = \frac{1}{S}\left(\frac{1}{10}\right)^{P} \quad \text{in } [0,b]\times[T_{6*},\infty).$$

Here, let

$$\underline{v}_*(x;t) \;=\; \begin{cases} \dfrac{8}{9}\left(\dfrac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} A^{\frac{2}{m-1}}t^{-\frac{1}{m+1}} \\ \qquad\qquad \text{if } \mid x\mid \in\; [At^{\frac{1}{m+1}}-1,\infty) \\ v_*^{(1)}(x,t) \text{ if } \mid x \mid \in\; (b, At^{\frac{1}{m+1}}-1) \\ w^*(x,t)\left(1+\dfrac{1}{S}(\dfrac{1}{10})^{P}\right)^{\frac{1}{m-1}} \text{ if } \mid x\mid \in\; [0,b] \end{cases}$$

and let

$$\bar{u}^*(x;t) \;=\; \begin{cases} t^{-2-\frac{1}{m+1}} \text{ if } \mid x\mid \in [b,\infty) \\ w^*(x,t)\left(1-\dfrac{1}{S}(\dfrac{1}{10})^{P}\right)^{\frac{1}{m-1}} \text{ if } \mid x \mid \in\; [0,b). \end{cases}$$

Then, by (8.68) and (8.91), there exists a positive constant $T_{7*} \geq T_{6*}$ such that for $t \geq T_{7*}$ $\underline{v}_*(\cdot;t)$ and $\bar{u}^*(\cdot;t)$ satisfy the properties of this lemma except these regularities.

Let $\underline{v}(\cdot;t)$ be the convolution of $\underline{v}_*(\cdot;t)$ and an appropriate mollifier and let $\bar{u}(\cdot;t)$ be the convolution of $\bar{u}^*(\cdot;t)$ and an appropriate mollifier. Then, for $t \geq T_{7*}$, $\underline{v}(\cdot;t)$ and $\bar{u}(\cdot;t)$ satisfy the properties of this lemma.

Q.E.D.

Lemma 8.6. Let T_6 be the constant such that

$$T_6 \;\geq\; \max\{T_5,1\}\ ,\ T_6^{\frac{m+2-2n}{m+1}}\left(\frac{1}{2}H\right)^{2n-1} \;>\; \frac{1}{m+1}\ ,$$
$$T_6^{-\frac{1}{m+1}-2} \;<\; \frac{1}{9}H\ ,\ \frac{1}{m+1} \;>\; 2T_6^{-2n+\frac{m}{m+1}} \mid v_0 \mid_{\infty,\mathbf{R}}^{n-1},$$
$$AT_6^{\frac{1}{m+1}} \;>\; N\zeta(T_6) \text{ and } b,$$

where A, N and b are the constants in Lemma 8.1, Lemma 8.3 and Lemma 8.4 respectively, ζ is the constant in Lemma 8.1 and $H = ((m-1)/(2m(m+1)))^{1/(m-1)} A^{2/(m-1)}$.

Let the constant T in (I.B.) be T_6 and set $u_{0b} = \bar{u}(\cdot; T_6)$ and $v_{0b} = \underline{v}(\cdot; T_6)$ in (I.B.), where $\bar{u}$ and $\underline{v}$ are the functions in Lemma 8.5.

Then, there exists a unique pair of positive classical solutions u_b and $v_b \in H_{loc}^{2+\beta, 1+\beta/2}(G_{T_6})$ of (I.B.), satisfying the following properties :

$$u(x,t) \leq u_b(x,t) \leq v_b(x,t) \leq v(x,t)$$

$$\text{for } t \geq T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}]$$

and there exists a positive constant h_* such that

$$v_b(x,t) \geq h_* t^{-\frac{1}{m+1}}$$

$$\text{for } t \geq T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}],$$

where u and v are the solutions of (1.1) and (1.2).

Proof. Since $\bar{u}(\cdot; T_6)$ and $\underline{v}(\cdot; T_6)$ are positive functions satisfying the compatibility condition of first order for (I.B.), then by Theorem 7.1 of Section 7 in [14] and the change of variables there exists a unique pair of positive classical solutions u_b and $v_b \in H_{loc}^{2+\beta, 1+\beta/2}(G_{T_6})$ of (I.B.).

Let us consider the following initial boundary value problem :

$$w_t = (w^m)_{xx} - w^{2n} \text{ in } G_{T_6}, \tag{8.92}$$

$$w(\pm At^{\frac{1}{m+1}}, t) = \frac{1}{2} H t^{-\frac{1}{m+1}} \text{ on } [T_6, \infty), \tag{8.93}$$

$$w(\cdot, T_6) = w_{0b} \text{ on } [-AT_6^{\frac{1}{m+1}}, AT_6^{\frac{1}{m+1}}]. \tag{8.94}$$

There exists a positive function $w_{0b} \in H^{2+\beta}([-AT_6^{1/(m+1)}, AT_6^{1/(m+1)}])$ satisfying the compatibility condition of first order for (8.92) - (8.94) and the following property :

$$\bar{u}(\cdot; T_6) \leq w_{0b} \leq \underline{v}(\cdot; T_6) \text{ on } [-AT_6^{\frac{1}{m+1}}, AT_6^{\frac{1}{m+1}}]. \tag{8.95}$$

Then, by Theorem 4.1 of Section 4 in [14] and the change of variables, there exists a unique positive classical solution $w_b \in H_{loc}^{2+\beta,1+\beta/2}(G_{T_6})$ for (8.92) - (8.94). By Lemma 8.5 and (8.95) we see

$$w(\cdot, T_6) \ \leq \ w_{0b} \ \text{ on } [-AT_6^{\frac{1}{m+1}}, AT_6^{\frac{1}{m+1}}],$$

and hence, by Lemma 4.5,

$$w \ \leq \ w_b \ \text{ in } G_{T_6}. \tag{8.96}$$

Let $\mathcal{L}(p; q)$ be the differential operator defined in (8.49).

By Theorem 4.1 of Section 4 in [14] and the change of variables there exists a unique positive classical solution $v_1 \in H_{loc}^{2+\beta,1+\beta/2}(G_{T_6})$

$$\mathcal{L}(w \ ; \ w_b) = 0 \ \text{ in } Q_{T_6} \tag{8.97}$$

$$w(\pm At^{\frac{1}{m+1}}, t) = \frac{8}{9} H t^{-\frac{1}{m+1}} \ \text{ on } [T_6, \infty), \tag{8.98}$$

$$w(\cdot; T_6) \ = \ \underline{v}(\cdot; T_6) \text{ on } [-AT_6^{\frac{1}{m+1}}, AT_6^{\frac{1}{m+1}}]. \tag{8.99}$$

Since v_1 is a subsolution of $\mathcal{L}(v_1; u) = 0$ in G_{T_6}, Lemma 4.4, Lemma 8.1, Lemma 8.6, the comparison theorem and (8.95) give

$$w_b(x,t) \ \leq \ v_1(x,t) \ \leq \ v(x,t) \tag{8.100}$$

$$\text{for } t \geq T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}].$$

By a similar argument, there exists a unique positive classical solution $u_1 \in H_{loc}^{2+\beta,1+\beta/2}(G_{T_6})$ of

$$\mathcal{L}(w \ ; \ v_1) = 0 \ \text{ in } G_{T_6}, \tag{8.101}$$

$$w(\pm At^{\frac{1}{m+1}}, t) \ = \ t^{-\frac{1}{m+1}-2} \ \text{ on } [T_6, \infty), \tag{8.102}$$

$$w(\cdot, T_6) \ = \ \bar{u}(\cdot; T_6) \text{ on } [-AT_6^{\frac{1}{m+1}}, AT_6^{\frac{1}{m+1}}]. \tag{8.103}$$

By (8.100) u_1 is a supersolution for $\mathcal{L}(u_1\ ;\ v) = 0$ in G_{T_6}, and Lemma 4.5, Lemma 8.6, the comparison theorem and (8.96) give

$$(8.104) \qquad u(x,t) \ \le\ u_1(x,t) \ \le\ w_b(x,t)$$

$$\text{for } t \ge T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}].$$

By a similar argument, there exist two sequences of positive functions u_j and $v_j \in H^{2+\beta,1+\beta/2}_{loc}(G_{T_6})$ satisfying the following property :
For each $j \ge 2$, v_j is a unique classical solution for $\mathcal{L}(v_j\ ;\ u_{j-1}) = 0$ with (8.98) and (8.99) in G_{T_6} and u_j is a unique classical solution for $\mathcal{L}(u_j; v_j) = 0$ with (8.102) and (8.103) in G_{T_6}.

By (8.100), (8.104), Lemma 4.4, Lemma 4.5 and the comparison theorem we see

$$(8.105) \quad u(x,t) \ \le\ \cdots \le\ u_2(x,t) \ \le\ u_1(x,t) \ \le\ w_b(x,t)$$

$$\le\ v_1(x,t) \ \le\ v_2(x,t) \ \le \cdots \le\ v(x,t)$$

$$\text{for } t \ge T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}].$$

Similarly as in (8.57), (8.58) we can prove

$$v_b(x,t) = \lim_{j\to\infty} v_j(x,t) \quad \text{for } (x,t) \in G_{T_6},$$
$$u_b(x,t) = \lim_{j\to\infty} u_j(x,t) \quad \text{for } (x,t) \in G_{T_6},$$

and

$$(8.106) \qquad u(x,t) \ \le\ u_b(x,t) \ \le\ w_b(x,t) \ \le\ v_b(x,t) \ \le\ v(x,t)$$

$$\text{for } t \ge T_6 \text{ and a.e. } x \in [-At^{\frac{1}{m+1}}, At^{\frac{1}{m+1}}].$$

Setting

$$\hat{v}(y,s) = e^{\frac{1}{m+1}s} v_b(e^{\frac{1}{m+1}s} y, e^s)$$

and

$$\hat{u}(y,s) = e^{\frac{1}{m+1}s} u_b(e^{\frac{1}{m+1}s} y, e^s),$$

we see that the functions $\hat{v}$ and $\hat{u} \in H_{loc}^{2+\beta,1+\beta/2}([-A,A]\times[\log T_6,\infty))$ satisfy

$$\begin{cases} \hat{v}_s = (\hat{v}^m)_{yy} + \dfrac{y}{m+1}\hat{v}_y + \dfrac{1}{m+1}\hat{v} - e^{\frac{m+2-2n}{m+1}s}\hat{u}^n\hat{v}^n \\ \qquad\qquad \text{in } [-A,A]\times[\log T_6,\infty), \\ \hat{u}_s = (\hat{u}^m)_{yy} + \dfrac{y}{m+1}\hat{u}_y + \dfrac{1}{m+1}\hat{u} - e^{\frac{m+2-2n}{m+1}s}\hat{v}^n\hat{u}^n \\ \qquad\qquad \text{in } [-A,A]\times[\log T_6,\infty), \\ \hat{v}(\pm A,s) \;=\; \dfrac{8}{9}H \;\text{ on } [\log T_6,\infty), \\ \hat{u}(\pm A,s) \;=\; e^{-2s} \;\text{ on } [\log T_6,\infty), \\ \hat{v}(y,\,\log T_6) \;=\; T_6^{\frac{1}{m+1}}\underline{v}(yT_6^{\frac{1}{m+1}}\;;\,T_6) \;\text{ on } [-A,A], \\ \hat{u}(y,\,\log T_6) \;=\; T_6^{\frac{1}{m+1}}\bar{u}(yT_6^{\frac{1}{m+1}}\;;\,T_6) \;\text{ on } [-A,A]. \end{cases}$$

Let S^* be an arbitrary positive constant such that $S^* > \log T_6$ and let λ be an arbitrary positive constant such that

$$1-\frac{\lambda}{m+1} \;\geq\; e^{-\frac{2}{m+1}\lambda},\; 2 \;>\; e^{\frac{m+2-2n}{m+1}\lambda}, \tag{8.107}$$
$$\lambda 2ne^{\frac{m-2n+2}{m+1}S^*}\left(|\, v_0\,|_{\infty,\mathbf{R}}\, e^{\frac{2}{m+1}S^*}\right)^{2n-1} \;<\; 1.$$

Let $S_j = \lambda j + \log T_6$. Then, there exists a finite sequence of positive classical solutions $v_\lambda(\cdot,S_j)\in H^{2+\beta}([-A,A])$, $j=1,\cdots,[\lambda^{-1}(S^*-\log T_6)]$, of the problem

$$\hat{v}_\lambda(\cdot,S_j) \;-\; \lambda(\hat{v}_\lambda^m)_{yy}(\cdot,S_j) \;-\; \frac{\lambda}{m+1}y\hat{v}_{\lambda y}(\cdot,S_j) \tag{8.108}$$
$$-\;\frac{\lambda}{m+1}\hat{v}_\lambda(\cdot,S_j) \;=\; \hat{v}_\lambda(\cdot,S_{j-1}) \;-\; \lambda e^{\frac{m+2-2n}{m+1}S_j}\;\times$$
$$\times\; \hat{v}_\lambda^n(\cdot,S_{j-1})\hat{u}_\lambda^n(\cdot,S_{j-1}) \;\text{ on } [-A,A]$$

with

$$\hat{v}_\lambda(\pm A,S_j) \;=\; \frac{8}{9}H, \tag{8.109}$$

where $\hat{v}_\lambda(\cdot,S_0) \;=\; \hat{v}(\cdot,S_0)$ on $[-A,A]$. Also, for $j=1,\cdots,$ $[\lambda^{-1}(S^*-\log T_6)]$, there exists a positive classical solution $\hat{u}_\lambda(\cdot,S_j)\in$

$H^{2+\beta}([-A,A])$ of the problem

$$(8.110) \qquad \hat{u}_\lambda(\cdot,S_j) - \lambda(\hat{u}_\lambda^m)_{yy}(\cdot,S_j) - \frac{\lambda}{m+1}y\hat{u}_{\lambda y}(\cdot,S_j) - \frac{\lambda}{m+1}\hat{u}_\lambda(\cdot,S_j) = \hat{u}_\lambda(\cdot,S_{j-1}) - \lambda e^{\frac{m+2-2n}{m+1}S_j} \times \times \hat{v}_\lambda^n(\cdot,S_{j-1})\hat{u}_\lambda^n(\cdot,S_{j-1}) \text{ on } [-A,A]$$

with

$$(8.111) \qquad \hat{u}_\lambda(\pm A,S_j) = e^{-2S_j},$$

where $\hat{u}_\lambda(\cdot,S_0) = \hat{u}(\cdot,S_0)$ on $[-A,A]$.

In fact, by (8.109), we see

$$\hat{v}_\lambda(y,S_0) - \lambda e^{\frac{m+2-2n}{m+1}S_0}\hat{v}_\lambda^n(y,S_0)\hat{u}_\lambda^n(y,S_0) > 0 \quad \text{for } y \in [-A,A],$$

$$\hat{u}_\lambda(y,S_0) - \lambda e^{\frac{m+2-2n}{m+1}S_0}\hat{v}_\lambda^n(y,S_0)\hat{u}_\lambda^n(y,S_0) > 0 \quad \text{for } y \in [-A,A]$$

and

$$0 < \hat{u}_\lambda(y,S_0) \le \hat{v}_\lambda(y,S_0) \le |\, v_0\,|_{\infty,\mathbf{R}}\, T_6^{\frac{1}{m+1}} \quad \text{for } y \in [-A,A].$$

Hence by Theorem 5.1 of Section 8 in [18] and the comparison theorem we find a required unique positive classical solution $\hat{v}_\lambda(\cdot,S_1), \hat{u}_\lambda(\cdot,S_1) \in H^{2+\beta}([-A,A])$ satisfying

$$0 < \hat{u}_\lambda(y,S_1) \le \hat{v}_\lambda(y,S_1) \le e^{\frac{2}{m+1}\lambda}\,|\, v_0\,|_{\infty,\mathbf{R}}\, T_6^{\frac{1}{m+1}} \qquad \text{for } y \in [-A,A].$$

By induction, there exist two finite sequence of the positive classical solutions $\hat{u}_\lambda(\cdot,S_j)$ and $\hat{v}_\lambda(\cdot,S_j) \in H^{2+\beta}([-A,A])$ with the property

$$(8.112) \qquad 0 < \hat{u}_\lambda(y,S_j) \le \hat{v}_\lambda(y,S_j)$$

$$\leq \; e^{\frac{2}{m+1}\lambda j} \mid v_0 \mid_{\infty,\mathbf{R}} T_6^{\frac{1}{m+1}}$$

for $j = 1, 2, \cdots, [\lambda^{-1}(S^* - \log T_6)]$

and $y \in [-A, A]$.

Further, we can show

(8.113) $$\mid \hat{u}_\lambda(\cdot, S_j) \mid_{\infty,[-A,A]} \leq \frac{1}{2}H$$

for $j = 0, 1, \cdots, [\lambda^{-1}(S^* - \log T_6)]$.

Indeed, setting $E_j = \mid \hat{u}_\lambda(\cdot, S_j) \mid_{\infty,[-A,A]}$ for $j = 0, 1, \cdots, [\lambda^{-1}(S^* - \log T_6)]$, we see by (8.109), (8.112), Lemma 8.5 and the definitions of T_6 that

$$E_1 \leq \frac{1}{2}H,$$

and by induction, we have (8.113).

For each $j = 0, 1, \cdots, [\lambda^{-1}(S^* - \log T_6)]$, we set $d_j = \min\{\hat{v}_\lambda(y, S_j) : y \in [-A, A]\}$.

We shall show that

(8.114) $$d_j - E_j \geq \delta_0 \quad \text{for } j = 0, 1, \cdots, [\lambda^{-1}(S^* - \log T_6)],$$

where $\delta_0 = \min(d_0 - E_0, \; 7H/18)$.
Let y_{j*} and y_j^* be the points satisfying $d_j = \hat{v}_\lambda(y_{j*}, S_j)$ and $E_j = \hat{u}_\lambda(y_j^*, S_j)$, respectively.

In case of y_j^* and $y_{j*} \in (-A, A)$, we observe by (8.109) and (8.112) that

(8.115) $$(d_j - E_j)\left(1 - \frac{\lambda}{m+1}\right)$$

$$\geq \left(d_{j-1} - \lambda e^{\frac{m+2-2n}{m+1}S_j} E_{j-1}^n d_{j-1}^m\right) - \left(E_{j-1} - \lambda e^{\frac{m+2-2n}{m+1}S_j} \times \right.$$

$$\left. \times \, d_{j-1}^m E_{j-1}^n\right) = d_{j-1} - E_{j-1}.$$

In case of $y_{j*} = \pm A$, we get by (8.113)

$$(8.116) \qquad d_j - E_j \geq \frac{8}{9}H - \frac{1}{2}H = \frac{7}{18}H.$$

In case of $y_{j*} \in (-A, A)$ and $y_j^* = \pm A$, we have by (8.112) and the definitions of λ and T_6,

$$\begin{aligned} & e^{\frac{m+2-2n}{m+1}S_j}\hat{u}_\lambda(x, S_{j-1})^n \hat{v}_\lambda(x, S_{j-1})^{n-1} \\ \leq \ & 2e^{\frac{m+2-2n}{m+1}S_{j-1}}e^{-2S_{j-1}n} \mid v_0 \mid_{\infty,\mathbf{R}}^{n-1} T_6^{\frac{n-1}{m+1}} \times e^{2\frac{n-1}{m+1}(S_{j-1}-\log T_6)} \\ < \ & \frac{1}{m+1} \text{ for } y \in [-A, A] \end{aligned}$$

and that $E_{j-1} \geq e^{-2S_{j-1}} > e^{-2S_j} = E_j$

Then, since

$$\begin{aligned} d_j\left(1 - \frac{\lambda}{m+1}\right) &\geq d_{j-1} - \lambda 2e^{\frac{m}{m+1}S_{j-1}-2nS_{j-1}} \mid v_0 \mid_{\infty,\mathbf{R}}^{n-1} d_{j-1} \\ &> d_{j-1}\left(1 - \frac{\lambda}{m+1}\right), \end{aligned}$$

we have

$$(8.117) \qquad d_j - E_j > d_{j-1} - E_j > d_{j-1} - E_{j-1}$$

By (8.115) - (8.117) we conclude (8.114).

For $j = 1, 2, \cdots, [\lambda^{-1}(S^* - \log T_6)]$, let

$$\begin{aligned} f_j(y) = & \int_{S_{j-1}}^{S_j} (\hat{v}_s(y,s) - \hat{v}_s(y, S_j))ds \\ & + \lambda e^{\frac{m-2n+2}{m+1}S_j}\{\hat{u}^n(y, S_{j-1})\hat{v}^n(y, S_{j-1}) \\ & \qquad - \hat{u}^n(y, S_j)\hat{v}^n(y, S_j)\} \end{aligned}$$

and let

$$\begin{aligned} g_j(y) = & \int_{S_{j-1}}^{S_j} (\hat{u}_s(y,s) - \hat{u}_s(y, S_j))ds \\ & + \lambda e^{\frac{m-2n+2}{m+1}S_j}\{\hat{v}^n(y, S_{j-1})\hat{u}^n(y, S_{j-1}) \\ & \qquad - \hat{v}^n(y, S_j)\hat{u}^n(y, S_j)\}. \end{aligned}$$

Then we observe that for $y \in [-A, A]$

$$(8.118) \qquad \hat{v}(y,S_j) - \lambda(\hat{v}^m)_{yy}(y,S_j) - \frac{\lambda}{m+1} y\hat{v}_y(y,S_j)$$

$$- \frac{\lambda}{m+1}\hat{v}(y,S_j) = \hat{v}(y,S_{j-1}) - \lambda e^{\frac{m-2n+2}{m+1}S_j} \times$$

$$\times \hat{u}^n(y,S_{j-1})\hat{v}^n(y,S_{j-1}) + f_j(y)$$

$$\text{for } j = 1, 2, \cdots, [\lambda^{-1}(S^* - \log T_6)]$$

and

$$(8.119) \, \hat{u}(y,S_j) - \lambda(\hat{u}^m)_{yy}(y,S_j) - \frac{\lambda}{m+1} y\hat{u}_y(y,S_j)$$

$$- \frac{\lambda}{m+1}\hat{u}(y,S_j) = \hat{u}(y,S_{j-1}) - \lambda e^{\frac{m-2n+2}{m+1}S_j}\hat{v}^n(y,S_{j-1}) \times$$

$$\times \hat{u}^n(y,S_{j-1}) + g_j(y).$$

Let

$$\varphi_k(y) = \begin{cases} -1 & \text{for } y \in (-\infty \, , \, 0], \\ ky & \text{for } y \in [-\frac{1}{k} \, , \, \frac{1}{k}], \\ 1 & \text{for } y \in [\frac{1}{k} \, , \, \infty). \end{cases}$$

We subtract (8.108) from (8.118) and muliply $\varphi_k(\hat{v}(y,S_j) - \hat{v}_\lambda(y,S_j))$. We shall denote by (8.120) the resulted equation. Similarly we subtract (8.109) from (8.119) and muliply $\varphi_k(\hat{u}(y,S_j) - \hat{u}_\lambda(y,S_j))$. We shall denote by (8.121) the resulted equation. Adding (8.120) and (8.121) and integrating over $[-A, A]$ we have by

$$\int_{-A}^{A} (\hat{v}(y,S_j) - \hat{v}_\lambda(y,S_j))\varphi_k(\hat{v}(y,S_j) - \hat{v}_\lambda(y,S_j))dy$$

$$+ \int_{-A}^{A} (\hat{u}(y,S_j) - \hat{u}_\lambda(y,S_j))\varphi_k(\hat{u}(y,S_j) - \hat{u}_\lambda(y,S_j))dy$$

$$\leq (1 - \frac{\lambda}{m+1})^{-1} \{ | \hat{v}(\cdot,S_{j-1}) - \hat{v}_\lambda(\cdot,S_{j-1}) |_{1,[-A,A]}$$

$$+ \mid \hat{u}(\cdot, S_{j-1}) \; - \; \hat{u}_\lambda(\cdot, S_{j-1}) \mid_{1,[-A,A]} \Big\}$$

$$+ \; (1 - \tfrac{\lambda}{m+1})^{-1} \Big\{ \mid f_j \mid_{1,[-A,A]} \; + \; \mid g_j \mid_{1,[-A,A]} \Big\}.$$

Letting $k \to \infty$, we have that

$$(8.122) \mid \hat{v}(\cdot, S_j) \; - \; \hat{v}_\lambda(\cdot, S_j) \mid_{1,[-A,A]} \; + \; \mid \hat{u}(\cdot, S_j) \; - \; \hat{u}_\lambda(\cdot, S_j) \mid_{1,[-A,A]}$$

$$\leq \; (1 - \frac{\lambda}{m+1})^{-1} \Big\{ \mid \hat{v}(\cdot, S_{j-1}) \; - \; \hat{v}_\lambda(\cdot, S_{j-1}) \mid_{1,[-A,A]}$$

$$+ \; \mid \hat{u}(\cdot, S_{j-1}) \; - \; \hat{u}_\lambda(\cdot, S_{j-1}) \mid_{1,[-A,A]} \Big\}$$

$$+ \; (1 - \frac{\lambda}{m+1})^{-1} \Big\{ \mid f_j \mid_{1,[-A,A]} \; + \; \mid g_j \mid_{1,[-A,A]} \Big\}.$$

Setting

$$L_0 \; = \; \max \Big\{ \mid \hat{u} \mid^{(2+\beta)}_{[-A,A]\times[\log T_6, S^*]}, \; \mid \hat{v} \mid^{(2+\beta)}_{[-A,A]\times[\log T_6, S^*]} \Big\}.$$

We obtain by (8.107) that

$$(8.123) \qquad \mid f_j \mid_{1,[-A,A]}, \; \mid g_j \mid_{1,[-A,A]} \leq \; \frac{2L_0 A}{1+\frac{\beta}{2}} \lambda^{1+\frac{\beta}{2}}$$

$$+ \; 4nAe^{\frac{m-2n+2}{m+1}S^*} L_0^{2n} \lambda^2 \; \leq \; L\lambda^{1+\frac{\beta}{2}},$$

where

$$L \; = \; 2(1+\frac{\beta}{2})^{-1} L_0 A \; + \; 4ne^{\frac{m-2n+2}{m+1}S^*} L_0^{2n} (m+1)^{1-\frac{\beta}{2}} A.$$

By (8.107), (8.122) and (8.123) we have

$$(8.124) \mid \hat{v}(\cdot, S_j) \; - \; \hat{v}_\lambda(\cdot, S_j) \mid_{1,[-A,A]} \; + \; \mid \hat{u}(\cdot, S_j) \; - \; \hat{u}_\lambda(\cdot, S_j) \mid_{1,[-A,A]}$$

$$\leq \; 2S^* e^{\frac{2}{m+1}S^*} L \lambda^{\frac{\beta}{2}} \;\; \text{for } j = 1, 2, \cdots, [\lambda^{-1}(S^* - \log T_6)].$$

Letting $\lambda \to 0$ we have by (8.114), (8.124)

$$\min \{ \hat{v}(y, s) \; : \; y \in [-A, A] \} \; - \; \mid \hat{u}(\cdot, s) \mid_{\infty,[-A,A]}$$

$$\geq \; \delta_0 \; > \; 0 \quad \text{for } s \geq \log T_6.$$

Thus, by changing the variables $e^S = t$, $ye^{S/(m+1)} = x$, we conclude

$$\min\{v_b(x,t)\ :\ x \in I(t)\} \ -\ |\, u_b(\cdot,t)\,|_{\infty,I(t)}$$
$$\geq\ \delta_0 t^{-\frac{1}{m+1}} \quad \text{for any } t \geq T_6,$$
$$\text{where } I(t) = [-At^{1/(m+1)}, At^{1/(m+1)}],$$

which proves the lemma.

Q.E.D.

By Lemma 8.1 and Lemma 8.6, there exists two positive constants h_* and h^* such that

$$h_* t^{-\frac{1}{m+1}} \ \leq\ v(x,t)\ \leq\ h^* t^{-\frac{1}{m+1}} \tag{8.125}$$

$$\text{for } t \geq T_6 \text{ and a.e. } x \in \text{ supp } u(\cdot,t).$$

Let u_0^* and $u_{0*} \in C_0(\mathbf{R})$ be the functions such that

$$0 \leq u_{0*}(x) \leq u(x,T_6) \leq u_0^*(x) \text{ a.e. } x \in \mathbf{R},$$
$$u_{0*}\ \not\equiv\ 0 \text{ in } \mathbf{R}.$$

Let u^* be the generalized solution of

$$u_t^*\ =\ (u^{*m})_{xx}\ -\ (h^* t^{-\frac{1}{m+1}})^n u^{*n} \text{ in } \mathbf{R} \times [T_6,\infty) \tag{8.126}$$

with

$$u^*(\cdot,T_6) = u_0^* \text{ in } \mathbf{R}$$

and let u_* be the generalized solution of

$$u_{*t} = (u_*^m)_{xx} - (h_* t^{-\frac{1}{m+1}})^n u_*^n \text{ in } \mathbf{R} \times [T_6,\infty) \tag{8.127}$$

with

$$u_*(\cdot,T_6)\ =\ u_{0*} \text{ in } \mathbf{R}.$$

By (8.125) u is a subsolution of (8.127) in $\mathbf{R} \times [T_6, \infty)$ and a supersolution of (8.126) in $\mathbf{R} \times [T_6, \infty)$, and we obtain by Lemma 4.3

$$u_*(x,t) \;\leq\; u(x,t) \;\leq\; u^*(x,t)$$

$$\text{for } t \geq T_6 \text{ and a.e. } x \in \mathbf{R}.$$

Therefore, by Lemma 5.2, we arrive at the desired estimates of $|\ u(\cdot,t)\ |_\infty$ and supp $u(\cdot,t)$.

By Lemma 8.1, we obtain also the desired estimates of $|\ v(\cdot,t)\ |_\infty$ and supp $v(\cdot,t)$.

The proof of Theorem 1.1 is now complete.

Q.E.D. of Theorem 1.1.

9. Proof of Corollary 1.2.

It suffices to prove

$$\bigcup_{t\geq 0} \text{supp } u(\cdot,t) = \mathbf{R} \text{ in case of } m = 2n-2.$$

There exists a positive constant h^* such that

$$v(x,t) \leq h^*(t+1)^{-\frac{1}{m+1}} \text{ for } t \geq 0 \text{ and a.e. } x \in \mathbf{R}.$$

Let u_* be the generalized solution of

$$u_{*t} = (u_*^m)_{xx} - h^{*n}(t+1)^{-\frac{n}{m+1}} u_*^n \text{ in } \mathbf{R} \times [0,\infty)$$

with

$$u_*(\cdot,0) = \min(u_0(x),1) \text{ in } \mathbf{R}$$

Let u_{**} be the generalized solution of

$$u_{**t} = (u_{**}^m)_{xx} - h^{*n}(t+1)^{-\frac{\hat{n}}{m+1}} u^{\hat{n}} \text{ in } \mathbf{R} \times [0,\infty)$$

with

$$u_{**}(\cdot,0) = \min(u_0(x),1) \text{ in } \mathbf{R},$$

where $\hat{n} = n - 1/4$.

By Lemma 4.3, we obtain

$$u_{**}(x,t) \leq u_*(x,t) \leq u(x,t) \quad (9.1)$$

$$\text{for } t \geq 0 \text{ and a.e. } x \in \mathbf{R}.$$

Therefore, by Lemma 5.2 we have $\bigcup_{t\geq 0} \text{supp } u_{**}(\cdot,t) = \mathbf{R}$, and by (9.1) we conclude that $\bigcup_{t\geq 0} \text{supp } u(\cdot,t) = \mathbf{R}$.

Q.E.D.

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