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Explosion problem for holomorphic diffusion processes
and its applications

（正則拡散過程の爆発問題とその応用）

谷口 説男
EXPLOSION PROBLEM FOR HOLOMORPHIC DIFFUSION PROCESSES
AND ITS APPLICATIONS

SETSUO TANIGUCHI

(Recieved

1. Introduction

A holomorphic diffusion process on an n-dimensional complex
manifold $M$ is a diffusion process $\{ (Z_t, \xi, P_Z) : \xi \in M \}$ on $M$, $\xi$ being the
life time, such that $h(Z_{t+\tau})$ is a local martingale for each stopping
time $\tau < \xi$ and $h \in \text{Hol}(M)$, the space of holomorphic functions on $M$. Such
diffusion processes connect martingales with holomorphic functions.
Thus, holomorphic diffusion processes enable us to discuss topics of
complex analysis in probabilistic terms. The aim of this paper is to
see that the conservativeness of holomorphic diffusion processes is
closely related to domains of holomorphy.

Several classes of holomorphic diffusion processes were studied
by Debiard-Gaveau[4], Fukushima-Okada[8],[9] and Kaneko-Taniguchi[16].
Especially, Fukushima and Okada [8] showed that there is a one to one
correspondence between a family of symmetric holomorphic diffusion
processes on $M$ and the totality of admissible pairs $(\Theta, m)$ on $M$ of
closed positive current $\Theta$ of bidegree $(n-1, n-1)$ and everywhere dense
positive Radon measure $m$ on $M$ (for the definition of admissible
pairs, see Section 2). For a bounded domain $D$ in $\mathbb{C}^n$, one can
construct the admissible pairs $(\Theta_b, m_b)$ on $D$ and $(\Theta_c, m_c)$ on $D \times \mathbb{C}^n \equiv
D \times (\mathbb{C}^n \backslash \{0\})$ from the Bergman kernel function $K(z; D)$ and the
Carathéodry infinitesimal metric $c(z, \xi; D)$, $z \in D$, $\xi \in \mathbb{C}^n \backslash \{0\}$, respectively.

- 1 -
For details, see Section 4. Let $M_b$ (resp. $M^\theta_C$) be the holomorphic diffusion process on $D$ (resp. $D \times \mathbb{C}^n_\times$) associated with $(\theta_b, m_b)$ (resp. $(\theta_C^\theta, m_C^\theta)$). One of the main objects of the present paper is to show that the conservativeness of either $M_b$ or $M^\theta_C$ implies that $D$ is a domain of holomorphy under suitable assumptions on the boundary. In fact, we will see that $D$ is a domain of holomorphy if either (i) $M_b$ is conservative and $\text{Cap}(U \setminus D) > 0$ for any open $U$ with $U \cap \partial D = \emptyset$, where $\text{Cap}$ stands for the Newtonian (logarithmic if $n = 1$) capacity, or (ii) $M^\theta_C$ is conservative and $D^0 = D$. See Theorem 4.

It is well known ([2], [19]) that $K(z; D)$ and $c(z, \xi; D)$ can be extended to a holomorphic extension $M$ of $D$ and its holomorphic tangent bundle $TM$, respectively, and hence so are $(\theta_b, m_b)$ and $(\theta_C^\theta, m_C^\theta)$. Therefore, in the proof of the above assertion, a key role is played by the observation that if an admissible pair $(\theta, m)$ on $M$ satisfies the "ellipticity" condition, then the conservativeness of the part on an open set $G \subset M$ of the holomorphic diffusion process associated with $(\theta, m)$ implies the smallness of $M \setminus G$. For detailed statement, see Theorem 2.

Before stating another object of this paper, let us consider an example. Let $D$ be an bounded strictly pseudoconvex domain in $\mathbb{C}^n$.

Then, the Bergman metric $\beta(D)$ of $D$ is Kählerian ([2], [18]). Denote by $M(D)$ the Brownian motion on the Kähler manifold $(D, \beta(D))$, i.e. the minimal diffusion process generated by $\Delta/2$, where $\Delta$ is the Laplace-Beltrami operator (for the definition of minimal diffusion processes, see [15]). As will be seen in Section 4, the holomorphic diffusion process $M_b$ discussed in the above paragraph coincides with $M(D)$ up to time change $t \rightarrow t/2n$. Since $D$ is strictly pseudoconvex,
\((D, \beta(D))\) looks like a space of constant holomorphic sectional curvature near the boundary ([17]). Hence the Ricci curvature is bounded from the below by a constant. Therefore \(M(D)\) is conservative, i.e. the lifetime is infinity a.s. ([13]) and so is \(M_b\).

On account of this example, it is natural to ask whether the contrary to the first assertion holds, i.e. whether either \(M_b\) and \(M_C^\phi\) is conservative if \(D\) is a domain of holomorphy. In order to answer to this question, we will establish that \(M_b\) (resp. \(M_C^\phi\)) is conservative if there is a nice exceptional set \(E \subset \partial D\) such that

\[
\limsup_k K(z_k; D) = +\infty \quad (\text{resp. } \limsup_k K(z_k, \xi_k; D) = +\infty)
\]

for every \(z_k \rightarrow z^* \in \partial D \setminus E\) (resp. \((z_k, \xi_k) \rightarrow (z^*, \xi^*) \in (\partial D \setminus E) \times \mathbb{C}_*^n\)). For details, see Theorem 3. As will be seen in Examples 4.1 and 4.2, this assertion yields that \(M_b\) and \(M_C^\phi\) are conservative if \(D\) is a domain of holomorphy with nice boundary.

This assertion follows essentially from more general criteria for conservativeness and explosion for symmetric holomorphic diffusion processes, see Theorem 1. Since we can not expect the smoothness of admissible pairs \((\theta, m)\) in general (this is the case when \((\theta, m) = (\theta_C^\phi, m_C^\phi)\)), we have no nice expression of the generator of the diffusion process. Therefore, the results due to Hasminskii [12], Ichihara [13],[14] are not applicable in our situation. We will establish our criteria by using the stochastic analysis for plurisubharmonic functions. As will be seen at the end of Section 2, our criteria yields also a unified way to test the explosion and conservativeness of these specific symmetric holomorphic diffusion...

The organization of this paper is as follows. We will begin Section 2 with giving a brief review on the symmetric holomorphic diffusion processes. We will then give the above mentioned general criteria for them. In Section 3, we will see that if the part on $G$ of the holomorphic diffusion process on $M$ is conservative then $M \setminus G$ is small. Section 4 will be devoted to showing that the conservativeness of either $M_b^\phi$ or $M_C^\phi$ implies that $D$ is a domain of holomorphy. A criterion for $M_b^\phi$ or $M_C^\phi$ to be conservative will be also given in the same section. Several examples will be presented at the end of the section to illustrate our results.

2. Conservativeness

In this section, we will discuss the conservativeness of symmetric holomorphic diffusion processes. We first give a brief review on symmetric holomorphic diffusion processes, following Fukushima and Okada [8],[9]. Let $M$ be a $\sigma$-compact connected complex manifold of complex dimension $n$. A pair $(\theta,m)$ of closed positive current $\theta$ of bidegree $(n-1,n-1)$ and everywhere dense positive Radon measure $m$ on $M$ is said to be admissible if the symmetric form

$$E^\theta(u,v) = \int_M du \wedge d^c v \wedge \theta, \quad u,v \in C_0^\infty(M)$$

is closable on $L^2(M;m)$, the space of $m$-square integrable functions,
where $d=\partial+\bar{\partial}$ is the exterior derivative and $d^C=i(\partial-\bar{\partial})$. We denote by $U(M)$ the totality of admissible pairs on $M$. For $(\theta,m)\in U(M)$, the minimal closed extension $E(\theta,m)$ of $E^\theta$ with the domain $F(\theta,m)$ is a $C^\infty_0$-regular local Dirichlet form on $L^2(M;m)$. Then, every $h\in\text{Hol}(M)$ is $E(\theta,m)$-harmonic. As usual, the associated capacity for compact set $K\subset M$ is defined by

$$\text{Cap}(\theta,m)(K) = \inf\left\{ \int_M u^2 dm + E(\theta,m)(u,u) : u\in C^\infty_0(M), u\geq 1 \text{ on } K \right\}$$

and is extended to the capacity for any set as a Choquet capacity, which we call the $E(\theta,m)$-1-capacity. Throughout this paper, by "$E(\theta,m)$-q.e." we mean "except for a set of $E(\theta,m)$-1-capacity zero".

By virtue of the theory of Dirichlet spaces [5], we obtain a diffusion process $M(\theta,m)=\{(Z_t,\xi,p^\theta_z) : z\in M\}$ associated with this $E(\theta,m)$, up to equivalence, where $\xi$ is the life time. Then, each $h(Z_{t\wedge \tau})$, $h\in\text{Hol}(M)$, is a local martingale for every stopping time $\tau<\xi$ under $p^\theta_z$, $E(\theta,m)$-q.e. $z\in M$. We call $M(\theta,m)$ the holomorphic diffusion process associated with $(\theta,m)\in U(M)$. Moreover, we say that $M(\theta,m)$ is conservative if $p^\theta_z(\xi=\infty) = 1$, $E(\theta,m)$-q.e. $z\in M$ and that it explodes if $p^\theta_z(\xi<\infty) = 1$, $E(\theta,m)$-q.e.

A function $u:M\to[-\infty,\infty)$ is called a plurisubharmonic (abbreviated to psh) if $u$ is upper semicontinuous and the derivative $dd^C u$ in the distribution sense is a positive current. A subset $N$ of $M$ is said to be pluripolar if there is a psh function $\varphi$ such that $N \subset \varphi^{-1}(-\infty)$. Finally, for locally bounded psh $u$ and closed positive $(n-1,n-1)$-current $\theta$ on $M$, we define a positive Radon measure $dd^C u\wedge \theta$ on $M$ by
\[ \int_M f \, \dd c u \wedge \theta = \int_M u \, \dd c f \wedge \theta, \quad f \in C^\infty_0(M). \]

For details, see [21]. We are now ready to state our result on conservativeness and explosion:

Theorem 1. Let \( D \) be a bounded domain in \( C^n \) and \((\theta, m) \in U(D)\). Denote by \( M(\theta, m) \) the associated holomorphic diffusion.

(i) Assume that there exists a locally bounded psh function \( p \) such that \( m \leq \dd c p \wedge \theta \). Then, \( M(\theta, m) \) is conservative, if either of the following conditions is satisfied:

(i.a) there is a sequence \( \{A_j\} \) of analytic sets in \( C^n \) such that \( A_j \cap D = \emptyset \) and for every \( z_k \rightarrow z^* \in \partial D \setminus \bigcup_{j=1}^{\infty} A_j \), it holds that

\[ (2.1) \quad \limsup_k p(z_k) = +\infty \quad \text{and} \quad \liminf_k p(z_k) > -\infty, \]

(i.b) \( m \) is equivalent to the Lebesgue measure \( V \) on \( D \) and there is a pluripolar set \( N \) in \( C^n \) such that \( N \subset \partial D \) and (2.1) holds for any \( z_k \rightarrow z^* \in \partial D \setminus N \).

(ii) Assume that there exists a locally bounded psh function \( q \) such that \( m \leq \dd c q \wedge \theta \). Then, \( M(\theta, m) \) explodes if either of the following conditions is fulfilled:

(ii.a) there is a sequence \( \{A_j\} \) of analytic sets in \( C^n \) such that \( A_j \cap D = \emptyset \) and for every \( z_k \rightarrow z^* \in \partial D \setminus \bigcup_{j=1}^{\infty} A_j \), it holds that

\[ (2.2) \quad \limsup_k q(z_k) < +\infty, \]
(2.4) \[ p_z(\theta, m, \{N[u]_{\xi^-} < +\infty, I(u) > -\infty\} \setminus \{S(u) < +\infty\}) = 0, \ E(\theta, m)\text{-q.e.}, \]

where \( A \Delta B = (A \setminus B) \cup (B \setminus A) \). If \( m \geq d c p \wedge \theta \) for some locally bounded psh \( p \) on \( M \), then it holds

(2.5) \[ p_z(\theta, m, \{\xi < +\infty \text{ and } I(p) > -\infty\} \setminus \{S(p) < +\infty\}) = 0, \ E(\theta, m)\text{-q.e. } z \in M. \]

Finally, if \( m \geq d c q \wedge \theta \) for some locally bounded psh \( q \) on \( M \), then

(2.6) \[ p_z(\theta, m, \{S(q) < +\infty\} \setminus \{\xi < +\infty\}) = 0, \ E(\theta, m)\text{-q.e. } z \in M. \]

**Proof.** By using a standard time change argument (cf. [15]), it follows from (2.3) that

(2.7) \[ u(Z_t) - u(z) = B(\langle M[u]_t \rangle) + N[t, u], \quad t < \xi, \]

under \( p_z(\theta, m) \), where \( B(t) \) is an \( \mathbb{R}^1 \)-valued Brownian motion with \( B(0) = 0 \). Since \( N[t, u] \) is a nonnegative increasing process, this implies that

\[ \limsup_{t \uparrow \xi} B(\langle M[u]_t \rangle) \leq S(u) - u(z), \]

\[ \liminf_{t \uparrow \xi} B(\langle M[u]_t \rangle) \geq I(u) - u(z) - N[\xi]. \]

Recalling that \( \limsup_{t \uparrow \infty} B(t) = +\infty \) and \( \liminf_{t \uparrow \infty} B(t) = -\infty \), we deduce from these inequalities that \( \langle M[u]_\xi \rangle \) is finite \( p_z(\theta, m)\text{-a.s. on } \{S(u) < +\infty\} \cup \{N[u]_{\xi^-} < +\infty, I(u) > -\infty\} \). Plugging this into (2.7), we obtain that (2.4) holds, because \( B(t), t \in [0, \infty) \), is continuous.
(ii.b) $m$ is equivalent to $V$ and there is a pluripolar set $N$ in $C^n$ such that $N \subset \partial D$ and (2.2) holds for any $z_k \to z^* \in \partial D \setminus N$.

For the proof of Theorem 1, we prepare two lemmas. We first recall that locally bounded psh $u$ is locally in $F(\theta, m)$, the domain of $E(\theta, m)$. See [8]. Moreover, there exist a continuous local martingale additive functional $M_t^{[u]}$ and a continuous positive additive functional $N_t^{[u]}$, $t \in [0, \xi)$ such that

(i) the Revuz measure of $N_t^{[u]}$ is $dd^Cu \wedge \theta$ and

(ii) the semimartingale $u(Z_t)$ has a decomposition

$$u(Z_t) - u(z) = M_t^{[u]} + N_t^{[u]}, \quad t < \xi,$$

under $p_z^{(\theta, m)}$, $E(\theta, m)$-q.e. $z \in M$ (cf. [8],[9]). We now proceed to the first lemma.

**Lemma 1.** Let us consider an $n$-dimensional complex manifold $M$. Let $(\theta, m) \in U(M)$ and $M(\theta, m) = \{(Z_t, \xi, p_{Z_t}^{(\theta, m)}): z \in M\}$ be the associated symmetric holomorphic diffusion process. For $R$-valued function $f$ on $M$, we denote by $I(f)$ and $S(f)$ the random variables given by

$$I(f) = \liminf_{t \uparrow \xi} f(Z_t), \quad S(f) = \limsup_{t \uparrow \xi} f(Z_t).$$

For every locally bounded psh function $u$ on $M$, it holds that
To see the second and the third assertions, it suffices to mention that $\xi \geq N_{\xi}^{[p]}$ (resp. $\leq N_{\xi}^{[q]}$) if $m \geq \text{dd}^{c}p \wedge \theta$ (resp. $\leq \text{dd}^{c}q \wedge \theta$). The proof is complete.

Lemma 2. Let us consider a bounded domain $D$ in $\mathbb{C}^{n}$. Let $(\theta, m) \in U(D)$ and $M(\theta, m) = \{(Z_{t}, \xi, p_{Z}(\theta, m)): \xi \in D\}$ be the associated holomorphic diffusion process. Then,

$$\begin{equation}
(2.8) \quad p_{z}(\theta, m)[\lim_{t \uparrow \xi} Z_{t} \text{ exists}] = 1, \quad E(\theta, m)-\text{q.e. } \xi \in D.
\end{equation}$$

If $A$ is an analytic set in $\mathbb{C}^{n}$ such that $A \cap D = \emptyset$, then

$$\begin{equation}
(2.9) \quad p_{z}(\theta, m)[\lim_{t \uparrow \xi} Z_{t} \in A] = 0, \quad E(\theta, m)-\text{q.e.}
\end{equation}$$

Finally, if $m$ is equivalent to the Lebesgue measure $\nu$ on $D$ and $N$ is a pluripolar set in $\mathbb{C}^{n}$ such that $N \subset \partial D$, then

$$\begin{equation}
(2.10) \quad p_{z}(\theta, m)[\lim_{t \uparrow \xi} Z_{t} \in N] = 0, \quad E(\theta, m)-\text{q.e.}
\end{equation}$$

Proof. Since $D$ is bounded, each component of $Z_{t}$ is a bounded martingale on $[0, \xi)$ under $p_{Z}(\theta, m)$, $E(\theta, m)$-q.e. Thus, the martingale convergence theorem implies that $\lim_{t \uparrow \xi} Z_{t}$ exists $p_{Z}(\theta, m)$-a.s.

To see the second assertion, let $w_{0} \in A \cap \partial D$. By the definition of analytic sets, there are an open set $U$ in $\mathbb{C}^{n}$ and $w^{1}, w^{2}, \ldots, w^{k} \in \text{Hol}(U)$ such that $w_{0} \in U$ and $U \cap A = \{w^{1} = \cdots = w^{k} = 0\}$. By shrinking $U$ if necessary, we may and will assume that $w^{i}$'s are all bounded on $U$. Put
\[ \tau = \inf\{t > 0 : Z_t \notin U \cap D\}. \]

Then, \( \omega^i(Z_t) \) is a \( C^1 \)-valued continuous martingale on \( [0, \xi \wedge \tau) \) such that \( \langle \omega^i(Z), \omega^i(Z) \rangle_t = 0 \). By a standard time change argument (cf. [15]), we have

\[ \omega^i(Z_t) = \omega^i(z) + B^i(\langle \omega^i(Z), \omega^i(Z) \rangle_t), \quad t < \xi \wedge \tau, \]

under \( p^{(\theta, m)}_z \), \( E^{(\theta, m)} \)-q.e., where \( B^i(t) \) is a \( C^1 \)-valued Brownian motion with \( B^i(0) = 0 \). By the argument similar to that in the proof of Lemma 1, we see that \( \langle \omega^i(Z), \omega^i(Z) \rangle_{(\xi \wedge \tau)} < +\infty \) a.s. Since \( A \cap D = \emptyset \), \( \omega^i(z) \neq 0 \) for some \( 1 \leq i = i(z) \leq k \) for every \( z \in U \cap D \). Moreover, \( C^1 \)-valued Brownian motion never hits \( -\omega^i(z) \). Hence

\[ p^{(\theta, m)}_z [\lim_{t \uparrow \xi} Z_t \in U \cap A, \xi < \tau] = 0, \quad E^{(\theta, m)} \)-q.e. \( z \in U \cap D \).

Therefore, by [5: Theorem 4.2.1], there exists a Borel set \( \bar{N} \subset D \) such that

\begin{align*}
(2.11) \quad & \text{Cap}^{(\theta, m)}(\bar{N}) = 0, \\
(2.12) \quad & p^{(\theta, m)}_z [\lim_{t \to \xi} Z_t \in U \cap A, \xi < \tau] = 0, \quad z \in U \cap (D \setminus \bar{N}), \\
(2.13) \quad & p^{(\theta, m)}_z [Z_t \in D \setminus \bar{N}, 0 \leq t < \xi] = 1, \quad z \in D \setminus \bar{N}.
\end{align*}

Let

\[ A_r = \{ \lim_{t \uparrow \xi} Z_t \in U \cap A \text{ and } Z_t \in U \cap D \text{ for } r \leq t < \xi \}. \]
Then, (2.12) implies

\[(2.14) \quad p_{z}^{(\theta, m)}(A_{0}) = 0 \quad \text{for} \quad z \in U \cap (D \setminus N).\]

Combining (2.11), (2.13) and (2.14) with the Markov property, we have

\[(2.15) \quad p_{z}^{(\theta, m)}(A_{r}) = E_{z}^{(\theta, m)}[p_{z}^{(\theta, m)}(A_{0}) ; \{Z_{r} \in U \cap D, r < \xi}\]

\[= 0,
\]

for \(E^{(\theta, m)}\)-q.e. \(z \in D\), where \(E_{z}^{(\theta, m)}\) stands for the expectation with respect to \(p_{z}^{(\theta, m)}\). Note that

\[\{\lim_{t \uparrow t_{\xi}} Z_{t} \in U \cap A\} \subset \bigcup_{r} A_{r},\]

where the union is taken over all nonnegative rational numbers \(r\).

Thus, (2.15) yields

\[(2.16) \quad p_{z}^{(\theta, m)}[\lim_{t \uparrow t_{\xi}} Z_{t} \in U \cap A] = 0 \quad \text{\(E^{(\theta, m)}\)-q.e.}\]

Covering \(A \cap \partial D\) with countably many \(U\)'s as above, we can conclude from (2.16) that (2.9) holds.

We finally verify the third assertion. For this purpose, we modify the argument in the proof of [7:Theorem 1]. For a bounded domain \(\Omega \subset C^{n}\), the extremal function \(u^{*}_{E}(z; \Omega)\) of a set \(E \subset \Omega\) is defined by
\[ u_E(z; \Omega) = \sup(\nu(z); \nu \text{ is a nonpositive psh function on } \Omega \text{ with } \nu \leq -1 \text{ on } E) \]
\[ u_E^*(z; \Omega) = \limsup_{w \to z} u_E(w; \Omega). \]

We set
\[ C_#(E; \Omega) = -\int_{\Omega} u_E^*(z; \Omega) \nu(dz). \]

It is known ([1],[9]) that

(i) \[ C_#(E; \Omega) = \inf\{C_#(O; \Omega); \text{ } O \text{ is open, } O \supset E\}, \]
(ii) \[ C_#(E; \Omega) \leq C_#(E; \Omega') \text{ if } \Omega \subset \Omega'. \]

Let \( N \subset \mathcal{E}D \) be a pluripolar set in \( C^n \) and \( \Omega \) be a bounded domain such that \( \overline{D} \subset \Omega \). Then, \( C_#(N; \Omega) = 0 \) ([1]). Hence there exists a sequence \( \{O_k\} \) of open sets in \( C^n \) such that \( N \subset O_k \subset O_{k-1} \), \( k \geq 2 \) and
\[ C_#(O_k; \Omega) \to 0 \quad \text{as } k \to \infty. \]

Since \( m \) is equivalent to the Lebesgue measure \( \nu \), by [8:Lemma 4], we have
\[ \int_D v_{z}^{(\theta, m)}[\sigma_k < +\infty] \nu(dz) \leq C_#(O_k; D) \leq C_#(O_k; \Omega), \quad k=1,2,\ldots, \]

where \( \sigma_k = \inf\{t > 0; Z_t \in D \cap O_k \} \). Letting \( k \to \infty \), we have
for V-a.e. \( z \in D \). Since \( m \) is equivalent to \( V \), (2.17) holds for \( m \)-a.e. \( z \in D \). Note that \( u(z) = \text{P}_z[\cap_{k=1}^{\infty}\{\sigma_k < +\infty\}] \) is excessive (for the definition of excessive functions, see [5:p.99]). Due to [5:Lemma 4.2.5], we see that (2.17) holds for \( E^{(\theta, m)} \)-q.e. \( z \in D \). Thus, (2.10) holds. The proof of Lemma 2 completes.

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. We first assume the existence of locally bounded psh function \( p \) with \( m \geq \text{dd}^C p \wedge \theta \). Since \( M^{(\theta, m)} \) has no killing inside, due to (2.8), we have

\[
p_z^{(\theta, m)}[\{z < +\infty\} \setminus \{\lim_{t \uparrow z} Z_t \in \partial D\}] = 0, \quad E^{(\theta, m)} \text{-q.e.}
\]

Combining this with (2.1) and Lemmas 1 and 2, we have

\[
p_z^{(\theta, m)}[z < +\infty] = p_z^{(\theta, m)}[z < +\infty, I(p) > -\infty, S(p) < +\infty] = 0, \quad E^{(\theta, m)} \text{-q.e.}
\]

Hence the first assertion of Theorem 1 has been seen.

To see the second assertion, assume that there is a locally bounded psh function \( q \) on \( D \) such that \( m \leq \text{dd}^C q \wedge \theta \). Since \( q \) is upper semicontinuous, \( S(q) < +\infty \) on \( \{\lim_{t \uparrow z} Z_t \in D\} \). It follows from (2.6) and (2.8) that
\[
p^{(\theta,m)}_{z}\{\lim_{t\uparrow \xi} Z_t \in D\} = p^{(\theta,m)}_{z}\{\xi < +\omega\} \cap \{\lim_{t\uparrow \xi} Z_t \in D\}
\]

for \(E^{(\theta,m)}\)-q.e. \(z \in D\). Thus, since \(M^{(\theta,m)}\) has no killing inside, we obtain

\[
p^{(\theta,m)}_{z}\lim_{t\uparrow \xi} Z_t \in \partial D = 1, \quad E^{(\theta,m)}\)-q.e.
\]

Combining this with (2.2) and Lemmas 1 and 2, we obtain

\[
1 = p^{(\theta,m)}_{z}\{S^{(q)} < +\omega\} \leq p^{(\theta,m)}_{z}\{\xi < +\omega\}, \quad E^{(\theta,m)}\)-q.e.
\]

The second assertion of Theorem 1 has been verified. The proof is completed.

Remark 2.1. Let \(u\) be a locally bounded psh function on \(D\). By the same reasoning as at the end of the proof of Lemma 1, \(M^{(\theta,m)}\) is conservative (resp. explodes) if \(dd^{C}u \wedge \theta \leq m\) (resp. \( \geq m\)) and

\[
p^{(\theta,m)}_{z}[N(u) < +\omega] = 0\text{ (resp. } =1).\]

Let \(h\) be a locally bounded upper semicontinuous function with \(dd^{C}h = 0\). Note that \(dd^{C}u = dd^{C}(u+h)\) and hence \(dd^{C}(u+h) \wedge \theta\) enjoys the same inequality as \(dd^{C}u \wedge \theta\) does. Moreover, \(N(u) = N(u+h)\). Therefore, as far as we discuss the conservativeness and explosion problem after evaluating \(p^{(\theta,m)}_{z}[N(u) < +\omega]\), there is no difference between choosing \(u\) or \(u+h\). However, Lemma 1 gives a way to estimate \(p^{(\theta,m)}_{z}[N(u) < +\omega]\) in terms of \(I(u)\) and \(S(u)\). Thus, to estimate \(p^{(\theta,m)}_{z}[N(u) < +\omega]\), a particular \(u\) will be much easier to handle than the others.
In the remainder of this section, we will apply Theorem 1 to some known examples. In what follows, D is a bounded domain in $\mathbb{C}^n$ and $\mathcal{M}(\theta, m)$ is a holomorphic diffusion process associated with $(\theta, m) \in U(D)$. Moreover, for locally bounded plurisubharmonic $u_1, \ldots, u_n$ on $D$, we will use $\ddc u_1 \wedge \cdots \wedge \ddc u_k$ to denote the $(k,k)$-closed positive currents defined inductively by

$$
\int_D f \wedge \ddc u_1 \wedge \cdots \wedge \ddc u_k = \int_D u_k \ddc f \wedge \ddc u_1 \wedge \cdots \wedge \ddc u_{k-1}
$$

for every $C^\infty (n-k,n-k)$ form $f$ with compact support.

Case(1). Fukushima and Okada [8] showed that $\mathcal{M}(\theta, m)$ explodes if

$$
\theta = (\ddc p)^n \text{ and } m = \ddc |z|^2 \wedge \theta,
$$

where $p$ is a bounded psh function on $D$ such that $m(dz) \geq g(z) V(dz)$ for some positive continuous $g$. In this case, Assumption (ii.a) in Theorem 1 is satisfied with $q(z) = |z|^2$. Hence, Theorem 1 also implies that $\mathcal{M}(\theta, m)$ explodes.

Case(2). In [9], Fukushima and Okada showed that $\mathcal{M}(\theta, m)$ explodes if $m(D) < +\infty$ and the Poincaré type inequality holds:

$$
\int_D \varphi^2 \, dm \leq C \mathcal{E}(\theta, m)(\varphi, \varphi) \quad \text{ for every } \varphi \in C^\infty_0(D)
$$

for some constant $C > 0$. Suppose that there exists a bounded psh function $u$ on $D$ such that $m \leq \ddc u \wedge \theta$. Then, the Poincaré type inequality holds:
\[ \int_D \varphi^2 \, dm \leq 8\|u\|_\infty \, K(\theta,m)(\varphi,\varphi) \quad \text{for every } \varphi \in C^0_0(D) \]

where \( \|u\|_\infty = \sup\{|u(z)|: z \in D\} \) (see [9],[21]). In this case, Assumption (ii.a) is fulfilled with \( q = u \). Thus, even if \( m(D) = +\infty \), \( M(\theta,m) \) explodes.


\[ \theta = \{dd^c \Sigma^1_i (-\log(-\varphi_i))\}^{n-1} \quad \text{and} \quad m = \{dd^c \Sigma^1_i (-\log(-\varphi_i))\}^n \]

for some bounded plurisubharmonic negative functions \( \varphi_i \) with \( \varphi_i(z) \to 0 \) as \( z \to \partial D \). Then, it was seen that \( M(\theta,m) \) is conservative. In this case, Assumption (i.a) is satisfied with \( p = -\Sigma^1_i \log(-\varphi_i) \).

Thus, Theorem 1 also yields that \( M(\theta,m) \) is conservative.

Case(4). Finally we consider an example. Let

\[ D = \{z=(z^1,z^2) \in \mathbb{C}^2: |z^1| < 1, |z^2| < 1\}, \]

\[ \theta = dd^c |z^2|^2 \quad \text{and} \quad m = dd^c |z|^2 \wedge \theta. \]

Then, Assumption (ii.a) is fulfilled with \( q(z) = |z|^2 \). Thus, by virtue of Theorem 1, \( M(\theta,m) \) explodes. Furthermore, it is straightforward to see that \( P(z,t) = z^2 \) for \( t \geq 0 \) = 1. Hence the \( \alpha \)-order Green measures \( G_\alpha(z,\cdot) = \int_0^\infty e^{-\alpha t} P(t,z,\cdot) dt \), \( P(t,z,\cdot) \) being the transition probability of \( M(\theta,m) \), are not equivalent. Thus, the criteria due to
Ichihara [14] for explosion are not applicable directly. However, it should be noted that if we restrict ourselves to the submanifold $D_w = \{(z,w): z \in C^1 \text{ and } |z|<1\}$, then the criteria by Ichihara are applicable and yield in the end that $M(\theta,m)$ explodes.

### 3. Smallness of sets

Let $G$ be an open subset of a $\sigma$-compact connected $n$-dimensional complex manifold $M$ and $(\theta,m) \in U(M)$. The part $M_G(\theta,m)$ of $M(\theta,m) = \{(Z_t,\xi,P_z(\theta,m)): z \in M\}$ on $G$ is by definition the holomorphic diffusion process on $G$ given by

$$M_G(\theta,m) = \{(Z_t,\sigma \wedge \xi, P_z(\theta,m)): z \in G\},$$

where $\sigma = \inf\{t > 0: Z_t \notin M\setminus G\}$. Our aim of this section is to see that the conservativeness of $M_G(\theta,m)$ implies the smallness of $M\setminus G$. To state our result, we prepare some notions. We say that $A \subset M$ is of measure zero (resp. of capacity zero) if, for every coordinate neighbourhood $U$ and diffeomorphism $\varphi: U \to \varphi(U) \subset C^n$, $\varphi(A \cap U)$ is of Lebesgue measure zero (resp. of Newtonian (logarithmic if $n=1$) capacity zero). For $(k,k)$-currents $u,v$, we mean by "$u \geq v$" that $u - v$ is a positive current. Our goal of this section will be

Theorem 2. Let $G$ be an open set in a $\sigma$-compact connected $n$-dimensional complex manifold $M$ and $(\theta,m) \in U(M)$. Assume that

$$\theta \geq C(U)(dd^c \sum_{i=1}^{\infty} |z_i|^2)^{n-1} \quad \text{and} \quad m \geq C(U)(dd^c \sum_{i=1}^{\infty} |z_i|^2)^n,$$
on each relatively compact coordinate neighbourhood $U$ with a
coordinate system $z^1, \ldots, z^n$ for some $C(U)>0$. If the part $M_G^{(\theta,m)}$ of
$M^{(\theta,m)}$ on $G$ is conservative, then $M \setminus G$ is of measure zero. If,
furthermore, $m(A) = 0$ for any $A \subset M$ of measure zero, then $M \setminus G$ is of
capacity zero.

Proof Let $M^{(\theta,m)} = \{(Z_t, \xi, p_z^{(\theta,m)}): z \in M\}$ be the holomorphic
diffusion process associated with $(\theta,m)$. By (3.1), the
conservativeness of $M_G^{(\theta,m)}$ implies that

$$
(3.3) \quad p_z^{(\theta,m)}[\sigma < +\infty] = 0 \quad E^{(\theta,m)} -\text{q.e. } z \in G,
$$

because $A \subset G$ is of $E_G^{(\theta,m)}$-1-capacity zero if and only if it is of
$E^{(\theta,m)}$-1-capacity zero (see [5: Theorem 4.4.2]). Recall that

$$
(3.4) \quad p_z^{(\theta,m)}[\sigma = 0] = 1 \quad E^{(\theta,m)} -\text{q.e. } z \in M \setminus G
$$

(see [5:p.94]). Thus, (3.3) and (3.4) imply the identity

$$
(3.5) \quad (f x_{M \setminus G})(z) = E_z^{(\theta,m)}[e^{-\sigma f(Z_\sigma)}] \quad E^{(\theta,m)} -\text{q.e. } z \in M,
$$

for every $f \in C^\infty_0(M)$, where $x_A(z) = 1$ or $0$ accordingly as $z \in A$ or not and
$E_z^{(\theta,m)}$ stands for the expectation with respect to $p_z^{(\theta,m)}$. By virtue
of [5: Theorem 4.4.1], we can conclude from (3.5) that

$$
(3.6) \quad f x_{M \setminus G} \in F^{(\theta,m)} \quad \text{for every nonnegative } f \in C^\infty_0(M),
$$
where \( F(\theta, m) \) is the domain of \( E(\theta, m) \).

Let \( z \in \overline{G} \) and \( z^1, \ldots, z^n \) be a local coordinate system on a relatively compact coordinate neighbourhood \( U \) of \( z \), where \( \overline{G} \) is the closure of \( G \) in \( M \). Then, \( U \cap G \neq \emptyset \). By identifying \( U \) with a bounded open set in \( \mathbb{C}^n \) through the coordinate system, we can construct the absorbing barrier Brownian motion on \( U \). We denote by \( (F', E') \) the corresponding Dirichlet space. Then, combined with Assumption (3.2), (3.6) yields that

\[
(3.7) \quad f x_{U \setminus G} \in F' \quad \text{for every nonnegative } f \in C_0^\infty(U)
\]

and hence

\[
(3.8) \quad x_{U \setminus G} \text{ is locally in } F'.
\]

Recall that \( (F', E') \) is irreducible, i.e., either \( A \) or \( U \setminus A \) is of Lebesgue measure zero if \( A \) is locally in \( F' \) ([6]). Thus, either \( U \cap G \) or \( U \setminus G \) is of Lebesgue measure zero. Because the open set \( U \cap G \) is not empty, \( U \setminus G \) is of Lebesgue measure zero. In particular, \( U \setminus \overline{G} = \emptyset \) and \( U \subset \overline{G} \). This implies that \( \overline{G} \) is open and closed and hence \( \overline{G} = M \), for \( M \) is connected. Therefore, \( M \setminus G \) is of measure zero.

We next assume, moreover, that \( m(A) = 0 \) for every \( A \subset M \) of measure zero. By the above observation, we have

\[
(3.9) \quad m(M \setminus G) = 0.
\]
This and (3.3) imply that $M \setminus G$ is of $E^{(\theta, m)}$-1-capacity zero ([5]).

Therefore, by Assumption (3.2), we see that $U \setminus G = U \cap (M \setminus G)$ is of $E'$-1-capacity zero, where $(F', E')$ is the Dirichlet space of the absorbing barrier Brownian motion on $U$ as in the proceeding paragraph.

Recall that $A \subset U$ is of capacity zero if and only if it is of $E'$-1-capacity zero. Hence $M \setminus G$ is of capacity zero. The proof is completed.

Remark 3.1. The generator $A$ of $M^{(\theta, m)}$ is expressed formally as

$$A \varphi = \frac{dd^C \varphi \wedge \theta}{dm}, \quad \varphi \in C^\infty(M)$$

(cf.[21]). Let $U$ be a coordinate neighbourhood with a coordinate system $z^1, \ldots, z^n$. We denote by $V$ the Lebesgue measure on $U$ induced through this coordinate system. Suppose that it holds for some $a^{ij}$, $b \in C^\infty(U)$ that

$$dm = b \, dV \quad \text{and} \quad \theta(idz^i \wedge dz^j) = a^{ij} \, dV.$$ 

Then, we have

$$A \varphi = \frac{1}{b} \sum_{i,j=1}^n a^{ij} \frac{\partial^2 \varphi}{\partial z^i \partial z^j}.$$ 

Moreover, Assumption (3.1) is equivalent to that

$$(a^{ij})_{1 \leq i, j \leq n} \geq C (\delta^{ij})_{1 \leq i, j \leq n} \quad \text{and} \quad b \geq C \text{ on } U \text{ for some } C > 0.$$
Thus, Assumption (3.1) can be thought of as an assumption on the ellipticity of the generator $A$.

Without the "ellipticity" assumption (3.2), the assertion of Theorem 2 does not hold in general. For example, let $M = \mathbb{C}^2$ and $G = \{z \in \mathbb{C}^2 : |z| > 1\}$. Then, $M \setminus G$ is not of measure zero. Take $\varphi \in C_0^\infty(\mathbb{R}^1)$ such that $\varphi \geq 0$, $\text{supp}[\varphi] \subseteq [1,2]$ and $\int_{-\infty}^{+\infty} \varphi(x) \, dx = 1$. Define $\psi(x) = \int_1^{x} \int_1^{y} \varphi(w) \, dw$, $p(z) = \psi(|z|^2)$ and $\theta = dd^C p$. Then, $(\theta, V)$, $V$ being the Lebesgue measure on $\mathbb{C}^2$, is an admissible pair and the associated holomorphic diffusion is generated by

$$A = 16\sum_{i,j=1}^{2} a^{ij} (\partial^2 / \partial z^i \partial \bar{z}^j),$$

where

$$a^{ij} = \begin{pmatrix}
\psi'(|z|^2) + \varphi(|z|^2)|z|^2 & -\varphi(|z|^2) z \bar{z}^{1-2} \\
-\varphi(|z|^2) z^{-1} \bar{z}^{2} & \psi'(|z|^2) + \varphi(|z|^2) |z|^1 \bar{z}^{2}
\end{pmatrix}.$$

Since

$$\sup \left\{ \frac{|\psi'(x)|}{|x-1|^k}, \frac{|\varphi(x)|}{|x-1|^k} : x > 1 \right\} < +\infty, \ k = 0, 1, \ldots$$

$\psi'(x) = 1$ and $\varphi(x) = 0$ for $x \in [2, +\infty)$,

it is straightforward to see that
for every $z \in G$ and $T > 0$. Thus, we see that the part of $M(\theta, m)$ on $G$ is conservative.

4. Domains of holomorphy

This section is devoted to the study of domains of holomorphy as an application of Theorems 1 and 2. Let $D \subset \mathbb{C}^n$ be a bounded domain. The Bergman kernel function $K(z; D)$ of $D$ is defined by

$$K(z; D) = \sup\{ |f(z)|^2 / \|f\|^2 : f \in L^2_h(D) \},$$

where $L^2_h(D)$ is the space of holomorphic functions on $D$ with $\|f\|^2 \equiv \int_D |f(z)|^2 V(dz) < \infty$. We set

$$\rho_b(z) = \log K(z; D), \quad \theta_b = (dd^c \rho_b)^{n-1} \quad \text{and} \quad m_b = (dd^c \rho_b)^n.$$

As we will see later, $(\theta_b, m_b) \in U(D)$. Therefore, a holomorphic diffusion process associated with $(\theta_b, m_b)$ is defined. In what follows, for the sake of simplicity, we write $M_b = \{(Z_t, s, p^b_{z}) : z \in D\}$ instead of $M_{(\theta_b, m_b)} = \{(Z_t, s, p^b_{z}) : z \in D\}$. Now let us show that $(\theta_b, m_b) \in U(D)$. To do this, recall that $\rho_b$ is $C^\infty$ and strictly psh on $D$ ([2],[18]). Hence, $\theta_b$ is a closed positive current on $D$ of bidegree
(n-1,n-1) and $m_b$ is a everywhere dense positive Radon measure on $D$. Thus, it suffices to show that $E^{b}_\theta$ is closable. To this end, take a sequence \( \{u_n\} \) in $C_0^\infty(D)$ such that $\int_D u_n^2 \, dm_b \to 0$ as $n \to \infty$. Let $\varphi \in C_0^\infty(D)$. Since $p_b$ is strictly psh, we have

$$-C \chi_{\text{supp} \varphi} \, dm_b \leq dd^c \varphi \wedge \theta_b \leq C \chi_{\text{supp} \varphi} \, dm_b$$

on $D$ for some $C > 0$. This implies that

$$\int_D u_n \, dd^c \varphi \wedge \theta_b \to 0 \quad \text{as } n \to \infty. \quad \text{(4.1)}$$

On the other hand, the closedness of $\theta_b$ implies that

$$E^{b}_\theta(u_n, \varphi) = -\int_D u_n \, dd^c \varphi \wedge \theta_b.$$

Therefore,

$$E^{b}_\theta(u_n, \varphi) \to 0 \quad \text{as } n \to \infty \text{ for every } \varphi \in C_0^\infty(D)$$

and hence $E^{b}_\theta$ is closable on $L^2(D;m)$ (cf. [5]).

An important property of $M_b$ is that it coincides with the Brownian motion associated with a Kähler metric on $D$. To state more precisely, let us introduce the Bergman metric $\zeta(D)$ on $D$ defined by
\[ \beta(D) = \sum_{i,j=1}^{\infty} \frac{\partial^2 p_{ij}}{\partial z_i \partial \bar{z}_j} \, dz_i d\bar{z}_j. \]

It is known ([2], [18]) that \( \beta(D) \) is Kählerian. We observe that the time changed process \( \tilde{M}_b = \{(z_{2nt}, \xi/2n, p^b_z): z \in D\} \) is the Brownian motion on the Kähler manifold \((D, \beta(D))\). In fact, it is easy to see that

\[ dd^c \varphi \wedge \theta_b = 2^{n-2} (n-1)! \Delta \varphi \, dv \]
\[ dm_b = 2^n n! \, dv, \]

where \( \Delta \) is the Laplace-Beltrami operator and \( v \) is the volume element on \((D, \beta(D))\). Thus the generator \( A \) of \( M_b \) is expressed as

\[ A \varphi = \Delta \varphi / 4n \quad \text{for } \varphi \in C^\omega_0(D). \]

This implies that \( \tilde{M}_b \) is the Brownian motion on \((D, \beta(D))\).

To define symmetric holomorphic diffusion processes corresponding to the Carathéodory infinitesimal metric, we introduce some more notations. The Carathéodory infinitesimal metric \( c(z, \xi; D), z \in D, \xi \in C^*_+ \) is given by

\[ c(z, \xi; D) = \sup \{ |\sum_{i=1}^{\infty} \frac{\partial f}{\partial z_i}(z) \xi_i| : f: D \to \Delta \text{ is holomorphic and } f(z) = 0 \}, \]

where \( \Delta = \{ z \in C^1_+: |z| < 1 \} \). It is well known ([19]) that \( c(\cdot, \cdot; D) \) is a non-negative continuous psh function on \( D \times C^*_+ \). Let \( \text{Ext}(D) \) be the totality of pairs \((X, \pi)\) of \( \sigma \)-compact connected \( n \)-dimensional complex manifold.
X and local biholomorphism $\pi: X \to \mathbb{C}^n$ such that

(i) $D \subset X$ and $D$ is open in $X$,

(ii) $\pi(z) = z, z \in D$

(iii) every $f \in \text{Hol}(D)$ has a $g \in \text{Hol}(X)$ such that $f = g$ on $D$.

For a complex manifold $M$, we set

$$\text{SP}(M) = \{ p: M \to \mathbb{R} : (1) \text{ p is locally bounded and psh} \]

(2) on each relatively compact coordinate neighbourhood with coordinate system $z = (z^1, \cdots, z^n), p - \delta |z|^2$ is psh for some $\delta > 0 \}.$$

It is straightforward to show that, for $p \in \text{SP}(M)$, $(dd^c p)^n$ and $(dd^c p)^n$ satisfy Assumption (3.2) in Theorem 2. We use $E(D)$ to denote the totality of nonnegative $\varphi \in \text{SP}(DX^n)$ such that there is a $\tilde{\varphi} \in \text{SP}(TX^n)$ so that $\varphi = \tilde{\varphi}$ on $DX^n = TD^{n} \subset TX^n$ for every $(X, \pi) \in \text{Ext}(D)$, where $TX^n$ is an open subset of the holomorphic tangent bundle $TX$ over $X$ consisting of nonzero tangent vectors. It is easily seen that $f(|z|^2) + g(|\xi|^2) \in E(D)$ for any $C^2$-functions $f, g$ on $[0, \infty)$ with positive first and nonnegative second derivatives. For $\varphi \in E(D)$, let

$$p^\varphi_c(z, \xi) = c(z, \xi; D) + \varphi(z, \xi), \quad q^\varphi_c = (dd^c p^\varphi_c)^{2n-1} \quad \text{and} \quad m^\varphi_c = (dd^c p^\varphi_c)^{2n}.$$

By the same argument as we saw that $(\theta^\varphi_b, m^\varphi_b) \in U(D)$, we can see that $(\theta^\varphi_c, m^\varphi_c) \in U(DX^n)$. We denote by $M^\varphi_c = \{(Z_t, \xi_t), Z_c, (z, \xi) : (z, \xi) \in DX^n\}$ the
holomorphic diffusion process on $\mathbb{D} \times \mathbb{C}^n$ associated with $(\theta_c^{\phi}, m_c^{\phi})$.

We will establish the criteria for conservativeness of $M_b$ and $M_c^{\phi}$ as follows.

Theorem 3

(i) $M_b$ is conservative if there is a pluripolar set $N$ in $\mathbb{C}^n$ such that $N \subset \partial D$ and $\limsup_k K(z_k; D) = +\infty$ for every $z_k \to z^* \in \partial D \setminus N$.

(ii) $M_c^{\phi}$ is conservative if, for every $a > 0$, there exists a $c > 0$ such that

$$\{(z, \xi) \in \mathbb{D} \times \mathbb{C}^n : \phi(z, \xi) \leq a\} \subset \{(z, \xi) \in \mathbb{D} \times \mathbb{C}^n : |\xi| \leq c\}$$

and there exist analytic sets $A_j$ in $\mathbb{C}^n$ such that $A_j \cap D = \emptyset$ and $\limsup_k c(z_k, \xi_k; D) = +\infty$ for every $(z_k, \xi_k) \to (z^*, \xi^*) \in (\partial D \setminus \bigcup_j A_j) \times \mathbb{C}^n$.

Combining Theorem 2 with the well known fact that $K(\cdot; D)$ (resp. $c(\cdot, \cdot; D)$) can be extended to $X$ (resp. $TX$), $(X, \pi) \in \text{Ext}(D)$, we will have the following theorem, which is a generalized version of the result announced in [23].

Theorem 4. A bounded domain $D$ in $\mathbb{C}^n$ is a domain of holomorphy if either of the followings is satisfied:

(i) $M_b$ is conservative and $U \setminus D$ is of positive Newtonian (logarithmic for $n=1$) capacity for every open $U$ with $U \cap \partial D \neq \emptyset$,

(ii) $M_c^{\phi}$ is conservative for some $\phi \in E(D)$ and $D^0 = D$.

The difference between the assumptions on the boundary in (i) and
(ii) of Theorem 4 comes from that $K(z; D)$ is $C^\omega$ but $c(z, \xi; D)$ is only continuous in general. Before proceeding to the proofs of Theorems 3 and 4, we remark an immediate consequence of these theorems.

Corollary. A bounded domain $D \subset C^n$ is a domain of holomorphy if either of the followings holds:

(i) for each open $U$ with $U \cap \partial D \neq \emptyset$, $U \setminus D$ is of positive Newtonian (logarithmic for $n=1$) capacity and there is a pluripolar set $N$ in $C^n$ such that $N \subset \partial D$ and $\limsup_k K(z_k; D)^{z_k \rightarrow z^*} \in \partial D \setminus N$,

(ii) $D^0 = D$ and there is a sequence $\{A_j\}$ of analytic sets in $C^n$ such that $A_j \cap D = \emptyset$ and $\limsup_k c(z_k, \xi_k; D)^{z_k \rightarrow z^*, \xi_k \rightarrow \xi^*}$ for every $(z_k, \xi_k) \rightarrow (z^*, \xi^*) \in (\partial D \setminus (\bigcup_j A_j)) \times C^n$.

Proof. The first assertion is an immediate consequence of Theorems 3 and 4. To see the second one, it suffices to take $\varphi(z, \xi) = \left| z \right|^2 + \left| \xi \right|^2$.

We now proceed to the proofs of Theorems 3 and 4.

Proof of Theorem 3. Suppose that the assumption in the first assertion of Theorem 3 is satisfied. Because $\mu_b \in \text{SP}(D) \cap C^\omega(D)$, $\mu_b$ is equivalent to the Lebesgue measure $\nu$ on $D$. Moreover, since $1 \in L^2_h(D)$, it follows from the definition that

$$K(z; D) \geq 1/\nu(D).$$

Thus, Assumption (i.b) in Theorem 1 is satisfied with $p = p_b$. Hence,
by Theorem 1, we see that $M_b$ is conservative.

We next suppose that the assumption in the second assertion is fulfilled. Since $\{\varphi \leq a\} \subset \{|\xi| \leq c\}$ and $p^\varphi_c \geq 0$, by (2.5) in Lemma 1, it suffices to show that

$$
(4.2) \quad p^c_{(z, \xi)}[\{\xi < +\infty\} \cap \{\limsup_{t \uparrow s} (c(Z_t, \gamma_t; D) + I|\xi_t| < +\infty\}] = 0.
$$

Notice that the $i$-th component $\xi^i_t$ of $\xi_t$ is a continuous local martingale on $[0, s]$ with values in $C^n$. Then, by a standard time change argument (cf. [15]), we have

$$
\xi^i_t = \xi^i_0 + B^i(\xi^i, \xi^i_t), \quad t < s,
$$

for some $C^1$-valued Brownian motion with $B^i(0) = 0$. Since $\limsup_{t \uparrow s} |B^i(t)| = +\infty$, we obtain that

$$
\langle \xi^i, \xi^i \rangle_{s} < +\infty, \quad i = 1, \ldots, n, \text{ a.s. on } \{\limsup_{t \uparrow s} |\xi_t| < +\infty\}.
$$

Recall that $B^i(t)$ never hit $-\xi^i$. Hence it holds

$$
(4.3) \quad p^c_{(z, \xi)}[\{\limsup_{t \uparrow s} |\xi_t| < +\infty\}] = p^c_{(z, \xi)}[\{\limsup_{t \uparrow s} |\xi_t| < +\infty\} \cap \{\lim_{t \uparrow s} \xi_t \text{ exists in } C^n\}]
$$

Furthermore, since $M^\varphi_c$ has no killing inside, $\lim_{t \uparrow s} Z_t$ exists in $\partial D$, $p^c_{(z, \xi)}$-a.s. on $\{\xi < +\infty, \limsup_{t \uparrow s} |\xi_t| < +\infty\}$. Moreover, the argument similar to that in the proof of Lemma 2 implies that
\[(4.4) \quad P^{c, \phi}_{(z, \xi)}[\lim_{t \uparrow \xi} Z_t \text{ exists in } \bigcup_{j=1}^{\infty} A_j] = 0.\]

Therefore, we have

\[P^{c, \phi}_{(z, \xi)}[\{\xi < +\infty\} \cap \{\limsup_{t \uparrow \xi} (c(Z_t, \xi_t; D) + |\xi_t|) < +\infty\}]\]

\[\leq P^{c, \phi}_{(z, \xi)}[\{\limsup_{t \uparrow \xi} c(Z_t, \xi_t; D) < +\infty\} \cap \lim_{t \uparrow \xi} (Z_t, \xi_t) \in \partial D \setminus \{\bigcup_{j=1}^{\infty} A_j \times C^*_n\}].\]

From this and the assumption, (4.2) follows and hence the proof of the second assertion is complete.

**Proof of Theorem 4.** It suffices to show that

(i) if \(D\) is not a domain of holomorphy and \(M_b\) is conservative, then there is an open set \(U\) such that \(U \cap \partial D \neq \emptyset\) but \(U \setminus D\) is of Newtonian (logarithmic) capacity zero and

(ii) if \(D\) is not a domain of holomorphy and \(M^\phi_c\) is conservative, then \(\overline{D^0} \neq D\).

To do this, in the remainder of this proof, we assume that \(D\) is not a domain of holomorphy. Then, there are a connected open set \(U \subset C^1\) and a connected component \(V\) of \(U \cap D\) such that \(U \cap D \neq \emptyset\), \(U \setminus D \neq \emptyset\) and each \(f \in \text{Hol}(D)\) has a \(g \in \text{Hol}(U)\) with \(f = g\) on \(V\). By attaching \(U\) and \(D\) at \(V\), we obtain \((X, \pi) \in \text{Ext}(D)\) such that \(\pi(X) = U \cup D\), \(\pi(X \setminus D) = U \setminus V\) and \(\pi \circ X \setminus (D \setminus V) \to U\) is biholomorphic.

Assume that \(M_b\) is conservative. It was seen by Bremermann [2] that there is a \(q \in \text{SP}(X) \cap C^\infty(X)\) such that \(q = p_b\) on \(D\). We put \(\theta = (dd^c q)^{-1}\) and \(m = (dd^c q)^n\). Then, since \(q \in \text{SP}(X)\), the argument similar
to that used to see that $(\theta_b, m_b) \in \U(D)$ implies that $(\theta, m) \in \U(X)$.

Obviously, $\theta_b = \theta|_D$ and $m_b = m|_D$. Thus, by [9:Proposition 9.1], we see that $M_b$ is the part of $M(\theta, m)$ on $D$. Moreover, $\theta$ and $m$ satisfy Assumption (3.2) and $m(A) = 0$ if $A \subset X$ is of measure zero, because $q \in \SP(X)$. Hence, by Theorem 2, we have that $X \setminus D$ is of capacity zero. Because $\pi: X \setminus (D \setminus V) \to U$ is biholomorphic, this implies that $U \setminus D$ is of Newtonian (logarithmic if $n=1$) capacity zero. Moreover, the connectedness of $U$ and the assumption that $U \cap D \neq \emptyset$ and $U \setminus D \neq \emptyset$ imply $U \cap \partial D \neq \emptyset$. Thus, the first assertion has been verified.

Next assume that $M_C^\varphi$ is conservative. Let $c(z, \xi)$ be the Carathéodory infinitesimal metric of $X$:

$$c(z, \xi) = \sup\{\| (f_*)_z \xi \| : f \in \text{Hol}(X) \text{ taking values in } \Delta\},$$

$z \in X$, $\xi \in T_zX$, where $(f_*)_z$ is the differential of $f$ at $z$, $\| \cdot \|$ is the norm associated with the Poincaré metric on $\Delta = \{ z \in C^1 : |z| < 1 \}$ and $T_zX$ denotes the space of holomorphic tangent vectors at $z$. It is well known ([19]) that $c(z, \xi)$ is a non-negative continuous psh function on $TX$. Let

$$q(z, \xi) = c(z, \xi) + \tilde{\varphi}(z, \xi), \quad (z, \xi) \in TX_*,$$

where $\tilde{\varphi} \in \SP(TX_*)$ is the function appearing in the definition of $E(D)$. We set

$$\theta = (dd^c q)^{2n-1} \quad \text{and} \quad m = (dd^c q)^{2n}.$$
Then \((\theta, m)\) is an admissible pair on \(\text{T}_{2\times} \) and satisfies Assumption (3.2) in Theorem 2. We observe that, by identifying \(\text{T}_{D} \subset \text{T}_{2\times} \) with \(D \times C_{2\times}^{n}\), the following identities hold:

\[(4.5) \quad \theta_{C}^{\phi} = \theta \mid D \times C_{2\times}^{n} \quad \text{and} \quad m_{C}^{\phi} = m \mid D \times C_{2\times}^{n}.\]

Indeed, since \(\tilde{\phi}(z) = \phi(z), z \in D\), (4.5) follows immediately from the well known fact ([19]) that

\[(4.6) \quad c(z, \xi; D) = c(z, \xi) \quad \text{for} \quad (z, \xi) \in D \times C_{2\times}^{n} = \text{T}_{D} \subset \text{T}_{2\times}.

Thus, \(M_{C}^{\phi}\) is the part of \(M_{2\times}^{(\theta, m)}\) on \(\text{T}_{D} ([9])\). By Theorem 2, we conclude that \(\text{T}_{2\times} \setminus \text{T}_{D}\) is of zero measure. In particular, \(X \setminus \overline{D} = \emptyset\) and hence \(U \setminus \overline{D} = \emptyset\). This implies that \(U \subset \overline{D}\) and which implies that \(\overline{D}^{0} \neq D\), for \(U \setminus D \neq \emptyset\). Therefore, the second assertion has been seen.

In the remainder, we will give five examples to illustrate our result.

**Example 4.1.** We say that the generalized conic condition is satisfied at \(z^{*} \in \partial D\) if there are a sequence \(\{w_{k}\} \subset C_{n} \setminus D\), \(\alpha \geq 1\) and \(0 < r \leq 1\) such that \(w_{k} \neq z^{*}\), \(w_{k} \to z^{*}\) and \(D \cap \{z \in C_{n} : |z - w_{k}| < r |z^{*} - w_{k}|^{\alpha}\} = \emptyset\) for every \(k\). It was shown in [22] that \(\limsup_{k} \left| \left| K(z_{k}; D) \right| \right| = \infty\) for every \(z_{k} \to z^{*}\) if \(D\) is a domain of holomorphy and the generalized conic condition is satisfied at \(z^{*}\). Therefore, by virtue of Theorem 3, we conclude that \(M_{d}\) is conservative if \(D\) is a domain of holomorphy and there is a
pluripolar set $N$ in $C^h$ such that $N \subset \partial D$ and the generalized conic condition is satisfied at every $z^* \in \partial D \setminus N$.

Example 4.2. If $D$ is a strictly pseudoconvex bounded domain in $C^h$, then $c(z, \xi) \geq C|\xi|/d(z)^{1/2}$ for every $(z, \xi) \in D \times C^h$ for some $C > 0$, where $d(z)=\inf\{|z-w|: w \in \partial D\}$ (see [10]). Thus, $M^\varphi_c$ is conservative if $\{\varphi < a\} \subset \{|\xi| < c\}$ for some $c$ for each $a \geq 0$. Next, let $r \in C^\infty(C^2)$ be psh and $D=\{z=(z^1, z^2) \in C^2: r > 0\}$ be bounded. Obviously, $D$ is a bounded domain of holomorphy in $C^2$. We define

$$L=\frac{\partial r}{\partial z^2} \frac{\partial}{\partial z^1} - \frac{\partial r}{\partial z^1} \frac{\partial}{\partial z^2}, \quad \lambda(z)=\partial r([L, L^\ast])(z) \quad \text{and} \quad C_k(z)=(L^\ast L)^{k-1} \lambda(z),$$

where $[L, L^\ast]=L^\ast L-LL$. Assume that there are a sequence $\{A_j\}$ of analytic sets in $C^h$ and $k: \partial D \setminus \bigcup_j A_j \rightarrow \{1, 2, \cdots\}$ such that $D \cap A_j = \emptyset$, $j = 1, 2, \cdots$, $C_k(w)(w) \neq 0$ and $C_k(w) = 0$, $1 \leq k < k(w)$. Then, for each $w \in \partial D \setminus \bigcup_j A_j$, there is a constant $C > 0$ such that $c(z, \xi; D) \geq C|\xi|/d(z)^{1/2k(w)}$ near $w$ (see [3]). In this case, $M^\varphi_c$ is also conservative if $\{\varphi < a\} \subset \{|\xi| < c\}$ for some $c$ for every $a \geq 0$.

Example 4.3. Without the assumption on the boundary, the assertion in Theorem 4 does not hold in general. For example, let $D_0 = \{z \in C^2: |z| < 1\}$ and $D = D_0 \setminus \{0\}$. Then, not only $\overline{D}^0 \neq D$ but also $U \setminus D$ is of Newtonian capacity zero, where $U = \{z \in C^2: |z| < 1/2\}$. Remark that every $f \in Hol(D)$ can be extended to a holomorphic function on $D_0$ and hence $D$ is not a domain of holomorphy. Let us show that $M_0$ and $M^\varphi_c$ with $\varphi(z, \xi) = |z|^2 + |\xi|^2$ are both conservative. By the
above remark, it holds that $K(z; D) = K(z; D_0)$ and $c(z, \xi; D) = c(z, \xi; D_0)$, $z \in D$. It is known that $K(z_k; D_0) \to \infty$, $c(z_k, \xi_k; D_0) \to \infty$ for $(z_k, \xi_k) \to (z^*, \xi^*) \in \partial D_0 \times C^n$ (for the former, see [2] and for the latter, see [3]). Thus, due to Theorem 3, we see that $M^b$ and $M^c$ are both conservative.

Example 4.4. Theorem 4 is a stochastic analogy of the well-known result that $D$ is a domain of holomorphy if $\beta(D)$ is complete ([2]). In this example, we see that there is a domain of holomorphy which is not complete with respect to $\beta(D)$ but Assumption (i) in Theorem 4 is satisfied.

Let $D = \{(z^1, z^2) \in C^2 : |z^1| < |z^2| < 1\}$. It is obvious that $D_0 = D$ and, especially, $U \cap D$ is of positive Newtonian capacity for every open $U$ with $U \cap \partial D \neq \emptyset$. If we define $D' = \{(w^1, w^2) \in C^2 : 0 < |w^1| < 1, |w^2| < 1\}$ and $F: D' \to D$ by $F(w^1, w^2) = (w^1 w^2, w^1)$, then $F$ is a biholomorphism. It is well known ([2]) that

\begin{equation}
(4.7) \quad K((w^1, w^2); D') = 1/(\pi(1 - |w^1|^2)(1 - |w^2|^2))^2,
\end{equation}

This yields that $D'$ is not complete with respect $\beta(D')$ and hence $D$ is not complete with respect to $\beta(D)$, for $\beta(D)$ is the pullback of $\beta(D')$ by $F^{-1}$ ([2]). To see that $M^b$ is conservative, recall that $K(F(w); D) = K(w; D')|\det(\frac{\partial F}{\partial w})|^{-2}$ ([2]). Hence, due to (4.7), we have

\begin{equation}
(4.8) \quad K((z^1, z^2); D) = 1/(\pi(1 - |z^1|^2)(1 - |z^2|^2 - |z^1|^2))\|^2.
\end{equation}

Combining this with Theorem 3, we can conclude that $M^b$ is
conservative.

Example 4.5. If a bounded domain D of holomorphy in $\mathbb{C}^n$ has a $C^1$ boundary, then the Bergman metric $\beta(D)$ is complete ([20]). Moreover, as mentioned in Example 4.1, then $M_b$ is conservative. In this example, we consider the case when D is not known a priori to be a domain of holomorphy but $\beta(D)$ is complete and $M_b$ is conservative. In the remainder, for the sake of simplicity, we write $\beta$ for $\beta(D)$.

Assume that D is simply connected and that there is a Kähler metric $g$ on D which makes D a complete Kähler manifold of non-positive sectional curvature. Assume, furthermore, that

$$ (4.9) \quad -B \leq \text{sectional curvature} \leq -A \quad \text{on } D \quad \text{for some } A, B > 0. $$

Then, Greene and Wu [11] showed that

$$ (4.10) \quad \beta \geq C g \quad \text{on } D \text{ for some } C > 0. $$

Since $g$ is complete on D, so is $\beta$. Especially, D is a domain of holomorphy. In this case, we can also show that $M_b$ is conservative.

The proof of the conservativeness of $M_b$ will be completed once we show the existence of $a > 0$ and nonnegative $u \in C^\infty(\{r > a\})$ such that

$$ (4.11) \quad u(z) \to +\infty \text{ as } r(z) \to +\infty, $$

$$ (4.12) \quad \Delta \beta u \leq \tilde{C} u \quad \text{on } \{r > a\} \quad \text{for some } \tilde{C} > 0, $$
where $\Delta_b$ is the Laplace-Beltrami operator associated with $b$. In fact, as we saw after the definition of $M_b$, the time changed process
\[ \tilde{M}_b = \{(Z_{2nt}, t/2n, P^b_z): z \in D\} \] is the Brownian motion on $(D, b)$. For the sake of simplicity, we denote $\tilde{M}_b = \{(X_t, n, P^b_z): z \in D\}$. Let $\sigma = \inf\{t > 0: r(X_t) \leq a\}$. Then, (4.12) yields that, for any stopping time $\tau < n$,
\[
0 \leq E_z[e^{-\tilde{C}(\tau \wedge \sigma)}u(X_{\tau \wedge \sigma})] \leq u(z) \quad \text{for } z \text{ with } r(z) > a,
\]
where $E_z$ stands for the expectation with respect to $P^b_z$. Thus, by (4.11), we can conclude
\[
(4.13) \quad P_z[\sigma = +\omega, \eta < +\omega] = 0 \quad \text{for } z \in \{r > a\}.
\]
Let
\[
\tau_0 = \inf\{t > 0: r(X_t) > a + 1\},
\]
\[
\sigma_k = \inf\{t > \tau_k: r(X_t) \leq a\},
\]
\[
\tau_{k+1} = \inf\{t > \sigma_k: r(X_t) > a + 1\}.
\]
By the strong Markov property of $\tilde{M}_b$ and (4.13), we have
\[
P_z[\sigma_k = +\omega, \tau_k < \eta < +\omega] = E_z[P_{X_{\tau_k}}[\sigma = +\omega, \eta < +\omega]; \tau_k < \eta] = 0.
\]
Note that
Thus, \( P_z[\eta < +\infty] = 0 \). Therefore, \( \mathcal{M}_b \) is conservative and so is \( \mathcal{M}_b \).

Let us see the existence of such \( a \) and \( u \). Fix \( o \in D \) and let \( r = r(z) \) be the distance from \( o \) to \( z \). It suffices to show that

\[
(4.14) \quad \Delta_\beta r \leq C' \coth(B^{1/2} r) \quad \text{on } \{ r > 0 \} \quad \text{for some } C' > 0.
\]

To do this, let \( (D', o') \) be a real \( 2n \)-dimensional model with the radial curvature function \( k(s) = -B \) (for definition, see [11]). It was seen in [11] that \( \Delta' r' = (2n-1)B^{1/2} \coth(B^{1/2} r') \), where \( \Delta' \) is the Laplace-Beltrami operator on \( (D', o') \) and \( r' \) is the distance from \( o' \).

For normal geodesics \( \gamma(t) \) and \( \gamma'(t) \) starting at \( o \in D \) and \( o' \in D' \), respectively, it follows from (4.9) that

each radial curvature of \( D \) at \( \gamma(t) \)

\[ \geq -B = \text{every radial curvature of } D' \text{ at } \gamma'(t). \]

By applying the Hessian comparison theorem ([11]), we have

\[
(4.15) \quad \Delta_g r(\gamma(t)) \leq \Delta' r'(\gamma'(t)) = (2n-1)B^{1/2} \coth(B^{1/2} t).
\]

Since \( \beta \geq C_g \) and \( r \) is strictly psh on \( D \) ([11]), this implies

\[
\Delta_\beta r \leq C^{-1}(2n-1)B^{1/2} \coth(B^{1/2} r)
\]

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and which yields (4.14).

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