Explosion problem for holomorphic diffusion processes and its applications

（正則拡散過程の爆発問題とその応用）

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EXPLOSION PROBLEM FOR HOLOMORPHIC DIFFUSION PROCESSES AND ITS APPLICATIONS

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(Recieved

1. Introduction

A holomorphic diffusion process on an n-dimensional complex manifold M is a diffusion process $\{(Z_t, \zeta, P_z): z \in M\}$ on M, ζ being the life time, such that $h(Z_{t\wedge\tau})$ is a local martingale for each stopping time $\tau<\zeta$ and $h \in Hol(M)$, the space of holomorphic functions on M. Such diffusion processes connect martingales with holomorphic functions. Thus, holomorphic diffusion processes enable us to discuss topics of complex analysis in probabilistic terms. The aim of this paper is to see that the conservativeness of holomorphic diffusion processes is closely related to domains of holomorphy.

Several classes of holomorphic diffusion processes were studied by Debiard-Gaveau[4], Fukushima-Okada[8],[9] and Kaneko-Taniguchi[16]. Especially, Fukushima and Okada [8] showed that there is a one to one correspondence between a family of symmetric holomorphic diffusion processes on M and the totality of admissible pairs (θ, m) on M of closed positive current θ of bidegree $(n-1,n-1)$ and everywhere dense positive Radon measure m on M (for the definition of admissible pairs, see Section 2). For a bounded domain D in C^n, one can constract the admissible pairs (θ_b, m_b) on D and $(\theta_c^{\varphi}, m_c^{\varphi})$ on $D\times C_*^n \equiv D\times\left(C^n\setminus\{0\}\right)$ from the Bergman kernel function $K(z;D)$ and the Carathéodry infinitesimal metric $c(z,\xi;D)$, $z\in D$, $\xi\in C_*^n$, respectively.

For details, see Section 4. Let M_b (resp. M_c^{φ}) be the holomorphic diffusion process on D (resp. $D\times C_*^n$) associated with (θ_b, m_b) (resp. $(\theta_c^{\varphi}, m_c^{\varphi})$). One of the main objects of the present paper is to show that the conservativeness of either M_b or M_c^{φ} implies that D is a domain of holomorphy under suitable assumptions on the boundary. In fact, we will see that D is a domain of holomorphy if either (i) M_b is conservative and $\mathrm{Cap}(U\setminus D)>0$ for any open U with $U\cap\partial D\neq\phi$, where Cap stands for the Newtonian (logarithmic if n=1) capacity, or (ii) M_c^{φ} is conservative and $\overline{D}^0=D$. See Theorem 4.

It is well known ([2],[19]) that $K(z;D)$ and $c(z,\xi;D)$ can be extended to a holomorphic extension M of D and its holomorphic tangent bundle TM, respectively, and hence so are (θ_b, m_b) and $(\theta_c^{\varphi}, m_c^{\varphi})$. Therefore, in the proof of the above assertion, a key role is played by the observation that if an admissible pair (θ, m) on M satisfies the "ellipticity" condition, then the conservativeness of the part on an open set $G\subset M$ of the holomorphic diffusion process associated with (θ, m) implies the smallness of $M\setminus G$. For detailed statement, see Theorem 2.

Before stating another object of this paper, let us consider an example. Let D be an bounded strictly pseudoconvex domain in C^n. Then, the Bergman metric $\beta(D)$ of D is Kählerian ([2], [18]). Denote by M(D) the Brownian motion on the Kähler manifold $(D,\beta(D))$, i.e. the minimal diffusion process generated by $\Delta/2$, where Δ is the Laplace-Beltrami operator (for the definition of minimal diffusion processes, see [15]). As will be seen in Section 4, the holomorphic diffusion process M_b discussed in the above paragraph coincides with M(D) up to time change $t\longrightarrow t/2n$. Since D is strictly pseudoconvex,

$(D,\beta(D))$ looks like a space of constant holomorphic sectional curvature near the boundary ([17]). Hence the Ricci curvature is bounded from the below by a constant. Therefore $M(D)$ is conservative, i.e. the life time is infinity a.s. ([13]) and so is M_b.

On account of this example, it is natural to ask whether the contrary to the first assertion holds, i.e. whether either M_b and M_c^{φ} is conservative if D is a domain of holomorphy. In order to answer to this question, we will establish that M_b (resp. M_c^{φ}) is conservative if there is a nice exceptional set $E \subset \partial D$ such that

$$\limsup_k K(z_k;D)=+\infty \quad (\text{resp. } \limsup_k c(z_k,\xi_k;D)=+\infty)$$

for every $z_k \longrightarrow z^* \in \partial D \setminus E$ (resp. $(z_k,\xi_k) \longrightarrow (z^*,\xi^*) \in \left(\partial D \setminus E\right) \times C_*^n$). For details, see Theorem 3. As will be seen in Examples 4.1 and 4.2, this assertion yields that M_b and M_c^{φ} are conservative if D is a domain of holomorphy with nice boundary.

This assertion follows essentially from more general criteria for conservativeness and explosion for symmetric holomorphic diffusion processes, see Theorem 1. Since we can not expect the smoothness of admissible pairs (θ,m) in genaral (this is the case when $(\theta,m)=(\theta_c^{\varphi},m_c^{\varphi})$), we have no nice expression of the generator of the diffusion process. Therefore, the results due to Hasminskii [12], Ichihara [13],[14] are not applicable in our situation. We will establish our criteria by using the stochastic analysis for plurisubharmonic functions. As will be seen at the end of Section 2, our criteria yields also a unified way to test the explosion and conservativeness of these specific symmetric holomorphic diffusion processes already studied by Fukushima-Okada [8],[9], Debiard-Gaveau [4] and Kaneko-Taniguchi [16] respectively.

The organization of this paper is as follows. We will begin Section 2 with giving a brief review on the symmetric holomorphic diffusion processes. We will then give the above mentioned general criteria for them. In Section 3, we will see that if the part on G of the holomorphic diffusion process on M is conservative then $M \setminus G$ is small. Section 4 will be devoted to showing that the conservativeness of either M_b or M_c^{φ} implies that D is a domain of holomorphy. A criterion for M_b or M_c^{φ} to be conservative will be also given in the same section. Several examples will be presented at the end of the section to illustrate our results.

2. Conservativeness

In this section, we will discuss the conservativeness of symmetric holomorphic diffusion processes. We first give a brief review on symmetric holomorphic diffusion processes, following Fukushima and Okada [8],[9]. Let M be a σ-compact connected complex manifold of complex dimension n. A pair (θ,m) of closed positive current θ of bidegree $(n-1,n-1)$ and everywhere dense positive Radon measure m on M is said to be admissible if the symmetric form

$$E^{\theta}(u,v) = \int_M du \wedge d^c v \wedge \theta, \qquad u,v \in C_0^{\infty}(M)$$

is closable on $L^2(M:m)$, the space of m-square integrable functions, where $d=\partial+\bar{\partial}$ is the exterior derivative and $d^c=i(\bar{\partial}-\partial)$. We denote by $U(M)$ the totallity of admissible pairs on M. For $(\theta,m)\in U(M)$, the minimal closed extension $E^{(\theta,m)}$ of E^{θ} with the domain $F^{(\theta,m)}$ is a C_0^{∞}-regular local Dirichlet form on $L^2(M:m)$. Then, every $h\in Hol(M)$ is $E^{(\theta,m)}$-harmonic. As usual, the associated capacity for compact set $K \subset M$ is defined by

$$Cap^{(\theta,m)}(K) = \inf\{\int_M u^2 dm + E^{(\theta,m)}(u,u) : u\in C_0^{\infty}(M),\ u\geq 1 \text{ on } K\}$$

and is extended to the capacity for any set as a Choquet capacity, which we call the $E^{(\theta,m)}$-1-capacity. Throughout this paper, by "$E^{(\theta,m)}$-q.e." we mean "except for a set of $E^{(\theta,m)}$-1- capacity zero". By virtue of the theory of Dirichlet spaces [5], we obtain a diffusion process $M^{(\theta,m)} = \{(Z_t,\zeta,P_z^{(\theta,m)}):z\in M\}$ associated with this $E^{(\theta,m)}$, up to equivalence, where ζ is the life time. Then, each $h(Z_{t\wedge\tau})$, $h\in Hol(M)$, is a local martingale for every stopping time $\tau<\zeta$ under $P_z^{(\theta,m)}$, $E^{(\theta,m)}$-q.e. $z\in M$. We call $M^{(\theta,m)}$ the holomorphic diffusion process associated with $(\theta,m)\in U(M)$. Moreover, we say that $M^{(\theta,m)}$ is conservative if $P_z^{(\theta,m)}[\zeta=+\infty] = 1$, $E^{(\theta,m)}$-q.e. $z\in M$ and that it explodes if $P_z^{(\theta,m)}[\zeta<+\infty] = 1$, $E^{(\theta,m)}$-q.e.

A function $u:M\longrightarrow[-\infty,+\infty)$ is called a plurisubharmonic (abbreviated to psh) if u is upper semicontinuous and the derivative $dd^c u$ in the distribution sense is a positive current. A subset N of M is said to be pluripolar if there is a psh function φ such that $N \subset \varphi^{-1}(-\infty)$. Finally, for locally bounded psh u and closed positive $(n-1,n-1)$-current θ on M, we define a positive Radon measure $dd^c u\wedge\theta$ on M by

$$\int_M f\, dd^c u\wedge\theta = \int_M u\, dd^c f\wedge\theta, \qquad f\in C_0^\infty(M).$$

For details, see [21]. We are now ready to state our result on conservativeness and explosion:

Theorem 1. Let D be a bounded domain in C^n and $(\theta,m)\in U(D)$. Denote by $M^{(\theta,m)}$ the associared holomorphic diffusion.

(i) Assume that there exists a locally bounded psh function p such that $m\geq dd^c p\wedge\theta$. Then, $M^{(\theta,m)}$ is conservative, if either of the following conditions is satisfied:

(i.a) there is a sequence $\{A_j\}$ of analytic sets in C^n such that $A_j \cap D = \phi$ and for every $z_k \longrightarrow z^* \in \partial D\setminus\{\cup_{j=1}^\infty A_j\}$, it holds that

$$(2.1) \qquad \limsup_k p(z_k) = +\infty \quad \text{and} \quad \liminf_k p(z_k) > -\infty,$$

(i.b) m is equivalent to the Lebesgue measure V on D and there is a pluripolar set N in C^n such that $N\subset\partial D$ and (2.1) holds for any $z_k\longrightarrow z^*\in\partial D\setminus N$.

(ii) Assume that there exists a locally bounded psh function q such that $m\leq dd^c q\wedge\theta$. Then, $M^{(\theta,m)}$ explodes if either of the following conditions is fulfilled:

(ii.a) there is a sequence $\{A_j\}$ of analytic sets in C^n such that $A_j \cap D = \phi$ and for every $z_k \longrightarrow z^* \in \partial D\setminus\{\cup_{j=1}^\infty A_j\}$, it holds that

$$(2.2) \qquad \limsup_k q(z_k) < +\infty,$$

(2.4) $P_z^{(\theta,m)}[\{N_{\xi-}^{[u]}<+\infty, I^{(u)}>-\infty\}\Delta\{S^{(u)}<+\infty\}] = 0$, $E^{(\theta,m)}$-q.e.,

where $A\Delta B = (A\setminus B)\cup(B\setminus A)$. If $m\geq dd^c p\wedge\theta$ for some locally bounded psh p on M, then it holds

(2.5) $P_z^{(\theta,m)}[\{\xi<+\infty \text{ and } I^{(p)}>-\infty\}\setminus\{S^{(p)}<+\infty\}] = 0$, $E^{(\theta,m)}$-q.e. $z\in M$.

Finally, if $m\leq dd^c q\wedge\theta$ for some locally bounded psh q on M, then

(2.6) $P_z^{(\theta,m)}[\{S^{(q)}<+\infty\}\setminus\{\xi<+\infty\}] = 0$, $E^{(\theta,m)}$-q.e. $z\in M$.

Proof. By using a standard time change argument (cf.[15]), it follows from (2.3) that

(2.7) $u(Z_t)-u(z) = B(\langle M^{[u]}\rangle_t) + N_t^{[u]}$, $t<\xi$,

under $P_z^{(\theta,m)}$, where B(t) is an R^1-valued Brownian motion with B(0)=0. Since $N_t^{[u]}$ is a nonnegative increasing process, this implies that

$$\limsup_{t\uparrow\xi} B(\langle M^{[u]}\rangle_t) \leq S^{(u)} - u(z),$$

$$\liminf_{t\uparrow\xi} B(\langle M^{[u]}\rangle_t) \geq I^{(u)} - u(z) - N_{\xi-}^{[u]}.$$

Recalling that $\limsup_{t\uparrow\infty} B(t) = +\infty$ and $\liminf_{t\uparrow\infty} B(t) = -\infty$, we deduce from these inequarities that $\langle M^{[u]}\rangle_{\xi-}$ is finite $P_z^{(\theta,m)}$-a.s. on $\{S^{(u)}<+\infty\}\cup\{N_{\xi-}^{[u]}<+\infty, I^{(u)}>-\infty\}$. Pluging this into (2.7), we obtain that (2.4) holds, because $B(t), t\in[0,\infty)$, is continuous.

(ii.b) m is equivalent to V and there is a pluripolar set N in C^n such that $N \subset \partial D$ and (2.2) holds for any $z_k \longrightarrow z^* \in \partial D \setminus N$.

For the proof of Theorem 1, we prepare two lemmas. We first recall that locally bounded psh u is locally in $F^{(\theta,m)}$, the domain of $E^{(\theta,m)}$. See [8]. Moreover, there exist a continuous local martingale additive functional $M_t^{[u]}$ and a continuous positive additive functional $N_t^{[u]}$, $t \in [0,\xi)$ such that

(i) the Revuz measure of $N_t^{[u]}$ is $dd^c u \wedge \theta$ and

(ii) the semimartingale $u(Z_t)$ has a decomposition

$$(2.3) \qquad u(Z_t)-u(z) = M_t^{[u]} + N_t^{[u]}, \qquad t<\xi,$$

under $P_z^{(\theta,m)}$, $E^{(\theta,m)}$-q.e. $z \in M$ (cf. [8],[9]). We now proceed to the first lemma.

Lemma 1. Let us consider an n-dimensional complex manifold M. Let $(\theta,m) \in U(M)$ and $M^{(\theta,m)} = \{(Z_t,\xi,P_z^{(\theta,m)}):z \in M\}$ be the associated symmetric holomorphic diffusion process. For R-valued function f on M, we denote by $I^{(f)}$ and $S^{(f)}$ the random variables given by

$$I^{(f)} = \liminf_{t\uparrow\xi} f(Z_t), \qquad S^{(f)} = \limsup_{t\uparrow\xi} f(Z_t).$$

For every locally bounded psh function u on M, it holds that

To see the second and the third assertions, it suffices to mention that $\xi \geq N_{\xi-}^{[p]}$ (resp. $\leq N_{\xi-}^{[q]}$) if $m \geq dd^c p \wedge \theta$ (resp. $\leq dd^c q \wedge \theta$). The proof is complete.

Lemma 2. Let us consider a bounded domain D in C^n. Let $(\theta,m) \in U(D)$ and $M^{(\theta,m)} = \{(Z_t,\xi,P_z^{(\theta,m)}) : z \in D\}$ be the associated holomorphic diffusion process. Then,

$$(2.8) \qquad P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \text{ exists}] = 1, \qquad E^{(\theta,m)}\text{-q.e. } z \in D.$$

If A is an analytic set in C^n such that $A \cap D = \phi$, then

$$(2.9) \qquad P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \in A] = 0, \quad E^{(\theta,m)}\text{-q.e.}$$

Finally, if m is equivalent to the Lebesgue measure V on D and N is a pluripolar set in C^n such that $N \subset \partial D$, then

$$(2.10) \qquad P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \in N] = 0, \qquad E^{(\theta,m)}\text{-q.e.}$$

Proof. Since D is bounded, each component of Z_t is a bounded martingale on $[0,\xi)$ under $P_z^{(\theta,m)}$, $E^{(\theta,m)}$-q.e. Thus, the martingale convergence theorem implies that $\lim_{t\uparrow\xi} Z_t$ exists $P_z^{(\theta,m)}$-a.s.

To see the second assertion, let $w_0 \in A \cap \partial D$. By the definition of analytic sets, there are an open set U in C^n and $w^1, w^2, \cdots, w^k \in \mathrm{Hol}(U)$ such that $w_0 \in U$ and $U \cap A = \{w^1 = \cdots = w^k = 0\}$. By shrinking U if necessary, we may and will assume that w^i's are all bounded on U. Put

$$\tau = \inf\{t>0 : Z_t \notin U \cap D\}.$$

Then, $w^i(Z_t)$ is a C^1-valued continuous martingale on $[0,\xi\wedge\tau)$ such that $\langle w^i(Z_\cdot), w^i(Z_\cdot)\rangle_t \equiv 0$. By a standard time change argument (cf.[15]), we have

$$w^i(Z_t) = w^i(z) + B^i(\langle w^i(Z_\cdot), \overline{w}^i(Z_\cdot)\rangle_t), \quad t<\xi\wedge\tau,$$

under $P_z^{(\theta,m)}$, $E^{(\theta,m)}$-q.e., where $B^i(t)$ is a C^1-valued Brownian motion with $B^i(0)=0$. By the argument similar to that in the proof of Lemma 1, we see that $\langle w^i(Z_\cdot), \overline{w}^i(Z_\cdot)\rangle_{(\xi\wedge\tau)-} < +\infty$ a.s. Since $A \cap D = \phi$, $w^i(z) \neq 0$ for some $1 \leq i = i(z) \leq k$ for every $z \in U \cap D$. Moreover, C^1-valued Brownian motion never hits $-w^i(z)$. Hence

$$P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \in U \cap A,\ \xi<\tau] = 0, \quad E^{(\theta,m)}\text{-q.e. } z \in U \cap D.$$

Therefore, by [5:Theorem 4.2.1], there exists a Borel set $\tilde{N} \subset D$ such that

(2.11) $\quad \mathrm{Cap}^{(\theta,m)}(\tilde{N}) = 0,$

(2.12) $\quad P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \in U \cap A,\ \xi<\tau] = 0, \qquad z \in U \cap (D\setminus\tilde{N}),$

(2.13) $\quad P_z^{(\theta,m)}[Z_t \in D\setminus\tilde{N},\ 0 \leq t < \xi] = 1, \qquad z \in D\setminus\tilde{N}.$

Let

$$A_r = \{\lim_{t\uparrow\xi} Z_t \in U \cap A \text{ and } Z_t \in U \cap D \text{ for } r \leq t < \xi\}.$$

Then, (2.12) implies

$$(2.14) \qquad P_z^{(\theta,m)}(A_0) = 0 \qquad \text{for } z \in U \cap (D\setminus\tilde{N}).$$

Combining (2.11), (2.13) and (2.14) with the Markov property, we have

$$(2.15) \qquad \begin{aligned} P_z^{(\theta,m)}(A_r) &= E_z^{(\theta,m)}[P_{Z_r}^{(\theta,m)}(A_0);\{Z_r \in U \cap D,\ r < \xi\}] \\ &= 0, \end{aligned}$$

for $E^{(\theta,m)}$-q.e. $z \in D$, where $E_z^{(\theta,m)}$ stands for the expectation with respect to $P_z^{(\theta,m)}$. Note that

$$\{\lim_{t\uparrow\xi} Z_t \in U \cap A\} \subset \cup_r A_r,$$

where the union is taken over all nonnegative rational numbers r. Thus, (2.15) yields

$$(2.16) \qquad P_z^{(\theta,m)}[\lim_{t\uparrow\xi} Z_t \in U \cap A] = 0 \qquad E^{(\theta,m)}\text{-q.e.}$$

Covering $A \cap \partial D$ with countably many U's as above, we can conclude from (2.16) that (2.9) holds.

We finally verify the third assertion. For this purpose, we modify the argument in the proof of [7:Theorem 1]. For a bounded domain $\Omega \subset C^n$, the extremal function $u_E^*(z;\Omega)$ of a set $E \subset \Omega$ is defined by

$$u_E(z;\Omega) = \sup\{v(z): v \text{ is a nonpositive psh function on } \Omega \text{ with } v \le -1 \text{ on } E\}$$

$$u_E^*(z;\Omega) = \limsup_{w \longrightarrow z} u_E(w;\Omega).$$

We set

$$C_\#(E;\Omega) = -\int_\Omega u_E^*(z;\Omega)V(dz).$$

It is known ([1],[9]) that

(i) $C_\#(E;\Omega) = \inf\{C_\#(O;\Omega): O \text{ is open, } O \supset E\}$,

(ii) $C_\#(E;\Omega) \le C_\#(E;\Omega')$ if $\Omega \subset \Omega'$.

Let $N \subset \partial D$ be a pluripolar set in C^n and Ω be a bounded domain such that $\overline{D} \subset \Omega$. Then, $C_\#(N;\Omega) = 0$ ([1]). Hence there exists a sequence $\{O_k\}$ of open sets in C^n such that $N \subset O_k \subset O_{k-1}$, $k \ge 2$ and

$$C_\#(O_k;\Omega) \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

Since m is equivalent to the Lebesgue measure V, by [8:Lemma 4], we have

$$\int_D P_z^{(\theta,m)}[\sigma_k < +\infty]V(dz) \le C_\#(O_k;D) \le C_\#(O_k;\Omega), \quad k=1,2,\cdots,$$

where $\sigma_k = \inf\{t>0: Z_t \in D \cap O_k\}$. Letting $k \longrightarrow \infty$, we have

(2.17) $$P_z^{(\theta,m)}[\cap_{k=1}^{\infty}\{\sigma_k < +\infty\}] = 0$$

for V-a.e. $z\in D$. Since m is equivalent to V, (2.17) holds for m-a.e. $z\in D$. Note that $u(z)=P_z[\cap_{k=1}^{\infty}\{\sigma_k<+\infty\}]$ is excessive (for the definition of excessive functions, see [5:p.99]). Due to [5:Lemma 4.2.5], we see that (2.17) holds for $E^{(\theta,m)}$-q.e. $z\in D$. Thus, (2.10) holds. The proof of Lemma 2 completes.

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. We first assume the existence of locally bounded psh function p with $m\geq dd^c p\wedge\theta$. Since $M^{(\theta,m)}$ has no killing inside, due to (2.8), we have

$$P_z^{(\theta,m)}[\{\zeta<+\infty\}\setminus\{\lim_{t\uparrow\zeta}Z_t \in \partial D\}] = 0, \quad E^{(\theta,m)}\text{-q.e.}$$

Combining this with (2.1) and Lemmas 1 and 2, we have

$$P_z^{(\theta,m)}[\zeta<+\infty] = P_z^{(\theta,m)}[\zeta<+\infty,\ I^{(p)}>-\infty,\ S^{(p)}<+\infty] = 0, \quad E^{(\theta,m)}\text{-q.e.}$$

Hence the first assertion of Theorem 1 has been seen.

To see the second assertion, assume that there is a locally bounded psh function q on D such that $m\leq dd^c q\wedge\theta$. Since q is upper semicontinuous, $S^{(q)} <+\infty$ on $\{\lim_{t\uparrow\zeta}Z_t \in D\}$. It follows from (2.6) and (2.8) that

$$P_z^{(\theta,m)}[\{\lim_{t\uparrow\zeta} Z_t \in D\}] = P_z^{(\theta,m)}[\{\zeta<+\infty\}\cap\{\lim_{t\uparrow\zeta} Z_t \in D\}]$$

for $E^{(\theta,m)}$-q.e. $z\in D$. Thus, since $M^{(\theta,m)}$ has no killing inside, we obtain

$$P_z^{(\theta,m)}[\lim_{t\uparrow\zeta} Z_t \in \partial D] = 1, \qquad E^{(\theta,m)}\text{-q.e.}$$

Combining this with (2.2) and Lemmas 1 and 2, we obtain

$$1 = P_z^{(\theta,m)}[S^{(q)}<+\infty] \le P_z^{(\theta,m)}[\zeta<+\infty], \qquad E^{(\theta,m)}\text{-q.e.}$$

The second assertion of Theorem 1 has been verified. The proof is completed.

Remark 2.1. Let u be a locally bounded psh function on D. By the same reasoning as at the end of the proof of Lemma 1, $M^{(\theta,m)}$ is conservative (resp. explodes) if $dd^c u\wedge\theta \le m$ (resp. $\ge m$) and $P_z^{(\theta,m)}[N_{\zeta-}^{[u]}<+\infty] = 0$ (resp. $=1$). Let h be a locally bounded upper semicontinuous function with $dd^c h = 0$. Note that $dd^c u = dd^c(u+h)$ and hence $dd^c(u+h)\wedge\theta$ enjoys the same inequality as $dd^c u\wedge\theta$ does. Moreover, $N^{[u]} = N^{[u+h]}$. Therefore, as far as we discuss the conservativeness and explosion problem after evaluating $P_z^{(\theta,m)}[N_{\zeta-}^{[u]}<+\infty]$, there is no difference between choosing u or $u+h$. However, Lemma 1 gives a way to estimate $P_z^{(\theta,m)}[N_{\zeta-}^{[u]}<+\infty]$ in terms of $I^{(u)}$ and $S^{(u)}$. Thus, to estimate $P_z^{(\theta,m)}[N_{\zeta-}^{[u]}<+\infty]$, a particular u will be much easier to handle than the others.

In the remainder of this section, we will apply Theorem 1 to some known examples. In what follows, D is a bounded domain in C^n and $M^{(\theta,m)}$ is a holomorphic diffusion process associated with $(\theta,m)\in U(D)$. Moreover, for locally bounded plurisubharmonic $u_1,\cdots,u_n$ on D, we will use $dd^c u_1\wedge\cdots\wedge dd^c u_k$ to denote the (k,k)-closed positive currents defined inductively by

$$\int_D f\wedge dd^c u_1\wedge\cdots\wedge dd^c u_k = \int_D u_k dd^c f\wedge dd^c u_1\wedge\cdots\wedge dd^c u_{k-1}$$

for every C^∞ (n-k,n-k) form f with compact support.

Case(1). Fukushima and Okada [8] showed that $M^{(\theta,m)}$ explodes if

$$\theta = (dd^c p)^n \text{ and } m = dd^c|z|^2\wedge\theta,$$

where p is a bounded psh function on D such that $m(dz)\geq g(z)V(dz)$ for some positive continuous g. In this case, Assumption (ii.a) in Theorem 1 is satisfied with $q(z) = |z|^2$. Hence, Theorem 1 also implies that $M^{(\theta,m)}$ explodes.

Case(2). In [9], Fukushima and Okada showed that $M^{(\theta,m)}$ explodes if $m(D) < +\infty$ and the Poincaré type inequality holds:

$$\int_D \varphi^2\, dm \leq C\, E^{(\theta,m)}(\varphi,\varphi) \qquad \text{for every } \varphi\in C_0^\infty(D)$$

for some constant C>0. Suppose that there exists a bounded psh function u on D such that $m \leq dd^c u\wedge\theta$. Then, the Poincaré type inequality holds:

$$\int_D \varphi^2 \, dm \leq 8\|u\|_\infty \, E^{(\theta,m)}(\varphi,\varphi) \qquad \text{for every } \varphi \in C_0^\infty(D)$$

where $\|u\|_\infty = \sup\{|u(z)| : z \in D\}$ (see [9],[21]). In this case, Assumption (ii.a) is fulfilled with $q = u$. Thus, even if $m(D) = +\infty$, $M^{(\theta,m)}$ explodes.

Case(3). Debiard-Gaveau[4] and Kaneko-Taniguchi[16] treated the case when

$$\theta = \{dd^c\textstyle\sum_1^k(-\log(-\varphi_i))\}^{n-1} \text{ and } m = \{dd^c\textstyle\sum_1^k(-\log(-\varphi_i))\}^n$$

for some bounded plusisubharmonic negative functions φ_i with $\pi_{i=1}^k\varphi_i(z) \longrightarrow 0$ as $z \longrightarrow \partial D$. Then, it was seen that $M^{(\theta,m)}$ is conservative. In this case, Assumption (i.a) is satisfied with $p = -\sum_{i=1}^k \log(-\varphi_i)$. Thus, Theorem 1 also yields that $M^{(\theta,m)}$ is conservative.

Case(4). Finally we consider an example. Let

$$D = \{z = (z^1, z^2) \in C^2 : |z^1| < 1, |z^2| < 1\},$$

$$\theta = dd^c|z^2|^2 \text{ and } m = dd^c|z|^2 \wedge \theta.$$

Then, Assumption (ii.a) is fulfilled with $q(z) = |z|^2$. Thus, by virtue of Theorem 1, $M^{(\theta,m)}$ explodes. Furthermore, it is straightforward to see that $P_z^{(\theta,m)}[Z_t^2 = z^2 \text{ for } t \geq 0] = 1$. Hence the α-order Green measures $G_\alpha(z,\cdot) = \int_0^\infty e^{-\alpha t}P(t,z,\cdot)dt$, $P(t,z,\cdot)$ being the transition probability of $M^{(\theta,m)}$, are not equivalent. Thus, the criteria due to

Ichihara [14] for explosion are not applicable directly. However, it should be noted that if we restrict ourselves to the submanifold $D_w = \{(z,w): z\in C^1 \text{ and } |z|<1\}$, then the criteria by Ichihara are applicable and yield in the end that $M^{(\theta,m)}$ explodes.

3. Smallness of sets

Let G be an open subset of a σ-compact connected n-dimensional complex manifold M and $(\theta,m)\in U(M)$. The part $M_G^{(\theta,m)}$ of $M^{(\theta,m)} = \{(Z_t,\xi,P_z^{(\theta,m)}): z\in M\}$ on G is by definition the holomorphic diffusion process on G given by

$$(3.1) \qquad M_G^{(\theta,m)} = \{(Z_t,\sigma\wedge\xi,P_z^{(\theta,m)}): z\in G\},$$

where $\sigma = \inf\{t>0: Z_t \in M\setminus G\}$. Our aim of this section is to see that the conservativeness of $M_G^{(\theta,m)}$ implies the smallness of $M\setminus G$. To state our result, we prepare some notions. We say that $A \subset M$ is of measure zero (resp. of capacity zero) if, for every coordinate neighbourhood U and diffeomorphism $\varphi: U \longrightarrow \varphi(U) \subset C^n$, $\varphi(A\cap U)$ is of Lebesgue measure zero (resp. of Newtonian (logarithmic if n=1) capacity zero). For (k,k)-currents u,v, we mean by "$u\geq v$" that u-v is a positive current. Our goal of this section will be

Theorem 2. Let G be an open set in a σ-compact connected n-dimensional complex manifold M and $(\theta,m)\in U(M)$. Assume that

$$(3.2) \qquad \theta \geq C(U)(dd^c\textstyle\sum_{i=1}^n|z^i|^2)^{n-1} \text{ and } m \geq C(U)(dd^c\sum_{i=1}^n|z^i|^2)^n,$$

on each relatively compact coordinate neighbourhood U with a coordinate system $z^1,\cdots,z^n$ for some $C(U)>0$. If the part $M_G^{(\theta,m)}$ of $M^{(\theta,m)}$ on G is conservative, then $M\setminus G$ is of measure zero. If, furthermore, $m(A) = 0$ for any $A \subset M$ of measure zero, then $M\setminus G$ is of capacity zero.

Proof Let $M^{(\theta,m)}=\{(Z_t,\xi,P_z^{(\theta,m)}):z\in M\}$ be the holomorphic diffusion process associated with (θ,m). By (3.1), the conservativeness of $M_G^{(\theta,m)}$ implies that

$$(3.3) \qquad P_z^{(\theta,m)}[\sigma < +\infty] = 0 \qquad E^{(\theta,m)}\text{-q.e. } z\in G,$$

because $A \subset G$ is of $E_G^{(\theta,m)}$-1-capacity zero if and only if it is of $E^{(\theta,m)}$-1-capacity zero (see [5:Theorem 4.4.2]). Recall that

$$(3.4) \qquad P_z^{(\theta,m)}[\sigma = 0] = 1 \qquad E^{(\theta,m)}\text{-q.e. } z\in M\setminus G$$

(see [5:p.94]). Thus, (3.3) and (3.4) imply the identity

$$(3.5) \qquad (f\chi_{M\setminus G})(z) = E_z^{(\theta,m)}[e^{-\sigma}f(Z_\sigma)] \qquad E^{(\theta,m)}\text{-q.e. } z\in M,$$

for every $f \in C_0^\infty(M)$, where $\chi_A(z)=1$ or 0 accordingly as $z\in A$ or not and $E_z^{(\theta,m)}$ stands for the expectation with respect to $P_z^{(\theta,m)}$. By virtue of [5:Theorem 4.4.1], we can conclude from (3.5) that

$$(3.6) \qquad f\chi_{M\setminus G} \in F^{(\theta,m)} \qquad \text{for every nonnegative } f\in C_0^\infty(M),$$

where $F^{(\theta,m)}$ is the domain of $E^{(\theta,m)}$.

Let $z \in \overline{G}$ and $z^1,\cdots,z^n$ be a local coordinate system on a relatively compact coordinate neighbourhood U of z, where $\overline{G}$ is the closure of G in M. Then, $U \cap G \neq \phi$. By identifying U with a bounded open set in C^n through the coodinate system, we can construct the absorbing barrier Brownian motion on U. We denote by (F',E') the corresponding Dirichlet space. Then, combined with Assumption (3.2), (3.6) yields that

(3.7) $\quad f\chi_{U\setminus G} \in F' \qquad$ for every nonnegative $f \in C_0^\infty(U)$

and hence

(3.8) $\quad \chi_{U\setminus G}$ is locally in F'.

Recall that (F',E') is irreducible, i.e., either A or $U\setminus A$ is of Lebesgue measure zero if A is locally in F' ([6]). Thus, either $U \cap G$ or $U\setminus G$ is of Lebesgue measure zero. Because the open set $U \cap G$ is not empty, $U\setminus G$ is of Lebesgue measure zero. In particular, $U\setminus\overline{G} = \phi$ and $U \subset \overline{G}$. This implies that $\overline{G}$ is open and closed and hence $\overline{G} = M$, for M is connected. Therefore, $M\setminus G$ is of measure zero.

We next assume, moreover, that $m(A) = 0$ for every $A \subset M$ of measure zero. By the above observation, we have

(3.9) $\quad m(M\setminus G) = 0.$

This and (3.3) imply that $M \setminus G$ is of $E^{(\theta,m)}$-1-capacity zero ([5]). Therefore, by Assumption (3.2), we see that $U \setminus G = U \cap \{M \setminus G\}$ is of E'-1-capacity zero, where (F', E') is the Dirichlet space of the absorbing barrier Brownian motion on U as in the proceding paragraph. Recall that $A \subset U$ is of capacity zero if and only if it is of E'-1-capacity zero. Hence $M \setminus G$ is of capacity zero. The proof is completed.

Remark 3.1. The generator A of $M^{(\theta,m)}$ is expressed formally as

$$A\varphi = \frac{dd^c\varphi \wedge \theta}{dm}, \qquad \varphi \in C^\infty(M)$$

(cf.[21]). Let U be a coordinate neighbourhood with a coordinate system $z^1, \cdots, z^n$. We denote by V the Lebesgue measure on U induced through this coordinate system. Suppose that it holds for some a^{ij}, $b \in C^\infty(U)$ that

$$dm = b\ dV \text{ and } \theta(idz^i \wedge d\bar{z}^j) = a^{ij}\ dV.$$

Then, we have

$$A\varphi = \frac{1}{b} \Sigma_{i,j=1}^n\ a^{ij} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}.$$

Moreover, Assumption (3.1) is equivalent to that

$$\left(a^{ij}\right)_{1 \le i,j \le n} \ge C\left(\delta^{ij}\right)_{1 \le i,j \le n} \quad \text{and } b \ge C \text{ on } U \text{ for some } C>0.$$

Thus, Assumption (3.1) can be thought of as an assumption on the ellipticity of the generator A.

Without the "ellipticity" assumption (3.2), the assertion of Theorem 2 does not hold in general. For example, let $M = C^2$ and $G = \{z \in C^2 : |z| > 1\}$. Then, $M \setminus G$ is not of measure zero. Take $\varphi \in C_0^\infty(R^1)$ such that $\varphi \geq 0$, $\mathrm{supp}[\varphi] \subset [1,2]$ and $\int_{-\infty}^{+\infty} \varphi(x)dx = 1$. Define $\psi(x) = \int_1^x dy \int_1^y \varphi(w)dw$, $p(z) = \psi(|z|^2)$ and $\theta = dd^c p$. Then, (θ, V), V being the Lebesgue measure on C^2, is an admissible pair and the associated holomorphic diffusion is generated by

$$A = 16\sum_{i,j=1}^{2} a^{ij}(\partial^2/\partial z^i \partial \bar{z}^j),$$

where

$$(a^{ij}) = \begin{pmatrix} \psi'(|z|^2)+\varphi(|z|^2)|z^2|^2 & -\varphi(|z|^2)z^1\bar{z}^2 \\ -\varphi(|z|^2)\bar{z}^1 z^2 & \psi'(|z|^2)+\varphi(|z|^2)|z^1|^2 \end{pmatrix}.$$

Since

$$\sup\left\{\frac{|\psi'(x)|}{|x-1|^k}, \frac{|\varphi(x)|}{|x-1|^k} : x > 1\right\} < +\infty, \quad k=0,1,\cdots$$

$$\psi'(x) = 1 \text{ and } \varphi(x) = 0 \text{ for } x \in [2,+\infty),$$

it is straightforward to see that

$$E_z^{(\theta,m)}[\sup_{0\le t\le T} 1/(|Z_t|^2-1)] < +\infty$$

$$E_z^{(\theta,m)}[\sup_{0\le t\le T}|Z_t|^2] < +\infty,$$

for every $z \in G$ and $T > 0$. Thus, we see that the part of $M^{(\theta,m)}$ on G is conservative.

4.Domains of holomorphy

This section is devoted to the study of domains of holomorphy as an application of Theorems 1 and 2. Let $D \subset C^n$ be a bounded domain. The Bergman kernel function $K(z;D)$ of D is defined by

$$K(z;D)=\sup\{|f(z)|^2/\|f\|^2:\ f\in L_h^2(D)\},$$

where $L_h^2(D)$ is the space of holomorphic functions on D with $\|f\|^2 \equiv \int_D |f(z)|^2 V(dz)<+\infty$. We set

$$p_b(z) = \log K(z;D),\quad \theta_b = (dd^c p_b)^{n-1} \quad\text{and}\quad m_b = (dd^c p_b)^n.$$

As we will see later, $(\theta_b,m_b) \in U(D)$. Therefore, a holomorphic diffusion process associated with (θ_b,m_b) is defined. In what follows, for the sake of simplicity, we write $M_b=\{(Z_t,\xi,P_z^b):z\in D\}$ instead of $M^{(\theta_b,m_b)}=\{(Z_t,\xi,P_z^{(\theta_b,m_b)}):z\in D\}$. Now let us show that $(\theta_b,m_b)\in U(D)$. To do this, recall that p_b is C^∞ and strictly psh on D ([2],[18]). Hence, θ_b is a closed positive current on D of bidegree

$(n-1,n-1)$ and m_b is a everywhere dense positive radon measure on D. Thus, it suffices to show that E^{θ_b} is closable. To this end, take a sequence $\{u_n\}$ in $C_0^\infty(D)$ such that $\int_D u_n^2\, dm_b \longrightarrow 0$ as $n \longrightarrow \infty$. Let $\varphi \in C_0^\infty(D)$. Since p_b is strictly psh, we have

$$-C\chi_{\mathrm{supp}\varphi} dm_b \le dd^c\varphi \wedge \theta_b \le C\chi_{\mathrm{supp}\varphi} dm_b \qquad \text{on } D$$

for some $C > 0$. This implies that

$$(4.1) \qquad \int_D u_n\, dd^c\varphi \wedge \theta_b \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

On the other hand, the closedness of θ_b implies that

$$E^{\theta_b}(u_n,\varphi) = -\int_D u_n\, dd^c\varphi \wedge \theta_b.$$

Therefore,

$$E^{\theta_b}(u_n,\varphi) \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ for every } \varphi \in C_0^\infty(D)$$

and hence E^{θ_b} is closable on $L^2(D:m)$ (cf.[5]).

An important property of M_b is that it coincides with the Brownian motion associated with a Kähler metric on D. To state more precisely, let us introduce the Bergman metric $\beta(D)$ on D defined by

$$\beta(D) = \sum_{i,j=1}^{n} \frac{\partial^2 p_b}{\partial z^i \partial \bar{z}^j} \, dz^i d\bar{z}^j .$$

It is known ([2],[18]) that $\beta(D)$ is Kählerian. We observe that the time changed process $\tilde{M}_b = \{(Z_{2nt}, \zeta/2n, P_z^b) : z \in D\}$ is the Brownian motion on the Kähler manifold $(D, \beta(D))$. In fact, it is easy to see that

$$dd^c \varphi \wedge \theta_b = 2^{n-2}(n-1)! \Delta\varphi \ dv$$

$$dm_b = 2^n \ n! \ dv,$$

where Δ is the Laplace-Beltrami operator and v is the volume element on $(D, \beta(D))$. Thus the generator A of M_b is expressed as

$$A\varphi = \Delta\varphi/4n \quad \text{for } \varphi \in C_0^\infty(D).$$

This implies that $\tilde{M}_b$ is the Brownian motion on $(D, \beta(D))$.

To define symmetric holomorphic diffusion processes corresponding to the Carathéodory infinitesimal metric, we introduce some more notations. The Carathéodory infinitesimal metric $c(z, \xi; D)$, $z \in D$, $\xi \in C_*^n$ is given by

$$c(z,\xi;D) = \sup\{|\sum_{i=1}^{n} \frac{\partial f}{\partial z^i}(z)\xi^i| : f : D \longrightarrow \Delta \text{ is holomorphic and } f(z)=0\},$$

where $\Delta = \{z \in C^1 : |z| < 1\}$. It is well known ([19]) that $c(\cdot,\cdot;D)$ is a non-negative continuous psh function on $D \times C_*^n$. Let Ext(D) be the totality of pairs (X, π) of σ-compact connected n-dimensional complex manifold

X and local biholomorphism $\pi: X \longrightarrow C^n$ such that

(i) $D \subset X$ and D is open in X,

(ii) $\pi(z) = z$, $z \in D$

(iii) every $f \in Hol(D)$ has a $g \in Hol(X)$ such that $f = g$ on D.

For a complex manifold M, we set

$$SP(M) = \{p: M \longrightarrow R: (1)\ p \text{ is locally bounded and psh}$$
$$(2) \text{ on each relatively compact coordinate neighbourhood with coordinate system } z=(z^1, \cdots, z^n),\ p-\delta|z|^2 \text{ is psh for some } \delta>0 \}.$$

It is straightforward to show that, for $p \in SP(M)$, $(dd^c p)^{n-1}$ and $(dd^c p)^n$ satisfy Assumption (3.2) in Theorem 2. We use $E(D)$ to denote the totality of nonnegative $\varphi \in SP(D \times C_*^n)$ such that there is a $\tilde{\varphi} \in SP(TX_*)$ so that $\varphi = \tilde{\varphi}$ on $D \times C_*^n = TD_* \subset TX_*$ for every $(X, \pi) \in Ext(D)$, where TX_* is an open subset of the holomorphic tangent bundle TX over X consisting of nonzero tangent vectors. It is easyly seen that $f(|z|^2)+g(|\xi|^2) \in E(D)$ for any C^2-functions f, g on $[0, \infty)$ with positive first and nonnegative second derivatives. For $\varphi \in E(D)$, let

$$p_c^{\varphi}(z,\xi)=c(z,\xi;D)+\varphi(z,\xi), \quad \theta_c^{\varphi}=(dd^c p_c^{\varphi})^{2n-1} \text{ and } m_c^{\varphi}=(dd^c p_c^{\varphi})^{2n}.$$

By the same argument as we saw that $(\theta_b, m_b) \in U(D)$, we can see that $(\theta_c^{\varphi}, m_c^{\varphi}) \in U(D \times C_*^n)$. We denote by $M_c^{\varphi}=\{((Z_t,\xi_t),\zeta,P_{(z,\xi)}^{c,\varphi}):(z,\xi) \in D \times C_*^n\}$ the holomorphic diffusion process on $D \times C_*^n$ associated with $(\theta_c^{\varphi}, m_c^{\varphi})$.

We will establish the criteria for conservativeness of M_b and M_c^{φ} as follows.

Theorem 3

(i) M_b is conservative if there is a pluripolar set N in C^n such that $N \subset \partial D$ and $\limsup_k K(z_k;D)=+\infty$ for every $z_k \to z^* \in \partial D \setminus N$.

(ii) M_c^{φ} is conservative if, for every a > 0, there exists a c >0 such that

$$\{(z,\xi)\in D\times C_*^n : \varphi(z,\xi)\le a\} \subset \{(z,\xi)\in D\times C_*^n : |\xi|\le c\}$$

and there exist analytic sets A_j in C^n such that $A_j \cap D = \phi$ and $\limsup_k c(z_k,\xi_k;D)=+\infty$ for every $(z_k,\xi_k) \longrightarrow (z^*,\xi^*) \in (\partial D\setminus\{\cup_j A_j\})\times C_*^n$.

Combining Theorem 2 with the well known fact that $K(\cdot;D)$ (resp. $c(\cdot,\cdot;D)$) can be extended to X (resp. TX), $(X,\pi)\in \mathrm{Ext}(D)$, we will have the following theorem, which is a generalized version of the result announced in [23].

Theorem 4. A bounded domain D in C^n is a domain of holomorphy if either of the followings is satisfied:

(i) M_b is conservative and $U \setminus D$ is of positive Newtonian (logarithmic for n=1) capacity for every open U with $U \cap \partial D \neq \phi$,

(ii) M_c^{φ} is conservative for some $\varphi \in E(D)$ and $\bar{D}^0 = D$.

The difference between the assumptions on the boundary in (i) and

(ii) of Theorem 4 comes from that $K(z;D)$ is C^{∞} but $c(z,\xi;D)$ is only continuous in general. Before proceeding to the proofs of Theorems 3 and 4, we remark an immediate consequence of these theorems.

Corollary. A bounded domain $D \subset C^n$ is a domain of holomorphy if either of the followings holds:

(i) for each open U with $U \cap \partial D \neq \phi$, $U \setminus D$ is of positive Newtonian (logarithmic for n=1) capacity and there is a pluripolar set N in C^n such that $N \subset \partial D$ and $\limsup_k K(z_k;D)=+\infty$ for every $z_k \longrightarrow z^* \in \partial D \setminus N$,

(ii) $\overline{D}^0 = D$ and there is a sequence $\{A_j\}$ of analytic sets in C^n such that $A_j \cap D = \phi$ and $\limsup_k c(z_k,\xi_k;D)=+\infty$ for every $(z_k,\xi_k) \longrightarrow (z^*,\xi^*) \in (\partial D \setminus \{\cup_j A_j\}) \times C^n_*$.

Proof. The first assetion is an immediate consequence of Theorems 3 and 4. To see the second one, it suffices to take $\varphi(z,\xi)=|z|^2+|\xi|^2$.

We now proceed to the proofs of Theorems 3 and 4.

Proof of Theorem 3. Suppose that the assumption in the first assertion of Theorem 3 is satisfied. Because $p_b \in SP(D) \cap C^{\infty}(D)$, m_b is equivalent to the Lebesgue measure V on D. Moreover, since $1 \in L^2_h(D)$, it follows from the definition that

$$K(z:D) \geq 1/V(D).$$

Thus, Assumption (i.b) in Theorem 1 is satisfied with $p = p_b$. Hence, by Theorem 1, we see that M_b is conservative.

We next suppose that the assumption in the second assertion is fulfilled. Since $\{\varphi\le a\} \subset \{|\xi|\le c\}$ and $p_c^{\varphi} \ge 0$, by (2.5) in Lemma 1, it suffices to show that

$$(4.2)\qquad P^{c,\varphi}_{(z,\xi)}[\{\zeta<+\infty\}\cap\{\limsup_{t\uparrow\zeta}(c(Z_t,\xi_t;D)+|\xi_t|)<+\infty\}] = 0.$$

Notice that the i-th component ξ_t^i of ξ_t is a continuous local martingale on $[0,\zeta)$ with values in C^n. Then, by by a standard time change argument(cf.[15]), we have

$$\xi_t^i = \xi_0^i + B^i(\langle\xi^i,\bar{\xi}^i\rangle_t),\qquad t<\zeta,$$

for some C^1-valued Brownian motion with $B^i(0) = 0$. Since $\limsup_{t\uparrow\infty}|B^i(t)| = +\infty$, we obtain that

$$\langle\xi^i_\cdot,\bar{\xi}^i_\cdot\rangle_{\zeta-} < +\infty,\ i=1,\cdots,n,\ \text{a.s. on } \{\limsup_{t\uparrow\zeta}|\xi_t|<+\infty\}.$$

Recall that $B^i(t)$ never hit $-\xi^i$. Hence it holds

$$(4.3)\qquad \begin{aligned} &P^{c,\varphi}_{(z,\xi)}[\{\limsup_{t\uparrow\zeta}|\xi_t|<+\infty\}] \\ &= P^{c,\varphi}_{(z,\xi)}[\{\limsup_{t\uparrow\zeta}|\xi_t|<+\infty\}\cap\{\lim_{t\uparrow\zeta}\xi_t \text{ exists in } C^n_*\}] \end{aligned}$$

Furthermore, since M_c^{φ} has no killing inside, $\lim_{t\uparrow\zeta}Z_t$ exists in ∂D, $P^{c,\varphi}_{(z,\xi)}$-a.s. on $\{\zeta<+\infty,\ \limsup_{t\uparrow\zeta}|\xi_t|<+\infty\}$. Moreover, the argument similar to that in the proof of Lemma 2 implies that

(4.4) $\quad P^{c,\varphi}_{(z,\xi)}[\lim_{t\uparrow\xi} Z_t$ exists in $\cup_{j=1}^{\infty} A_j] = 0.$

Therefore, we have

$$P^{c,\varphi}_{(z,\xi)}[\{\xi<+\infty\}\cap\{\limsup_{t\uparrow\xi}(c(Z_t,\xi_t;D)+|\xi_t|)<+\infty\}]$$

$$\le P^{c,\varphi}_{(z,\xi)}[\{\limsup_{t\uparrow\xi} c(Z_t,\xi_t;D)<+\infty\}\cap\{\lim_{t\uparrow\xi}(Z_t,\xi_t)\in\partial D\setminus\{\cup_{j=1}^{\infty}A_j\}\times C_*^n\}].$$

From this and the assumption, (4.2) follows and hence the proof of the second assertion is complete.

Proof of Theorem 4. It suffices to show that

(i) if D is not a domain of holomorphy and M_b is conservative, then there is an open set U such that $U \cap \partial D \neq \phi$ but $U\setminus D$ is of Newtonian (logarithmic) capacity zero and

(ii) if D is not a domain of holomorphy and M_c^{φ} is conservative, then $\bar{D}^0 \neq D$.

To do this, in the remainder of this proof, we assume that D is not a domain of holomorphy. Then, there are a connected open set $U \subset C^n$ and a connected component V of $U \cap D$ such that $U \cap D \neq \phi$, $U\setminus D \neq \phi$ and each $f \in Hol(D)$ has a $g \in Hol(U)$ with $f = g$ on V. By attaching U and D at V, we obtain $(X,\pi)\in Ext(D)$ such that $\pi(X) = U \cup D$, $\pi(X\setminus D) = U\setminus V$ and $\pi: X\setminus(D\setminus V) \longrightarrow U$ is biholomorphic.

Assume that M_b is conservative. It was seen by Bremermann [2] that there is a $q \in SP(X) \cap C^{\infty}(X)$ such that $q = p_b$ on D. We put $\theta = (dd^c q)^{n-1}$ and $m=(dd^c q)^n$. Then, since $q \in SP(X)$, the argument similar to that used to see that $(\theta_b, m_b) \in U(D)$ implies that $(\theta, m) \in U(X)$. Obviously, $\theta_b = \theta_{|D}$ and $m_b = m_{|D}$. Thus, by [9:Proposition 9.1], we see that M_b is the part of $M^{(\theta,m)}$ on D. Moreover, θ and m satisfy Assumption (3.2) and $m(A) = 0$ if $A \subset X$ is of measure zero, because $q \in SP(X)$. Hence, by Theorem 2, we have that $X \setminus D$ is of capasity zero. Because $\pi : X \setminus (D \setminus V) \longrightarrow U$ is biholomorphic, this implies that $U \setminus D$ is of Newtonian (logarithmic if n=1) capacity zero. Moreover, the connectedness of U and the assumption that $U \cap D \neq \phi$ and $U \setminus D \neq \phi$ imply $U \cap \partial D \neq \phi$. Thus, the first assertion has been verified.

Next assume that M_c^{φ} is conservative. Let $c(z,\xi)$ be the Carathéodory infinitesimal metric of X:

$$c(z,\xi) = \sup\{\|(f_*)_z \xi\| : f \in \mathrm{Hol}(X) \text{ taking values in } \Delta\},$$

$z \in X$, $\xi \in T_z X$, where $(f_*)_z$ is the differential of f at z, $\|\cdot\|$ is the norm associated with the Poincaré metric on $\Delta = \{z \in C^1 : |z| < 1\}$ and $T_z X$ denotes the space of holomorphic tangent vectors at z. It is well known ([19]) that $c(z,\xi)$ is a non-negative continuous psh function on TX. Let

$$q(z,\xi) = c(z,\xi) + \tilde{\varphi}(z,\xi), \qquad (z,\xi) \in TX_*,$$

where $\tilde{\varphi} \in SP(TX_*)$ is the function appearing in the definition of E(D). We set

$$\theta = (dd^c q)^{2n-1} \text{ and } m = (dd^c q)^{2n}.$$

Then (θ, m) is an admissible pair on TX_* and satisfies Assumption (3.2) in Theorem 2. We observe that, by identifying $TD_* \subset TX_*$ with $D \times C^n_*$, the following identities hold:

$$(4.5) \qquad \theta^{\varphi}_{c} = \theta|_{D\times C^n_*} \qquad \text{and} \qquad m^{\varphi}_{c} = m|_{D\times C^n_*}.$$

Indeed, since $\tilde{\varphi}(z) = \varphi(z)$, $z \in D$, (4.5) follows immediately from the well known fact ([19]) that

$$(4.6) \qquad c(z,\xi;D) = c(z,\xi) \qquad \text{for } (z,\xi) \in D\times C^n_* = TD_* \subset TX_*.$$

Thus, M^{φ}_{c} is the part of $M^{(\theta,m)}$ on TD_* ([9]). By Theorem 2, we conclude that $TX_* \setminus TD_*$ is of zero measure. In particular, $X\setminus\bar{D} = \phi$ and hence $U\setminus\bar{D} = \phi$. This implies that $U \subset \bar{D}$ and which implies that $\bar{D}^0 \neq D$, for $U\setminus D \neq \phi$. Therefore, the second assertion has been seen.

In the remainder, we will give five examples to illustrate our result.

Example 4.1. We say that the generalized conic condition is satisfied at $z^* \in \partial D$ if there are a sequence $\{w_k\} \subset C^n\setminus D$, $\alpha \geq 1$ and $0<r\leq 1$ such that $w_k \neq z^*$, $w_k \longrightarrow z^*$ and $D \cap \{z\in C^n: |z-w_k|<r|z^*-w_k|^{\alpha}\} = \phi$ for every k. It was shown in [22] that $\limsup_k K(z_k;D) = +\infty$ for every $z_k \longrightarrow z^*$ if D is a domain of holomorphy and the generalized conic condition is satisfied at z^*. Therefore, by virtue of Theorem 3, we conclude that M_b is conservative if D is a domain of holomorphy and there is a

pluripolar set N in C^n such that $N \subset \partial D$ and the generalized conic condition is satisfied at every $z^* \in \partial D \setminus N$.

Example 4.2. If D is a strictly pseudoconvex bounded domain in C^n, then $c(z,\xi) \geq C|\xi|/d(z)^{1/2}$ for every $(z,\xi) \in D \times C^n$ for some $C>0$, where $d(z) = \inf\{|z-w| : w \in \partial D\}$ (see [10]). Thus, M_c^{φ} is conservative if $\{\varphi<a\} \subset \{|\xi|<c\}$ for some c for each $a \geq 0$. Next, let $r \in C^{\infty}(C^2)$ be psh and $D = \{z=(z^1,z^2) \in C^2 : r<0\}$ be bounded. Obviously, D is a bounded domain of holomorphy in C^2. We define

$$L = \frac{\partial r}{\partial z^2}\frac{\partial}{\partial \bar{z}^1} - \frac{\partial r}{\partial z^1}\frac{\partial}{\partial \bar{z}^2},\ \lambda(z) = \partial r([L,\bar{L}])(z) \text{ and } C_k(z) = (L\bar{L})^{k-1}\lambda(z),$$

where $[L,\bar{L}] = L\bar{L} - \bar{L}L$. Assume that there are a sequence $\{A_j\}$ of analytic sets in C^n and $k : \partial D \setminus \{\cup_j A_j\} \longrightarrow \{1,2,\cdots\}$ such that $D \cap A_j = \phi$, $j=1,2,\cdots$, $C_{k(w)}(w) \neq 0$ and $C_k(w) = 0$, $1 \leq k < k(w)$. Then, for each $w \in \partial D \setminus \{\cup_j A_j\}$, there is a constant $C>0$ such that $c(z,\xi;D) \geq C|\xi|/d(z)^{1/2k(w)}$ near w (see [3]). In this case, M_c^{φ} is also conservative if $\{\varphi<a\} \subset \{|\xi|<c\}$ for some c for every $a \geq 0$.

Example 4.3. Without the assumption on the boundary, the assretion in Theorem 4 does not holds in general. For example, let $D_0 = \{z \in C^2 : |z| < 1\}$ and $D = D_0 \setminus \{0\}$. Then, not only $\bar{D}^0 \neq D$ but also $U \setminus D$ is of Newtonian capacity zero, where $U = \{z \in C^2 : |z| < 1/2\}$. Remark that every $f \in Hol(D)$ can be extended to a holomorphic function on D_0 and hence D is not a domain of holomorphy. Let us show that M_b and M_c^{φ} with $\varphi(z,\xi) = |z|^2 + |\xi|^2$ are both conservative. By the above remark, it holds that $K(z;D) = K(z;D_0)$ and $c(z,\xi;D) = c(z,\xi;D_0)$, $z\in D$. It is known that $K(z_k;D_0)\longrightarrow\infty$, $c(z_k,\xi_k;D_0)\longrightarrow\infty$ for $(z_k,\xi_k)\longrightarrow(z^*,\xi^*) \in \partial D_0\times C^n_*$ (for the former, see [2] and for the latter, see [3]). Thus, due to Theorem 3, we see that M_b and M_c^φ are both conservative.

Example 4.4. Theorem 4 is a stochastic analogy of the well known result that D is a domain of holomorphy if $\beta(D)$ is complete ([2]). In this example, we see that there is a domain of holomorphy which is not complete with respect to $\beta(D)$ but Assumption (i) in Theorem 4 is satisfied.

Let $D=\{(z^1,z^2)\in C^2:|z^1|<|z^2|<1\}$. It is obvious that $\overline{D}^0 = D$ and, especially, $U\setminus D$ is of positive Newtonian capacity for every open U with $U\cap\partial D \neq \phi$. If we define $D'= \{(w^1,w^2)\in C^2:0<|w^1|<1,\ |w^2|<1\}$ and $F:D'\longrightarrow D$ by $F(w^1,w^2)=(w^1w^2,w^1)$, then F is a biholomorphism. It is well known ([2]) that

$$(4.7)\qquad K((w^1,w^2);D') = 1/\{\pi(1-|w^1|^2)(1-|w^2|^2)\}^2,$$

This yields that D' is not complete with respect $\beta(D')$ and hence D is not complete with respect to $\beta(D)$, for $\beta(D)$ is the pullback of $\beta(D')$ by F^{-1} ([2]). To see that M_b is conservative, recall that $K(F(w);D)= K(w;D')|\det(\frac{\partial F}{\partial w})|^{-2}$ ([2]). Hence, due to (4.7), we have

$$(4.8)\qquad K((z^1,z^2);D) = 1/\{\pi(1-|z^2|^2)(|z^2|^2-|z^1|^2)\}^2.$$

Combining this with Theorem 3, we can conclude that M_b is conservative.

Example 4.5. If a bounded domain D of holomorphy in C^n has a C^1 boundary, then the Bergman metric $\beta(D)$ is complete ([20]). Moreover, as mentioned in Example 4.1, then M_b is conservative. In this example, we consider the case when D is not known a priori to be a domain of holomorphy but $\beta(D)$ is complete and M_b is conservative. In the remainder, for the sake of simplicity, we write β for $\beta(D)$.

Assume that D is simply connected and that there is a Kähler metric g on D which makes D a complete Kähler manifold of non positive sectional curvature. Assume, furthermore, that

(4.9) $\quad -B \leq$ sectional curvature $\leq -A \quad$ on D $\quad$ for some $A,B>0$.

Then, Greene and Wu [11] showed that

(4.10) $\quad \beta \geq C\, g \quad$ on D for some $C>0$.

Since g is complete on D, so is β. Especially, D is a domain of holomorphy. In this case, we can also show that M_b is conservative.

The proof of the conservativeness of M_b will be completed once we show the existence of $a>0$ and nonnegative $u\in C^\infty(\{r>a\})$ such that

(4.11) $\quad u(z)\longrightarrow+\infty$ as $r(z)\longrightarrow+\infty$,

(4.12) $\quad \Delta_\beta u \leq \tilde{C}u \quad$ on $\{r > a\}$ for some $\tilde{C} > 0$, where Δ_β is the Laplace-Beltrami operator associated with β. In fact, as we saw after the definition of M_b, the time changed process $\tilde{M}_b=\{(Z_{2nt},\xi/2n,P_z^b):z\in D\}$ is the Brownian motion on (D,β). For the sake of simplicity, we denote $\tilde{M}_b=\{(X_t,\eta,P_z):z\in D\}$. Let $\sigma = \inf\{t>0:r(X_t) \le a\}$. Then, (4.12) yields that, for any stopping time $\tau < \eta$,

$$0 \le E_z[e^{-\tilde{C}(\tau\wedge\sigma)}u(X_{\tau\wedge\sigma})] \le u(z) \quad \text{for } z \text{ with } r(z) > a,$$

where E_z stands for the expectation with respect to P_z. Thus, by (4.11), we can conclude

$$\text{(4.13)} \qquad P_z[\sigma = +\infty,\ \eta < +\infty] = 0 \quad \text{for } z \in \{r > a\}.$$

Let

$$\tau_0 = \inf\{t>0:r(X_t) > a+1\},$$

$$\sigma_k = \inf\{t>\tau_k:r(X_t) \le a\},$$

$$\tau_{k+1} = \inf\{t>\sigma_k:r(X_t) > a+1\}.$$

By the strong Markov property of $\tilde{M}_b$ and (4.13), we have

$$P_z[\sigma_k=+\infty,\ \tau_k<\eta<+\infty] = E_z[P_{X_{\tau_k}}[\sigma=+\infty,\eta<+\infty];\tau_k<\eta] = 0.$$

Note that

$$\{\eta < +\infty\} \subset \cup_{k=1}^{\infty} \{\tau_k < \eta < +\infty,\ \sigma_k = +\infty\}.$$

Thus, $P_z[\eta<+\infty] = 0$. Therefore, $\tilde{M}_b$ is conservative and so is M_b.

Let us see the existence of such a and u. Fix $o\in D$ and let $r=r(z)$ be the distance from o to z. It suffices to show that

$$(4.14) \qquad \Delta_\beta r \le C' \coth(B^{1/2}r) \quad \text{on } \{r > 0\} \quad \text{for some } C' > 0.$$

To do this, let (D',o') be a real 2n-dimensional model with the radial curvature function $k(s)\equiv-B$ (for definition, see [11]). It was seen in [11] that $\Delta'r'=(2n-1)B^{1/2}\coth(B^{1/2}r')$, where Δ' is the Laplace-Beltrami operator on (D',o') and r' is the distance from o'. For normal geodesics $\gamma(t)$ and $\gamma'(t)$ starting at $o\in D$ and $o'\in D'$, respectively, it follows from (4.9) that

each radial curvature of D at $\gamma(t)$

$\ge -B =$ every radial curvature of D' at $\gamma'(t)$.

By applying the Hessian comparison theorem ([11]), we have

$$(4.15) \qquad \Delta_g r(\gamma(t)) \le \Delta' r'(\gamma'(t)) = (2n-1)B^{1/2}\coth(B^{1/2}t).$$

Since $\beta\ge Cg$ and r is strictly psh on D ([11]), this implies

$$\Delta_\beta r \le C^{-1}(2n-1)B^{1/2}\coth(B^{1/2}r)$$

and which yields (4.14).

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