STUDIES OF SECRETARY PROBLEM

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1979

CONTENTS

1. Introduction

The problems we consider in this report are often referred to as the secretary problem, the beauty contest problem, the dowry problem and the marriage problem and they fall under the general heading of problems of optimal stopping. In each of these problems the decision-maker takes observations one by one sequentially and at each stage he must decide whether to accept (choose) it or reject it and take another observation so that he may maximize the specified stopping reward. Before discussing our problems, we shall give a brief outline of the variants of the secretary problem studied so far.

Gilbert and Mosteller [3], who investigate the secretary problem and several related variants extensively, distinguish between the two extreme cases, i.e., no-information and full-information.

1. The "no-information" case: The decision-maker has no information about the distribution of the observed values. The only information he can know about each object is the relative rank among those that have already appeared.

2. The "full-information" case: The decision-maker can observe a sequence of the independently and identically distributed random variables each with a known cumulative distribution function.

The original problem in the no-information case, described by Lindley [5] for example, is as follows.

There are N objects. They are supposed to appear before the decision-maker sequentially in a random order and can be ranked according to some quality (1 being the best and N the worst). The decision-maker observes the rank of the present object relative to those preceding it and decides either to accept it or to reject it and take another observation. (There is no recall of objects already passed over.) He makes only one choice and his goal is to find a stopping rule which maximizes the probability of choosing the best object.

Gilbert and Mosteller [3], Gusein-Zade [4] and Mucci [6] treat the problem in which the decision-maker regards the choice as successful if the chosen object is one of r best among N objects. In the problem we shall consider in Section 2.1 the decision-maker makes two choices and succeeds if either of his choices is the best <u>or</u> the second best. We give the name "1-candidate" to any object which is best among all that have already appeared and also give the name "2-candidate" to any object which is second best among all that have already appeared. When we need not distinguish between 1-candidate and 2-candidate, we call them by the name of "candidate". Then it is obviously seen that any object which is not a candidate can not be chosen. For $1 \leq t \leq N$, we denote by (t,i), $i=1,2$, the state where the decision-maker is facing the t-th object which happens to be a i-candidate and he is allowed two choices, and by (t,ij), $i,j=1,2$, the state where the decision-maker, who has already chosen the i-candidate among the first t-1 objects as the first choice, is facing the t-th object which happens to be a j-candidate. Then our result can be stated as follows. There exists a pair of integers

d_1^* and d_2^*, $1 \leq d_1^* \leq d_2^* \leq N$, such that the optimal strategy in state (t,i), $i=1,2$, is to accept the candidate if $t \geq d_i^*$, and there also exists another pair of integers s_1^* and s_2^*, $1 \leq s_1^* \leq s_2^* \leq N$, such that the optimal strategy in state $(t,2j)$, $j=1,2$, is to accept the candidate if $t \geq s_j^*$. In state $(t,1j)$, $j=1,2$, it does no good to accept the candidate.

Recently Nikolaev [8] considers the problem of choosing both the best and the second best. In Section 2.2 we shall solve the same problem by using the OLA (one stage lookahead) policy which was proposed by Ross [12] to solve a certain class of problems in Markov decision processes for which action space essentially consists of two alternatives - acceptance and rejection. We easily see that the first choice should be restricted to a 1-candidate and that, once the first choice is made and if the earliest candidate thereafter is a 1-candidate, it should be unconditionally accepted as the second choice. Hence, the decision-maker is interested in only states $(t,1)$ and $(t,12)$. It will be shown that there exists a pair of integers r_1^* and r_2^*, $1 \leq r_2^* \leq r_1^* \leq N$, such that the optimal strategy is to accept the candidate if $t \geq r_2^*$ in state $(t,1)$, and accept the candidate if $t \geq r_1^*$ in state $(t,12)$.

Though in most of the problems the number of objects N is known, it is natural to study the situation where N is not known in advance but is a random variable whose distribution is given beforehand. Problems of this type have been studied by Presman and Sonin [9] and Rasmussen and Robbins [11] . We also consider the problem, in which knowing an a priori distribution of the actual number of the objects the decision-maker makes r choices and succeeds if either of his choices is the best. In Section 2.3, the explicit solution is obtained in the case where the decision-maker makes two choices and a uniform a priori distribution is assumed for N and can be stated as follows. Let $\langle t,r \rangle$, r=1,2, be the state where the decision-maker is facing the t-th object which happens to be a 1-candidate and there remain r choice(s) allowed to be made. Then there exists a pair of integers m_1^* and m_2^*, $1 \leq m_2^* \leq m_1^* \leq N$, such that the optimal strategy in state $\langle t,r \rangle$, r=1,2, is to accept the candidate if $t \geq m_r^*$.

Another modification of interest is to allow more flexibility in recalling an object which has been passed over at a previous stage. Smith and Deely [21] treat the problem with finite-memory of size m, where the decision-maker is allowed a backward solicitation to any one of the last m objects. Yang [28] assumes that at every stage a backward solicitation to any object which was passed over is allowed but with a known probability of success. Rubin and Samuels [13] also consider a different type of the finite-memory problem. A variant in Smith [20] and Sakaguchi [18] assumes that each of the objects has the right to refuse an offer of acceptance with a fixed probability. In Section 2.4, we shall investigate a mixed model where the decision-maker, who is allowed a backward solicitation at any stage, succeeds if his choice is the best among all, while each of the objects has the right to refuse an offer of acceptance with a known and fixed probability. If an object is chosen and it accepts the offer, we call it available. Suppose that the decision-maker is at stage k. Let m_k be the relative position (to k) of the available 1-candidate if it exists. We call the available 1-candidate a "current candidate" if $m_k=0$, and a "potential candidate" if $1 \leq m_k \leq k-1$. Now assume that, if at stage k our attempt is made to procure the available 1-candidate, this attempt will be successful with the specified probability $q(m_k)$, where $q(0)=1$. Then our result becomes as follows. In the case of $q(m)=q, m \geq 1$, there exists an integer n_1^* such that the optimal strategy is to pass over the first n_1^*-1 objects and thereafter accept the earliest current candidate. If it happens that stage N is reached and the absolute best is among the first n_1^*-1 objects, we attempt to procure it. On the other hand, in the case of $q(m)=q^m$, $m \geq 1$, there exists an integer n_2^* such that the optimal strategy is to pass over the first n_2^*-1 objects, solicit the available 1-candidate, if possible (no matter where it is) at stage n_2^*, and if the procurement at stage n_2^* is unsuccessful, continue the observations of the remaining objects and accept the earliest current candidate thereafter.

Mucci [6] [7] treat the problem with a generalized payoff function and obtain the asymptotic form for the optimal payoff and the stopping rule by the analysis of related differential equations. Game theoretic approach is incorporated into the secretary problem by Presman and Sonin [10] . This approach is also employed in Sakaguchi [17] . Although the secretary problems stated above are mainly concerned with maximizing the probability of choosing the desired object, Lindley [5] and Chow et al. [1] discuss the problem in which the decision-maker wishes to minimize the expected absolute (not relative) rank of the object he chooses.

We now turn to the full-information case. Let X_i, $i=1,2,\ldots, N$, be the value attached to the i-th object and suppose that $X_1, X_2, \ldots, X_N$ be independent and identically distributed random variables that can be observed sequentially. Then the decision-maker, observing the random variable at each stage, must decide whether to accept it or not. His objective is to find the stopping rule which maximizes the probability of choosing the largest random variable. Gilbert and Mosteller [3] originally studied this problem with one choice. Since their derivation depends on some ingenious and heuristic method and the optimality of strategy is not necessarily clear, Sakaguchi [16] solves the same problem via the dynamic programming method and provides much insight into the optimal strategy. The problem treated in Section 3.1 is that the decision-maker makes two choices and succeeds if either of his choices is the largest of the sequentially presented random variables. We give the name candidate to any object which has a maximum value observed so far and let $(x;n,r)$, $r=1,2$, be the state where the decision-maker is facing a candidate whose value is x and he is allowed to make r choice(s) from the remaining $n+1$ observations (including the present one). We can assume without loss of generality that each of the observed values has a uniform distribution on the interval $(0, 1)$. Our result in this case can be summarized as follows. There exists a pair of increasing sequences $\{s_n\}$ and $\{d_n\}$, such that the optimal strategy is to accept the candidate if $x \geqq s_n$ in state $(x;n,1)$ and accept the candidate if $x \geqq d_n$ in state $(x;n,2)$.

Apart from the two extreme cases - no-information and full-information, Sakaguchi [14] and Stewart [22] consider the intermediate case in which the value attached to each object is a random variable whose distribution has a parameter unknown a priori.

Secretary problem has so far received a great deal of attention for its simple structure. This report is composed of five papers [23, 24, 25, 26, 27], and the author wishes it will be a step to further research.

2. Secretary Problems with No-information

2.1 Recognizing the Best or the Second Best

The problem we consider here is that the decision-maker makes two choices and succeeds if either of his choices is the best or the second best. If he has not accepted until the last two objects then he is forced to accept both of them. He wishes to find the stopping rule which maximizes the probability of success.

Given the t-th object is a candidate, the conditional probability that the s-th ($s>t$) is the earliest candidate and, at the same time, earliest i-candidate ($i=1,2$) is $t(t-1)/s(s-1)(s-2)$ which we indicate by π_{ts}. Hence, the probability that no candidate will appear after stage t is $1-2\sum_{s=t+1}^{N} \pi_{ts} = t(t-1)/N(N-1)$. We denote by $u_t^{(i)}$ and $v_t^{(ij)}$ the probabilities of success under an optimal policy starting from states (t,i) and (t,ij) respectively. The probability of success is $t(2N-t-1)/N(N-1)$ when the decision-maker chooses 1-candidate among the first t objects and terminates the process and the probability of success is $t(t-1)/N(N-1)$ when he chooses 2-candidate and terminates the process. So we now obtain the following recurrence relations

$$(2.1.1)\qquad v_t^{(21)} = \max\begin{cases} A: & \dfrac{t(2N-t-1)}{N(N-1)} \\ R: & \displaystyle\sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \end{cases}$$

$$(3 \leq t \leq N-1;\ v_N^{(21)}=1) \quad ,$$

$$(2.1.2)\qquad v_t^{(22)} = \max \begin{cases} A: & \dfrac{t(t-1)}{N(N-1)} \\ R: & \displaystyle\sum_{s=t+1}^{N} \pi_{ts}\left(v_s^{(21)} + v_s^{(22)}\right) \end{cases}$$

$$(3 \leq t \leq N-1;\ v_N^{(22)} = 1)\quad ,$$

$$(2.1.3)\qquad v_t^{(11)} = \max \begin{cases} A: & \dfrac{t(2N-t-1)}{N(N-1)} \\ R: & \dfrac{t(t-1)}{N(N-1)} + \displaystyle\sum_{s=t+1}^{N} \pi_{ts}\left(v_s^{(21)} + v_s^{(22)}\right) \end{cases}$$

$$(2 \leq t \leq N-1;\ v_N^{(11)} = 1)\quad ,$$

$$(2.1.4)\qquad v_t^{(12)} = \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}\left(v_s^{(11)} + v_s^{(12)}\right)$$

$$(2 \leq t \leq N-1;\ v_N^{(12)} = 1)\quad ,$$

$$(2.1.5)\qquad u_t^{(1)} = \max \begin{cases} A: & \dfrac{t(t-1)}{N(N-1)} + \displaystyle\sum_{s=t+1}^{N} \pi_{ts}\left(v_s^{(11)} + v_s^{(12)}\right) \\ R: & \displaystyle\sum_{s=t+1}^{N} \pi_{ts}\left(u_s^{(1)} + u_s^{(2)}\right) \end{cases}$$

$$(1 \leq t \leq N-1;\ u_N^{(1)} = 1)\quad ,$$

$$(2.1.6) \quad u_t^{(2)} = \max\begin{cases} A: \dfrac{t(t-1)}{N(N-1)} + \displaystyle\sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \\ R: \displaystyle\sum_{s=t+1}^{N} \pi_{ts}(u_s^{(1)} + u_s^{(2)}) \end{cases}$$

$$(2 \leqq t \leqq N-1;\ u_N^{(2)} = 1)$$

where A and R symbolically represent acceptance and rejection respectively. (2.1.4) comes from the fact that it does no good to stop in state (t,12). The system of equations (2.1.1)-(2.1.6) can be solved recursively to yield the optimal stopping rule and the maximum probability $u_1^{(1)}$.

We put

$$(2.1.7) \quad v_t \equiv \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)}) \quad ,$$

then we have by (2.1.1) and (2.1.2)

$$(2.1.8) \quad v_{t-1} = \frac{1}{t}(v_t^{(21)} + v_t^{(22)}) + \frac{t-2}{t} v_t$$

$$= v_t + \frac{1}{t}[\max(\frac{t(2N-t-1)}{N(N-1)} - v_t, 0) + \max(\frac{t(t-1)}{N(N-1)} - v_t, 0)].$$

Hence, v_t is non-increasing in t. Considering that $t(2N-t-1)/N(N-1) > t(t-1)/N(N-1)$, for $1<t<N$, and that both of these functions are increasing in t, we can summarize the optimal policy in state (t,2j), j=1,2, as follows.

Theorem 1.1. There exists a pair of integers s_1^* and s_2^*, $1 \leq s_1^* \leq s_2^* \leq N$, such that the optimal strategy in state $(t,2j)$, $j=1,2$, is to accept the candidate and stop immediately if and only if $t \geq s_j^*$, where

$$(2.1.9) \qquad s_1^* = \min [t \mid \frac{t(2N-t-1)}{N(N-1)} \geq v_t],$$

and

$$(2.1.10) \qquad s_2^* = \min [t \mid \frac{t(t-1)}{N(N-1)} \geq v_t].$$

Gilbert and Mosteller [3] and Gusein-Zade [4] have shown this theorem. The explicit expression of v_t was given in [4] and the values of s_1^* and s_2^* were given in Table 6 of [3].

Lemma 1.2. $v_t^{(11)}$ is increasing in t and we have, for $1<t\leq N$,

$$(2.1.11) \qquad v_t^{(11)} = \frac{t(t-1)}{N(N-1)} + v_t \geq \frac{t(2N-t-1)}{N(N-1)}.$$

Proof: It is easily seen by (2.1.1), (2.1.2) and (2.1.7)

$$\frac{t(t-1)}{N(N-1)} + v_t = \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts}(v_s^{(21)} + v_s^{(22)})$$

$$\geq \frac{t(t-1)}{N(N-1)} + \sum_{s=t+1}^{N} \pi_{ts} [\frac{s(2N-s-1)}{N(N-1)} + \frac{s(s-1)}{N(N-1)}]$$

$$= \frac{t(2N-t-1)}{N(N-1)}.$$

Thus, by (2.1.3), the latter half of the lemma is shown.

Showing that $v_t^{(11)}$ is increasing in t is equivalent to showing that $v_t^{(11)} - v_{t-1}^{(11)} > 0$. (2.1.7) and (2.1.11) give

$$(2.1.12) \qquad v_t^{(11)}-v_{t-1}^{(11)} = \frac{1}{t}\left[(v_t^{(11)}-v_t^{(21)})+(v_t^{(11)}-v_t^{(22)})\right].$$

Hence, considering (2.1.9) and (2.1.10), we can rewrite (2.1.12) as

$$(2.1.13) \qquad v_t^{(11)}-v_{t-1}^{(11)} = \begin{cases} \frac{1}{t}\ \frac{2t(t-1)}{N(N-1)}, & 2\leq t<s_1^* \\ \frac{1}{t}\left[(v_t^{(11)} - \frac{t(2N-t-1)}{N(N-1)})+\frac{t(t-1)}{N(N-1)}\right], & s_1^*\leq t<s_2^* \\ \frac{1}{t}\left[(v_t^{(11)} - \frac{t(2N-t-1)}{N(N-1)})+ v_t\right], & s_2^*\leq t\leq N \end{cases}.$$

In either case, (2.1.13) is positive. Thus the lemma is proved.

Lemma 1.2 and (2.1.4) show that, as far as the object chosen as the first choice remains a candidate, the decision-maker should not accept a new candidate as the second choice. We can rewrite (2.1.4) as

$$(2.1.14) \qquad v_t^{(12)} = \frac{1}{t+1}\left(v_{t+1}^{(11)} + tv_{t+1}^{(12)}\right).$$

We have the following lemma.

Lemma 1.3. $v_t^{(12)}$ is increasing in t and we have, for $1<t<N$, $v_t^{(12)} > v_t^{(11)}$.

Proof: We show $v_t^{(12)} > v_t^{(11)}$ by backward induction. It is easily checked $v_{N-1}^{(12)} > v_{N-1}^{(11)}$. Suppose that $v_{t+1}^{(12)} > v_{t+1}^{(11)}$, then, by (2.1.14) and Lemma 1.2, we soon have $v_t^{(12)} > v_{t+1}^{(11)} > v_t^{(11)}$. Hence, the induction is completed. Applying this result to (2.1.14), we also have $v_t^{(12)} < v_{t+1}^{(12)}$. Thus the lemma is proved.

Now put

$$(2.1.15) \qquad u_t \equiv \sum_{s=t+1}^{N} \pi_{ts}(u_s^{(1)} + u_s^{(2)}).$$

Then, by (2.1.5) and (2.1.6), u_t turns out to be non-increasing in t. Hence, (2.1.5) and (2.1.6), combined with Lemma 1.3, lead us to the following theorem.

Theorem 1.4. There exists a pair of integers d_1^* and d_2^*, $1 \leq d_1^* \leq d_2^* \leq N$, such that the optimal strategy in state (t,i), i=1,2, is to accept the candidate if any only if $t \geqq d_i^*$, where

$$(2.1.16) \qquad d_1^* = \min[t \mid v_t^{(12)} \geq u_t]$$

and

$$(2.1.17) \qquad d_2^* = \min[t \mid v_t^{(11)} \geq u_t].$$

Ultimately optimal stopping policy of our problem is given by Theorem 1.1, 1.4, Lemma 1.2 and (2.1.4). Table 1 gives the values of s_1^*, s_2^*, d_1^*, d_2^* and the maximum probability for some values of N.

Table 1

N	s_1^*	s_2^*	d_1^*	d_2^*	$u_1^{(1)}$
4	2	(3)4	(1)2	(2)3	0.9167
5	(2)3	4	2	3	0.9167
6	3	5	2	3	0.9000
7	3	5	2	4	0.8810
8	4	6	2	4	0.8705
9	4	7	3	5	0.8585
10	4	7	3	5	0.8551
20	8	14	5	9	0.8248
50	18	34	11	22	0.8056
100	35	67	22	43	0.7995
200	70	134	44	85	0.7965
500	174	334	108	211	0.7946
1000	348	667	216	421	0.7940
∞	0.3470N	0.6667N	0.2150N	0.4201N	0.7934

The values of s_1^* and s_2^* are reproduced from Table 6 in Gilbert and Mosteller (1966).

Following the method proposed by [6] , [7] and [18] , we finally derive the asymptotic values of s_1^*, s_2^*, d_1^* and d_2^*, when N tends to infinity, by the analysis of the corresponding differential and integral equations. Write $v_t^{(ij)}$, $u_t^{(i)}$, v_t and u_t, as $f^{(ij)}(t/N)$, $g^{(i)}(t/N)$, $p(t/N)$ and $q(t/N)$ respectively, and let N and t go to infinity, setting $t/N=x$. Then, by (2.1.1),(2.1.2),(2.1.11),(2.1.12) and (2.1.14), we have, for $o \leq x \leq 1$,

$$(2.1.18)\qquad f^{(21)}(x) = \max(x(2-x), p(x)),$$

$$(2.1.19)\qquad f^{(22)}(x) = \max(x^2, p(x)),$$

$$(2.1.20)\qquad p(x) = f^{(11)}(x) - x^2,$$

$$(2.1.21)\qquad f'^{(11)}(x) = \frac{2}{x}f^{(11)}(x) - \frac{1}{x}(f^{(21)}(x) + f^{(22)}(x)),$$

$$(2.1.22)\qquad f'^{(12)}(x) = \frac{1}{x}(f^{(12)}(x) - f^{(11)}(x)),$$

where $f^{(21)}(1)=f^{(22)}(1)=f^{(11)}(1)=f^{(12)}(1)=1$ and $p(1)=1$. These equations (2.1.18)-(2.1.22) can be solved easily and especially $f^{(11)}(x)$ and $f^{(12)}(x)$ become

$$(2.1.23)\qquad f^{(11)}(x) = \begin{cases} x^2+\alpha_1(2-\alpha_1), & 0\leqq x\leqq\alpha_1 \\ 2x(x-\ln x+\ln \alpha_2), & \alpha_1\leqq x\leqq\alpha_2 \\ x(2-x), & \alpha_2\leqq x \leq 1 \end{cases}$$

and

$$(2.1.24)\qquad f^{(12)}(x) = \begin{cases} -x^2+(\alpha_1^2-2\alpha_1+1-2\ln \alpha_2)x+\alpha_1(2-\alpha_1), & 0\leqq x\leqq\alpha_1 \\ x(2-2x-2\ln \alpha_2+(\ln x/\alpha_2)^2), & \alpha_1\leqq x\leqq\alpha_2 \\ x(x-2\ln x), & \alpha_2\leqq x\leq 1 \end{cases}$$

where $\alpha_2=2/3$ and $\alpha_1\cong 0.3470$ is the unique root in $(0,\alpha_2)$ of the equation

$$(2.1.25)\qquad x-\ln x=1-\ln \alpha_2.$$

This coincides with the results of [3] and [4] . Similarly we have, by (2.1.5), (2.1.6) and (2.1.15), for $0\leqq x\leqq 1$,

$$(2.1.26)\qquad g^{(1)}(x) = \max(f^{(12)}(x), q(x)),$$

$$(2.1.27)\qquad g^{(2)}(x) = \max(f^{(11)}(x), q(x)),$$

$$(2.1.28)\qquad q(x) = \int_x^1 \frac{x^2}{y^3}\,(g^{(1)}(y)+g^{(2)}(y))dy,$$

where, $g^{(1)}(1)=g^{(2)}(1)=1$ and $q(1)=0$.

Applying (2.1.23) and (2.1.24) to (2.1.26)-(2.1.28), we have, after tedious calculations,

$$(2.1.29)\quad q(x)=\begin{cases} q(\beta_1), & 0\leqq x\leqq\beta_1 \\ x^2+[(\alpha_1-1)^2\ln\alpha_1-2\ln\alpha_2\ln\beta_2-(\ln\alpha_1-\ln\beta_2)\ln^2\alpha_2 & \\ +(\ln^2\alpha_1-\ln^2\beta_2)\ln\alpha_2-\frac{1}{3}(\ln^3\alpha_1-\ln^3\beta_2)]\,x & \\ +(1-\alpha_1^2+2\ln\alpha_1)x\ln x+\alpha_1(2-\alpha_1), & \beta_1\leqq x\leqq\alpha_1 \\ 2x^2+(\ln^2\alpha_2\ln\beta_2+2\ln\alpha_2-\ln\alpha_2\ln^2\beta_2 & \\ -2\ln\alpha_2\ln\beta_2+\frac{1}{3}\ln^3\beta_2)x-\frac{1}{3}x\ln^3 x & \\ +(\ln\alpha_2)x\ln^2 x+(2\ln\alpha_2-2-\ln^2\alpha_2)x\ln x, & \alpha_1\leqq x\leqq\beta_2 \\ -3x^2+(\ln^2\alpha_2-2\ln\alpha_2+2)x & \\ +x\ln^2 x-2(\ln\alpha_2)x\ln x, & \beta_2\leqq x\leqq\alpha_2 \\ -2x\ln x, & \alpha_2\leqq x\leqq 1 \end{cases},$$

where $\beta_2\cong 0.4201$ is the unique root in (0,1) of the equation

$$(2.1.30)\qquad (\ln x-\ln \alpha_2)^2+2-5x-4\ln \alpha_2+2\ln x=0$$

and $\beta_1 \cong 0.2150$ is also the unique root in $(0,\beta_2)$ of the equation

$$(2.1.31)\qquad \begin{aligned} &2x+(1-\alpha_1^2+2\ln \alpha_1)\ln x+[(1-\alpha_1^2)+(\alpha_1^2-2\alpha_1+3)\ln \alpha_1 \\ &-2\ln \alpha_2 \ln \beta_2-(\ln \alpha_1-\ln \beta_2)\ln^2\alpha_2 \\ &+(\ln^2\alpha_1-\ln^2\beta_2)\ln \alpha_2-\tfrac{1}{3}(\ln^3\alpha_1-\ln^3\beta_2)]=0. \end{aligned}$$

Summarizing the above results we reach the following lemma.

Lemma 1.5. We have asymptotically $s_1^* \cong \alpha_1 N$, $s_2^* \cong \alpha_2 N$, $d_1^* \cong \beta_1 N$ and $d_2^* \cong \beta_2 N$. And then $q(\beta_1)$ in (2.1.29), the maximum probability of success, becomes approximately 0.7934.

2.2 Recognizing the Best and the Second Best

In this section, we consider the problem of choosing both the best and the second best by using the OLA policy which was proposed by Ross [12] to solve a certain class of problems in Markov decision processes for which action space essentially consists of two alternatives - acceptance and rejection.

Suppose that the Markov decision process has the countably infinite state space $\{i \mid i=0,1,2,\ldots\}$ and the transition probability $\{p_{ij} \mid i,j=0,1,2,\ldots\}$ and that, in each state, the decision-maker must decide whether to accept a terminal reward R(i) and stop or, by paying C(i), to proceed to the next state according to the above transition probability p_{ij}. Then we have the following optimality equation

$$V(i) = \max \begin{cases} A : R(i) \\ R : -C(i) + \sum_{j=0}^{\infty} p_{ij} V(j) \end{cases} \quad (i=0,1,2,\ldots),$$

where V(i) is the expected reward under an optimal pllicy starting from the initial state i. We define B as the set of states for which stopping immediately is at least as good as proceeding for exactly one more period and then stopping. That is

$$B \equiv \{ i \mid R(i) \geq -C(i) + \sum_{j=0}^{\infty} p_{ij} R(j) \}$$

and we also define the OLA policy as the one which tells us to stop immediately in state i if and only if $i \in B$. Say that B is closed, if $p_{ij}=0$ for all $i \in B$ and $j \notin B$. Then Ross shows that, under some reasonable conditions which assure stability of the decision process, if B is closed, then the OLA policy is optimal.

It is easy to see that the first choice should be restricted to a 1-candidate and that, once the first choice is made and if the earliest candidate thereafter is a 1-candidate, it should be unconditionally accepted as the second choice. Hence, the decision-maker is interested in only states (t,1) and (t,12). If he has not accepted until the last two objects, he is forced to accept them. Let g(t) and f(t) be the probabilities of success under an optimal policy starting from states (t,1) and (t,12), respectively. Then considering that the conditional probability that the s-th (s>t) object is the earliest 1-candidate given that the t-th is a candidate is t/s(s-1) and the joint probability that the 1-candidate and the 2-candidate among the first t objects are, respectively, the best and the second best among N all is t(t-1)/N(N-1), we obtain the following recurrence relations

$$(2.2.1) \qquad f(t) = \max\begin{cases} A : \dfrac{t(t-1)}{N(N-1)} \\ R : \displaystyle\sum_{s=t+1}^{N} \pi_{ts} \left[\frac{s(s-1)}{N(N-1)} + f(s) \right] \end{cases}$$

$$(2 \leq t \leq N-1;\ f(N)=1),$$

$$(2.2.2) \qquad g(t) = \max \begin{cases} A : \sum_{s=t+1}^{N} \pi_{ts} \left[\frac{s(s-1)}{N(N-1)} + f(s) \right] \\ R : \sum_{s=t+1}^{N} \frac{t}{s(s-1)} g(s) \end{cases}$$

$$(2 \leqq t \leqq N-1;\ g(N)=0),$$

$$(2.2.3) \qquad g(1) = \max \begin{cases} A : \frac{1}{2} \left[\frac{2(2-1)}{N(N-1)} + f(2) \right] \\ R : \sum_{s=2}^{N} \frac{1}{s(s-1)} g(s) \end{cases},$$

where $\pi_{ts}=t(t-1)/s(s-1)(s-2)$. The system of equations (2.2.1)-(2.2.3) can be solved recursively to yield the optimal stopping rule and the maximum probability g(1). The following theorem shows the optimal strategy in state (t,12).

Theorem 2.1. In state (t,12), the optimal strategy is to accept the candidate if and only if $t \geqq r_1^*$, where

$$(2.2.4) \qquad r_1^* = \min \left\{ t \;\middle|\; \sum_{s=t}^{N-1} \frac{1}{s-1} \leqq \frac{1}{2} \right\}.$$

Proof: For (2.2.1), we let

$$(2.2.5) \quad B_1 \equiv \{ (t,12) \mid \frac{t(t-1)}{N(N-1)} \geq \sum_{s=t+1}^{N} \pi_{ts} \left[\frac{s(s-1)}{N(N-1)} + \frac{s(s-1)}{N(N-1)} \right] \} .$$

Then the set B_1, which can be written as $B_1=\{(t,12) \mid r_1^* \leq t \leq N-1\}$, turns out to be closed. Thus the theorem is proved.

Let $F(t)=f(t)/t(t-1)$, we can rewrite (2.2.1) as

$$(2.2.6) \qquad F(t) = \max \begin{cases} A : & \frac{1}{N(N-1)} \\ R : & \sum_{s=t+1}^{N} \frac{1}{s-2} \left[\frac{1}{N(N-1)} + F(s) \right] \end{cases}$$

and since we already know the type of the stopping rule, we obtain by (2.2.5)

$$(2.2.7) \qquad F(t) = \begin{cases} \frac{r_1^*-2}{t-1} \left[F(r_1^*-1) + \frac{r_1^*-t-1}{N(N-1)(r_1^*-2)} \right], & 2 \leq t \leq r_1^*-1 \\ \frac{1}{N(N-1)} , & r_1^* \leq t \leq N , \end{cases}$$

where

$$(2.2.8) \qquad F(r_1^*-1) = \frac{2}{N(N-1)} \sum_{s=r_1^*-1}^{N-1} \frac{1}{s-1} .$$

It is important to note that $F(r_1^*-1) > 1/N(N-1)$. Moreover, with $G(t)=g(t)/t(t-1)$, we can rewrite (2.2.2) and (2.2.3) as

$$(2.2.9)\qquad G(t) = \max \begin{cases} A : \sum_{s=t+1}^{N} \frac{1}{s-2} \left[\frac{1}{N(N-1)} + F(s) \right] \\ R : \frac{1}{t-1} \sum_{s=t+1}^{N} G(s) \end{cases} ,$$

$$(2.2.10)\qquad g(1) = \max \begin{cases} A : \frac{1}{N(N-1)} + F(2) \\ R : \sum_{s=2}^{N} G(s) \end{cases} .$$

Let $H(t)= \frac{1}{t-2} [F(t) + \frac{1}{N(N-1)}]$ for $t \geqq 3$, then we have by (2.2.7)

$$(2.2.11)\qquad H(t) = \begin{cases} \frac{r_1^*-2}{(t-1)(t-2)} [F(r_1^*-1) + \frac{1}{N(N-1)}] , & 3 \leqq t \leqq r_1^*-1 \\ \frac{2}{N(N-1)(t-2)} , & r_1^* \leqq t \leqq N . \end{cases}$$

Now we define $\phi(t)$ as

$$(2.2.12)\qquad \phi(t) = \begin{cases} \sum_{s=2}^{N} (2-s)H(s) + F(2) + \frac{1}{N(N-1)} , & t=1 \\ \sum_{s=t+1}^{N} (2t-s)H(s) , & 2 \leqq t \leqq N-1 . \end{cases}$$

Theorem 2.2. In state (t,1), the optimal strategy is to accept the candidate if and only if $t \geqq r_2^*$,

where

$$(2.2.13)\qquad r_2^* = \min \{ t \mid \phi(t) \geqq 0 \} .$$

Proof: For (2.2.9) and (2.2.10), we let

$$(2.2.14)\quad B_2 \equiv \left\{ (t,1) \;\middle|\; \begin{array}{ll} F(2) + \frac{1}{N(N-1)} \geqq \sum_{s=2}^{N} (s-2)H(s), & t=2 \\ \sum_{s=t+1}^{N} H(s) \geqq \frac{1}{t-1} \sum_{s=t+1}^{N-1} \sum_{k=s+1}^{N} H(k), & 2 \leqq t \leqq N-2 \end{array} \right\}.$$

It is sufficient to show that B_2 is closed. By (2.2.12), we can rewrite (2.2.14) as

$$(2.2.15)\quad B_2 \equiv \{ (t,1) \mid \phi(t) \geqq 0 \}.$$

Showing that B_2 is closed is equivalent to showing that, if there exists an integer r_2^* such that $\phi(r_2^*) \geqq 0$, then $\phi(t)$ is also non-negative for any $t \geqq r_2^*$. It is easy to see that, by

$$(2.2.16)\quad \phi(t) = \begin{cases} (2.2.11) \\ \quad + \frac{1}{N(N-1)} \left[2(2r_1^*-N-t-2) - (r_1^*-2) \sum_{s=t}^{r_1^*-2} \frac{1}{s} \right] \\ \quad + (r_1^*-2)F(r_1^*-1)\left(2 - \sum_{s=t}^{r_1^*-2} \frac{1}{s}\right), & 2 \leqq t \leqq r_1^*-2 \\ \frac{2}{N(N-1)} \sum_{s=t}^{N-1} \left[\frac{2(t-1)}{s-1} - 1 \right], & r_1^*-1 \leqq t \leqq N-1 \end{cases}.$$

Hence, by the definition of r_1^*, we have for $t \geqq r_1^*-1$

$$(2.2.17)\quad \phi(t) - \phi(t+1) = \frac{4}{N(N-1)} \left(\frac{1}{2} - \sum_{s=t+1}^{N-1} \frac{1}{s-1} \right) \geqq 0$$

and for $2 \leq t \leq r_1^*-3$

$$(2.2.18) \quad \phi(t) - \phi(t+1) = -\left[\frac{r_1^*-2}{tN(N-1)} + \frac{r_1^*-2}{t} \quad F(r_1^*-1) - \frac{2}{N(N-1)} \right]$$

$$< -\frac{2}{N(N-1)} \left(\frac{r_1^*-2}{t} -1 \right) < 0 \ .$$

By (2.2.17) and (2.2.18), $\phi(t)$ proves to be unimodal. Since $\phi(N-1)=2/N(N-1)>0$ we see that, from second stage onward, B_2 is closed and that state $(r_1^*,1)$ belongs to B_2. To complete the proof that B_2 is closed, we need to show that, if $\phi(2)<0$, then $\phi(1)<0$. We show this in the case of $r_1^* \geq 4$. Since, for $r_1^* \leq 3$, the proof is easy and can be done in the similar way, we omit it. By the assumption, we have

$$(2.2.19) \qquad \phi(2) = \sum_{s=3}^{N} (4-s)H(s) < 0 \ .$$

Hence, by using the fact that $F(2)=2F(3)+1/N(N-1)$, $\phi(1)$ can be rewritten in the following form

$$(2.2.20) \qquad \phi(1) = \phi(2) + \left[F(2) + \frac{1}{N(N-1)} - 2 \sum_{s=3}^{N} H(s) \right]$$

$$= \phi(2) - 2 \sum_{s=4}^{N} H(s) \ .$$

Since $H(s)>0$ for all s, the right hand side of (2.2.20) is negative. Thus the proof is completed.

We have by (2.2.9)

$$(2.2.21) \qquad G(t) = \begin{cases} \frac{1}{t-1} \sum_{s=t+1}^{N} G(s), & 2 \leq t \leq r_2^*-1 \\ \sum_{s=t+1}^{N} H(s), & r_2^* \leq t \leq N-1 \end{cases} .$$

Therefore, by (2.2.10), we can calculate g(1), which is the probability of success under the optimal strategy. Combining Theorem 2.1 with Theorem 2.2, we can now summarize the optimal stopping rule as follows.

Pass over the first r_2^*-1 objects and thereafter accept the earliest 1-candidate(as the first choice) and after that, if the earliest candidate is a 1-candidate accept it unconditionally(as the second choice), but if the earliest candidate is a 2-candidate accept it only when it appears after stage r_1^*.

Finally we shall give several asymptotic results.

Theorem 2.3. Let $\gamma_i = \lim_{N\to\infty} r_i^*/N$, i=1,2. Then we have

$$(2.2.22) \qquad \gamma_1 = e^{-1/2} \simeq 0.6065$$

and $\gamma_2 \simeq 0.2291$ is the unique root x of the equation

$$(2.2.23) \qquad (1 + x)e^{1/2} - \ln x = 7/2 .$$

The asymptotic value g^* of the maximum probability of choosing both the best and the second best is given by

$$(2.2.24) \qquad g^* = \lim_{N\to\infty} g(1) = \gamma_2(2\gamma_1-\gamma_2) \simeq 0.2254.$$

Proof: The proofs of (2.2.22) and (2.2.23) are straightforward by (2.2.4) and (2.2.13), respectively. By (2.2.21), we have for $2 \leq t \leq r_2^*-1$

$$(2.2.25) \qquad G(t) = \frac{(r_2^*-2)(r_2^*-1)}{(t-1)t} G(r_2^*-1)$$

and, for large N, by (2.2.10) and (2.2.25)

$$(2.2.26) \qquad g(1) = \sum_{s=2}^{N-1} G(s) = (r_2^*-1)(r_2^*-2)G(r_2^*-1) \ .$$

Noting that

$$(2.2.27) \qquad (r_2^*-2)G(r_2^*-1) = \sum_{t=r_2^*}^{N-1} \sum_{s=t+1}^{N} H(s) \ ,$$

we get, by (2.2.11), (2.2.26) and (2.2.27)

$$(2.2.28) \qquad g(1) = (r_2^*-1) \Big[(r_1^*-2)F(r_1^*-1) \Big\{ \sum_{s=r_2^*}^{r_1^*-1} \frac{1}{s-1} - 1 \Big\}$$

$$+ \frac{1}{N(N-1)} \Big\{ 2(N+1) + r_2^* - 3r_1^* + (r_1^*-2) \sum_{s=r_2^*}^{r_1^*-1} \frac{1}{s-1} \Big\} \Big] .$$

We can immediately obtain (2.2.24) by (2.2.28).

Table 2 gives the values of r_1^*, r_2^* and $g(1)$ for some values of N.

Table 2

N	r_1^*	r_2^*	g(1)
3	3	1	0.5000
4	3	1	0.3333
5	4	2	0.3333
6	5	2	0.3139
7	5	2	0.2956
8	6	2	0.2800
9	7	3	0.2739
10	7	3	0.2714
20	13	5	0.2461
50	31	12	0.2333
100	62	24	0.2293
200	122	46	0.2273
500	304	115	0.2262
1000	608	230	0.2258
∞	0.6065N	0.2291N	0.2254

Remark. In a similar way, it is easy to extend our model to the one in which the decision-maker is allowed to make k choices and succeeds only when what he chooses are exactly k best.

2.3 Random Number of Objects

The problem we treat in this section is as follows. At most N objects appear before the decision-maker but he does not know exactly how many objects will appear. He has only an a priori distribution $p_m = \Pr\{M=m\}$ on the actual number M of the objects, where $\sum_{m=1}^{N} p_m = 1$. The decision-maker is allowed to make r choices and succeeds if either of his choices is the best object. If the decision-maker has not accepted last r objects, he he forced to accept them. Our purpose is to find the optimal strategy which maximizes the probability of success. Let $\phi_r(t)$, $r \geq 1$, be the probability of success under an optimal policy starting from state $\langle t,r \rangle$, and also let $\pi_s = \sum_{m=s}^{N} p_m$, $1 \leq s \leq N$. Then the conditional probability that the s-th ($s>t$) object is the earliest 1-candidate given that the t-th is a 1-candidate is $t\pi_s/s(s-1)\pi_t$ and hence the probability that no 1-candidate will appear after stage t is $1 - \sum_{s=t+1}^{N} t\pi_s/s(s-1)\pi_t = \sum_{s=t}^{N} tp_s/s\pi_t$.

Therefore we have the following recurrence relation

$$
(2.3.1)\quad \phi_r(t) = \max \begin{cases} A : \displaystyle\sum_{s=t+1}^{N} \frac{t}{s(s-1)} \frac{\pi_s}{\pi_t} \phi_{r-1}(s) + \sum_{s=t}^{N} \frac{t}{s} \frac{p_s}{\pi_t} \\ R : \displaystyle\sum_{s=t+1}^{N} \frac{t}{s(s-1)} \frac{\pi_s}{\pi_t} \phi_r(s) \end{cases}
$$

$$
(1 \leq t \leq N-1; \quad r \geq 1; \quad \phi_0(t) \equiv 0)
$$

where $\phi_r(N)=1$ $(r\geq 1)$.

We do not consider the case of general a priori distribution on M, but only the case of uniform distribution; $p_m = 1/N$, $1\leq m\leq N$. Then (2.3.1) becomes

$$(2.3.2)\quad \phi_r(t) = \max\begin{cases} A: \dfrac{t}{N-t+1}\left[\displaystyle\sum_{s=t+1}^{N} \frac{N-s+1}{s(s-1)}\phi_{r-1}(s) + \sum_{s=t}^{N}\frac{1}{s}\right] \\ R: \dfrac{t}{N-t+1}\displaystyle\sum_{s=t+1}^{N}\frac{N-s+1}{s(s-1)}\phi_r(s)\ . \end{cases}$$

Now define $\Phi_r(t) = N-t+1/t \cdot \phi_r(t)$, then (2.3.2) becomes

$$(2.3.3)\quad \Phi_r(t) = \max\begin{cases} A: \displaystyle\sum_{s=t+1}^{N}\frac{1}{s-1}\Phi_{r-1}(s) + \sum_{s=t}^{N}\frac{1}{s} \\ R: \displaystyle\sum_{s=t+1}^{N}\frac{1}{s-1}\Phi_r(s)\ . \end{cases}$$

When r=1, we have the following theorem.

Theorem 3.1. In state $\langle t,1\rangle$, the optimal strategy is to accept the candidate if and only if $t\geq m_1^*$, where

$$(2.3.4)\quad m_1^* = \min\left\{t \;\middle|\; \sum_{s=t}^{N}\frac{1}{s} \geq \sum_{s=t}^{N-1}\frac{1}{s}\sum_{k=s+1}^{N}\frac{1}{k}\right\}.$$

Proof: For r=1 in (2.3.3), we let

$$(2.3.5)\quad B_1 \equiv \left\{\langle t,1\rangle \;\middle|\; \sum_{s=t}^{N}\frac{1}{s} \geq \sum_{s=t+1}^{N}\frac{1}{s-1}\sum_{k=s}^{N}\frac{1}{k}\right\} = \{\langle t,1\rangle \mid g_1(t)\geq 0\},$$

where

$$(2.3.6)\quad g_1(t) = \sum_{s=t}^{N}\frac{1}{s} - \sum_{s=t}^{N-1}\frac{1}{s}\sum_{k=s+1}^{N}\frac{1}{k}\ .$$

Note that

(2.3.7) $$g_1(t+1)-g_1(t) = \frac{1}{t}\left(\sum_{s=t+1}^{N} \frac{1}{s} - 1\right).$$

It is easily seen that $g_1(t)$ is unimodal. Since $g_1(N-1)=2/N>0$, we can say that B_1 is closed. Hence, we can rewrite (2.3.5) as

(2.3.8) $$B_1 \equiv \{\langle t,1\rangle \mid m_1^* \leq t \leq N-1\}.$$

Thus the theorem is proved.

This theorem has been already shown by Rasmussen and Robbins [11]. They also give the table of m_1^* and $\phi_1(1)$, the probability of success under the optimal policy. We now have

(2.3.9) $$\Phi_1(t) = \begin{cases} \sum_{s=t}^{N-1} \frac{1}{s}\Phi_1(s+1), & 1\leq t\leq m_1^*-1 \\ \sum_{s=t}^{N} \frac{1}{s}, & m_1^*\leq t\leq N \end{cases}.$$

Let $\Psi_1(t)=\sum_{s=t}^{N-1} \Phi_1(s+1)/s + \sum_{s=t}^{N} 1/s$. Then we have

(2.3.10) $$\Psi_1(t)-\Phi_1(t)= \begin{cases} \sum_{s=t}^{N} \frac{1}{s}, & 1\leq t\leq m_1^*-1 \\ \sum_{s=t}^{N-1} \frac{1}{s}\Phi_1(s+1)= \sum_{s=t}^{N-1} \frac{1}{s} \sum_{k=s+1}^{N} \frac{1}{k}, & m_1^*\leq t\leq N-1 \end{cases}.$$

And hence

$$(2.3.11)\quad \sum_{s=t}^{N-1} \frac{1}{s}[\Psi_1(s+1)-\Phi_1(s+1)] = \begin{cases} \sum_{s=t}^{m_1^*-2} \frac{1}{s} \sum_{k=s+1}^{N} \frac{1}{k} + \sum_{s=m_1^*-1}^{N-2} \frac{1}{s} \\ \sum_{k=s+1}^{N-1} \frac{1}{k} \sum_{\ell=k+1}^{N} \frac{1}{\ell}, \quad 1\leq t\leq m_1^*-2 \\ \sum_{s=t}^{N-2} \frac{1}{s} \sum_{k=s+1}^{N-1} \frac{1}{k} \sum_{\ell=k+1}^{N} \frac{1}{\ell}, \quad m_1^*-1\leq t\leq N-1 \end{cases}.$$

If we let

$$(2.3.12)\quad g_2(t)=\sum_{s=t}^{N} \frac{1}{s} - \sum_{s=t}^{N-1} \frac{1}{s}[\Psi_1(s+1)-\Phi_1(s+1)] \quad ,$$

we have the following theorem for r=2.

Theorem 3.2. In state $\langle t,2\rangle$, the optimal strategy is to accept the candidate if and only if $t\geq m_2^*$, where

$$(2.3.13)\quad m_2^* = \min \{ t \mid g_2(t)\geq 0 \}.$$

Proof: For r=2 in (2.3.3), we let

$$(2.3.14)\quad B_2 \equiv \{\langle t,2\rangle \mid \Psi_1(t) \geq \sum_{s=t}^{N-1} \Psi_1(s+1)/s \}.$$

It is sufficient to show that B_2 is closed. By (2.3.12), we can rewrite (2.3.14) as

$$(2.3.15)\quad B_2 = \{ \langle t,2\rangle \mid g_2(t) \geq 0 \}.$$

By (2.3.10) and (2.3.12), we see that

$$g_2(t+1)-g_2(t)= \frac{1}{t} [\Psi_1(t+1)-\Phi_1(t+1)-1]$$

$$(2.3.16)\qquad = \begin{cases} \frac{1}{t}\left(\sum_{s=t+1}^{N} \frac{1}{s} - 1\right), & 1 \leq t \leq m_1^* - 2 \\ \frac{1}{t}\left(\sum_{s=t+1}^{N-1} \frac{1}{s} \sum_{k=s+1}^{N} \frac{1}{k} - 1\right), & m_1^* - 1 \leq t \leq N-2. \end{cases}$$

Since, for $t \leq m_1^* - 2$, by (2.3.7) and the definition of m_1^*, $g_2(t+1) - g_2(t) = g_1(t+1) - g_1(t) > 0$ and for $t \geq m_1^* - 1$, $\sum_{s=t+1}^{N-1} 1/s \sum_{k=s+1}^{N} 1/k - 1$ is a decreasing function in t, $g_2(t)$ is unimodal. Thus, considering $g_2(N-1) = (2N-1)/N(N-1) > 0$, we can say that B_2 is closed. Hence, we can rewrite (2.3.15) as

$$(2.3.17)\qquad B_2 \equiv \{ \langle t, 2 \rangle \mid m_2^* \leq t \leq N-1 \} .$$

Thus the proof is completed.

We now show that $m_2^* \leq m_1^*$. This is equivalent to showing that $g_2(m_1^*) \geq 0$. Since, by (2.3.6), $\sum_{s=t}^{N} 1/s \geq \sum_{s=t}^{N-1} 1/s \sum_{k=s+1}^{N} 1/k$ for $t \geq m_1^*$, we soon obtain, using (2.3.11)

$$g_2(s_1^*) = \sum_{s=m_1^*}^{N} \frac{1}{s} - \sum_{s=m_1^*}^{N-2} \frac{1}{s} \sum_{k=s+1}^{N-1} \frac{1}{k} \sum_{\ell=k+1}^{N} \frac{1}{\ell}$$

$$\geq \sum_{s=m_1^*}^{N} \frac{1}{s} - \sum_{s=m_1^*}^{N-1} \frac{1}{s} \sum_{k=s+1}^{N} \frac{1}{k} \geq 0 .$$

For the computation of $\phi_2(1)$, the probability of success under the optimal policy, we can utilize the following relation

$$(2.3.18)\quad \Phi_2(t) = \begin{cases} \sum_{s=t}^{N-1} \frac{1}{s}\Phi_2(s+1), & 1\leq t\leq m_2^*-1 \\ \sum_{s=t}^{N-1} \frac{1}{s}\Phi_1(s+1) + \sum_{s=t}^{N} \frac{1}{s}, & m_2^*\leq t\leq N-1 \end{cases} .$$

Finally we give the asymptotic results.

Theorem 3.3. Let $\delta_i = \lim_{N\to\infty} m_i^*/N$, $i=1,2$. Then we have

$$(2.3.19)\quad \delta_1 = e^{-2} \simeq 0.1353$$

and

$$(2.3.20)\quad \delta_2 = e^{-(3+\sqrt{21})/3} \simeq 0.0799.$$

The asymptotic values $\phi_r(1)$, $r=1,2$, are given by

$$(2.3.21)\quad \lim_{N\to\infty} \phi_1(1) = 2e^{-2} \simeq 0.2707$$

and

$$(2.3.22)\quad \lim_{N\to\infty} \phi_2(1) = 2e^{-2}+(3+\sqrt{21})/3 \cdot e^{-(3+\sqrt{21})/3} \simeq 0.4725.$$

Proof: The proofs of (2.3.19) and (2.3.20) are straightforward by (2.3.4) and (2.3.12), respectively. By (2.3.9), we have

$$(2.3.23)\quad \phi_1(1) = \frac{1}{N}\Phi_1(1) = \frac{m_1^*-1}{N}\ \Phi_1(m_1^*-1) = \frac{m_1^*-1}{N} \sum_{s=m_1^*-1}^{N-1} \frac{1}{s} \sum_{k=s+1}^{N} \frac{1}{k} .$$

(2.3.21) soon becomes from this relation.

Moreover, we get, by (2.3.9) and (2.3.18)

$$(2.3.24)\quad \Phi_2(m_2^*-1)=\frac{m_1^*-1}{m_2^*-1}\,\Phi_1(m_1^*-1)+\sum_{s=m_2^*-1}^{N}\frac{1}{s}\sum_{k=s+1}^{N}\frac{1}{k}$$

$$+\sum_{s=m_1^*-1}^{N-2}\frac{1}{s}\sum_{k=s+1}^{N-1}\frac{1}{k}\sum_{\ell=k+1}^{N}\frac{1}{\ell}\,.$$

Hence

$$(2.3.25)\quad \phi_2(1)=\frac{1}{N}\,\Phi_2(1)=\frac{m_2^*-1}{N}\,\Phi_2(m_2^*-1)$$

$$=\phi_1(1)+\frac{m_2^*-1}{N}\left\{\sum_{s=m_2^*-1}^{m_1^*-2}\frac{1}{s}\sum_{k=s+1}^{N}\frac{1}{k}+\sum_{s=m_1^*-1}^{N-2}\frac{1}{s}\sum_{k=s+1}^{N-1}\frac{1}{k}\sum_{\ell=k+1}^{N}\frac{1}{\ell}\right\}.$$

We can immediately have (2.3.22) by (2.3.25)

Table 3 gives the values of m_1^*, m_2^*, $\phi_1(1)$ and $\phi_2(1)$ for some values of N.

Table 3

N	m_1^*	m_2^*	$\phi_1(1)$	$\phi_2(1)$
3	1	1	0.61111	0.94444
4	1	1	0.52083	0.88542
5	1	1	0.45667	0.83167
6	1	1	0.40833	0.78426
7	2	1	0.37222	0.74263
8	2	1	0.36621	0.70594
9	2	1	0.35907	0.67340
10	2	1	0.35145	0.64435
20	3	2	0.30760	0.54821
50	7	5	0.28491	0.49984
100	14	9	0.27779	0.48613
500	68	40	0.27208	0.47522
1000	136	80	0.27137	0.47386
∞	0.13533N	0.07986N	0.27067	0.47251

The values of m_1^* and $\phi_1(1)$ are reproduced from Table 1 in Rasmussen and Robbins (1975).

Remark. The recurrence relation (2.3.2) for $r \geqq 3$ can also be solved similarly and we can obtain the critical numbers m_r^* $(r \geqq 3)$.

2.4 Backward Solicitation with Rejection Probability

We shall investigate a model where the dicision-maker, who is allowed a backward solicitation at any stage, succeeds if his choice is the best among all, but each of the objects has the right to refuse an offer of acceptance with a known and fixed probability 1-p, $0<p<1$, independent of its rank and the arrangement of the other objects. If an object is chosen and it accepts (refuses) the offer, we call it available (unavailable). We assume that a backward solicitation may be attempted at any stage, but that solicitation may not be successful, whereupon the decision-maker must resume his observation of the remaining objects. Suppose that the decision-maker is at stage k and has recognized that the j-th ($1\leq j\leq k$) object is the relative best among those k objects. We set $m_k=k-j$, if j-th object is available and no attempt has been made to procure it and $m_k=\infty$, if j-th is unavailable or an unsuccessful attempt has been made to procure it. Thus m_k is interpreted as the relative position (to k) of the best available object and it can take values 0,1, ..., k-1, ∞ . We denote this state by (k, m_k). In state (k, m_k), we call the $(k-m_k)$-th object a "current candidate" if $m_k=0$ and a "potential candidate" if $1\leq m_k\leq k-1$. When we need not distinguish between a current candidate and a potential candidate, we call them by the name of candidate.

We suppose that the decision-maker is now in state (k, m_k). Let $\pi_1(k, m_k)$ be the probability of obtaining the best object if he decides to observe the next object without solicitation of the candidate and $\pi_2(k, m_k)$ be the probability of that if he decides to solicite the candidate. Then $\pi(k, m_k)$, probability of obtaining the best object under an optimal policy starting from state (k, m_k), becomes

(2.4.1) $$\pi(k, m_k) = \max[\pi_1(k, m_k), \pi_2(k, m_k)],$$

and, for all m_k

(2.4.2) $$\pi_1(k, m_k) = \frac{p}{k+1}\pi(k+1,0) + \frac{1-p}{k+1}\pi(k+1, \infty) + \frac{k}{k+1}\pi(k+1, m_k+1),$$

(2.4.3) $$\pi_2(k, m_k) = \frac{k}{N} q(m_k) + (1-q(m_k))\pi(k, \infty),$$

(2.4.4) $$\pi(k, \infty) = \frac{p}{k+1}\pi(k+1,0) + \frac{k+1-p}{k+1}\pi(k+1, \infty),\ 1 \leq k \leq N-1,$$

(2.4.5) $$\pi(N, m_N) = q(m_N).$$

Since $\pi(N,0)=1$ and $\pi(N, \infty)=0$, by repeated applications of (2.4.4), we have

(2.4.6) $$\pi(k, \infty) = p\,k \sum_{j=k+1}^{N} \frac{\pi(j, 0)}{j(j-1)} \prod_{\ell=k}^{j-2} \left(1 + \frac{1-p}{\ell}\right).$$

In this section, we assume $\prod_{\ell=a}^{b} \left(1 + \frac{1-p}{\ell}\right) = 1$, if $b<a$.

Once the set of values of $\{q(m)\}$ is given it is easy to solve recursively the system of equations (2.4.1)-(2.4.3) and obtain the optimal policy. The case of constant probability and geometric probability of successful procurement are discussed in detail in (i) and (ii), respectively. Our model reduces to the Yang model if p=1, and also to the Smith model if the appropriate limit cases are taken in (i) and (ii). We owe the proofs of our theorems and lemmas to Yang [28] .

(i) Constant probability; $q(m)=q$, $m \geq 1$.

Here we consider the case of constant probability of successful procurement $q(0)=1$, $q(m)=q$, for $1 \leq m \leq N-1$. Then we have the following result.

Theorem 4.1. Pass over the first n_1^*-1 objects and thereafter accept the earliest current candidate. If it happens that stage N is reached and the absolute best is among the first n_1^*-1 objects, attempt to procure it. n_1^* is the smallest integer s such that

$$(2.4.7) \qquad \prod_{\ell=s}^{N-1} \left(1 + \frac{1-p}{\ell}\right) \leq 1 + \frac{(1-p)(1-q)}{p} .$$

The maximum probability of obtaining the best object is given by

$$(2.4.8) \qquad U = \frac{(n_1^*-1)pq}{N} + \frac{(n_1^*-1)p}{N(1-p)} \left[\prod_{\ell=n_1^*-1}^{N-1} \left(1 + \frac{1-p}{\ell}\right) - 1 \right] .$$

It is also shown that

$$(2.4.9) \qquad \lim_{N\to\infty} n_1^*/N = \left[\frac{p}{1 - q(1-p)}\right]^{1/1-p} ,$$

$$(2.4.10) \qquad \lim_{N\to\infty} U = \left[\frac{p}{\{1-q(1-p)\}^{p}}\right]^{1/1-p} .$$

Proof: First we show that $\pi_1(k, m_k) \geqq \pi_2(k, m_k)$ for all $k \leqq N-1$, when $m_k \neq 0$. We prove this by backward induction. When $m_k = \infty$, it is evident that $\pi_1(k, m_k) = \pi_2(k, m_k)$. For $k = N-1$, it is easy to see that

$$\pi_1(N-1, m_{N-1}) - \pi_2(N-1, m_{N-1}) = \frac{pq}{N} \geqq 0, \text{ for } \quad m_{N-1} \neq 0 .$$

Now assume that $\pi_1(k, m_k) \geqq \pi_2(k, m_k)$ for all $k = s+1, s+2, \ldots, N-2, N-1$, when $m_k \neq 0$. Then, for $m_s \neq \infty$, repeating (2.4.2) we obtain

$$(2.4.11) \qquad \pi_1(s, m_s) = ps \sum_{j=s+1}^{N} \frac{\pi(j,0)}{j(j-1)} + (1-p)s \sum_{j=s+1}^{N} \frac{\pi(j,\infty)}{j(j-1)} + \frac{s}{N}q ,$$

and by (2.4.3), for $m_s \neq 0$

$$(2.4.12) \qquad \pi_2(s, m_s) = \frac{s}{N} q + (1-q)\pi(s, \infty).$$

Hence, substituting (2.4.6) into (2.4.11) and (2.4.12), we have

$$\begin{aligned} &\pi_1(s, m_s) - \pi_2(s, m_s) \\ &= ps \sum_{j=s+1}^{N} \frac{\pi(j,0)}{j(j-1)} + (1-p)ps \sum_{j=s+1}^{N} \frac{1}{j-1} \sum_{i=j+1}^{N} \frac{\pi(i,0)}{i(i-1)} \prod_{\ell=j}^{i-2} (1+ \frac{1-p}{\ell}) \\ &\quad -(1-q)ps \sum_{j=s+1}^{N} \frac{\pi(j,0)}{j(j-1)} \prod_{\ell=s}^{j-2} (1 + \frac{1-p}{\ell}) \\ &= pqs \sum_{j=s+1}^{N} \frac{\pi(j,0)}{j(j-1)} \prod_{\ell=s}^{j-2} (1 + \frac{1-p}{\ell}) \geqq 0 . \end{aligned}$$

Thus we get $\pi_1(k, m_k) \geq \pi_2(k, m_k)$ for $m_k \neq 0$.

Moreover, we must show that if there exists s such that $\pi(s,0)=s/N$, then $\pi(s+1,0)=(s+1)/N$. Assume that, when $\pi(s,0)=s/N$, $\pi(s+1,0)=\pi_1(s+1,0)>(s+1)/N$. Since, for $m_k \neq \infty$, $\pi_1(k, m_k)$ does not depend on m_k, we have by (2.4.2)

$$\pi_1(s,0)=\frac{p}{s+1}\pi_1(s+1,0)+\frac{1-p}{s+1}\pi(s+1,\infty)+\frac{s}{s+1}\pi_1(s+1,1)$$

$$\geq \frac{s+p}{s+1}\pi_1(s+1,0)>\frac{s+p}{s+1}\;\frac{s+1}{N}=\frac{s+p}{N}.$$

Therefore, $\pi(s,0)=\max\,[\pi_1(s,0),\; s/N\,] > s/N$. This is a contradiction. Since we have known the form of the optimal policy, the maximum probability of obtaining the best object can easily be computed. When we set $u_s=\pi_1(s,0)$, the decision-maker accepts the s-th object if and only if s-th is a current candidate and $u_s \leq s/N$. Let $n_1^* = \min\{s \mid u_s \leq s/N\}$, then we have by (2.4.2), (2.4.5) and (2.4.6), for $n_1^* \leq s \leq N$

$$(2.4.13)\quad u_{s-1} = \frac{p}{s}\frac{s}{N}+\frac{1-p}{s}\pi(s,\infty)+\frac{s-1}{s}u_s = \frac{p}{N}\prod_{\ell=s}^{N-1}\left(1+\frac{1-p}{\ell}\right)+\frac{s-1}{s}u_s,$$

with $u_{N-1} = p/N + (N-1)q/N$. The solution of (2.4.13) is

$$(2.4.14)\quad u_s=\frac{sq}{N}+\frac{sp}{N(1-p)}\left[\prod_{\ell=s}^{N-1}\left(1+\frac{1-p}{\ell}\right)-1\right],\ \text{for } s=n_1^*-1,\ \ldots,\ N-1.$$

Hence, n_1^* can be expressed by (2.4.7) and the maximum probability U, which is equal to $pu_{n_1^*-1}+(1-p)\pi(n_1^*-1,\infty)$.

becomes as given by (2.4.8). We can immediately obtain (2.4.9) and (2.4.10), by (2.4.7) and (2.4.8), respectively.

Table 4 gives the values of n_1^* and U for some values of p,q and N.

Table 4

p=0.5

q	0.2		0.5		0.8	
N	n_1^*	U	n_1^*	U	n_1^*	U
2	1	0.3750	1	0.3750	2	0.4500
3	2	0.3250	2	0.3750	3	0.4333
4	2	0.3219	2	0.3594	3	0.4292
5	2	0.3122	3	0.3563	4	0.4275
6	3	0.3016	3	0.3516	5	0.4250
7	3	0.3015	4	0.3489	5	0.4232
8	3	0.2987	4	0.3472	6	0.4230
9	3	0.2946	5	0.3450	7	0.4222
10	4	0.2938	5	0.3444	8	0.4212
20	7	0.2858	10	0.3385	14	0.4189
50	16	0.2809	23	0.3354	35	0.4176
100	32	0.2793	45	0.3344	70	0.4171
1000	309	0.2779	445	0.3334	695	0.4167
∞	0.309N	0.2778	0.444N	0.3333	0.694N	0.4167

p=0.8

q	0.2		0.5		0.8	
N	n_1^*	U	n_1^*	U	n_1^*	U
2	2	0.4800	2	0.6000	2	0.7200
3	2	0.4800	2	0.5600	3	0.6933
4	2	0.4480	3	0.5467	4	0.6800
5	3	0.4352	3	0.5312	4	0.6720
6	3	0.4284	4	0.5296	5	0.6720
7	4	0.4177	5	0.5221	6	0.6705
8	4	0.4170	5	0.5213	7	0.6686
9	4	0.4120	6	0.5178	8	0.6667
10	5	0.4098	6	0.5164	8	0.6656
20	9	0.3976	12	0.5077	16	0.6620
50	21	0.3905	28	0.5026	40	0.6596
100	41	0.3881	56	0.5011	79	0.6589
1000	403	0.3860	556	0.4996	784	0.6582
∞	0.402N	0.3858	0.555N	0.4994	0.784N	0.6582

Remark. When $q \to 0$, (2.4.7) - (2.4.10) agree with those of Smith [20] , and when $p \to 1$, (2.4.7) - (2.4.10) reduce to Theorem 4 of Yang [28] and in this case n_1^* is the smallest integer s such that $\sum_{\ell=s}^{N-1} 1/\ell \leq 1-q$ and $U = (n_1^*-1)/N \cdot (q+\sum_{\ell=n_1^*-1}^{N-1} 1/\ell)$. Hence, $N \to \infty$, $n_1^* \simeq Ne^{q-1}$, and $U \simeq e^{q-1}$.

(ii) Geometric probability ; $q(m)=q^m$.

Here, we consider the case where the probability of successful procurement is geometric, that is, $q(m)=q^m$. We must prepare two lemmas before deriving the optimal strategy in this case. Let n^* be the smallest integer s such that

$$\prod_{\ell=s}^{N-1} \left(1+ \frac{1-p}{\ell}\right) \leq \frac{1}{p} .$$

Lemma 4.2. For any $\{q(m)\}$, no solicitation should be made prior to stage n^*.

Proof: Let

$$a(s) = \frac{1}{N} - p \sum_{j=s}^{N} \frac{\pi(j,0)}{j(j-1)} \prod_{\ell=s-1}^{j-2} \left(1 + \frac{1-p}{\ell}\right), \quad 1 < s \leq N.$$

Since $a(s)$ is increasing in s, we can define σ as $\max\{s \mid a(s) \leq 0\}$.

Now put

(2.4.15) $\quad D(s, m) = \pi(s, m) - \pi_2(s, m).$

Then we have

$$\pi_1(s-1, m) - \pi_2(s-1, m) = \frac{1}{s}[p\pi(s,0) + (1-p)\pi(s,\infty)]$$

$$+ \frac{s-1}{s}[D(s,m+1) + \pi_2(s, m+1)] - \frac{s-1}{N}q(m) - (1-q(m))\,\pi(s-1,\infty).$$

Using (2.4.3) on $\pi_2(s, m+1)$ and (2.4.4) in simplifying, we find

$$\pi_1(s-1, m) - \pi_2(s-1, m) = \frac{s-1}{s} D(s, m+1) + (s-1)[a(s+1)q(m+1)$$
$$- a(s)q(m)].$$

For $s \leq \sigma$, $a(s) \leq 0$, we have $a(s+1)q(m+1) - a(s)q(m) \geq 0$. $D(s, m+1)$ is non-negative, we see that the right side of (2.4.16) is non-negative for all m and $s \leq \sigma$. Hence, no solicitation should be made prior to stage σ. To complete the proof, it suffices to show that $\sigma \geq n^*$. From the definition of n^*, this is equivalent to showing that $\prod_{\ell=\sigma}^{N-1} \left(1 + \frac{1-p}{\ell}\right) \leq \frac{1}{p}$. By the fact that $0 < a(\sigma+1)$ and $\pi(j,0) \geq \frac{j}{N}$, we find

$$0 < \frac{1}{N} - \frac{p}{N} \sum_{j=\sigma+1}^{N} \frac{1}{j-1} \prod_{\ell=\sigma}^{j-2} (1 + \frac{1-p}{\ell}) .$$

This can be rewritten as

$$\prod_{\ell=\sigma}^{N-1} (1 + \frac{1-p}{\ell}) \leq \frac{1}{p} .$$

Now let $b(r) = p \prod_{\ell=r-1}^{N-1} (1+ \frac{1-p}{\ell})$. Then n^* is the largest integer such that $1-b(r) \leq 0$.

Lemma 4.3. Suppose $q(m) > 0$ for all finite m. If there exists a τ such that τ is the smallest $r(\geq n^*)$ satisfying

(2.4.17) $$\frac{q(m+1)}{q(m)} \leq \frac{1 - b(r+1)}{1 - b(r+2)} \quad \text{for all } m < \infty ,$$

then the decision-maker should make solicitation to the earliest candidate as it appears if the process has not terminated before stage τ.

Proof: Note that relation (2.4.17) holds for any $r<\tau$, because $1 - b(r+1)/1-b(r+2)$ is a monotonically increasing function for $r \geq n^*$. We prove our lemma by backward induction. It is easily seen that $\pi_1(N-1, m_{N-1}) \leq \pi_2(N-1, m_{N-1})$. Suppose that $\pi_1(k, m_k) - \pi_2(k, m_k) \leq 0$ for all m_k and $k \geq s+1 > \tau$. Then $D(k, m_k) = 0$ and $a(s) = 1/N - [\,b(s) - p\,] /N(1-p)$ for all $k \geq s+1$. By (2.4.16)

$$\pi_1(s, m_s) - \pi_2(s, m_s)$$
$$= s[a(s+2)q(m_s+1) - a(s+1)q(m_s)]$$

$$= \frac{s}{N(1-p)} [\{1 - b(s+2)\}q(m_{s+1}) - \{1 - b(s+1)\}q(m_s)] \leq 0.$$

This induction terminates at $k = \tau$. Thus our lemma is proved.

For the geometric case we can derive the optimal strategy of obtaining the best object by the next theorem.

Theorem 4.4. In the geometric case, the optimal strategy is to pass over the first n_2^*-1 objects, solicit the candidate, if possible (no matter where it is) at stage n_2^*, and if the procurement at stage n_2^* is unsuccessful, continue the observations of the remaining objects and accept the earliest current candidate thereafter. The value of n_2^* is the largest t which satisfies

$$\text{(2.4.18)} \qquad \frac{(1-p)c(t)}{(t-1)[1-c(t)]} > 1-q ,$$

where

$$\text{(2.4.19)} \qquad c(t) = p \prod_{\ell=t}^{N-1} (1 + \frac{1-p}{\ell}) .$$

The maximum probability of obtaining the best object is given by

$$\text{(2.4.20)} \quad V = \frac{p}{N(1-p)} \cdot \frac{1-q^{n_2^*}}{1-q} [1 - c(n_2^*)] + \frac{n_2^*}{N(1-p)} [c(n_2^*) - p] .$$

Proof: n^* satisfies the relation (2.4.18) for all q. Thus n_2^* exists and $n_2^* \geq n^*$, which is guaranteed by Lemma 4.2. It is easy to see that, when we put $\tau = n_2^*$, (2.4.17) and (2.4.18) are equivalent in the case of $q(m) = q^m$.

Therefore, by Lemma 4.3, all we have to do to complete the proof is showing that $\pi_1(k, m_k) \geq \pi_2(k, m_k)$ for all $k < n_2^*$. We prove this by backward induction. From the definition of n_2^*, we obtain

$$q > \frac{1 - b(n_2^*)}{1 - b\,(n_2^*+1)} = \frac{a(n_2^*)}{a(n_2^*+1)} .$$

Also, by (2.4.16) and the fact $D(n_2^*, m) = 0$ for any m, we have

(2.4.21) $\pi_1(n_2^*-1, m) - \pi_2(n_2^*-1, m) = (n_2^*-1)q^m\,[a(n_2^*+1)q - a(n_2^*)]$.

Since $n_2^* = \tau \geq \sigma$, $a(n_2^* + 1) > 0$, (2.4.21) is positive. Now suppose that $\pi_1(k, m) \geq \pi_2(k, m)$ for all $k = s+1, s+2, \ldots, n_2^*-2, n_2^*-1$. Then we have

(2.4.22) $$\pi_1(s, m) = ps \sum_{j=s+1}^{n_2^*} \frac{\pi(j, 0)}{j(j-1)} + (1-p)\, s \sum_{j=s+1}^{n_2^*} \frac{\pi(j, \infty)}{j(j-1)}$$

$$+ \frac{s}{n_2^*}\, \pi(n_2^*, m + n_2^* - s) .$$

By (2.4.6), second term in the right hand side of (2.4.22) becomes

$$(1-p)\, s \sum_{j=s+1}^{n_2^*} \frac{\pi(j, \infty)}{j(j-1)}$$

$$= p\, s \sum_{j=s+2}^{N} \frac{\pi(j, 0)}{j(j-1)} \left[\prod_{\ell=s}^{j-2} \left(1 + \frac{1-p}{\ell}\right) - 1 \right]$$

$$- p\, s \sum_{j=n_2^*+2}^{N} \frac{\pi(j, 0)}{j(j-1)} \left[\prod_{\ell=n_2^*}^{j-2} \left(1 + \frac{1-p}{\ell}\right) - 1 \right]$$

$$(2.4.23) = \pi(s, \infty) - \frac{s}{n_2^*} \pi(n_2^*, \infty) - p\, s \sum_{j=s+1}^{n_2^*} \frac{\pi(j, 0)}{j(j-1)} .$$

By (2.4.6) and the fact that $\pi(j, 0) = j/N$ for $j > n_2^*$, we have

$$(2.4.24)\quad \pi(n_2^*, \infty) = \frac{pn_2^*}{N} \sum_{j=n_2^*+1}^{N} \frac{1}{j-1} \prod_{\ell=n_2^*}^{j-2} \left(1 + \frac{1-p}{\ell}\right) = \frac{n_2^*}{N(1-p)} [c(n_2^*)-p] .$$

Therefore, (2.4.23) becomes

$$(1-p)\, s \sum_{j=s+1}^{n_2^*} \frac{\pi(j, \infty)}{j(j-1)} = \pi(s, \infty) - \frac{s}{N(1-p)} [c(n_2^*) - p]$$

$$(2.4.25) \qquad - p\, s \sum_{j=s+1}^{n_2^*} \frac{\pi(j, 0)}{j(j-1)} .$$

By using (2.4.24), third term of (2.4.22) becomes

$$\frac{s}{n_2^*} \pi(n_2^*, m + n_2^* - s) = \frac{s}{n_2^*} \left[\frac{n_2^*}{N} q^{m+n_2^*-s} + (1 - q^{m+n_2^*-s}) \pi(n_2^*, \infty)\right]$$

$$(2.4.26) = \frac{s}{N} q^{m+n_2^*-s} + \frac{s}{N(1-p)} (1 - q^{m+n_2^*-s}) [c(n_2^*) - p] .$$

Hence (2.4.22), combined with (2.4.25) and (2.4.26), can be rewritten in the following form

$$(2.4.27)\quad \pi_1(s, m) = \pi(s, \infty) + \frac{s}{N} q^{m+n_2^*-s} - \frac{s}{N(1-p)} q^{m+n_2^*-s} [c(n_2^*)-p] .$$

On the other hand, we get

$$(2.4.28)\qquad \pi_2(s, m) = \frac{s}{N} q^m + (1 - q^m)\pi(s, \infty).$$

Thus by (2.4.27) and (2.4.28), to prove $\pi_1(s, m) \geq \pi_2(s, m)$ is equivalent to showing that

(2.4.29) $g(s) = [1 - c(n_2^*)]\, q^{n_2^*-s} - (1-p) + \frac{N}{s}(1-p)\pi(s,\infty) \geq 0.$

Since we already know that $g(n_2^* - 1) \geq 0$, it suffices to show that $g(s)$ is a decreasing function in s. By (2.4.29) we have

$$g(s+1) - g(s) = (1-q)q^{n_2^*-s-1}\,[1 - c(n_2^*)]$$

(2.4.30) $$- N(1-p)\,\left[\frac{\pi(s,\infty)}{s} - \frac{\pi(s+1,\infty)}{s+1}\right].$$

It is easily seen by backward induction that, for $s+1 \leq j \leq n_2^* -1$

$$\pi(j, 0) = \frac{p}{n_2^*} \sum_{m=0}^{n_2^*-j-1} \pi(n_2^*, m) + \frac{j}{n_2^*}\,\pi(n_2^*, n_2^* - j)$$

(2.4.31) $$+ \frac{(1-p)(n_2^*-j)}{n_2^*}\,\pi(n_2^*, \infty)$$

and

(2.4.32) $\pi(n_2^*, m) = \frac{n_2^*}{N} q^m + (1 - q^m)\ \pi(n_2^*, \infty).$

Hence, by substituting (2.4.24) and (2.4.32) into (2.4.31), we can reduce (2.4.31) to

(2.4.33) $$\pi(j, 0) = \frac{1-c(n_2^*)}{N(1-p)}\ [p\,\frac{1-q^{n_2^*-j}}{1-q} + j\cdot q^{n_2^*-j}] + \frac{n_2^*[c(n_2^*)-p]}{N(1-p)}.$$

By (2.4.6), we have

(2.4.34) $$- N(1-p)\ \left[\frac{\pi(s,\infty)}{s} - \frac{\pi(s+1,\infty)}{s+1}\right]$$

$$= - N(1-p)p\ \frac{\pi(s+1, 0)}{s(s+1)} - \frac{N(1-p)^2 p}{s} \sum_{j=s+2}^{N} \frac{\pi(j, 0)}{j(j-1)} \prod_{\ell=s+1}^{j-2} (1+\frac{1-p}{\ell}).$$

Since $p < c(n_2^*) \leq 1$, by (2.4.33) the first term in the right side of (2.4.34) becomes

$$(2.4.35) \quad - N(1-p)p \frac{\pi(s+1,\ 0)}{s(s+1)}$$

$$= - \frac{N(1-p)p}{s(s+1)} \Big[\frac{c(n_2^*)-p}{N(1-p)} \{ n_2^* - (s+1)q^{n_2^*-s-1} \}$$

$$+ \frac{p[1-c(n_2^*)]}{N(1-p)} \frac{1-q^{n_2^*-s-1}}{1-q} + \frac{1}{N}(s+1)q^{n_2^*-s-1} \Big] \leq - \frac{(1-p)p}{s} q^{n_2^*-s-1} .$$

By the fact that $\pi(j,\ 0) \geq j/N$, the second term of (2.4.34) becomes

$$(2.4.36) \quad -N(1-p)^2 \frac{p}{s} \sum_{j=s+2}^{N} \frac{\pi(j,\ 0)}{j(j-1)} \prod_{\ell=s+1}^{j-2} (1 + \frac{1-p}{\ell})$$

$$\leq - \frac{1-p}{s} [p \prod_{\ell=s+1}^{N-1} (1 + \frac{1-p}{\ell}) - p] \leq - \frac{1-p}{s} [c(n_2^*) - p]$$

$$\leq - \frac{1-p}{s} q^{n_2^*-s-1} [c(n_2^*) - p] .$$

From the definition of n_2^*, we have $(1-q)[1 - c(n_2^*)] < (1-p)c(n_2^*)/s$. Therefore (2.4.30), combined with (2.4.34)-(2.4.36), becomes

$$(2.4.37) \quad g(s+1) - g(s) \leq q^{n_2^*-s-1} [(1-q)(1-c(n_2^*)) - \frac{(1-p)c(n_2^*)}{s}] \leq 0 ,$$

which shows that g(s) is decreasing in s. The maximum probability V is given by

$$(2.4.38) \quad V = \sum_{m=0}^{n_2^*-1} \frac{p}{n_2^*} \pi(n_2^*,\ m) + (1-p)\pi(n_2^*, \infty).$$

Substituting (2.4.24) and (2.4.32) into (2.4.38), we have (2.4.20). Thus the proof is completed.

Table 5 gives the values of n_2^* and V for some values of p, q and N.

Table 5

p=0.5

q	0.2		0.5		0.8	
N	n_2^*	V	n_2^*	V	n_2^*	V
2	1	0.3750	1	0.3750	2	0.4500
3	2	0.3167	2	0.3542	3	0.4067
4	2	0.3104	2	0.3307	3	0.3792
5	2	0.2994	2	0.3102	4	0.3583
6	2	0.2878	3	0.3030	4	0.3459
7	3	0.2804	3	0.2962	4	0.3337
8	3	0.2785	3	0.2889	5	0.3249
9	3	0.2753	4	0.2832	5	0.3183
10	3	0.2713	4	0.2809	5	0.3115
20	6	0.2604	6	0.2630	8	0.2804
50	13	0.2539	14	0.2545	16	0.2595
100	26	0.2519	27	0.2521	29	0.2536
1000	251	0.2502	252	0.2502	255	0.2502
∞	0.250N	0.2500	0.250N	0.2500	0.250N	0.2500

p=0.8

q	0.2		0.5		0.8	
N	n_2^*	V	n_2^*	V	n_2^*	V
2	1	0.4800	2	0.6000	2	0.7200
3	2	0.4587	2	0.5067	3	0.6507
4	2	0.4203	3	0.4567	4	0.5904
5	3	0.3912	3	0.4336	4	0.5379
6	3	0.3860	3	0.4091	5	0.5098
7	3	0.3754	4	0.3977	5	0.4828
8	4	0.3659	4	0.3883	6	0.4647
9	4	0.3639	4	0.3774	6	0.4491
10	4	0.3593	5	0.3733	7	0.4355
20	8	0.3419	8	0.3468	10	0.3757
50	17	0.3332	18	0.3341	21	0.3411
100	34	0.3304	35	0.3306	37	0.3326
1000	329	0.3280	329	0.3280	332	0.3280
∞	0.328N	0.3277	0.328N	0.3277	0.328N	0.3277

Remark. When $q \to o$, (2.4.18) and (2.4.20) agree with those of Smith [20], and when $p \to 1$, (2.4.18) and (2.4.20) reduce to Theorem 5 of Yang [28] and, in this case, n_2^* is the largest t which satisfies $\{1/(1-q)(t-1)\} + \sum_{\ell=t}^{N-1} 1/\ell > t$ and $V = (1-q^{n_2^*})(1- \sum_{\ell=n_2^*}^{N-1} 1/\ell) / N(1-q) + (n_2^* \sum_{\ell=n_2^*}^{N-1} 1/\ell) / N$.
Since, as $N \to \infty$, n_2^* is large and the relative position $m_{n_2^*}$ of the candidate is also large with high probability.
Hence, it does not matter whether we make backward solicitation or not. Therefore the asymptotic results for $0 \leqq q < 1$ become $n_2^* \simeq Np^{1/1-p}$ and $V \simeq p^{1/1-p}$, which are independent of q. When $q=1$, it is trivial that $n_2^* = N$ and $V = p$.

3. A Secretary Problem with Full-information

3.1 Recognizing the Best with Two Choices

Let X_i, i=1, 2, ..., N, be the value attached to the i-th object and suppose that $X_1, X_2, \ldots, X_N$ be independent and identically distributed random variables with a common distribution function F(x). F(x) is assumed to be known to the decision-maker and is continuous and strictly increasing on the set where $0<F(x)<1$. After a random variable has been observed, the decision-maker must decide to accept it or reject it and take another observation. The decision-maker is allowed to make two choices, and if he has not accepted until the last two observations, then he is forced to choose both of them. If either of his choices is the largest of the sequentially presented random variables, he succeeds. The problem in this section consists of finding a policy which maximizes the probability of success. Since the distribution function F(x) is continuous and strictly increasing, and since the largest measurement in a sample remains the largest under all monotonic transformations of its variable, we can assume without loss of generality that F(x) is a uniform distribution on the interval (0, 1). Following Sakaguchi [16], we employ dynamic programming approach.

Let $p_n^{(r)}(y)$, r= 1,2, $1\leq n\leq N-1$, be the probability of success under an optimal policy, given that y was the largest value observed so far and was rejected, and that the decision-maker is still allowed to make r choices from the remaining n observations. Let also $q_n^{(1)}(y)$ be the probability of success under an optimal policy, given that y was the largest value observed so far and was already chosen, and that the decision-maker is still allowed to make one choice from the remaining n observations. We have, from the principle of optimality

$$(3.1.1)\quad p_n^{(1)}(y) = yp_{n-1}^{(1)}(y) + \int_y^1 \max\{x^{n-1}, p_{n-1}^{(1)}(x)\}\, dx$$

$$(1 \le n \le N-1,\ p_0^{(1)}(y) \equiv 0),$$

$$(3.1.2)\quad q_n^{(1)}(y) = yq_{n-1}^{(1)}(y) + \int_y^1 \max\{x^{n-1}, p_{n-1}^{(1)}(x)\}\, dx$$

$$(1 \le n \le N-1,\ q_0^{(1)}(y) \equiv 1),$$

$$(3.1.3)\quad p_n^{(2)}(y) = yp_{n-1}^{(2)}(y) + \int_y^1 \max\{q_{n-1}^{(1)}(x), p_{n-1}^{(2)}(x)\}\, dx$$

$$(1 \le n \le N-1,\ p_0^{(2)}(y) \equiv 0).$$

It is easily seen, by (3.1.1) and (3.1.2)

$$(3.1.4)\quad q_n^{(1)}(y) \equiv y^n + p_n^{(1)}(y).$$

(3.1.1) is the fundamental equation for one choice problem and is completely solved in Sakaguchi [16] . We now review some remarkable results in [16] . If we define the functions, for $n \ge 1$, over $[0,1]$

$$(3.1.5)\quad \phi_n(y) = \frac{1}{n} + \frac{1}{n-1}y + \frac{1}{n-2}y^2 + \dots + y^{n-1} - \sum_{k=1}^{n}(k^{-1})\, y^n,$$

then the equation

$$(3.1.6) \qquad \phi_n(y) = y^n \quad (n \geq 1)$$

has a unique root s_n in the interval $(s_{n-1}, 1)$ where we interprete s_0 as 0. Thus the sequence $\{s_n\}$ is strictly increasing. It is also proved that, for $y \geq s_{n-1}$

$$(3.1.7) \qquad p_n^{(1)}(y) = \phi_n(y) .$$

Inspection of (3.1.1) shows that the optimal strategy in the state described by this equation is to

$$(3.1.8) \qquad \left\{ \begin{matrix} \text{reject} \\ \\ \text{accept} \end{matrix} \right\} \text{ the observation, if } x \left\{ \begin{matrix} \leq \\ > \end{matrix} \right\} \max (y, s_{n-1}) .$$

Hence (3.1.1) can be rewritten as

$$(3.1.9) \; p_n^{(1)}(y) - y p_{n-1}^{(1)}(y) = \begin{cases} \displaystyle\int_y^{s_{n-1}} p_{n-1}^{(1)}(x)dx + \frac{1}{n} (1 - s_{n-1}^n) & \text{if } y \leq s_{n-1} \\ \frac{1}{n} (1 - y^n) & \text{if } y > s_{n-1} . \end{cases}$$

Let $(a \vee b)$ be the larger of the numbers a and b. Since the sequence $\{ (s_n \vee y) \}$ is non-decreasing in n for each y, we can give the explicit expression of $p_n^{(1)}(y)$ by the argument proposed by Gilbert and Mosteller [3] . Let, for $n \geq 1$

$$(3.1.10) \qquad s_j(n) = \begin{cases} s_j & \text{if} \quad j=0, 1\ 2, \ldots, n-1 \\ 1 & \text{if} \quad j=n \end{cases} .$$

Then we have the following lemma. We interprete $\sum_{a}^{b} = 0$ if $b<a$ in the remaining part of this section.

Lemma 1.1. For $s_j(n) \leq y \leq s_{j+1}(n)$

$$(3.1.11) \quad p_n^{(1)}(y) = \frac{1}{n}\left[1-(j+1)y^n + \sum_{i=1}^{j} y^{n-i}\phi_i(y) + \sum_{i=1}^{j}\sum_{r=1}^{i} \frac{1}{n-r} y^{n-r} \right.$$

$$\left. + \sum_{i=j+1}^{n-1}\sum_{r=1}^{i} \frac{1}{n-r} s_i^{n-r} \right] \qquad (0 \leq j \leq n-1, \quad n \geq 1) .$$

Proof: By [3; Section 3c] and (2.5), it follows that for $0 \leq y \leq 1$

$$p_n^{(1)}(y) = \frac{1}{n}\{ 1-(s_{n-1}\vee y)^n \} + \sum_{r=1}^{n-1}\left[\frac{1}{n-r} \sum_{i=1}^{r} \left\{ \frac{1}{r}(s_{n-i}\vee y)^r \right.\right.$$

$$\left.\left. - \frac{1}{n}(s_{n-i}\vee y)^n \right\} - \frac{1}{n}(s_{n-r-1}\vee y)^n \right]$$

$$= \frac{1}{n}\left[1-(s_{n-1}\vee y)^n + \sum_{i=1}^{n-1}\left\{ \sum_{r=1}^{i}\left(\frac{1}{n-r}+\frac{1}{r}\right)(s_i\vee y)^{n-r} \right.\right.$$

$$\left.\left. -(s_i\vee y)^n \sum_{k=1}^{i} k^{-1} - (s_{i-1}\vee y)^n \right\}\right]$$

$$= \frac{1}{n}\left[1-y^n + \sum_{i=1}^{n-1}\left\{ (s_i\vee y)^{n-i}\phi_i(s_i\vee y) - (s_i\vee y)^n \right.\right.$$

$$\left.\left. + \sum_{r=1}^{i} \frac{1}{n-r}(s_i\vee y)^{n-r} \right\}\right] .$$

Hence, by (3.1.6), we soon obtain (3.1.11).

It is easily checked that, for $s_{n-1} \leq y \leq 1$, (3.1.11) coincides with (3.1.7). In fact, by (3.1.11) and (3.1.5), we have

$$p_n^{(1)}(y) = \frac{1}{n} - y^n + \frac{1}{n} \sum_{i=1}^{n-1} y^{n-i} \{ \sum_{r=1}^{i} \frac{1}{r} y^{i-r} - \sum_{r=1}^{i} \frac{1}{r} y^i \}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \sum_{r=1}^{i} \frac{1}{n-r} y^{n-r}$$

$$= \frac{1}{n} - y^n + \sum_{i=1}^{n-1} \sum_{r=1}^{i} \frac{1}{r(n-r)} y^{n-r} - \frac{1}{n} \sum_{i=1}^{n-1} \sum_{r=1}^{i} \frac{1}{r} y^n$$

$$= \frac{1}{n} - y^n + \sum_{r=1}^{n-1} \frac{1}{r} y^{n-r} - \frac{1}{n} \sum_{r=1}^{n-1} \frac{n-r}{r} y^n$$

$$= \sum_{r=1}^{n} \frac{1}{r} y^{n-r} - \sum_{r=1}^{n} \frac{1}{r} y^n$$

$$= \phi_n(y) .$$

Although it is obvious by its definition that $q_n^{(1)}(y)$ is increasing in y, we can prove this by using (3.1.11).

Lemma 1.2. For $n \geq 1$,

$q_n^{(1)}(y)$ is increasing in y.

Proof: By (3.1.4), (3.1.5) and (3.1.11), we have for $s_j(n) \leq y \leq s_{j+1}(n)$

$$q_n^{(1)}(y) = \frac{1}{n}[1 + (n-1-j)y^n + \sum_{i=1}^{j} y^{n-i}\{\sum_{r=1}^{i} \frac{1}{r} y^{i-r} - \sum_{r=1}^{i} \frac{1}{r} y^i\}$$

$$+ \sum_{i=1}^{j} \sum_{r=1}^{i} \frac{1}{n-r} y^{n-r} + \sum_{i=j+1}^{n-1} \sum_{r=1}^{i} \frac{1}{n-r} s_i^{n-r}] .$$

Differentiating both sides, we get

$$q_n^{(1)'}(y) = \frac{1}{n} [n(n-1-j)y^{n-1} + n \sum_{i=1}^{j} \{\sum_{r=1}^{i} \frac{1}{r} y^{n-r-1} - y^{n-1} \sum_{r=1}^{i} \frac{1}{r}\}]$$

$$= (n-1-j)y^{n-1} + \sum_{i=1}^{j} y^{n-1-i}\{\sum_{r=1}^{i} \frac{1}{r} y^{i-r} - y^i \sum_{r=1}^{i} \frac{1}{r}\}$$

$$= (n-1-j)y^{n-1} + \sum_{i=1}^{j} y^{n-1-i} \phi_i(y) ,$$

and hence, for $0<y<1$, $q_n^{(1)'}(y) > 0$.

Since $p_n^{(2)}(x)$ is non-increasing in x by its definition, and $q_n^{(1)}(x)$ is increasing in x by Lemma 1.2, the equation

(3.1.12) $$p_n^{(2)}(x) = q_n^{(1)}(x) \qquad (n\geq 1)$$

has a unique root which we denote by d_n, where we interprete d_0 as 0. Then (3.1.3) shows that, when the decision-maker is allowed to make two choices, the optimal strategy in the state described by (3.1.3) is to

$$(3.1.13)\quad \left\{\begin{matrix}\text{reject}\\ \text{accept}\end{matrix}\right\} \text{ the observation, if } x\left\{\begin{matrix}\leq\\ >\end{matrix}\right\}\max(y, d_{n-1}) .$$

As for the first observation, interprete the above strategy as y=0. Now (3.1.3), with (3.1.4), can be rewritten as

$$(3.1.14)\quad p_n^{(2)}(y)-yp_{n-1}^{(2)}(y)=\begin{cases}\int_y^{d_{n-1}} p_{n-1}^{(2)}(x)dx + \int_{d_{n-1}}^{1} p_{n-1}^{(1)}(x)dx + \frac{1}{n}(1-d_{n-1}^n) & \text{if } y\leq d_{n-1}\\ \int_y^1 p_{n-1}^{(1)}(x)dx + \frac{1}{n}(1-y^n) & \text{if } y>d_{n-1} .\end{cases}$$

Since we have obtained $p_n^{(1)}(x)$ explicitly, we can solve $p_n^{(2)}(y)$ recursively by (3.1.12) and (3.1.14). Differentiating (3.1.14) for $y\leq d_{n-1}$, we have

$$(3.1.15)\qquad p_n^{(2)\prime}(y) = yp_{n-1}^{(2)\prime}(y) ,$$

with the boundary condition

$$(3.1.16)\quad p_n^{(2)}(d_{n-1}) = \frac{1}{n} + (1-\frac{1}{n})d_{n-1}^n + d_{n-1}p_{n-1}^{(1)}(d_{n-1}) + \int_y^1 p_{n-1}^{(1)}(x)dx .$$

Lemma 1.3. $d_1 < d_2 < \dots < d_{n-1} < d_n < \dots$.

Proof: In order to prove $d_{n-1} < d_n$, it is sufficient to show

$$(3.1.17)\qquad p_n^{(2)}(d_{n-1}) > q_n^{(1)}(d_{n-1}) .$$

We obtain, by (3.1.4) and (3.1.16)

$$p_n^{(2)}(d_{n-1}) - q_n^{(1)}(d_{n-1}) = \frac{1}{n}(1-d_{n-1}^n) + \int_{d_{n-1}}^{1} p_{n-1}^{(1)}(x)dx$$

(3.1.18)

$$- \{ p_n^{(1)}(d_{n-1}) - d_{n-1}p_{n-1}^{(1)}(d_{n-1}) \} .$$

Well, by (3.1.9), the right hand side of (3.1.18) becomes

$$(3.1.19)\quad \frac{1}{n}(1-d_{n-1}^n) + \int_{d_{n-1}}^{1} p_{n-1}^{(1)}(x)dx - \begin{cases} \int_{d_{n-1}}^{s_{n-1}} p_{n-1}^{(1)}(x)dx + \frac{1}{n}(1-s_{n-1}^n) & \text{if } d_{n-1} \leq s_{n-1} \\ \frac{1}{n}(1-d_{n-1}^n) & \text{if } d_{n-1} > s_{n-1} \end{cases}$$

$$= \begin{cases} \frac{1}{n}(s_{n-1}^n - d_{n-1}^n) + \int_{s_{n-1}}^{1} p_{n-1}^{(1)}(x)dx & \text{if } d_{n-1} \leq s_{n-1} \\ \int_{d_{n-1}}^{1} p_{n-1}^{(1)}(x)dx & \text{if } d_{n-1} > s_{n-1} . \end{cases}$$

In either case, (3.1.19) is positive. Hence, the lemma is proved.

Lemma 1.4. For $y \geqq d_{n-1}$,

$$(3.1.20)\qquad p_n^{(2)}(y) = \phi_n(y) + \sum_{i=1}^{n-1} y^{n-1-i} \int_{y}^{1} p_i^{(1)}(x)dx \quad (n \geqq 1) .$$

Proof: We show this by induction. Since, by (3.1.3), $p_1^{(2)}(y)=1-y$, the lemma is true for $n=1$. For $n \geqq 2$, if $y \geqq d_n$, then $y > d_{n-1}$ by Lemma 1.3.

Therefore we have, by induction hypothesis and (3.1.14),

$$p_{n+1}^{(2)}(y) = yp_n^{(2)}(y) + \int_y^1 p_n^{(1)}(x)dx + \frac{1}{n+1}(1-y^{n+1})$$

$$= y\{\phi_n(y) + \sum_{i=1}^{n-1} y^{n-1-i}\int_y^1 p_i^{(1)}(x)dx\} + \int_y^1 p_n^{(1)}(x)dx$$

$$+ \frac{1}{n+1}(1-y^{n+1})$$

$$= \phi_{n+1}(y) + \sum_{i=1}^{n} y^{n-i} \int_y^1 p_i^{(1)}(x)dx \quad ,$$

where we used the relation $\phi_{n+1}(y)=y\phi_n(y) + (1-y^{n+1})/n+1$.

By Lemma 1.3, 1.4 and (3.1.12), we can successively define the value d_n as the unique root in the interval of $(d_{n-1}, 1)$ of the equation

(3.1.21) $$\phi_n(y) + \sum_{i=1}^{n-1} y^{n-1-i}\int_y^1 p_i^{(1)}(x)dx = y^n + p_n^{(1)}(y) .$$

Theorem 1.5. The optimal stopping policy of our problem is given by (3.1.8) and (3.1.13).

The following lemma shows a relation between the sequences $\{s_n\}$ and $\{d_n\}$.

Lemma 1.6. For $n \geq 1$, $s_n > d_n$.

Proof: We prove this lemma by induction. Since it is easily shown that $s_1=1/2$ and $d_1=0$, the lemma is true for $n=1$. Assume that the lemma is true for $n=k-1$. Since $p_k^{(2)}(y)$ is non-increasing and $q_k^{(1)}(y)$ is strictly increasing, it suffices to show

$$(3.1.22) \qquad q_k^{(1)}(s_k) > p_k^{(2)}(s_k)$$

in order to prove $s_k > d_k$. We observe by (3.1.4), (3.1.6) and (3.1.7)

$$(3.1.23) \qquad q_k^{(1)}(s_k) - p_k^{(2)}(s_k) = 2s_k^k - p_k^{(2)}(s_k).$$

The sequence $\{s_n\}$ is strictly increasing, so $s_k>d_{k-1}$ by induction hypothesis. Hence, with Lemma 1.4, (3.1.23) can be rewritten as

$$(3.1.24) \quad 2s_k^k - \{\phi_k(s_k)+\sum_{i=1}^{k-1} s_k^{k-1-i}\int_{s_k}^{1} p_i^{(1)}(x)dx\} = s_k^k - \sum_{i=1}^{k-1} s_k^{k-1-i}\int_{s_k}^{1} p_i^{(1)}(x)dx.$$

Since $p_i^{(1)}(x)<x^i$, for all $i=1,2, \ldots, k-1$, and $x>s_k$,

$$\sum_{i=1}^{k-1} s_k^{k-1-i}\int_{s_k}^{1} p_i^{(1)}(x)dx < \sum_{i=1}^{k-1} s_k^{k-1-i}\int_{s_k}^{1} x^i dx = \phi_k(s_k) - s_k^{k-1} + s_k^k$$

$$= 2s_k^k - s_k^{k-1} .$$

Applying this inequality to (3.1.24), we have

$$q_k^{(1)}(s_k) - p_k^{(2)}(s_k) > s_k^{k-1}(1-s_k) > 0$$

which is the desired result.

We finally calculate the probability of success, which we denote by P_n. As easily seen by Lemma 1.3, 1.4 and (3.1.13), P_n can be written as

$$P_n \equiv \int_0^1 \max\{ p_{n-1}^{(2)}(x),\ q_{n-1}^{(1)}(x) \} dx$$

$$= \int_0^{d_{n-1}} p_{n-1}^{(2)}(x)dx + \int_{d_{n-1}}^1 \{ x^{n-1} + p_{n-1}^{(1)}(x) \} dx$$

(3.1.25)
$$= \int_0^{d_{n-2}} p_{n-1}^{(2)}(x)dx + \int_{d_{n-2}}^{d_{n-1}} \{ \phi_{n-1}(x) + \sum_{i=1}^{n-2} x^{n-2-i} \int_x^1 p_i^{(1)}(y)dy \} dx$$

$$+ \int_{d_{n-1}}^1 p_{n-1}^{(1)}(x)dx + \frac{1}{n}(1-d_{n-1}^n) .$$

To calculate the first term of (3.1.25), we prepare the following lemma which is similar to that obtained in Sakaguchi [16; Section 3].

Lemma 1.7. For any positive integers m and k

(3.1.26)
$$\int_0^{d_{m-1}} p_m^{(2)}(x)dx^k = \int_0^{d_{m-2}} p_{m-1}^{(2)}(x)dx^{k+1} + \int_{d_{m-2}}^{d_{m-1}} \{ \phi_{m-1}(x) + \sum_{i=1}^{m-2} x^{m-2-i} \cdot \int_x^1 p_i^{(1)}(y)dy \} dx^{k+1} + \frac{1}{m} d_{m-1}^k (1-d_{m-1}^m) + d_{m-1}^k \int_{d_{m-1}}^1 p_{m-1}^{(1)}(x)dx .$$

Proof: (3.1.4), (3.1.12), (3.1.15), (3.1.16), Lemma 1.4 and integration by parts give

$$\int_0^{d_{m-1}} p_m^{(2)}(x)dx^k = [p_m^{(2)}(x)x^k]_0^{d_{m-1}} - \int_0^{d_{m-1}} p_m^{\prime(2)}(x)x^k dx$$

$$= p_m^{(2)}(d_{m-1})d_{m-1}^k - \int_0^{d_{m-1}} p_{m-1}^{\prime(2)}(x)x^{k+1}dx$$

$$= p_m^{(2)}(d_{m-1})d_{m-1}^k - [p_{m-1}^{(2)}(x)x^{k+1}]_0^{d_{m-1}} + \int_0^{d_{m-1}} p_{m-1}^{(2)}(x)dx^{k+1}$$

$$= \{\frac{1}{m} + (1-\frac{1}{m})d_{m-1}^m + d_{m-1}p_{m-1}^{(1)}(d_{m-1}) + \int_{d_{m-1}}^1 p_{m-1}^{(1)}(x)dx\}d_{m-1}^k$$

$$- \{d_{m-1}^{m-1} + p_{m-1}^{(1)}(d_{m-1})\}d_{m-1}^{k+1} + \int_0^{d_{m-2}} p_{m-1}^{(2)}(x)dx^{k+1} + \int_{d_{m-2}}^{d_{m-1}} p_{m-1}^{(2)}(x)dx^{k+1}$$

$$= \frac{1}{m}d_{m-1}^k(1-d_{m-1}^m) + d_{m-1}^k\int_{d_{m-1}}^1 p_{m-1}^{(1)}(x)dx + \int_0^{d_{m-2}} p_{m-1}^{(2)}(x)dx^{k+1}$$

$$+ \int_{d_{m-2}}^{d_{m-1}} \{\phi_{m-1}(x) + \sum_{i=1}^{m-2} x^{m-2-i}\int_x^1 p_i^{(1)}(y)dy\}dx^{k+1} .$$

Thus the lemma is proved.

Theorem 1.8. For $n \geq 3$,

$$P_n = \sum_{i=2}^{n-1} [(n-i)\{\int_{d_{i-1}}^{d_i} x^{n-1-i}\phi_i(x)dx + \sum_{j=1}^{i-1}\int_{d_{i-1}}^{d_i} x^{n-2-j}dx\int_x^1 p_j^{(1)}(y)dy\}$$

(3.1.27)

$$+ \frac{1}{i+1}(d_i^{n-1-i} - d_i^n) + d_i^{n-1-i}\int_{d_i}^1 p_i^{(1)}(x)dx] .$$

Proof: Applying (3.1.26) repeatedly, we have

$$\int_0^{d_{n-2}} p_{n-1}^{(2)}(x)dx = \int_0^{d_{n-3}} p_{n-2}^{(2)}(x)dx^2 + \int_{d_{n-3}}^{d_{n-2}} \{\phi_{n-2}(x) + \sum_{i=1}^{n-3} x^{n-3-i}\int_x^1 p_i^{(1)}(y)dy\}dx^2$$

$$+ \frac{1}{n-1}(d_{n-2}-d_{n-2}^n) + d_{n-2}\int_{d_{n-2}}^1 p_{n-2}^{(1)}(x)dx$$

$$= \int_0^{d_{n-4}} p_{n-3}^{(2)}(x)dx^3 + \int_{d_{n-4}}^{d_{n-3}} \{\phi_{n-3}(x) + \sum_{i=1}^{n-4} x^{n-4-i}\int_x^1 p_i^{(1)}(y)dy\}dx^3$$

$$+ \int_{d_{n-3}}^{d_{n-2}} \{\phi_{n-2}(x) + \sum_{i=1}^{n-3} x^{n-3-i}\int_x^1 p_i^{(1)}(y)dy\}dx^2 + \frac{1}{n-2}(d_{n-3}^2-d_{n-3}^n)$$

$$+ \frac{1}{n-1}(d_{n-2}-d_{n-2}^n) + d_{n-3}^2\int_{d_{n-3}}^1 p_{n-3}^{(1)}(x)dx + d_{n-2}\int_{d_{n-2}}^1 p_{n-2}^{(1)}(x)dx$$

(3.1.28)

$$= \cdots\cdot$$

$$= \sum_{i=2}^{n-2} [\int_{d_{i-1}}^{d_i} \{\phi_i(x) + \sum_{j=1}^{i-1} x^{i-1-j}\int_x^1 p_j^{(1)}(y)dy\}dx^{n-i} + \frac{1}{i+1}(d_i^{n-1-i}-d_i^n)$$

$$+ d_i^{n-1-i}\int_{d_i}^1 p_i^{(1)}(x)dx\,] \;.$$

Note that

$$\int_{d_{i-1}}^{d_i}\{\phi_i(x)+\sum_{j=1}^{i-1}x^{i-1-j}\int_x^1 p_j^{(1)}(y)dy\}dx^{n-i}=(n-i)\{\int_{d_{i-1}}^{d_i}x^{n-1-i}\phi_i(x)dx$$

$$+\sum_{j=1}^{i-1}\int_{d_{i-1}}^{d_i}x^{n-2-j}dx\int_x^1 p_j^{(1)}(y)dy\}\ .$$

Substituting this into (3.1.28) and combining the result with (3.1.25), we have the desired expression (3.1.27).

Table 6 gives the values of s_n, d_n and P_n for n=1(1)30.

Table 6

n	s_n	d_n	P_n
1	0.5000	0.0000	1.0000
2	0.6899	0.4083	1.0000
3	0.7759	0.5676	0.9430
4	0.8246	0.6605	0.9170
5	0.8560	0.7202	0.9021
6	0.8778	0.7623	0.8923
7	0.8939	0.7933	0.8854
8	0.9063	0.8172	0.8803
9	0.9160	0.8361	0.8763
10	0.9240	0.8515	0.8731
11	0.9305	0.8642	0.8705
12	0.9361	0.8749	0.8683
13	0.9408	0.8841	0.8665
14	0.9448	0.8920	0.8649
15	0.9484	0.8989	0.8636
16	0.9515	0.9050	0.8624
17	0.9542	0.9104	0.8614
18	0.9567	0.9152	0.8604
19	0.9589	0.9195	0.8596
20	0.9609	0.9234	0.8589
21	0.9627	0.9269	0.8582
22	0.9644	0.9302	0.8576
23	0.9659	0.9331	0.8570
24	0.9673	0.9358	0.8565
25	0.9686	0.9383	0.8560
26	0.9697	0.9406	0.8556
27	0.9708	0.9428	0.8552
28	0.9719	0.9448	0.8548
29	0.9728	0.9466	0.8545
30	0.9737	0.9484	0.8541

The values of s_n are reproduced from Table 7 of Gilbert and Mosteller (1966).

Remark. Our problem can be generalized to the case with r choices. The decision-maker is allowed to make r choices and succeeds if the either of his choices is the largest of the sequentially presented random variables. Let $p_n^{(m)}(y)$, $1 \leq m \leq r$, $1 \leq n$, be the probability of success under an optimal policy, given that y was the largest value observed so far and was rejected, and that the decision-maker is still allowed to make m choices from the remaining n observations. Let also $q_n^{(m)}(y)$, $1 \leq m \leq r-1$, $1 \leq n$, be the probability of success under an optimal policy, given that y was the largest observed so far and was already chosen, and that the decision-maker is still allowed to make m choices from the remaining n observations. Then we have

$$p_n^{(m)}(y) = y p_{n-1}^{(m)}(y) + \int_y^1 \max\{q_{n-1}^{(m-1)}(x),\ p_{n-1}^{(m)}(x)\}dx \qquad (1 \leq m \leq r,\ 1 \leq n)$$

$$\left(\begin{array}{l} p_n^{(0)}(y) \equiv 0, \text{ for all } y \text{ and } 0 \leq n \\ p_0^{(m)}(y) \equiv 0, \text{ for all } y \text{ and } 0 \leq m \leq r \end{array} \right),$$

$$q_n^{(m)}(y) = y q_{n-1}^{(m)}(y) + \int_y^1 \max\{q_{n-1}^{(m-1)}(x),\ p_{n-1}^{(m)}(x)\}dx \qquad (1 \leq m \leq r-1,\ 1 \leq n)$$

$$\left(\begin{array}{l} q_n^{(0)}(y) \equiv y^n, \text{ for all } y \text{ and } 0 \leq n \\ q_0^{(m)}(y) \equiv 1, \text{ for all } y \text{ and } 0 \leq m \leq r-1 \end{array} \right).$$

We immediately have, using above equations

$$q_n^{(m)}(y) = y^n + p_n^{(m)}(y) \qquad (0 \leq m \leq r-1,\ 0 \leq n) \quad .$$

Since $q_n^{(m-1)}(x)$ is non-decreasing in x and $p_n^{(m)}(x)$ is non-increasing in x, by their definitions, the equation

$$p_n^{(m)}(x) = q_n^{(m-1)}(x)$$

has at least one root. Let $s_n^{(m)}$ be the smallest root of this equation (of course, $s_n^{(1)} = s_n$, $s_n^{(2)} = d_n$). Then we can easily prove, in a similar way as used in Lemma 1.3, that the sequence $\{ s_n^{(m)} \}$ is increasing in n for any given m. But it seems to be difficult for the author to prove the conjecture that the sequence $\{ s_n^{(m)} \}$ is decreasing in m for each n.

Acknowledgments

The author wishes to thank Professor M. Sakaguchi for several ideas he greatly contributed to the study of the optimal stopping problems.

The author wishes to thank Professor S. Takamatsu for his continuing guidance and warm encouragement.

The author is thankful to Professors H. Nakamichi, S. Watanabe, K. Nakayama and Lecturer T. Ohta for their frequent and helpful discussions and also to T. Ohta, K. Kikuta and S. Tamai for their computer work in preparing tables.

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