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CHARACTERIZATIONS OF p-NILPOTENT GROUPS

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Introduction

Let G be a finite group and p a prime. For a p-block B of G, let $\mathrm{Irr}^0(B)$ be the set of irreducible characters of height 0 in B. Most results in this paper are related with the characters of height 0 in the principal p-block $B_0(G)$. In section 1 we shall show that G is p-nilpotent if and only if every $\chi \in \mathrm{Irr}^0(B_0(G))$ is modularly irreducible (Theorem 1.3). This result is in a sense analogous to a theorem of Okuyama and Tsushima [8]. We shall give also a characterization of p-nilpotent groups via weights [1]. In section 2 several normal subgroups associated to $\mathrm{Ker}\,\chi$, $\chi \in \mathrm{Irr}^0(B)$, are shown to be p-nilpotent. Also p-nilpotent groups are characterized via their character values (Corollary 2.10). In section 3 a question arising from a paper of Ono [9] will be discussed. Throughout this paper (K, R, k) denotes a p-modular system. We assume that K contains the $|G|$-th roots of unity. The maximal ideal of R is denoted by (π).

1. Characterizations of p-nilpotent groups

Let

$$\Lambda(G) = \{\chi; \chi \in \mathrm{Irr}^0(B_0(G)), \ o(\det \chi) \not\equiv 0 \ (\mathrm{mod} \ p)\} \ ,$$

where $o(\det \chi)$ denotes the determinantal order of χ. For an irreducible Brauer character ϕ of G and a subset Λ of $\Lambda(G)$, let $\delta(\Lambda, \phi) = \Sigma\, d(\chi, \phi)\, \chi(1)$, where $d(\chi, \phi)$ is the decomposition number and the sum is taken over all $\chi \in \Lambda$. For brevity, put $\delta(G, \phi) = \delta(\Lambda(G), \phi)$.

The following lemma will be used frequently in the sequel.

Lemma 1.1. *If* $\delta(G, \phi) \not\equiv 0 \ (\mathrm{mod} \ p)$ *for some irreducible Brauer character* ϕ *in* $B_0(G)$ *with* $\phi(1) \not\equiv 0 \ (\mathrm{mod} \ p)$, *then* G *is* p-*nilpotent.*

Proof. Put $N = O^p(G)$. Since $\phi(1)$ is prime to p, $\psi := \phi_N$ is irreducible. The same is true for $\chi \in \Lambda(G)$, and the restriction gives a bijection from $\Lambda(G)$ onto the subset Ξ of G-invariant members of $\Lambda(N)$, cf. Corollary 6.28 in Isaacs [5]. From this it follows that $\delta(G, \phi) = \delta(\Xi, \psi)$. Now let Ψ be the character

of the projective cover of the module affording ψ. Since Ψ and ψ are G-invariant, it follows that $\Psi(1) \equiv \delta(\Xi, \psi) \pmod{p}$ (consider the natural action of G on Irr (N)). Hence we get $\Psi(1) \not\equiv 0 \pmod{p}$, which shows that N is a p'-group. This completes the proof.

Remark 1.2. Although the above lemma was inspired by the proof of Theorem 12.1 in Isaacs [5], it turned out that a similar idea had appeared, cf. the proof of Theorem 2 in Pahlings [10].

Theorem 1.3. *The following conditions are equivalent.*
(i)   *G is p-nilpotent.*
(ii) *$l(B_0(G)) = 1$.*
(iii) *Every irreducible character of height 0 in $B_0(G)$ is linear.*
(iv) *Every irreducible character of height 0 in $B_0(G)$ is modularly irreducible.*
(v)   *$\Lambda(G) = \{1_G\}$.*

Proof. (i)$\Rightarrow$(ii):   This is obvious

(ii)$\Rightarrow$(iii):   Let $\chi \in \text{Irr}(B_0(G))$ and assume that $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$ (Burnside). If s is the p'-part of g, then $\chi(g) \equiv \chi(s) \pmod{\pi}$, and $\chi(s) = \chi(1)$ by (ii). Hence $\chi(1) \equiv 0 \pmod{p}$, completing the proof.

(iii)$\Rightarrow$(iv):   Obvious.

(iv)$\Rightarrow$(i):   Let $\chi \in \Lambda(G)$. If $d(\chi, 1_G) \neq 0$, (iv) implies that when considered as a Brauer character, χ is the trivial Brauer character. In particular $\chi(1) = 1$ and then $o(\det \chi) \not\equiv 0 \pmod{p}$ implies $\chi = 1_G$. So we get $\delta(G, 1_G) = 1$ and G is p-nilpotent by Lemma 1.1.

(i)$\Rightarrow$(v):   This is obvious.

(v)$\Rightarrow$(i):   (v) implies $\delta(G, 1_G) = 1$, so G is p-nilpotent as above.

Remark. 1.4. The condition (ii) is due to R. Brauer and the condition (iii), which strengthens Thompson's condition [12], is due to Isaacs and Smith [6]. See also Pahlings [10]. In [6] the implication (iii)$\Rightarrow$(i) is proved through a characterization of groups of p-length 1. For a generalization of their characterization, cf. [7]. The condition (iv) may be considered as a special case of nonabelian version of a theorem of Okuyama and Tsushima [8].

We give still another characterization of p-nilpotent groups, which is related to the notion of weight introduced by Alperin [1].

Theorem 1.5. *The following conditions are equivalent.*
(i)   *G is p-nilpotent.*
(ii) *$N_G(P)$ is p-nilpotent for a p-Sylow subgroup P of G, and there is no weight (Q, S) for G with $Q < P$ and $S \in B_0(N_G(Q))$.*

Proof. (i)$\Rightarrow$(ii):   For any p-subgroup Q, $N_G(Q)$ is p-nilpotent. If there

exists a simple $kN_G(Q)$-module S with vertex Q lying in $B_0(N_G(Q))$, then S must be the trivial module. So Q is a p-Sylow subgroup of $N_G(Q)$ and hence of G.

(ii)$\Rightarrow$(i): Assume false and let G be a counterexample of minimal order. Take a p-subgroup $Q \neq 1$ which is maximal under the condition that $N_G(Q)$ is not p-nilpotent. (Recall that G is p-nilpotent if $N_G(Q)$ is p-nilpotent for all p-subgroups $Q \neq 1$ of G.) Put $H = N_G(Q)$. By the choice of Q, $Q = O_p(H)$. We claim that H satisfies the same assumption as G. To see this let R be a p-Sylow subgroup of H. Then $R > Q$, so $N_H(R) \leq N_G(R)$ is p-nilpotent. Next let (R, S) be a weight for H with S in $B_0(N_H(R))$. If $R \leq Q$, $N_Q(R) \leq O_p(N_H(R)) = R$, cf. [1]. Hence $R = Q$, which contradicts the assumption (ii). Thus $RQ \neq Q$, so $N_H(R) \leq N_G(QR)$ is p-nilpotent and R is a p-Sylow subgroup of H, cf. the proof of (i)$\Rightarrow$(ii). Thus the claim is proved. By the choice of G, we get that $G = H$. It is not difficult to see that $G/O_{p'}(G)$ and G/Q satisfy the same assumption as G. Hence $O_{p'}(G) = 1$ and G/Q is p-nilpotent. In particular, G is p-solvable and $C_G(Q) \leq Q$. Let $R/Q \neq 1$ be any p-subgroup of G/Q and K the normal p-complement of $N_G(R)$, then $[K, Q] = 1$ and $K \leq C_G(Q) \leq Q$, so $K = 1$. Hence $N_{G/Q}(R/Q)$ is a p-group. This shows that G/Q is a Frobenius group whose Frobenius complement is a p-Sylow subgroup. So G/Q has a simple kG/Q-module of p-defect 0. Since G has a unique block, this contradicts the assumption (ii). This completes the proof.

2. Block kernels

Throughout this section P is a p-Sylow subgroup of the group G.

Lemma 2.1. *Let N be a normal subgroup of G and B a block of G covering $B_0(N)$.*

(i) *Assume the following:*

(*) *there exists $\zeta \in \mathrm{Irr}^0(B)$ with $N \leq \mathrm{Ker}\, \zeta$.*

Let ξ be a P-invariant member of $\Lambda(N)$. Then for some $\chi \in \mathrm{Irr}^0(B)$, we have $(\chi, \xi)_N \neq 0$.

(ii) *Assume that for any P-invariant member $\xi \neq 1_N$ of $\Lambda(N)$, $d(\xi, 1_N) = 0$. Then N is p-nilpotent.*

Proof. (i) There exists an extension $\overset{\wedge}{\xi}$ of ξ to PN, as before. With ζ as in (*), let θ be the class function on G defined by

$$\theta(g) = \begin{cases} p^d\, \zeta(g) & \text{if } g \text{ is } p\text{-regular,} \\ 0 & \text{otherwise,} \end{cases}$$

where d is the defect of B. We have $(\overset{\wedge}{\xi}{}^G, \theta)_G = (\overset{\wedge}{\xi}, \theta)_{PN} = p^d\, \zeta(1)\, |PN|^{-1}\, a$, where a denotes $\Sigma\, \xi(y)$ (y runs through $N_{p'}$). As is well-known ([2]) $a \not\equiv 0$ (mod π), so $(\overset{\wedge}{\xi}{}^G, \theta)_G \not\equiv 0$ (mod π). Hence $(\overset{\wedge}{\xi}{}^G, \chi)_G \neq 0$ for some $\chi \in \mathrm{Irr}^0(B)$ ([2]).

By Frobenius reciprocity, ξ appears as an irreducible constituent of χ_N. This completes the proof.

(ii) As in the proof of Lemma 1.1, we have that $\delta(N, 1_N) \equiv \delta(\Xi, 1_N)$ (mod p), where Ξ is the set of P-invariant members of $\Lambda(N)$. By assumption, $\delta(\Xi, 1_N) = 1$, so the result follows from Lemma 1.1.

REMARK 2.2. The condition (*) is always satisfied and the assertion (i) itself could be extended, cf. Corollary 4.6 and Theorem 4.4 in [7].

For an arbitrary block B of G, we let $\mathrm{Ker}^0(B) = \cap \mathrm{Ker}\, \chi$, where χ runs through $\mathrm{Irr}^0(B)$.

Theorem 2.3. $\mathrm{Ker}^0(B)$ *is p-nilpotent.*

Proof. Put $N = \mathrm{Ker}^0(B)$. Let P be as above and ξ a P-invariant member of $\Lambda(N)$ and choose $\chi \in \mathrm{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$ (Lemma 2.1 (i)). Since $N \leq \mathrm{Ker}\, \chi$, $\xi = 1_N$. So N is p-nilpotent by Lemma 2.1 (ii).

Let $\mathfrak{N}(G)$ be the set of normal subgroups N of G such that for any $\chi \in \mathrm{Irr}^0(B_0(G))$, χ_N is a sum of linear characters of N. The following theorem gives a characterization of $O_{p',p}(G)$ via (ordinary) irreducible characters. We remark that $O_{p',p}(G)$ has been characterized by R. Brauer via irreducible modular representations.

Theorem 2.4. $O_{p',p}(G)$ *is the unique maximal member of $\mathfrak{N}(G)$.*

Proof. Let $N \in \mathfrak{N}(G)$. Let ξ be a P-invariant member of $\Lambda(N)$. Choose $\chi \in \mathrm{Irr}^0(B_0(G))$ with $(\chi, \xi)_N \neq 0$. (The condition (*) in Lemma 2,1 (i) is satisfied with $\zeta = 1_G$.) By definition of $\mathfrak{N}(G)$, ξ must be linear, and then $o(\det \xi) \not\equiv 0$ (mod p) implies that $d(\xi, 1_N) = 0$ unless $\xi = 1_N$. So N is p-nilpotent by Lemma 2.1 (ii), and $N \subseteq O_{p',p}(G)$. Conversely, let ξ be an irreducible constituent of χ_N, where $N = O_{p',p}(G)$ and $\chi \in \mathrm{Irr}^0(B_0(G))$. Then ξ lies in $B_0(N)$ and $\xi(1)$ is prime to p, so $\xi(1) = 1$, since N is p-nilpotent. This completes the proof.

REMARK 2.5. The implication (iii)$\Rightarrow$(i) in Theorem 1.3 follows also from the above theorem.

We can restate Theorem 2.4 as follows:

Corollary 2.6. $O_{p',p}(G)/\mathrm{Ker}^0(B_0(G))$ *is the unique maximal normal abelian subgroup of $G/\mathrm{Ker}^0(B_0(G))$.*

For the principal block, let

$$Z^0(G) = \{g \in G; |\chi(g)| = \chi(1) \quad \text{for any} \quad \chi \in \mathrm{Irr}^0(B_0(G))\}\,,$$

where $|\cdot|$ denotes the absolute value. Then $Z^0(G) \in \mathfrak{N}(G)$, so we get:

Corollary 2.7. $Z^0(G)$ *is p-nilpotent.*

REMARK 2.8. This corollary could be used in the proof of Z^*-Theorem, cf. Step VI of the proof of Theorem 1 in Glauberman [3].

Theorem 2.9. $O_{p'}(G/Z^0(G))=1$.

Proof. Put $Z=Z^0(G)$. Let N be the inverse image in G of $O_{p'}(G/Z)$. We claim that N is p-nilpotent. Assume this, then N/Z is a p-group, since $O_{p'}(N)=O_{p'}(G)\leq Z$. Hence $N/Z=1$, as required. To prove the claim, let ξ be any P-invariant member of $\Lambda(N)$ and choose $\mathcal{X}\in\mathrm{Irr}^0(B_0(G))$ such that $(\mathcal{X},\xi)_N\neq 0$ as above. By definition of Z, $\mathcal{X}_Z$ is a multiple of a linear character. So $\xi_Z=e\,\eta$, where $e=\xi(1)$ and η is a linear character of Z. Since $\mathcal{X}$ is trivial on $O_{p'}(G)=O_{p'}(N)$, so is ξ. Hence $(\det\xi)_Z$ (which equals η^e) and η are inflated from $Z/O_{p'}(N)$. Since this group is a p-group by Corollary 2.7 and $o(\det\xi)$ is prime to p, it follows that $(\det\xi)_Z=1_Z$. Then $\eta=1_Z$, since e is prime to p. So ξ is inflated from N/Z. Hence $d(\xi,1_N)=0$ unless $\xi=1_N$, since N/Z is a p'-group. This implies that N is p-nilpotent as before.

Now we give a characterization of p-nilpotent groups via their character values, from which the implication (iii)$\Rightarrow$(i) in Thoerem 1.3 follows again.

Corollary 2.10. *The following conditions are equivalent.*
(i) *G is p-nilpotent.*
(ii) *$|\mathcal{X}(u)|=\mathcal{X}(1)$ for all p-elements u of G and all $\mathcal{X}\in\mathrm{Irr}^0(B_0(G))$.*

Proof. (i)$\Rightarrow$(ii): This is obvious.
(ii)$\Rightarrow$(i): (ii) implies that $G/Z^0(G)$ is a p'-group, so $G=Z^0(G)$ by Theorem 2.9. Then G is p-nilpotent by Corollary 2.7.

3. Conjugacy classes of Ono type

For any irreducible character $\mathcal{X}$ of the group G, we let, as usual, $\omega_{\mathcal{X}}$ be the central character associated to $\mathcal{X}$.

DEFINITION 3.1. Let α be an element of the center of ZG, where Z is the ring of integers. α is said to be of *Ono type* if for any $\mathcal{X}\in\mathrm{Irr}(G)$ there holds either $\omega_{\mathcal{X}}(\alpha)=0$ or $|\omega_{\mathcal{X}}(\alpha)|=\varepsilon(\alpha)$, where $\varepsilon\colon ZG\to Z$ is the augmentation map. A conjugacy class K is said to be of Ono type if the class sum $\hat{K}$ is of Ono type. (If $g\in K$, the condition is the same as saying that either $\mathcal{X}(g)=0$ or $|\mathcal{X}(g)|=\mathcal{X}(1)$ for all $\mathcal{X}\in\mathrm{Irr}(G)$.) A group G is said to be of Ono type if every conjugacy class of G is of Ono type.

Groups of Ono type has appeared in Ono [9]. First we prove:

Psoposition 3.2. *Groups of Ono type are nilpotent.*

Proof.   Let G be a group of Ono type and p any prime.   For any p-element g of G, either $\mathcal{X}(g)=0$ or $|\mathcal{X}(g)|=\mathcal{X}(1)$ holds, for any $\mathcal{X}\in\mathrm{Irr}^0(B_0(G))$.   Since $\mathcal{X}(g)\equiv\mathcal{X}(1)\not\equiv 0$ (mod π), the latter holds.   So G is p-nilpotent by Corollary 2.10.   Since p is arbitrary, G is nilpotent.

REMARK 3.3.   The above proposition could be proved by induction on the group order (without using block theory).

One may well conjecture that the subgroup generated by a conjugacy class of Ono type is solvable, as will be explained below.

For a subset H of G, let $\hat{H}=\Sigma_{h\in H} h$.   For an element $\alpha=\Sigma_g \alpha_g g$ of $\mathbf{Z}G$, put $\mathrm{Supp}\,\alpha=\{g\in G;\ \alpha_g\neq 0\}$.

Lemma 3.4.   *For an element $\alpha(\neq 0)$ of the center of $\mathbf{Z}G$ with $\alpha_g>0$ for all $g\in\mathrm{Supp}\,\alpha$, the following conditions are equivalent.*
(i)   *α is of Ono type.*
(ii)   *$\alpha=mg\hat{H}$, for a positive integer m and a subgroup H of G.*
(iii)   *$\alpha=mg\hat{H}$, for a positive integer m and a normal subgroup H of G.*

Proof.   (i)$\Rightarrow$(ii):   This is proved by induction on $|G|$.   First assume that there is $\mathcal{X}\in\mathrm{Irr}(G)$ such that $|\omega_{\mathcal{X}}(\alpha)|=\mathcal{E}(\alpha)$ and that $\mathcal{X}(1)>1$.   Then $\mathrm{Supp}\,\alpha\subseteq Z(\mathcal{X})\neq G$, where $Z(\mathcal{X})=\{g\in G;\ |\mathcal{X}(g)|=\mathcal{X}(1)\}$.   For any $\zeta\in\mathrm{Irr}(Z(\mathcal{X}))$, take $\mathcal{X}\in\mathrm{Irr}(G)$ such that $(\mathcal{X},\zeta)_N\neq 0$, then $\omega_{\mathcal{X}}(\alpha)=\omega_{\zeta}(\alpha)$.   So we get the conclusion by the induction hypothesis applied to $Z(\mathcal{X})$.   So we may assume that $\omega_{\mathcal{X}}(\alpha)=0$ for any $\mathcal{X}\in\mathrm{Irr}(G)$ with $\mathcal{X}(1)>1$.   Then $\alpha=\Sigma\,\omega_{\lambda}(\alpha)\,e_{\lambda}$, where the sum is taken over the linear characters λ of G and e_{λ} is the central idempotent associated to λ.   Replacing α by $g^{-1}\alpha$, $g\in\mathrm{Supp}\,\alpha$, if necessary, we may further assume $1\in\mathrm{Supp}\,\alpha$.   Assume that for some λ, $|\omega_{\lambda}(\alpha)|=\mathcal{E}(\alpha)$.   Then for any $g\in\mathrm{Supp}\,\alpha$, $\lambda(g)=\lambda(1)$, so $g\in\mathrm{Ker}\,\lambda$, and if $\mathrm{Ker}\,\lambda\neq G$, we get the conclusion by induction as above.   So we may assume that $\omega_{\lambda}(\alpha)=0$ for $\lambda\neq 1_G$.   This implies α is a multiple of e_{1_G}, so (ii) holds.

(ii)$\Rightarrow$(iii):   Let $\alpha=mg\hat{H}$ as above.   For any $x\in G$, $m\,g^x\,\hat{H}^x=\alpha^x=\alpha=mg\hat{H}$.   So $g^x\in gH$ and $g^x\,\hat{H}=g\hat{H}=g^x\,\hat{H}^x$.   Hence $\hat{H}^x=\hat{H}$, so H is normal.

(iii)$\Rightarrow$(i):   Let $\mathcal{X}\in\mathrm{Irr}(G)$.   If $\mathrm{Ker}\,\mathcal{X}\geqq H$, $|\omega_{\mathcal{X}}(\alpha)|=\mathcal{E}(\alpha)$, because gH is central in G/H.   Otherwise, $\omega_{\mathcal{X}}(\alpha)=0$, as is well-known.   This completes the proof.

From this lemma we get:

Corollary 3.5.   *A conjugacy class K of G is of Ono type if and only if $K=gH$ for some $g\in G$ and a (normal) subgroup H of G.*

Lemma 3.6.   *Let K be a conjugacy class of Ono type consisting of p-elements for some prime p.   Then the subgroup generated by K is p-nilpotent.*

Proof. Let $g \in K$. As in the proof of Proposition 3.2, we get that $g \in Z^0(G)$, and the conclusion follows from Corollary 2.7.

Now we have:

Theorem 3.7. *The following assertions are equivalent.*

(i) *Any conjugacy class of Ono type consisting of elements of prime power order generates a solvable subgroup.*

(ii) *Let G be a semi-direct product of groups A and N with N normal. If $C_N(A) = 1$ and A is cyclic of prime power order, then G is solvable.*

Proof. (i)$\Rightarrow$(ii): Let g be a generator of A and K the conjugacy class containing g of G. Obviously $K \subseteq gN$ and we have $|K| = |N|$, since $C_N(g) = 1$. So $K = gN$, and K is of Ono type by Corollary 3.5. The subgroup generated by K is G, so G is solvable. (ii)$\Rightarrow$(i): Let K be a conjugacy class of a group G consisting of p-elements for a prime p. The proof is done by induction on $|G|$. Since K is of Ono type, $K = gH$ for a normal subgroup H of G and $g \in K$. We see that $\langle K \rangle = \langle g \rangle H$ and that $H = \{g^{-1}g^x; x \in G\}$. Let N be the normal p-complement of H, cf. Lemma 3.6. We may assume that $N \neq 1$. Let C be the inverse image in G of $C_{G/N}(gN)$. We claim that the conjugacy class K' containing g of C is of Ono type. Since $N \leq H$, it follows that $N = \{g^{-1}g^x; x \in C\}$. Then $K' = gN$ and the claim follows from Corollary 3.5. If $C \neq G$, $\langle K' \rangle$ and hence N is solvable by induction. Since the image of K in G/N is of Ono type, the image of $\langle K \rangle$ in G/N is solvable by induction. So $\langle K \rangle$ is solvable. Now assume $C = G$. Then $N = H$ by the above. On the other hand, we must have $G = C_G(g)N$, since N is a p'-group. This implies $C_G(g) \cap N = 1$, since $|N| = |K|$. Taking $A = \langle g \rangle$ in (ii), we get that G is solvable.

The assertion (ii) is a longstanding conjecture (see for example [4], p 487)

From the above (proof) and a theorem of Thompson [11], we get:

Corollary 3.8. *Let K be a conjugacy class of Ono type consisting of elements of prime order. Then K generates a solvable subgroup.*

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