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Studies on examples of one-parameter families of Calabi-Yau threefolds: Generic Torelli theorem and global monodromy

1-パラメータの3次元カラビ-ヤウ多様体の族の例の研究: 弱大域トレリの定理と大域モノドロミー

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Contents

Introduction

In this thesis, the author studies examples of one-parameter families of Calabi-Yau threefolds and provides two results. The first result is a generic Torelli theorem for one-parameter mirror families to weighted hypersurfaces. The second result is a presentation of the global monodromy group of quintic-mirror family in the general linear group of degree 4 over the ring of integers modulo 5.

According to the above two results, this thesis consists of two parts.

In the first part, we will focus generic Torelli problem for four specific examples of one-parameter families of Calabi-Yau threefolds, which are listed in [M1]. For a given smooth projective family, we have the period map by associating with each parameter the Hodge filtration on the third cohomology group of the fiber over it. Global Torelli problem asks the injectivity of the period map, and generic Torelli problem, which is a weak version of global Torelli problem, asks the injectivity of the restriction of the period map to a Zariski open set.

One of the families in the list of [M1] is the quintic mirror family. For this family, Usui proves the generic Torelli theorem in [U2]. He gives this result as an application of the logarithmic Hodge theory of Kato and Usui. By using their theory in [KU], the period map $\mathbb{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D$ arising from the quintic-mirror family extends to a morphism $\varphi : \mathbb{P}^1 \to \Gamma\backslash D_\Xi$ of logarithmic ringed spaces. Here $\Gamma\backslash D_\Xi$ is a logarithmic Hodge partial compactification of $\Gamma\backslash D$, and is endowed with a geometric structure of a logarithmic manifold, which is a nearly logarithmic analytic space. In general, it is known that under certain conditions the images of extended period maps are analytic spaces by [U1]. In the present case, $\varphi(\mathbb{P}^1)$ is an analytic curve. For this extended period map, the following theorem is proved.

Theorem 0.1 ([U2])**.** *The extended period map $\varphi : \mathbb{P}^1 \to \Gamma\backslash D_\Xi$ of the quintic-mirror family is the normalization of analytic spaces over its image.*

There are two proofs of this theorem as follows:

- $\varphi^{-1}(\varphi(1)) = \{1\}$ and the ramification index at 1 of φ is 1.
- $\varphi^{-1}(\varphi(\infty)) = \{\infty\}$ and the ramification index at ∞ of φ is 1.

The other three families in the list of [M1] are also one-parameter families, whose parameter spaces are $\mathbb{P}^1$, and the geometric situations of these families are similar to that of the quintic-mirror family. In the main theorem of the first part, we prove the same theorem for these families after the proof of Usui.

Theorem 0.2 ([Sh1])**.** *For the above three families, which are listed in* [M1], *the generic Torelli theorem holds.*

In the second part, we will be concerned with a description of the global monodromy of the quintic-mirror family. The restriction $f : (W_\lambda)_{\lambda \in U} \to U$ of the quintic-mirror family to $U := \mathbb{P}^1 - \{0, 1, \infty\}$ is a smooth projective family of Calabi-Yau threefolds. Fix a base point $b \in U$. Then the representation $\pi_1(U, b) \to \mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ arise from the local system $R^3 f_* \mathbb{Z}$, whose fiber $H^3(W_b, \mathbb{Z})$ over b is endowed with cup product $\langle\ ,\ \rangle$. The global monodromy Γ of the quintic-mirror family is the image of this representation. By taking a symplectic basis, $\mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ is identified with $\mathrm{Sp}(4, \mathbb{Z})$.

Matrix presentations of the generators of Γ are well studied and it is also known that Γ is Zariski dense in $\mathrm{Sp}(4, \mathbb{Z})$ (e.g. [COGP], [D]). However, it is not known whether the index of Γ in $\mathrm{Sp}(4, \mathbb{Z})$ is finite or not (e.g. [CYY]). A direct approach for this problem is to describe Γ explicitly. In the main theorem of the second part, we give a presentation of Γ in $\mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$, which is a small attempt toward a description of Γ.

Theorem 0.3 ([Sh2])**.** *Let* $\rho : \mathrm{GL}(4, \mathbb{Z}) \to \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$ *be the natural projection. There exists a subgroup* Γ' *of* $\mathrm{GL}(4, \mathbb{Z})$ *such that* $\Gamma' \simeq \Gamma$ *as a group and*

$$\rho(\Gamma') = \left\{ \begin{pmatrix} 1 & n & 3n^2 + 2n & a \\ 0 & 1 & n & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z}) \;\middle|\; n, a, b, c \in \mathbb{Z}/5\mathbb{Z} \right\}.$$

On the other hand, Chen, Yang and Yui find a congruence subgroup $\Gamma(5, 5)$ of $\mathrm{Sp}(4, \mathbb{Z})$ of finite index, which contains Γ.

Theorem 0.4 ([CYY])**.** *Let*

$$\Gamma(5, 5) := \left\{ X \in \mathrm{Sp}(4, \mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \mod 5 \right\}.$$

Then, $\Gamma(5, 5)$ *is a congruence subgroup of* $\mathrm{Sp}(4, \mathbb{Z})$ *of finite index, which contains* Γ.

Combining their result and the main theorem of the second part, we can construct a smaller congruence subgroup $\tilde{\Gamma}(5,5)$ of $\mathrm{Sp}(4,\mathbb{Z})$ of finite index, which contains Γ. The difference between $\Gamma(5,5)$ and $\tilde{\Gamma}(5,5)$ is that although $\Gamma(5,5)$ contains the principal congruence group $\Gamma(5) := \mathrm{Ker}(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/5\mathbb{Z}))$, $\tilde{\Gamma}(5,5)$ do not. $\tilde{\Gamma}(5,5)$ contains the principal congruence group $\Gamma(25)$ instead of $\Gamma(5)$. However, this result is merely the fact that $\tilde{\Gamma}(5,5)$ contains Γ. Although $\tilde{\Gamma}(5,5)$ is of finite index in $\mathrm{Sp}(4,\mathbb{Z})$, Γ itself may not be so. After all, the index of Γ in $\mathrm{Sp}(4,\mathbb{Z})$ is still unknown.

Part I

Generic Torelli theorem for one-parameter mirror families to weighted hypersurfaces

1 Fundamentals of Hodge theory

In this section, we recall basic facts of Hodge theory after [G], [KU].

Let $w \in \mathbb{Z}$, and let $(h^{p,q})_{p,q\in\mathbb{Z}}$ be a family of non-negative integers such that $h^{p,q} = 0$ unless $p + q = w$, $h^{p,q} \neq 0$ for only finitely many (p, q), and such that $h^{p,q} = h^{q,p}$ for all p, q.

Definition 1.1. A *Hodge structure* of weight w and of Hodge type $(h^{p,q})_{p,q\in\mathbb{Z}}$ is a pair $(H_\mathbb{Z}, F)$ consisting of a free $\mathbb{Z}$-module $H_\mathbb{Z}$ of finite rank and of a decreasing filtration F on $H_\mathbb{C} := \mathbb{C} \otimes_\mathbb{Z} H_\mathbb{Z}$, which satisfies the following conditions.

$$\dim_\mathbb{C}(F^p/F^{p+1}) = h^{p,q} \quad (p + q = w). \tag{1}$$

$$H_\mathbb{C} = \bigoplus_{p+q=w} (F^p \cap \bar{F}^q). \tag{2}$$

Definition 1.2. A *polarized Hodge structure* of weight w and of Hodge type $(h^{p,q})_{p,q\in\mathbb{Z}}$ is a triple $(H_\mathbb{Z}, \langle\ ,\ \rangle, F)$ consisting of a Hodge structure $(H_\mathbb{Z}, F)$ of weight w and of a non-degenerate $\mathbb{Q}$-bilinear form $\langle\ ,\ \rangle$ on $H_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} H_\mathbb{Z}$, symmetric for even w and skew-symmetric for odd w, which satisfies the following two conditions.

$$\langle F^p, F^q\rangle = 0 \quad (p + q > w). \tag{3}$$

(4) The Hermitian form

$$H_\mathbb{C} \times H_\mathbb{C} \to \mathbb{C}, \quad (x, y) \mapsto \langle C_F(x), \bar{y}\rangle,$$

is positive definite.

Here $\langle\ ,\ \rangle$ is regarded as the natural extension to $\mathbb{C}$-bilinear form, $^-$ is the complex conjugation with respect to $H_{\mathbb{Z}}$, and C_F is the Weil operator which is a $\mathbb{C}$-linear map and defined by $C_F(x) := i^{p-q}x$ for $x \in F^p \cap \bar{F}^q$ with $p+q=w$. The condition (3) (resp. (4)) is called the *Riemann-Hodge first* (resp. *second*) *bilinear relation*.

Let w and $(h^{p,q})_{p,q\in\mathbb{Z}}$ be as before. We fix a 4-tuple $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbb{Z}}, H_0, \langle\ ,\ \rangle_0)$, where H_0 is a free $\mathbb{Z}$-module of rank $\Sigma_{p,q}h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a non-degenerate bilinear form on $H_{0,\mathbb{Q}} := \mathbb{Q}\otimes_{\mathbb{Z}} H_0$, which is symmetric if w is even and skew-symmetric if w is odd.

Definition 1.3. The classifying space D of type Φ_0 is the set of all decreasing filtrations F on $H_{0,\mathbb{C}} := \mathbb{C}\otimes_{\mathbb{Z}} H_0$ such that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})_{p,q\in\mathbb{Z}}$. The compact dual $\check{D}$ of D is defined to be the set of all decreasing filtrations on $H_{\mathbb{C}}$ which satisfies the above conditions (1) and (3).

Example 1.1. We consider the case that $w=3$, $h^{3,0}=h^{2,1}=h^{1,2}=h^{0,3}=1$ and $h^{p,q}=0$ otherwise. Let H_0 be a free $\mathbb{Z}$-module with basis $(e_i)_{1\le j\le 4}$ and define a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle_0 : H_0\times H_0 \to \mathbb{Z}$ by

$$J := \big(\langle e_i, e_j\rangle_0\big)_{1\le i,j\le 4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Define the decreasing filtration $F_0 = \{F_0^p\}_{p\in\mathbb{Z}}$ on $H_{0,\mathbb{C}}$ by

$F_0^4 = \{0\}$,
$F_0^3 =$ (the $\mathbb{C}$-subspace of $H_{0,\mathbb{C}}$ spanned by $-ie_2+e_4$),
$F_0^2 =$ (the $\mathbb{C}$-subspace of $H_{0,\mathbb{C}}$ spanned by ie_1+e_3 and $-ie_2+e_4$),
$F_0^1 = (F_0^3)^{\perp}$,
$F_0^0 = H_{0,\mathbb{C}}$.

Then we have $F_0 \in D \subset \check{D}$.

Now, $\mathrm{Aut}(H_{0,\mathbb{C}}, \langle\ ,\ \rangle_0) = \mathrm{Sp}(4,\mathbb{C})$ acts freely on $\check{D}$, and $\mathrm{Aut}(H_{0,\mathbb{R}}, \langle\ ,\ \rangle_0) = \mathrm{Sp}(4,\mathbb{R})$ acts freely on D. So, if we take

$$P = \{g \in \mathrm{Sp}(4,\mathbb{C}) \mid gF_0 = F_0\},\ B = P\cap \mathrm{Sp}(4,\mathbb{R}),$$

then we have the following expressions of D and $\check{D}$ as homogeneous spaces:

$$\begin{aligned} \mathrm{Sp}(4,\mathbb{C})/P &\simeq \check{D}\ ,\ gP \mapsto gF_0\ , \\ \mathrm{Sp}(4,\mathbb{R})/B &\simeq D\ ,\ gB \mapsto gF_0\ . \end{aligned}$$

From a simple calculation, we have

$$B = \left\{ \left. \begin{pmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & \sin\theta_2 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \right| \theta_1, \theta_2 \in \mathbb{R} \right\} \simeq U(1) \times U(1).$$

Their Lie algebras are described as follows:

$$\begin{aligned} &\mathfrak{g}_{\mathbb{C}} := \mathrm{Lie}(\mathrm{Sp}(4,\mathbb{C})) = \{X \in \mathrm{M}(4,\mathbb{C}) \mid \mathrm{Trace}\, X = 0,\ {}^tXJ + JX = 0\}, \\ &\mathfrak{g}_{\mathbb{R}} := \mathrm{Lie}(\mathrm{Sp}(4,\mathbb{R})) = \{X \in \mathrm{M}(4,\mathbb{R}) \mid \mathrm{Trace}\, X = 0,\ {}^tXJ + JX = 0\}, \\ &\mathfrak{p} := \mathrm{Lie}(P) = \left\{ \left. \begin{pmatrix} a & b & c & ib \\ d & e & ib & f \\ -2ia - c & id & -a & -d \\ id & 2ie - f & -b & -e \end{pmatrix} \right| a,b,c,d,e,f \in \mathbb{C} \right\}, \\ &\mathfrak{b} := \mathrm{Lie}(B) = \left\{ \left. \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{pmatrix} \right| a,b \in \mathbb{R} \right\}, \\ &\mathrm{Lie}(\mathrm{Sp}(4,\mathbb{C})/P) = \mathfrak{g}_{\mathbb{C}}/\mathfrak{p},\ \mathrm{Lie}(\mathrm{Sp}(4,\mathbb{R})/P) = \mathfrak{g}_{\mathbb{R}}/\mathfrak{b}. \end{aligned}$$

The dimension of them are listed as follows:

$$\begin{aligned} &\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = 10,\ \dim_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} = 10,\ \dim_{\mathbb{C}} \mathfrak{p} = 6,\ \dim_{\mathbb{R}} \mathfrak{b} = 2, \\ &\dim_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}) = 4,\ \dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}/\mathfrak{b}) = 8. \end{aligned}$$

Here the natural homomorphism $\mathfrak{g}_{\mathbb{R}}/\mathfrak{b} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ is injective. Since $\dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p})$ $= \dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{R}}/\mathfrak{b})$, $\mathfrak{g}_{\mathbb{R}}/\mathfrak{b} \to \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ is bijective. Thus, it follows from the above descriptions that $\check{D}$ is a complex manifold, whose dimension over $\mathbb{C}$ is 4, and D is an open submanifold of $\check{D}$.

On the other hand, by the Riemann-Hodge first bilinear relation, $\check{D}$ is a closed subspace of the flag manifold

$$\left\{ F : \text{decreasing filtration on } H_{0,\mathbb{C}} \ \middle|\ \begin{array}{ll} \dim_{\mathbb{C}}(F^p/F^{p+1}) = 1 & \text{if } 0 \le p \le 3, \\ \dim_{\mathbb{C}}(F^p/F^{p+1}) = 0 & \text{otherwise} \end{array} \right\},$$

which is a compact complex manifold. Therefore, $\check{D}$ is also a compact complex manifold.

Definition 1.4. Let X be a complex manifold. A *variation of Hodge structure* on X of weight w is a pair $(H_{\mathbb{Z}}, F)$ consisting of a locally constant sheaf $H_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of finite rank on X and of a decreasing filtration F of $H_{\mathcal{O}} := \mathcal{O}_X \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ by $\mathcal{O}_X$-submodules which satisfy the following three conditions.

(1) $F^p = H_{\mathcal{O}}$ for $p \ll 0$, $F^p = 0$ for $p \gg 0$, and F^p/F^{p+1} is a locally free $\mathcal{O}_X$-module for any p.

(2) For any $x \in X$, the fiber $(H_{\mathbb{Z},x}, F(x))$ is a Hodge structure of weight w.

(3) $(d \otimes 1_{H_{\mathbb{Z}}})(F^p) \subset \Omega^1_X \otimes_{\mathcal{O}_X} F^{p-1}$ for all p.

A polarization of variation Hodge structure $(H_{\mathbb{Z}}, F)$ of weight w on X is a bilinear form $\langle\ ,\ \rangle : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}$, which yields for each $x \in X$ a polarization $\langle\ ,\ \rangle_x$ on the fibre $(H_{\mathbb{Z},x}, F(x))$. In this case, the triple $(H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$ is called a *variation of polarized Hodge structure*.

Let $f : \mathcal{X} \to S$ be a proper smooth family with projective or Kähler fibers of dimension n. For each $s \in S$, we use the notation $X_s := f^{-1}(s)$. The direct image $H_{\mathbb{Z}} := R^n f_* \mathbb{Z}$ is a locally constant sheaf of free $\mathbb{Z}$-modules of finite rank on S. For each $s \in S$, if we take a sufficient small open neighborhood U of s, an isomorphism $H_{\mathbb{Z}}|_U \simeq H^n(X_s, \mathbb{Z})$ is given by a $\mathcal{C}^\infty$ trivialization $f^{-1}(U) \simeq X_s \times U$. On the $H^n(X_s, \mathbb{C})$, we have the Hodge decomposition

$$H^n(X_s, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X_s)$$

with $H^{p,q}(X_s) \simeq H^q(X_s, \Omega^p_{X_s})$. Let

$$F^p H^n(X_s, \mathbb{C}) := \bigoplus_{p' \geq p} H^{p', n-p'}(X_s),$$

and let $H_{\mathcal{O}} := \mathcal{O}_S \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$. We then have the following theorem.

Theorem 1.1 ([G])**.** *Assigning $F^p H^n(X_s) \subset H^n(X_s, \mathbb{C}) = H_{\mathcal{O}}(s)$ for all s in S defines a holomorphic subbundle $F^p H_{\mathcal{O}} \subset H_{\mathcal{O}}$.*

In addition, we have the following theorem.

Theorem 1.2 ([G])**.** *The Gauss-Manin connection* $\nabla := d \otimes 1_{H_{\mathbb{Z}}}$ *satisfies the following transversality condition:*

$$\nabla F^p H_{\mathcal{O}} \subset \Omega^1_S \otimes_{\mathcal{O}} F^{p-1} H_{\mathcal{O}}.$$

Thus the above pair $(H_{\mathbb{Z}}, F)$ becomes a variation of Hodge structure on S of weight n. In the case that the fibers of f are Calabi-Yau threefolds, i.e.

X_s : compact Kähler threefold, $K_{X_s} = \mathcal{O}_{X_s},\ h^{2,0}(X_s) = h^{1,0}(X_s) = 0\ (s \in S)$,

a polarization $\langle\ ,\ \rangle$ of $(H_{\mathbb{Z}}, F)$ is defined by the cup product

$$\langle\ ,\ \rangle_s : H^3(X_s, \mathbb{Z}) \times H^3(X_s, \mathbb{Z}) \to \mathbb{Z},\ (\xi, \eta) \mapsto \int_{X_s} \xi \wedge \eta \ \ (s \in S),$$

where ξ, η are regarded as elements of the image of

$$H^3(X_s, \mathbb{Z}) \to H^3(X_s, \mathbb{C}) \simeq H^3_{\mathrm{DR}}(X_s) \otimes_{\mathbb{R}} \mathbb{C}.$$

Hence $(H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$ becomes a variation of polarized Hodge structure.

Finally, we recall that a holomorphic map arises from a variation of polarized Hodge structure.

Let $(H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$ be a variation of polarized Hodge structure on S of weight w. Fix a base point $0 \in S$. Then the representation $\pi_1(S, 0) \to \mathrm{Aut}(H_{\mathbb{Z},0}, \langle\ ,\ \rangle_0)$ arise from the locally constant sheaf $H_{\mathbb{Z}}$. The image of this representation is called the *global monodromy*, and we denote it by Γ. Let $(h^{p,q})_{p,q \in \mathbb{Z}}$ be the Hodge type of the polarized Hodge structure on $0 \in S$, $\Phi_0 := (w, (h^{p,q})_{p,q \in \mathbb{Z}}, H_{\mathbb{Z},0}, \langle\ ,\ \rangle_0)$, and let D be the classifying space of type Φ_0. Then we can define the holomorphic map

$$\varphi : S \to \Gamma \backslash D,\ s \mapsto F(s) \bmod \Gamma.$$

Here $F(s)$ $(s \in S)$ are regarded as filtrations on $H_{\mathbb{C},0}$ by isomorphisms $H_{\mathbb{C},s} \simeq H_{\mathbb{C},0}$ $(s \in S)$ arising from local trivializations of $H_{\mathbb{Z}}$. The above holomorphic map is called the *period map*.

Global Torelli problem asks injectivity of the period map, and generic Torelli problem asks injectivity of the period map on a Zariski open set in the case that S is an algebraic variety.

2 Our objects

In this section, we recall the construction of our objects, for which we will consider generic Torelli problem, after [M1].

We consider four types one-parameter families of Calabi-Yau hypersurfaces in complex weighted projective four-space. For $\psi \in \mathbb{P}^1$, they are

$$Q^1_\psi = \left\{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1x_2x_3x_4x_5 = 0\right\} \subset \mathbb{P}^{(1,1,1,1,1)},$$

$$Q^2_\psi = \left\{2x_1^3 + x_2^6 + x_3^6 + x_4^6 + x_5^6 - 6\psi x_1x_2x_3x_4x_5 = 0\right\} \subset \mathbb{P}^{(2,1,1,1,1)},$$

$$Q^3_\psi = \left\{4x_1^2 + x_2^8 + x_3^8 + x_4^8 + x_5^8 - 8\psi x_1x_2x_3x_4x_5 = 0\right\} \subset \mathbb{P}^{(4,1,1,1,1)},$$

$$Q^4_\psi = \left\{5x_1^2 + 2x_2^5 + x_3^{10} + x_4^{10} + x_5^{10} - 10\psi x_1x_2x_3x_4x_5 = 0\right\} \subset \mathbb{P}^{(5,2,1,1,1)}.$$

The weights of the above list are characterized by the following proposition.

Proposition 2.1 ([CLS], [I]). *Let $k_1 \geq \cdots \geq k_5$ are positive integers, $\mathbb{P}^{(k_1,\cdots,k_5)}$ be well formed, i.e.*

$$\gcd(k_1, \cdots, \hat{k_i}, \cdots, k_5) = 1 \textit{ for each } i,$$

and let X be a hypersurface defined by a polynomial of degree $k_1 + \cdots + k_5$. If X is nonsingular, then $(k_1, \cdots, k_5)$ is either of the following types:

$$(1, 1, 1, 1, 1),\ (2, 1, 1, 1, 1),\ (4, 1, 1, 1, 1),\ (5, 2, 1, 1, 1).$$

Hodge numbers of the above hypersurfaces, which are nonsingular, are listed as follows:

$$h^{3,0}(Q^i_\psi) = h^{1,1}(Q^i_\psi) = 1\ (i = 1, 2, 3, 4),\ h^{2,1}(Q^i_\psi) = \begin{cases} 101 & i = 1, \\ 103 & i = 2, \\ 149 & i = 3, \\ 145 & i = 4. \end{cases}$$

We shall construct Calabi-Yau threefolds W^i_ψ, whose Hodge numbers are $h^{1,1}(W^i_\psi) = h^{2,1}(Q^i_\psi)$, $h^{2,1}(W^i_\psi) = h^{1,1}(Q^i_\psi)$, by taking quotients of Q^i_ψ by finite abelian groups and desingularizing the quotients.

Let μ_k be the group of k-th root of $1 \in \mathbb{C}$. Finite abelian groups G^i $(i = 1, 2, 3, 4)$ are defined as follows:

$$
\begin{aligned}
G^1 &= \{(\alpha_1, \cdots, \alpha_5) \in (\mu_5)^5 \mid \alpha_1 \cdots \alpha_5 = 1\}/\mu_5, \\
G^2 &= \{(\alpha_1, \cdots, \alpha_5) \in \mu_3 \times (\mu_6)^4 \mid \alpha_1 \cdots \alpha_5 = 1\}/\mu_6, \\
G^3 &= \{(\alpha_1, \cdots, \alpha_5) \in \mu_2 \times (\mu_8)^4 \mid \alpha_1 \cdots \alpha_5 = 1\}/\mu_8, \\
G^4 &= \{(\alpha_1, \cdots, \alpha_5) \in \mu_2 \times \mu_5 \times (\mu_{10})^3 \mid \alpha_1 \cdots \alpha_5 = 1\}/\mu_{10},
\end{aligned}
$$

where we embed $\mu_5, \mu_6, \mu_8, \mu_{10}$ in $(\mu_5)^5, \mu_3 \times (\mu_6)^4, \mu_2 \times (\mu_8)^4, \mu_2 \times \mu_5 \times (\mu_{10})^3$ respectively by

$$
\alpha \mapsto \begin{cases} (\alpha, \alpha, \alpha, \alpha, \alpha) & \text{if } i = 1, \\ (\alpha^2, \alpha, \alpha, \alpha, \alpha) & \text{if } i = 2, \\ (\alpha^4, \alpha, \alpha, \alpha, \alpha) & \text{if } i = 3, \\ (\alpha^5, \alpha^2, \alpha, \alpha, \alpha) & \text{if } i = 4. \end{cases}
$$

G^i $(i = 1, 2, 3, 4)$ are abstractly isomorphic to $(\mu_5)^3,\ \mu_3 \times (\mu_6)^2,\ (\mu_8)^3,\ (\mu_{10})^2$ as groups.

When we divide these hypersurfaces Q^i_ψ by G^i, quotient singularities appear. For $\psi \in \mathbb{C} \subset \mathbb{P}^1$, it is known that there are simultaneous desingularizations of these singularities, and we have four families $(W^i_\psi)_{\psi \in \mathbb{P}^1}$ $(i = 1, 2, 3, 4)$ of the mirrors to the above hypersurfaces in each case.

Let

$$
\nu_i = \begin{cases} \mu_5 & \text{if } \ i = 1, \\ \mu_6 & \text{if } \ i = 2, \\ \mu_8 & \text{if } \ i = 3, \\ \mu_{10} & \text{if } \ i = 4. \end{cases}
$$

$(W^i_\psi)_{\psi \in \mathbb{P}^1}$ is parametrized by ψ. The singular fibers of $(W^i_\psi)_{\psi \in \mathbb{P}^1}$ are as follows:

- When ψ belongs to $\nu_i \subset \mathbb{C} \subset \mathbb{P}^1$, W^i_ψ has one ordinary double point.
- W^i_∞ is a normal crossing divisor in the total space.

The other fibers of $(W^i_\psi)_{\psi \in \mathbb{P}^1}$ are smooth with Hodge numbers

$$
h^{3,0}(W^i_\psi) = h^{2,1}(W^i_\psi) = 1,\ h^{1,1}(W^i_\psi) = h^{2,1}(Q^i_\psi) \ \ (i = 1, 2, 3, 4).
$$

By the action of

$$
\alpha \in \nu_i,\ (x_1, \cdots, x_5)\ \mapsto (x_1, \cdots, x_4, \alpha^{-1} x_5),
$$

we have the isomorphism from the fiber over ψ to the fiber over $\alpha\psi$. Let λ be

$$\begin{cases} \psi^5 & \text{if } i = 1, \\ \psi^6 & \text{if } i = 2, \\ \psi^8 & \text{if } i = 3, \\ \psi^{10} & \text{if } i = 4, \end{cases}$$

and let

$$\begin{array}{ccc} (W^i_\lambda)_{\lambda \in \mathbb{P}^1} & = & ((W^i_\psi)_{\psi \in \mathbb{P}^1})/\nu_i \\ \downarrow & & \downarrow \\ (\lambda\text{-plane}) & = & (\psi\text{-plane})/\nu_i. \end{array}$$

These families are our objects, for which we will give the proof of a generic Torelli theorem later. $(W^1_\lambda)_{\lambda \in \mathbb{P}^1}$ is the so-called quintic-mirror family. (For more details of the above families, see e.g. [M1], [M2].)

Now, we shall summarize our situation and notation.

Let $(W_\lambda)_{\lambda \in \mathbb{P}^1} \to \mathbb{P}^1$ be one of the families $(W^i_\lambda)_{\lambda \in \mathbb{P}^1} \to \mathbb{P}^1$ $(i = 1, 2, 3, 4)$. The restriction $(W_\lambda)_{\lambda \in \mathbb{P}^1 - \{0,1,\infty\}} \to \mathbb{P}^1 - \{0, 1, \infty\}$ is a smooth projective family of Calabi-Yau threefolds. Therefore, we have the variation of polarized Hodge structure $(H_\mathbb{Z}, \langle\ ,\ \rangle, F)$ on $\mathbb{P}^1 - \{0, 1, \infty\}$ of weight 3 associated to this family as we have seen its construction in §1. From this polarized Hodge structure, we also have the period map $\varphi_0 : \mathbb{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D$. The main theme of the first part is about the injectivity of this period map $\varphi_0 : \mathbb{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D$.

We use the following notation frequently after this section.

Notation 2.1.

$(W_\lambda)_{\lambda \in \mathbb{P}^1} \to \mathbb{P}^1$: one of the families $(W^i_\lambda)_{\lambda \in \mathbb{P}^1} \to \mathbb{P}^1$ $(i = 1, 2, 3, 4)$,

$(H_\mathbb{Z}, \langle\ ,\ \rangle, F)$: the variation of polarized Hodge structure on $\mathbb{P}^1 - \{0, 1, \infty\}$ of weight 3 associated to the family $(W_\lambda)_{\lambda \in \mathbb{P}^1 - \{0,1,\infty\}} \to \mathbb{P}^1 - \{0, 1, \infty\}$,

$b \in \mathbb{P}^1 - \{0, 1, \infty\}$: the base point,

$H_0 := H_{\mathbb{Z},b} = H^3(W_b, \mathbb{Z})$ the free $\mathbb{Z}$-module of rank 4,

$$h^{p,q} := \begin{cases} 1 & p+q=3,\ p,q \geq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{the Hodge type of } (H_{\mathbb{Z},b}, F(b)),$$

$\langle\ ,\ \rangle_0 := \langle\ ,\ \rangle_b$ the non-degenerate skew-symmetric bilinear form on $H^3(W_b, \mathbb{Z})$ defined by the cup product,

$D := \big($ the classifying space of type $(3, (h^{p,q})_{p,q\in\mathbb{Z}}, H_0, \langle\ ,\ \rangle_0)\,\big)$,

$\Gamma := \mathrm{Im}(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0))$ the global monodromy,

$A, T, T_\infty \in \Gamma :$ the local monodromies around $\lambda = 0, 1, \infty$,

$\varphi_0 : \mathbb{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D$ the period map arising from the above $(H_{\mathbb{Z}}, \langle\ ,\ \rangle, F)$.

3 Local monodromies of our objects

In the previous section, we saw the construction of our objects $(W^i_\lambda)_{\lambda\in\mathbb{P}^1}$ ($i = 1,2,3,4$). In this section, we review the local monodromies of our four families, which are important to study behaviors of the period maps on neighborhoods of boundary points. For the quintic-mirror family, Candelas, de la Ossa, Green and Parks gave the matrix presentations of the local monodromies for the symplectic basis in [COGP]. For the other 3 families, Klemm and Theisen gave it in [KT]. We recall their results.

We use Notation 2.1. Then, there exists a symplectic basis e_1, e_2, e_3, e_4 of $H_0 = H^3(W_b, \mathbb{Z})$ and the matrix presentations of the local monodromies A, T, T_∞ around $\lambda = 0, 1, \infty$ for this basis are listed as follows:

In all the cases,

$$[\,T(e_1), T(e_2), T(e_3), T(e_4)\,] = [\,e_1, e_2, e_3, e_4\,]\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

In the case of $i = 1$,

$$[\,A(e_1), A(e_2), A(e_3), A(e_4)\,] = [\,e_1, e_2, e_3, e_4\,]\begin{pmatrix}-9&-3&5&3\\0&1&0&-1\\-20&-5&11&5\\-15&5&8&-4\end{pmatrix},$$

$$[\,T_\infty(e_1), T_\infty(e_2), T_\infty(e_3), T_\infty(e_4)\,] = [\,e_1, e_2, e_3, e_4\,]\begin{pmatrix}11&8&-5&0\\0&1&0&0\\20&15&-9&0\\5&-5&-3&1\end{pmatrix}.$$

In the case of $i = 2$,

$$[\,A(e_1), A(e_2), A(e_3), A(e_4)\,] = [\,e_1, e_2, e_3, e_4\,]\begin{pmatrix}1&-1&0&1\\0&1&0&-1\\-3&-3&1&3\\-6&4&1&-3\end{pmatrix},$$

$$[\,T_\infty(e_1), T_\infty(e_2), T_\infty(e_3), T_\infty(e_4)\,] = [\,e_1, e_2, e_3, e_4\,]\begin{pmatrix}1&1&0&0\\0&1&0&0\\3&6&1&0\\3&-4&-1&1\end{pmatrix}.$$

In the case of $i = 3$,

$$[\,A(e_1), A(e_2), A(e_3), A(e_4)\,] = [\,e_1, e_2, e_3, e_4\,] \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & -2 & 1 & 2 \\ -4 & 4 & 1 & -3 \end{pmatrix},$$

$$[\,T_\infty(e_1), T_\infty(e_2), T_\infty(e_3), T_\infty(e_4)\,] = [\,e_1, e_2, e_3, e_4\,] \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 2 & -4 & -1 & 1 \end{pmatrix}.$$

In the case of $i = 4$,

$$[\,A(e_1), A(e_2), A(e_3), A(e_4)\,] = [\,e_1, e_2, e_3, e_4\,] \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 3 & 1 & -2 \end{pmatrix},$$

$$[\,T_\infty(e_1), T_\infty(e_2), T_\infty(e_3), T_\infty(e_4)\,] = [\,e_1, e_2, e_3, e_4\,] \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}.$$

The above matrix presentations of A and T are the matrices A and T in the lists of [COGP], [KT] respectively, and the 5,6,8,10-th power of the matrix presentation of T_∞ are listed in [COGP], [KT] for each case.

4 Nilpotent orbits and extended classifying spaces as sets

Later, we consider the extended period maps of our objects. So we need to recall ambient spaces of the images of extended period maps. In this section, we recall the definition of nilpotent orbits and extended classifying spaces after [KU].

Let H_0 be a free $\mathbb{Z}$-module of finite rank, $\langle\ ,\ \rangle_0$ be a non-degenerate bilinear form on $\mathbb{Q} \otimes_{\mathbb{Z}} H_0$, which is symmetric or skew-symmetric,

$$G_{\mathbb{Z}} := \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let

$$\begin{aligned} H_{0,R} &:= R \otimes_{\mathbb{Z}} H_0, \\ G_R &:= \mathrm{Aut}(H_{0,R}, \langle\ ,\ \rangle_0), \\ \mathfrak{g}_R &:= \mathrm{Lie}(G_R) \\ &= \{N \in \mathrm{End}_R(H_{0,R}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0 \text{ for all } x, y \in H_{0,R}\}. \end{aligned}$$

Definition 4.1 ([KU, 1.3.1])**.** A subset σ of $\mathfrak{g}_{\mathbb{R}}$ is said to be a *nilpotent cone*, if the following conditions are satisfied.

(1) $\sigma = \mathbb{R}_{\geq 0}N_1 + \cdots + \mathbb{R}_{\geq 0}N_n$ for some $n \geq 1$ and for some $N_1, \cdots, N_n \in \sigma$.
(2) Any element of σ is nilpotent as an endomorphism of $H_{\mathbb{R}}$.
(3) $[N, N'] = 0$ for any $N,\ N' \in \sigma$ as endomorphisms of $H_{\mathbb{R}}$,
where $[N, N'] := NN' - N'N$.

A nilpotent cone is said *rational*, if we can take $N_1, \cdots, N_n \in \mathfrak{g}_{\mathbb{Q}}$ in 4.1 (1).

For a nilpotent cone σ, a *face* of σ is a non-empty subset τ of σ which satisfies the following two conditions.

(1) If $x, y \in \tau$ and $a \in \mathbb{R}_{\geq 0}$, then $x + y,\ ax \in \tau$.
(2) If $x, y \in \sigma$ and $x + y \in \tau$, then $x, y \in \tau$.

Definition 4.2 ([KU, 1.3.3])**.** A *fan* in $\mathfrak{g}_{\mathbb{Q}}$ is a non-empty set Σ of rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ satisfying the following three conditions:

(1) If $\sigma \in \Sigma$, any face of σ belongs to Σ.
(2) If $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of σ and of σ'.
(3) Any $\sigma \in \Sigma$ is sharp. That is, $\sigma \cap (-\sigma) = \{0\}$.

Example 4.1. We use Notation 2.1. Let $N_1 := \log T,\ N_\infty := \log T_\infty \in \mathfrak{g}_\mathbb{Q}$. It follows from the list of §3 that

$$N_1 \neq 0,\ (N_1)^2 = 0\ ,$$
$$(N_\infty)^k \neq 0\ (k = 1, 2, 3),\ (N_\infty)^4 = 0\ .$$

Define

$$\sigma_1 := \mathbb{R}_{\geq 0} N_1,\ \sigma_\infty := \mathbb{R}_{\geq 0} N_\infty,$$
$$\Xi := \{\mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_1, \sigma_\infty,\ g \in \Gamma\},$$

where $\mathrm{Ad}(g)\sigma := g\sigma g^{-1}$. Then Ξ is a fan in $\mathfrak{g}_\mathbb{Q}$.

Let σ be a nilpotent cone in $\mathfrak{g}_\mathbb{R}$. For $R = \mathbb{R}, \mathbb{C}$, we denote by σ_R the R-linear span of $\sigma \subset \mathfrak{g}_\mathbb{R}$.

Definition 4.3 ([KU, 1.3.7])**.** Let $\sigma = \Sigma_{1 \leq j \leq r}(\mathbb{R}_{\geq 0})N_j$ be a rational nilpotent cone. A subset Z of $\check{D}$ is said to be a *σ-nilpotent orbit* if there is $F \in \check{D}$ which satisfies $Z = \exp(\sigma_\mathbb{C})F$ and satisfies the following two conditions.

(1) $N_j F^p \subset F^{p-1}$ $(1 \leq j \leq r,\ p \in \mathbb{Z})$.
(2) $\exp(\sum_{1 \leq j \leq r} z_j N_j)F \in D$ if $z_j \in \mathbb{C}$ and $\mathrm{Im}(z_j) \gg 0$.

The conditions (1) and (2) are called *Griffiths transversality* and *positivity*, respectively.

We say that the pair (σ, F), consisting of a rational nilpotent cone $\sigma \subset \mathfrak{g}_\mathbb{R}$ and of $F \in \check{D}$, generates a *nilpotent orbit* if $Z = \exp(\sigma_\mathbb{C})F$ is a σ-nilpotent orbit.

Definition 4.4 ([KU, 1.3.8])**.** Let Σ be a fan in $\mathfrak{g}_\mathbb{Q}$. As a set, we define D_Σ by

$$D_\Sigma := \{(\sigma, Z) \mid \sigma \in \Sigma, Z \subset \check{D} \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

Note that we have the inclusion map

$$D \hookrightarrow D_\Sigma,\ F \mapsto (\{0\}, \{F\}).$$

For a rational sharp nilpotent cone σ in $\mathfrak{g}_\mathbb{R}$, we denote

$$D_\sigma := D_{\{\text{face of } \sigma\}}.$$

Then, for a fan Σ in $\mathfrak{g}_\mathbb{Q}$, we have

$$D_\Sigma = \bigcup_{\sigma \in \Sigma} D_\sigma.$$

Example 4.2. We use Notation 2.1 and Example 4.1. We consider the case of $i = 2$.

Let e_1, e_2, e_3, e_4 be the symplectic basis of H_0 in §3, $w \in \mathbb{C}$ and let

$$\begin{aligned} v_1(w) &:= e_2 - \frac{7}{4}e_3 + we_4, \\ v_2 &:= N_\infty(v_1(w)) = e_1 + \frac{9}{2}e_3 - \frac{7}{4}e_4, \\ v_3 &:= (N_\infty)^2(v_1(w)) = 3e_3. \end{aligned}$$

We define $F(w) \in \check{D}$ as follows:

$$\begin{aligned} &F^4(w) := \{0\}, \\ &F^3(w) := (\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } v_1(w)), \\ &F^2(w) := (\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } v_1(w) \text{ and } v_2), \\ &F^1(w) := (F^3(w))^\perp = (\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } v_1(w), v_2 \text{ and } v_3), \\ &F^0(w) := H_{0,\mathbb{C}}\ . \end{aligned}$$

Then $(\sigma_\infty, \exp(\mathbb{C}N_\infty)F(w))$ is a nilpotent orbit. We shall check it. By the definition of $F(w)$, we have

$$N_\infty F^p(w) \subset F^{p-1}(w) \ \text{ for all } \ p \in \mathbb{Z}.$$

Hence Griffiths transversality holds. We check the positivity, that is,

$$\exp(iyN_\infty)F(w) \in D \ \text{ for } \ y \gg 0.$$

For $z \in \mathbb{C}$, we have

$$\begin{aligned} &\exp(zN_\infty)(v_1(w)) = ze_1 + e_2 + \left(\frac{3}{2}z^2 + \frac{9}{2}z - \frac{7}{4}\right)e_3 + \left(-\frac{1}{2}z^3 - \frac{7}{4}z + w\right)e_4, \\ &\exp(zN_\infty)(v_2) = e_1 + \left(3z + \frac{9}{2}\right)e_3 + \left(-\frac{3}{2}z^2 - \frac{7}{4}\right)e_4, \\ &\exp(zN_\infty)(v_3) = 3e_3 - 3ze_4. \end{aligned}$$

Now,

$$\begin{aligned} &\exp(iyN_\infty)F^3(w) \bigcap \overline{\exp(iyN_\infty)F^0(w)} \\ &= (\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } \exp(iyN_\infty)(v_1(w)))\,, \end{aligned}$$

$$\begin{aligned}&\exp(iyN_\infty)F^2(w)\bigcap \overline{\exp(iyN_\infty)F^1(w)}\\&= (\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } u(w))\,,\\&\text{where } u(w) := \exp(iyN_\infty)(v_1(w)) + \left(\frac{-w+\bar{w}}{6y^2} - \frac{2}{3}iy\right)\exp(iyN_\infty)(v_2).\end{aligned}$$

From a simple calculation, we obtain

$$\begin{aligned}&\left\langle i^3\exp(iyN_\infty)(v_1(w)), \overline{\exp(iyN_\infty)(v_1(w))}\right\rangle_0 = 4y^3 - i(w-\bar{w}) > 0,\\&\left\langle iu(w), \overline{u(w)}\right\rangle_0 = \frac{4}{3}y^3 + \frac{i}{3}(w-\bar{w}) + \frac{1}{6y^3}(w-\bar{w})^2 > 0 \ \text{ for } \ y \gg 0.\end{aligned}$$

Hence the positivity holds. Thus we see that $(\sigma_\infty, \exp(\mathbb{C}N_\infty)F(w))$ is a nilpotent orbit.

From the above calculation, we also see that the map

$$\mathbb{C} \to D_{\sigma_\infty} - D,\ w \to (\sigma_\infty, \exp(\mathbb{C}N_\infty)F(w))$$

is injective. We show that this map is surjective. Let $(\sigma_\infty, Z) \in D_{\sigma_\infty} - D$ and $F \in Z$. We take the basis

$$v := v_1(0),\ N_\infty(v),\ (N_\infty)^2(v),\ (N_\infty)^3(v)$$

of $H_{0,\mathbb{C}}$ and express a base u of F^3 by

$$u = a_0v + a_1N_\infty(v) + a_2(N_\infty)^2(v) + a_3(N_\infty)^3(v),\ a_0, a_1, a_2, a_3 \in \mathbb{C}$$

It follows from $F \in \check{D}$ and Griffiths transversality that

$$0 = \langle u, N_\infty(u)\rangle_0 = 6a_0a_2 - 3(a_1)^2.$$

If $a_0 = 0$, then $a_1 = 0$ and

$$\exp(iyN_\infty)(v) = 3a_2e_3 + (-3iya_2 - 3a_3)e_4.$$

Then we have

$$\left\langle i^3\exp(iyN_\infty)(u), \overline{\exp(iyN_\infty)(u)}\right\rangle_0 = 0.$$

This contradicts the positivity. Therefore, $a_0 \neq 0$. So we assume that $a_0 = 1$. Then we have

$$u = v + a_1N_\infty(v) + \frac{1}{2}(a_1)^2(N_\infty)^2(v) + a_3(N_\infty)^3(v).$$

From Griffiths transversality and the fact that $u, N_\infty(u), (N_\infty)^2(u), (N_\infty)^3(u)$ are linearly independent,

$F^{3-j} = \big(\text{the } \mathbb{C}\text{-subspace of } H_{0,\mathbb{C}} \text{ spanned by } u, \cdots, (N_\infty)^j(u)\big)$ $(j = 0, 1, 2, 3)$.

Moreover, we have

$$\begin{aligned}
&\exp(-a_1 N_\infty)(u) = v_1\left(\frac{1}{2}(a_1)^3 - 3a_3\right),\\
&\exp(-a_1 N_\infty)(N_\infty(u)) = v_2,\\
&\exp(-a_1 N_\infty)((N_\infty)^2(u)) = v_3.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\exp(-a_1 N_\infty)F &= F\left(\frac{1}{2}(a_1)^3 - 3a_3\right),\\
Z &= \exp(\mathbb{C}N_\infty)F\left(\frac{1}{2}(a_1)^3 - 3a_3\right).
\end{aligned}$$

Thus, we have the bijection

$$\mathbb{C} \simeq D_{\sigma_\infty} - D.$$

By [KU, 12.3], it is also known that

$$\{(w, z) \in \mathbb{C}^2 \mid \operatorname{Im}(z) < 0\} \simeq D_{\sigma_1} - D.$$

For the other cases, there are similar descriptions of $D_{\sigma_\infty} - D$ and $D_{\sigma_1} - D$.

Definition 4.5 ([KU, 1.3.10])**.** Let Σ be a fan in $\mathfrak{g}_\mathbb{Q}$ and let Γ be a subgroup of $G_\mathbb{Z}$.

(i) We say Γ is *compatible* with Σ if the following condition (1) is satisfied.

(1) If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\operatorname{Ad}(\gamma)(\sigma) \in \Sigma$. Here, $\operatorname{Ad}(\gamma)(\sigma) := \gamma\sigma\gamma^{-1}$. Note that, if Γ is compatible with Σ, Γ acts on D_Σ by

$$\gamma : (\sigma, Z) \mapsto (\operatorname{Ad}(\gamma)(\sigma), \gamma Z)\ (\gamma \in \Gamma).$$

(ii) We say Γ is *strongly compatible* with Σ if it is compatible with Σ and the following condition (2) is also satisfied. For $\sigma \in \Sigma$, define

$$\Gamma(\sigma) := \Gamma \cap \exp(\sigma).$$

(2) The cone σ is generated by $\log \Gamma(\sigma)$, that is, any element of σ can be written as a sum of $c \log(\gamma)$ $(c \in \mathbb{R}_{\geq 0},\ \gamma \in \Gamma(\sigma))$.

Example 4.3. We use Notation 2.1 and Example 4.1. Then the global monodromy Γ is obviously compatible with Ξ from the construction of Ξ. For $\sigma \in \Xi$, we can express σ by

$$\mathbb{R}_{\geq 0}(gNg^{-1}),\ N = 0, N_1, N_\infty,\ g \in \Gamma.$$

From this expression of σ, we have

$$\Gamma(\sigma) = \begin{cases} \{0\} & \text{in the case of } N = 0, \\ \{gT^n g^{-1} \mid n \in \mathbb{N}\} & \text{in the case of } N = N_1, \\ \{g(T_\infty)^n g^{-1} \mid n \in \mathbb{N}\} & \text{in the case of } N = N_\infty, \end{cases}$$

where we denote the monoid of non-negative integers by $\mathbb{N}$. Hence Γ is strongly compatible with Ξ.

Thus we have the extended classifying space D_Ξ and the quotient $\Gamma\backslash D_\Xi$ by the global monodromy. By Example 4.2, we have the bijections

$$(\Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D_{\sigma_\infty}) - (\Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D) \simeq \mathbb{C},$$
$$(\Gamma(\sigma_1)^{\mathrm{gp}}\backslash D_{\sigma_1}) - (\Gamma(\sigma_1)^{\mathrm{gp}}\backslash D) \simeq \{(w, z) \in \mathbb{C}^2 \mid \mathrm{Im}(z) < 0\}.$$

Finally, we shall summarize the notation under the situation arising from our objects.

Notation 4.1. We prepare the following notation under Notation 2.1.

e_1, e_2, e_3, e_4 : the symplectic basis of H_0 in §3,

$N_1 := \log T,\ N_\infty := \log T_\infty \in \mathfrak{g}_\mathbb{Q}$ the logarithms of local monodromies around $\lambda = 1, \infty$ of our object $(W_\lambda)_{\lambda \in \mathbb{P}^1} \to \mathbb{P}^1$,

$\sigma_1 := \mathbb{R}_{\geq 0} N_1,\ \sigma_\infty := \mathbb{R}_{\geq 0} N_\infty$ the rational nilpotent cones,

$\Xi := \{\mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_1, \sigma_\infty, g \in \Gamma\}$ the fan in $\mathfrak{g}_\mathbb{Q}$,

D_Ξ: the extended classifying space defined by Ξ,

D_σ ($\sigma \in \Xi$): the extended classifying space defined by the fan $\{\{0\}, \sigma\}$,

$\Gamma\backslash D_\Xi$: the quotient of D_Ξ by the action of the global monodromy,

$\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ ($\sigma \in \Xi$): the quotient of D_σ by the action of the local monodromy.

5 Logarithmic structures

In the previous section, we saw the construction of the extended classifying space D_Ξ and the quotient $\Gamma\backslash D_\Xi$ by the global monodromy. We endow $\Gamma\backslash D_\Xi$ with a logarithmic ringed space later. So we shall recall logarithmic structures after [KU].

Definition 5.1 ([KU, 2.1.1])**.** Let X be a ringed space with structure sheaf $\mathcal{O}_X$. A *pre-logarithmic structure* on X is a sheaf of monoids M together with a homomorphism $\alpha : M \to \mathcal{O}_X$, where $\mathcal{O}_X$ is regarded as a sheaf of monoids by multiplication.

A *logarithmic structure* on X is a pre-logarithmic structure (M, α) on X which satisfies

$$\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times \text{ via } \alpha.$$

A ringed space endowed with a logarithmic structure is called a *logarithmic ringed space.*

Example 5.1 ([KU, 2.1.2])**.** Let X be a complex manifold and let Y be a divisor on X with normal crossings, and let

$$M := \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } Y\} \subset \mathcal{O}_X.$$

Then, M with the inclusion map $\alpha : M \hookrightarrow \mathcal{O}_X$ is a logarithmic structure, and is called the logarithmic structure on X associated to Y.

Definition 5.2 ([KU, 2.1.1])**.** Let $(M,\ \alpha)$ be a pre-logarithmic structure on X. The *associated logarithmic structure* $(\tilde{M},\ \tilde{\alpha})$ is defined as the push-out $\tilde{M}$ of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \xrightarrow{\subset} & M \\ \alpha \downarrow & & \\ \mathcal{O}_X^\times & & \end{array}$$

in the category of sheaves of monoids on X, together with the homomorphism $\tilde{\alpha} : \tilde{M} \to \mathcal{O}_X$ induced by $\alpha : M \to \mathcal{O}_X$ and the inclusion $\mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$. More explicitly, $\tilde{M}$ is the sheafification of the presheaf $(M \times \mathcal{O}_X^\times)/\sim$, where $(m, f) \sim (m', f')$ if and only if there exists $g_1, g_2 \in \alpha^{-1}(O_X^\times)$ such that $mg_1 = m'g_2$ and $f\alpha(g_2) = f'\alpha(g_1)$.

A morphism $(X, M) \to (Y, N)$ of pre-logarithmic ringed spaces is defined to be a pair (f, h) of a morphism of ringed spaces $f : X \to Y$ and a homomorphism $h : f^{-1}(N) \to M$ such that the diagram

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{h} & M \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_Y) & \longrightarrow & \mathcal{O}_X \end{array}$$

is commutative. A morphism of logarithmic ringed spaces is defined as a morphism of pre-logarithmic ringed spaces.

Example 5.2. Let $X := \mathbb{C}$ and let $Y := \mathbb{C} \times Z$, where Z is a complex manifold. We endow X with the logarithmic structure M_X associated to the divisor $\{x \in X \mid x = 0\}$ and endow Y with the logarithmic structure M_Y associated to the divisor $\{0\} \times Z = \{(y, z) \in Y \mid y = 0\}$. Then we have

$$M_X = \bigcup_{n \in \mathbb{N}} \mathcal{O}_X^\times x^n, \; M_Y = \bigcup_{n \in \mathbb{N}} \mathcal{O}_Y^\times y^n.$$

Let $\psi : X \to Z$ be a holomorphic map. Define the holomorphic map $f : X \to Y, \; x \mapsto (x, \psi(x))$. Then the homomorphism $h : f^{-1}(M_Y) \to M_X$ is induced as follows:

$$f^{-1}(M_Y) \ni gy^n \mapsto (g \circ f) \cdot (y^n \circ f) = (g \circ f)x^n \in M_X.$$

This (f, h) obviously satisfies the above commutative diagram. Hence (f, h) is a morphism of logarithmic ringed spaces.

Definition 5.3 ([KU, 2.1.3])**.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and (M, α) be a logarithmic structure on Y. Then the sheaf-theoretic inverse image $f^{-1}M$ together with the composite morphism $f^{-1}M \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ form a pre-logarithmic structure on X. The *inverse image* $f^*(M, \alpha)$ of (M, α) is defined as the logarithmic structure on X associated to the above pre-logarithmic structure.

Definition 5.4 ([KU, 2.1.4])**.** An *fs monoid* is a commutative monoid S having the following three properties:

(1) S is finitely generated.

(2) If $a, b, c \in S$ and $ab = ac$, then $b = c$.
(Hence S is embedded in the group $S^{\mathrm{gp}} = \{\frac{a}{b} \mid a, b \in S\}$.)

(3) If $a \in S^{\mathrm{gp}}$ and $a^n \in S$ for some integer $n \geq 1$, then $a \in S$.

Definition 5.5 ([KU, 2.1.5])**.** A logarithmic structure (M, α) on a ringed space X is *fs* if there exist an open covering $(U_\lambda)_\lambda$ of X and a family of pairs $(S_\lambda, \theta_\lambda)_\lambda$ consisting of an fs monoid S_λ, regarded as a constant sheaf on U_λ, and of a homomorphism $\theta_\lambda : S_\lambda \to M|_{U_\lambda}$ of sheaves of monoids which induces an isomorphism $\tilde{S}_\lambda \simeq M|_{U_\lambda}$. Here $\tilde{S}_\lambda$ denotes the logarithmic structure associated to the pre-logarithmic structure $S_\lambda \to M|_{U_\lambda} \to \mathcal{O}_{U_\lambda}$. In this case, $(S_\lambda, \theta_\lambda)$ is called a *chart* of $M|_{U_\lambda}$.

A ringed space endowed with an fs logarithmic structure is called an *fs logarithmic ringed space*. In particular, an analytic space endowed with an fs logarithmic structure is called an *fs logarithmic analytic space*. By an *fs logarithmic point*, we mean an fs logarithmic analytic space whose underling ringed space over $\mathbb{C}$ is $\mathrm{Spec}(\mathbb{C})$.

Definition 5.6 ([KU, 2.1.11])**.** An fs logarithmic analytic space X is said to be *logarithmically smooth* if there are an open covering$(U_\lambda)_\lambda$ of X and an fs monoid S_λ for each λ such that each U_λ is isomorphic to an open subset of $Z_\lambda := \mathrm{Spec}(\mathbb{C}[S_\lambda])_{\mathrm{an}}$ endowed with the restrictions of $\mathcal{O}_{Z_\lambda}$ and M_{Z_λ}. Here $\mathrm{Spec}(\mathbb{C}[S_\lambda])_{\mathrm{an}}$ denotes the analytic space associated to $\mathrm{Spec}(\mathbb{C}[S_\lambda])$, and M_{Z_λ} denotes the canonical fs logarithmic structure associated to the pre-logarithmic structure $S_\lambda \to \mathbb{C}[S_\lambda] \subset \mathcal{O}_{Z_\lambda}$.

Example 5.3. Let $Z = \mathbb{C}^2$ and let M_Z be the logarithmic structure associated to the divisor $\{0\} \times \mathbb{C}$. Then Z is a logarithmically smooth fs analytic space. This is obvious, but we shall check it. Let

$$S := \{x^m y^n \in \mathbb{C}[x, y] \mid m, n \in \mathbb{N}\} \simeq \mathbb{N}^2,$$
$$Z' := \mathrm{Spec}(\mathbb{C}[S])_{\mathrm{an}} = \mathbb{C}^2,$$

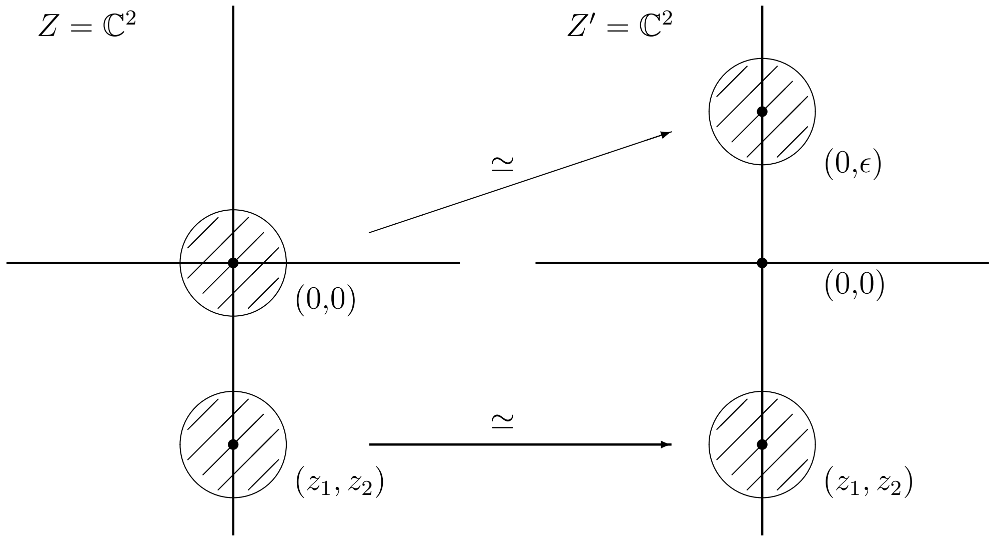
and let $M_{Z'}$ be the canonical fs logarithmic structure associated to the pre-logarithmic structure $S \hookrightarrow \mathcal{O}_{Z'}$. Then we have

$$M_{Z'} = \bigcup_{m,n \in \mathbb{N}} \mathcal{O}_{Z'}^\times x^m y^n.$$

On the other hand, we have

$$M_Z = \bigcup_{m \in \mathbb{N}} \mathcal{O}_Z^\times x^m.$$

Hence, an open neighborhood of $(z_1, z_2) \in Z$ such that $z_2 \neq 0$ is isomorphic to an open neighborhood of $(z_1, z_2) \in Z'$ as a logarithmic ringed space, and an open neighborhood of $(z_1, 0) \in Z$ is isomorphic to an open neighborhood of $(z_1, \epsilon) \in Z'$ as a logarithmic ringed space, where $\epsilon \neq 0$. Thus we see that Z is logarithmic smooth.



Let $z = (z_1, z_2) \in Z$. Define the structure sheaf on z and the logarithmic structure on z by

$$\mathcal{O}_z := \mathcal{O}_{Z,z} / \left(\mathcal{O}_{Z,z}(x - z_1) + \mathcal{O}_{Z,z}(y - z_2)\right) \simeq \mathbb{C},$$
$$M_z := \iota^*(M_Z),$$

where $\iota : z \hookrightarrow Z$ is the inclusion map. Then we have

$$M_z = \begin{cases} \mathbb{C}^\times & \text{if } \ z_1 \neq 0, \\ \bigcup_{n \in \mathbb{N}} \mathbb{C}^\times x^n & \text{if } \ z_1 = 0. \end{cases}$$

Thus, z is an fs logarithmic ringed space.

Definition 5.7 ([KU, 2.1.7])**.** For an analytic space X, let $\Omega_X^1 := \Delta^*(\mathcal{I}/\mathcal{I}^2)$ be the sheaf of Kähler differentials on X, where $\mathcal{I}$ is the sheaf of ideals of $\mathcal{O}_{X \times X}$ defining the image of the diagonal morphism $\Delta : X \to X \times X$. For $f \in \mathcal{O}_X$, the class of $\mathrm{pr}_1^*(f) - \mathrm{pr}_2^*(f)$ in Ω_X^1 is denoted by df.

Let X be an fs logarithmic analytic space. The *sheaf of logarithmic differential 1-form* on X is defined by

$$\omega_X^1 := (\Omega_X^1 \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{\mathrm{gp}}))/N_X,$$

where N_X is the $\mathcal{O}_X$-submodule generated by

$$\{(-d\alpha(f), \alpha(f) \otimes f) \mid f \in M_X\}.$$

For $f \in M_X^{\mathrm{gp}}$, the image of $(0, 1 \otimes f)$ in ω_X^1 is denoted by $d\log(f)$.

Example 5.4 ([KU, 2.1.8])**.** In the standard example 5.1, ω_X^1 is nothing but the sheaf $\Omega_X^1(\log(Y))$ of differential forms with logarithmic poles along Y.

Definition 5.8 ([KU, 3.1.1])**.** Let X be an analytic space and S be a subset of X. The *strong topology* of S in X is defined as follows: A subset U of S is open if, for any analytic space Y and for any morphism $\lambda : Y \to X$ of analytic spaces such that $\lambda(Y) \subset S$, $\lambda^{-1}(U)$ is open in Y.

Definition 5.9 ([KU, 3.5.7])**.** By a *logarithmic manifold*, we mean a logarithmic local ringed space over $\mathbb{C}$ which has an open covering $(U_\lambda)_\lambda$ with the following property: For each λ, there exist a logarithmically smooth fs analytic space Z_λ, a finite subset I_λ of $\Gamma(Z_\lambda, \omega_{Z_\lambda}^1)$, and an isomorphism of logarithmic local ringed spaces over $\mathbb{C}$ between U_λ and an open set of

$$S_\lambda := \{z \in Z_\lambda \mid \text{the image of } I_\lambda \text{ in } \omega_z^1 \text{ is zero}\},$$

where S_λ is endowed with the strong topology in Z_λ and with the inverse images $\mathcal{O}_{Z_\lambda}$ and M_{Z_λ}.

Example 5.5. We use the notation in Example 5.3. From the descriptions of M_Z and M_z, we have

$$\begin{aligned}
N_Z &= \mathcal{O}_Z \left\{(-df, f \otimes f) \mid f \in \mathcal{O}_Z^\times\right\} + \mathcal{O}_Z(-dx, x \otimes x), \\
\omega_Z^1 &= \left(\Omega_Z^1 \oplus \left(\mathcal{O}_Z \otimes_{\mathbb{Z}} \mathcal{O}_Z^\times + \mathcal{O}_Z(1 \otimes x)\right)\right)/N_Z \\
&\simeq \left(\Omega_Z^1 \oplus \mathcal{O}_Z(1 \otimes x)\right)/\mathcal{O}_Z(-dx, x \otimes x) \\
&\simeq \mathcal{O}_Z(0, 1 \otimes x) + \mathcal{O}_Z(dy, 0) \\
&\simeq \mathcal{O}_Z d\log(x) \oplus \mathcal{O}_Z dy,
\end{aligned}$$

and

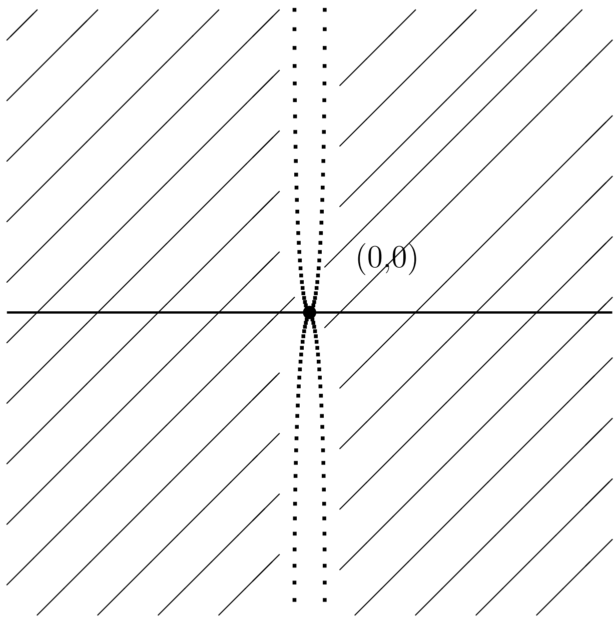
$$\begin{aligned}
\omega_z^1 &= (\mathcal{O}_z \otimes M_z^{\mathrm{gp}})/(\mathcal{O}_z \otimes \mathcal{O}_z^\times) \\
&= \begin{cases} (\mathbb{C} \otimes \mathbb{C}^\times)/(\mathbb{C} \otimes \mathbb{C}^\times) & \text{if } \ z_1 \neq 0, \\ (\mathbb{C} \otimes \mathbb{C}^\times + \mathbb{C}(1 \otimes x))\,/(\mathbb{C} \otimes \mathbb{C}^\times) & \text{if } \ z_1 = 0, \end{cases} \\
&\simeq \begin{cases} 0 & \text{if } \ z_1 \neq 0, \\ \mathbb{C}(1 \otimes x) & \text{if } \ z_1 = 0, \end{cases} \\
&\simeq \begin{cases} 0 & \text{if } \ z_1 \neq 0, \\ \mathbb{C} d\log(x) & \text{if } \ z_1 = 0. \end{cases}
\end{aligned}$$

Let $\eta := yd\log(x) \in \Gamma(Z, \omega_Z)$ and let

$$U := \{z \in Z \mid \text{the image of } yd\log(x) \text{ in } \omega_z^1 \text{ is zero}\}.$$

Then we have

$$U = \big(\mathbb{C}^2 - (\{0\} \times \mathbb{C})\big) \cup \{(0,0)\}.$$



Endow U with the strong topology and the inverse images $\mathcal{O}_Z$ and M_Z. Then U becomes a logarithmic manifold.

6 Geometric structure of $\Gamma\backslash D_\Xi$ and extended period map

We use Notation 2.1 and Notation 4.1. In §4, we saw the construction of the extended classifying space D_Ξ and the quotient $\Gamma\backslash D_\Xi$ by the global monodromy. In this section, we endow $\Gamma\backslash D_\Xi$ with a geometric structure of a logarithmic manifold after [KU].

Let $\{0\} \neq \sigma \in \Xi$. Then it follows from Example 4.3 that $\Gamma(\sigma) \simeq \mathbb{N}$ as a monoid. So, let γ be its generator and let $N = \log(\gamma)$. Define

$$E_\sigma := \left\{ (q,F) \in \mathbb{C} \times \check{D} \;\middle|\; \begin{array}{ll} \exp\left(\dfrac{\log(q)}{2\pi i}N\right) F \in D & \text{if } q \neq 0, \\ \exp(\mathbb{C}N)F : \sigma\text{-nilpotent orbit} & \text{if } q = 0 \end{array} \right\},$$

and the map

$$\rho_\sigma : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma,$$

$$(q,F) \mapsto \begin{cases} \exp\left(\dfrac{\log(q)}{2\pi i}N\right) F \mod \Gamma(\sigma)^{\mathrm{gp}} & \text{if } q \neq 0, \\ (\sigma, \exp(\mathbb{C}N)F) \mod \Gamma(\sigma)^{\mathrm{gp}} & \text{if } q = 0. \end{cases}$$

Here $\Gamma(\sigma)^{\mathrm{gp}}$ is the subgroup of Γ generated by $\Gamma(\sigma)$. This group is the local monodromy group , which is isomorphic to $\mathbb{Z}$ as a group. In fact, by Example 4.3, we have

$$\Gamma(\sigma)^{\mathrm{gp}} = \begin{cases} \{\, gT^n g^{-1} \mid n \in \mathbb{Z} \,\} & \text{if } \sigma = \mathrm{Ad}(g)\sigma_1,\ g \in \Gamma, \\ \{\, g(T_\infty)^n g^{-1} \mid n \in \mathbb{Z} \,\} & \text{if } \sigma = \mathrm{Ad}(g)\sigma_\infty,\ g \in \Gamma. \end{cases}$$

First we shall endow $\Gamma\backslash D_\Xi$ with a topology. Endow E_σ with the strong topology of E_σ in $\mathbb{C} \times \check{D}$. By $\rho_\sigma : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, the quotient topology is introduced on $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$. We then endow $\Gamma\backslash D_\Xi$ with the strongest topology for which the natural maps $\rho'_\sigma : \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Xi$ are continuous for all $\sigma \in \Xi$. Then [KU, Theorem A, (v)] asserts

$$\Gamma\backslash D_\Xi \text{ and } \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \ (\sigma \in \Xi) \text{ are Hausdorff.}$$

Next we shall endow $\Gamma\backslash D_\Xi$ with a structure of a logarithmic ringed space. Let $M_{\mathbb{C}\times\check{D}}$ be the logarithmic structure on $\mathbb{C} \times \check{D}$ associated to the divisor

$\{0\} \times \check{D}$. By the inclusion map $E_\sigma \overset{\iota}{\hookrightarrow} \mathbb{C} \times \check{D}$, the structure sheaf $\mathcal{O}_{E_\sigma} := \iota^{-1}\mathcal{O}_{\mathbb{C}\times\check{D}}$ and the logarithmic structure $M_{E_\sigma} := \iota^* M_{\mathbb{C}\times\check{D}}$ are introduced on E_σ. Let $\pi_\sigma := \rho'_\sigma \circ \rho_\sigma : E_\sigma \to \Gamma\backslash D_\Xi$. Then the structure of logarithmic ringed space on $\Gamma\backslash D_\Xi$ is defined as follows: For any open set U of $\Gamma\backslash D_\Xi$, define

$$\mathcal{O}_{\Gamma\backslash D_\Xi}(U) := \left\{ \text{map } f : U \to \mathbb{C} \mid f \circ \pi_\sigma \in \mathcal{O}_{E_\sigma}(\pi_\sigma^{-1}(U)) \text{ for any } \sigma \in \Xi \right\},$$

$$M_{\Gamma\backslash D_\Xi}(U) := \left\{ \text{map } f : U \to \mathbb{C} \mid f \circ \pi_\sigma \in M_{E_\sigma}(\pi_\sigma^{-1}(U)) \text{ for any } \sigma \in \Xi \right\}.$$

The structure sheaf $\mathcal{O}_{\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma}$ and the logarithmic structure $M_{\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma}$ are introduced on $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ similarly. By [KU, theorem A],

$$E_\sigma \text{ and } \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \ (\sigma \in \Xi) \text{ are logarithmic manifolds.}$$

If Γ is neat, i.e. for each $\gamma \in \Gamma$, the subgroup $\mathbb{C}^\times$ generated by all the eigenvalues of γ is torsion free, then $\Gamma\backslash D_\Xi$ is also a logarithmic manifold and $\rho'_\sigma : \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Xi$ ($\sigma \in \Xi$) are locally isomorphisms of logarithmic ringed spaces. In the present case, the global monodromy group Γ is not neat. In fact, the order of A ,which is the local monodromy around 0, is

$$\begin{cases} 5 & \text{if } \ i = 1, \\ 6 & \text{if } \ i = 2, \\ 8 & \text{if } \ i = 3, \\ 10 & \text{if } \ i = 4. \end{cases}$$

However, if we take a sufficient small open neighborhood for each point on $(\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma) - (\Gamma(\sigma)^{\mathrm{gp}}\backslash D)$, then the restriction of $\rho'_\sigma : \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Xi$ to the open neighborhood is an isomorphism of logarithmic ringed spaces.

Next, we recall the extended period map in the present case.

Endow $\mathbb{P}^1$ with the logarithmic structure associated to the divisor $\{1, \infty\}$. Then, by [KU, 4.3.1, (i)], the period map $\varphi_0 : \mathbb{P}^1 - \{0, 1, \infty\} \to \Gamma\backslash D$ extends to a morphism

$$\varphi : \mathbb{P}^1 \to \Gamma\backslash D_\Xi$$

of logarithmic ringed spaces. We shall see this extension.

Let Δ be the unit disc, whose center is $0 \in \mathbb{C}$, and $\mathfrak{h} \to \Delta^*,\ z \mapsto e^{2\pi i z}$ be the universal covering, where Δ^* denotes the punctured disk $\Delta - \{0\}$. We identify Δ with the unit disc, whose center is $\infty \in \mathbb{P}^1$, and we also denote the restriction of the period map φ_0 to Δ^* by $\varphi_0 : \Delta^* \to \Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D$. Let $\tilde{\varphi}_0 : \mathfrak{h} \to D$ be a lifting of φ_0.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\tilde{\varphi}_0} & D \\ {\scriptstyle e^{2\pi i z}}\downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\varphi_0} & \Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D \end{array}$$

Define

$$\tilde{\psi} : \mathfrak{h} \to \check{D},\ z \mapsto \exp(-zN_\infty)\tilde{\varphi}_0.$$

Then $\tilde{\psi}$ drops down to the holomorphic map $\psi : \Delta^* \to \check{D}$.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\tilde{\psi}} & \check{D} \\ {\scriptstyle e^{2\pi i z}}\downarrow & & \| \\ \Delta^* & \xrightarrow{\psi} & \check{D} \end{array}$$

The nilpotent orbit theorem of Schmid in [Sc] asserts that

$$\begin{cases} \psi \text{ extends to } \psi : \Delta \to \check{D} \text{ as a holomorphic map,} \\ (\sigma_\infty, \exp(\mathbb{C}N_\infty)\psi(0)) \text{ is a nilpotent orbit.} \end{cases}$$

So, we define the map $\psi' : \Delta \to \mathbb{C} \times \check{D},\ w \to (w, \psi(w))$. Then, ψ' is a morphism of logarithmic ringed spaces by Example 5.2. From what we have just mentioned, the image of ψ' is contained in E_{σ_∞}, and

$$\rho_{\sigma_\infty} \circ \psi'(w) = \begin{cases} \exp\left(\dfrac{\log(w)}{2\pi i} N_\infty\right) \psi(w) \bmod \Gamma(\sigma_\infty)^{\mathrm{gp}} & \text{if } w \neq 0 \\ (\sigma_\infty,\ \exp(\mathbb{C}N_\infty)) \bmod \Gamma(\sigma_\infty)^{\mathrm{gp}} & \text{if } w = 0. \end{cases}$$

Hence we obtain the following commutative diagram of logarithmic manifolds.

$$\begin{array}{ccc} \Delta & \xrightarrow{\psi'} & E_{\sigma_\infty} \\ \cup & & \\ \Delta^* & & \downarrow {\scriptstyle \rho_{\sigma_\infty}} \\ & {\searrow}^{\varphi_0} & \\ & & \Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D_{\sigma_\infty} \end{array}$$

Thus we have the extended period map

$$\varphi : \Delta \to \Gamma(\sigma_\infty)^{\mathrm{gp}}\backslash D_{\sigma_\infty} \to \Gamma\backslash D_\Xi.$$

For an open neighborhood of $1 \in \mathbb{C} \subset \mathbb{P}^1$, we can also extend the period map similarly.

We shall see the extension on a neighborhood of $0 \in \mathbb{C} \subset \mathbb{P}^1$. We consider the case of $i = 1$. For the other cases, the following argument works well. Let Δ' be the unit disc, whose center is $0 \in \mathbb{C}$. We denote the cyclic group of order 5, which is generated by the local monodromy A around $\lambda = 0$, by $\langle A\rangle$, and we also denote the restriction of the period map φ_0 to $(\Delta')^*$ by $\varphi_0 : (\Delta')^* \to \langle A\rangle\backslash D$. Then, we have the following commutative diagram.

$$\begin{array}{ccc} (\Delta')^* & \xrightarrow{\tilde{\varphi}_0} & D \\ {\scriptstyle \psi^5}\downarrow & & \downarrow \\ (\Delta')^* & \xrightarrow{\varphi_0} & \langle A\rangle\backslash D \end{array}$$

Here $(\Delta')^* \to (\Delta')^*$, $\psi \mapsto \psi^5$ is the covering map, and $\tilde{\varphi}_0$ is a lifting of φ_0. $\tilde{\varphi}_0$ is nothing but the restriction of the period map arising from the family $(W_\psi)_{\psi\in\mathbb{P}^1} \to$ (ψ-plane) to $(\Delta')^*$. The restriction of this family to Δ' is a smooth projective family. Therefore $\tilde{\varphi}_0$ extends to a holomorphic map $\tilde{\varphi} : \Delta' \to D$ naturally. Hence φ_0 also extends to a morphism

$$\varphi : \Delta' \to \langle A\rangle\backslash D \to \Gamma\backslash D$$

of analytic spaces.

Thus we have the extended period map $\varphi : \mathbb{P}^1 \to \Gamma\backslash D$, which is a morphism of logarithmic ringed spaces, and the images of boundary points $0, 1, \infty \in \mathbb{P}^1$ are

$$\begin{aligned} \varphi(0) &= \big(\text{point mod } \Gamma\big) \in \Gamma\backslash D, \\ \varphi(1) &= (\sigma_1\text{-nilpotent orbit mod } \Gamma), \\ \varphi(\infty) &= (\sigma_\infty\text{-nilpotent orbit mod } \Gamma). \end{aligned}$$

Let

$$X := \Gamma\backslash D_\Xi,\ P_1 := 1,\ P_\infty := \infty \in \mathbb{P}^1,$$

and let

$$Q_1 := \varphi(P_1),\ Q_\infty := \varphi(P_\infty) \in X.$$

Then, by the above correspondence, we have

$$\varphi^{-1}(Q_\lambda) = \{P_\lambda\} \quad \text{for} \quad \lambda = 1, \infty.$$

For the image of the extended period map, there is a very useful theorem.

Theorem 6.1 ([U1])**.** *Let $h : Y \to M$ be a morphism from an analytic space Y to the underlying ringed space of a logarithmic manifold M. If Y is compact, then the image* $\mathrm{Im}(h) \subset M$ *is a compact analytic subspace.*

By this theorem, $\varphi(\mathbb{P}^1) \subset X$ is an analytic curve.

7 Generic Torelli theorem

We use the notation in the previous sections. In this section, we give the proof of the generic Torelli theorem for $(W^i_\lambda)_{\lambda\in\mathbb{P}^1}$ $(i=2,3,4)$. The proof for $(W^1_\lambda)_{\lambda\in\mathbb{P}^1}$ is already given by Usui in [U2], and the proofs for the other three families are similar to that in [U2].

Theorem 7.1. *For each $i=2,3,4$, the extended period map $\varphi:\mathbb{P}^1\to X$ in §6 is the normalization of analytic spaces over its image.*

The argument by using the fs logarithmic points P_1 and Q_1 at the boundaries for $i=1$ in [U2, §4] works also well for $i=2,3,4$, and gives the above theorem.

We give another proof of the above theorem by using the fs logarithmic points P_∞ and Q_∞ at the boundaries.

Proof. The method of the proof is similar to that for $i=1$ given in [U2, §5]. We give the full proof in each case $i=2,3,4$.

Since $\varphi^{-1}(Q_\infty)=\{P_\infty\}$, it is enough to show the following:

Claim 7.1. $(M_X/\mathcal{O}_X^\times)_{Q_\infty}\to(M_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1}^\times)_{P_\infty}$ *is surjective.*

Before the proof of Claim, we prepare a Lemma. Let $N:=N_\infty$.

Lemma 7.1. *In each case, there exists a symplectic basis g_3,g_2,g_1,g_0 of H_0 for which the matrix presentation of N is listed as follows:*

In the case of $i=2$,

$$[\ N(g_3),N(g_2),N(g_1),N(g_0)\ ]=[\ g_3,g_2,g_1,g_0\ ]\begin{pmatrix}0&1&0&0\\0&0&0&0\\3&9/2&0&0\\9/2&-7/2&-1&0\end{pmatrix}.$$

In the case of $i=3$,

$$[\ N(g_3),N(g_2),N(g_1),N(g_0)\ ]=[\ g_3,g_2,g_1,g_0\ ]\begin{pmatrix}0&1&0&0\\0&0&0&0\\2&3&0&0\\3&-11/3&-1&0\end{pmatrix}.$$

In the case of $i = 4$,

$$[\, N(g_3), N(g_2), N(g_1), N(g_0) \,] = [\, g_3, g_2, g_1, g_0 \,] \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 0 \\ 1/2 & -17/6 & -1 & 0 \end{pmatrix}.$$

Proof of Lemma. The basis g_3, g_2, g_1, g_0 is given as follows:

$$\begin{cases} \text{In the case of } i = 2, 3, & g_3 = e_1,\ g_2 = e_2,\ g_1 = e_3,\ g_0 = e_4. \\ \text{In the case of } i = 4, & g_3 = -e_3,\ g_2 = e_2,\ g_1 = e_1,\ g_0 = e_4. \end{cases}$$

□

Proof of Claim. Let $\tilde{q} \in M_{\mathbb{P}^1, P_\infty}$ be a local coordinate on an open neighborhood Δ of $P_\infty \in \mathbb{P}^1$. Then we have

$$(M_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1}^\times)_{P_\infty} = \left\{ \tilde{q}^n \in (M_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1}^\times)_{P_\infty} \mid n \in \mathbb{N} \right\}.$$

We shall find $q \in M_{X, Q_\infty}$ such that $q \circ \varphi = \tilde{q} \in (M_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1}^\times)_{P_\infty}$.

Define the map $\check{q} : E_{\sigma_\infty} \to \mathbb{C}$ by

$$\left(w, \left(F^0 \supset F^1 \supset F^2 \supset F^3 = \mathbb{C} \left(\sum_{0 \le i \le 3} a_i g_i \right) \supset \{0\} \right) \right) \mapsto w \exp \left(2\pi i \frac{a_3}{a_2} \right)$$

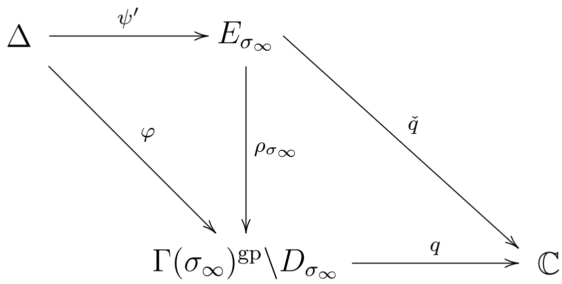
and define the map

$q : \Gamma(\sigma_\infty)^{\mathrm{gp}} \backslash D_{\sigma_\infty} \to \mathbb{C}$,

$$\begin{cases} \left(F^0 \supset F^1 \supset F^2 \supset F^3 = \mathbb{C} \left(\sum_{0 \le i \le 3} b_i g_i \right) \supset \{0\} \right) \bmod \Gamma(\sigma_\infty)^{\mathrm{gp}}, \\ (\sigma_\infty, Z) \bmod \Gamma(\sigma_\infty)^{\mathrm{gp}} \end{cases}$$

$$\mapsto \begin{cases} \exp \left(2\pi i \dfrac{b_3}{b_2} \right), \\ 0. \end{cases}$$

Then, by the definition of $\check{q}$, we have $\check{q} \in M_{E_{\sigma_\infty}, \psi'(P_\infty)}$. Moreover, we have the following commutative diagram.



We shall check $q \circ \rho_{\sigma_\infty} = \check{q}$ in the above diagram in the case of $i = 2$. For the other cases, we can check it similarly. Let

$$(w, F) \in E_{\sigma_\infty},\ w \neq 0,\ z := \frac{\log(w)}{2\pi i}.$$

Then we have

$$\rho_{\sigma_\infty}(w, F) = (\exp(zN)F \mod \Gamma(\sigma_\infty)^{\mathrm{gp}}), \text{ and}$$

$$\exp(zN)F^3 = \mathbb{C}\left([g_3, g_2, g_1, g_0]\begin{pmatrix} 1 & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3z & \frac{9}{2}z + \frac{3}{2}z^2 & 1 & 0 \\ \frac{9}{2}z - \frac{3}{2}z^2 & -\frac{7}{2}z - \frac{1}{2}z^3 & -z & 1 \end{pmatrix}\begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix}\right)$$

$$= \mathbb{C}\Bigg((a_3 + za_2)g_3 + a_2 g_2 + \left(3za_3 + \left(\frac{9}{2}z + \frac{3}{2}z^2\right)a_2 + a_1\right)g_1 + \left(\left(\frac{9}{2}z - \frac{3}{2}z^2\right)a_3 + \left(-\frac{7}{2}z - \frac{1}{2}z^3\right)a_2 - za_1 + a_0\right)g_0\Bigg).$$

From this calculation, we have

$$\begin{aligned} q \circ \rho_{\sigma_\infty}(w, F) &= \exp\left(2\pi i \frac{a_3 + za_2}{a_2}\right) \\ &= \exp(2\pi i z)\exp\left(2\pi i \frac{a_3}{a_2}\right) \\ &= w \exp\left(2\pi i \frac{a_3}{a_2}\right) \\ &= \check{q}\ (w, F). \end{aligned}$$

If $w = 0$, then

$$\begin{aligned} q \circ \rho_{\sigma_\infty}(0, F) &= q\left((\sigma_\infty, \exp(\mathbb{C}N)F) \mod \Gamma(\sigma_\infty)^{\mathrm{gp}}\right) \\ &= 0 \\ &= \check{q}\,(0, F). \end{aligned}$$

Thus, we see the commutativity of the above diagram. Here, it follows from $q \circ \rho_{\sigma_\infty} = \check{q}$ that $q \in M_{X,Q_\infty}$. By the above commutative diagram,

$$\begin{aligned} &q \circ \varphi(w) \\ &= \check{q} \circ \psi'(w) \\ &= \check{q}\,(w, \psi(w)) \\ &= \check{q}\left(w, \left(F^0(w) \supset F^1(w) \supset F^2(w) \supset F^3(w) = \mathbb{C}\left(\sum_{0 \le i \le 3} a_i(w) g_i\right) \supset \{0\}\right)\right) \\ &= w \exp\left(2\pi i \frac{a_3(w)}{a_2(w)}\right) \\ &= (\tilde{q}u)(w), \end{aligned}$$

where $u = \exp\left(2\pi i \dfrac{a_3(w)}{a_2(w)}\right) \in \mathcal{O}^\times_{\mathbb{P}^1, P_\infty}$. Therefore $q \circ \varphi = \tilde{q}$ in $(M_{\mathbb{P}^1}/\mathcal{O}^\times_{\mathbb{P}^1})_{P_\infty}$. $\square$

We thus have proven the theorem in this section. $\square$

Remark 7.1. $g_0, g_1 \in \mathrm{Im}(N^2)$ in Lemma 7.1 is called a good integral basis in [M2]. Moreover, we can see that the above $\tilde{q}u$ is a so-called canonical coordinate. Let ω be a local frame of the free $\mathcal{O}_\Delta$-module F^3 arising from the holomorphic map $\psi : \Delta \to \check{D}$. Then we have

$$\begin{aligned} (\tilde{q}u)(w) &= \exp\left(2\pi i z + 2\pi i \frac{a_3(w)}{a_2(w)}\right) \\ &= \exp\left(2\pi i \frac{z\langle g_0, \omega(w)\rangle_0 + \langle g_1, \omega(w)\rangle_0}{\langle g_0, \omega(w)\rangle_0}\right) \\ &= \exp\left(2\pi i \frac{\langle g_1 + z g_0, \omega(w)\rangle_0}{\langle g_0, \omega(w)\rangle_0}\right) \\ &= \exp\left(2\pi i \frac{\langle \exp(-zN) g_1, \omega(w)\rangle_0}{\langle g_0, \omega(w)\rangle_0}\right). \end{aligned}$$

The last term of this equation is just a canonical coordinate in [M2].

Part II
Global monodromy modulo 5 of quintic-mirror family

8 Partial normalization of monodromy

Let $\Gamma \subset \mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ be the global monodromy of the quintic mirror family $(W_\lambda)_{\lambda \in \mathbb{P}^1}$. (See §2 and §3 for the definition of the quintic mirror family and its global monodromy.) By taking e_1, e_2, e_3, e_4 in §3 as the basis of $H^3(W_b, \mathbb{Z})$, we regard Γ as a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$. We denote the matrix presentations of the local monodromies around $\lambda = 0, 1$ in §3 by A, T. Note that Γ is generated by A and T.

We can partially normalize A and T simultaneously as follows.

Lemma 8.1. *There exists $P \in \mathrm{GL}(4, \mathbb{Q})$ such that*

$$P^{-1}AP = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & 5 & 5 & -4 \end{pmatrix}, \quad P^{-1}TP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. We take $P = \begin{pmatrix} 5 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The assertion follows. □

9 Presentation modulo 5 of global monodromy

Let

$$\Gamma' := \{P^{-1}XP \in \mathrm{GL}(4,\mathbb{Z}) \mid X \in \Gamma\},$$

and let $\rho : \mathrm{GL}(4,\mathbb{Z}) \to \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z})$ be the natural projection. Define

$$\tilde{\Gamma} := \rho(\Gamma').$$

We study $\tilde{\Gamma}$.

Let $\tilde{A} := \rho(P^{-1}AP)$, $\tilde{T} := \rho(P^{-1}TP) \in \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z})$. By a simple calculation, we obtain

$$\tilde{A}^n = \begin{pmatrix} 1 & n & 3n(n+4) & n(n+1)(4n+1) \\ 0 & 1 & n & 2n(n+1) \\ 0 & 0 & 1 & 4n \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z}).$$

Let $\hat{\Gamma}$ be

$$\left\{ \begin{pmatrix} 1 & n & 3n^2+2n & a \\ 0 & 1 & n & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z}) \;\middle|\; n,a,b,c \in \mathbb{Z}/5\mathbb{Z} \right\}.$$

$\hat{\Gamma}$ is a subgroup of $\mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z})$ which contains $\tilde{A}$ and $\tilde{T}$. The following Theorem and Corollary are the main results of the second part.

Theorem 9.1. $\tilde{\Gamma} = \hat{\Gamma}$.

Proof. $\tilde{\Gamma} \subset \hat{\Gamma}$ follows from what we just mentioned. So we shall prove the inverse inclusion.

From the presentations of elements of $\hat{\Gamma}$, we see that $\hat{\Gamma}$ is generated by

$$\tilde{A},\ \tilde{T},\ E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, it is enough to show E_1 and E_2 belong to $\tilde{\Gamma}$. In fact, we have

$$E_2 = \tilde{A}\tilde{T}\tilde{A}^4\tilde{T}^4,\ E_1 = (E_2^2\tilde{A}^2\tilde{T}^4\tilde{A}^3\tilde{T})^4.$$

Hence $E_1,\ E_2 \in \tilde{\Gamma}$. □

Corollary 9.1. *Let $X \in \Gamma$. Then the eigenpolynomial of X is*

$$x^4 + (5m+1)x^3 + (5n+1)x^2 + (5m+1)x + 1,$$

where m, n are some integers. In particular, if X is not the unit matrix and the order of X is finite, then the order of X is 5 and the eigenvalues of X are $\exp(2\pi i/5)$, $\exp(4\pi i/5)$, $\exp(6\pi i/5)$, $\exp(8\pi i/5)$.

Proof. We shall prove the first part. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of X. Then the the eigenpolynomial $p(X)$ of X is

$$x^4 - \left(\sum_{1 \le i \le j \le k \le 4} \lambda_i \lambda_j \lambda_k\right) x^3 + \left(\sum_{1 \le i \le j \le 4} \lambda_i \lambda_j\right) x^2 - \left(\sum_{1 \le i \le 4} \lambda_i\right) x + 1.$$

On the other hand, the the eigenpolynomial $p(X^{-1})$ of X^{-1} is

$$\begin{aligned} & x^4 - \left(\sum_{1 \le i \le j \le k \le 4} \frac{1}{\lambda_i}\frac{1}{\lambda_j}\frac{1}{\lambda_k}\right) x^3 + \left(\sum_{1 \le i \le j \le 4} \frac{1}{\lambda_i}\frac{1}{\lambda_j}\right) x^2 - \left(\sum_{1 \le i \le 4} \frac{1}{\lambda_i}\right) x + 1 \\ =& x^4 - \left(\sum_{1 \le i \le 4} \lambda_i\right) x^3 + \left(\sum_{1 \le i \le j \le 4} \lambda_i \lambda_j\right) x^2 - \left(\sum_{1 \le i \le j \le k \le 4} \lambda_i \lambda_j \lambda_k\right) x + 1. \end{aligned}$$

Since $X \in \mathrm{Sp}(4, \mathbb{Z})$, $p(X) = p(X^{-1})$. So we can express $p(X)$ by

$$x^4 + ax^3 + bx^2 + ax + 1,$$

where $a, b \in \mathbb{Z}$. It follows from the theorem that $a \equiv -4,\ b \equiv 6 \bmod 5$. Hence the claim of the first part follows.

Next we shall prove the latter part. Let λ be an eigenvalue of X. By the property for eigenvalues of elements of the symplectic group, $\bar{\lambda},\ 1/\lambda,\ 1/\bar{\lambda}$ are also eigenvalues of X. If 1 or -1 is an eigenvalue of X, its multiplicity is even. Since the order of X is finite, we can express eigenvalues of X by $\exp(i\theta_1)$, $\exp(-i\theta_1)$, $\exp(i\theta_2)$, $\exp(-i\theta_2)$ $(0 \le \theta_1, \theta_2 \le \pi)$. Then the eigenpolynomial of X is

$$x^4 - 2(\cos\theta_1 + \cos\theta_2)x^3 + 2(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) + 1)x^2 - 2(\cos\theta_1 + \cos\theta_2)x + 1.$$

By the claim of the first part of the Corollary, we have

$$-2(\cos\theta_1 + \cos\theta_2) = 5m+1,\ 2(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) + 1) = 5n+1,\ m, n \in \mathbb{Z}.$$

By the addition theorem, we have

$$2(\cos\theta_1 + \cos\theta_2) = -5m - 1, \; 4\cos\theta_1\cos\theta_2 = 5n - 1.$$

It follows from $-4 \leq 2(\cos\theta_1 + \cos\theta_2) \leq 4$ that $m = 0, -1$. If $m = -1$, then $\cos\theta_1, \cos\theta_2 = 1$ and all eigenvalues of X are 1. Since the order of X is finite, X is the unit matrix. It contradicts the assumption that X is not the unit matrix. Hence $m = 0$ and

$$\cos\theta_1 + \cos\theta_2 = -\frac{1}{2}.$$

It follows from from $-4 \leq 4\cos\theta_1\cos\theta_2 \leq 4$ that $n = 0, 1$. If $n = 1$, then $\cos\theta_1 = \pm 1$, $\cos\theta_2 = \pm 1$. It contradicts the fact that $\cos\theta_1 + \cos\theta_2 = -1/2$. Hence $n = 0$ and

$$\cos\theta_1\cos\theta_2 = -\frac{1}{4}.$$

Combining these two equations, we have

$$\cos^2\theta_1 + \frac{1}{2}\cos\theta_1 - \frac{1}{4} = 0.$$

When we solve this equation for $\cos\theta_1$,

$$\cos\theta_1 = \frac{-1 \pm \sqrt{5}}{4}, \; \sin\theta_1 = \frac{\sqrt{10 \pm 2\sqrt{5}}}{4},$$

$$\cos\theta_2 = \frac{-1 \mp \sqrt{5}}{4}, \; \sin\theta_2 = \frac{\sqrt{10 \mp 2\sqrt{5}}}{4}.$$

Then we can verify easily that $(\exp(i\theta_1))^5, \; (\exp(i\theta_2))^5 = 1$. Hence we have

$$(\theta_1, \theta_2) = \left(\frac{2\pi}{5}, \frac{4\pi}{5}\right) \text{ or } \left(\frac{4\pi}{5}, \frac{2\pi}{5}\right).$$

□

10 Relation to the other result

In this section, we shall compare the main result of this part with the result in [CYY]. Chen, Yang and Yui find the congruence subgroup $\Gamma(5,5)$ which contains the global monodromy Γ. Combining their result and our theorem, we can find a smaller group which contains Γ.

The congruence subgroup $\Gamma(5,5)$ is defined by

$$\Gamma(5,5) := \left\{ X \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \mod 5 \right\}.$$

Let $X \in \Gamma(5,5)$ and express X by

$$\begin{pmatrix} 5x_{11}+1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22}+1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33}+1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44}+1 \end{pmatrix}, \; x_{ij} \in \mathbb{Z} \; (1 \leq i,j \leq 4).$$

Then we have

$$\mathrm{GL}(4,\mathbb{Z}) \ni P^{-1}XP \equiv \begin{pmatrix} 1 & -9x_{31} & -x_{12}+3x_{32} & -x_{14}+3x_{34} \\ 0 & 1 & -2x_{12} & -2x_{14} \\ 0 & 0 & 1 & x_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \mod 5.$$

By the theorem 9.1, if $X \in \Gamma$, then $\rho(P^{-1}XP) \in \tilde{\Gamma}$ and

$$-9x_{31} \equiv n, \; -2x_{12} \equiv n, \; -x_{12}+3x_{32} \equiv 3n^2+2n \mod 5.$$

where n is some integer. From a simple calculation, the above equation is equivalent to

$$x_{31} \equiv 3x_{12}, \; x_{32} \equiv 4x_{12}^2 + 4x_{12} \mod 5.$$

So we define

$$\tilde{\Gamma}(5,5)$$
$$:= \left\{ \begin{pmatrix} 5x_{11}+1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22}+1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33}+1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44}+1 \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \begin{array}{l} x_{31} \equiv 3x_{12}, \\ x_{32} \equiv 4x_{12}^2 + 4x_{12} \\ \mod 5 \end{array} \right\}.$$

Then we have the following Corollary.

Corollary 10.1.
(i) $\tilde{\Gamma}(5,5)$ *is a subgroup of* $\Gamma(5,5)$.
(ii) $\Gamma \subset \tilde{\Gamma}(5,5) \subsetneq \Gamma(5,5)$.
(iii) $\tilde{\Gamma}(5,5)$ *is a congruence subgroup of* $\mathrm{Sp}(4,\mathbb{Z})$ *of finite index.*

Proof. Let

$$\rho' : \Gamma(5,5) \to \mathrm{GL}(4,\mathbb{Z}),\ X \mapsto P^{-1}XP$$

and

$$\pi := \rho \circ \rho' : \Gamma(5,5) \to \mathrm{GL}(4,\mathbb{Z}) \to \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z}).$$

Then, $\tilde{\Gamma}(5,5) = \pi^{-1}(\tilde{\Gamma})$ follows from what we just mentioned. Since π is a group homomorphism, $\pi^{-1}(\tilde{\Gamma})$ is a subgroup of $\Gamma(5,5)$. Hence the claim of (i) follows.

We can verify easily that A and T belong to $\tilde{\Gamma}(5,5)$. Therefore $\tilde{\Gamma}(5,5)$ contains Γ.

We shall show $\tilde{\Gamma}(5,5)$ is a proper subgroup of $\Gamma(5,5)$. We take

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have $X \in \Gamma(5,5)$ and $X \notin \tilde{\Gamma}(5,5)$.

$\tilde{\Gamma}(5,5)$ contains the principal congruence subgroup

$$\Gamma(25) := \mathrm{Ker}(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/25\mathbb{Z})).$$

Hence we obtain

$$\mid \tilde{\Gamma}(5,5) : \mathrm{Sp}(4,\mathbb{Z}) \mid < \mid \Gamma(25) : \mathrm{Sp}(4,\mathbb{Z}) \mid = \mid \mathrm{Sp}(4,\mathbb{Z}/25\mathbb{Z})) \mid < \infty.$$

□

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