STUDIES ON DYNAMICS OF ROBOT MANIPULATORS

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Abstract

Dynamics of robot manipulators is to be discussed in this dissertation, focusing on inertial parameters of kinematic chains of the robot manipulators and identification of them for dynamic modeling.

There are three problems solved, in this dissertation, concerning the inertial parameters of the kinematic chains and the identification of them.

- All the values of link inertial parameters (mass, the product of the mass and the location of center of mass, and moments of inertia for each link) of a kinematic chain are redundant to determine its dynamic equations uniquely, hence they can not be identified independently and some parameters can not be identified completely from input data and motion data. Then it is important to investigate a minimum set of inertial parameters whose values can determine the dynamic equations uniquely. Such a set of inertial parameters is called *A base parameter set* below. The investigation of a base parameter set gives many insights into the structure of the dynamic equations. Each element of a base parameter set is also an identifiable parameter.

- It is needed to establish an efficient identification method of the base parameters.

- The identified parameters are inevitably biased more or less, hence, it may happen that some sets of the obtained base-parameter values are physically impossible. Such set of base-parameter values should be avoided.

In Chapter 2, a base parameter set is investigated for each of three types of manipulators. Then, two identification methods of the base parameters, which have been proposed, are experimentally examined and compared about some items in Chapter 3. In Chapter 4, one method is proposed to judge if a set of base-parameter values for a kinematic chain determines the inertial matrix of the dynamic equations to be positive definite or not for each configuration of the manipulator. If not, it is physically impossible.

The results obtained in this dissertation would have direct contribution to the identification problem of the inertial parameters for robot manipulators. Moreover, knowledge obtained through the detailed examination of the dynamic equations (e.g. redundancy of the link inertial parameters or physical impossibility of a set of base-parameter values) would help us to better understanding of the dynamics of robot manipulators.

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Contents

Chapter 1

Introduction

How wonderful the motions of animals are ! Human beings can walk and run by very skillful use of their two legs. Horses, cheetahs, and other animals can run in very sophisticated manners. When they move without any coupling to an external body, e.g. in the air or in the space, we can observe curious phenomena. Cats, dropped from upside-down with no angular momentum, change their shape in such a way as to land on their feet. The other examples are too numerous to mention. It is needless to say that the control by their brains plays very important role in their wonderful motions. However, the keys of wonderful motions of the animals might be potentially in the dynamics of their bodies. Animal body composed of obviously nonrigid members may well be treated as a system of interconnected rigid bodies when its gross motion is of interest. Such a system which is composed of rigid bodies, however, in the joints connecting the bodies nonrigid members such as springs and dampers are allowed will be called *multi-body system* [1]. Then, we could find out the keys of wonderful motions in features that are proper to the dynamics of multi-body systems. Actually, the dynamics of multi-body systems would show features that are proper to its dynamics, hence reveal to be quite unique. We have had some evidences for its uniqueness through the control of robot manipulators; kinematic chains of the robot manipulators are examples of multi-body systems. (Here, the robot manipulators are considered to consist of kinematic chains with driving systems and the way they are used.) The dynamic models (dynamic equations) of the robot manipulators play very important role in position control and force control. There have been model-based position controllers proposed for robot manipulators; *feedforward control* [2] and *computed torque control* [3] which are based on joint coordinates, and *resolved acceleration position control* [4] which is specified in term of Cartesian coordinates. For force control Cartesian-based force controllers such as *impedance control* [5]-[7] and *operational space method* [8] have been proposed. Also a unified control scheme of position, force, and impedance has been proposed [9]. All these control schemes use dynamic models of the robot manipulators *explicitly*, incorporating them to improve the control. Hence, we need values of parameters appearing in the dynamic model to use these control schemes. Through identification problem of inertial parameters, a fact has been obtained that link inertial parameters (mass, the product of the mass and the location of center of mass, and moments of inertia of each link) appear only in the form of linear combinations in the dynamic equations, and the dynamic equations are linear in the terms of inertial parameters. This phenomenon would arise in any multi-body system. Taking advantage of that fact adaptive control scheme for robot manipulators have been proposed [10]-[12]. On the other hands, there has been *learning control* for robot manipulators proposed by Arimoto et.al. [13]-[17]. The learning control scheme does not use dynamic models of the manipulators. The key issues are the stability and convergence of iterative process to desired trajectory. In the proof of them, some features of the dynamic equations of robot manipulators play an important role. The learning control scheme uses the knowledge of the dynamic model, hence we can say it uses the dynamic model *implicitly*. Arimoto et.al. have found out what are essential for learning control i.e., remarkable features of the dynamic equations for robot manipulators. Thus, the dynamic model of robot manipulators plays very important role when we control them, and some features which characterize the dynamics of the robot manipulators have been found out. Hence, it would be worth while to examine the dynamic equations for the robot manipulators in detail and obtain more knowledge about their dynamics, especially the features proper to their dynamics. Then we extend the examination of the dynamic equations to the multi-body systems, thereby we would get to know the keys of wonderful motions of animals. Some results in this dissertation would be a clue for the goal.

In this dissertation, dynamics of robot manipulators is to be discussed , focusing on inertial parameters of kinematic chains of the robot manipulators and identification of them for dynamic modeling. As mentioned above, for the model-based control of a robot manipulator, it is very crucial to obtain an accurate dynamic model of the manipulator. The dynamic model of the manipulator consisting of rigid links is described as a set of nonlinear differential equations involving various constant parameters: kinematic parameters, link inertial parameters of its kinematic chain, and dynamic parameters of driving systems. If all the values of these parameters are known, we can determine the dynamic model. Hence, accurate values of the parameters are required to obtain an accurate dynamic model. The values of the kinematic parameters can be obtained from design data or by kinematic calibration. The most practical way to obtain the values of the link inertial parameters and driving system parameters is to make test motions of the manipulator and to estimate them from the input data and joint motion data which are taken while the manipulator is in the test motions.

However, unfortunately, it is impossible to estimate all the link inertial parameter values from the input data and the joint motion data in general since they are redundant to determine the dynamic model uniquely. This fact has driven us to investigate nonredundant inertial parameters sufficient to determine the dynamic model uniquely, then, in **Chapter 2**, we show a base parameter set which is defined to be a minimum set of inertial parameters whose values can determine the dynamic model uniquely for each of three types of manipulators. The investigation of a base parameter set would give us many insights into the structure of the dynamic equations. The definitions for a base parameter set would be valid for multi-body systems. We might say that the base parameters are physical existence in a sense in the dynamics of multi-body systems. Then, they suggest us a new formulation of dynamics that are suitable to describe the dynamics of the multi-body systems.

The base parameters are also the parameters that can be identified independently from input data and joint motion data. We describe each element of the base parameter set in a linear combination of the link inertial parameters directly and completely in closed form, also we give the exact number of the base parameters.

Next, it would be very important to have a good identification method to obtain the values of the base parameters for the modeling. Then, in **Chapter 3**, we experimentally examine to estimate the base parameters for an industrial manipulator applying the identification methods: "step-by-step method", "simultaneous method", and "advanced simultaneous method". We compare the methods about the accuracy of estimates. To evaluate the accuracy of them, we simulate the manipulator motion using the estimates and compare the simulated trajectories with measured trajectories. We also describe in detail the contents of the work which is needed to obtain the estimates about each identification method, and compare them about the amount of labour and consuming time on a computer.

If we could obtain the true values of the parameters, no problem would happen. However we are forced to have the estimates biased more or less, and determine the dynamic models using them. Thereby it may happen that the inertial matrix of the dynamic model is not always positive definite for arbitrary configuration of the manipulator, though it is the fact that the inertial matrix is positive definite for arbitrary configuration of the manipulator. If a set of estimated base-parameter values determines such inertial matrix, it is physically impossible. Hence, in **Chapter 4** we propose a method to judge if a set of base-parameter values determines the inertial matrix to be positive definite for arbitrary configuration of the manipulator or not, when we approximately consider the continuous change of each joint variable of the manipulator as a finite set of discrete points. The method can be executed on computers. Using this method we can judge if a set of estimated base-parameter values is "possible" or not. Here,we use "possible" in the sense that the set of base-parameter values determines the inertial matrix to be always positive definite. We also propose one method to modify the estimated base-parameter values for the set of them to be at least "possible" if we judge it is not.

The results in this dissertation would have direct contribution to the identification problem of the inertial parameters for robot manipulators. Moreover, through the detailed examination of the dynamic equations we have had a fact that some link inertial parameters appear in the form of linear combinations in dynamic equations. Also we have noticed that some sets of base-parameter values for the dynamic model are physically impossible. Some other features of the dynamics of robot manipulators have been found to be quite important by several researchers. Those would help us to better understanding of the dynamics of robot manipulators.

Chapter 2

Base Parameters for the Dynamic Models of Robot Manipulators

2.1 Introduction

For the model-based control of a robot manipulator, it is very crucial to obtain an accurate dynamic model of the manipulator [18]. The dynamic model of the manipulator consisting of rigid links is described as a set of nonlinear differential equations involving various constant parameters: kinematic parameters(link lengths, twist angles of adjacent joint axes, and types of joints–rotational or translational), link inertial parameters of its kinematic chain (mass, the product of the mass and the location of center of mass, and moments of inertia for the links), and dynamic parameters of driving systems, since the dynamic model of the manipulator is obtained by means of the combination of the dynamic equations of motion for the kinematic chain and the dynamic models of the driving systems. If all the values of these parameters are known, we can determine the dynamic model. The values of the kinematic parameters can be obtained from design data or by kinematic calibration. The most practical way to obtain the values of the link inertial parameters and driving system parameters is to make test motions of the manipulator and to estimate them from the input data (joint torques or forces) and joint motion data(joint positions, velocities, and accelerations if needed) which are taken while the manipulator is in the test motions.

However, unfortunately, it is impossible to estimate all the link inertial parameter values from the input data and the joint motion data in general since they are redundant to determine the dynamic model uniquely. Thè redundancy is caused by the fact that relative motions of two adjacent links are restricted to one degree-of-freedom, and the first link of the manipulator is connected to fixed base by a joint. It is well recognized that making clear nonredundant inertial parameters sufficient to determine the dynamic model uniquely is fundamentally important for the identification of the dynamic model [18]-[23]. Such nonredundant inertial parameters can be termed a minimum set of inertial parameters whose values can determine the dynamic model uniquely. Such a set of inertial parameters is called a base parameter set for certain reasons to be shown later.

A base parameter set is useful for more efficient and accurate identification of the dynamic model [18]-[23] since we can reduce the number of the inertial parameter values to be estimated; the elements of a base parameter set are also the inertial parameters which can be estimated independently from the input data and joint motion data. To solve the inverse dynamics problem which is a key procedure in the computed torque control and the simulation of the manipulator motions, we need all the link inertial parameter values [24]-[26]. If a base parameter set is made clear, we can show that a certain number of link inertial parameter values can be always supposed to be 0 or 1 assigning appropriate values to the rest of the link inertial parameters without contradiction to estimated values of the inertial parameters in the base parameter set [27]-[29]. Taking advantage of this result, we can reduce the amount of calculations in the Luh et al. algorithm [24] for inverse dynamics problem by about 20 % [28]. The same idea will be valid to other algorithm [25], [26]. Thus, an investigation of the base parameter set is also useful in obtaining efficient algorithms to solve the inverse dynamics problem. Since the notion of base parameter set is so fundamental, the study of it must be helpful for a better understanding of the dynamic models and must give insights to many other problems concerning the dynamic models.

General methods to find a base parameter set has been addressed by several authors. Khosla [21],[22] and Khalil and Kleinfinger [29] have developed computer-aided methods by symbolic procedures for Newton-Euler formulation. Gautier and Khalil [31] have examined a direct determination of a base parameter set for tree structured manipulators by differentiating the energy function of the manipulator. They have used recursive symbolic expressions of the inertial parameters. However this method does not give complete closed form solutions and gives only upper limit of the number of inertial parameters in a base parameter set. Since the method uses the energy function, it is difficult to see directly the affections of the inertial parameters in the obtained base parameter set to the manipulator motions. Sheu and Walker [32] have proposed a method to find out a base parameter set that can be applied to both general open-loop kinematic chains and closed-loop kinematic chains. The method is based on a numerical analysis of the possible changes in the energy contents of the kinematic chain by using sampled motion data. Their method gives a way of determining which base parameter is more effective on the energy. However the choice of motion data would be a problem. Ghodoussi and Nakamura [33] have developed a method to find out a base parameter set for both open and closed kinematic chains, investigating the dynamic equations of the kinematic chains. Their method also includes numerical analysis but whole admissible motion set of the kinematic chain is considered in the numerical analysis. They further have defined the set of the Principal Base Parameters as a set of the base parameters that are orthogonal to each other and numbered in the order of sensitivity to joint torque. Kawasaki et al.[34] have given a method to determine a base parameter set for tree structured manipulators examining the dynamic equations in Newton-Eular formulation. Though the method gives the exact number of inertial parameters in a base parameter set, it uses recursive symbolic expressions of the inertial parameters.

The methods by Khosla, Khalil and Klainfinger, Sheu and Walker, and Ghodoussi and Nakamura are to be applied to each type of manipulator and require long time to execute when the number of links is increased. The methods give by nature fewer insights about physical meaning of the inertial parameters in the obtained base parameter set.

On the other hand, we show a base parameter set, every inertial parameter in which is described in a linear combination of the link inertial parameters directly and completely in closed form. Also, thereby, we give the exact number of the base parameters that is the minimum number of inertial parameters whose values can determine the dynamic model uniquely. The complete closed form expression of base parameters gives many insights into the physical meaning of the parameters, hence the redundancy of the link inertial parameters. The method we use to find out a base parameter set is based on a coordinate free expression of the dynamic equations for kinematic chains, hence it gives many insights about the structure of the dynamic equations. It is also shown that any base parameter set can be obtained by a nonsingular linear transformation of all the inertial parameters in a base parameter set.

To investigate a base parameter set, first of all, in the next section we give some definitions and properties to discuss the redundancy of the link inertial parameters strictly and to show exact meaning of base parameter set. A base parameter set is proved to be a base set for linear vector space of all the identifiable inertial parameters. Several useful results are derived from this. The definitions and results in this section are valid to any kinematic chain. Then, in section 3 a base parameter set is shown for general open-loop parallel and perpendicular manipulators(successive axes of which are parallel or perpendicular) with rotational joints only. In section 4 the results of section 3 is extended to general open-loop parallel and perpendicular manipulators with rotational and translational joints. The results of section 3 and 4 have been extended to general open-loop kinematic chains [35]. However, there are many examples in mechanisms of robot arms or walking machines and in manipulations by multi-finger hands or multi-arms, where we need to treat closed link mechanisms. In particular, there is an important class of industrial manipulators that have closed kinematic chain mechanisms. This mechanism has the advantage that the inertia of links and gravitational loads can be reduced. Hence, it is important to investigate a base parameter set for closed-loop kinematic chains. There have been studies to investigate the base parameter set for such kind of kinematic chains. Based on the same definitions as in the section 1, Mayeda et al.[36] extended the investigation of a base parameter set and giving the complete closed-form solutions of it to a planar closed link mechanism with rotational joints only. They have given the exact number of the base parameters. Bennis and Khalil [37] have examined a direct method to determine a base parameter set by differentiating the energy function of manipulators with parallelogram closed-loop.

They used recursive symbolic expressions of the inertial parameter. However, this method does not give complete closed-form solutions. They have given only the upper limits of a number of parameters in a base parameter set. Kawasaki et al.[38] have proposed a computer-aided method by symbolic procedure to find out a base parameter set for closed kinematic chains. The methods by Sheu and Walker[32], Ghodoussi and Nakamura [33], and Kawasaki et al.[38] are applicable to the close-loop kinematic chains, however they are to be applied to each type of kinematic chains, then they have the same demerits as mentioned above. Hence, it is very important to extend the results in [36] to general closed-loop kinematic chains for the purpose of control and getting a better understanding of the dynamics of the kinematic chains. In section 5 we make a small extension of the results in [36] to manipulators with a planar parallelogram link mechanism. The results of the section would cover most of commercially available industrial manipulators with closed chain mechanisms such as OKURA A930, MITSUBISHI RV-S100A and so on. Finally, in section 6 a conclusion is given.

2.2 Definition of Base Parameters

We investigate the nonredundant inertial parameters sufficient to determine the dynamic model uniquely. Hence, we need not consider the dynamic models of driving systems. Then, we consider only the dynamic equations of the kinematic chains of manipulators and take them as the dynamic models of the manipulators below in this chapter.

First, describing a structure of the dynamic equations of open-loop kinematic chains, we make clear the redundancy of the link inertial parameters.

Each rigid link of the kinematic chain has 10 link inertial parameters: the link mass $\mathbf{m}$, the six independent elements of inertial tensor $\mathbf{I}^x$, $\mathbf{I}^y$, $\mathbf{I}^z$, $\mathbf{I}^{xy}$, $\mathbf{I}^{xz}$, $\mathbf{I}^{yz}$, and the three elements of the center of mass vector multiplied by the mass: $\mathbf{mr}^x$, $\mathbf{mr}^y$, $\mathbf{mr}^z$, which are represented about the coordinate system fixed on the rigid link. Hence, an N degree-of-freedom manipulator has $10N$ link inertial parameters. However, all $10N$ link inertial parameters are redundant to determine the dynamic equations for the kinematic chain.

As is well known, the dynamic equations of open-loop N degree-of-freedom manipulators can be represented in the following form:

$$\boldsymbol{\tau} = \boldsymbol{H}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \dot{\boldsymbol{H}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{\theta}}(\dot{\boldsymbol{\theta}}\boldsymbol{H}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) + \boldsymbol{G}(\boldsymbol{\theta}) \tag{2.1}$$

where $\boldsymbol{\tau} = [\tau_1 \cdots\cdots \tau_N]^t$ is the joint torque and force vector and $\boldsymbol{\theta} = [\theta_1 \cdots\cdots \theta_N]^t$ is the joint variable vector. The superscript $(\cdot)^t$ indicates transposition. $\boldsymbol{H}(\boldsymbol{\theta})$ is $N \times N$ inertial term matrix, and $\boldsymbol{G}(\boldsymbol{\theta})$ is N-dimensional gravity term vector. $\frac{\partial}{\partial \boldsymbol{\theta}}(\dot{\boldsymbol{\theta}}\boldsymbol{H}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}})$ represents the vector $[\frac{\partial}{\partial \theta_1}(\dot{\boldsymbol{\theta}}\boldsymbol{H}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) \ \frac{\partial}{\partial \theta_2}(\dot{\boldsymbol{\theta}}\boldsymbol{H}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}) \ \cdots \ \frac{\partial}{\partial \theta_N}(\dot{\boldsymbol{\theta}}\boldsymbol{H}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}})]^t$. Hence, it is evident that the dynamic equations (2.1) can be determined if and only if each element of $\boldsymbol{H}(\boldsymbol{\theta})$ and $\boldsymbol{G}(\boldsymbol{\theta})$ is determined as functions of $\boldsymbol{\theta}$. Here, let $\boldsymbol{q}$ denote the vector whose entries are all translational joint variables of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_r$ the vector whose entries are all rotational joint variables of $\boldsymbol{\theta}$. Then, we can describe each element of $\boldsymbol{H}(\boldsymbol{\theta})$ and $\boldsymbol{G}(\boldsymbol{\theta})$ in the following form:

$$\sum_{v=1}^{T} \mathbf{p}_v f_v(\boldsymbol{\theta}_r,\ \boldsymbol{q}) \tag{2.2}$$

where $\mathbf{p}_v$ is a linear combination of the link inertial parameters and f_v is a polynomial of $\boldsymbol{q}$ and trigonometric functions of $\boldsymbol{\theta}_r$. (f_v is allowed to be a constant function.) The form (2.2) will be called a function of $\boldsymbol{\theta}$ generated by $\mathbf{p}_1,\ \mathbf{p}_2,\ \cdots,\ \mathbf{p}_T$. These forms are determined if the values of the kinematic parameters are given. These facts will become evident in later discussions. Thus, assuming that the values of kinematic parameters are known, if we give values to all the link inertial parameters, we can determine all the elements of $\boldsymbol{H}(\boldsymbol{\theta})$ and $\boldsymbol{G}(\boldsymbol{\theta})$ as functions of $\theta_1,\ \theta_2,\ \ldots,\ \theta_N$ and hence the dynamic equations (2.1). If two choices of values to all the link inertial parameters determine some different elements of $\boldsymbol{H}(\boldsymbol{\theta})$ or $\boldsymbol{G}(\boldsymbol{\theta})$, different dynamic equations are determined from them, and vice versa.

For the purpose of determining the dynamic equations (2.1) uniquely, the link inertial parameters are redundant in the sense that same dynamic equations might be determined even if some link inertial parameters take different values. Therefore, it is unfortunately impossible to estimate all the link inertial parameter values from link motion and joint torque or force data. The redundancy is caused by linear dependencies among $f_1, f_2, \cdots, f_T$ and also among $\mathbf{p}_1,\ \mathbf{p}_2,\ \cdots,\ \mathbf{p}_T$ in (2.1). In any kinematic chain similar phenomenon would arise. For modeling of the manipulator motion and its identification, it is a very fundamental problem to find nonredundant parameters that are sufficient to determine the dynamic equations uniquely and can be identified independently from motion and torque or force data. Any linear combination of the link inertial parameters is defined to be an inertial parameter as a candidate of nonredundant parameters and will be written in upright bold face letters. Since the set of all the inertial parameters includes every $\mathbf{p}_v$ in (2.1), it is no use to consider any broader class of parameters as candidates of the nonredundant parameters. The set of all inertial parameters obviously constitutes a linear vector space.

To investigate the problem we give following definitions and properties.

Definition 2.2.1 An inertial parameter $\mathbf{p}$ is called a fundamental parameter if any two choices of values to all the link inertial parameters, that give different values to $\mathbf{p}$, never determine same dynamic equations.

A fundamental parameter corresponds to an identifiable parameter from motion data and joint torque or force data since there exists an appropriate joint torque or force for two different dynamic equations, which generates different link motions for them.

Definition 2.2.2 A set $\boldsymbol{F}$ of inertial parameters is said to generate the dynamic equations if same dynamic equations are always determined by any choices of values to all the link inertial parameters as long as they give same value to each inertial parameter in $\boldsymbol{F}$.

Definition 2.2.3 A set $\boldsymbol{F}$ of linearly independent fundamental parameters that generates the dynamic equations is called a base parameter set and each fundamental parameter in $\boldsymbol{F}$ is called a base parameter.

Property 2.2.1 The set of all the fundamental parameters constitutes a linear vector space, and a base parameter set forms a base set of the linear vector space.

Proof Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be fundamental parameters, and let α be a scaler. For different values of $\alpha\mathbf{p}_1$ or $\mathbf{p}_1 + \mathbf{p}_2$, $\mathbf{p}_1$ or either $\mathbf{p}_1$ or $\mathbf{p}_2$ must take different values. Hence same dynamic equations can not be determined if $\alpha\mathbf{p}_1$ or $\mathbf{p}_1 + \mathbf{p}_2$ takes different values, and hence, $\alpha\mathbf{p}_1$ and $\mathbf{p}_1 + \mathbf{p}_2$ are also fundamental parameters. The set of all the fundamental parameters constitutes a linear vector space.

Suppose that a base parameter set $\boldsymbol{F}$ does not form a base set of the linear vector space. Then, there exists a fundamental parameter $\mathbf{p}$ which is linearly independent to the base parameters in $\boldsymbol{F}$. It is easy to find two choices of values to all the link inertial parameters, that give different values to $\mathbf{p}$ and same value to each base parameter in $\boldsymbol{F}$. Since $\mathbf{p}$ is a fundamental parameter, these two choices of link inertial parameter values determine different dynamic equations. This contradicts the definition of base parameter set. ∎

Property 2.2.2 A base parameter set is a minimum set of inertial parameters, that can generate the dynamic equations.

Proof Assume that there exists a set $\boldsymbol{F}$ of inertial parameters that can generate the dynamic equations and such that the number of the inertial parameters in $\boldsymbol{F}$ is less than that in a base parameter set. Since $\boldsymbol{F}$ can not span the linear vector space of all the fundamental parameters, there exists a fundamental parameter $\mathbf{p}$ that is linearly independent to the inertial parameters in $\boldsymbol{F}$. It is easy to find two choices of link inertial parameter values that give different values to $\mathbf{p}$ and the same value to each inertial parameter in $\boldsymbol{F}$. These two choices of link inertial parameter values determine different dynamic equations since $\mathbf{p}$ is a fundamental parameter. This contradicts the assumption that $\boldsymbol{F}$ can generate the dynamic equations. ∎

From **Property 2.2.2**, we can regard a base parameter set as the nonredundant parameters that are sufficient to determine the dynamic equations. Values of base parameters can be estimated independently from motion data and torque or force data of the manipulator since base parameters are fundamental parameters. **Property 2.2.1** shows the reason why we adopt the name of base parameter. Note that once one base parameter set $\boldsymbol{F}$ is found, any fundamental parameter is a linear combination of the base parameters in $\boldsymbol{F}$ and any other base parameter set can be obtained by a nonsingular linear transformation of all the base parameters in $\boldsymbol{F}$. Thus, the problem to be solved is to find a base parameter set.

2.3 Base Parameters for Manipulators with Rotational Joints Only

In ordinary manipulators, any two adjacent joint axes are parallel or perpendicular. In this section we consider a general manipulator of this type with N links and assume that every joint is rotational for simplicity. We show a base parameter set for the manipulators, describing every base parameter in a linear combination of the link inertial parameters directly and completely in closed form. Also, thereby, we give the exact number of the base parameters.

2.3.1 Dynamic Models of the Manipulators

To describe the manipulator motions, we number the links successively from 0 to N. (0 is assigned to the base.) Joint i connects links $i-1$ and i. We attach a coordinate system $(o_i; \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ to each link i in the way shown in Fig.2.3.1. This is similar to Craig's convention [39] except that the origin o_i of $(o_i; \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ is chosen to be the intersection of joint i axis and common normal to axes of joints $i-1$ and i. Joint angle θ_i is the angle between $\boldsymbol{x}_{i-1}$ and $\boldsymbol{x}_i$ measured around $\boldsymbol{z}_i$. Taking advantage of every joint axis that is perpendicular to the predecessor, we divide the whole N links into link clusters as shown in Fig.2.3.2. More precisely, let $\alpha_1 = 1$ and let $(2 \leq)\alpha_2 < \alpha_3 < \cdots < \alpha_K$ be link numbers such that joint α_d axis is perpendicular to joint $\alpha_d - 1$ axis for $2 \leq d \leq K$. Define $\beta_d = \alpha_{d+1} - 1$ for $1 \leq d \leq K-1$ and $\beta_K = N$. Then, links $\alpha_d,\ \alpha_d + 1,\ \cdots\ ,\ \beta_d$ constitute link cluster d where axes of joints $\alpha_d,\ \alpha_d + 1,\ \cdots\ ,\ \beta_d$ are parallel. K is the number of link clusters in the manipulator. When link i is included in link cluster d, we define $c(i)$ as $c(i) = d$.

Let $\boldsymbol{m}_i$ be the mass of link i, $\boldsymbol{I}_i$ be the moment of inertia matrix of link i around o_i, and $\boldsymbol{r}_i$ and $\boldsymbol{L}_i$ be the vectors from o_i to the center of mass of link i and o_{i+1}, respectively. We consider any vector $\boldsymbol{v}$ and any tensor $\boldsymbol{T}$ are represented about the base coordinate system $(o_0; \boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{z}_0)$. The representations of $\boldsymbol{v}$ and $\boldsymbol{T}$ about $(o_i; \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ are denoted by ${}^i\boldsymbol{v}$ and ${}^i\boldsymbol{T}$, respectively. ${}^i\boldsymbol{L}_i$, ${}^i\boldsymbol{r}_i$ and ${}^i\boldsymbol{I}_i$ are constant vectors and a constant matrix and will be denoted by

$$ {}^i\boldsymbol{L}_i = \begin{bmatrix} [L]_i^x & 0 & [L]_i^z \end{bmatrix}^t \tag{2.3} $$

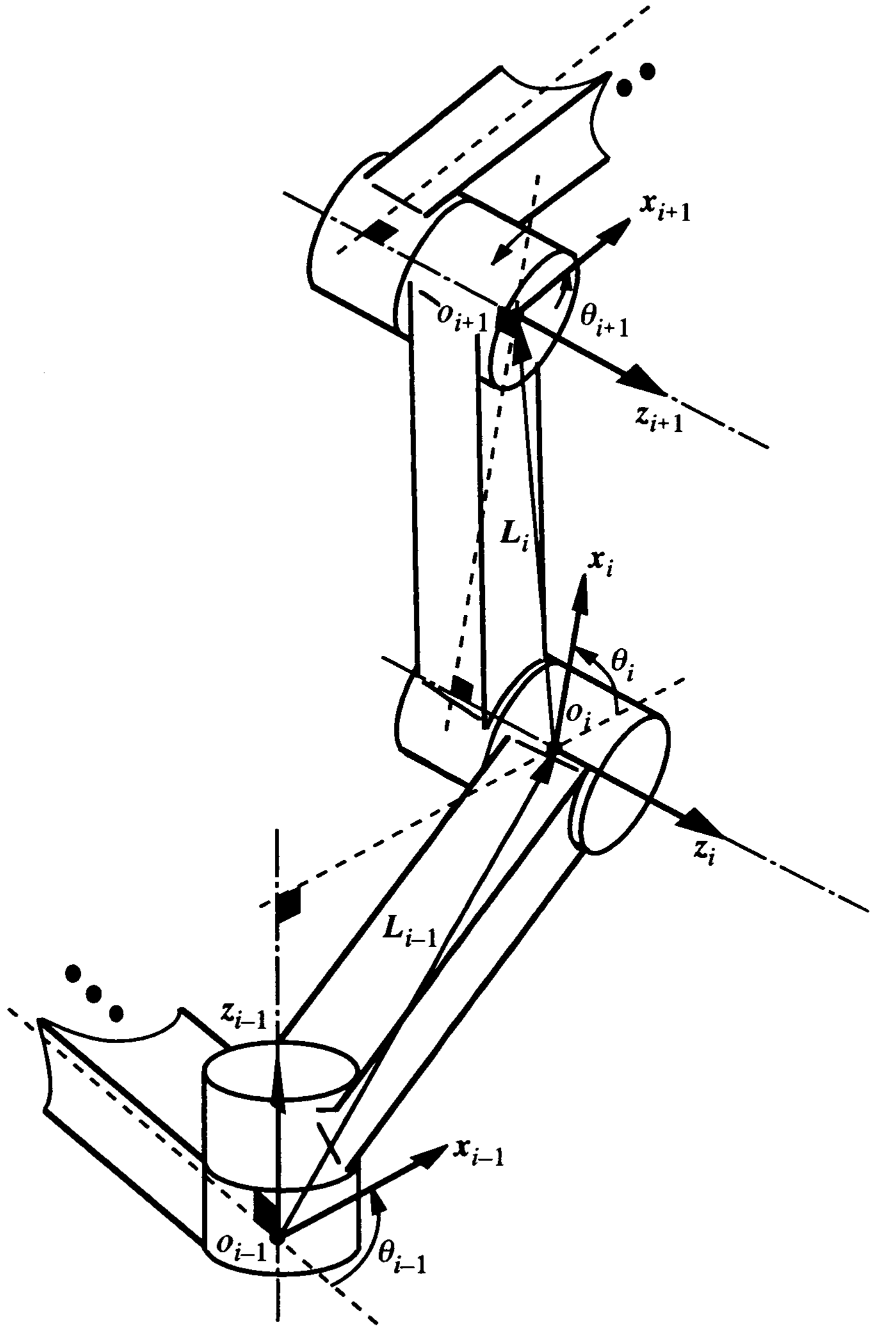


Fig. 2.3.1 Parallel and Perpendicular Manipulator and its Coordinate Systems

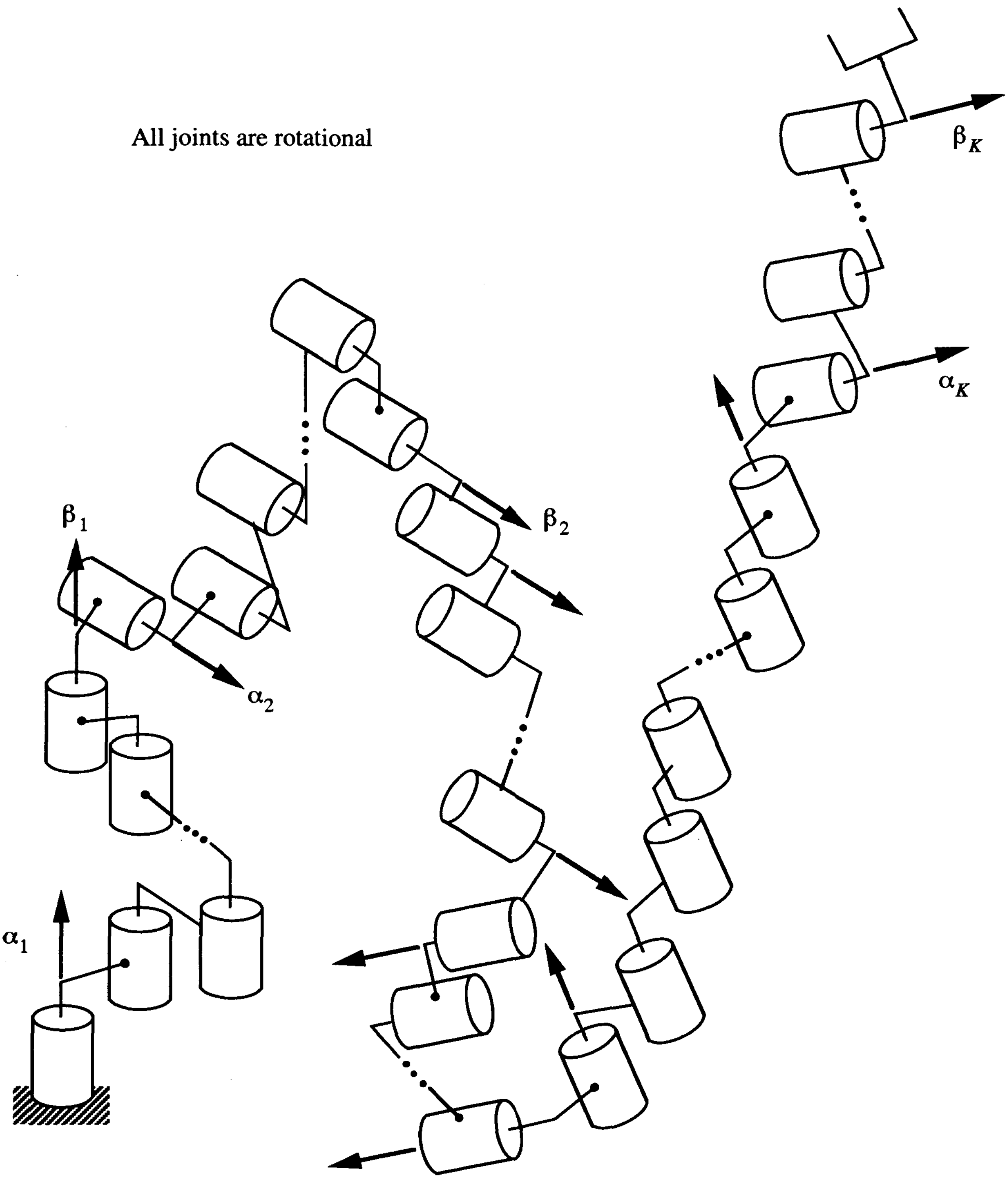


Fig. 2.3.2 Link Clusters in Parallel and Perpendicular Manipulator

$$ {}^{i}\boldsymbol{r}_i = \begin{bmatrix} r_i^x & r_i^y & r_i^z \end{bmatrix}^t \tag{2.4} $$

$$ {}^{i}\boldsymbol{I}_i = \begin{bmatrix} \mathrm{I}_i^x & \mathrm{I}_i^{xy} & \mathrm{I}_i^{xz} \\ \mathrm{I}_i^{xy} & \mathrm{I}_i^y & \mathrm{I}_i^{yz} \\ \mathrm{I}_i^{xz} & \mathrm{I}_i^{yz} & \mathrm{I}_i^z \end{bmatrix}. \tag{2.5} $$

The superscript $(\cdot)^t$ denotes the transposition of $(\cdot)$. ${}^{i}\boldsymbol{L}_i$ means the length of link i and is assumed to be known. ($\boldsymbol{y}_i$ component of ${}^{i}\boldsymbol{L}_i$ is zero when the link coordinate systems in Fig.2.3.1 are adopted. This simplifies expressions of the base parameters in later discussions.) As mentioned in preceding section, for each link i we have 10 link inertial parameters: m_i, $\mathrm{m}_i\mathrm{r}_i^x$, $\mathrm{m}_i\mathrm{r}_i^y$, $\mathrm{m}_i\mathrm{r}_i^z$, I_i^x, I_i^y, I_i^z, I_i^{xy}, I_i^{xz}, I_i^{yz} ($\mathrm{m}_i \neq 0$ is assumed.)

After attaching coordinate systems to links, we have rotation matrices. 3×3 matrix $\boldsymbol{A}_i = [\boldsymbol{x}_i\ \boldsymbol{y}_i\ \boldsymbol{z}_i]$ represents the orientation of $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ about the base coordinate system and ${}^{j}\boldsymbol{A}_i = [{}^{j}\boldsymbol{x}_i\ {}^{j}\boldsymbol{y}_i\ {}^{j}\boldsymbol{z}_i]$ represents rotation of $(\boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{z}_j)$ to $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$. Entries of ${}^{j}\boldsymbol{A}_i$ are functions of θ_{j+1}, θ_{j+2}, $\cdots$, θ_i when $i > j$ and will be denoted by

$$ {}^{j}\boldsymbol{A}_i = \begin{bmatrix} ({}^{j}A_i)_{11} & ({}^{j}A_i)_{12} & ({}^{j}A_i)_{13} \\ ({}^{j}A_i)_{21} & ({}^{j}A_i)_{22} & ({}^{j}A_i)_{23} \\ ({}^{j}A_i)_{31} & ({}^{j}A_i)_{32} & ({}^{j}A_i)_{33} \end{bmatrix}. \tag{2.6} $$

It is well known that ${}^{j}\boldsymbol{v} = {}^{j}\boldsymbol{A}_i {}^{i}\boldsymbol{v}$ for any vector $\boldsymbol{v}$, $({}^{j}\boldsymbol{A}_i)^t = {}^{i}\boldsymbol{A}_j$, ${}^{j}\boldsymbol{A}_i = {}^{j}\boldsymbol{A}_S {}^{S}\boldsymbol{A}_i$, and that

$$ {}^{i-1}\boldsymbol{A}_i = \begin{cases} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{, if } \boldsymbol{z}_i \text{ is parallel to } \boldsymbol{z}_{i-1} \\ \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ 0 & 0 & -1 \\ \sin\theta_i & \cos\theta_i & 0 \end{bmatrix} & \text{, if } \boldsymbol{z}_i \text{ is perpendicular to } \boldsymbol{z}_{i-1}. \end{cases} \tag{2.7} $$

Let us define $\theta(i,j) = \theta_i + \theta_{i+1} + \cdots + \theta_j$. It is easily derived that the rotation matrices have following properties.

Property 2.3.1

(i) For $\alpha_d \leq j < i \leq \beta_d$ where $1 \leq d \leq K$,

$$ {}^j\boldsymbol{A}_i = \begin{bmatrix} \cos\theta(j+1,i) & -\sin\theta(j+1,i) & 0 \\ \sin\theta(j+1,i) & \cos\theta(j+1,i) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.8} $$

(ii) For $\alpha_2 \leq i \leq N$,

$$ {}^{\beta_{C(i)-1}}\boldsymbol{A}_i = \begin{bmatrix} \cos\theta(\alpha_{C(i)},i) & -\sin\theta(\alpha_{C(i)},i) & 0 \\ 0 & 0 & -1 \\ \sin\theta(\alpha_{C(i)},i) & \cos\theta(\alpha_{C(i)},i) & 0 \end{bmatrix}. \tag{2.9} $$

(iii) When $c(j) < c(i)$,

$$ {}^j\boldsymbol{A}_i = \begin{bmatrix} ({}^jA_i)_{11} & ({}^jA_i)_{12} & -({}^jA_{\beta_{C(i)-1}})_{12} \\ ({}^jA_i)_{21} & ({}^jA_i))_{22} & -({}^jA_{\beta_{C(i)-1}})_{22} \\ ({}^jA_i)_{31} & ({}^jA_i)_{32} & -({}^jA_{\beta_{C(i)-1}})_{32} \end{bmatrix}. \tag{2.10} $$

Denoting operations of inner product, cross product, and tensor product of two vectors by $\cdot$, $\times$, and $\otimes$, respectively, we introduce the following notation:

$$ \mathbf{M}_i = \sum_{j=i}^{N} \mathbf{m}_j \tag{2.11} $$

$$ \boldsymbol{R}_i = \mathbf{M}_{i+1}\boldsymbol{L}_i + \mathbf{m}_i\boldsymbol{r}_i \tag{2.12} $$

$$ \boldsymbol{SR}_i = \sum_{j=i}^{N} \boldsymbol{R}_j \tag{2.13} $$

$$\boldsymbol{J}_i = \boldsymbol{I}_i + \mathrm{M}_{i+1}[(\boldsymbol{L}_i \cdot \boldsymbol{L}_i)\boldsymbol{E} - \boldsymbol{L}_i \otimes \boldsymbol{L}_i] \tag{2.14}$$

$$\boldsymbol{L}_{j,i} = \sum_{S=j}^{i-1} \boldsymbol{L}_S \tag{2.15}$$

where $\boldsymbol{E}$ is unit tensor of rank 2. $\mathbf{m}_i$, $\boldsymbol{r}_i$, and $\boldsymbol{J}_i$ are the moments of order 0, 1 and 2 for the augmented link [25] of link i around o_i, respectively. $\boldsymbol{L}_{j,i}$ means vector from o_j to o_i. Note that $\mathbf{m}_i$ and all the entries of ${}^i\boldsymbol{R}_i$ and ${}^i\boldsymbol{J}_i$ are inertial parameters.

For the dynamic model of the manipulator, we adopt following simple dynamic equations in vector-tensor form, which can be derived from the *Lagrange equations* by elementary operations:

$$\sum_{j=1}^{N} H(i,j)\ddot{\theta}_j + \sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial H(i,j)}{\partial \theta_k} - \frac{\partial H(k,j)}{2\partial \theta_i}\right)\dot{\theta}_k\dot{\theta}_j - \boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i) = \tau_i \tag{2.16}$$

for $i = 1,\ 2,\ \ldots\ ,\ N$, where τ_i is torque imposed around joint i axis, $\boldsymbol{g}$ is gravity vector, and $\dot{\theta}_i$ and $\ddot{\theta}_i$ are first and second time derivative of θ_i,

$$\begin{aligned} H(i,j) = {} & \boldsymbol{z}_i \cdot \Big(\sum_{S=i}^{N} \boldsymbol{J}_S\Big)\boldsymbol{z}_j \\ & + \boldsymbol{z}_i \cdot \Big\{\sum_{S=i}^{N-1}[2(\boldsymbol{L}_S \cdot \boldsymbol{SR}_{S+1})\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{SR}_{S+1} - \boldsymbol{SR}_{S+1} \otimes \boldsymbol{L}_S]\Big\}\boldsymbol{z}_j \\ & + \boldsymbol{z}_i \cdot [(\boldsymbol{L}_{j,i} \cdot \boldsymbol{SR}_i)\boldsymbol{E} - \boldsymbol{L}_{j,i} \otimes \boldsymbol{SR}_i]\boldsymbol{z}_j \end{aligned} \tag{2.17}$$

for $1 \le j \le i \le N$, and

$$H(i,j) = H(j,i) \tag{2.18}$$

for $i \le j$. These dynamic equations can also be derived by Newton-Euler approach. In the approach, $H(i,j)$ and $\boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ would be understood more intuitively.

The dynamic equations (2.16) can be determined if and only if $H(i,j)$ and $\boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ are determined as functions of $\theta_1,\ \theta_2,\ \cdots\ ,\ \theta_N$ for $1 \le i,j \le N$. Evaluating each of them about an appropriate coordinate system, we can describe them in the following form:

$$\sum_{V=1}^{T} \mathrm{p}_V f_V \tag{2.19}$$

where $\mathbf{p}_v$ is an inertial parameter and f_v is a polynomial of trigonometric functions of θ_i for $1 \leq i \leq N$. (f_v is allowed to be a constant function.) The form (2.19) will be called a function of $\boldsymbol{\theta}$ generated by $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_T$ in later discussions.

2.3.2 A Base Parameter Set

In this subsection we show a base parameter set for the dynamic model (2.16), each base parameter is described by the link inertial parameters directly and completely in closed form. First, to describe the base parameters, we introduce the following notation. ${}^i\boldsymbol{R}_i$ and ${}^i\boldsymbol{J}_i$ are constant vector and matrix, respectively, hence, they will be represented as follows:

$$
{}^i\boldsymbol{R}_i = \left[\; \mathbf{R}_i^x \;\; \mathbf{R}_i^y \;\; \mathbf{R}_i^z \; \right]^t \tag{2.20}
$$

$$
{}^i\boldsymbol{J}_i = \begin{bmatrix} \mathbf{J}_i^x & \mathbf{J}_i^{xy} & \mathbf{J}_i^{xz} \\ \mathbf{J}_i^{xy} & \mathbf{J}_i^y & \mathbf{J}_i^{yz} \\ \mathbf{J}_i^{xz} & \mathbf{J}_i^{yz} & \mathbf{J}_i^z \end{bmatrix}. \tag{2.21}
$$

To simplify descriptions of the base parameters, following inertial parameters are introduced:

$$
\mathbf{RZ}(i) = \begin{cases} 0, & \text{if } i = \beta_{c(i)} \\ \sum\limits_{j=i+1}^{\beta_{c(i)}} \mathbf{R}_j^z, & \text{otherwise} \end{cases} \tag{2.22}
$$

$$
\mathbf{RZB}(i) = \begin{cases} \sum\limits_{j=\alpha_{c(i)+1}}^{\beta_{c(i)+1}} \mathbf{R}_j^z, & \text{if } i = \beta_{c(i)} \text{ and } c(i) \neq K \\ 0, & \text{otherwise} \end{cases} \tag{2.23}
$$

$$
\mathbf{JYB}(i) = \begin{cases} \sum\limits_{j=\alpha_{c(i)+1}}^{\beta_{c(i)+1}} (\mathbf{J}_j^y + 2[L]_j^z \mathbf{RZ}(j)), & \text{if } i = \beta_{c(i)} \text{ and } c(i) \neq K \\ 0, & \text{otherwise} \end{cases} \tag{2.24}
$$

Here we assume that $[L]_i^x \neq 0$ for $1 \leq i \leq \beta_1 - 1$. (Removal of this assumption is possible but makes no sense for practical cases.) Then, a base parameter set is given in the following theorem.

Theorem 2.3.1 The following inertial parameters constitute a base parameter set for the manipulator dynamic model (2.16). For the case that $\boldsymbol{z}_1$ is not parallel to gravity vector $\boldsymbol{g}$,

$$\mathbf{J}_i^z + \mathbf{JYB}(i), \quad \mathbf{R}_i^x, \quad \mathbf{R}_i^y - \mathbf{RZB}(i) \tag{2.25}$$

for $1 \leq i \leq N$, and

$$\begin{aligned} &\mathbf{J}_i^x - \mathbf{J}_i^y + \mathbf{JYB}(i), \quad \mathbf{J}_i^{xz} - [L]_i^x\mathbf{RZ}(i), \\ &\mathbf{J}_i^{xy} + [L]_i^x\mathbf{RZB}(i), \quad \mathbf{J}_i^{yz} + [L]_i^z\mathbf{RZB}(i) \end{aligned} \tag{2.26}$$

for $\alpha_2 \leq i \leq N$. For the case that $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$, delete $\mathbf{R}_1^x$ and $\mathbf{R}_1^y - \mathbf{RZB}(1)$ from the above inertial parameters.

The total number of base parameters in the base parameter set is $7N - 4\beta_1$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$ or $7N - 4\beta_1 - 2$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$.

We first show the following lemmas necessary for the proof of **Theorem 2.3.1**.

Lemma 2.3.1 Suppose that $H(i,j)$ or $\boldsymbol{g} \cdot (\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ is described as

$$\sum_{v=1}^{U_1} \mathbf{p}_v f v + \sum_{v=U_1+1}^{U_2} \mathbf{p}_v f v \tag{2.27}$$

where $\mathbf{p}_v$ is an inertial parameter and f_v is a polynomial of trigonometric functions of $\theta_1, \ldots, \theta_N$ for $1 \leq v \leq T$. If $f_1, f_2, \ldots, f_{U_1}$ are linearly independent functions and $\mathbf{p}_{U_1+1}, \mathbf{p}_{U_1+2}, \ldots, \mathbf{p}_{U_2}$ are fundamental parameters, then $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{U_1}$ are also fundamental parameters.

Proof Assume that some $\mathbf{p}_a$ such that $1 \leq a \leq T_1$ is not a fundamental parameter. Then, we can consider two choices of values to all the link inertial parameters, which give different values p_a^1 and p_a^2 to $\mathbf{p}_a$ and determine same dynamic equations. Let p_v^1 and p_v^2 be the values of $\mathbf{p}_v$ corresponding to the two choices of the link inertial parameter values. Then $p_v^1 = p_v^2$ for $T_1 + 1 \leq v \leq T$ and hence $(p_a^1 - p_a^2)fa = \sum_{\substack{v=1 \\ v \neq a}}^{T_1} (p_v^2 - p_v^1)fv$ must be satisfied since $\mathbf{p}_v$ for $T_1 + 1 \leq v \leq T$ are fundamental parameters, and the two choices of the link inertial parameter values determine the same dynamic equations. This contradicts the condition that $f_1, \ldots, f_{T_1}$ are linearly independent since $(p_a^1 - p_a^2) \neq 0$. ∎

Lemma 2.3.2 ${}^i\boldsymbol{SR}_i$ can be represented by

$$
{}^{i}\boldsymbol{SR}_{i} = \begin{bmatrix} \mathbf{R}_i^x \\ \\ \mathbf{R}_i^y - \mathbf{RZB}(i) \\ \\ \mathbf{R}_i^z + \mathbf{RZ}(i) \end{bmatrix} + \boldsymbol{G}_1 \tag{2.28}
$$

for $1 \le i \le N$ where $\boldsymbol{G}_1$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{R}_S^x$ and $\mathbf{R}_S^y - \mathbf{RZB}(s)$ for $i+1 \le s \le N$.

Proof ${}^{i}\boldsymbol{SR}_{i}$ can be described as $\sum_{S=i}^{N} {}^{i}\boldsymbol{A}_S {}^{S}\boldsymbol{R}_S$, and wth entry of ${}^{i}\boldsymbol{SR}_{i}$ is given by

$$
\sum_{S=i}^{N}(({}^{i}A_S)_{w1}\mathbf{R}_S^x + ({}^{i}A_S)_{w2}\mathbf{R}_S^y + ({}^{i}A_S)_{w3}\mathbf{R}_S^z) \tag{2.29}
$$

$w = 1, 2, 3$. Since $({}^{i}A_S)_{w3} = -({}^{i}A_{\beta_{c(s)-1}})_{w2}$ for $\alpha_{c(i)+1} \le s$ from **Property 2.3.1**, we can deform (2.29) as

$$
\begin{aligned}
&\sum_{S=i}^{\beta_{c(i)}}({}^{i}A_S)_{w3}\mathbf{R}_S^z + \sum_{S=i}^{N}(({}^{i}A_S)_{w1}\mathbf{R}_S^x + ({}^{i}A_S)_{w2}\mathbf{R}_S^y) - \sum_{d=c(i)}^{K-1}(({}^{i}A_{\beta_d})_{w2}\sum_{S=\alpha_{d+1}}^{\beta_{d+1}}\mathbf{R}_S^z) \\
&= \sum_{S=i}^{\beta_{c(i)}}({}^{i}A_S)_{w3}\mathbf{R}_S^z + \sum_{S=i}^{N}(({}^{i}A_S)_{w1}\mathbf{R}_S^x + ({}^{i}A_S)_{w2}(\mathbf{R}_S^y - \mathbf{RZB}(s))).
\end{aligned} \tag{2.30}
$$

We can obtain (2.28) from (2.30) using the fact that $({}^{i}A_i)_{wt} = 1$ if $w = t$, $({}^{i}A_i)_{wt} = 0$ if $w \ne t$, and $({}^{i}A_S)_{33} = 1$, $({}^{i}A_S)_{13} = 0$, and $({}^{i}A_S)_{23} = 0$ for $i \le s \le \beta_{c(i)}$ from **Property 2.3.1**. ∎

Lemma 2.3.3

$$
\begin{aligned}
H(i,i) &= \mathbf{J}_i^z + \mathbf{JYB}(i) + \sum_{S=\alpha_{c(i)+1}}^{\beta_{c(i)+1}}(\mathbf{J}_S^x - \mathbf{J}_S^y + \mathbf{JYB}(s))({}^{i}A_S)_{31}^2 \\
&\quad +2\sum_{S=\alpha_{c(i)+1}}^{\beta_{c(i)+1}}(\mathbf{J}_S^{xy} + [L]_S^x\mathbf{RZB}(s))({}^{i}A_S)_{31}({}^{i}A_S)_{32} + G_2
\end{aligned} \tag{2.31}
$$

for $1 \le i \le N$, where G_2 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{J}_S^z + \mathbf{JYB}(s)$, $\mathbf{R}_S^x$, and $\mathbf{R}_S^y - \mathbf{RZB}(s)$ for $s \ge i+1$, $\mathbf{J}_S^{yz} + [L]_S^z\mathbf{RZB}(s)$ and $\mathbf{J}_S^{xz} - [L]_S^x\mathbf{RZ}(s)$ for $s \ge \alpha_{c(i)+1}$, and $\mathbf{J}_S^x - \mathbf{J}_S^y + \mathbf{JYB}(s)$ and $\mathbf{J}_S^{xy} + [L]_S^x\mathbf{RZB}(s)$ for $s \ge \alpha_{c(i)+2}$.

Lemma 2.3.4

$$
\begin{aligned}
H(i,\beta_{c(i)-1}) &= \sum_{s=i}^{\beta_{c(i)}}[(\mathbf{J}_s^{xz} - [L]_s^x\mathbf{RZ}(s))\sin\theta(\alpha_{c(i)},s) \\
&\quad +(\mathbf{J}_s^{yz} + [L]_s^z\mathbf{RZB}(s))\cos\theta(\alpha_{c(i)},s)] \;+\; G_3
\end{aligned}
\tag{2.32}
$$

for $\alpha_2 \le i \le N$, where G_3 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_s^x$ and $\mathbf{R}_s^y - \mathbf{RZB}(s)$ for $s \ge i$ and $\mathbf{J}_s^z + \mathbf{JYB}(s)$, $\mathbf{J}_s^x - \mathbf{J}_s^y + \mathbf{JYB}(s)$, $\mathbf{J}_s^{xy} + [L]_s^x\mathbf{RZB}(s)$, $\mathbf{J}_s^{yz} + [L]_s^z\mathbf{RZB}(s)$ and $\mathbf{J}_s^{xz} - [L]_s^x\mathbf{RZ}(s)$ for $s \ge \alpha_{c(i)+1}$.

Lemma 2.3.5

$$
\begin{aligned}
H(i,j) &= H(i,i) \;+\; \mathbf{R}_i^x\sum_{s=j}^{i-1}[L]_s^x\cos\theta(s+1,i) \\
&\quad -(\mathbf{R}_i^y - \mathbf{RZB}(i))\sum_{s=j}^{i-1}[L]_s^x\sin\theta(s+1,i) \;+\; G_4
\end{aligned}
\tag{2.33}
$$

for $\alpha_{c(i)} \le j < i$, $1 \le i \le N$, where G_4 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_s^x$ and $\mathbf{R}_s^y - \mathbf{RZB}(s)$ for $s \ge i+1$.

Lemma 2.3.6 All the $H(i,j)$ and $\boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ for $1 \le j \le i \le N$ are functions of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1**.

The proofs of **Lemma 2.3.3, 2.3.4, 2.3.5,** and **2.3.6** are given in the Appendix.

(**Proof** of **Theorem 2.3.1**) Represent ${}^i\boldsymbol{g}$ as ${}^i\boldsymbol{g} = [{}^ig_x \;\; {}^ig_y \;\; {}^ig_z]^t$. Using **Lemma 2.3.2** and ${}^i\boldsymbol{z}_i = \boldsymbol{e}_3$ ($\boldsymbol{e}_3 = [0\;\;0\;\;1]^t$) , we can evaluate $\boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ about $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ as

$$
{}^i\boldsymbol{g}\cdot({}^i\boldsymbol{z}_i \times {}^i\boldsymbol{SR}_i) = \mathbf{R}_i^x\;{}^ig_y - (\mathbf{R}_i^y - \mathbf{RZB}(i))\;{}^ig_x \;+\; G \tag{2.34}
$$

where G is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_s^x$ and $\mathbf{R}_s^y - \mathbf{RZB}(s)$ for $s \ge i+1$. It is easy to show that ${}^ig_x$ and ${}^ig_y$ are nonzero independent functions of $\theta_1,\ \theta_2,\ \ldots\ ,\ \theta_i$ for $1 \le i \le N$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$ or for $\alpha_2 \le i \le N$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$. Therefore, if $\mathbf{R}_s^x$ and $\mathbf{R}_s^y - \mathbf{RZB}(s)$ for $s \ge i+1$ are assumed to be fundamental parameters, $\mathbf{R}_i^x$ and $\mathbf{R}_i^y - \mathbf{RZB}(i)$ are also fundamental parameters by **Lemma 2.3.1**. When $i = N$, $G = 0$ in (2.34). It can be derived in the same way that $\mathbf{R}_N^x$ and $\mathbf{R}_N^y$ are fundamental parameters. By the mathematical induction, it is concluded that $\mathbf{R}_i^x$ and $\mathbf{R}_i^y - \mathbf{RZB}(i)$ are fundamental parameters for $1 \le i \le N$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$ or for $\alpha_2 \le i \le N$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$.

It is derived from (2.17) and ${}^N\boldsymbol{z}_N = \boldsymbol{e}_3$ that $H(N,N) = {}^N\boldsymbol{z}_N^t {}^N\boldsymbol{J}_N {}^N\boldsymbol{z}_N = \mathbf{J}_N^z$. $\mathbf{J}_N^z$ is obviously a fundamental parameter. For arbitrary d such that $2 \leq d \leq K$, assume that

$$\mathbf{J}_i^z + \mathbf{JYB}(i) \tag{2.35}$$

for $i \geq \beta_d$ and

$$\mathbf{J}_i^{yz} + [L]_i^z \mathbf{RZB}(i), \quad \mathbf{J}_i^{xz} - [L]_i^x \mathbf{RZ}(i) \tag{2.36}$$

$$\mathbf{J}_i^x - \mathbf{J}_i^y + \mathbf{JYB}(i), \quad \mathbf{J}_i^{xy} + [L]_i^x \mathbf{RZB}(i) \tag{2.37}$$

for $i \geq \alpha_{d+1}$ are proved to be fundamental parameters. Let $\boldsymbol{F}$ be the set of all these fundamental parameters and $\mathbf{R}_i^x$ and $\mathbf{R}_i^y - \mathbf{RZB}(i)$ for $\alpha_2 \leq i \leq N$.

By **Lemma 2.3.4**,

$$\begin{aligned} H(\beta_d, \beta_{d-1}) \; &= (\mathbf{J}_{\beta_d}^{xz} - [L]_{\beta_d}^x \mathbf{RZ}(\beta_d)) \sin\theta(\alpha_d, \beta_d) \\ &+ \; (\mathbf{J}_{\beta_d}^{yz} + [L]_{\beta_d}^z \mathbf{RZB}(\beta_d)) \cos\theta(\alpha_d, \beta_d) \; + \; G \end{aligned} \tag{2.38}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in F. Since $\sin\theta(\alpha_d, \beta_d)$ and $\cos\theta(\alpha_d, \beta_d)$ are linearly independent functions, it is concluded by **Lemma 2.3.1** that $\mathbf{J}_{\beta_d}^{xz} - [L]_{\beta_d}^x \mathbf{RZ}(\beta_d)$ and $\mathbf{J}_{\beta_d}^{yz} + [L]_{\beta_d}^z \mathbf{RZB}(\beta_d)$ are fundamental parameters. Next, we can derive by **Lemma 2.3.3** that

$$H(\beta_d - 1, \beta_d - 1) = \mathbf{J}_{\beta_d - 1}^z + \mathbf{JYB}(\beta_d - 1) \; + \; G \tag{2.39}$$

where G is a function of $\boldsymbol{\theta}$ generated by the fundamental parameters in F. By **Lemma 2.3.1**, $\mathbf{J}_{\beta_d-1}^z + \mathbf{JYB}(\beta_d - 1)$ is a fundamental parameter. Add these new fundamental parameters to the set $\boldsymbol{F}$. Investigating $H(i, \beta_{d-1})$ and $H(i-1, i-1)$ for $i = \beta_d - 1, \ \beta_d - 2, \ \ldots \ , \ \alpha_d + 1$ and $H(\alpha_d, \beta_{d-1})$ in order, we can prove that the inertial parameters in (2.36) for $\alpha_d \leq i \leq \beta_d - 1$ and the inertial parameters in (2.35) for $\alpha_d \leq i \leq \beta_d - 2$ are fundamental parameters by use of the same arguments as above. Add all these new fundamental parameters to $\boldsymbol{F}$.

Next, it can be shown by **Lemma 2.3.3** that

$$\begin{aligned} H(\beta_{d-1}, \beta_{d-1}) \; &= \mathbf{J}_{\beta_{d-1}}^z + \mathbf{JYB}(\beta_{d-1}) \\ &+ \sum_{S=\alpha_d}^{\beta_d} (\mathbf{J}_S^x - \mathbf{J}_S^y + \mathbf{JYB}(s))({}^{\beta_{d-1}}A_S)_{31}^2 \\ &+ 2 \sum_{S=\alpha_d}^{\beta_d} (\mathbf{J}_S^{xy} + [L]_S^x \mathbf{RZB}(s))({}^{\beta_{d-1}}A_S)_{31} ({}^{\beta_{d-1}}A_S)_{32} \\ &+ \; G \end{aligned} \tag{2.40}$$

where G is a function of $\boldsymbol{\theta}$ generated by the fundamental parameters in $\boldsymbol{F}$. It is obvious from **Property 2.3** that $({}^{\beta_{d-1}}A_S)_{31} = \sin\theta(\alpha_d, s)$ and $({}^{\beta_{d-1}}A_S)_{32} = \cos\theta(\alpha_d, s)$ for $\alpha_d \le s \le \beta_d$. Constant function, $\sin^2\theta(\alpha_d, s)$ and $\sin\theta(\alpha_d, s)\cos\theta(\alpha_d, s)$ for $\alpha_d \le s \le \beta_d$ can be easily proved to be linearly independent functions. We can conclude by **Lemma 2.3.1** that $\mathbf{J}^z_{\beta_{d-1}} + \mathbf{JYB}(\beta_{d-1})$ and the inertial parameters in (2.37) for $\alpha_d \le i \le \beta_d$ are fundamental parameters. Add these new fundamental parameters to $\boldsymbol{F}$.

Now, the inertial parameters in (2.35) for $i \ge \beta_{d-1}$ and the inertial parameters in (2.36) and (2.37) for $i \ge \alpha_d$ have been shown to be fundamental parameters. Using the mathematical induction from $d = K$ to $d = 2$, we can prove that the inertial parameters in (2.35) for $i \ge \beta_1$ and the inertial parameters in (2.36) and (2.37) for $i \ge \alpha_2$ are fundamental parameters. Add all these fundamental parameters to $\boldsymbol{F}$.

Next, it is derived by **Lemma 2.3.5** that

$$\begin{aligned} H(\beta_1, \beta_1 - 1) &= H(\beta_1, \beta_1) + \mathbf{R}^x_{\beta-1}[L]^x_{\beta_1-1}\cos\theta_{\beta_1} \\ &\quad - (\mathbf{R}^y_{\beta-1} - \mathbf{RZB}(\beta_1))[L]^x_{\beta_1-1}\sin\theta_{\beta_1} + G \end{aligned} \tag{2.41}$$

where G is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}^x_S$ and $\mathbf{R}^y_S - \mathbf{RZB}(s)$ for $s \ge \alpha_2$ that are included in $\boldsymbol{F}$. From the above arguments and (2.40), $H(\beta_1, \beta_1)$ is a function of $\boldsymbol{\theta}$ generated by the fundamental parameters in $\boldsymbol{F}$. Since $\cos\theta_{\beta_1}$ and $\sin\theta_{\beta_1}$ are linearly independent functions and $[L]^x_{\beta_1-1} \ne 0$ is assumed, $\mathbf{R}^x_{\beta-1}$ and $\mathbf{R}^y_{\beta-1} - \mathbf{RZB}(\beta_1)$ are fundamental parameters by **Lemma 2.3.1**. Add these fundamental parameters to $\boldsymbol{F}$. It can be derived by **Lemma 2.3.3** that

$$H(\beta_1 - 1, \beta_1 - 1) = \mathbf{J}^z_{\beta_1-1} + \mathbf{JYB}(\beta_1 - 1) + G \tag{2.42}$$

where G is a function of $\boldsymbol{\theta}$ generated by the fundamental parameters in $\boldsymbol{F}$. $\mathbf{J}^z_{\beta_1-1} + \mathbf{JYB}(\beta_1 - 1)$ is obviously a fundamental parameter by **Lemma 2.3.1**. Add this to $\boldsymbol{F}$. Investigating $H(i, i-1)$ and $H(i-1, i-1)$ for $i = \beta_1 - 1,\ \beta_1 - 2,\ \ldots\ , 2$ in order, we can prove by iterative use of the above argument that $\mathbf{J}^z_i + \mathbf{JYB}(i)$ for $1 \le i \le \beta_1$ and $\mathbf{R}^x_i$ and $\mathbf{R}^y_i - \mathbf{RZB}(i)$ for $2 \le i \le \beta_1$ are fundamental parameters. In the case that $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$, it has been already proved that $\mathbf{R}^x_i$ and $\mathbf{R}^y_i - \mathbf{RZB}(i)$ for $i \ge 1$ are fundamental parameters.

Now, we have proved that all the inertial parameters given in **Theorem 2.3.1** are fundamental parameters. These inertial parameters are obviously linearly independent since each of them includes at least one link inertial parameter which does not appear in the others. We can conclude by **Lemma 2.3.6** that these inertial parameters generate the dynamic equations (2.16). Thus, the set of all the inertial parameters given in **Theorem 2.3.1** constitutes a base parameter set. The number of base parameters in this base parameter set is evident. ∎

From the **Theorem 2.3.1** and **Property 2.3.11** we can conclude that the minimum number of the inertial parameters whose values can determine the dynamic model uniquely is $7N - 4\beta_1$ ($7N - 4\beta_1 - 2$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$). This minimum number can be interpreted as follows. The mass $\boldsymbol{m}_i$ appears alone in neither $H(i,j)$ nor $\boldsymbol{g} \cdot (\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ in the sense of zeroth order moment since all the joints are rotational. $\mathbf{R}_i^z$ or $\mathbf{J}_i^y$ can always be grouped with other inertial parameters. The mechanism of the grouping will be shown in the proofs of **Lemmas 2.3, 2,4,** and **2.5** in the Appendix. Since the links in the first-link cluster rotate only around the joint axes parallel to $\boldsymbol{z}_1$, second-order moments except for $\mathbf{J}_i^z$, do not appear in the dynamic equations. If $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$, $\mathbf{R}_1^x$ and $\mathbf{R}_1^y - \mathbf{RZB}(1)$ appears in neither $H(i,j)$ nor $\boldsymbol{g} \cdot (\boldsymbol{z}_i \times \boldsymbol{SR}_i)$.

$H(i,j)$ relates $\ddot{\theta}_j$ to τ_i and it is a function of $\boldsymbol{\theta}$ which represents the attitude of the manipulator. From the proof of **Theorem 2.3.1**, we can guess a procedure to estimate the base parameter values given in **Theorem 2.3.1** as well as desirable joint motions and manipulator attitudes for accurate estimations.

2.3.3 Conclusion

The base parameter set as a minimum set of inertial parameters which can generate the dynamic model is investigated for a general parallel and perpendicular manipulator with rotational joints only. This is also regarded as a parametrization of the manipulator dynamic model. Base parameters can be identified from link motion and joint torque data. A base parameter set such that every base parameter is described by the link inertial parameters directly and completely in closed form is given, and the exact number of the base parameters in the set is also evaluated. Any base parameter set can be obtained from the base parameter set by a nonsingular linear transformation.

The proof of **Theorem 2.3.1** gives good understanding of the relation between the base parameters and the manipulator motions, which is useful for efficient and accurate identification of the dynamic model. The notion of base parameter is so fundamental that it will be helpful for other problems related to the manipulator dynamic models like the inverse dynamics problem.

2.4 Base Parameters for Manipulators with Rotational and Translational Joints

The manipulators considered in preceding section are assumed to have rotational joints only. Then, in this section we extend the results of the preceding section to manipulators with rotational and translational joints. We show a base parameter set for the manipulators, describing every base parameter in a linear combination of the link inertial parameters directory and completely in closed form. Also, thereby, we give the exact number of the base parameters.

2.4.1 Dynamic Models of the Manipulators

We consider the manipulator that has a open-loop kinematic chain, each link of which is connected to the predecessor (base for the first link) by a rotational or translational joint and is called rotational or translational link, respectively. Let N be the number of all the rotational links in the manipulator. The n-th rotational link from the base is labeled as $(n, 0)$ for $1 \leq n \leq N$. The base is labeled as $(0, 0)$, regarding it as a rotational link. Base coordinate system $(o_0; \boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{z}_0)$ is set arbitrarily, and $\boldsymbol{z}_0$ is supposed as $(0, 0)$ joint axis. Let T_n be the number of all the translational links between rotational links $(n, 0)$ and $(n+1, 0)$ for $0 \leq n \leq N-1$ or succeeding to rotational link $(N, 0)$ for $n = N$. The t-th translational link from rotational link $(n, 0)$ is labeled as (n, t) for $0 \leq n \leq N, \quad 1 \leq t \leq T_n$. Thus the total number of links $\overline{N}$ is $\sum_{n=0}^{N}(1 + T_n) - 1$. Here we assume that the axes of rotational joints $(n, 0)$ and $(n+1, 0)$ are parallel or perpendicular for $1 \leq n \leq N-1$. Note that relation between the direction of the axes of joints $(n, 0)$ and $(n+1, 0)$ never change for whatever displacements along the translational joint axes between the two rotational joints axes.

To describe manipulator motions, we attach coordinate system $(o_n; \boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n)$ to rotational link $(n, 0)$ and coordinate system $(o_{n,t}; \boldsymbol{x}_{n,t}, \boldsymbol{y}_{n,t}, \boldsymbol{z}_{n,t})$ to translational link (n, t) in the way shown in Fig. 2.4.1. For translational links, $\boldsymbol{z}_{n,t}$ is along joint (n, t) axis, and $\boldsymbol{x}_{n,t}$ and $\boldsymbol{y}_{n,t}$ are chosen arbitrarily to complete a right-hand coordinate system. The origin $o_{n,t}$ of $(o_{n,t}; \boldsymbol{x}_{n,t}, \boldsymbol{y}_{n,t}, \boldsymbol{z}_{n,t})$ is chosen on joint (n, t) axis arbitrarily. We set a reference point $p_{n,t}$ on joint (n, t) axis arbitrarily, which is fixed to link $(n, t-1)$. The amount of translation about translational joint (n, t) is considered as the distance $q_{n,t}$ from $p_{n,t}$ to $o_{n,t}$. For rotational links, setting the position of every translational link $(n-1, t)$ between rotational links $(n-1, 0)$ and $(n, 0)$ on its reference point i.e. $p_{n-1,t} = o_{n-1,t}$ or $q_{n-1,t} = 0$ for $1 \leq t \leq T_{n-1}$, we set the origin o_n of $(o_n; \boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n)$ at the intersection of joint $(n, 0)$ axis and the common normal to the axes of joints $(n-1, 0)$ and $(n, 0)$. $\boldsymbol{z}_n$ direct along joint $(n, 0)$ axis. $\boldsymbol{x}_n$ directs along the common normal from joint $(n, 0)$ axis to joint $(n+1, 0)$ axis. $\boldsymbol{y}_n$ is chosen to complete a right-hand coordinate system. Rotational joint angle θ_n is the angle between $\boldsymbol{x}_{n-1}$ and $\boldsymbol{x}_n$ mesured in the right-hand sense about $\boldsymbol{z}_n$. For $0 \leq n \leq N, \quad 0 \leq t \leq T_n$ and $t \neq T_n$ the vectors from $o_{n,t}$ (o_n if $t = 0$) to $p_{n,t+1}$(o_{n+1} if $t = T_n$) or to $o_{n,t+1}$ (o_{n+1} if $t = T_n$) are denoted by $\boldsymbol{L}_{n,t}$ or $\overline{\boldsymbol{L}}_{n,t}$, respectively. Note that $\boldsymbol{L}_{n,t}$ is a constant vector and the following relations are satisfied:

$$\overline{\boldsymbol{L}}_{n,t} = \boldsymbol{L}_{n,t} + q_{n,t+1}\boldsymbol{z}_{n,t+1} \tag{2.43}$$

for $0 \leq n \leq N, \quad 0 \leq t \leq T_{n-1}$,

$$\overline{\boldsymbol{L}}_{n,T_n} = \boldsymbol{L}_{n,T_n} \tag{2.44}$$

for $0 \leq n \leq N-1$.

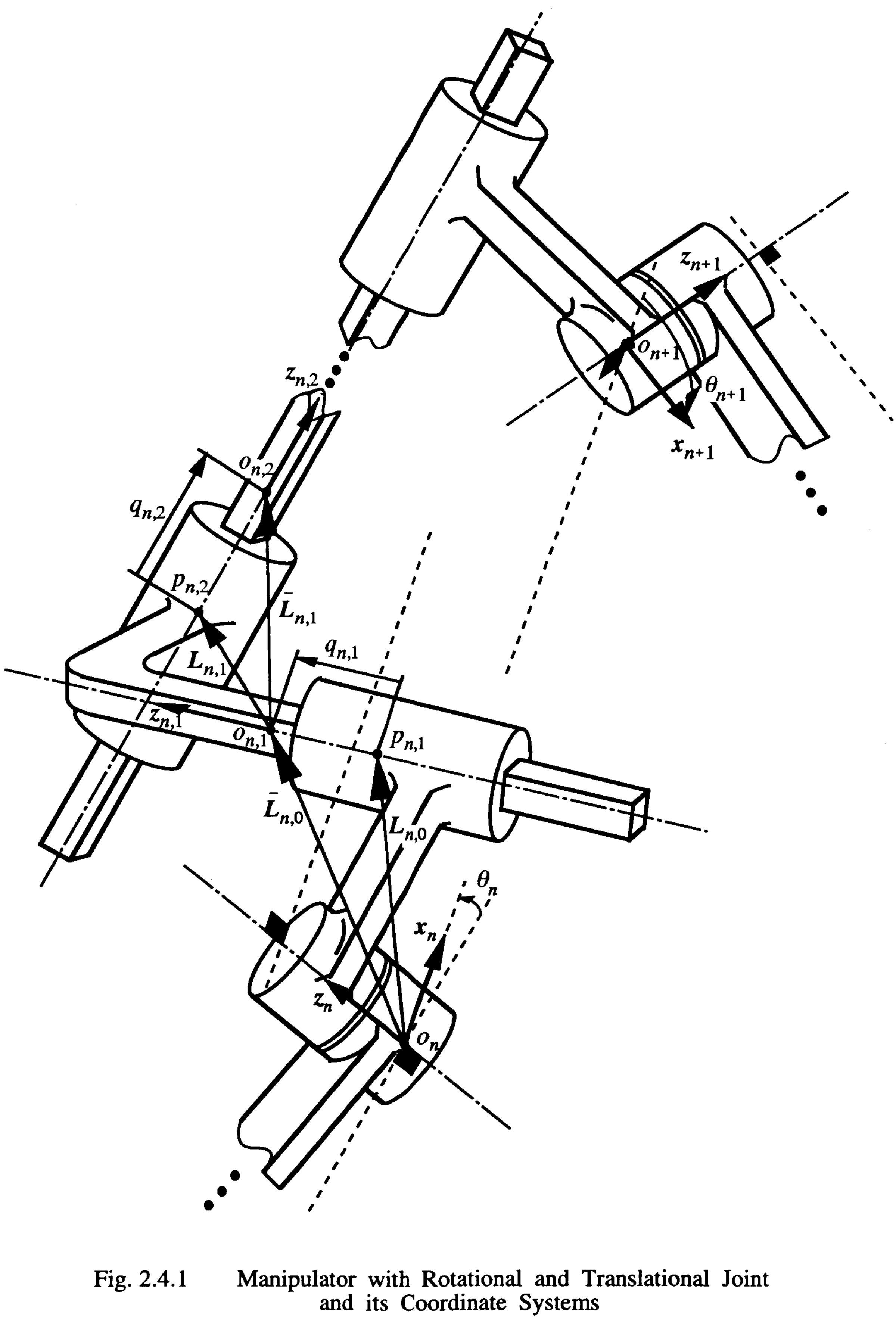


Fig. 2.4.1 Manipulator with Rotational and Translational Joint and its Coordinate Systems

For each link (n, t) let $\mathbf{m}_{n,t}$ be the mass, $\boldsymbol{I}_{n,t}$ be the moment of inertia tensor around the origin $o_{n,t}$ and $\boldsymbol{r}_{n,t}$ be the vector from $o_{n,t}$ to the center of mass.

Since the axes of rotational joints $(n, 0)$ and $(n+1, 0)$ are parallel or perpendicular for $1 \leq n \leq N-1$, showing in Fig. 2.4.2, we can divide the whole manipulator links into link clusters as follows; Let $a(1) = 1$ and $2 \leq a(2) < \cdots < a(K)$ be numbers such that joint $(a(d), 0)$ axis is perpendicular to joint $(a(d)-1, 0)$ axis for $2 \leq d \leq K$, and define $b(d) = a(d+1) - 1$ for $1 \leq d \leq K-1$ and $b(K) = N$. Then all the links from $(a(d), 0)$ to $(b(d), T_{b(d)})$ constitute d-th link cluster in which all the rotational joint axes are parallel. Translational links $(0,1), (0,2), \ldots, (0, T_0)$ are considered to be 0-th link cluster. $(d-1)$-th and d-th link clusters are connected by joint $(a(d), 0)$. Thus the number of link clusters in the manipulator is $K+1$. When rotational link $(n, 0)$ is included in d-th link cluster , we define $k(n)$ as $k(n) = d$.

After attaching coordinate systems to links, we have rotation matrices. ${}^n\boldsymbol{A}_{n,t} = [{}^n\boldsymbol{x}_{n,t}$ ${}^n\boldsymbol{y}_{n,t}$ ${}^n\boldsymbol{z}_{n,t}]^t$ represents the rotation of $(\boldsymbol{x}_{n,t}, \boldsymbol{y}_{n,t}, \boldsymbol{z}_{n,t})$ to $(\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n)$. Each element of the matrix will be denoted by

$$
{}^n\boldsymbol{A}_{n,t} = \begin{bmatrix} +[A_{n,t}]_{11} & [A_{n,t}]_{12} & [A_{n,t}]_{13} \\ +[A_{n,t}]_{21} & [A_{n,t}]_{22} & [A_{n,t}]_{23} \\ +[A_{n,t}]_{31} & [A_{n,t}]_{32} & [A_{n,t}]_{33} \end{bmatrix}. \tag{2.45}
$$

${}^n\boldsymbol{A}_{n,t}$ is a constant matrix since relative motions between $(n, 0)$ and (n, t) links are only translational. The matrix ${}^j\boldsymbol{A}_i = \begin{bmatrix} {}^j\boldsymbol{x}_i & {}^j\boldsymbol{y}_i & {}^j\boldsymbol{z}_i \end{bmatrix}^t$ represents the rotation of $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ to $(\boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{z}_j)$, both coordinate systems are attached to rotational links. Hence, they are same as the rotation matrices defined in section 3, then each elements of the matrices will be denoted by same symbols as in the section 3. Let us define $\theta(i, j) = \theta_i + \theta_{i+1} + \cdots \theta_j$ for $i \leq j$. Then, it can be easily shown that the rotation matrices have following property.

Property 2.4.1

i) For $a(d) \leq i < j < b(d)$ where $1 \leq d \leq K$,

$$
{}^i\boldsymbol{A}_j = \begin{bmatrix} \cos\theta(i+1, j) & -\sin\theta(i+1, j) & 0 \\ \sin\theta(i+1, j) & \cos\theta(i+1, j) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.46}
$$

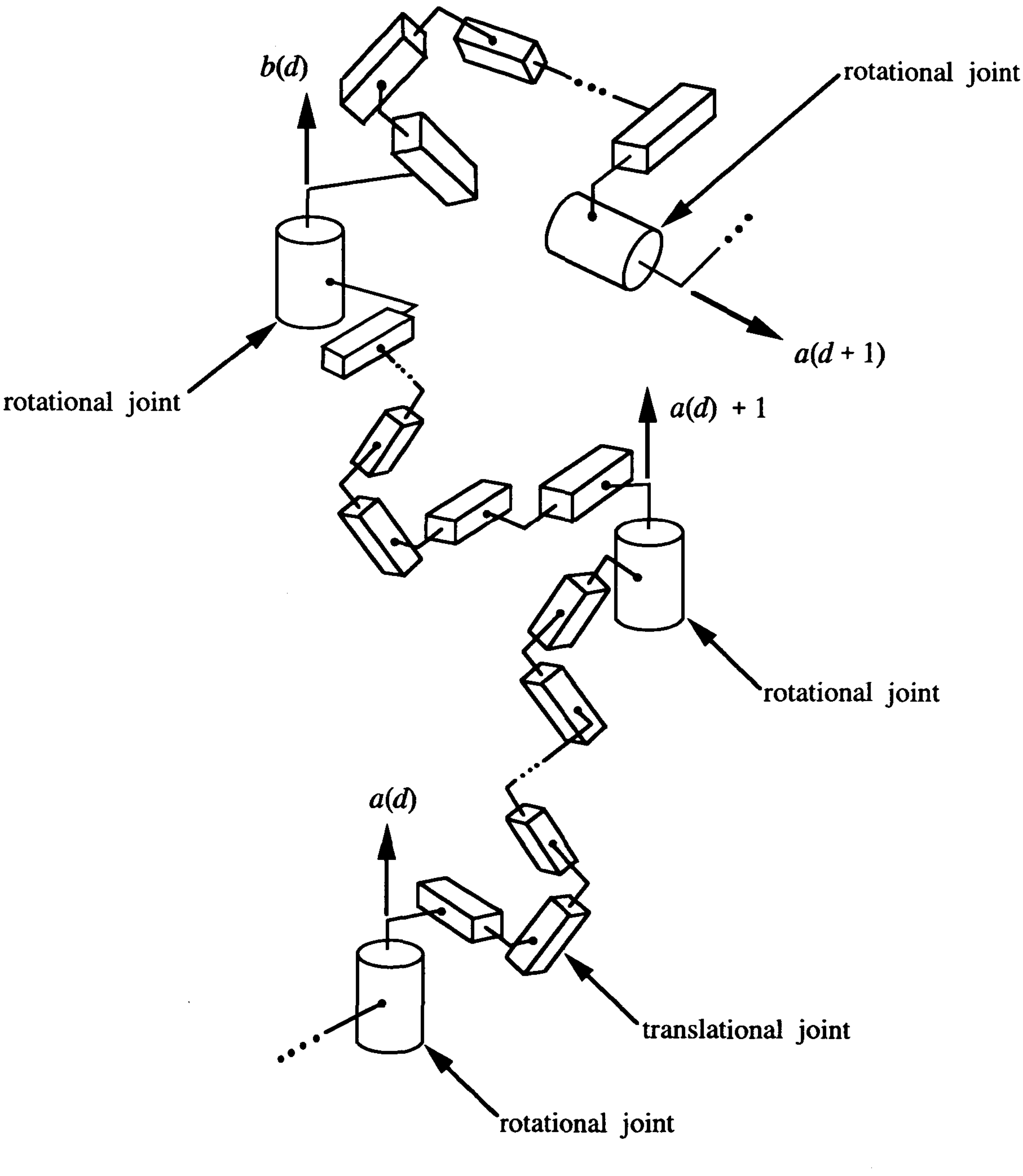


Fig. 2.4.2 d-th Link Cluster in the Manipulator

ii) For $a(2) \le i \le N$,

$$
{}^{b(k(i)-1)}\boldsymbol{A}_i = \begin{bmatrix} \cos\theta(a(k(i)),i) & -\sin\theta(a(k(i)),i) & 0 \\ 0 & 0 & -1 \\ \sin\theta(a(k(i)),i) & \cos\theta(a(k(i)),i) & 0 \end{bmatrix}. \tag{2.47}
$$

iii) When $k(i) < k(j)$,

$$
(\text{3rd column of } {}^{i}\boldsymbol{A}_j) = -\begin{bmatrix} ({}^{i}A_{b(k(j)-1)})_{12} \\ ({}^{i}A_{b(k(j)-1)})_{22} \\ ({}^{i}A_{b(k(j)-1)})_{32} \end{bmatrix}. \tag{2.48}
$$

${}^{(n,t)}\boldsymbol{L}_{n,t}, {}^{n}\boldsymbol{A}_{n,t}$ for $0 \le n \le N, 0 \le t \le T_n$ and type of ${}^{n-1}\boldsymbol{A}_n$ for $1 \le n \le N$ are obtained from the kinematic parameter values of the manipulator, and they all are assumed to be known.

We introduce following notation for later argument.

$$
\mathrm{M}_{n,t} = \sum_{t_1=t}^{T_n} \mathrm{m}_{n,t_1} + \sum_{i=n+1}^{N} \sum_{t_1=0}^{T_i} \mathrm{m}_{i,t_1} \tag{2.49}
$$

$$
\boldsymbol{R}_{n,t} = \mathrm{m}_{n,t}\boldsymbol{r}_{n,t} + \mathrm{M}_{n,t+1}\boldsymbol{L}_{n,t}, \quad \overline{\boldsymbol{R}}_{n,t} = \mathrm{m}_{n,t}\boldsymbol{r}_{n,t} + \mathrm{M}_{n,t+1}\overline{\boldsymbol{L}}_{n,t} \tag{2.50}
$$

$$
\boldsymbol{J}_{n,t} = \boldsymbol{I}_{n,t} + \mathrm{M}_{n,t+1}(\boldsymbol{L}_{n,t} \cdot \boldsymbol{L}_{n,t}\boldsymbol{E} - \boldsymbol{L}_{n,t} \otimes \boldsymbol{L}_{n,t}) \tag{2.51}
$$

$\mathrm{M}_{n,t}$, $\boldsymbol{R}_{n,t}$, $\boldsymbol{J}_{n,t}$ are moments of order $0, 1, 2$ for augmented link of link (n,t) around $o_{n,t}$ (o_n if $t = 0$), respectively. Moreover we define followings:

$$
\boldsymbol{RC}_n = \sum_{t_1=0}^{T_n} \boldsymbol{R}_{n,t_1}, \qquad \overline{\boldsymbol{RC}}_n = \sum_{t_1=0}^{T_n} \overline{\boldsymbol{R}}_{n,t_1} \tag{2.52}
$$

$$
\boldsymbol{RC}_{n,t} = \sum_{t_1=t}^{T_n} \boldsymbol{R}_{n,t_1}, \qquad \overline{\boldsymbol{RC}}_{n,t} = \sum_{t_1=t}^{T_n} \overline{\boldsymbol{R}}_{n,t_1} \tag{2.53}
$$

$$
\boldsymbol{SR}_n = \sum_{i=n}^{N} \boldsymbol{RC}_i, \qquad \overline{\boldsymbol{SR}}_n = \sum_{i=n}^{N} \overline{\boldsymbol{RC}}_i \tag{2.54}
$$

$$
\boldsymbol{SR}_{n,t} = \boldsymbol{RC}_{n,t} + \boldsymbol{SR}_{n+1}, \qquad \overline{\boldsymbol{SR}}_{n,t} = \overline{\boldsymbol{RC}}_{n,t} + \overline{\boldsymbol{SR}}_{n+1} \tag{2.55}
$$

$$\boldsymbol{L}_n = \sum_{t=0}^{T_n} \boldsymbol{L}_{n,t}, \qquad \overline{\boldsymbol{L}}_n = \sum_{t=0}^{T_n} \overline{\boldsymbol{L}}_{n,t}. \tag{2.56}$$

$\overline{\boldsymbol{L}}_n$ is the vector from o_n to o_{n+1}, and $\boldsymbol{L}_n$ is that when $q_{n,t} = 0$ for $1 \le t \le T_n$. $\boldsymbol{L}_n$ is a constant vector. For (n_1, t_1) and (n_2, t_2) links we denote $(n_1, t_1) < (n_2, t_2)$ if (n_1, t_1) link is predecessor to (n_2, t_2) link i.e. $n_1 < n_2$ or $t_1 < t_2$ when $n_1 = n_2$. In this case the vector from $o_{n_1,t_1}(o_{n_1}$ if $t_1 = 0)$ to $o_{n_2,t_2}(o_{n_2}$ if $t_2 = 0)$ is defined as

$$\overline{\boldsymbol{L}}_{(n_1,t_1)(n_2,t_2)} = \sum_{t=t_1}^{T_{n_1}} \overline{\boldsymbol{L}}_{n_1,t} + \sum_{n=n_1+1}^{n_2-1} \overline{\boldsymbol{L}}_n + \sum_{t=0}^{t_2-1} \overline{\boldsymbol{L}}_{n_2,t} \tag{2.57}$$

or, when $q_{n_1,t} = 0$ for $t_1 + 1 \le t \le T_{n_1}$, $q_{n,t} = 0$ for $n_1 + 1 \le n \le n_2 - 1$, $1 \le t \le T_n$ and $q_{n_2,t} = 0$ for $1 \le t \le t_2$,

$$\boldsymbol{L}_{(n_1,t_1)(n_2,t_2)} = \sum_{t=t_1}^{T_{n_1}} \boldsymbol{L}_{n_1,t} + \sum_{n=n_1+1}^{n_2-1} \boldsymbol{L}_n + \sum_{t=0}^{t_2-1} \boldsymbol{L}_{n_2,t}. \tag{2.58}$$

Finally, we define

$$\begin{aligned} \boldsymbol{J}_n = \sum_{t=0}^{T_n} \boldsymbol{J}_{n,t} + \sum_{t=0}^{T_n-1} (\ & 2\boldsymbol{L}_{n,t} \cdot \boldsymbol{RC}_{n,t+1}\boldsymbol{E} \\ & -\boldsymbol{L}_{n,t} \otimes \boldsymbol{RC}_{n,t+1} - \boldsymbol{RC}_{n,t+1} \otimes \boldsymbol{L}_{n,t}). \end{aligned} \tag{2.59}$$

When $o_{n,t}$ is fixed on $p_{n,t}$ for all $1 \le t \le T_n$, Links $(n,0), (n,1), \ldots, (n, T_n)$ can be considered as a rigid rotational link. It can be proved that $\boldsymbol{J}_n$ is the moment of inertia tensor of this composite rotational link. To construct a dynamic model of the manipulator, we adopt Lagrangian formulation. By length but straightforward operations, following coordinate free expressions for the dynamic equations of manipulator motions are derived:

(i) For input torque τ_n generated in rotational joint $(n, 0)$ around the joint axis

$$\begin{aligned}
\tau_n &= \sum_{i=1}^{N} H_1(n,i)\ddot{\theta}_i + \sum_{i=0}^{N}\sum_{t=1}^{T_i} H_2(n,(i,t))\ddot{q}_{i,t} \\
&+\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial H_1(n,i)}{\partial\theta_j} - \frac{\partial H_1(j,i)}{2\partial\theta_n}\right)\dot{\theta}_i\dot{\theta}_j \\
&+\sum_{i=0}^{N}\sum_{t=1}^{T_i}\sum_{j=1}^{N}\left(\frac{\partial H_2(n,(i,t))}{\partial\theta_j} - \frac{\partial H_2(j,(i,t))}{\partial\theta_n}\right)\dot{\theta}_j\dot{q}_{i,t} \\
&+\sum_{i=1}^{N}\sum_{j=0}^{N}\sum_{t=1}^{T_j}\left(\frac{\partial H_1(n,i)}{\partial q_{j,t}}\right)\dot{\theta}_i\dot{q}_{j,t} \\
&+\sum_{i=0}^{N}\sum_{t_1=1}^{T_i}\sum_{j=0}^{N}\sum_{t_2=1}^{T_j}\left(\frac{\partial H_2(n,(i,t_1))}{\partial q_{j,t_2}} - \frac{\partial H_3((j,t_2),(i,t_1))}{\partial\theta_n}\right)\dot{q}_{i,t_1}\dot{q}_{j,t_2} \\
&-g\cdot(\boldsymbol{z}_n \times \overline{\boldsymbol{SR}_n})
\end{aligned} \tag{2.60}$$

for $1 \le n \le N$.

(ii) For force $f_{n,t}$ generated in translational joint (n, t) along the joint axis

$$\begin{aligned}
f_{n,t} &= \sum_{i=1}^{N} H_2(i,(n,t))\ddot{\theta}_i + \sum_{i=0}^{N}\sum_{t=1}^{T_i} H_3((i,t_1),(n,t))\ddot{q}_{i,t_1} \\
&+\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial H_2(i,(n,t))}{\partial\theta_j} - \frac{\partial H_1(j,i)}{2\partial q_{n,t}}\right)\dot{\theta}_i\dot{\theta}_j \\
&+\sum_{i=0}^{N}\sum_{t_1=1}^{T_i}\sum_{j=1}^{N}\left(\frac{\partial H_3((i,t_1),(n,t))}{\partial\theta_j} - \frac{\partial H_2(j,(i,t_1))}{\partial q_{n,t}}\right)\dot{\theta}_j\dot{q}_{i,t_1} \\
&+\sum_{i=1}^{N}\sum_{j=0}^{N}\sum_{t_1=1}^{T_j}\left(\frac{\partial H_2(i,(n,t))}{\partial q_{j,t_1}}\right)\dot{\theta}_i\dot{q}_{j,t_1} \\
&+\sum_{i=0}^{N}\sum_{t_1=1}^{T_i}\sum_{j=0}^{N}\sum_{t_2=1}^{T_j}\left(\frac{\partial H_3((i,t_1),(n,t))}{\partial q_{j,t_2}} - \frac{\partial H_3((j,t_2),(i,t_1))}{2\partial q_{n,t}}\right)\dot{q}_{i,t_1}\dot{q}_{j,t_2} \\
&-\mathbf{M}_{n,t}\,\boldsymbol{g}\cdot\boldsymbol{z}_{n,t}
\end{aligned} \tag{2.61}$$

for $0 \le n \le N$, $1 \le t \le T_n$.

where

$$
\begin{aligned}
H_1(i,j) &= \boldsymbol{z}_i \cdot (\sum_{n=i}^{N} \boldsymbol{J}_n)\boldsymbol{z}_j \\
&+ \boldsymbol{z}_i \cdot [\sum_{n=i}^{N-1} (2\boldsymbol{L}_n \cdot \boldsymbol{SR}_{n+1}\boldsymbol{E} - \boldsymbol{L}_n \otimes \boldsymbol{SR}_{n+1} - \boldsymbol{SR}_{n+1} \otimes \boldsymbol{L}_n)]\boldsymbol{z}_j \\
&+ \boldsymbol{z}_i \cdot [\boldsymbol{L}_{(j,0)(i,0)} \cdot \boldsymbol{SR}_i\boldsymbol{E} - \boldsymbol{L}_{(j,0)(i,0)} \otimes \boldsymbol{SR}_i]\boldsymbol{z}_j \\
&+ \boldsymbol{z}_i \cdot [\sum_{n=i}^{N} \sum_{t=1}^{T_n} q_{n,t}(2\boldsymbol{z}_{n,t} \cdot \boldsymbol{SR}_{n,t}\boldsymbol{E} - \boldsymbol{z}_{n,t} \otimes \boldsymbol{SR}_{n,t} \\
&\quad - \boldsymbol{SR}_{n,t+1} \otimes \boldsymbol{z}_{n,t})]\boldsymbol{z}_j \\
&+ \boldsymbol{z}_i \cdot [\sum_{n=j}^{i-1} \sum_{t=1}^{T_n} q_{n,t}(\boldsymbol{z}_{n,t} \cdot \boldsymbol{SR}_i\boldsymbol{E} - \boldsymbol{z}_{n,t} \otimes \boldsymbol{SR}_i)]\boldsymbol{z}_j \\
&+ \sum_{n=i}^{N} \sum_{t=1}^{T_n} \alpha_{n,t}\mathbf{M}_{n,t}
\end{aligned}
\tag{2.62}
$$

for $j \leq i$ and $H_1(i,j) = H_1(j,i)$ for $i < j$, where $\alpha_{n,t}$ is a scaler generated from $\boldsymbol{z}_i, \boldsymbol{z}_j$, and $q_{n,t}\boldsymbol{z}_{n,t}$ for $j \leq n \leq N,\ 1 \leq t \leq T_n$ and $\boldsymbol{L}_{n,t}$ for $j \leq n \leq N,\ 0 \leq t \leq T_n$ by vector operations.

$$
H_2(i,(j,t)) = \begin{cases} (\boldsymbol{z}_i \times \overline{\boldsymbol{SR}}_i) \cdot \boldsymbol{z}_{j,t}, & \text{if } j < i \\ [\boldsymbol{z}_i \times (\mathbf{M}_{j,t}\overline{\boldsymbol{L}}_{(i,0)(j,t)} + \overline{\boldsymbol{SR}}_{j,t})] \cdot \boldsymbol{z}_{j,t}, & \text{if } i \leq j \end{cases}
\tag{2.63}
$$

for $1 \leq i \leq N,\ 0 \leq j \leq N,\ 1 \leq t \leq T_j$,

$$
H_3((i,t_1),(j,t_2)) = \mathbf{M}_{i,t_1}\boldsymbol{z}_{i,t_1} \cdot \boldsymbol{z}_{j,t_2}
\tag{2.64}
$$

for $0 \leq i,j \leq N,\ 1 \leq t_1 \leq T_i,\ 1 \leq t_2 \leq T_j,\ (j,t_j) \leq (i,t_i)$ and $H_3((i,t_i),(j,t_j)) = H_3((j,t_j),(i,t_i))$ for $(i,t_i) \leq (j,t_j)$, and $\boldsymbol{g}$ is the gravity vector.

It is obvious that the dynamic equations (2.60),(2.61) can be determined if terms $H_1(i,j)$ for $1 \leq i,j \leq N$, $H_2(i,(j,t))$ for $1 \leq i \leq N$, $0 \leq j \leq N$, $1 \leq t \leq T_j$, $H_3((i,t_1),(j,t_2))$ for $0 \leq i,j \leq N$, $1 \leq t_1 \leq T_i$, $1 \leq t_2 \leq T_j$, $\boldsymbol{g} \cdot (z_i \times \overline{\boldsymbol{SR}}_i)$ for $1 \leq i \leq N$ and

$\mathbf{M}_{i,t}\boldsymbol{g} \cdot z_{i,t}$ for $0 \leq i \leq N, \quad 1 \leq t \leq T_i$ are given as functions of θ_n for $1 \leq n \leq N$ and $q_{n,t}$ for $0 \leq n \leq N, \quad 1 \leq t \leq T_n$.

Evaluating each of them about an appropriate coordinate system, we can describe them in the following form:

$$\sum_{u=1}^{U} \boldsymbol{p}_u f_u \tag{2.65}$$

where $\boldsymbol{p}_u$ is an inertial parameter and f_u is a polynomial of $q_{n,t}$ for $0 \leq n \leq N, \quad 1 \leq t \leq t_n$ and trigonometric function of θ_n for $1 \leq n \leq N$. (f_u is allowed to be a constant function.) The form (2.65) will be called a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_U$ in later discussions.

2.4.2 A Base Parameter Set

In this subsection we show a base parameter set for the dynamic model (2.60) and (2.61). To describe the base parameters we introduce the following notation; Since ${}^{(n,0)}\boldsymbol{L}_n$ is a constant vector, ${}^{(n,0)}\boldsymbol{L}_n$ is denoted by

$${}^{(n,0)}\boldsymbol{L}_n = \begin{bmatrix} [L_n]^x & 0 & [L_n]^z \end{bmatrix}^t. \tag{2.66}$$

Since ${}^{(n,t)}\boldsymbol{R}_{n,t}$ is a constant vector, ${}^{(n,t_1)}\boldsymbol{R}_{n,t_2} = {}^{n}\boldsymbol{A}_{n,t_1}^t \, {}^{n}\boldsymbol{A}_{n,t_2} {}^{(n,t_2)}\boldsymbol{R}_{n,t_2}$ is a constant vector. ${}^{(n,t)}\boldsymbol{RC}_{n,t}$ and ${}^{(n,0)}\boldsymbol{RC}_n$ are also constant vectors, then they will be denoted by

$$\begin{aligned} {}^{(n,t)}\boldsymbol{RC}_{n,t} &= \begin{bmatrix} \mathrm{RC}_{n,t}^x & \mathrm{RC}_{n,t}^y & \mathrm{RC}_{n,t}^z \end{bmatrix}^t \\ {}^{(n,0)}\boldsymbol{RC}_n &= \begin{bmatrix} \mathrm{RC}_n^x & \mathrm{RC}_n^y & \mathrm{RC}_n^z \end{bmatrix}^t. \end{aligned} \tag{2.67}$$

${}^{(n,0)}\boldsymbol{J}_n$ is also a constant matrix, and will be denoted by

$${}^{(n,0)}\boldsymbol{J}_n = \begin{bmatrix} \mathrm{J}_n^x & \mathrm{J}_n^{xy} & \mathrm{J}_n^{xz} \\ \mathrm{J}_n^{xy} & \mathrm{J}_n^y & \mathrm{J}_n^{yz} \\ \mathrm{J}_n^{xz} & \mathrm{J}_n^{yz} & \mathrm{J}_n^z \end{bmatrix}. \tag{2.68}$$

To simplify descriptions of the base parameters, following inertial parameters are introduced:

$$\mathbf{RCZ}(n) = \begin{cases} 0, & \text{if } n = b(d)\ (1 \le d \le K) \\ \sum_{i=n+1}^{b(k(n))} \mathbf{RC}_i^z, & \text{otherwise} \end{cases} \tag{2.69}$$

$$\mathbf{RCZB}(n) = \begin{cases} \sum_{i=a(d+1)}^{b(d+1)} \mathbf{RC}_i^z, & \text{if } n = b(d)\ (1 \le d \le K) \\ 0, & \text{otherwise} \end{cases} \tag{2.70}$$

Using these we modify some parameters:

$$\hat{\mathbf{J}}_n^{xy} = \mathbf{J}_n^{xy} + [L_n]^x \mathbf{RCZB}(n) \tag{2.71}$$

$$\hat{\mathbf{J}}_n^{xz} = \mathbf{J}_n^{xz} + [L_n]^x \mathbf{RCZ}(n) \tag{2.72}$$

$$\hat{\mathbf{J}}_n^{yz} = \mathbf{J}_n^{yz} + [L_n]^z \mathbf{RCZB}(n) \tag{2.73}$$

$$\begin{bmatrix} \hat{\mathbf{RC}}_{n,t}^x \\ \hat{\mathbf{RC}}_{n,t}^y \\ \hat{\mathbf{RC}}_{n,t}^z \end{bmatrix} = \begin{bmatrix} \mathbf{RC}_{n,t}^x \\ \mathbf{RC}_{n,t}^y \\ \mathbf{RC}_{n,t}^z \end{bmatrix} + {}^n\boldsymbol{A}_{n,t}^t \begin{bmatrix} 0 \\ -\mathbf{RCZB}(n) \\ \mathbf{RCZ}(n) \end{bmatrix} \tag{2.74}$$

$$\hat{\mathbf{RC}}_n^y = \mathbf{RC}_n^y - \mathbf{RCZB}(n) \tag{2.75}$$

and define

$$\mathbf{J}\hat{\mathbf{Y}}(n) = \sum_{i=a(k(n)+1)}^{b(k(n)+1)} \hat{\mathbf{J}}_i^y. \tag{2.76}$$

we show base parameters using these inertial parameters. Define rotational link number Q less than $a(2)$ as follows;$Q = 0$ if $\boldsymbol{z}_1$ or some $\boldsymbol{z}_{0,t}$ is not parallel to gravity vector $\boldsymbol{g}$. If $Q \neq 0$, Q is the minimum number such that $\boldsymbol{z}_{Q+1}$ or $\boldsymbol{z}_{Q,t}$ is not parallel to $\boldsymbol{g}$ or $[L_Q]^x \neq 0$.

Theorem 2.4.1 The following inertial parameters constitute a base parameter set for the dynamic model (2.60) and (2.61).

$$\mathbf{J}_n^z + \hat{\mathbf{J}}\hat{\mathbf{Y}}(n) \tag{2.77}$$

for $1 \le n \le N$,

$$\mathbf{RC}_n^x, \qquad \hat{\mathbf{RC}}_n^y \tag{2.78}$$

for $Q + 1 \le n \le N$,

$$\mathbf{J}_n^x - \mathbf{J}_n^y + \hat{\mathbf{J}}\hat{\mathbf{Y}}(n), \qquad \hat{\mathbf{J}}_n^{xy}, \qquad \hat{\mathbf{J}}_n^{yz}, \qquad \hat{\mathbf{J}}_n^{xz} \tag{2.79}$$

for $a(2) \le n \le N$,

$$\mathbf{M}_{n,t} \tag{2.80}$$

for $0 \le n \le N$, $1 \le t \le T_n$,

$$\hat{\mathbf{RC}}_{n,t}^x, \qquad \hat{\mathbf{RC}}_{n,t}^y, \qquad \hat{\mathbf{RC}}_{n,t}^z \tag{2.81}$$

for $a(2) \le n \le N$, $1 \le t \le T_n$, and

$$\begin{gathered} [A_{n,t}]_{32}\hat{\mathbf{RC}}_{n,t}^x - [A_{n,t}]_{31}\hat{\mathbf{RC}}_{n,t}^y \\ \hat{\mathbf{RC}}_{n,t}^z - [A_{n,t}]_{33}([A_{n,t}]_{31}\hat{\mathbf{RC}}_{n,t}^x + [A_{n,t}]_{32}\hat{\mathbf{RC}}_{n,t}^y + [A_{n,t}]_{33}\hat{\mathbf{RC}}_{n,t}^z) \end{gathered} \tag{2.82}$$

for $1 \le n \le b(1)$, $1 \le t \le T_n$. The parameters in (2.82) are disappeared for (n,t) if and only if translational joint axis $\boldsymbol{z}_{n,t}$ is parallel to rotational joints axis $\boldsymbol{z}_n$.

For $1 \le n \le b(1)$ let W_n be the number of translational joint axes between $\boldsymbol{z}_n$ and $\boldsymbol{z}_{n+1}$, that are parallel to $\boldsymbol{z}_n$. Then the total number B of base parameters in a base parameter set is given as $B = B_0 - B_1$ where $B_0 = 7N - 4 + T_0 + 3T_1 + 4\sum_{n=2}^{N} T_n$ and $B_1 = 2Q + 4(b(1) - 1) + \sum_{n=2}^{b(1)} T_n + \sum_{n=1}^{b(1)} 2W_n$. $B = B_0$ for most general cases, and we can reduce this by B_1 which depends on kinematic structure of the manipulator and relation between the directions of $\boldsymbol{z}_1$ and $\boldsymbol{g}$.

We give following lemmas for proof of **Theorem 2.4.1**.

Lemma 2.4.1 Suppose that $H_1(i,j)$, $H_2(i,(j,t))$, $H_3((i,t_1),(i,t_2))$, $\boldsymbol{g}\cdot(\boldsymbol{z}_i\times\overline{\boldsymbol{SR}}_i)$ or $\mathbf{M}_{i,t}\boldsymbol{g}\cdot\boldsymbol{z}_{i,t}$ is described as

$$\sum_{u=1}^{U_1}\mathbf{p}_u f_u+\sum_{u=U_1+1}^{U_2}\mathbf{p}_u f_u \tag{2.83}$$

where $\mathbf{p}_u$ is an inertial parameter and f_u is a polynomial of $q_{n,t}$ for $0\le n\le N$, $1\le t\le T_n$ and trigonometric function of θ_n for $1\le n\le N$. If $f_1,f_2,\cdots,f_{U_1}$ are mutually independent and $\mathbf{p}_{U_1},\mathbf{p}_{U_1+1},\cdots,\mathbf{p}_{U_2}$ are fundamental parameters, then $\mathbf{p}_u$ for $1\le u\le U_1$ are fundamental parameters .

Lemma 2.4.2

$$^{(n,0)}\boldsymbol{SR}_n=\begin{bmatrix}\mathbf{RC}_n^x\\ \hat{\mathbf{RC}}_n^y\\ \mathbf{RC}_n^z+\mathbf{RCZ}(n)\end{bmatrix}+\boldsymbol{G}_1 \tag{2.84}$$

for $1\le n\le N$,

$$^{(n,t)}\boldsymbol{SR}_{n,t}=\begin{bmatrix}\hat{\mathbf{RC}}_{n,t}^x\\ \hat{\mathbf{RC}}_{n,t}^y\\ \hat{\mathbf{RC}}_{n,t}^z\end{bmatrix}+\boldsymbol{G}_2 \tag{2.85}$$

for $0\le n\le N, 1\le t\le T_n$ where $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $n+1\le s\le N$.

The proofs of **Lemma 2.4.1** and **2.4.2** are almost same as those of **Lemma 2.3.1** and **2.3.2**, respectively, hence we omit them.

(Proof of Theorem 2.4.1) It is evident from (2.64) that $H_3((n,t),(n,t))=\mathbf{M}_{n,t}$. Hence $\mathbf{M}_{n,t}$ is a fundamental parameter for $0\le n\le N$, $1\le t\le T_n$.

Represent $^{(n,0)}\boldsymbol{g}$ as $^{(n,0)}\boldsymbol{g}=[^n g_x \quad ^n g_y \quad ^n g_z]^t$. Since

$$\overline{\boldsymbol{SR}}_n=\boldsymbol{SR}_n+\sum_{i=n}^{N}\sum_{t=1}^{T_i}\mathbf{M}_{i,t}q_{i,t}\boldsymbol{z}_{i,t}, \tag{2.86}$$

using **Lemma 2.2.2** and $^{(n,0)}\boldsymbol{z}_n=\boldsymbol{e}_3$ $(\boldsymbol{e}_3=[0 \quad 0 \quad 1]^t)$, we can evaluate $\boldsymbol{g}\cdot(\boldsymbol{z}_n\times\overline{\boldsymbol{SR}}_n)$ about $(\boldsymbol{x}_n,\boldsymbol{y}_n,\boldsymbol{z}_n)$ as

$$^{(n,0)}\boldsymbol{g}\cdot(^{(n,0)}\boldsymbol{z}_n\times{}^{(n,0)}\overline{\boldsymbol{SR}}_n)={}^n g_y\mathbf{RC}_n^x-{}^n g_x\hat{\mathbf{RC}}_n^y+G \tag{2.87}$$

where G is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\mathbf{RC}_s^x, \hat{\mathbf{RC}}_s^y$ for $n+1 \le s \le N$ and $\mathbf{M}_{s,t}$ for $n \le s \le N, \quad 1 \le t \le T_s$. It is easily shown that ${}^n g_x$ and ${}^n g_y$ are nonzero mutually independent functions for $a(2) \le n \le N$ in general case and for $1 \le n \le N$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$. Hence if $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $s \ge n+1$ are assumed to be fundamental parameters, $\mathbf{RC}_n^x$ and $\hat{\mathbf{RC}}_n^y$ are fundamental parameters by **Lemma 2.2.1** since all $\mathbf{M}_{n,t}$ are fundamental parameters. For $n = N$, $\mathbf{RC}_N^x$ and $\hat{\mathbf{RC}}_N^y$ can be shown to be fundamental parameters in the same way. By the mathematical induction it is concluded that $\mathbf{RC}_n^x$ and $\hat{\mathbf{RC}}_n^y$ are fundamental parameters for $a(2) \le n \le N$ in general and for $1 \le n \le N$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$.

For n such that $n > j$, we have

$$H_2(n,(j,t)) = \boldsymbol{z}_{j,t} \cdot (\boldsymbol{z}_n \times \overline{\boldsymbol{SR}}_n) \tag{2.88}$$

in (2.63). This is same form as $\boldsymbol{g} \cdot (\boldsymbol{z}_n \times \overline{\boldsymbol{SR}}_n)$ if $\boldsymbol{z}_{j,t}$ is regarded as $\boldsymbol{g}$. By the same arguments in the above, we conclude that $\mathbf{RC}_n^x$ and $\hat{\mathbf{RC}}_n^y$ are fundamental parameters for $(1 \le) w \le n \le N$ if there exist any $\boldsymbol{z}_{w-1,t}$ for $1 \le t \le T_{w-1}$ which is not parallel to $\boldsymbol{z}_w$.

Consider terms:

$$\boldsymbol{z}_i \cdot [\sum_{n=j}^{i-1} \sum_{t=1}^{T_n} q_{n,t} (\boldsymbol{z}_{n,t} \cdot \boldsymbol{SR}_i \boldsymbol{E} - \boldsymbol{z}_{n,t} \otimes \boldsymbol{SR}_i)] \boldsymbol{z}_j \tag{2.89}$$

in $H_1(i,j)$. In the case that $1 \le j \le n < i \le b(1)$, then ${}^{(i,0)}\boldsymbol{z}_i = \boldsymbol{e}_3, {}^{(i,0)}\boldsymbol{z}_j = \boldsymbol{e}_3$. Denoting ${}^{(n,0)}\boldsymbol{z}_{n,t}$ as ${}^{(n,0)}\boldsymbol{z}_{n,t} = [\alpha \quad \beta \quad \gamma]^t ({}^{(n,0)}\boldsymbol{z}_{n,t}$ is a constant vector), we obtain from **Property 2.4.1** that

$${}^{(i,0)}\boldsymbol{z}_{n,t} = {}^n\boldsymbol{A}_i^t \, {}^{(n,0)}\boldsymbol{z}_{n,t} = \begin{bmatrix} \alpha \cos\theta(n+1,i) + \beta \sin\theta(n+1,i) \\ -\alpha \sin\theta(n+1,i) + \beta\cos\theta(n+1,i) \\ \gamma \end{bmatrix}. \tag{2.90}$$

Using **Lemma 2.4.2** we can easily derive that

$$\begin{aligned} & q_{n,t} {}^{(i,0)}\boldsymbol{z}_i ({}^{(i,0)}\boldsymbol{z}_{n,t} \cdot {}^{(i,0)}\boldsymbol{SR}_i \boldsymbol{E} - {}^{(i,0)}\boldsymbol{z}_{n,t} \otimes {}^{(i,0)}\boldsymbol{SR}_i) {}^{(i,0)}\boldsymbol{z}_j \\ & = \mathbf{RC}_i^x q_{n,t} (\alpha\cos\theta(n+1,i) + \beta\sin\theta(n+1,i)) \\ & + \hat{\mathbf{RC}}_i^y q_{n,t} (-\alpha\sin\theta(n+1,i) + \beta\cos\theta(n+1,i)) + G. \end{aligned} \tag{2.91}$$

where G is a function of $\boldsymbol{\theta}$ generated by $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $s \ge i+1$.

$$q_{n,t}(\alpha\cos\theta(n+1,i) + \beta\sin\theta(n+1,i)) \tag{2.92}$$

and

$$q_{n,t}(-\alpha \sin\theta(n+1,i) + \beta\cos\theta(n+1,i)) \tag{2.93}$$

are mutually independent functions unless $\alpha = 0$ and $\beta = 0$, and $\alpha = 0$ and $\beta = 0$ are satisfied if and only if $\boldsymbol{z}_{n,t}$ is parallel to $\boldsymbol{z}_n$. If $\boldsymbol{z}_{n,t}$ is not parallel to $\boldsymbol{z}_n$, then $\boldsymbol{z}_{n,t}$ is not parallel to $\boldsymbol{z}_{n+1}$. We have proved $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ are fundamental parameters for $(1 \leq)w \leq n \leq N$ if there exist any $\boldsymbol{z}_{w-1,t}$ for $1 \leq t \leq T_{w-1}$ which is not parallel to $\boldsymbol{z}_w$. Hence, it is shown that the term (2.89) can be generated by fundamental parameters.

For n such that $n \geq i$,

$$H_2(i,(n,t)) = [\boldsymbol{z}_i \times (\mathbf{M}_{n,t}\overline{\boldsymbol{L}}_{(i,0)(n,t)} + \overline{\boldsymbol{SR}}_{n,t})] \cdot \boldsymbol{z}_{n,t} \tag{2.94}$$

in (2.63). Since we can easily derive that

$$\overline{\boldsymbol{SR}}_{n,t} = \boldsymbol{RC}_{n,t} + \boldsymbol{SR}_{n+1} + \sum_{t_1=t+1}^{T_n} \mathbf{M}_{n,t_1} q_{n,t_1}\boldsymbol{z}_{n,t_1} + \sum_{s=n+1}^{n}\sum_{t_1=1}^{T_s} \mathbf{M}_{s,t_1} q_{s,t_1}\boldsymbol{z}_{s,t_1}, \tag{2.95}$$

we obtain

$$H_2(i,(n,t)) = -\boldsymbol{z}_i \cdot [\boldsymbol{z}_{n,t} \times (\boldsymbol{RC}_{n,t} + \boldsymbol{SR}_{n+1})] + G \tag{2.96}$$

where G is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\mathbf{M}_{s,w}$. Since ${}^{(n,0)}\boldsymbol{SR}_{n+1} = {}^{(n,0)}\boldsymbol{SR}_n - {}^{(n,0)}\boldsymbol{RC}_n$, it can be shown by using **Lemma 2.4.2**, that

$${}^{(n,0)}\boldsymbol{SR}_{n+1} = [0 \quad -\mathbf{RCZB}(n) \quad \mathbf{RCZ}(n)]^t + \boldsymbol{G}' \tag{2.97}$$

where $\boldsymbol{G}'$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $n+1 \leq s \leq N$. Thus, using (2.74) we obtain

$$\begin{aligned} {}^{(n,t)}\boldsymbol{RC}_{n,t} + {}^{(n,t)}\boldsymbol{SR}_{n+1} &= \begin{bmatrix} \mathbf{RC}_{n,t}^x \\ \mathbf{RC}_{n,t}^y \\ \mathbf{RC}_{n,t}^z \end{bmatrix} + {}^n\boldsymbol{A}_{n,t}^t \begin{bmatrix} 0 \\ -\mathbf{RCZB}(n) \\ \mathbf{RCZ}(n) \end{bmatrix} + \boldsymbol{G}' \\ &= \begin{bmatrix} \hat{\mathbf{RC}}_{n,t}^x \\ \hat{\mathbf{RC}}_{n,t}^y \\ \hat{\mathbf{RC}}_{n,t}^z \end{bmatrix} + \boldsymbol{G}'. \end{aligned} \tag{2.98}$$

Since ${}^{(n,t)}\boldsymbol{z}_{n,t} = \boldsymbol{e}_3$ and ${}^{(n,t)}\boldsymbol{z}_i = {}^n\boldsymbol{A}_{n,t}^t\,{}^i\boldsymbol{A}_n^t\boldsymbol{e}_3$, it is easily derived that

$$H_2(i,(n,t)) = {}^i\boldsymbol{A}_n^t\boldsymbol{e}_3 \cdot {}^n\boldsymbol{A}_{n,t}\begin{bmatrix} -\hat{\mathbf{RC}}_{n,t}^y \\ \hat{\mathbf{RC}}_{n,t}^x \\ 0 \end{bmatrix} + G'' \tag{2.99}$$

where G'' is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $n+1 \le s \le N$ and $\mathbf{M}_{s,w}$ for $i \le s \le N,\quad 1 \le w \le T_s$. If $2 \le k(i) = k(n)$, then ${}^i\boldsymbol{A}_n^t\boldsymbol{e}_3 = \boldsymbol{e}_3$, hence 3rd component of ${}^n\boldsymbol{A}_{n,t}[-\hat{\mathbf{RC}}_{n,t}^y \quad \hat{\mathbf{RC}}_{n,t}^x \quad 0]^t$ is a fundamental parameter by **Lemma 2.2.1** since $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $s \ge a(2)$ and $\mathbf{M}_{s,w}$ for $1 \le s \le N,\quad 1 \le w \le T_s$ are fundamental parameters. If $2 \le k(n) = k(i)+1$, it is derived by using **Property 2.4.1** that

$${}^i\boldsymbol{A}_n^t\boldsymbol{e}_3 = [\sin\theta(a(k(n)),n) \quad \cos\theta(a(k(n)),n) \quad 0]^t. \tag{2.100}$$

In this case, since $\sin\theta(a(k(n)),n)$ and $\cos\theta(a(k(n)),n)$ are mutually independent functions, 1st and 2nd components of ${}^n\boldsymbol{A}_{n,t}[-\hat{\mathbf{RC}}_{n,t}^y \quad \hat{\mathbf{RC}}_{n,t}^x \quad 0]^t$ are fundamental parameters from the same reasons. Any linear combination of fundamental parameters is a fundamental parameter . Thus we can conclude that $\hat{\mathbf{RC}}_{n,t}^x$ and $\hat{\mathbf{RC}}_{n,t}^y$ are fundamental parameters for $a(2) \le n \le N,\quad 1 \le t \le T_n$ since ${}^n\boldsymbol{A}_{n,t}$ is a nonsingular constant matrix. In the case that $1 = k(i) = k(n)$, since ${}^i\boldsymbol{A}_n^t\boldsymbol{e}_3 = \boldsymbol{e}_3$, it is derived from (2.99) that

$$H_2(i,(n,t)) = [A_{n,t}]_{32}\hat{\mathbf{RC}}_{n,t}^x - [A_{n,t}]_{31}\hat{\mathbf{RC}}_{n,t}^y + G''. \tag{2.101}$$

$[A_{n,t}]_{32}\hat{\mathbf{RC}}_{n,t}^x - [A_{n,t}]_{31}\hat{\mathbf{RC}}_{n,t}^y$ is disappeared if and only if $[A_{n,t}]_{32} = 0$ and $[A_{n,t}]_{31} = 0$ i.e. $\boldsymbol{z}_{n,t}$ is parallel to $\boldsymbol{z}_n$ (since ${}^n\boldsymbol{A}_{n,t}$ is a rotational matrix). If not the case, since $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ for $n+1 \le s$ have been shown to be fundamental parameters, by **Lemma 2.2.1** we can conclude from the same reasons that $[A_{n,t}]_{32}\hat{\mathbf{RC}}_{n,t}^x - [A_{n,t}]_{31}\hat{\mathbf{RC}}_{n,t}^y$ is a fundamental parameter for $1 \le n \le b(1),\ 1 \le t \le T_n$.

Next, consider term:

$$\boldsymbol{z}_i \cdot [\sum_{n=i}^{N}\sum_{t=1}^{T_n} q_{n,t}(2\boldsymbol{z}_{n,t}\cdot \boldsymbol{SR}_{n,t}\boldsymbol{E} - \boldsymbol{z}_{n,t}\otimes \boldsymbol{SR}_{n,t} - \boldsymbol{SR}_{n,t}\otimes \boldsymbol{z}_{n,t})]\boldsymbol{z}_j \tag{2.102}$$

in $H_1(i,j)$. Evaluating this term about coordinate system (n,t) and using **Lemma 2.4.2** we can easily derive that

$$2^{(n,t)}\boldsymbol{z}^t_{n,t}{}^{(n,t)}\boldsymbol{SR}_{n,t}\boldsymbol{E} - {}^{(n,t)}\boldsymbol{z}_{n,t}{}^{(n,t)}\boldsymbol{SR}^t_{n,t} - {}^{(n,t)}\boldsymbol{SR}_{n,t}{}^{(n,t)}\boldsymbol{z}^t_{n,t}$$

$$= \begin{bmatrix} 2\hat{\mathbf{RC}}^z_{n,t} & 0 & -\hat{\mathbf{RC}}^x_{n,t} \\ 0 & 2\hat{\mathbf{RC}}^z_{n,t} & -\hat{\mathbf{RC}}^y_{n,t} \\ -\hat{\mathbf{RC}}^x_{n,t} & -\hat{\mathbf{RC}}^y_{n,t} & 0 \end{bmatrix} + \boldsymbol{G} \tag{2.103}$$

where $\boldsymbol{G}$ is a matrix whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{RC}^x_s$ and $\hat{\mathbf{RC}}^y_s$ for $s \geq n+1$. If $n \geq a(2)$ and $i = j = b(k(n)-1)$, then,

$$^{(n,0)}\boldsymbol{z}_i = {}^{(n,0)}\boldsymbol{z}_j = {}^i\boldsymbol{A}^t_n\boldsymbol{e}_3 = [\sin\theta(a(k(n)),n) \quad \cos\theta(a(k(n)),n) \quad 0]^t. \tag{2.104}$$

In this case, using $^{(n,t)}\boldsymbol{z}_i = {}^{(n,t)}\boldsymbol{z}_j = {}^n\boldsymbol{A}^t_{n,t}{}^{(n,0)}\boldsymbol{z}_i$ we can derive directly from (2.103) that

$$q_{n,t}\boldsymbol{z}_i \cdot (2\boldsymbol{z}_{n,t} \cdot \boldsymbol{SR}_{n,t}\boldsymbol{E} - \boldsymbol{z}_{n,t} \otimes \boldsymbol{SR}_{n,t} - \boldsymbol{SR}_{n,t} \otimes \boldsymbol{z}_{n,t})\boldsymbol{z}_j = \hat{\mathbf{RC}}^z_{n,t}f + G' \tag{2.105}$$

where G' is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\mathbf{RC}^x_s$ and $\hat{\mathbf{RC}}^y_s$ for $s \geq n+1$ and $\hat{\boldsymbol{RC}}^x_{s,t}$ and $\hat{\mathbf{RC}}^y_{s,t}$ for $s \geq n$, $1 \leq t \leq T_s$, and

$$\begin{aligned} f &= 2q_{n,t}[([A_{n,t}]_{11}\sin\theta(a(k(n)),n) + [A_{n,t}]_{21}\cos\theta(a(k(n)),n))^2 \\ &\quad +([A_{n,t}]_{12}\cos\theta(a(k(n)),n) + [A_{n,t}]_{22}\sin\theta(a(k(n)),n))^2]. \end{aligned} \tag{2.106}$$

We can easily show that $f \not\equiv 0$. It is evident that $q_{n,t}$ in (2.103) does not appear in other terms of $H_1(i,i)$ except the term that is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ ganarated by fundamental parameters $\mathbf{M}_{s,w}$s only. Hence, it is concluded by **Lemma 2.4.1** that $\hat{\mathbf{RC}}^z_{n,t}$ is a fundamental parameter for $n > a(2)$, $1 \leq t \leq T_n$ since $\mathbf{RC}^x_s$ and $\hat{\mathbf{RC}}^y_s$ for $s \geq a(2)$, $\hat{\mathbf{RC}}^t_{s,t}$ and $\hat{\mathbf{RC}}^y_{s,t}$ for $s \geq a(2)$ and $\mathbf{M}_{s,t}$ for $0 \leq s \leq N$, $1 \leq t \leq T_s$ are already shown to be fundamental parameters. If $1 \leq j \leq i \leq n \leq b(1)$, then $^{(n,0)}\boldsymbol{z}_i = {}^{(n,0)}\boldsymbol{z}_j = \boldsymbol{e}_3$. In this case, using $^{(n,t)}\boldsymbol{z}_i = {}^{(n,t)}\boldsymbol{z}_j = {}^n\boldsymbol{A}^t_{n,t}\boldsymbol{e}_3$ and $[A_{n,t}]^2_{31} + [A_{n,t}]^2_{32} + [A_{n,t}]^2_{33} = 1$, we can easily derive that

$$q_{n,t}\boldsymbol{z}_i \cdot (2\boldsymbol{z}_{n,t} \cdot \boldsymbol{SR}_{n,t}\boldsymbol{E} - \boldsymbol{z}_{n,t} \otimes \boldsymbol{SR}_{n,t} - \boldsymbol{SR}_{n,t} \otimes \boldsymbol{z}_{n,t})\boldsymbol{z}_j = 2\boldsymbol{p}q_{n,t} + G' \tag{2.107}$$

where G' is a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$ generated by $\mathbf{RC}^x_s$ and $\hat{\mathbf{RC}}^y_s$ for $s \geq n+1$, and

$$\boldsymbol{p} = \hat{\mathbf{RC}}^z_{n,t} - [A_{n,t}]_{33}([A_{n,t}]_{31}\hat{\mathbf{RC}}^x_{n,t} + [A_{n,t}]_{32}\hat{\mathbf{RC}}^y_{n,t} + [A_{n,t}]_{33}\hat{\mathbf{RC}}^z_{n,t}). \tag{2.108}$$

Since ${}^{n}\boldsymbol{A}_{n,t}$ is a rotation matrix, $\boldsymbol{p}$ is disappeared if and only if $[A_{n,t}]_{31} = [A_{n,t}]_{32} = 0$ and $[A_{n,t}]_{33} = 1$ i.e. $\boldsymbol{z}_{n,t}$ is parallel to $\boldsymbol{z}_n$. If not the case, since $\mathbf{RC}_s^x$ and $\hat{\mathbf{RC}}_s^y$ have been proved to be fundamental parameters, thus, in the same way, we can conclude by **Lemma 2.4.1** that $\boldsymbol{p}$ is a fundamental parameter for $1 \le n \le b(1)$ and $1 \le t \le T_n$.

The terms of $H_1(i,j)$ in the first 3 lines in (2.62) have the same form as the case that the manipulator has rotational joints only and any two adjacent rotational joint axes are parallel or perpendicular. This case is treated in preceding section. Using exactly same arguments and **Lemma 2.4.1**, we can show that all the parameters in (2.77),(2.78) and (2.79) are fundamental parameters and they can generate these terms completely. In the case $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$, $[L_n]^x \neq 0$ was assumed for $1 \le n \le b(1) - 1$ in the preceding section. It makes no sense to remove the assumption in the preceding section, but not for the manipulators in this section. Modifying the arguments in the preceding section slightly we can easily remove the assumption and modify the results concerning $\mathbf{RC}_n^x$ and $\hat{\mathbf{RC}}_n^y$ for $1 \le n \le b(1)$ as follows; $\mathbf{RC}_i^x$ and $\hat{\mathbf{RC}}_i^y$ for $n \le i \le b(1)$ are fundamental parameters if $[L_{n-1}]^x \neq 0$.

It has been proved that all the parameters given in **Theorem 2.4.1** are fundamental parameters. Observation of the above arguments shows that these parameters generate all $H_1(i,j)$, $H_2(i,(j,t))$, $H_3((i,t_1),(j,t_2))$, $\boldsymbol{g} \cdot (\boldsymbol{z}_n \times \overline{\boldsymbol{SR}_n})$ and $\mathrm{M}_{n,t}\boldsymbol{g} \cdot \boldsymbol{z}_{n,t}$. It is evident that these parameters are mutually independent since each of them includes a link inertial parameter which is not appear in the others. Thus the set of all the inertial parameters given in the **Theorem 2.4.1** constitutes a base parameter set. The number of base parameters in this base parameter set is evident. ∎

2.4.3 Conclusion

We have shown a base parameter set, which is a minimum set of inertial parameters that can generate the dynamic models uniquely, for general parallel and perpendicular manipulators with rotational and translational joints. We have described every base parameter in a linear combination of the link inertial parameters directly and completely in closed form. Also, we have given the exact number of the base parameters.

Any base parameters can be obtained from these base parameters by nonsingular linear transformation of them.

The assumption that any pair of two adjacent rotational joint axes is parallel or perpendicular is not restrictive since all existing industrial manipulators satisfy it.

The investigation of a base parameter set and giving the complete closed-form solutions of it has been extended to general open-loop kinematic chains without essential change of the results in this section [35].

2.5 Base Parameters for Manipulators with a Planar Parallelogram Link Mechanism

In this section, we extend the investigation of a base parameter set and giving the complete closed-form solutions of it to manipulators with a planar parallelogram link mechanism. We also give the exact number of the base parameters. The results of this section would cover most of commercially available industrial manipulators with closed chain mechanisms.

2.5.1 Description of Manipulators

In this section, manipulators with only one planar parallelogram link mechanism are treated. Also, only revolute joints are considered. As shown in Fig.2.5.1, link 0 is a stationary pillar. Link i is connected to link $i-1$ through joint i for $1 \le i \le \xi$. Link ξ and link ζ are connected to link $\xi-1$ through joint ξ and joint ζ, respectively. The axes of joint ξ and joint ζ coincide with each other. Link $\xi+1$ is connected to link ξ through joint $\xi+1$ and link $\zeta+1$ is connected to link ζ through joint $\zeta+1$ and to link $\xi+1$ through joint $\zeta+2$. The axes of joints $\xi, \xi+1, \zeta, \zeta+1$, and $\zeta+2$ are parallel. Link ξ has same length as that of $\zeta+1$ and link ζ has same length as the length between joints $\xi+2$ and $\zeta+2$. Then, links $\xi, \xi+1, \zeta, \zeta+1$ form a planar parallelogram. We assume that joint ξ and joint ζ are actuated and joints $\xi+1, \zeta+1$, and $\zeta+2$ are passive. Finally, link i is connected to link $i-1$ through joint i for $\xi+2 \le i \le N$ where N denotes the last link number.

Suppose that the parallelogram were cut open at joint $\zeta+2$, then the kinematic mechanisms of the manipulator would have a tree structured open kinematic chain. For open kinematic chain, it is possible to apply the same method as in the sections 1,2 to assign and attach a coordinate system $(o_i; \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ to link i where o_i denotes the origin of the coordinate system. As shown in Fig.2.5.1, we can make o_ζ coincide with o_ξ since the axes of joints ξ and ζ coincide. We can set origins $o_\xi, o_{\xi+1}, o_\zeta, o_{\zeta+1}$ such that they determine a plane since the axes of joint $\xi, \xi+1, \zeta, \zeta+1$ are parallel and we set a point $o_{\zeta+2}$ on the axis of joint $\zeta+2$ such that it is in the plane. Let θ_i denote the joint angle that is measured from $\boldsymbol{x}_{i-1}$-axis to $\boldsymbol{x}_i$-axis about $\boldsymbol{z}_i$-axis for $1 \le i \le N$ or $i = \zeta+1$. θ_ζ is measured from $\boldsymbol{x}_{\xi-1}$-axis to $\boldsymbol{x}_\zeta$-axis about $\boldsymbol{z}_\zeta$-axis which coincide with $\boldsymbol{z}_\xi$-axis. Let α_i denote the twist angle between $\boldsymbol{z}_{i-1}$-axis and $\boldsymbol{z}_i$-axis about $\boldsymbol{x}_{i-1}$-axis and let $\boldsymbol{L}_i$ denote the vector from o_i to o_{i+1}.

For each link i let $\mathbf{m}_i$ denote the mass, $\boldsymbol{I}_i$ denote the inertial tensor around o_i and $\boldsymbol{r}_i$ denote the vector from o_i to the center of mass. Then, ${}^i\boldsymbol{r}_i$, and ${}^i\boldsymbol{I}_i$ are a constant vector and a constant matrix, respectively.

After attaching coordinate systems, we can obtain rotation matrices. Let ${}^j\boldsymbol{A}_i$ denote a

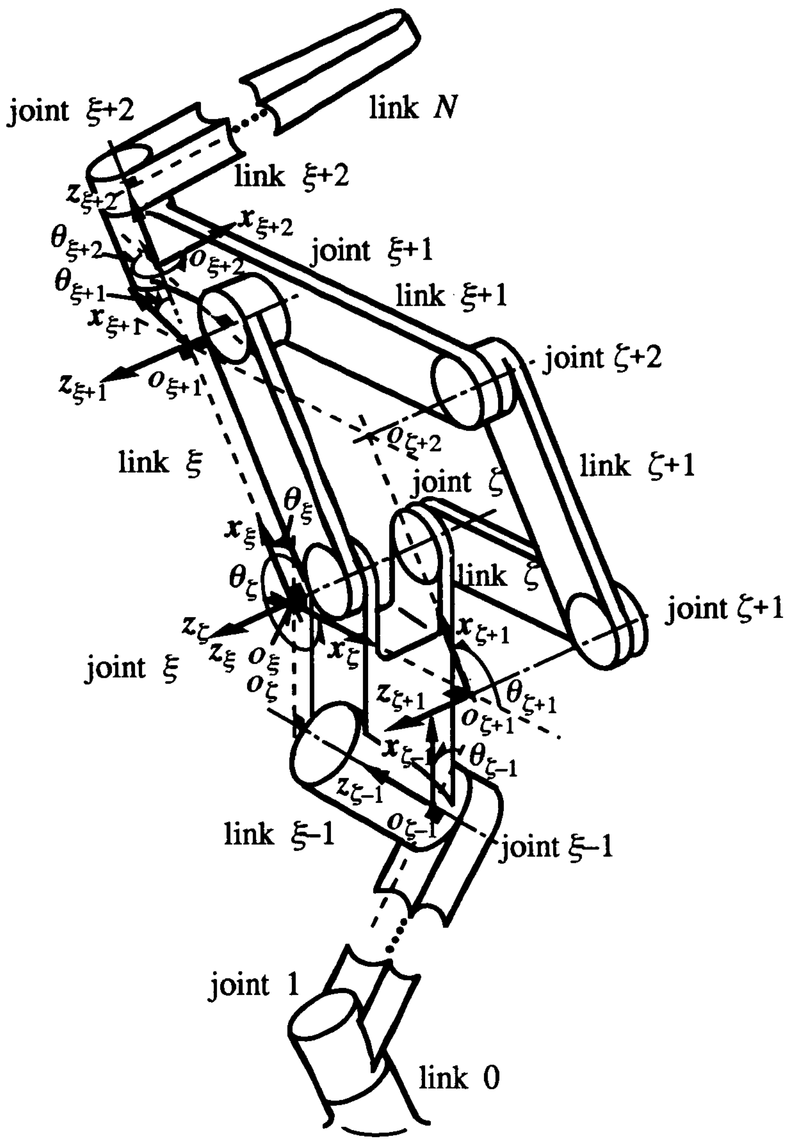


Fig. 2.5.1 Manipulator with a Planar Parallelogram Link Mechanism

rotation matrix that represents $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ with reference to $(\boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{z}_j)$. Then, ${}^{i-1}\boldsymbol{A}_i$ is described as

$$
{}^{i-1}\boldsymbol{A}_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i \cos\alpha_i & \cos\theta_i \cos\alpha_i & -\sin\alpha_i \\ \sin\theta_i \sin\alpha_i & \cos\theta_i \sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{2.109}
$$

and ${}^j\boldsymbol{A}_i$ $(j < i)$ will be denoted as

$$
{}^j\boldsymbol{A}_i = \begin{bmatrix} ({}^jA_i)_{11} & ({}^jA_i)_{12} & ({}^jA_i)_{13} \\ ({}^jA_i)_{21} & ({}^jA_i)_{22} & ({}^jA_i)_{23} \\ ({}^jA_i)_{31} & ({}^jA_i)_{32} & ({}^jA_i)_{33} \end{bmatrix}. \tag{2.110}
$$

Each $\boldsymbol{L}_i$ and each element of ${}^j\boldsymbol{A}_i$ can be determined from kinematic parameter values of the manipulator. We assume that all they are known.

2.5.2 Constraints

Fig.2.5.2 depicts the schematic diagram of the planar parallelogram. The planar parallelogram must satisfy two equality constraints:

$$
\theta_a = \theta_{\zeta+1} \tag{2.111}
$$

$$
\theta_b = \theta_{\zeta+1} \tag{2.112}
$$

where θ_a and θ_b are the angles shown in Fig.2.5.2. From the two equalities, we obtain

$$
\sin\theta_\xi = \sin(\theta_\zeta + \theta_{\zeta+1}) \tag{2.113}
$$

$$
\cos\theta_\xi = \cos(\theta_\zeta + \theta_{\zeta+1}) \tag{2.114}
$$

$$
\sin\theta_\zeta = -\sin(\theta_\xi + \theta_{\xi+1} + \rho) \tag{2.115}
$$

$$
\cos\theta_\zeta = -\cos(\theta_\xi + \theta_{\xi+1} + \rho) \tag{2.116}
$$

where ρ denotes the angle that is measured from $\boldsymbol{x}_{\xi+1}$-axis to the direction of $\boldsymbol{L}_{\zeta+2}$ which is defined as the vector from $o_{\zeta+2}$ to $o_{\xi+1}$. We can consider that two variables of $\theta_\zeta, \theta_{\zeta+1}, \theta_\xi$, and $\theta_{\xi+1}$ are independent variables and the rest are functions of the independent variables.

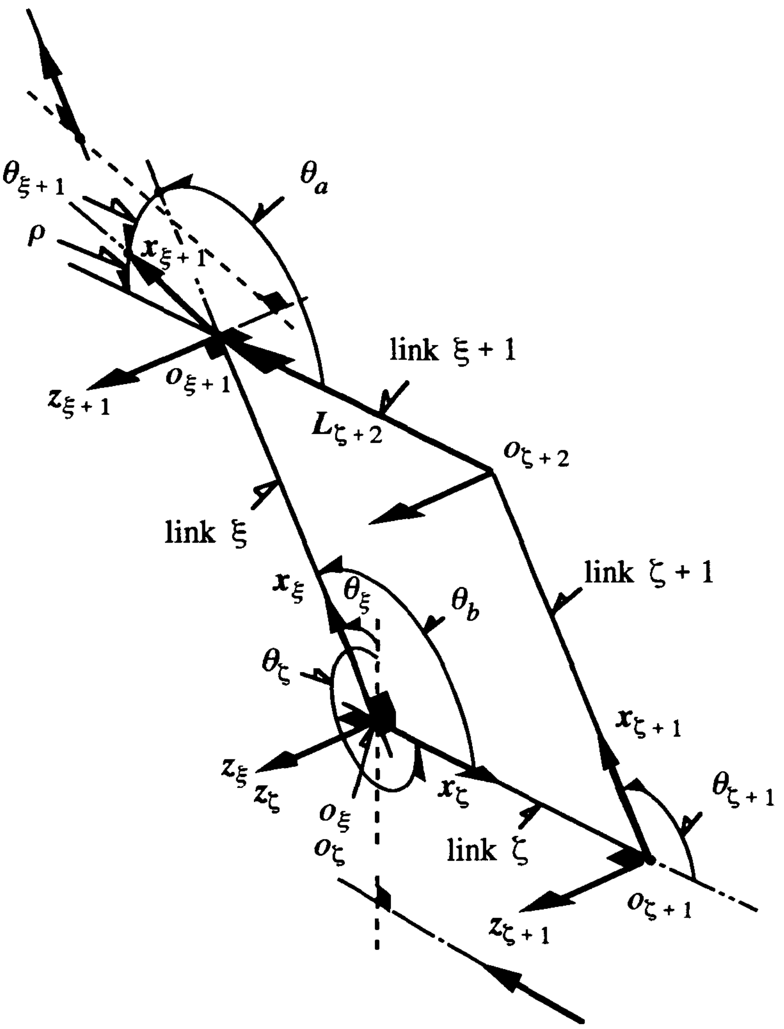


Fig. 2.5.2 Schematic Diagram of the Planar Parallelogram

2.5.3 Dynamic Models of Manipulators

In this subsection, we derive the dynamic model of the manipulator shown in Fig.2.5.1. First, we introduce the following notations:

$$\boldsymbol{R}_S = \begin{cases} \mathrm{m}_S \boldsymbol{r}_S + (\sum_{j=S+1}^{N} \mathrm{m}_j + \mathrm{m}_\zeta + \mathrm{m}_{\zeta+1}) \boldsymbol{L}_S, & \text{if } 1 \le s \le \xi - 1 \\ \mathrm{m}_S \boldsymbol{r}_S + \sum_{j=S+1}^{N} \mathrm{m}_j \boldsymbol{L}_S, & \text{if } \xi \le s \le N \end{cases} \tag{2.117}$$

$$\boldsymbol{SR}_i = \sum_{j=i}^{N} \boldsymbol{R}_j \tag{2.118}$$

$$\boldsymbol{R}_\zeta = \mathrm{m}_\zeta \boldsymbol{r}_\zeta + \mathrm{m}_{\zeta+1} \boldsymbol{L}_\zeta \tag{2.119}$$

$$\boldsymbol{R}_{\zeta+1} = \mathrm{m}_{\zeta+1} \boldsymbol{r}_{\zeta+1} \tag{2.120}$$

$$\boldsymbol{J}_S = \begin{cases} \boldsymbol{I}_S + (\sum_{j=S+1}^{N} \mathrm{m}_j + \mathrm{m}_\zeta + \mathrm{m}_{\zeta+1})[(\boldsymbol{L}_S \cdot \boldsymbol{L}_S)\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{L}_S], & \text{if } 1 \le s \le \xi - 1 \\ \boldsymbol{I}_S + (\sum_{j=S+1}^{N} \mathrm{m}_j)[(\boldsymbol{L}_S \cdot \boldsymbol{L}_S)\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{L}_S], & \text{if } \xi \le s \le N \end{cases} \tag{2.121}$$

$$\boldsymbol{J}_\zeta = \boldsymbol{I}_\zeta + \mathrm{m}_{\zeta+1}[(\boldsymbol{L}_\zeta \cdot \boldsymbol{L}_\zeta)\boldsymbol{E} - \boldsymbol{L}_\zeta \otimes \boldsymbol{L}_\zeta] \tag{2.122}$$

$$\boldsymbol{J}_{\zeta+1} = \boldsymbol{I}_{\zeta+1} \tag{2.123}$$

$$\boldsymbol{L}_{j,i} = \sum_{S=j}^{i-1} \boldsymbol{L}_S \tag{2.124}$$

Note that ${}^i\boldsymbol{R}_i$ and ${}^i\boldsymbol{J}_i$ are a constant vector and a constant matrix, respectively.

Next, we consider the manipulator shown in Fig.2.5.1 as two open kinematic chains. As shown in Fig.2.5.3, we will call the kinematic chain which consists of N links : link 0, link 1, link 2, ..., link ξ, link $\xi+1$, link $\xi+2$, ..., and link N, ξ-chain and we will call the kinematic chain which consists of $(\xi+1)$ links : link 0, link 1, ..., link $\xi-1$, link ζ, and link $\zeta+1$, ζ-chain. Then we define the variable vectors $\bar{\boldsymbol{\theta}}_\xi$ and $\bar{\boldsymbol{\theta}}_\zeta$ as

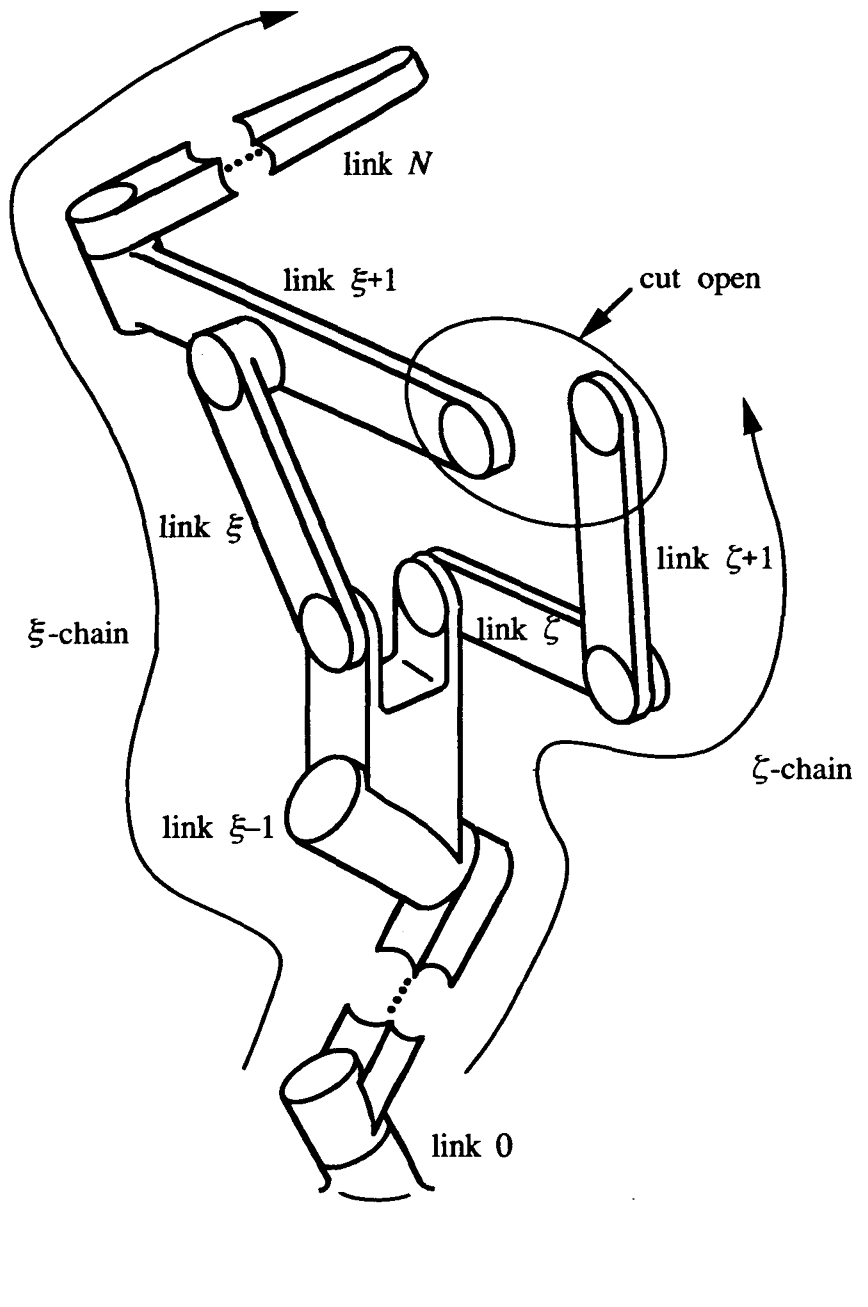


Fig. 2.5.3 Tree Structured Manipulator

$$\bar{\boldsymbol{\theta}}_\xi = \left[\begin{array}{ccccccc} \theta_1 & \cdots & \theta_{\xi-1} & \theta_\xi & \theta_{\xi+1} & \cdots & \theta_N \end{array}\right]^t, \tag{2.125}$$

$$\bar{\boldsymbol{\theta}}_\zeta = \left[\begin{array}{ccccc} \theta_1 & \cdots & \theta_{\xi-1} & \theta_\zeta & \theta_{\zeta+1} \end{array}\right]^t. \tag{2.126}$$

The kinetic energy K_ξ of ξ-chain is described as

$$K_\xi = \frac{1}{2}\dot{\bar{\boldsymbol{\theta}}}_\xi \cdot \boldsymbol{H}_\xi(\bar{\boldsymbol{\theta}}_\xi)\dot{\bar{\boldsymbol{\theta}}}_\xi \tag{2.127}$$

where $\boldsymbol{H}_\xi(\bar{\boldsymbol{\theta}}_\xi)$ is a positive definite symmetric matrix and (i,j)-th entry of it is described by coordinate free vector-tensor form as

$$\begin{aligned} H_\xi(i,j) &= \boldsymbol{z}_i \cdot [\sum_{s=i}^{N} \boldsymbol{J}_s]\boldsymbol{z}_j \\ &+\boldsymbol{z}_i \cdot [\sum_{s=i}^{N-1}\{2(\boldsymbol{L}_s \cdot \boldsymbol{SR}_{s+1})\boldsymbol{E} - \boldsymbol{L}_s \otimes \boldsymbol{SR}_{s+1} - \boldsymbol{SR}_{s+1} \otimes \boldsymbol{L}_s\}]\boldsymbol{z}_j \\ &+\boldsymbol{z}_i \cdot [(\boldsymbol{L}_{j,i} \cdot \boldsymbol{SR}_i)\boldsymbol{E} - \boldsymbol{L}_{j,i} \otimes \boldsymbol{SR}_i]\boldsymbol{z}_j. \end{aligned} \tag{2.128}$$

The kinetic energy K_ζ of ζ-chain is described as

$$K_\zeta = \frac{1}{2}\dot{\bar{\boldsymbol{\theta}}}_\zeta \cdot \boldsymbol{H}_\zeta(\bar{\boldsymbol{\theta}}_\zeta)\dot{\bar{\boldsymbol{\theta}}}_\zeta \tag{2.129}$$

where $\boldsymbol{H}_\zeta(\bar{\boldsymbol{\theta}}_\zeta)$ is also a positive definite symmetric matrix and (i,j)-th entry of it is described in the same way as shown just above.

The whole kinetic energy K of the manipulator is obtained by subtracting the kinetic energy of the common links to ζ-chain and ξ-chain from $K_\xi + K_\zeta$. Defining joint variable vector $\bar{\boldsymbol{\theta}}$ as

$$\bar{\boldsymbol{\theta}} = \left[\begin{array}{ccccccccccccc} \theta_1 & \cdots & \theta_{\xi-1} & \vdots & \theta_\xi & \theta_{\xi+1} & \vdots & \theta_\zeta & \theta_{\zeta+1} & \vdots & \theta_{\xi+2} & \cdots & \theta_N \end{array}\right]^t, \tag{2.130}$$

we can describe K as

$$K = \frac{1}{2}\dot{\bar{\boldsymbol{\theta}}} \cdot \boldsymbol{H}(\bar{\boldsymbol{\theta}})\dot{\bar{\boldsymbol{\theta}}} \tag{2.131}$$

where $\boldsymbol{H}(\bar{\boldsymbol{\theta}})$ is a $(N+2)\times(N+2)$ positive definite symmetric matrix and is given as

$$\boldsymbol{H}(\bar{\boldsymbol{\theta}}) = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{\xi_{11}} & \boldsymbol{H}_{\xi_{12}} & \boldsymbol{H}_{\xi_{13}} \\ \boldsymbol{H}_{\xi_{21}} & \boldsymbol{H}_{\xi_{22}} & \bigcirc_{2\times 2} & \boldsymbol{H}_{\xi_{23}} \\ \boldsymbol{H}_{\zeta_{21}} & \bigcirc_{2\times 2} & \boldsymbol{H}_{\zeta_{22}} & \bigcirc_{2\times p} \\ \boldsymbol{H}_{\xi_{31}} & \boldsymbol{H}_{\xi_{32}} & \bigcirc_{p\times 2} & \boldsymbol{H}_{\xi_{33}} \end{bmatrix} \tag{2.132}$$

where $\bigcirc_{q\times t}$ is a $q\times t$ zero matrix and $p = N-(\xi+1)$, and $\boldsymbol{H}_{\xi_{ij}}$ and $\boldsymbol{H}_{\zeta_{ij}}$ are block matrices in $\boldsymbol{H}_{\xi}(\bar{\boldsymbol{\theta}}_{\xi})$ and $\boldsymbol{H}_{\zeta}(\bar{\boldsymbol{\theta}}_{\zeta})$ when they are divided as

$$\boldsymbol{H}_{\xi}(\bar{\boldsymbol{\theta}}_{\xi}) = \begin{array}{c} \begin{bmatrix} \boldsymbol{H}_{\xi_{11}} & \boldsymbol{H}_{\xi_{12}} & \boldsymbol{H}_{\xi_{13}} \\ \boldsymbol{H}_{\xi_{21}} & \boldsymbol{H}_{\xi_{22}} & \boldsymbol{H}_{\xi_{23}} \\ \boldsymbol{H}_{\xi_{31}} & \boldsymbol{H}_{\xi_{32}} & \boldsymbol{H}_{\xi_{33}} \end{bmatrix} \\ \begin{matrix} \underbrace{}_{\xi-1} & \underbrace{}_{2} & \underbrace{}_{p} \end{matrix} \end{array} \begin{matrix} \}\ \xi-1 \\ \}\ 2 \\ \}\ p \end{matrix} \tag{2.133}$$

$$\boldsymbol{H}_{\zeta}(\bar{\boldsymbol{\theta}}_{\zeta}) = \begin{array}{c} \begin{bmatrix} \boldsymbol{H}_{\zeta_{11}} & \boldsymbol{H}_{\zeta_{12}} \\ \boldsymbol{H}_{\zeta_{21}} & \boldsymbol{H}_{\xi_{22}} \end{bmatrix} \\ \begin{matrix} \underbrace{}_{\xi-1} & \underbrace{}_{2} \end{matrix} \end{array} \begin{matrix} \}\ \xi-1 \\ \}\ 2 \end{matrix} \tag{2.134}$$

$\boldsymbol{H}_{11}$ is a $(\xi-1)\times(\xi-1)$ matrix, and the (i,j)-th entry of it is described as

$$
\begin{aligned}
H_{11}(i,j) &= \boldsymbol{z}_i \cdot [\sum_{S=i}^{N}(\boldsymbol{J}_S) + \boldsymbol{J}_\zeta + \boldsymbol{J}_{\zeta+1}]\boldsymbol{z}_j \\
&+\boldsymbol{z}_i \cdot [\sum_{S=i}^{\xi-1}\{2\boldsymbol{L}_S \cdot (\boldsymbol{S}\boldsymbol{R}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})\boldsymbol{E} \\
&\quad -\boldsymbol{L}_S \otimes (\boldsymbol{S}\boldsymbol{R}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1}) - (\boldsymbol{S}\boldsymbol{R}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1}) \otimes \boldsymbol{L}_S\} \\
&\quad + \sum_{S=\xi}^{N-1}\{2\boldsymbol{L}_S \cdot \boldsymbol{S}\boldsymbol{R}_{S+1}\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{S}\boldsymbol{R}_{S+1} - \boldsymbol{S}\boldsymbol{R}_{S+1} \otimes \boldsymbol{L}_S\} \\
&\quad +2\boldsymbol{L}_\zeta \cdot \boldsymbol{R}_{\zeta+1}\boldsymbol{E} - \boldsymbol{L}_\zeta \otimes \boldsymbol{R}_{\zeta+1} - \boldsymbol{R}_{\zeta+1} \otimes \boldsymbol{L}_\zeta]\boldsymbol{z}_j \\
&+\boldsymbol{z}_i \cdot [\boldsymbol{L}_{j,i} \cdot (\boldsymbol{S}\boldsymbol{R}_i + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})\boldsymbol{E} - \boldsymbol{L}_{j,i} \otimes (\boldsymbol{S}\boldsymbol{R}_i + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})]\boldsymbol{z}_j.
\end{aligned}
\tag{2.135}
$$

Let U denote the potential energy of the manipulator. Then, we can derive the dynamic model of the manipulator from *Lagrange equations.* It is described as

$$
\boldsymbol{\tau} = \boldsymbol{W}^t\boldsymbol{H}\boldsymbol{W}\ddot{\boldsymbol{\theta}} + \frac{d}{dt}(\boldsymbol{W}^t\boldsymbol{H}\boldsymbol{W})\dot{\boldsymbol{\theta}} - \frac{\partial}{\partial\boldsymbol{\theta}}(\frac{1}{2}\dot{\boldsymbol{\theta}} \cdot \boldsymbol{W}^t\boldsymbol{H}\boldsymbol{W}\dot{\boldsymbol{\theta}}) + \frac{\partial}{\partial\boldsymbol{\theta}}U \tag{2.136}
$$

where $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \cdots & \theta_\xi & \vdots & \theta_\zeta & \vdots & \theta_{\xi+2} & \cdots & \theta_N \end{bmatrix}^t$ whose entries are independent variables, and $\boldsymbol{\tau}$ is the generalized force vector (actuated joint torque) corresponding to $\boldsymbol{\theta}$. $\boldsymbol{W}$ is $(N+2) \times N$ Jacobian matrix which relates $\dot{\boldsymbol{\theta}}$ with $\dot{\bar{\boldsymbol{\theta}}}$ as

$$
\dot{\bar{\boldsymbol{\theta}}} = \boldsymbol{W}\dot{\boldsymbol{\theta}} \tag{2.137}
$$

and it is given as

$$
\boldsymbol{W} = \left[\begin{array}{cc|c|cc}
\multicolumn{3}{c|}{\boldsymbol{E}_{\xi\times\xi}} & \multicolumn{2}{c}{\bigcirc_{\xi\times p}} \\
\hline
0\cdots\cdots0 & \frac{\partial\theta_{\xi+1}}{\partial\theta_\xi} & \frac{\partial\theta_{\xi+1}}{\partial\theta_\zeta} & 0\cdots\cdots0 & \\
0\cdots\cdots0 & 0 & 1 & 0\cdots\cdots0 & \\
0\cdots\cdots0 & \frac{\partial\theta_{\zeta+1}}{\partial\theta_\xi} & \frac{\partial\theta_{\zeta+1}}{\partial\theta_\zeta} & 0\cdots\cdots0 & \\
\hline
\multicolumn{3}{c|}{\bigcirc_{p\times(\xi+1)}} & \multicolumn{2}{c}{\boldsymbol{E}_{p\times p}}
\end{array}\right] \tag{2.138}
$$

where

$$
\frac{\partial\theta_{\xi+1}}{\partial\theta_\xi} = -1, \quad \frac{\partial\theta_{\xi+1}}{\partial\theta_\zeta} = 1 \tag{2.139}
$$

$$
\frac{\partial\theta_{\zeta+1}}{\partial\theta_\xi} = 1, \quad \frac{\partial\theta_{\zeta+1}}{\partial\theta_\zeta} = -1 \tag{2.140}
$$

which are obtained from (2.113)-(3.2). $\boldsymbol{E}_{s\times s}$ denotes the $s \times s$ identity matrix. The (i,j)-th entry of $\boldsymbol{W}^t\boldsymbol{H}(\bar{\boldsymbol{\theta}})\boldsymbol{W}$ will be denoted by $H_r(i,j)$ in the followings.

It is obvious that the dynamic model (3.26) is determined if and only if $\boldsymbol{W}^t\boldsymbol{H}(\bar{\boldsymbol{\theta}})\boldsymbol{W}$ and $\frac{\partial}{\partial\boldsymbol{\theta}}U$ are determined as functions of $\boldsymbol{\theta}$.
Evaluating each element of them about an appropriate coordinate system, we can describe them also in the following form:

$$
\sum_{v=1}^{T} \boldsymbol{p}_v f_v \tag{2.141}
$$

where $\boldsymbol{p}_v$ is an inertial parameter and f_v is a polynomial of trigonometric functions of $\boldsymbol{\theta}$. (f_v is allowed to be a constant function.) This fact would be evident below. The form (3.15) will also be called a function of $\boldsymbol{\theta}$ generated by $\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_T$ in later discussions.

2.5.4 A Base Parameter Set

We introduce the following notation to describe a base parameter set. First, ${}^i\boldsymbol{L}_i$ is obviously constant vector, then, it will be denoted by

$$^{i}\boldsymbol{L}_{i} = \left[\ [L_i]^x \ \ 0 \ \ [L_i]^z \ \right]^t. \tag{2.142}$$

$^{i}\boldsymbol{L}_{i}$ denotes the length of the link i and is assumed to be known. Next, since we can derive from (2.113)-(3.2) that

$$\begin{aligned} ^{\xi+1}\boldsymbol{A}_{\zeta} &= \boldsymbol{A}_{\rho} \\ ^{\xi}\boldsymbol{A}_{\zeta+1} &= \boldsymbol{E}_{3\times 3} \end{aligned} \tag{2.143}$$

where $\boldsymbol{A}_{\rho}$ is a constant matrix which is given as

$$\boldsymbol{A}_{\rho} = \begin{bmatrix} -\cos\rho & \sin\rho & 0 \\ -\sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.144}$$

we obtain

$$^{\xi+1}\boldsymbol{R}_{\zeta} = \boldsymbol{A}_{\rho} \ ^{\zeta}\boldsymbol{R}_{\zeta} \tag{2.145}$$

$$^{\xi}\boldsymbol{R}_{\zeta+1} = \ ^{\zeta+1}\boldsymbol{R}_{\zeta+1}. \tag{2.146}$$

Then $^{\xi+1}\boldsymbol{R}_{\zeta}$ and $^{\xi}\boldsymbol{R}_{\zeta+1}$ are constant vectors since $^{n}\boldsymbol{R}_{n}$ is a constant vector. Hence, $^{\xi}\boldsymbol{R}_{\xi}$ and $^{\zeta+1}\boldsymbol{R}_{\zeta+1}$ are combined, also, $^{\xi+1}\boldsymbol{R}_{\xi+1}$ and $\boldsymbol{A}_{\rho}\,^{\zeta}\boldsymbol{R}_{\zeta}$ are combined. Entries of the combined vectors will be denoted by

$$^{\xi}\boldsymbol{R}_{\xi} + \ ^{\zeta+1}\boldsymbol{R}_{\zeta+1} = \left[\ \bar{\mathbf{R}}_{\xi}^{x} \ \ \bar{\mathbf{R}}_{\xi}^{y} \ \ \bar{\mathbf{R}}_{\xi}^{z} \ \right]^t \tag{2.147}$$

$$^{\xi+1}\boldsymbol{R}_{\xi+1} + \boldsymbol{A}_{\rho}\,^{\zeta}\boldsymbol{R}_{\zeta} = \left[\ \bar{\mathbf{R}}_{\xi+1}^{x} \ \ \bar{\mathbf{R}}_{\xi+1}^{y} \ \ \bar{\mathbf{R}}_{\xi+1}^{z} \ \right]^t. \tag{2.148}$$

For consistency, $^{n}\boldsymbol{R}_{n}$ will be denoted by

$$^{n}\boldsymbol{R}_{n} = \left[\ \bar{\mathbf{R}}_{n}^{x} \ \ \bar{\mathbf{R}}_{n}^{y} \ \ \bar{\mathbf{R}}_{n}^{z} \ \right]^t \tag{2.149}$$

for $1 \le n \le \xi - 1$ or $\xi + 2 \le n \le N$. Besides of (2.147) and (2.148), entries of $^{\xi+1}\boldsymbol{R}_{\xi+1}$ and $^{\zeta+1}\boldsymbol{R}_{\zeta+1}$ will be denoted by

$$^{\xi+1}\boldsymbol{R}_{\xi+1} = \left[\begin{array}{ccc} \mathbf{R}^x_{\xi+1} & \mathbf{R}^y_{\xi+1} & \mathbf{R}^z_{\xi+1} \end{array}\right]^t \tag{2.150}$$

$$^{\zeta+1}\boldsymbol{R}_{\zeta+1} = \left[\begin{array}{ccc} \mathbf{R}^x_{\zeta+1} & \mathbf{R}^y_{\zeta+1} & \mathbf{R}^z_{\zeta+1} \end{array}\right]^t \tag{2.151}$$

which are also needed below.

$^n\boldsymbol{J}_n$ is also a constant matrix and it will be denoted by

$$^n\boldsymbol{J}_n = \left[\begin{array}{ccc} \mathbf{J}^x_n & \mathbf{J}^{xy}_n & \mathbf{J}^{xz}_n \\ \mathbf{J}^{xy}_n & \mathbf{J}^y_n & \mathbf{J}^{yz}_n \\ \mathbf{J}^{xz}_n & \mathbf{J}^{yz}_n & \mathbf{J}^z_n \end{array}\right]. \tag{2.152}$$

We define

$$\bar{\mathbf{RZ}}(n) = \bar{\mathbf{R}}^z_{n+1} + \sum_{i=n+2}^{N} \left(\prod_{j=n+2}^{i} \cos\alpha_j \right) \bar{\mathbf{R}}^z_j \tag{2.153}$$

for $1 \le n \le \xi - 1$ or $\xi + 1 \le n \le N$,

$$\bar{\mathbf{RZ}}(\xi) = \mathbf{R}^z_{\xi+1} + \sum_{i=\xi+2}^{N} \left(\prod_{j=\xi+2}^{i} \cos\alpha_j \right) \bar{\mathbf{R}}^z_j, \tag{2.154}$$

$$\bar{\mathbf{RZ}}(\zeta) = \mathbf{R}^z_{\zeta+1}. \tag{2.155}$$

Using these we modify some parameters as follows:

$$\tilde{\mathbf{J}}^x_n = \mathbf{J}^x_n + 2[Ln]^z \cos\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.156}$$

$$\tilde{\mathbf{J}}^y_n = \mathbf{J}^y_n + 2[Ln]^z \cos\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.157}$$

$$\tilde{\mathbf{J}}^{xy}_n = \mathbf{J}^{xy}_n + [Ln]^x \sin\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.158}$$

$$\tilde{\mathbf{J}}^{xz}_n = \mathbf{J}^{xz}_n - [Ln]^x \cos\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.159}$$

$$\tilde{\mathbf{J}}^{yz}_n = \mathbf{J}^{yz}_n + [Ln]^z \sin\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.160}$$

$$\hat{\bar{\mathbf{R}}}^y_n = \bar{\mathbf{R}}^y_n - \sin\alpha_{n+1} \bar{\mathbf{RZ}}(n) \tag{2.161}$$

for $1 \leq n \leq N$ or $n = \zeta, n = \zeta + 1$. We remark here that $\hat{\hat{\mathbf{R}}}_{\xi}^{y} = \bar{\mathbf{R}}_{\xi}^{y}$ since $\alpha_{\xi+1} = 0$. Moreover we define

$$\hat{\mathbf{R}}_{\xi+1}^{y} = \mathbf{R}_{\xi+1}^{y} - \sin\alpha_{\xi+2}\bar{\mathbf{RZ}}(\xi+1). \tag{2.162}$$

Let ${}^{n}\tilde{\boldsymbol{J}}_{n}$ be a matrix whose entries are (2.157)-(2.161) and $\mathbf{J}_{n}^{z}$:

$${}^{n}\tilde{\boldsymbol{J}}_{n} = \begin{bmatrix} \tilde{\mathbf{J}}_{n}^{x} & \tilde{\mathbf{J}}_{n}^{xy} & \tilde{\mathbf{J}}_{n}^{xz} \\ \tilde{\mathbf{J}}_{n}^{xy} & \tilde{\mathbf{J}}_{n}^{y} & \tilde{\mathbf{J}}_{n}^{yz} \\ \tilde{\mathbf{J}}_{n}^{xz} & \tilde{\mathbf{J}}_{n}^{yz} & \mathbf{J}_{n}^{z} \end{bmatrix}. \tag{2.163}$$

Then we obtain from (2.143) and (2.144) that

$${}^{\xi+1}\tilde{\boldsymbol{J}}_{\zeta} = \boldsymbol{A}_{\rho}\ {}^{\zeta}\tilde{\boldsymbol{J}}_{\zeta}\ \boldsymbol{A}_{\rho}^{t} \tag{2.164}$$

$${}^{\xi}\tilde{\boldsymbol{J}}_{\zeta+1} = {}^{\zeta+1}\tilde{\boldsymbol{J}}_{\zeta+1}. \tag{2.165}$$

Hence, ${}^{\xi+1}\tilde{\boldsymbol{J}}_{\zeta}$ and ${}^{\xi}\tilde{\boldsymbol{J}}_{\zeta+1}$ are constant matrices. Therefore ${}^{\xi+1}\tilde{\boldsymbol{J}}_{\xi+1}$ and $\boldsymbol{A}_{\rho}\ {}^{\zeta}\tilde{\boldsymbol{J}}_{\zeta}\ \boldsymbol{A}_{\rho}^{t}$ are combined, also, ${}^{\xi}\tilde{\boldsymbol{J}}_{\xi}$ and ${}^{\zeta+1}\tilde{\boldsymbol{J}}_{\zeta+1}$ are combined. Entries of the combined matrices are denoted by

$${}^{\xi+1}\tilde{\boldsymbol{J}}_{\zeta} + \boldsymbol{A}_{\rho}\ {}^{\zeta}\tilde{\boldsymbol{J}}_{\zeta}\boldsymbol{A}_{\rho}^{t} = \begin{bmatrix} \hat{\mathbf{J}}_{\xi+1}^{x} & \hat{\mathbf{J}}_{\xi+1}^{xy} & \hat{\mathbf{J}}_{\xi+1}^{xz} \\ \hat{\mathbf{J}}_{\xi+1}^{xy} & \hat{\mathbf{J}}_{\xi+1}^{y} & \hat{\mathbf{J}}_{\xi+1}^{yz} \\ \hat{\mathbf{J}}_{\xi+1}^{xz} & \hat{\mathbf{J}}_{\xi+1}^{yz} & \hat{\mathbf{J}}_{\xi+1}^{z} \end{bmatrix} \tag{2.166}$$

$${}^{\xi}\tilde{\boldsymbol{J}}_{\xi} + {}^{\zeta+1}\tilde{\boldsymbol{J}}_{\zeta+1} = \begin{bmatrix} \hat{\mathbf{J}}_{\zeta}^{x} & \hat{\mathbf{J}}_{\zeta}^{xy} & \hat{\mathbf{J}}_{\zeta}^{xz} \\ \hat{\mathbf{J}}_{\zeta}^{xy} & \hat{\mathbf{J}}_{\zeta}^{y} & \hat{\mathbf{J}}_{\zeta}^{yz} \\ \hat{\mathbf{J}}_{\zeta}^{xz} & \hat{\mathbf{J}}_{\zeta}^{yz} & \hat{\mathbf{J}}_{\zeta}^{z} \end{bmatrix}. \tag{2.167}$$

For consistency, entries of ${}^{n}\tilde{\boldsymbol{J}}_{n}$ will be denoted by

$$
{}^{n}\tilde{J}_{n} = \begin{bmatrix} \hat{\mathbf{J}}_{n}^{x} & \hat{\mathbf{J}}_{n}^{xy} & \hat{\mathbf{J}}_{n}^{xz} \\ \hat{\mathbf{J}}_{n}^{xy} & \hat{\mathbf{J}}_{n}^{y} & \hat{\mathbf{J}}_{n}^{yz} \\ \hat{\mathbf{J}}_{n}^{xz} & \hat{\mathbf{J}}_{n}^{yz} & \hat{\mathbf{J}}_{n}^{z} \end{bmatrix} \tag{2.168}
$$

for $1 \leq n \leq \xi - 1$ or $\xi + 2 \leq n \leq N$. Moreover, we define

$$
\mathbf{J}\hat{\mathbf{Y}}(n) = \hat{\mathbf{J}}_{n+1}^{y} + \sum_{i=n+2}^{N} \Big(\prod_{j=n+2}^{i} \cos^2 \alpha_j \Big) \hat{\mathbf{J}}_{j}^{y}. \tag{2.169}
$$

Finally, we introduce some more symbols to state a theorem. Let K denote the maximum link number such that $\boldsymbol{z}_n$ is parallel to $\boldsymbol{z}_1$ that is $\alpha_2 = \cdots = \alpha_n = 0$ for $1 \leq n \leq K$. (If $\boldsymbol{z}_2$ is not parallel to $\boldsymbol{z}_1$, then $K = 1$.) We define link number Q less than $K+1$ as follows; $Q = 0$ if $\boldsymbol{z}_1$ is not parallel to gravity vector $\boldsymbol{g}$, otherwise Q is the minimum link number such that $\boldsymbol{z}_{Q+1}$ is not parallel to $\boldsymbol{g}$ or $[L_Q]^x \neq 0$.

We give a base parameter set.

Theorem 2.5.1 The following inertial parameters constitute a base parameter set of the dynamic model (3.26).

$$
\hat{\mathbf{J}}_{n}^{z} + \sin^2 \alpha_{n+1} \mathbf{J}\hat{\mathbf{Y}}(n) \tag{2.170}
$$

for $1 \leq n \leq N$,

$$
\bar{\mathbf{R}}_{n}^{x}, \ \hat{\bar{\mathbf{R}}}_{n}^{y} \tag{2.171}
$$

for $Q + 1 \leq n \leq N$,

$$
\hat{\mathbf{J}}_{n}^{x} - \hat{\mathbf{J}}_{n}^{y} + \sin^2 \alpha_{n+1} \mathbf{J}\hat{\mathbf{Y}}(n), \ \hat{\mathbf{J}}_{n}^{xy}, \ \hat{\mathbf{J}}_{n}^{xz}, \ \hat{\mathbf{J}}_{n}^{yz} + \sin \alpha_{n+1} \cos \alpha_{n+1} \mathbf{J}\hat{\mathbf{Y}}(n) \tag{2.172}
$$

for $K + 1 \leq n \leq N$, and

$$
\begin{aligned}
&[L_\xi]^x \mathbf{R}_{\xi+1}^{x} - \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^{x} - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^{y}, \\
&1[L_\xi]^x \hat{\mathbf{R}}_{\xi+1}^{y} - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^{x} + \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^{y}
\end{aligned} \tag{2.173}
$$

if $\xi + 1 \leq K$, otherwise

$$\mathbf{R}^{y}_{\zeta+1},\ \ [L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1},\ \ [L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} \qquad (2.174)$$

where $\bar{\mathbf{R}}^x_{\xi+1}$ and $\hat{\bar{\mathbf{R}}}^y_{\xi+1}$ are deleted if $\xi + 1 \leq K$ and $Q = \xi$.

The total number B of base parameters in a base parameter set is given as $B = 7N - 4K - 2Q + 3 - B_1$ where $B_1 = 3$ if $\xi + 1 \leq K$ and $Q = \xi$ or $B_1 = 1$ if $\xi + 1 \leq K$ and $Q \neq \xi$, otherwise it is 0.

Remark: Note that the size of a base parameter set is independent of virtually cut joint in the parallelogram.

We first show the following lemmas for the proof of the **Theorem 2.5.1** , introducing following notation. Let $\delta(n)$ be the link number such that $\delta(n) \leq n, \alpha_S = 0$ for $\delta(n) + 1 \leq s \leq n$, and $\alpha_{\delta(n)} \neq 0$. Similarly we define $\delta_{-k}(n)$ and $\delta_k(n)$ for $k = 1, 2, \ldots$ as $\delta_{-k}(n) = \delta(\delta_{-(k-1)}(n) - 1)$ regarding $\delta(n)$ as $\delta_0(n)$, and $\delta_k(n)$ is the link number such that $n + 1 \leq \delta_1(n) < \delta_2(n) < \cdots < \delta_k(n)$, $\alpha_{\delta_w(n)} \neq 0$ for $1 \leq w \leq k$, and $\alpha_S = 0$ for $s \neq \delta_w(n)$ and $n \leq s \leq \delta_k(n) - 1$. It is obvious that $\delta(n) = n$ when $\alpha_n \neq 0$. In case $K + 1 \leq n$, $\delta(n)$ always exists since $\alpha_{K+1} \neq 0$.(See Fig.2.5.4)

Lemma 2.5.1 Suppose any entry of $\boldsymbol{W}^T\boldsymbol{H}(\bar{\boldsymbol{\theta}})\boldsymbol{W}$ or $\frac{\partial}{\partial\boldsymbol{\theta}}U$ is described as

$$\sum_{u=1}^{U_1}\mathbf{p}_u f_u + \sum_{u=U_1+1}^{U_2}\mathbf{p}_u f_u \qquad (2.175)$$

where $\mathbf{p}_u$ is an inertial parameter and f_u is a polynomial of trigonometric functions of $\boldsymbol{\theta}$. If f_u for $1 \leq u \leq U_1$ are linearly independent functions and $\mathbf{p}_u$ for $U_1 + 1 \leq u \leq U_2$ are fundamental parameters, then $\mathbf{p}_u$ for $1 \leq u \leq U_1$ are fundamental parameters.

Lemma 2.5.2 ${}^i\boldsymbol{SR}_i +{}^i \boldsymbol{R}_\zeta +{}^i \boldsymbol{R}_{\zeta+1}$ can be represented as

$$ {}^i\boldsymbol{SR}_i +{}^i \boldsymbol{R}_\zeta +{}^i \boldsymbol{R}_{\zeta+1} = \begin{bmatrix} \bar{\mathbf{R}}^x_i \\ \\ \hat{\bar{\mathbf{R}}}^y_i \\ \\ \bar{\mathbf{R}}^z_i + \cos\alpha_{i+1}\bar{\mathbf{R}}\mathbf{Z}(i) \end{bmatrix} + \boldsymbol{G}_2 \qquad (2.176)$$

for $1 \leq i \leq N$ where

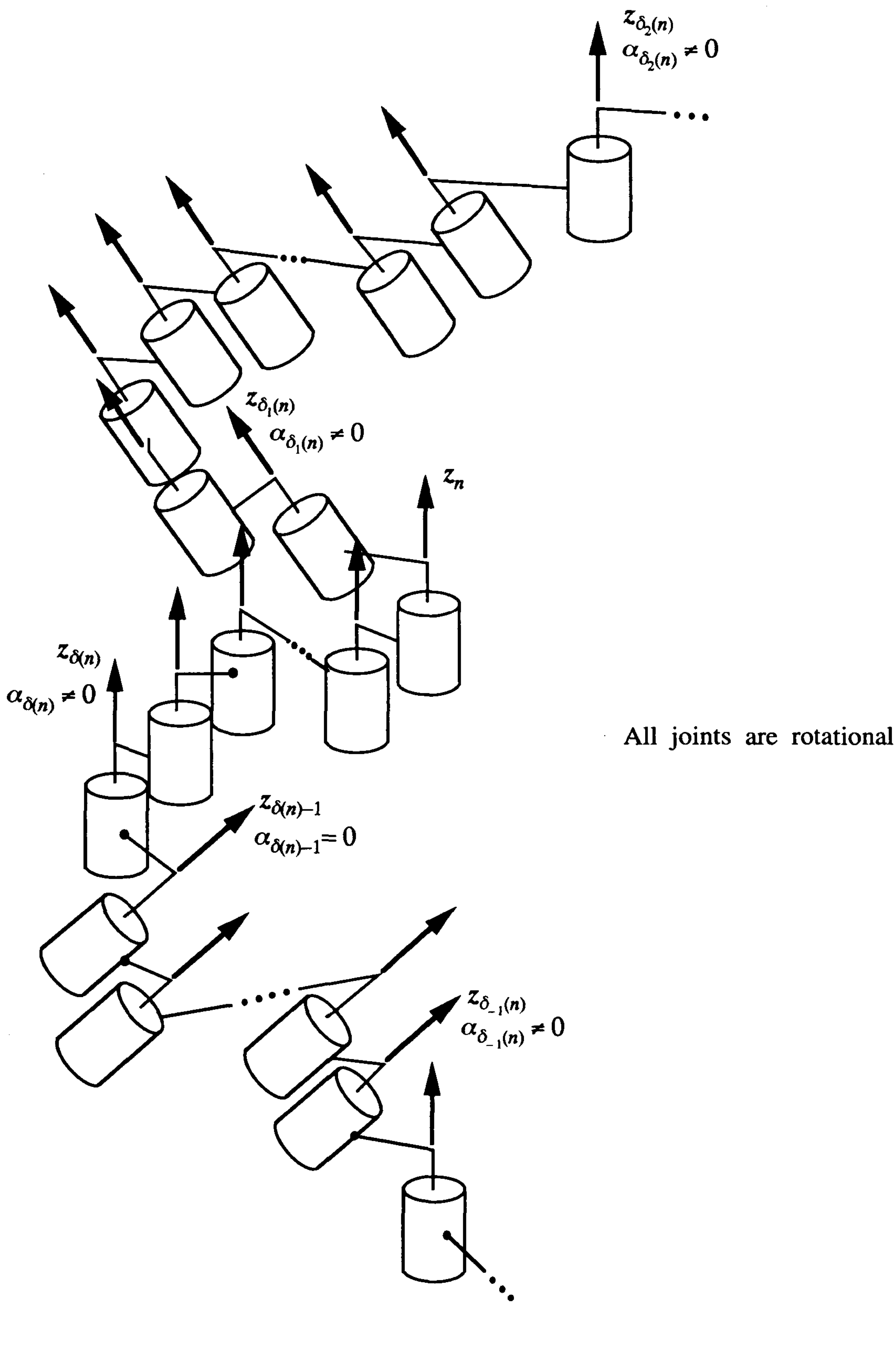


Fig. 2.5.4 Link Number: $\delta(n)$

$$G_2 = \begin{cases} {}^i\boldsymbol{R}_\zeta + {}^i\boldsymbol{R}_{\zeta+1} + \boldsymbol{G}', & \text{if} \quad \xi + 2 \leq i \\ {}^i\boldsymbol{R}_{\zeta+1} + \boldsymbol{G}', & \text{if} \quad i = \xi + 1 \\ \boldsymbol{G}', & \text{if} \quad i \leq \xi \end{cases} \tag{2.177}$$

where $\boldsymbol{G}'$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$.

Lemma 2.5.3

$$\begin{aligned} H_r(i,i) &= \sum_{S=i}^{\delta_1(i)-1} (\mathbf{J}_S^z + \sin^2 \alpha_{S+1} \hat{\mathbf{J}\mathbf{Y}}(s)) \\ &+ \sum_{S=\delta_1(i)}^{\delta_2(i)-1} [\sin^2 \theta(\delta_1(i), s) \sin^2 \alpha_{\delta_1(i)} (\hat{\mathbf{J}}_S^x - \hat{\mathbf{J}}_S^y + \sin^2 \alpha_{S+1} \mathbf{J}\hat{\mathbf{Y}}(s)) \\ &\qquad +2 \sin \theta(\delta_1(i), s) \cos \theta(\delta_1(i), s) \sin^2 \alpha_{\delta_1(i)} \hat{\mathbf{J}}_S^{xy}] \\ &+G_{3,1} + G_{3,2} \end{aligned} \tag{2.178}$$

for $1 \leq i \leq \xi - 1$ or $\xi \leq i \leq N$ where $G_{3,1}$ is a function of $\boldsymbol{\theta}$ generated by $\hat{\mathbf{J}}_S^z + \sin^2 \alpha_{S+1} \mathbf{J}\hat{\mathbf{Y}}(s)$ for $\delta_1(i) \leq s \leq N$, $\hat{\mathbf{J}}_S^{xz}$ and $\hat{\mathbf{J}}_S^{yz} + \sin \alpha_{S+1} \cos \alpha_{S+1} \mathbf{J}\hat{\mathbf{Y}}(s)$ for $\delta_1(i) \leq s \leq N$, $\hat{\mathbf{J}}_S^x - \hat{\mathbf{J}}_S^y + \sin^2 \alpha_{S+1} \mathbf{J}\hat{\mathbf{Y}}(s)$ and $\hat{\mathbf{J}}_S^{xy}$ for $\delta_2(i) \leq s \leq N$, and $G_{3,2}$ is as follows:

i) In case of $\xi + 1 \leq K$

a) If $k < i$, $G_{3,2}$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$

b) If $\xi + 1 \leq i \leq K$,

$$G_{3,2} = \sum_{S=i}^{K} [L_S]^x \bar{G}_S + G_{3,3} \tag{2.179}$$

where $\bar{G}_S$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_n^x$ and $\hat{\bar{\mathbf{R}}}_n^y$ for $s+1 \leq n \leq N$ and $G_{3,3}$ by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $K+2 \leq s \leq N$

c) If $1 \leq i \leq \xi - 1$,

$$\begin{aligned} G_{3,2} &= \sum_{s=i}^{\xi-1} [L_S]^x \bar{G}_S + G_{3,3} \\ &+2\sin\theta_{\xi+1}(-[L_\xi]^x \hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1} - \cos\rho[L_\zeta]^x \mathbf{R}^y_{\zeta+1}) \\ &+2\cos\theta_{\xi+1}([L_\xi]^x \mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x \mathbf{R}^y_{\zeta+1}) \end{aligned} \tag{2.180}$$

where $\bar{G}_S$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_n$ and $\hat{\bar{\mathbf{R}}}^y_n$ for $s + 1 \leq n \leq N$ and $G_{3,3}$ by $\bar{\mathbf{R}}^x_S$ and $\hat{\bar{\mathbf{R}}}^y_S$ for $\xi + 2 \leq s \leq N$.

ii) In case of $K < \xi$

a) If $\xi + 1 \leq i$, $G_{3,2}$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_S$ and $\hat{\bar{\mathbf{R}}}^y_S$ for $i + 1 \leq s \leq N$

b) If $K < \delta(\xi) \leq i \leq \xi - 1$,

$$\begin{aligned} G_{3,2} &= 2\sum_{s=i}^{\xi-1} [L_S]^x \bar{G}_S + G_{3,3} \\ &+2\sin\theta_{\xi+1}(-[L_\xi]^x \hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1} - \cos\rho[L_\zeta]^x \mathbf{R}^y_{\zeta+1}) \\ &+2\cos\theta_{\xi+1}([L_\xi]^x \mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x \mathbf{R}^y_{\zeta+1}) \end{aligned} \tag{2.181}$$

where $\bar{G}_S$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_S$ and $\hat{\bar{\mathbf{R}}}^y_S$ for $\xi + 2 \leq s \leq N$.

c) If $K \leq i \leq \delta(\xi) - 1$,

$$\begin{aligned} G_{3,2} &= f_1 \mathbf{R}^y_{\zeta+1} \\ &+f_2([L_\xi]^x \hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1}) \\ &+f_3([L_\xi]^x \mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x \mathbf{R}^x_{\zeta+1}) \\ &+G_{3,3} \end{aligned} \tag{2.182}$$

where $G_{3,3}$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_S$ and $\hat{\bar{\mathbf{R}}}^y_S$ for $i + 1 \leq s \leq N$.

d) If $1 \leq i < K$,

$$
\begin{aligned}
G_{3,2} &= 2\sum_{s=i}^{\xi-1}[L_s]^x \bar{G}_s + G_{3,3} \\
&+ f_1 \mathbf{R}^x_{\zeta+1} \\
&+ f_2([L_\xi]^x \hat{\bar{\mathbf{R}}}^y_{\xi+1} - \sin\rho [L_\zeta]^x \mathbf{R}^x_{\zeta+1}) \\
&+ f_3([L_\xi]^x \mathbf{R}^x_{\xi+1} - \cos\rho [L_\zeta]^x \mathbf{R}^x_{\zeta+1})
\end{aligned}
\tag{2.183}
$$

where $\bar{G}_s$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_n$ and $\hat{\bar{\mathbf{R}}}^y_n$ for $s+1 \leq n \leq N$ and $G_{3,3}$ by $\bar{\mathbf{R}}^x_s$ and $\hat{\bar{\mathbf{R}}}^y_s$ for $K+2 \leq s \leq N$.

In c) or d) f_1, f_2, f_3 are as follows:

$$
\begin{aligned}
f_1 = \;& [L_\xi]^x\{[({}^\xi A_i)^2_{13} + ({}^\xi A_i)^2_{33})](-2\sin\theta_{\xi+1}\cos\rho - 2\cos\theta_{\xi+1}\sin\rho) \\
&+ 2({}^\xi A_i)_{13}({}^\xi A_i)_{23}(-2\sin\theta_{\xi+1}\sin\rho + 2\cos\theta_{\xi+1}\cos\rho)\} \\
f_2 = \;& [({}^\xi A_i)^2_{23} + ({}^\xi A_i)^2_{33}](-2\sin\theta_{\xi+1}) + 2({}^\xi A_i)_{13}({}^\xi A_i)_{23}\cos\theta_{\xi+1} \\
f_3 = \;& [({}^\xi A_i)^2_{23} + ({}^\xi A_i)^2_{33}](2\cos\theta_{\xi+1}) + 2({}^\xi A_i)_{13}({}^\xi A_i)_{23}\sin\theta_{\xi+1}
\end{aligned}
\tag{2.184}
$$

and

$$
Hr(\xi,\xi) = \hat{\mathbf{J}}^z_\xi + \sin^2\alpha_{\xi+1}\mathbf{J}\hat{\mathbf{Y}}(\xi). \tag{2.185}
$$

Lemma 2.5.4

$$
\begin{aligned}
Hr(i,\delta(i)-1) = \;& \sum_{s=i}^{\delta_1(i)-1} [\cos\theta(\delta(i),s)\sin\alpha_{\delta(i)}(\hat{\mathbf{J}}^{yz}_i + \sin\alpha_{i+1}\cos\alpha_{i+1}\mathbf{J}\hat{\mathbf{Y}}(i)) \\
&+ \sin\theta(\delta(i),s)\sin\alpha_{\delta(i)}\hat{\mathbf{J}}^{xz}_i] \\
&+ G_{4,1} + G_{4,2}
\end{aligned}
\tag{2.186}
$$

for $K+1 \leq i \leq N$ and $i \neq \xi$ where $G_{4,1}$ is a function of $\boldsymbol{\theta}$ generated by $\hat{\mathbf{J}}^z_s + \sin^2\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s)$, $\bar{\mathbf{R}}^x_s$ and $\hat{\bar{\mathbf{R}}}^y_s$ for $i \leq s \leq N$, $\hat{\mathbf{J}}^{xz}_s$ and $\hat{\mathbf{J}}^{yz}_s + \sin\alpha_{s+1}\cos\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s)$ for $\delta_1(i) \leq s \leq N$, $\hat{\mathbf{J}}^x_s - \hat{\mathbf{J}}^y_s + \sin^2\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s)$ and $\hat{\mathbf{J}}^{xy}_s$ for $\delta_1(i) \leq s \leq N$, and $G_{4,2}$ is as follows:

i) In case of $\xi + 1 \leq K, G_{4,2}$ disappears

ii) In case of $K \leq \xi - 1$

a) If $\xi + 2 \leq i, G_{4,2}$ disappears

b) If $\delta(\xi) \leq i \leq \xi+1, G_{4,2}$ is a function of $\boldsymbol{\theta}$ generated by $(-[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \cos\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})$ and $([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})$

c) If $K+1 \leq i \leq \delta(\xi) - 1, G_{4,2}$ is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}^y_{\zeta+1}$, $([L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1})$, and $([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1})$,

and

$$
\begin{aligned}
Hr(\xi, \delta(\xi)-1) = \;& \cos\theta(\delta(\xi),\xi)\sin\alpha_{\delta(\xi)}(\hat{\mathbf{J}}^{yz}_\xi + \sin\alpha_{\xi+1}\cos\alpha_{\xi+1}\hat{\mathbf{J}\mathbf{Y}}(\xi)) \\
& + \sin\theta(\delta(\xi),\xi)\sin\alpha_{\delta(\xi)}\hat{\mathbf{J}}^{xz}_\xi + \cos\alpha_{\delta(\xi)}(\hat{\mathbf{J}}^z_\xi + \sin^2\alpha_{\xi+1}\hat{\mathbf{J}\mathbf{Y}}(\xi)) \\
& + [\cos\theta_{\xi+1}([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1}) \\
& - \sin\theta_{\xi+1}([L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} + \cos\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})]({}^{\delta(\xi)-1}A_\xi)_{33} \\
& + \bar{\mathbf{R}}^x_\xi([{}^\xi L_{\delta(\xi)-1,\xi}]^x({}^{\delta(\xi)-1}A_\xi)_{33} - [{}^\xi L_{\delta(\xi)-1,\xi}]^z({}^{\delta(\xi)-1}A_\xi)_{31}) \\
& + \hat{\bar{\mathbf{R}}}^y_\xi([{}^\xi L_{\delta(\xi)-1,\xi}]^y({}^{\delta(\xi)-1}A_\xi)_{33} - [{}^\xi L_{\delta(\xi)-1,\xi}]^z({}^{\delta(\xi)-1}A_\xi)_{32}) \\
& + G
\end{aligned}
\tag{2.187}
$$

where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}^x_s$ and $\hat{\bar{\mathbf{R}}}^y_s$ for $\xi + 2 \leq s \leq N$.

Lemma 2.5.5

$$
\begin{aligned}
Hr(i,j) = \;& Hr(i,i) \\
& + \bar{\mathbf{R}}^x_i \sum_{S=j}^{i-1}[L_S]^x\cos\theta(s+1,i) \\
& - \hat{\bar{\mathbf{R}}}^y_i \sum_{S=j}^{i-1}[L_S]^x\sin\theta(s+1,i) \\
& + G_5
\end{aligned}
\tag{2.188}
$$

for $1 \leq j \leq i \leq K$ and $j \neq \xi$, where G_5 is as follows:

i) In case of $K \leq \xi - 1$,G_5 is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$,

ii) In case of $\xi + 1 \leq K$

a) If $\xi + 2 \leq i \leq K, G_5$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$,

b) If $i = \xi$ or $i = \xi + 1$ and $1 \leq j \leq \xi - 1, G_5$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $\xi + 2 \leq s \leq N, [L_\xi]^x \mathbf{R}_{\xi+1}^x - \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^x - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^y$, and $[L_\xi]^x \hat{\mathbf{R}}_{\xi+1}^y - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^y + \cos\rho [L_\zeta]^x \hat{\mathbf{R}}_{\zeta+1}^y$,

c) If $1 \leq j \leq i \leq \xi - 1, G_5$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$, and

$$
\begin{aligned}
Hr(i,\xi) &= \cos\theta(\xi+1,i)[L_\xi]^x \bar{\mathbf{R}}_i^x \\
&\quad - \sin\theta(\xi+1,i)[L_\xi]^x \hat{\bar{\mathbf{R}}}_i^y \\
&\quad + G_{5,1}
\end{aligned}
\tag{2.189}
$$

for $\xi + 2 \leq i \leq K$ where $G_{5,1}$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$for $i+1 \leq s \leq N$.

Lemma 2.5.6 $\quad Hr(i,j)$ for $1 \leq i,j \leq N$ is a function of $\boldsymbol{\theta}$ generated by some inertial parameters given in **Theorem 2.5.1**

The proof of **Lemmas 2.5.1** is same as that of **Lemmas 2.3.1**, hence we omit it. The proofs of **Lemmas 2.5.2, 2.5.3, 2.5.4, 2.5.5** and **2.5.6** are given in Appendix.
(Proof of Theorem 2.5.1) Representing ${}^i\boldsymbol{g}$ as ${}^i\boldsymbol{g} = \begin{bmatrix} {}^ig_x & {}^ig_y & {}^ig_z \end{bmatrix}^T$ and using **Lemma 2.5.2**, we evaluate $\boldsymbol{g} \cdot (\sum_{S=1}^{N} \boldsymbol{R}_S + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})(= U)$ about coordinate system $(o_{i-1}; \boldsymbol{x}_{i-1}, \boldsymbol{y}_{i-1}, \boldsymbol{z}_{i-1})$. Then partially differentiating it by θ_i and using the relation ${}^i\boldsymbol{g} = {}^{i-1}\boldsymbol{A}_i^t\ {}^{i-1}\boldsymbol{g}$, we obtain

$$
\frac{\partial}{\partial\theta_i} U = {}^ig_y \bar{\mathbf{R}}_i^x - {}^ig_x \hat{\bar{\mathbf{R}}}_i^y + G \tag{2.190}
$$

for $1 \leq i \leq N$ and $i \neq \xi + 1$, and

$$
\frac{\partial}{\partial\theta_\zeta} U = {}^{\xi+1}g_y \bar{\mathbf{R}}_{\xi+1}^x - {}^{\xi+1}g_x \hat{\bar{\mathbf{R}}}_{\xi+1}^y + G \tag{2.191}
$$

where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \leq s \leq N$ if $i \neq \xi$, for $\xi + 2 \leq s \leq N$ if $i = \zeta$, and $G = 0$ if $i = \xi$ or $i = N$. It is easy to show that ${}^ig_x$ and

${}^{i}g_y$ are nonzero independent functions of θ_s for $1 \le s \le i$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$. or for $K+1 \le s \le i$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$. Hence, if $\bar{\mathbf{R}}_s^x$ and $\hat{\bar{\mathbf{R}}}_s^y$ for $i+1 \le s \le N$ are assumed to be fundamental parameters, $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ are also fundamental parameters by **Lemma 2.5.1**. It can be derived in the same way that $\bar{\mathbf{R}}_N^x$ and $\hat{\bar{\mathbf{R}}}_N^y$ are fundamental parameters. By mathematical induction, we can conclude that $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ are fundamental parameters for $1 \le i \le N$ if $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$ or for $K+1 \le i \le N$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$.

In the following, we first give a proof in case of $K < \xi$. For an arbitrary link number n such that $\xi + 2 \le n \le N$, assume that

$$\hat{\mathbf{J}}_i^z + \sin^2 \alpha_{i+1} \mathbf{J}\hat{\mathbf{Y}}(i) \tag{2.192}$$

for $n+1 \le i \le N$,

$$\hat{\mathbf{J}}_i^{xz}, \ \hat{\mathbf{J}}_i^{yz} + \sin \alpha_{i+1} \cos \alpha_{i+1} \mathbf{J}\hat{\mathbf{Y}}(i) \tag{2.193}$$

for $n+1 \le i \le N$, and

$$\hat{\mathbf{J}}_i^x - \hat{\mathbf{J}}_i^y + \sin^2 \alpha_{i+1} \mathbf{J}\hat{\mathbf{Y}}(i), \ \hat{\mathbf{J}}_i^{xy} \tag{2.194}$$

for $\delta_1(n) \le i \le N$ are proved to be fundamental parameters. Let $\boldsymbol{F}$ be the set of these fundamental parameters and $\bar{\mathbf{R}}_s^x$ and $\hat{\bar{\mathbf{R}}}_s^y$ for $K+1 \le s \le N$. From **Lemma 2.5.4** we can derive that

$$\begin{aligned} Hr(n, \delta(n)-1) &= \sin \alpha_{\delta(n)} \cos \theta_{\delta(n)} (\hat{\mathbf{J}}_n^{yz} + \sin \alpha_{n+1} \cos \alpha_{n+1} \mathbf{J}\hat{\mathbf{Y}}(n)) \\ &\quad + \sin \alpha_{\delta(n)} \sin \theta_{\delta(n)} \hat{\mathbf{J}}_n^{xz} + G \end{aligned} \tag{2.195}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. Since $\alpha_{\delta(n)} \ne 0$, then $\sin \alpha_{\delta(n)} \cos \theta_{\delta(n)}$ and $\sin \alpha_{\delta(n)} \sin \theta_{\delta(n)}$ are linearly independent functions. Hence we can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}_n^{yz} + \sin \alpha_{\delta(n+1)} \cos \alpha_{\delta(n+1)} \mathbf{J}\hat{\mathbf{Y}}(n)$ and $\hat{\mathbf{J}}_n^{xz}$ are fundamental parameters. Next, in case of $\delta(n) \le n-1$ we can derive from **Lemma 2.5.3** that

$$Hr(n-1, n-1) = \hat{\mathbf{J}}_{n-1}^z + \sin^2 \alpha_n \mathbf{J}\hat{\mathbf{Y}}(n-1) + G \tag{2.196}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. By **Lemma 2.5.1**, $\hat{\mathbf{J}}_{n-1}^z + \sin^2 \alpha_n \mathbf{J}\hat{\mathbf{Y}}(n-1)$ is proved to be a fundamental parameter. Add these new fundamental parameters to the set $\boldsymbol{F}$.

Investigating $Hr(i,\delta(n)-1), Hr(i-1,i-1)$ for $i = n, n-1, \ldots, \delta(n)+1$ and $Hr(\delta(n),\delta(n)-1)$ in order for an n such that $\delta_1(\xi+1) \le n \le N$, we can prove that the inertial parameters in (2.192) for $\delta(n) \le i \le n-1$ and the inertial parameters in (2.193) for $\delta(n) \le i \le n$ are fundamental parameters by use of the argument above. Add all these new fundamental parameters to $\boldsymbol{F}$. Next, we can derive from **Lemma 2.5.3** that

$$\begin{aligned} Hr(\delta(n)-1,\delta(n)-1) \;&= \hat{\mathbf{J}}^{z}_{\delta(n)-1} + \sin^2\alpha_{\delta(n)}\hat{\mathbf{J}\mathbf{Y}}(\delta(n)-1) \\ &+ \sum_{S=\delta(n)}^{\delta_1(n)-1} [\sin^2\alpha_{\delta(n)} \sin^2\theta(\delta(n),s)(\hat{\mathbf{J}}^{x}_{S} - \hat{\mathbf{J}}^{y}_{S} + \sin^2\alpha_{S+1}\hat{\mathbf{J}\mathbf{Y}}(s)) \\ &+2\sin^2\alpha_{\delta(n)} \sin\theta(\delta(n),s)\cos\theta(\delta(n),s)\hat{\mathbf{J}}^{xy}_{S}] \\ &+G \end{aligned} \tag{2.197}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. Since $\alpha_{\delta(n)} \neq 0$, then $\sin\alpha_{\delta(n)} \neq 0$. Hence $\sin^2\alpha_{\delta(n)}\sin^2\theta(\delta(n),s)$ and $\sin^2\alpha_{\delta(n)}\sin\theta(\delta(n),s)\cos\theta(\delta(n),s)$ for $\delta(n) \le s \le \delta_1(n)-1$ can be easily proved to be linearly independent functions. We can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}^{z}_{\delta(n)-1} + \sin^2\alpha_{\delta(n)}\hat{\mathbf{J}\mathbf{Y}}(\delta(n)-1)$ and the inertial parameters in (2.194) for $\delta(n) \le i \le \delta_1(n)-1$ are fundamental parameters. Add these new fundamental parameters to $\boldsymbol{F}$. Since ${}^{N}\boldsymbol{z}_N = \boldsymbol{e}_3$, it is derived that $Hr(N,N) = \hat{\mathbf{J}}^{z}_{N}$. $\hat{\mathbf{J}}^{z}_{N}$ is a fundamental parameter by **Lemma 2.5.1**. Using the mathematical induction from $n = N$ to $n = \delta_2(\xi+1)-1$ we can prove that the inertial parameters in (2.192) for $\delta_1(\xi+1)-1 \le i \le N$, the inertial parameters in (2.193) and (2.194) for $\delta_1(\xi+1) \le i \le N$ are fundamental parameters. Add these new fundamental parameters to the set $\boldsymbol{F}$.

Next, using (2.195), (2.196) again, we investigate $Hr(i,\delta(i)-1), Hr(i-1,i-1)$ for $i = \delta_1(\xi+1)-1, \ldots, \xi+2$ in order, then, since the inertial parameters in (2.192) for $\delta_1(\xi+1)-1 \le i \le N$ and the inertial parameters in (2.193) and (2.194) for $\delta_1(\xi+1) \le i \le N$ have been proved to be fundamental parameters, we can prove that the inertial parameters in (2.192) for $\xi+1 \le i \le \delta_1(\xi+1)-2$ and the inertial parameters in (2.193) for $\xi+1 \le i \le \delta_1(\xi+1)-1$ are fundamental parameters . Add these new fundamental parameters to the set $\boldsymbol{F}$.

Next, From **Lemma 2.5.3**, we can derive that

$$Hr(\xi,\xi) = \hat{\mathbf{J}}^{z}_{\xi} + \sin^2\alpha_{\xi+1}\hat{\mathbf{J}\mathbf{Y}}(\xi). \tag{2.198}$$

We can prove that it is a fundamental parameter by **Lemma 2.5.1**. Add it to $\boldsymbol{F}$. Next, for an arbitrary n such that $\delta(\xi) \le n \le \xi-1$, we can derive from **Lemma 2.5.3** that

$$\begin{aligned} Hr(n,n) &= \sum_{s=n}^{\delta_1(n)-1} (\hat{\mathbf{J}}_n^z + \sin^2\alpha_{n+1} + \mathbf{J}\hat{\mathbf{Y}}(n)) \\ &+2\sin\theta_{\xi+1}(-[L_\xi]^x\hat{\mathbf{R}}_{\xi+1}^y + \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x - \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y) \\ &+2\cos\theta_{\xi+1}([L_\xi]^x\mathbf{R}_{\xi+1}^x - \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x - \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y) \\ &+G \end{aligned} \tag{2.199}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. Investigating $Hr(n,n)$ when $n = \xi - 1$, we can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}_{\xi-1}^z + \sin^2\alpha_\xi + \mathbf{J}\hat{\mathbf{Y}}(\xi - 1)$, $(-[L_\xi]^x\hat{\mathbf{R}}_{\xi+1}^y + \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x - \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y)$, and $([L_\xi]^x\mathbf{R}_{\xi+1}^x - \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x - \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y)$ are fundamental parameters since $\sin\theta_{\xi+1}$ and $\cos\theta_{\xi+1}$ are linearly independent functions. Add these new fundamental parameters to $\boldsymbol{F}$. Here, assume that the inertial parameters in (2.192) for $n+1 \le i \le \xi - 2$ are proved to be fundamental parameters. Add these to $\boldsymbol{F}$. Then for an arbitrary n such that $\delta(\xi) \le n \le \xi - 2$, we can derive from (2.199) that

$$\begin{aligned} Hr(n,n) &= \hat{\mathbf{J}}_n^z + \sin^2\alpha_{n+1} + \mathbf{J}\hat{\mathbf{Y}}(n) \\ &+G \end{aligned} \tag{2.200}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$, hence we can prove $\hat{\mathbf{J}}_n^z + \sin^2\alpha_{n+1} + \mathbf{J}\hat{\mathbf{Y}}(n)$ to be a fundamental parameter by **Lemma 2.5.1**. Add it to $\boldsymbol{F}$. Using the mathematical induction from $n = \xi - 2$ to $n = \delta(\xi)$, we can conclude that the inertial parameters in (2.192) for $\delta(\xi) \le i \le \xi - 2$ are fundamental parameters. Add these to $\boldsymbol{F}$.

Next, for an arbitrary n such that $\delta(\xi) \le n \le \xi + 1$, assume that the inertial parameters in (2.193) for $n+1 \le i \le \xi + 1$, and add these to $\boldsymbol{F}$. Then, from **Lemma 2.5.4** we can derive that

$$\begin{aligned} Hr(n,\delta(n)-1) &= \cos\theta(\delta(n),n)\sin\alpha_{\delta(n)}(\hat{\mathbf{J}}_n^{yz} + \sin\alpha_{n+1}\cos\alpha_{n+1}\mathbf{J}\hat{\mathbf{Y}}(n)) \\ &+\sin\theta(\delta(n),n)\sin\alpha_{\delta(n)}\hat{\mathbf{J}}_n^{xz} \\ &+G \end{aligned} \tag{2.201}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. We can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}_n^{yz} + \sin\alpha_{n+1}\cos\alpha_{n+1}\hat{\mathbf{J}\mathbf{Y}}(n)$ and $\hat{\mathbf{J}}_n^{xz}$ are fundamental parameters because $\cos\theta(\delta(n),n)\sin\alpha_{\delta(\xi+1)}$ and $\sin\theta(\delta(n),n)\sin\alpha_{\delta(\xi+1)}$ are linearly independent functions since $\sin\alpha_{\delta(\xi+1)} \neq 0$. Add these new fundamental parameters to $\boldsymbol{F}$. Using the mathematical induction from $n = \xi+1$ to $n = \delta(\xi)$ in order, we can conclude that the inertial parameters in (2.193) for $\delta(\xi) \leq i \leq \xi+1$ are fundamental parameters.

Next, from **Lemma 2.5.3** we can derive that

$$
\begin{aligned}
Hr(\delta(\xi)-1,\delta(\xi)-1) \;&= \hat{\mathbf{J}}^z_{\delta(\xi)-1} + \sin^2\alpha_{\delta(\xi)}\hat{\mathbf{J}\mathbf{Y}}(\delta(\xi)-1) \\
&+ \sum_{s=\delta(\xi)}^{\delta_1(\xi)-1} [\sin^2\theta(\delta(\xi),s)\sin^2\alpha_{\delta(\xi)}(\hat{\mathbf{J}}^x_s - \hat{\mathbf{J}}^y_s + \sin^2\alpha_{s+1}\hat{\mathbf{J}\mathbf{Y}}(s)) \\
&\quad +2\sin\theta(\delta(\xi),s)\cos\theta(\delta(\xi),s)\sin^2\alpha_{\delta(\xi)}\hat{\mathbf{J}}^{xy}_s] \\
&-2f_1([L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1}) \\
&+2f_2([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1}) \\
&+2f_3\mathbf{R}^y_{\zeta+1} \\
&+G
\end{aligned}
\tag{2.202}
$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. f_1, f_2, f_3 and $(\sin^2\theta(\delta(\xi),s)\sin^2\alpha_{\delta(\xi)})$ and $(\sin\theta(\delta(\xi),s)\cos\theta(\delta(\xi),s)\sin^2\alpha_{\delta(\xi)})$ for $\delta(\xi) \leq s \leq \delta_1(\xi)-1$ can be proved to be linearly independent functions since $\alpha_{\delta(\xi)} \neq 0$, hence , we can conclude by **Lemma 2.5.1** that $\hat{\mathbf{J}}^z_{\delta(\xi)-1} + \sin^2\alpha_{\delta(\xi)}\hat{\mathbf{J}\mathbf{Y}}(\delta(\xi)-1)$, the inertial parameters in (2.194) for $\delta(\xi) \leq i \leq \delta_1(\xi)-1$, and $[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1}$, $[L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1}$, $\mathbf{R}^y_{\zeta+1}$ are fundamental parameters. Add these new fundamental parameters to $\boldsymbol{F}$.

We have proved that the inertial parameters in (2.192) for $\delta(\xi)-1 \leq i \leq N$, the inertial parameters in (2.193) and (2.194) for $\delta(\xi) \leq i \leq N$, $([L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} - \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1})$, $([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1})$, $\mathbf{R}^y_{\zeta+1}$, $(-[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \cos\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})$, $([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})$, $\bar{\mathbf{R}}^x_i$ and $\hat{\bar{\mathbf{R}}}^y_i$ for $K+1 \leq i \leq N$ are fundamental parameters by the argument above.

Next, for an arbitrary n such that $K+1 \leq n \leq \delta(\xi)-1$, assume that the inertial parameters in (2.192) for $n \leq i \leq \delta(\xi)-2$, the inertial parameters in (2.193) for $n+1 \leq i \leq \delta(\xi)-1$, and the inertial parameters in (2.194) for $\delta_1(n) \leq i \leq \delta(\xi)-1$ are proved to be fundamental parameters, and add these assumed fundamental parameters to $\boldsymbol{F}$. Then, we can derive (2.195) by **Lemma 2.5.4** and (2.196),(2.197) by **Lemma 2.5.3** . Hence, by the same argument as we used to prove the inertila parameters in (2.192) for $\delta_1(\xi+1)-1 \le i \le N$, the inertial parameters in (2.193) and (2.194) for $\delta_1(\xi+1) \le i \le N$ to be fundamental parameters, we can prove that the inertial parameters in (2.192) for $K \le i \le \delta(\xi)-2$ and the inertila parameters in (2.193) and (2.194) for $K+1 \le i \le \delta(\xi)-1$ are fundamental parameters. Add these fundamental parameters to $\boldsymbol{F}$.

Let S be a number such that $1 \le S \le K$, $[L_i]^x = 0$ for $1 \le i \le S-1$ and $[L_S]^x \ne 0$. ($S = 1$ when $[L_1]^x \ne 0$). For an arbitrary n such that $S < n < K$, assume that $Hr(n,n)$ is a function of $\boldsymbol{\theta}$ generated by fundamental parameters and $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $n+1 \le i \le K$ are proved to be fundamental parameters . Add these assumed fundamental parameters to $\boldsymbol{F}$. Then we can derive from **Lemma 2.5.5** that

$$\begin{aligned} Hr(n,1) \; &= Hr(n,n) \\ &+\bar{\mathbf{R}}_n^x \sum_{S=1}^{n-1} [L_S]^x \cos\theta(s+1,n) \\ &-\hat{\bar{\mathbf{R}}}_n^y \sum_{S=1}^{n-1} [L_S]^x \sin\theta(s+1,n) \\ &+G \end{aligned} \tag{2.203}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. Since $S < n$, then $\sum_{S=1}^{n-1}[L_S]^x \cos\theta(s+1,n)$ and $\sum_{S=1}^{n-1}[L_S]^x \sin\theta(s+1,n)$ are non-zero linearly independent functions. Hence, by **Lemma 2.5.1**, we can prove $\bar{\mathbf{R}}_n^x$ and $\hat{\bar{\mathbf{R}}}_n^y$ are fundamental parameters. Add these to $\boldsymbol{F}$. Next, we can derive from **Lemma 2.5.3** that

$$\begin{aligned} Hr(n-1,n-1) \; &= \hat{\mathbf{J}}_{n-1}^z + \sin^2\alpha_n \hat{\mathbf{JY}}(n-1) \\ &+2 \sum_{S=n-1}^{K} [L_S]^x \bar{G}_S \\ &+G \end{aligned} \tag{2.204}$$

for $2 \le n \le K$ where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$, $\bar{G}_S$ is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $s+1 \le i \le N$, then, we can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}_{n-1}^x + \sin^2\alpha_n \hat{\mathbf{JY}}(n-1)$ is a fundamental parameter. Hence, $Hr(n-1,n-1)$

is a function of $\boldsymbol{\theta}$ generated by fundamental parameters. $Hr(K,K)$ has been proved to be a function of $\boldsymbol{\theta}$ generated by fundamental parameters, and $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $K+1 \le i \le N$ have been proved to be fundamental parameters. Using the mathematical induction from $n = K$ to $S+1$, we can conclude that $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $S+1 \le i \le K$ and $\hat{\mathbf{J}}_i^z + \sin^2 \alpha_{i+1} \hat{\mathbf{J}\mathbf{Y}}(i)$ for $S \le i \le K-1$ are fundamental parameters. Add these new fundamental parameters to $\boldsymbol{F}$.

Next, for an arbitrary n such that $1 \le n \le S-1$, assume that $\hat{\mathbf{J}}_i^z + \sin^2 \alpha_{i+1} \hat{\mathbf{J}\mathbf{Y}}(i)$ for $n+1 \le i \le S-1$ are fundamental parameters. Add these assumed fundamental parameters to $\boldsymbol{F}$. Then we can derive from **Lemma 2.5.3** that

$$\begin{aligned} Hr(n,n) &= \hat{\mathbf{J}}_n^z + \sin^2 \alpha_{n+1} \hat{\mathbf{J}\mathbf{Y}}(n) \\ &+G \end{aligned} \tag{2.205}$$

where G is a function of $\boldsymbol{\theta}$ generated by fundamental parameters in $\boldsymbol{F}$. We can prove by **Lemma 2.5.1** that $\hat{\mathbf{J}}_n^z + \sin^2 \alpha_{n+1} \hat{\mathbf{J}\mathbf{Y}}(n)$ is a fundamental parameter. $\hat{\mathbf{J}}_S^z + \sin^2 \alpha_{S+1} \hat{\mathbf{J}\mathbf{Y}}(S)$ has been proved to be a fundamental parameter. Using the mathematical induction from $n = S-1$ to $n = 1$ we can conclude that $\hat{\mathbf{J}}_i^z + \sin^2 \alpha_{i+1} \hat{\mathbf{J}\mathbf{Y}}(i)$ for $1 \le i \le S-1$ are fundamental parameters.

$\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ have been proved to be fundamental parameters for $1 \le i \le N$ when $\boldsymbol{z}_1$ is not parallel to $\boldsymbol{g}$, for $K+1 \le i \le N$ when $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$, and for $S+1 \le i \le N$. Hence, it is obvious that they are fundamental parameters for $Q+1 \le i \le N$.

We have proved that the inertial parameters in (2.192) for $1 \le i \le N$, the inertial parameters in (2.193) and (2.194) for $K+1 \le i \le N$, $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $Q+1 \le i \le N$,

$$[L_\xi]^x \hat{\mathbf{R}}_{\xi+1}^y - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^x, \quad [L_\xi]^x \mathbf{R}_{\xi+1}^x - \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^x, \quad \mathbf{R}_{\zeta+1}^y, \tag{2.206}$$

and

$$\begin{aligned} &-[L_\xi]^x \hat{\mathbf{R}}_{\xi+1}^y + \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^x - \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^y, \\ &1[L_\xi]^x \mathbf{R}_{\xi+1}^x - \cos\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^x - \sin\rho [L_\zeta]^x \mathbf{R}_{\zeta+1}^y \end{aligned} \tag{2.207}$$

are fundamental parameters. The fundamental parameters in (2.192) for $1 \le i \le N$, in (2.193) and (2.194) for $K+1 \le i \le N$, and $\bar{\mathbf{R}}_i^x$ and $\hat{\bar{\mathbf{R}}}_i^y$ for $Q+1 \le i \le N$ are obviously linearly independent since each of them includes at least one link inertial parameter that does not appear in the others. Let $\boldsymbol{F}$' denote the set of these fundamental parameters. After we add the fundamental parameters (2.206) to $\boldsymbol{F}$', we can easily show that the fundamental parameters in $\boldsymbol{F}$' are linearly independent. However, each fundamental parameter in (2.207) can be obtained as a linear combination of the fundamental parameters in (2.206). Therefore, all the elements of $\boldsymbol{F}$' are linearly independent fundamental parameters in case of $K < \xi$. In case of $\xi + 1 \leq K$, we can prove the inertial parameters shown in **Theorem 2.5.1** are linearly independent fundamental parameters in the same manner as above. It is obvious from the argument above that $\frac{\partial}{\partial \boldsymbol{\theta}} U$ is a function of $\boldsymbol{\theta}$ generated by some inertial parameters in **Theorem 2.5.1**, hence, we can conclude with **Lemma 2.5.6** that the inertial parameters in **Theorem 2.5.1** generate the dynamic model (3.26). Thus, the set of the inertial parameters in **Theorem 2.5.1** constitute a base parameter set. The number of base parameters is evident. ∎

2.5.5 Conclusion

A base parameter set has been shown in complete closed form for manipulators with a planar parallelogram link mechanism. The exact number of base parameters has also been shown. The size of a base parameter set does not depend on a virtually cut joint in the planar parallelogram. The extension of the results to manipulators with general closed chain mechanisms remains. The results of this section would cover most of commercially available industrial manipulators with closed chain mechanisms.

2.6 Conclusion

The base parameter set which is defined to be a minimum set of inertial parameters that can generate a dynamic model uniquely was investigated for a general parallel and perpendicular manipulator with rotational joints only in section 3. The results of section 3 was extended to a general parallel and perpendicular manipulator with rotational and translational joints in section 4, and also the investigation of the base parameter set was extended to manipulators with a planar parallelogram link mechanism in section 5. The results in section 4 coincide with the results in section 3 by deleting the terms concerning translational links, but such operation would be so complicated that it would be easier to apply the results in section 3 to a manipulator if it has only rotational joints. Base parameters are also the inertial parameters which can be identified independently from link motion data and input data (joint torques or forces). We have given the definitions and properties concerning the base parameter set and made clear the meaning of the redundancy of the link inertial parameters. We have described each base parameter by a linear combination of the link inertial parameters directly and completely in closed form. We have given the exact number of base parameters in the set. Any base parameter set can be obtained from this base parameter set by a nonsingular linear transformation. The results of section 4 have been already extended to a general open-loop kinematic chain [35]. The investigation of base parameter set for a general closed-loop kinematic chain, which is the final extension of the results in section 5, still remains. The method we took in proofs of theorems is very laborious and complicated, especially in proofs of lemmas, hence, the extension of the investigation of the base parameters to complex kinematic chains would need another mathematical tool or idea. Through making clear the redundancy of the link inertial parameters, we have obtained the fact that some link inertial parameters can only be identified in linear combinations i.e., some link inertial parameters only appear in the form of linear combinations in the dynamic equations for manipulator. In multi-body systems, similar phenomenon would arise. Hence, the definitions we have given would be valid for multi-body systems.

Chapter 3

Experimental Examination of the Identification Methods of Base Parameters

3.1 Introduction

In this chapter, the identification methods of base parameters will be experimentally examined.

As mentioned in chapter 1, the dynamic model of the manipulator consisting of rigid links is described as a set of nonlinear differential equations involving various constant parameters: kinematic parameters, link inertial parameters, and dynamic parameters of driving systems. If all the values of these parameters are known, we can determine the dynamic model. Hence, accurate values of the parameters are required to obtain an accurate dynamic model. The values of the kinematic parameters can be obtained from design data or by kinematic calibrations. After they are obtained, for the purpose of determining the dynamic model, it is sufficient for us to obtain the values of base parameters and the dynamic parameters of the driving systems. The most practical way to obtain them will be to estimate them from the input data (joint torques or forces) and link motion data (joint positions, velocities, and accelerations if needed) which are taken while the manipulator is in test motions. The identifiability of base parameters from such data has been ensured in the previous chapter. Then it is very important to develop the identification method that gives us accurate parameter values. Several authors have proposed identification methods of the parameters.

Mayeda et al. [40] have first proposed a general systematic identification method of the parameters. The method consists of 3 types of simple test motions that move 1 or 2 joints simultaneously freezing the rest of the joints, and estimates a small number of parameter values at a time using the data of a test motion and the formerly estimated parameter values. Beginning to estimate the parameter values of the last link, it estimates parameter values step by step. Thus this method will be called step-by-step method. This step-by-step method was applied to a 3 degree-of-freedom industrial manipulator [41], and considerably good estimation of the parameter values have been reported. The method was also applied to a 6 degree-of-freedom Direct Drive manipulator and the results were also good [42].

Atokeson et al.[23] written the dynamic model of manipulators as a system of equations which are linear in terms of the inertial parameters that are identifiable independently. They have proposed a method to estimate all the parameter values at a time applying the least squares method to the system of linear equations using the input data and link motion data that are taken while the manipulator is in the test motion which moves all joints simultaneously in random enough way. They applied the method to a 3 degree-of-freedom Direct Drive manipulator. In the estimation, they have made clear the independently identifiable parameters of the manipulator in a closed form using a computer with a commercial software. The experimental results showed that good estimates of parameters values were obtained.

Khosla [21] has independently developed a identification method very similar to the method proposed by Atokeson et al. Khosla has also developed a computer-aided method to find out identifiable parameters of manipulators by symbolic procedures for the Newton-Eular formulation. Working with a 6 degree-of-freedom Direct Drive manipulator, he could show that the good estimates were obtained by the method.

Kawasaki et al.[43] have also developed a method similar to the methods proposed by Atokeson et al. and Khosla. Moreover they have proposed to take advantage of the instrumental variable method instead of the simple least squares method to avoid inconsistent estimation which is inevitable when the simple least squares method is applied with contaminated data. He applied the method to a 6 degree-of-freedom industrial manipulator . He has made clear the identifiable parameters of the manipulator by symbolic procedures. The experimental results showed that good estimates of parameters values were obtained.

The methods proposed by Atokeson et al., Khosla, and Kawasaki et al. are all based on the dynamic models of manipulators and estimate all the parameter values at a time using the data sampled while the manipulator is in the test motion that moves all joints simultaneously in random enough way. Hence they might be able to be grouped into one method which could be called simultaneous method.

Gautier and Khalil [44] have developed a different identification method which is based on the energy model of manipulators. From the energy model, they derived a energy difference equation which is linear in terms of identifiable parameters. They have proposed a method to estimate all the parameter values at a time applying the least squares method to the equation, using the input data, joint position data, and joint velocity data which are taken while the manipulator is in the test motion that moves all joints simultaneously in random enough way. They have also examined a direct determination of base parameter set by differentiating the energy functions of manipulators[31]. They recently applied their method to a 6 degree-of-freedom industrial manipulator[45] and have shown that good estimates of the parameter values were obtained.

The identification methods should be examined on validity through experiments. Mayeda et al. have examined their step-by-step method and Atokeson et al., Khosla, and Kawasaki. et.al. have examined the simultaneous method through the experiments applying them to their own manipulators. However, the validity of the identification methods may seriously depend on type, degree-of-freedom, and driving systems of manipulators. The two identification methods should be applied to a same manipulator and compared totally through experiments. Hence, in this chapter, we will experimentally examine the two methods on validity applying them to a typical industrial manipulator PUMA 560 and compare them. We will adopt also the instrumental variable method in applying the simultaneous method. To evaluate the accuracy of the estimates, we will simulate the motion of the manipulator using the estimates and compare the simulated trajectories with measured trajectories. As the results, it will be concluded that the step-by-step method is more precise way to estimate the parameters than the simultaneous method, however the simultaneous method taking advantage of the instrumental variables method is nearly as precise way as the step-by-step method. Moreover, we will describe in detail the contents of the work which is needed to obtain the estimates by each identification method, and compare the methods about the amount of labour (human involvement) and consuming time on a computer.

As we will show in below, we can obtain a good estimates adopting the instrumental variable method in the simultaneous method if we choose the instrumental matrix sequence appropriately. Then, it is very crucial how to choose the instrumental matrix sequence. Kawasaki et al.[43] used a instrumental model to make it. Their instrumental model is very simple, however, it is not easy to determine the values of parameters in the model. Afterwards, Kawasaki[46] proposed to use the dynamic models of manipulators as the instrumental models, in which all the parameter values that are to be estimated are needed. Hence, if we are to use the instrumental variable method with kawasaki's instrumental model, we have to obtain in advance a set of parameter values that is appropriate for the instrumental model. Kawasaki has not given any clear method to obtain it. Then, we will propose a method to obtain it. Though it is very time consuming, it is very easy.

3.2 PUMA 560 and Its Base Parameter Set

As shown in Fig.3.2.1, PUMA 560 is a 6 degree-of-freedom typical industrial manipulator and all joints are revolute type. To each link i of PUMA 560, a coordinate system $(o_i; \boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ is attached in the way shown in Fig.3.2.1. This is our convention adopted also in chapter 1. Let γ_i denote the twist angle between $\boldsymbol{z}_{i-1}$ and $\boldsymbol{z}_i$ measured around $\boldsymbol{x}_{i-1}$. In case of PUMA 560, $\gamma_i = 0$ for $i = 1, 3$ and $\gamma_i = \frac{\pi}{2}$ for $i = 2, 4, 5, 6$. Joint variable

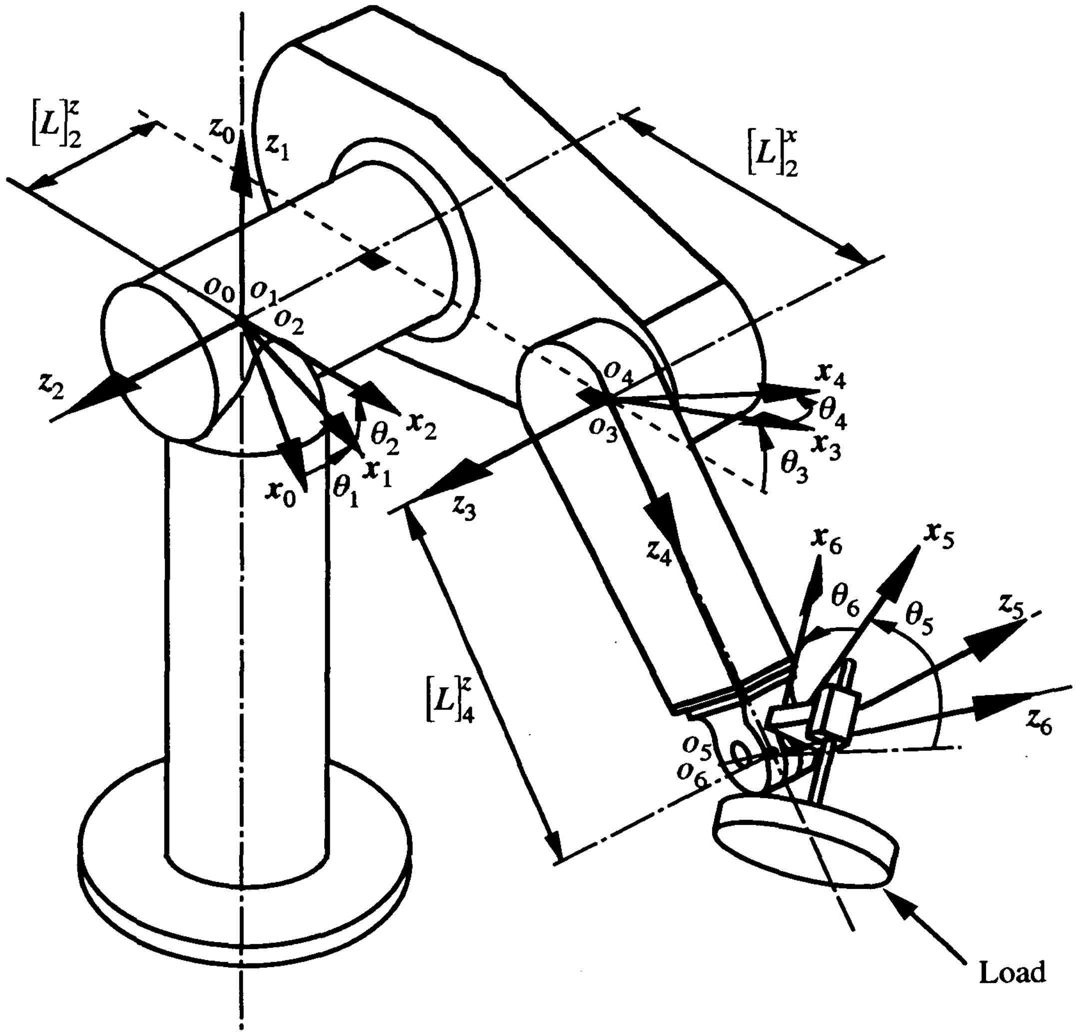


Fig. 3.2.1 PUMA 560 with Load and Its Coordinate Systems

θ_i is the angle from $\boldsymbol{x}_{i-1}$ to $\boldsymbol{x}_i$ measured around $\boldsymbol{z}_i$. $\boldsymbol{L}_i$ is the vector from o_i to o_{i+1}. ${}^i\boldsymbol{L}_i$ denotes the length of link i, and is described as

$$ {}^i\boldsymbol{L}_i = \left[\begin{array}{ccc} [L]_i^x & 0 & [L]_i^z \end{array} \right]^t \tag{3.1} $$

Then, for PUMA 560, only $[L]_2^x = 43.2[cm]$, $[L]_2^z = -15[cm]$, $[L]_4^z = 43.3[cm]$ are nonzero elements.

We attach a load to 6-th link, which is characterized by

Approximate mass:

$$ 1.6 \ [kg] $$

Approximate location of center of mass with respect to 6-th coordinate system:

$$ \left[\begin{array}{ccc} -7 & 7 & 11 \end{array} \right]^t \ [cm] $$

Approximate moment of inertia matrix around o_6 with respect to 6-th coordinate system:

$$ \left[\begin{array}{ccc} 3.67 \times 10^{-2} & 7.59 \times 10^{-3} & 1.23 \times 10^{-2} \\ 7.59 \times 10^{-3} & 3.67 \times 10^{-2} & -1.23 \times 10^{-2} \\ 1.23 \times 10^{-2} & -1.23 \times 10^{-2} & 2.87 \times 10^{-2} \end{array} \right] [kg \cdot m^2] $$

The 6-th link and the load will be regarded as one link.

Applying the results of section 1 in chapter 2 for this PUMA 560, we obtain following base parameter set consists of 36 inertial parameters.

$$ \mathbf{J1z} = \mathbf{J}_1^z + \mathbf{J}_2^y + \mathbf{J}_3^y + 2[L]_2^z\mathbf{R}_3^z, $$

$$
\begin{array}{lll}
\mathbf{J2z} = \mathbf{J}_2^z, & \mathbf{J2(x\text{-}y)} = \mathbf{J}_2^x - \mathbf{J}_2^y, & \\
\mathbf{J2xz} = \mathbf{J}_2^{xz} - [L]_2^x \mathbf{R}_3^z, & \mathbf{J2yz} = \mathbf{J}_2^{yz}, & \mathbf{J2xy} = \mathbf{J}_2^{xy}, \\
\mathbf{R2x} = \mathbf{R}_2^x, & \mathbf{R2y} = \mathbf{R}_2^y, & \\
 & & \\
\mathbf{J3z} = \mathbf{J}_3^z + \mathbf{J}_4^y, & \mathbf{J3(x\text{-}y)} = \mathbf{J}_3^x - \mathbf{J}_3^y + \mathbf{J}_4^y, & \\
\mathbf{J3xz} = \mathbf{J}_3^{xz}, & \mathbf{J3yz} = \mathbf{J}_3^{yz}, & \mathbf{J3xy} = \mathbf{J}_3^{xy}, \\
\mathbf{R3x} = \mathbf{R}_3^x, & \mathbf{R3y} = \mathbf{R}_3^y - \mathbf{R}_4^z, & \\
 & & \\
\mathbf{J4z} = \mathbf{J}_4^z + \mathbf{J}_5^y, & \mathbf{J4(x\text{-}y)} = \mathbf{J}_4^x - \mathbf{J}_4^y + \mathbf{J}_5^y, & \\
\mathbf{J4xz} = \mathbf{J}_4^{xz}, & \mathbf{J4yz} = \mathbf{J}_4^{yz} + [L]_4^z \mathbf{R}_5^z, & \mathbf{J4xy} = \mathbf{J}_4^{xy}, \\
\mathbf{R4x} = \mathbf{R}_4^x, & \mathbf{R4y} = \mathbf{R}_4^y - \mathbf{R}_5^z, & \\
 & & \\
\mathbf{J5z} = \mathbf{J}_5^z + \mathbf{J}_6^y, & \mathbf{J5(x\text{-}y)} = \mathbf{J}_5^x - \mathbf{J}_5^y + \mathbf{J}_6^y, & \\
\mathbf{J5xz} = \mathbf{J}_5^{xz}, & \mathbf{J5yz} = \mathbf{J}_5^{yz}, & \mathbf{J5xy} = \mathbf{J}_5^{xy}, \\
\mathbf{R5x} = \mathbf{R}_5^x, & \mathbf{R5y} = \mathbf{R}_5^y - \mathbf{R}_6^z, & \\
 & & \\
\mathbf{J6z} = \mathbf{J}_6^z, & \mathbf{J6(x\text{-}y)} = \mathbf{J}_6^x - \mathbf{J}_6^y, & \\
\mathbf{J6xz} = \mathbf{J}_6^{xz}, & \mathbf{J6yz} = \mathbf{J}_6^{yz}, & \mathbf{J6xy} = \mathbf{J}_6^{xy}, \\
\mathbf{R6x} = \mathbf{R}_6^x, & \mathbf{R6y} = \mathbf{R}_6^y. &
\end{array}
\tag{3.2}
$$

The dynamic equations for the kinematic chain of PUMA 560 can be described as

$$
\boldsymbol{\tau} = \boldsymbol{H}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{B}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{G}(\boldsymbol{\theta}) \tag{3.3}
$$

where $\boldsymbol{\tau} = [\tau_1 \ \cdots \ \tau_6]^t$ is joint torque vector, and $\boldsymbol{\theta} = [\theta_1 \ \cdots \ \theta_6]^t$ is joint angle vector. $\boldsymbol{H}(\boldsymbol{\theta})$, $\boldsymbol{B}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, and $\boldsymbol{G}(\boldsymbol{\theta})$ are the inertial matrix, the Coriolis and centrifugal force term matrix, and gravity term vector, respectively. Each element of these matrices and vector is a function of $\boldsymbol{\theta}$ generated by the base parameters shown above.

3.3 The Driving Systems of PUMA 560

The 6 joints of the manipulator are supplied torques by 6 motors via gear mechanisms. Let θ_{mi} denote the rotation angle of i-th motor and $\boldsymbol{\theta}_m = [\theta_{m1} \ \cdots \ \theta_{m6}]^t$. Then the joint angles are related to the motor angles by the gear mechanisms as

$$\boldsymbol{\theta} = \boldsymbol{K}\boldsymbol{\theta}_m \tag{3.4}$$

where

$$\boldsymbol{K} = \begin{bmatrix} k_1 & 0 & 0 & & & \\ 0 & k_2 & 0 & & \mathbf{0} & \\ 0 & 0 & k_3 & & & \\ & & & k_4 & 0 & 0 \\ & \mathbf{0} & & k_{5,4} & k_5 & 0 \\ & & & k_{6,4} & k_{6,5} & k_6 \end{bmatrix} \tag{3.5}$$

$$k_1 = 1.60 \times 10^{-2}, \quad k_2 = 9.26 \times 10^{-3}, \quad k_3 = 1.86 \times 10^{-2}$$

$$k_4 = -1.32 \times 10^{-2}, \quad k_5 = -1.39 \times 10^{-2}, \quad k_6 = -1.30 \times 10^{-2}$$

$$k_{5,4} = 1.80 \times 10^{-4}, \quad k_{6,4} = 1.40 \times 10^{-4}, \quad k_{6,5} = 2.51 \times 10^{-3}.$$

The gear mechanism for the last 3 degree-of-freedom of the wrist is sophisticated, and motors 4,5 and 6 have some interactions.

The dynamic models of the driving systems will be described as

$$\tau'_{mi} = \tau_{mi} - h_i\ddot{\theta}_{mi} - b_i\dot{\theta}_{mi} - c_i sgn\dot{\theta}_{mi} \tag{3.6}$$

for $1 < i < 6$ where τ_{mi} is i-th motor torque, τ'_{mi} is transmitted torque from i-th motor to gear mechanism, and h_i, b_i and c_i are the moment of inertia, the viscous friction coefficient and the Coulomb friction coefficient around i-th motor axis, respectively. The inertia, the viscous friction and the Coulomb friction of the gear mechanism are supposed to be concentrated around the motor axes. Let $\boldsymbol{\tau}'m = [\tau'm_1 \ \cdots \ \tau'm_6]^t$. Then we have

$$\boldsymbol{\tau}'m = \boldsymbol{K}^t\boldsymbol{\tau}. \tag{3.7}$$

Combining (3.3),(3.6) and (3.7), we can obtain the dynamic model of PUMA 560. h_1 and h_2 appear in the dynamic model always in the forms of

$$\mathbf{J1z} + (k_1)^{-2}h_1, \tag{3.8}$$

$$\mathbf{J2z} + (k_2)^{-2}h_2, \tag{3.9}$$

respectively. Therefore we can consider (3.11) and (3.12) as base parameters and abuse **J1z** and **J2z** to denote them, respectively. The parameters h_i,b_i and c_i for $1 < i < 6$ except for h_1 and h_2 will be called driving system parameters. To determine the dynamic model of PUMA 560, we have to estimate all the values of the 36 base parameters and the 16 driving system parameters. Those 52 parameters will be called model parameters, and column vector of the 52 model parameters will be denoted by $\boldsymbol{p}$.

It is easily shown that all the values of the model parameters can be estimated from motor torque and motor rotation data [40]. In this manipulators, each motor current and rotation angle are measurable by an equipment and an encoder, respectively. Thus we can obtain the motor torque and motor rotation data.

3.4 Identification by Step-by-step Method

The step-by-step method consists of 3 types of simple test motions, such that we only need to move one or two joints simultaneously, freezing the rest of the joint. The model parameters are divided into a certain number of subgroups, and values of model parameters in each subgroup are estimated from data of the test motions, use being made of formerly estimated model parameter values.

The extension of this method to general open-loop kinematic chains is given in [47]

3.4.1 Static Test

If the manipulator stands still, the gravity term is written as

$$\tau_i = -\boldsymbol{g} \cdot (\boldsymbol{z}_i \times \sum_{j=i}^{6} \boldsymbol{R}_j) \tag{3.10}$$

where $\boldsymbol{g}$ is the gravity vector. Denoting ${}^i\boldsymbol{g}$ by ${}^i\boldsymbol{g} = [{}^ig_x\ {}^ig_y\ {}^ig_z]^t$, we can easily derive that

$$\tau_i = {}^ig_x\mathbf{Riy} - {}^ig_y\mathbf{Rix} + G_i \tag{3.11}$$

where G_i is a term figured out from $\mathbf{Rsx}$ and $\mathbf{Rsy}$ for $i+1 < s < 6$. Using (3.11) for more than two angles of θ_i, we can estimate $\mathbf{Rix}$ and $\mathbf{Riy}$ provided that $\boldsymbol{z}_i$ is not parallel to $\boldsymbol{g}$. τ_i can be obtained from τ_{mi} using (3.5),(3.7). To avoid the effect of the Coulomb static friction around motor axis, we change τ_{mi} gradually and measure τ^+_{mi} and τ^-_{mi} at the instance when θ_{mi} begins to move + and − directions, respectively. Then τ_{mi} is estimated as $\tau_{mi} = (\tau^+_{mi} - \tau^-_{mi})/2$.

Performing this test from $i = 6$ to 2, we can estimate $\mathbf{Rsx}$ and $\mathbf{Rsy}$ in (3.2). Using the results obtained here, we can compensate the gravity term in (3.3). Thereby, we omit the gravity term in later discussions.

3.4.2 Constant Velocity Motion Test

Make i-th motor rotate in constant angular velocity, freezing other motors. Then, neglecting off-diagonal element of $\boldsymbol{K}$ since their effects are very small, we can derive from (3.6) that

$$\tau_{mi} = b_i\dot{\theta}_{mi} + c_i sgn\dot{\theta}_{mi}. \tag{3.12}$$

If we realize this motion for more than two angular velocities, it is easy to estimate b_i and c_i from (3.12).

Performing this test for every motor, b_i and c_i for $1 < i < 6$ can be estimated. By compensation, we omit the viscous friction and the Coulomb friction in later discussion.

3.4.3 Accelerated Motion Test

Make an accelerated rotations about 6-th joint freezing the other joint. The motion equation is described as

$$(k_6^2\,\mathbf{J6z} + h_6)\ddot{\theta}_{m6} = \tau_{m6}. \tag{3.13}$$

From this we can directly estimate $(k_6^2\,\mathbf{J6z} + h_6)$.

Next, make accelerated rotations about 5-th joint for three different θ_6s freezing the other joints. The motion equation about 5-th joint is easily derived as

$$\begin{aligned}&[k_5^2\,\mathbf{J5z} + h_5 + k_{6,5}^2 k_6^{-2} h_6 + k_5^2\,(\mathbf{J6(x\text{-}y)} \sin^2\theta_6 + 2\mathbf{J6xy} \sin\theta_6 \cos\theta_6)]\ddot{\theta}_{m5}\\ = & -\tau_{m5} - k_{6,5} k_6^{-1} \tau_{m6}.\end{aligned} \tag{3.14}$$

By solving linear equations obtained from (3.14) for three different θ_6s, we can estimate $\mathbf{J6(x\text{-}y)}$, $\mathbf{J6xy}$ and $k_5^2\,\mathbf{J5z} + h_5 + h_5^2 k_6^{-2} h_6$.

Next, make an accelerated rotation about 6-th and 5-th joints simultaneously freezing the other joints. The motion equation about 6-th joint is easily derived as

$$\begin{aligned}&k_6[k_5\,(\mathbf{J6xz} \sin\theta_6 + \mathbf{J6yz} \cos\theta_6) + k_{6,5}\,\mathbf{J6z}]\ddot{\theta}_{m5}\\ + & (k_6^2\,\mathbf{J6z} + h_6)\ddot{\theta}_{m6} - k_6 k_5^2 [\mathbf{J6(x\text{-}y)} \sin\theta_6 \cos\theta_6 + 2\mathbf{J6xy}(2\cos^2\theta_6 - 1)]\dot{\theta}_{m5}^2\\ = & \tau_{m6}.\end{aligned} \tag{3.15}$$

Since $k_6^2\mathbf{J6z} + h_6$, $\mathbf{J6(x\text{-}y)}$, and $\mathbf{J6xy}$ have been already estimated, by solving linear equations obtained from (3.15) for three different θ_6s, we can estimate $\mathbf{J6xz}$, $\mathbf{J6yz}$, and $\mathbf{J6z}$, and hence h_6.

Continuing same kind test motions for the rest of joints, we can estimate all the rested model parameters. 13 test motions are required for PUMA 560 (See Fig. 3.4.1). In this method, by integrating both sides of the motion equations, we can avoid to use angular acceleration data. The estimated values of the model parameter by the step-by-step method are shown in Table 1.

3.5 Identification by Simultaneous Method

Since the model parameters affect linearly to the motor torques, (3.3), (3.6)and (3.7) can be modified as

$$\tau_m = \boldsymbol{\Phi}(\ddot{\boldsymbol{\theta}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}, sgn\dot{\boldsymbol{\theta}})\boldsymbol{p} \tag{3.16}$$

where $\boldsymbol{\Phi}(\ddot{\boldsymbol{\theta}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}, sgn\dot{\boldsymbol{\theta}})$ is 6×52 block upper triangular matrix, each element of which is a function of $\ddot{\boldsymbol{\theta}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}$. As shown in Fig. 3.5.1, making random enough accelerated rotations for all the joints simultaneously and using the sampled data, we can estimate $\boldsymbol{p}$ by the least squares method by use of iterative formula.

The estimated values of the model parmeters are shown in Table 2.

In general, data contain errors which are caused by the dynamics of sensors, noises and so on. When we estimate the $\boldsymbol{p}$ by the least squares method using such contaminated data, bias may arise for the estimate[48]. Kawasaki et al.[43],[46] have proposed a method taking

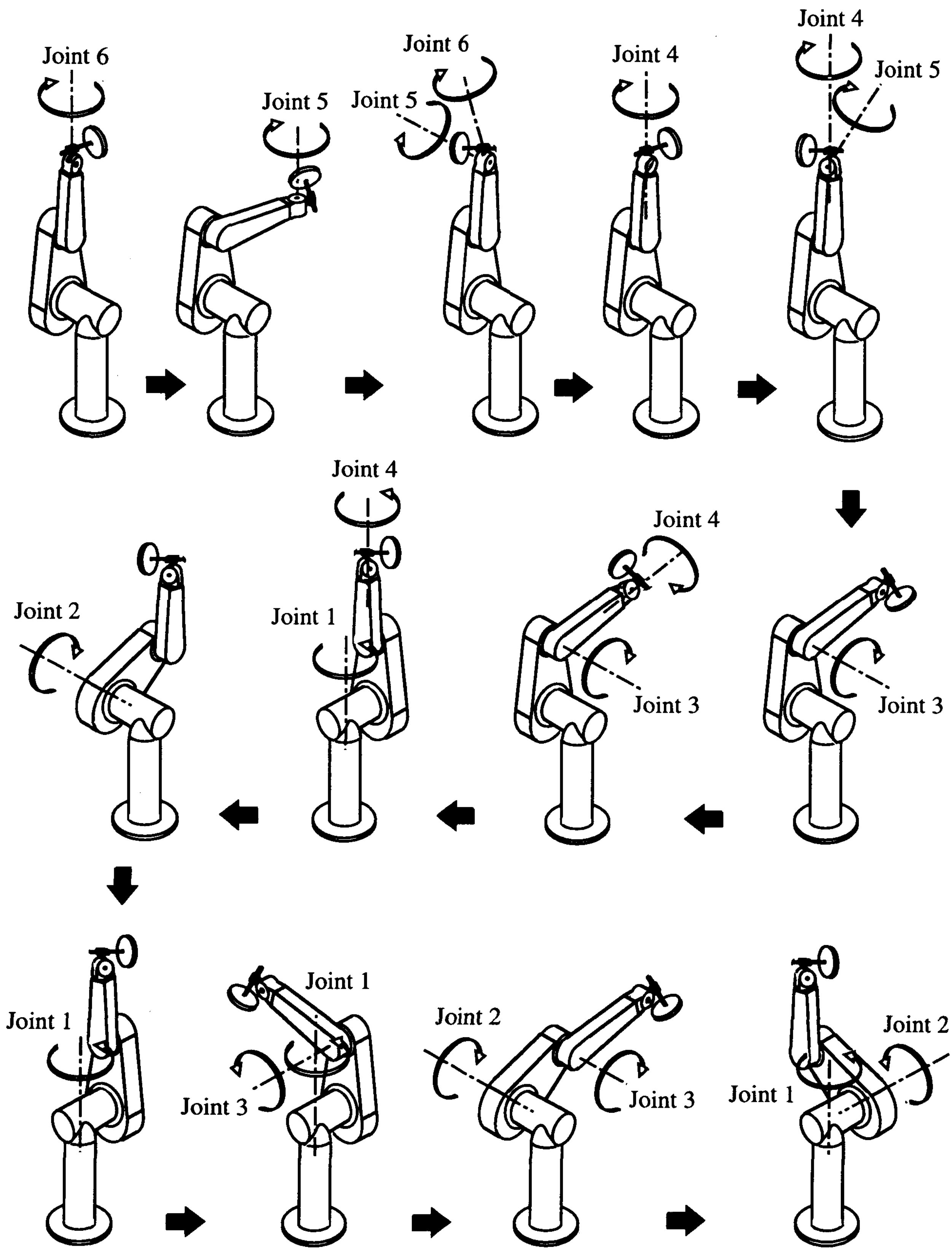


Fig. 3.4.1 The Accelerated Motion Tests for PUMA 560

Table 1 Model Parameter Values by Step-by-step Method

Parameter	Value	Parameter	Value
c_1	1.42×10^{-1}	**J1z**	5.31
c_2	1.05×10^{-1}	**J2z**	7.01
c_3	1.24×10^{-1}	**J3z**	1.11
c_4	2.36×10^{-2}	**J4z**	4.12×10^{-2}
c_5	1.43×10^{-2}	**J5z**	3.81×10^{-2}
c_6	2.25×10^{-2}	**J6z**	4.82×10^{-2}
	[Nm]		
b_1	8.63×10^{-4}	**J2(x-y)**	-2.58
b_2	4.00×10^{-4}	**J3(x-y)**	2.11×10^{-1}
b_3	3.67×10^{-4}	**J4(x-y)**	2.41×10^{-2}
b_4	5.55×10^{-5}	**J5(x-y)**	3.42×10^{-2}
b_5	4.27×10^{-5}	**J6(x-y)**	-3.85×10^{-3}
b_6	5.65×10^{-5}		
	[Nms]		
		J2xy	1.05×10^{-1}
h_3	3.12×10^{-4}	**J3xy**	2.12×10^{-2}
h_4	2.30×10^{-5}	**J4xy**	-7.57×10^{-3}
h_5	3.73×10^{-5}	**J5xy**	5.63×10^{-2}
h_6	3.16×10^{-5}	**J6xy**	-3.24×10^{-2}
	[kgm^2]		
R2x	7.27	**J2xz**	1.05
R3x	1.85×10^{-1}	**J3xz**	-4.85×10^{-1}
R4x	1.08×10^{-2}	**J4xz**	-3.58×10^{-2}
R5x	-1.16×10^{-3}	**J5xz**	1.41×10^{-2}
R6x	-9.86×10^{-2}	**J6xz**	4.47×10^{-2}
R2y	5.24×10^{-1}	**J2yz**	2.12×10^{-1}
R3y	-2.35	**J3yz**	3.07×10^{-2}
R4y	-8.60×10^{-3}	**J4yz**	-1.78×10^{-2}
R5y	-2.65×10^{-1}	**J5yz**	2.28×10^{-2}
R6y	1.16×10^{-1}	**J6yz**	-2.08×10^{-3}
	[kgm]		[kgm^2]

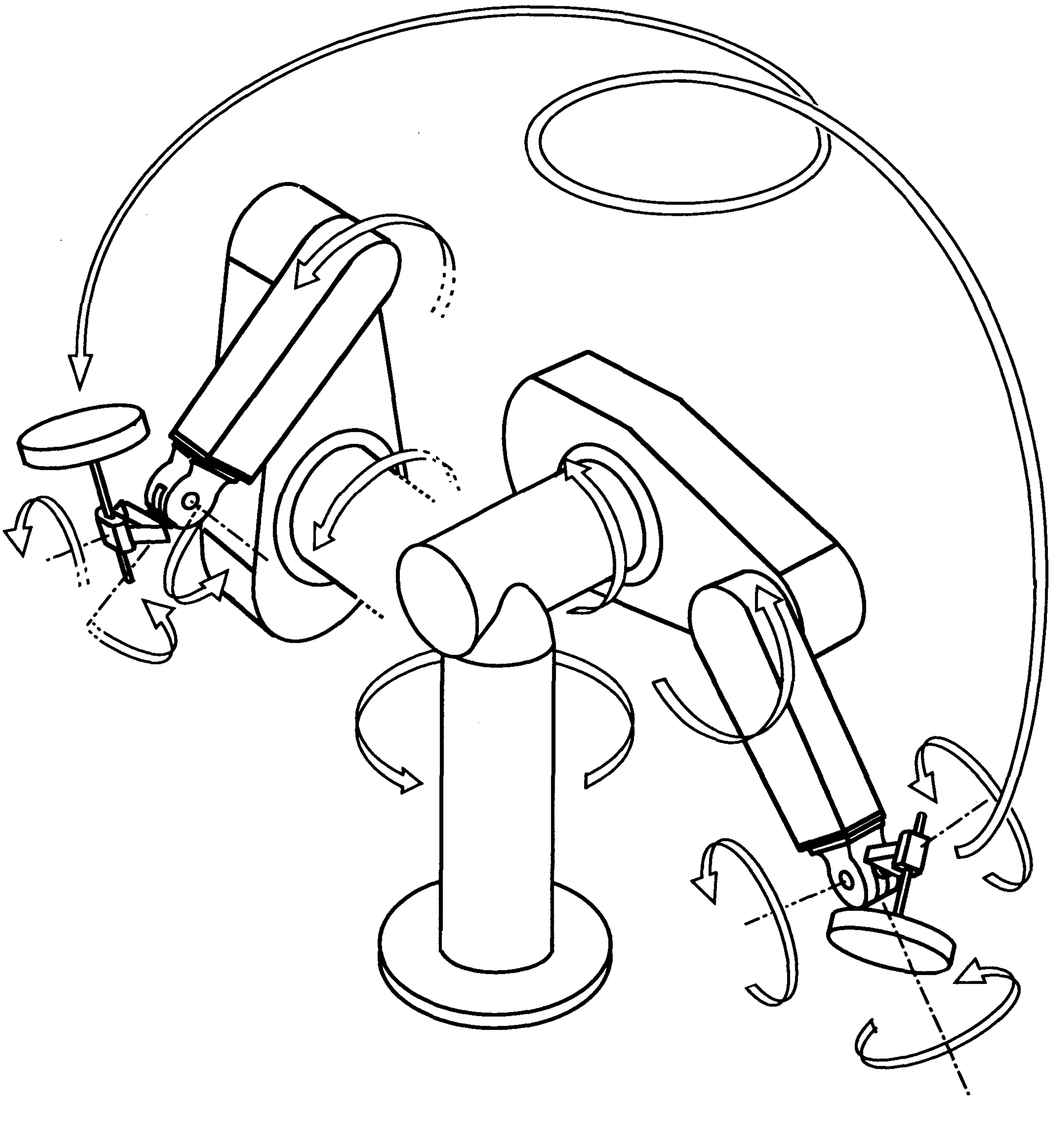


Fig. 3.5.1 A Test Motion for Simultaneous Method

Table 2 Model Parameter Values by Simultaneous Method

Parameter	Value	Parameter	Value
c_1	1.67×10^{-1}	**J1z**	1.61
c_2	1.21×10^{-1}	**J2z**	5.24
c_3	1.25×10^{-1}	**J3z**	-7.48×10^{-1}
c_4	2.89×10^{-2}	**J4z**	-2.86×10^{-3}
c_5	2.19×10^{-2}	**J5z**	-4.44×10^{-2}
c_6	2.35×10^{-2}	**J6z**	3.81×10^{-3}
	[Nm]		
b_1	4.30×10^{-4}	**J2(x-y)**	8.20×10^{-1}
b_2	6.22×10^{-4}	**J3(x-y)**	-1.94×10^{-1}
b_3	3.10×10^{-4}	**J4(x-y)**	-1.71×10^{-2}
b_4	1.73×10^{-5}	**J5(x-y)**	-2.21×10^{-2}
b_5	2.39×10^{-6}	**J6(x-y)**	6.96×10^{-3}
b_6	6.15×10^{-5}		
	[Nms]		
		J2xy	3.11×10^{-2}
h_3	7.08×10^{-4}	**J3xy**	7.25×10^{-2}
h_4	3.69×10^{-5}	**J4xy**	-2.16×10^{-2}
h_5	4.92×10^{-5}	**J5xy**	-4.76×10^{-3}
h_6	3.64×10^{-5}	**J6xy**	9.99×10^{-3}
	[kgm^2]		
R2x	8.94	**J2xz**	1.60×10^{-1}
R3x	-3.86×10^{-2}	**J3xz**	2.86×10^{-2}
R4x	5.63×10^{-2}	**J4xz**	1.40×10^{-2}
R5x	2.01×10^{-2}	**J5xz**	-1.38×10^{-2}
R6x	-8.44×10^{-2}	**J6xz**	6.91×10^{-4}
R2y	3.27×10^{-1}	**J2yz**	3.57×10^{-2}
R3y	-2.86	**J3yz**	3.93×10^{-1}
R4y	-3.91×10^{-2}	**J4yz**	-1.66×10^{-2}
R5y	-2.11×10^{-1}	**J5yz**	-2.60×10^{-3}
R6y	1.16×10^{-1}	**J6yz**	8.70×10^{-3}
	[kgm]		[kgm^2]

advantage of the instrumental variable method to avoid the bias. Then, we adopt also the instrumental variable method in the estimation by the simultaneous method. We will call this method as advanced simultaneous method in the later discussion. We, first, give brief explanation of the instrumental variable method.

3.5.1 Instrumental Variable Method [48]

We use the discrete time model of (3.16). It can be described by

$$\hat{\boldsymbol{\tau}}_m[n] = \hat{\boldsymbol{\Phi}}[n]\boldsymbol{p} + \boldsymbol{e}[n] \tag{3.17}$$

where $\hat{\boldsymbol{\tau}}_m[n]$ denotes the sampled value of $\boldsymbol{\tau}_m$ at sample time n, and $\hat{\boldsymbol{\theta}}[n]$, $\hat{\dot{\boldsymbol{\theta}}}[n]$, $\hat{\ddot{\boldsymbol{\theta}}}[n]$ denote the sampled values of $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, $\ddot{\boldsymbol{\theta}}$, respectively, on the same sample time. $\hat{\boldsymbol{\Phi}}[n]$ is $\boldsymbol{\Phi}$ which is determined by $\hat{\boldsymbol{\theta}}[n]$, $\hat{\dot{\boldsymbol{\theta}}}[n]$, $\hat{\ddot{\boldsymbol{\theta}}}[n]$. $\boldsymbol{e}[n]$ is error vector.

We define the following matrix and vectors.

$$\boldsymbol{\Psi}_N = [\hat{\boldsymbol{\Phi}}^t[1], \hat{\boldsymbol{\Phi}}^t[2], \cdots, \hat{\boldsymbol{\Phi}}^t[N]]^t \tag{3.18}$$

$$\boldsymbol{y}_N = [\hat{\boldsymbol{\tau}}_m^t[1], \hat{\boldsymbol{\tau}}_m^t[2], \cdots, \hat{\boldsymbol{\tau}}_m^t[N]]^t \tag{3.19}$$

$$\boldsymbol{\varepsilon}_N = [\boldsymbol{e}^t[1], \boldsymbol{e}^t[2], \cdots, \boldsymbol{e}^t[N]]^t. \tag{3.20}$$

Using the data from sample time 1 to sample time N (N $\geq$ 9), we can obtain following concatenated form of equation (3.17):

$$\boldsymbol{y}_N = \boldsymbol{\Psi}_N \boldsymbol{p} + \boldsymbol{\varepsilon}_N. \tag{3.21}$$

Then, it is possible to estimate the parameter $\boldsymbol{p}$ by weighted least squares method.

$$\bar{\boldsymbol{p}}_N = (\boldsymbol{\Psi}_N^t \boldsymbol{W}_N \boldsymbol{\Psi}_N)^{-1} \boldsymbol{\Psi}_N^t \boldsymbol{W}_N \boldsymbol{y}_N \tag{3.22}$$

where $\bar{\boldsymbol{p}}_N$ is the estimated parameter vector using the data from sample time 1 to N, and $\boldsymbol{W}_N$ is a weighted matrix. We can choose the $\boldsymbol{W}_N$ such that

$$\boldsymbol{W}_N = \boldsymbol{\Omega}_N \boldsymbol{\Omega}_N^t \tag{3.23}$$

where $\boldsymbol{\Omega}_N$ is same size as $\boldsymbol{\Psi}_N$.

In general, observed value vector $\boldsymbol{y}_N$ and matrix $\boldsymbol{\Psi}_N$ contain noises. We can write them as follows:

$$\boldsymbol{y}_N = \boldsymbol{\Psi}_N^* \boldsymbol{p}^* + \tilde{\boldsymbol{y}}_N \tag{3.24}$$

$$\boldsymbol{\Psi}_N = \boldsymbol{\Psi}_N^* + \tilde{\boldsymbol{\Psi}}_N \tag{3.25}$$

where $\boldsymbol{\Psi}_N^*$ and $\boldsymbol{p}^*$ are true value matrix and vector, respectively. $\tilde{\boldsymbol{y}}_N$ and $\tilde{\boldsymbol{\Psi}}_N$ are noise vector and matrix, respectively. Substituting (3.24) and (3.25) into (3.22), we consider the probability limit of the sequence $\bar{\boldsymbol{p}}_N$ (N= 9 to infinity). Applying the Slutsky's Theorem, it is derived under the assumption that $\boldsymbol{\Omega}_N^t \boldsymbol{\Psi}_N$ is nonsingular that

$$p \lim_{N\to\infty} \bar{\boldsymbol{p}}_N = \boldsymbol{p}^* + p \lim_{N\to\infty} \frac{1}{N}(\boldsymbol{\Omega}_N^t \boldsymbol{\Psi}_N)^{-1} \cdot p \lim_{N\to\infty} \frac{1}{N} \boldsymbol{\Omega}_N^t \boldsymbol{\nu}_N \tag{3.26}$$

where

$$\boldsymbol{\nu}_N = -\tilde{\boldsymbol{\Psi}}_N \boldsymbol{p}_N^* + \tilde{\boldsymbol{y}}_N. \tag{3.27}$$

Hence, if

$$p \lim_{N\to\infty} \frac{1}{N} \boldsymbol{\Omega}_N^t \boldsymbol{\Psi}_N \tag{3.28}$$

exists and is nonsingular, and

$$p \lim_{N\to\infty} \frac{1}{N} \boldsymbol{\Omega}_N^t \boldsymbol{\nu}_N = 0, \tag{3.29}$$

then, the instrumental variable estimate is a consistent estimate of $\boldsymbol{p}^*$. The matrix sequence $\boldsymbol{\Omega}_N$ (from N=9 to infinity) which satisfies (3.28),(3.29) is called an instrumental matrix sequence. It is clear that the estimated value is asymptotically unbiased.

3.5.2 Instrumental Matrix Sequence

In the instrumental variable method, we can choose the instrumental matrix sequence freely as long as it has no correlation to the noise vector $\boldsymbol{\nu}_N$.

Since PUMA 560 is equipped only encoders to measure the values about motions, we can obtain the joint angles with high precision. However we must estimate angular velocity and angular acceleration data based on the angle data. It is likely that the angular acceleration data is more contaminated rather than the angle and the angular velocity data. Hence, we assume followings:

$$\hat{\boldsymbol{\theta}}[n] = \boldsymbol{\theta}^*[n] \tag{3.30}$$

$$\hat{\dot{\boldsymbol{\theta}}}[n] = \dot{\boldsymbol{\theta}}^*[n] \tag{3.31}$$

$$\hat{\ddot{\boldsymbol{\theta}}}[n] = \ddot{\boldsymbol{\theta}}^*[n] + \tilde{\ddot{\boldsymbol{\theta}}}[n]. \tag{3.32}$$

We assume that $\tilde{\ddot{\boldsymbol{\theta}}}[n]$ is independent of $\ddot{\boldsymbol{\theta}}^*[n]$, $\hat{\dot{\boldsymbol{\theta}}}[n]$, and $\hat{\boldsymbol{\theta}}[n]$. We assume also $E[\tilde{\ddot{\boldsymbol{\theta}}}[n]] = 0$, for $n = 1, 2, 3, \ldots$. The symbols with * denote true values. $E[\cdot]$ denotes expected value of $(\cdot)$. We finally assume that

$$\hat{\boldsymbol{\tau}}_m[n] = \boldsymbol{\tau}^*_m[n]. \tag{3.33}$$

Then, we make a matrix sequence as follows. First, we estimate $\ddot{\boldsymbol{\theta}}^*[n]$ in the way described in the next subsection. Let $\bar{\ddot{\boldsymbol{\theta}}}[n]$ denote the estimated $\ddot{\boldsymbol{\theta}}^*[n]$. Next, since each element of $\boldsymbol{\Phi}(\ddot{\boldsymbol{\theta}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}, sgn\dot{\boldsymbol{\theta}})$ is a function of $\ddot{\boldsymbol{\theta}}, \dot{\boldsymbol{\theta}}, \boldsymbol{\theta}$, we can obtain the same size matrix $\boldsymbol{\Omega}[n]$ as $\hat{\boldsymbol{\Phi}}[n]$ using $\hat{\boldsymbol{\theta}}[n]$, $\hat{\dot{\boldsymbol{\theta}}}[n]$, and $\bar{\ddot{\boldsymbol{\theta}}}[n]$ instead of $\hat{\ddot{\boldsymbol{\theta}}}[n]$. Thereby, we obtain a matrix sequence

$$\boldsymbol{\Omega}_N = [\boldsymbol{\Omega}^t[1], \cdots, \boldsymbol{\Omega}^t[N]]^t \tag{3.34}$$

for $N = 9, 10, \ldots$. If the test motion is not controlled by feedback of trajectory, this matrix sequence has no correlation to the noise vector $\boldsymbol{\nu}_N$ which are caused only by $\tilde{\ddot{\boldsymbol{\theta}}}[n]$. Also, the test motion is made move all joints simultaneously random enough way.

Using this matrix sequence as an instrumental matrix sequence, we estimate $\boldsymbol{p}$ by iterative formula.

3.5.3 Instrumental Model

From equations (3.3),(3.4),(3.6) and (3.7), we obtain

$$\begin{aligned}&(\boldsymbol{H}+\boldsymbol{R}\boldsymbol{K}^{-1})\ddot{\boldsymbol{\theta}}\\&=(\boldsymbol{K}^t)^{-1}\boldsymbol{\tau}_m-(\boldsymbol{B}\dot{\boldsymbol{\theta}}+\boldsymbol{G}(\boldsymbol{\theta})+(\boldsymbol{K}^t)^{-1}(\boldsymbol{V}\boldsymbol{K}^{-1}\dot{\boldsymbol{\theta}}+\boldsymbol{C}sgn(\boldsymbol{K}^{-1}\dot{\boldsymbol{\theta}})))\end{aligned} \tag{3.35}$$

where

$$\boldsymbol{R}=diag[h1,h2,h3,h4,h5,h6] \tag{3.36}$$

$$\boldsymbol{V}=diag[b1,b2,b3,b4,b5,b6] \tag{3.37}$$

$$\boldsymbol{C}=diag[c1,c2,c3,c4,c5,c6]. \tag{3.38}$$

$\boldsymbol{K}$ is nonsingular.

Using the equation as an instrumental model, we obtain $\bar{\ddot{\boldsymbol{\theta}}}[n]$ by substituting $\hat{\boldsymbol{\theta}}[n]$, $\hat{\dot{\boldsymbol{\theta}}}[n]$ and $\hat{\tau}_m[n]$ into it.

The instrumental model (3.35) is nothing but the dynamic model of PUMA 560 and it is same as the instrumental model proposed by Kawasaki[46]. Hence, in advance, we need a set of the values of the model parameters for the instrumental model. Kawasaki has not given any method to obtain it. Then, we use the values of the model parameters which were estimated by the simultaneous method where the least squares method was adopted.

The estimated values of the model parmeters by the advanced simultaneous method are shown in Table 3.

3.6 Discussion and Comparison of the Identification Methods

In this section, we will describe in detail the contents of the work which is needed to obtain the estimates by each identification method, and compare the identification methods about the amount of labour (human involvement) and consuming time on a computer. We will also compare them about the accuracy of the estimates. To evaluate the accuracy of the estimates we will simulate the motion of the manipulator using the estimates and compare the simulated trajectories with measured trajectories.

First, we will describe in detail the contents of work in the estimation by each identification method.

The step-by-step method consists of 3 types of simple test motions, such that we only need to move one or two joints simultaneously, freezing the rest of the joints. The model parameters are divided into a certain number of subgroups, and values of model parameters

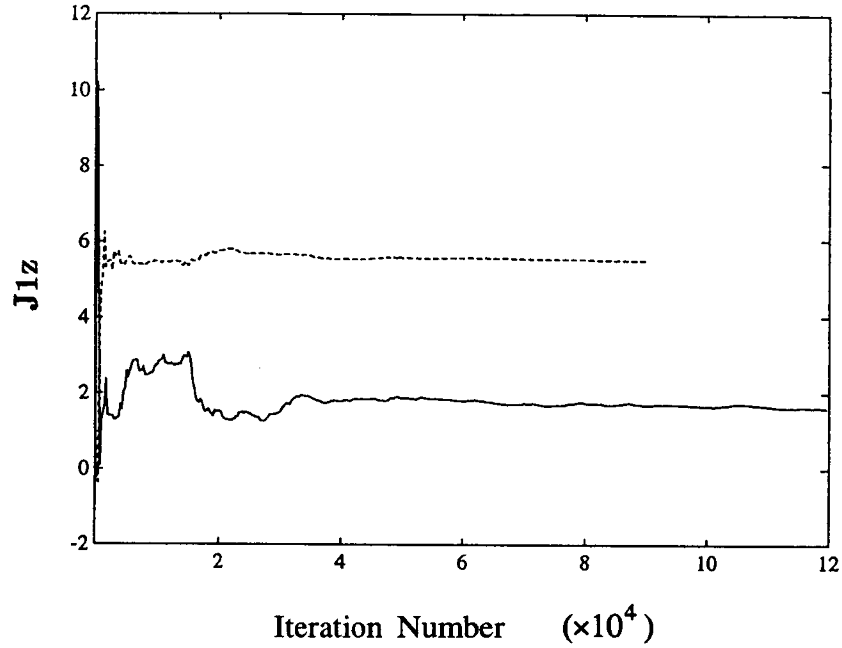
Table 3 Model Parameter Values by Advanced Simultaneous Method

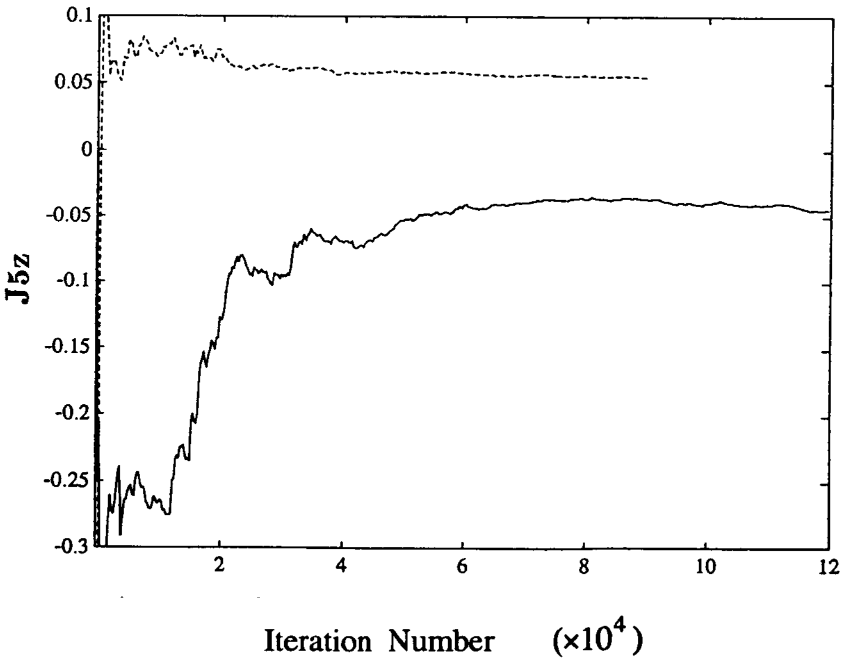
Parameter	Value	Parameter	Value
c_1	1.28×10^{-1}	**J1z**	5.56
c_2	1.23×10^{-1}	**J2z**	7.84
c_3	1.23×10^{-1}	**J3z**	1.00
c_4	2.68×10^{-2}	**J4z**	-1.13×10^{-2}
c_5	1.56×10^{-2}	**J5z**	5.76×10^{-2}
c_6	2.02×10^{-2}	**J6z**	2.11×10^{-2}
	[Nm]		
b_1	1.09×10^{-3}	**J2(x-y)**	-2.46
b_2	4.17×10^{-4}	**J3(x-y)**	7.40×10^{-1}
b_3	4.78×10^{-4}	**J4(x-y)**	2.79×10^{-3}
b_4	4.01×10^{-5}	**J5(x-y)**	4.68×10^{-2}
b_5	3.94×10^{-5}	**J6(x-y)**	-1.31×10^{-3}
b_6	8.31×10^{-5}		
	[Nms]		
		J2xy	-1.12×10^{-2}
h_3	3.62×10^{-4}	**J3xy**	3.78×10^{-2}
h_4	3.74×10^{-5}	**J4xy**	1.11×10^{-2}
h_5	3.22×10^{-5}	**J5xy**	6.13×10^{-4}
h_6	3.75×10^{-5}	**J6xy**	1.15×10^{-2}
	[kgm^2]		
R2x	7.76	**J2xz**	1.07
R3x	9.60×10^{-2}	**J3xz**	-8.09×10^{-3}
R4x	-2.69×10^{-2}	**J4xz**	-8.14×10^{-3}
R5x	-3.30×10^{-3}	**J5xz**	6.88×10^{-4}
R6x	-1.28×10^{-1}	**J6xz**	1.39×10^{-2}
R2y	3.15×10^{-1}	**J2yz**	2.80×10^{-2}
R3y	-2.23	**J3yz**	-1.75×10^{-2}
R4y	-5.10×10^{-3}	**J4yz**	-6.92×10^{-3}
R5y	-2.63×10^{-1}	**J5yz**	2.32×10^{-3}
R6y	1.41×10^{-1}	**J6yz**	-1.15×10^{-2}
	[kgm]		[kgm^2]

in each subgroup are estimated from data of the test motions, use being made of formerly estimated model parameter values. The test motions are very simple, and it is easy to understand how the target parameters in the test motion affect motor torques. Hence programming for data processing is simple and of small size, and we can contrive good test motions for the estimations. The estimated values of subgrouped parameters in each motion test are examined by a small size simulation, and we can improve the test motion and the accuracy of the estimation. On the other hand, this method requires a number of test motions; 6 static tests, 6 constant velocity motion tests, and 13 accelerated motion tests in case of PUMA 560. Hence, the step-by-step method requires considerable amount of labour.

The simultaneous method is simple to understand and requires only one random enough test motion. Hence, it is easy to execute, however, the programming of data process is more time consuming than that of the step-by-step method for the high degree-of-freedom manipulators. The most serious drawback of this method is difficulty of convergence judgment and a huge number of iterative calculations for convergence. In our estimation by this method, as a test motion, we made the manipulator stand still in a configuration first, and made the manipulator move by giving a command of torque values to each motor. The motor current and motor angle data are taken at every 5 ms for each motor while the manipulator is in the motion. The joint torques are computed from motor currents. To obtain the joint velocities and accelerations, the joint angles, which are computed from motor angles, are differentiated and double-differentiated, respectively. Fig.3.6.1 shows the convergence of model parameters **J1z** and **J5z** in iterative calculations when the model parameters were estimated by simultaneous method and advanced simultaneous method. In case of PUMA 560, the model parameters are grouped into two categories with respect to convergence rate. **J1z** is one of the model parameters which converge fast and have relatively large values. **J5z** is one of the model parameters which converge slowly and have relatively small values. The figure shows that it took about 80000 iterations for **J5z** to converge in simultaneous method, and hence about 80000 data points were needed. Also it took even about 40000 iterations for **J1z** to converge. Still, a few parameters could not be convinced to have converged after 120000 iterations. We stopped the estimation process at this point and obtained the model parameter values as the estimates. It took about 36 hours as total for sun SPARKstation IPC to do 120000 iterative calculations. Thus, the simultaneous method requires a huge number of data points and a very long time on a computer for obtaining the estimates.

The advanced simultaneous method is different from the simultaneous method only in data process. Hence, it is also easy to execute. However, it uses the instrumental variable method instead of the least squares method, hence, programming for data processing is more complicated than that of the simultaneous method. Moreover, it is a crucial problem to choose the instrumental matrix sequence. If we make the instrumental matrix sequence using the instrumental model, then it is a problem to construct the instrumental model.





------------------ : Advanced Simultaneous Method

———————— : Simultaneous Method

Fig. 3.6.1 Convergence of Model Parameters

This method also requires a big number of data points. As shown in Fig.3.6.1 it took about 50000 iterations for **J5z** to converge and about 30000 iterations for **J1z**. The other model parameters also could be judged to have converged after about 80000 iterations. Hence, the advanced simultaneous method requires a less number of data points than the simultaneous method, however 80000 data points is still huge. In our estimation by this method, we used the dynamic model of the manipulator as the instrumental model, hence we needed a set of model parameter values to construct the instrumental model in advance. We used the estimates by the simultaneous method for the instrumental model, which is our proposal. According to our proposal, you could construct the instrumental model very easily, however it would take a very long time.

Next, we will compare the three identification methods about the accuracy of the estimates. To evaluate the accuracy of the estimates we will simulate the motion of the manipulator using the estimates and compare the simulated trajectories with measured trajectories. As the motion for the simulation, we made the manipulator stand still in a configuration first, and made the manipulator move by giving a command of torque values to each motor. The motor current and motor angle data are taken at every 2 ms for each motor while the manipulator is in the motion. The joint torques are computed from motor currents. To obtain the joint velocities, the joint angles, which are computed from motor angles, are differentiated. The results are shown in Fig.3.6.2 - Fig.3.6.4. We can conclude from these results that the step-by-step method is more precise way to estimate the model parameter values than the simultaneous method, and the advanced simultaneous method is nearly as precise way as the step-by-step method.

The following is a summary of the above.

The step-by-step method is precise way to estimate the model parameters, however, it requires the most amount of labour among the three methods. The consuming time on a computer for data process is the shortest.

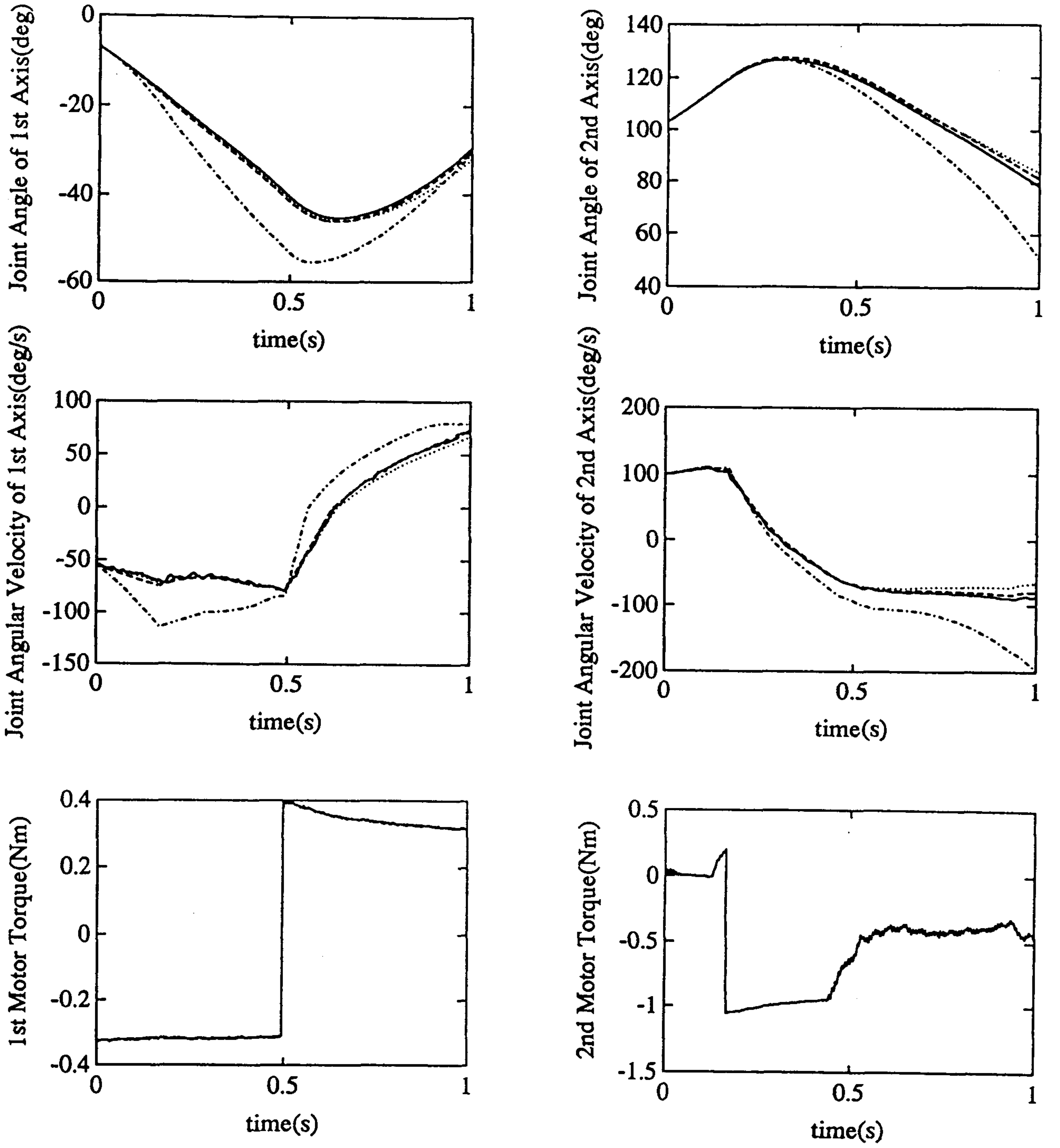
The simultaneous method is not so precise way as the other methods. It requires the least amount of labour among the three methods, however, a huge number of data points hence a very long time on a computer for obtaining the estimates.

The advanced simultaneous method is as precise way as the step-by-step method. It requires more amount of labour than the simultaneous method, however a less number of data points and a shorter time on a computer than the simultaneous method.

Finally, we give some comments on the experimental results.

The convergence rate of model parameters in the simultaneous method and the advanced simultaneous method depends on test motions. We did not take it into account at all. Hence, the demerit that the methods require a huge number of data points might be able to be improved. However, it is difficult to find a good test motion with regard to the convergence rate[49].

PUMA 560 has only encoders at each joint, hence we can obtain joint angle data with high precision, however we have to manage to obtain joint angular and joint acceleration



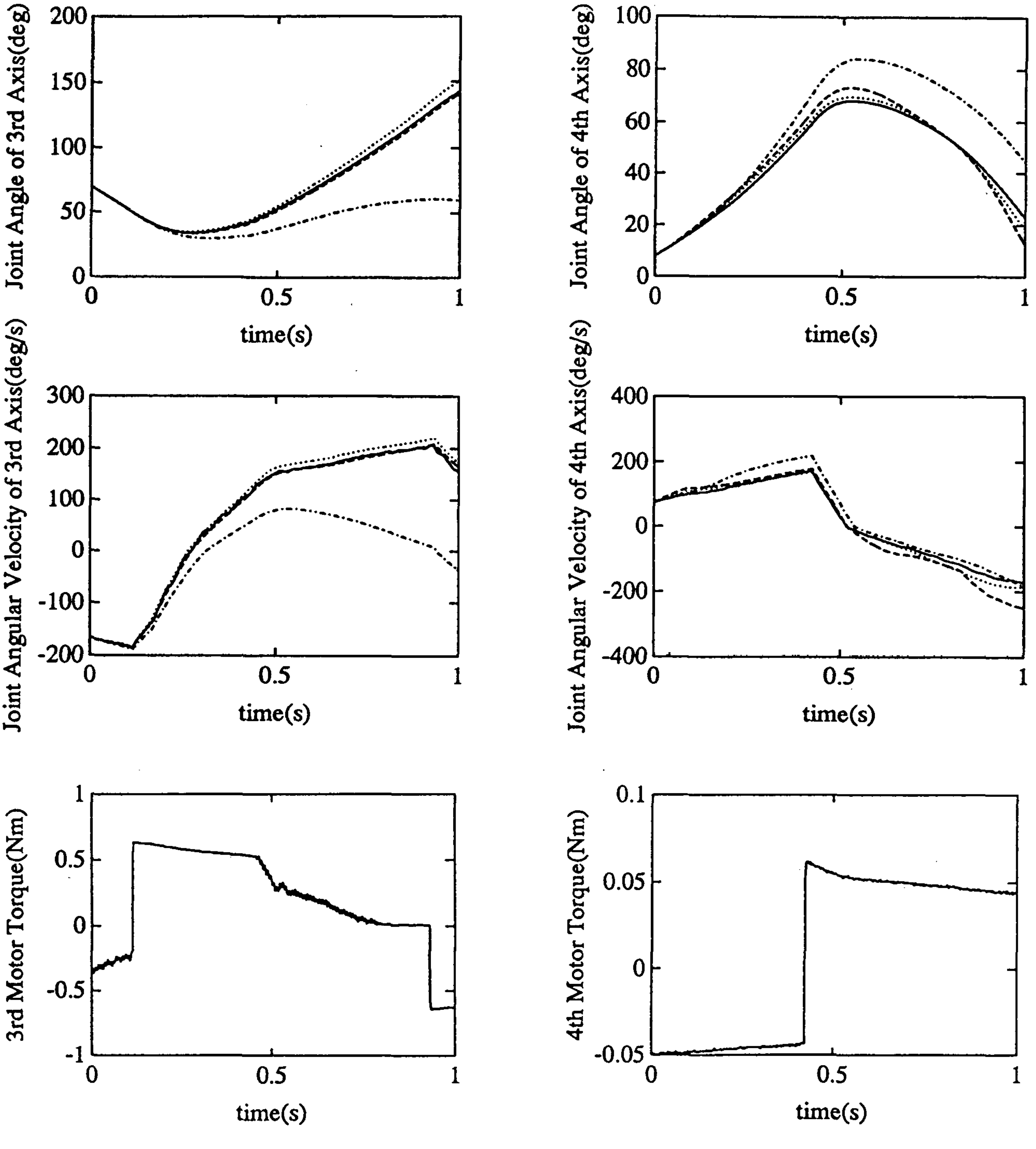
——————— : Simulated Trajectory using the estimates by **Step-by-step Method**

··················· : Simulated Trajectory using the estimates by **Advanced Simultaneous Method**

—·—·—·— : Simulated Trajectory using the estimates by **Simultaneous Method**

——————— : Measured Trajectory

Fig. 3.6.2 Comparison of Simulated Trajectories with Measured Trajectory of Joint 1 and Joint 2



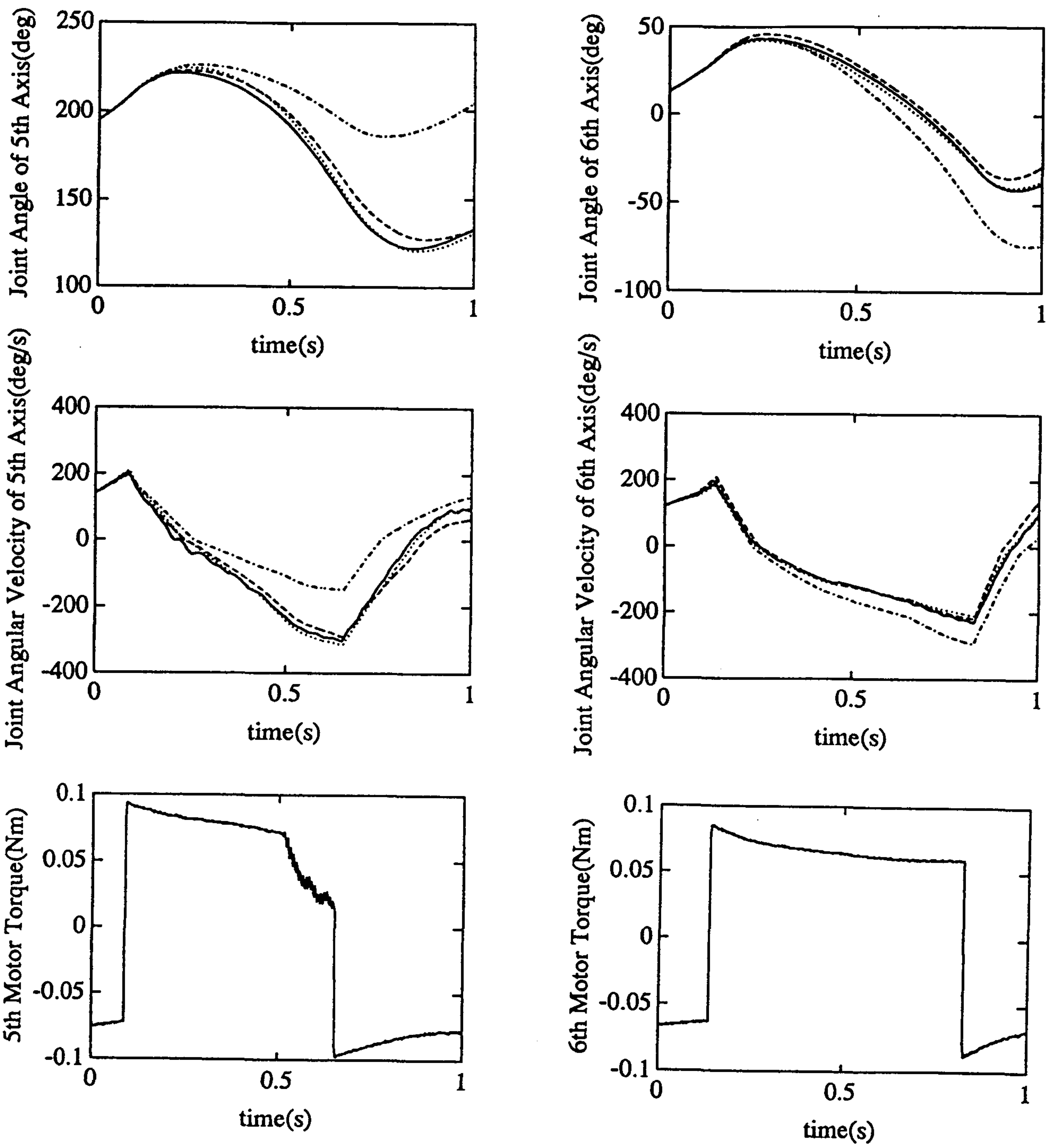
— — — — — — — — : Simulated Trajectory using the estimates by **Step-by-step Method**

······················ : Simulated Trajectory using the estimates by **Advanced Simultaneous Method**

—·—·—·— : Simulated Trajectory using the estimates by **Simultaneous Method**

———————— : Measured Trajectory

Fig. 3.6.3 Comparison of Simulated Trajectories with Measured Trajectory of Joint 3 and Joint 4



: Simulated Trajectory using the estimates by **Step-by-step Method**

: Simulated Trajectory using the estimates by **Advanced Simultaneous Method**

: Simulated Trajectory using the estimates by **Simultaneous Method**

: Measured Trajectory

Fig. 3.6.4 Comparison of Simulated Trajectories with Measured Trajectory of Joint 5 and Joint 6

data using the joint angle data. In our experiment, we obtained them by differentiating and double-differentiating the joint angles, respectively. Thus, the acceleration data contained a large amount of noise. It seems to have partially caused the slow convergence and the inaccuracy of the estimation in the simultaneous method.

PUMA 560 has a very sophisticated gear mechanism to move the 4-th, 5-th, and 6-th link, and torques are transmitted to it with long rods which may be a little elastic. Those factors may restrict the accuracy of the estimates, especially those of the 4-th, 5-th, and 6-th link.

Though we had several assumptions to fulfill the conditions to get estimates unbiased in applying the advanced simultaneous method, the instrumental matrix and the noise vector in (3.29) may have a correlation in practice, hence, the estimates may be biased. However, the simulation results show the bias are very small.

Strictly speaking, the origin of 4-th coordinate systems does not coincide with the origin of 3rd coordinate systems; there is a gap of a few cent meters in the direction of $\boldsymbol{x}_3$. The simulation results, however, show that the gap can be neglected in the dynamics of PUMA 560.

3.7 Conclusion

We have experimentally examined to estimate the model parameters of PUMA 560 applying the identification methods: the step-by-step method, the simultaneous method, and the advanced simultaneous method . To evaluate the accuracy of the estimates, we simulated the motion of the manipulator using the estimates and compared the simulated trajectories with measured trajectories. We described in detail the contents of the work which is needed to obtain the estimates by each identification method, and compared the identification methods about the amount of labour and consuming time on a computer.

We can conclude as follows. The step-by-step method is precise way to estimate the model parameters, however, it requires the most amount of labour among the three methods. The consuming time on a computer for data process is the shortest. The simultaneous method is not so precise way as the other methods. It requires the least amount of labour among the three methods, however, a huge number of data points hence a very long time on a computer for obtaining the estimates. The advanced simultaneous method is as precise way as the step-by-step method. It requires more amount of labour than the simultaneous method, however a less number of data points and a shorter time on a computer than the simultaneous method.

For the advanced simultaneous method, it is a crucial problem to choose a good instrumental matrix sequence. If we make the instrumental matrix sequence by using the instrumental model, it is a problem to construct the instrumental model. We have proposed one method to obtain a set of parameter values which is needed to the instrumental model that has been proposed by Kawasaki. Though the method is considerably time consuming, it is effective for accurate estimation and there is no difficulty.

Chapter 4

Physical Impossibility of the Set of Base-parameter Values

4.1 Introduction

In this chapter, we propose a method to judge if a set of base-parameter values for a manipulator determines the inertial matrix of its dynamic model to be positive definite or not. The set of base-parameter values that is judged not to do is physically impossible.

To obtain an accurate dynamic model of a manipulator for the model-based control, accurate values of the parameters that appear in the dynamic model of the manipulator are required. Then, it is important to have a good method to obtain the parameter values. Hence, as far as the base parameters are concerned, we have experimentally examined the identification methods of them, assuming the values of the kinematic parameters are known. If we could obtain the true values of the parameters, no problem would happen. However we are forced to use estimated values to determine the dynamic model. Thereby it may happen that the inertial matrix of the dynamic model is not always positive definite for each configuration of the manipulator, though it is the fact that the inertial matrix is positive definite for each configuration of the manipulator. If a set of base-parameter values determines such inertial matrix, it is physically impossible and it is needless to examine the accuracy of the values as far as we use the dynamic model derived under the assumption that all the links of the manipulator are rigid. The dynamic model that is determined by such a set of base-parameter values would express nothing in the physical world. If the manipulator were controlled by using such a set of base-parameter values, a good performance of the manipulator would not be ensured, and if the manipulator motion were simulated, the results would not be worth believing. Hence, in this chapter we propose a method to judge if a set of base-parameter values determines the inertial matrix to be positive definite for each configuration of the manipulator or not, when we approximately consider the continuous change of each joint variable of the manipulator as a finite set of discrete points. The method can be executed on computers. Using this method we can judge if a set of estimated base-parameter values is "possible" or not. Here we use "possible" in the sense that the set of base-parameter values determines the inertial matrix to be positive definite for each configuration of the manipulator. If a set of base-parameter values is not "possible", it is physically impossible. The physical impossibility of a set of estimated base-parameter values is caused by the biased base-parameter values of the set. We actually estimate a set of base-parameter values in the presence of a noise on data, modeling error, and so on, hence, we are forced to have biased base-parameter values more or less. Then, no estimation method would ensure to give a set of estimated base-parameter values to be always "possible". If we unfortunately estimate a set of base-parameter values not to be "possible", should we try to estimate again without any strategy ? Otherwise should we try to develop the ultimate estimation method that gives always "possible" set of base-parameter values ? Both would be just laborious and fruitless. Hence we will also propose one method to modify the estimated base-parameter values for the set of them to be at least "possible" if we judge it is not. In the modification we will make the best of the originally estimated base-parameter values. The modification of the estimated base-parameter values might reduce the accuracy of some values, however, it would be better that a set of base-parameter values that we use is "possible" than that it is not "possible", i.e., physically impossible.

Even if a set of base-parameter values is "possible", it is not always physically possible, since being "possible" of a set of base-parameter values only ensures the inertial matrix which is determined by the set of base-parameter values to be positive definite for each configuration of the manipulator. Hence the study in this chapter is the first step in the context of making it clear which set of base-parameter values is physically possible. Actually there has not been any study about the relationship between a set of base-parameter values and its physical possibility or impossibility in the dynamic model. It would be important to make it clear which set of base-parameter values is physically possible.

4.2 Inertial Matrix

The inertial matrix of the dynamic model of a manipulator will be explained briefly.

We consider the manipulator having open-loop kinematic chain with N links and revolute joints only. Then, the dynamic equations for the kinematic chain is described as

$$\boldsymbol{\tau} = \boldsymbol{H}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{B}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \boldsymbol{G}(\boldsymbol{\theta}) \tag{4.1}$$

where $\boldsymbol{\tau} = [\tau_1 \cdots\cdots \tau_N]^t$ is the joint torque vector and $\boldsymbol{\theta} = [\theta_1 \cdots\cdots \theta_N]^t$ is the joint variable vector. $\boldsymbol{H}(\boldsymbol{\theta})$ is an $N \times N$ inertial term matrix, $\boldsymbol{B}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ is the Coriolis and centrifugal force term matrix, and $\boldsymbol{G}(\boldsymbol{\theta})$ is an N-dimensional gravity term vector. The (i,j)-th element of inertial term matrix $\boldsymbol{H}(\boldsymbol{\theta})$ will be denoted by $H(i,j)$. Then it is described as

$$\begin{aligned} H(i,j) &= \boldsymbol{z}_i \cdot (\sum_{S=i}^{N} \boldsymbol{J}_S)\boldsymbol{z}_j \\ &+\boldsymbol{z}_i \cdot \{\sum_{S=i}^{N-1} [2(\boldsymbol{L}_S \cdot \boldsymbol{SR}_{S+1})\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{SR}_{S+1} - \boldsymbol{SR}_{S+1} \otimes \boldsymbol{L}_S]\}\boldsymbol{z}_j \\ &+\boldsymbol{z}_i \cdot [(\boldsymbol{L}_{j,i} \cdot \boldsymbol{SR}_i)\boldsymbol{E} - \boldsymbol{L}_{j,i} \otimes \boldsymbol{SR}_i]\boldsymbol{z}_j \end{aligned} \tag{4.2}$$

for $1 \le j \le i \le N$, and

$$H(i,j) = H(j,i) \tag{4.3}$$

for $i < j$. where the notation used in (4.2) is same as that in chapter 2. The equation (4.2) is quite same as that of the dynamic equations for the parallel and perpendicular manipulators with rotational joints only which is treated in section 3 of chapter 2. The dynamic equations that we have derived is a coordinate-free and vector-tensor form, hence, evaluating (4.2) about an appropriate coordinate system, we have difference between $H(i,j)$ for the parallel and perpendicular manipulators and $H(i,j)$ for general open-loop manipulators.

Using the rotation matrices and ${}^i\boldsymbol{z}_i = [0\ 0\ 1]^t$, we evaluate each $H(i,j)$ about an appropriate coordinate system, then we can describe it in the following form:

$$\sum_{v=1}^{T_b} \mathrm{p}_{bv} f_{bv}(\boldsymbol{\theta}) \tag{4.4}$$

where $f_{bv}(\boldsymbol{\theta})$ is a polynomial of trigonometric functions of θ_i for $1 \le i \le N$. (f_{bv} is allowed to be a constant function), and p_{bv} is a base parameter.

4.3 A Condition for a Set of Base-Parameter Values to be "possible"

In this section, first, a necessary and sufficient condition for the inertial matrix to be positive definite will be derived. As shown below, the condition is described as a system of N inequalities, each of the N inequalities can be described as

$$\sum_{j=1}^{n_i} p_j f_j(\boldsymbol{\theta}) > 0 \tag{4.5}$$

where p_j is a parameter generated by the base parameter set and $f_j(\boldsymbol{\theta})$ is a polynomial of trigonometric functions of θ_i for $1 \leq i \leq N$. Each inequality holds for each configuration of the manipulator. Conversely, based on this fact and assuming all p_j as unknowns, we can obtain the domain of the solutions of n_i-tuples:$(p_1, \dots, p_{n_i})$ for the inequality to hold for each θ_i and for each $1 \leq i \leq N$. Then a set of base-parameter values is "possible" if and only if the n_i-tuples that is determined by the set of base-parameter values is in the domain. The goal of this section is to obtain the domain.

4.3.1 A Condition for the Inertial Matrix to be Positive Definite

In this subsection, we will derive a necessary and sufficient condition for the inertial matrix to be positive definite using the elements of the inertial matrix. Taking advantage that the inertial matrix is symmetric, we apply *Sylvester's theorem* to the inertial matrix. Then, the inertial matrix $\boldsymbol{H}(\boldsymbol{\theta})$ is positive definite if and only if all the leading principal minors of it are positive. We have hence a system of N inequalities because we consider the N degree-of-freedom manipulator. Since each element of $\boldsymbol{H}(\boldsymbol{\theta})$ is described as shown in the preceding section and leading principal minors are calculated by only multiplication of elements and addition and/or subtraction of the products, each of the N inequalities can be described as

$$\sum_{v=1}^{T} p_v f_v(\boldsymbol{\theta}) > 0 \tag{4.6}$$

where p_v is a parameter generated by base parameter set and $f_v(\boldsymbol{\theta})$ is a polynomial of trigonometric functions of θ_i for $1 \leq i \leq N$. T is the number of the terms of a leading principal minor that is considered.

We remark here 3 items. First, after calculating a leading principal minor, we can delete the linear dependency among $f_1, \dots, f_n$ and $p_1, \dots, p_n$, thus we can modify (4.6) as

$$\sum_{v=1}^{T_m} p_{mv} f_{mv}(\boldsymbol{\theta}) > 0 \tag{4.7}$$

where p_{mv} is a parameter generated by base parameter set and $f_{m1}(\boldsymbol{\theta}), \dots, f_{mT_m}(\boldsymbol{\theta})$ are linearly independent. Secondly, $p_{mv}(v = 1, \dots, T_m)$ is a parameter that is a sum of products of some base parameters, for example, when the leading principal minor is 2nd order, p_{mv} may be described like $p_{mv} = (p_{b1})^2 - p_{b1}p_{b2} + p_{b3}p_{b4}$ where p_{b1}, p_{b2}, p_{b3}, and p_{b4} are base parameters. Hence, p_{mv} is continuous on the continuous change of base parameters. Finally, any $f_{mv}(\boldsymbol{\theta})$ is a continuous and bounded function because it is a polynomial of trigonometric functions.

Consequently, we obtain the following system of N inequalities as a necessary and sufficient condition for the inertial matrix to be positive definite.

$$\begin{gathered}
\sum_{v_1=1}^{n_1} p_{m1v_1} f_{m1v_1}(\boldsymbol{\theta}) > 0 \\
\sum_{v_2=1}^{n_2} p_{m2v_2} f_{m2v_2}(\boldsymbol{\theta}) > 0 \\
\cdots \\
\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}(\boldsymbol{\theta}) > 0 \\
\cdots \\
\sum_{v_N=1}^{n_N} p_{mNv_N} f_{mNv_N}(\boldsymbol{\theta}) > 0
\end{gathered} \tag{4.8}$$

where $p_{miv_i} (i = 1, \cdots, N, v_i = 1, \cdots, n_i)$ is a parameter generated by base parameter set, and $f_{miv_i}(\boldsymbol{\theta})(i = 1, \cdots, N, v_i = 1, \cdots, n_i)$ is a polynomial of trigonometric functions of θ_i for $1 \le i \le N$, besides $f_{mi1}, f_{mi2}, \ldots, f_{min_i}$ are linearly independent for $1 \le i \le N$.

The goal of this section is to show the subset of R^{n_i} of which the n_i-tuples $(p_{mi1}, \ldots, p_{min_i})$ must be the element when the n_i-tuples is determined by a set of base-parameter values. R^{n_i} denotes the vector space consisting of n_i-tuples. For the purpose, assuming $p_{mi1}, \cdots, p_{min_i}$ to be unknowns, we will show the set of the solutions for the inequality $\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}(\boldsymbol{\theta}) > 0$ to hold for each θ_i and for each $1 \le i \le N$.

4.3.2 Approximation of the Inequalities

Each of the N inequalities, derived in the preceding subsection, holds for each θ_i and for each $1 \le i \le N$. Then we must investigate the inequality when the functions change continuously. However, we will approximately consider each function as a finite set of discrete points for simplicity, thereby we obtain a system of finite linear inequalities instead of the inequality containing functions. In this subsection we will explain the approximation.

It is sufficient for us to consider the joint variables θ_i as $0 \le \theta_i < 2\pi$ for each $1 \le i \le N$, hence 2π is divided into m pieces and θ_i is approximately considered to take these discrete values. If θ_i takes $\frac{2\pi}{m}k$ where k is an integer $(0 \le k < m)$, it is denoted by $\theta_i[k]$. Thus,

$$\theta_i[0] = 0,\ \theta_i[1] = \frac{2\pi}{m},\ \cdots,\theta_i[m-2] = 2\pi - \frac{2\pi}{m} \times 2,\ \ \theta_i[m-1] = 2\pi - \frac{2\pi}{m}. \tag{4.9}$$

Then substituting $\theta_1[k1], \theta_2[k2], \theta_3[k3], \cdots, \theta_N[kN]$ into $f_{miv_i}(\boldsymbol{\theta})$, we can obtain a real value and will denote it by $f_{miv_i}^{(k1,\ k2,\cdots,kN)}$. Thus we obtain the system of finite linear inequalities:

$$\begin{aligned}
&\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}^{(0,\ 0,\cdots,0)} > 0 \\
&\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}^{(0,\ 0,\cdots,1)} > 0 \\
&\qquad \cdots \\
&\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}^{(m,\ m,\cdots,m)} > 0,
\end{aligned} \tag{4.10}$$

approximating the inequality:

$$\sum_{v_i=1}^{n_i} p_{miv_i} f_{miv_i}(\boldsymbol{\theta}) > 0 \tag{4.11}$$

If we take m large enough, we can obtain good approximation of (4.11) for each θ_i and for each $1 \le i \le N$.

We introduce some symbols for convenience. vectors $\boldsymbol{p}_i$ and $\boldsymbol{f}_i$ are defined as

$$\boldsymbol{p}_i = [p_{mi1} \quad p_{mi2} \quad \cdots \quad p_{min_i}]^t \tag{4.12}$$

$$\boldsymbol{f}_i = [f_{mi1}(\boldsymbol{\theta}) \quad f_{mi2}(\boldsymbol{\theta}) \quad \cdots \quad f_{min_i}(\boldsymbol{\theta})]^t \tag{4.13}$$

and vector $\boldsymbol{f}_i^1$ is defined as

$$\boldsymbol{f}_i^1 = [f_{mi1}^{(0,\, 0,\cdots,0)} \quad f_{mi2}^{(0,\, 0,\cdots,0)} \cdots f_{min_i}^{(0,\, 0,\cdots,0)}]^t. \tag{4.14}$$

Then the first inequality in (4.10) can be expressed as $\boldsymbol{f}_i^1 \cdot \boldsymbol{p}_i > 0$. In the same way, vector $\boldsymbol{f}_i^h$ is defined to express the h-th inequality from the top in (4.10), and vector $\boldsymbol{f}_i^q$ is defined as

$$\boldsymbol{f}_i^q = [f_{mi1}^{(m,\, m,\cdots,m)} \quad \cdots \quad f_{min_i}^{(m,\, m,\cdots,m)}]^t \tag{4.15}$$

where $q = m^m$. Finally, matrix $\boldsymbol{F}_i$ is defined as

$$\boldsymbol{F}_i = (\boldsymbol{f}_i^1 \ \vdots \ \boldsymbol{f}_i^2 \ \vdots \ \cdots \ \vdots \ \boldsymbol{f}_i^q). \tag{4.16}$$

Then the system of finite linear inequalities (4.10) is described as

$$\boldsymbol{F}_i^t \boldsymbol{p}_i > \mathbf{o}. \tag{4.17}$$

Each vector inequality holds for every component individually.

4.3.3 Existence of Solutions

In this subsection we will ensure that the system (4.17) has solutions.

We begin with a lemma concerning the dual systems

$$\boldsymbol{A}^t \boldsymbol{p} \geq \mathbf{o} \text{ and } \boldsymbol{A}\boldsymbol{x} = \mathbf{o}, \boldsymbol{x} \geq \mathbf{o} \tag{4.18}$$

where

$$\boldsymbol{A} = (\boldsymbol{a}_1 \vdots \boldsymbol{a}_2 \vdots \cdots \vdots \boldsymbol{a}_q) \tag{4.19}$$

is a q-columned matrix with arbitrary real entries, $\boldsymbol{p}$ and $\boldsymbol{x}$ denote vectors. Each vector inequality and each vector equality holds for every component individually.

Lemma 4.3.1 The system

$$\boldsymbol{A}^t\boldsymbol{p} > \mathbf{o} \tag{4.20}$$

possesses solutions if and only if the system of equations

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{o}, \boldsymbol{x} \geq \mathbf{o} \tag{4.21}$$

possesses only a solution $\boldsymbol{x} = \mathbf{o}$.

Proof As a corollary of **Tucker's theorem** [50] we can prove the lemma. ∎

Then, we have the following theorem.

Theorem 4.3.1 The system of finite linear inequalities (4.17) possesses solutions.

Proof We begin with showing that there is one constant function among

$$f_{mi1}(\boldsymbol{\theta}), \quad f_{mi2}(\boldsymbol{\theta}), \quad \cdots \quad , f_{min_i}(\boldsymbol{\theta})$$

for each $1 \leq i \leq N$.

The functions are obtained through calculating the leading principal minor of $\boldsymbol{H}(\boldsymbol{\theta})$. Let $det\boldsymbol{H}_r$ denote the r-th order leading principal minor ($1 \leq r \leq N$), then it is also described as

$$det\boldsymbol{H}_r = \sum_{\sigma \in S_r} sgn(\sigma)H(1,\sigma(1))\cdots H(r,\sigma(r)) \tag{4.22}$$

where σ denotes a permutation of the index set $1, \ldots, r$, S_r denotes the set consisting of all permutations, and $sgn(\sigma)$ denotes the sign of permutation σ. Then $det\boldsymbol{H}_r$ contains the term produced by $\prod_{s=1}^{r} H(s,s)$. On the other hand, from (4.2) each (i,i)-th entry of inertial matrix $\boldsymbol{H}(\boldsymbol{\theta})$ is described as

$$H(i,i) = \mathbf{J}_i^z + G \tag{4.23}$$

where G is a function of θ_s $(i+1 \le s \le N)$ generated by base parameters, and $\mathbf{J}_i^z$ is (3,3)-th element of ${}^i\boldsymbol{J}_i$. Hence, $det\boldsymbol{H}_r$ contains the term $\prod_{s=1}^{r} \mathbf{J}_s^z$ which is not multiplied by any function. This term will not be canceled by any other term, we will show it below. The parameter $\mathbf{J}_1^z$ only appears in $H(1,1)$, hence, if

$$sgn(\sigma')H(1,\sigma'(1)) \cdots H(r,\sigma'(r)) \quad (\sigma' \in S_r) \tag{4.24}$$

contains the term that cancels $\prod_{s=1}^{r} \mathbf{J}_s^z$, then (4.24) must contain $H(1,1)$. Hence, (4.24) must contain one of $H(2,2), H(2,3), \cdots, H(2,r)$ except $H(2,1)$, but $\mathbf{J}_2^z$ does not appear in $H(2,s)$ for $3 \le s \le r$, which we can see through lengthy calculation of $H(2,s)$ for $3 \le s \le r$. Therefore (4.24) must contain $H(2,2)$. By a similar argument, consequently, (4.24) must coincide with $\prod_{s=1}^{r} H(s,s)$, hence $\prod_{s=1}^{r} \mathbf{J}_s^z$ is not canceled. Then each leading principal minor contains the term that include $\prod_{s=1}^{r} \mathbf{J}_s^z$ and that is not multiplied by any function, hence we can make $f_{min_i}(\boldsymbol{\theta})$ to be constant function 1.

Then we have $\boldsymbol{f}_i^j$ as

$$\boldsymbol{f}_i^j = [f_{i1}^j \quad f_{i2}^j \cdots f_{i(q-1)}^j \quad 1]^t \tag{4.25}$$

where f_{ik}^j for $1 \le k \le q-1$ are real entries. Hence, we can easily show that the system

$$\boldsymbol{F}_i \boldsymbol{x} = \mathbf{o}, \boldsymbol{x} \ge \mathbf{o} \tag{4.26}$$

has only a solution $\boldsymbol{x} = \mathbf{o}$.

By **Lemma 4.3.1** the system (4.17) has solutions. ∎

4.3.4 A Set of Solutions

In this subsection we will show the set of the solutions of the system

$$\boldsymbol{F}_i^t \boldsymbol{p}_i > \mathbf{o} \tag{4.27}$$

for $i = 1, \cdots, N$. It will be shown as a subset of R^{n_i}.

First, we will investigate a subset of solutions of the system

$$\boldsymbol{F}_i^t \boldsymbol{p}_i \geq \mathbf{o} \tag{4.28}$$

for $i = 1, \cdots, N$ and will denote it by D_i. Secondly, we will investigate a subset of the solutions of the equality $\boldsymbol{f}_i^k \cdot \boldsymbol{p}_i = 0$ for $k = 1, \cdots, q$, which will be denoted by S_i^k. Then we will get rid of the intersection of D_i and the union of S_i^1, S_i^2, $\cdots$,S_i^q from D_i. We can ensure that the system (4.27) has solutions except $\boldsymbol{p}_i = \mathbf{o}$ because we have shown the existence of the solutions of the system $\boldsymbol{F}_i^t \boldsymbol{p}_i > \mathbf{o}$.

We introduce some terms and symbols [50]. Let A be a subset of R^{n_i}, then we denote by A^* the set of all $\boldsymbol{y} \in \mathsf{R}^{n_i}$ such that $\boldsymbol{x} \cdot \boldsymbol{y} \geq 0$ for each $\boldsymbol{x} \in \mathsf{A}$. The set A is called convex cone if it satisfies that:

1. $\lambda \boldsymbol{x} \in \mathsf{A}$ if $\lambda \in \mathsf{R}, \lambda \geq 0$, and $\boldsymbol{x} \in \mathsf{A}$;
2. $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathsf{A}$ if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2 \in \mathsf{A}$.

Let B be a subset of R^{n_i}. A set B is said to span a convex cone A if B is a subset of A and each vector of A can be expressed as a finite linear combination of vectors of B with non-negative coefficients. If convex cone A is spanned by a finite set, A is called polyhedral convex cone.

Let A be a finite subset of R^{n_i}, and $\mathsf{A} = \{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p\}$. We denote by $\mathsf{A}^{\angle}$ the set consisting of all vectors $\boldsymbol{y} \in \mathsf{R}^{n_i}$ such that

$$\boldsymbol{y} = \sum_{i=1}^{p} \lambda_i \boldsymbol{a}_i \text{ and } \lambda_i \geq 0 \ \ (i = 1, \ldots, p). \tag{4.29}$$

We can easily see that $\mathsf{A}^{\angle}$ is a convex cone. We can also easily see that A^* is a convex cone if A is a convex cone, then A^* is called dual cone of a convex cone A.

Let A be a convex cone. A is called vertex-convex cone if A does not contain subvector space of dimension r $(1 \leq r \leq n_i)$.

As we mentioned in subsection 4.3.1, $f_{miv_i}(\boldsymbol{\theta})(i = 1, \ldots, n_i)$ are linearly independent, hence the rank of the matrix

$$\boldsymbol{F}_i = (\boldsymbol{f}_i^1 \vdots \boldsymbol{f}_i^2 \vdots \cdots \vdots \boldsymbol{f}_i^q) \in \mathsf{R}^{n_i \times q}, \quad n_i < q \tag{4.30}$$

is n_i, which is the number of the terms of i-th order leading principal minor. Then the subvector space spanned by $\{\boldsymbol{f}_i^1 \vdots \boldsymbol{f}_i^2 \vdots \cdots \vdots \boldsymbol{f}_i^q\}$ is n_i-dimensional. Let Ω_i denote the set consisting of all indices of $\boldsymbol{f}_i^j$ $(1 \leq j \leq q)$:

$$\Omega = \{1,\ 2,\ \cdots,\ q\} \tag{4.31}$$

and φ_i a subset of Ω_i that satisfies the followings:

1. $\{\boldsymbol{f}_i^j \mid j \in \varphi_i\}$ spans a $(n_i - 1)$-dimensional subspace that will be denoted by $\boldsymbol{W}_{\varphi_i}$;
2. There is 1-dimensional vector $\boldsymbol{y}_{\varphi_i}$ in the orthogonal complement of $\boldsymbol{W}_{\varphi_i}$ $(= \boldsymbol{W}_{\varphi_i}^{\perp})$ such that $\boldsymbol{y}_{\varphi_i} \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$ and non-zero.

Let $\mathcal{B}_i$ denote the family consisting of all the set φ_i. $\mathcal{B}_i$ is a finite set that has at most 2^q elements.

We now proceed to a important theorem:

Theorem 4.3.2 The set D_i of the solutions of the system

$$\boldsymbol{F}_i^t \boldsymbol{p}_i \geq \mathbf{o} \tag{4.32}$$

is equal to the polyhedral vertex-convex cone

$$\{\boldsymbol{y}_{\varphi_i} \mid \varphi_i \in \mathcal{B}_i\}^{\angle}. \tag{4.33}$$

Proof We begin with the following lemma.

Lemma 4.3.2 $\mathsf{D}_i = \{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^*$ is equal to the dual cone of $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle}$, that is

$$\mathsf{D}_i = (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*. \tag{4.34}$$

Proof Each $\boldsymbol{y} \in \{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle}$ is described as $\boldsymbol{y} = \sum_{j=1}^{q} \boldsymbol{f}_i^j \lambda_j,\ \lambda_j \geq 0$. Then, $\boldsymbol{y} \cdot \boldsymbol{x} \geq 0$ holds for each $\boldsymbol{x} \in \mathsf{D}_i$. Hence, if $\boldsymbol{x} \in \mathsf{D}_i$, then $\boldsymbol{x} \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$, thereby $\mathsf{D}_i \subset (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$.

Conversely, if $\boldsymbol{x} \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$, then $\boldsymbol{y} \cdot \boldsymbol{x} \geq 0$ holds for each $\boldsymbol{y} \in \{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle}$. Hence, $\boldsymbol{f}_i^j \cdot \boldsymbol{x} \geq 0$ also holds for each $j = 1, \cdots, q$. Therefore if $\boldsymbol{x} \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$, then $\boldsymbol{x} \in \mathsf{D}_i$, thereby, $\mathsf{D}_i \supset (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$. Consequently, $\mathsf{D}_i = (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle})^*$. ∎

Therefore, it is sufficient for us to obtain the dual cone of polyhedral convex cone $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}^{\angle}$.

According to the following theorem:

Minkowsky-Farkas' Theorem[50]: The dual cone of polyhedral convex cone is polyhedral convex cone.

We can see that D_i is a polyhedral convex cone, and by **Weyl's Theorem**[50], it is shown that $\{\boldsymbol{y}_{\varphi_i} \mid \varphi_i \in \mathcal{B}_i\}$ span D_i.

Finally, we will show that D_i is polyhedral vertex-convex cone. We begin with following lemma.

Lemma 4.3.3 The dual cone of $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots, \boldsymbol{f}_i^q\}$ is polyhedral vertex-convex cone if and only if the rank of the matrix

$$\boldsymbol{F}_i = (\boldsymbol{f}_i^1 \vdots \boldsymbol{f}_i^2 \vdots \cdots \vdots \boldsymbol{f}_i^q) \in \mathsf{R}^{n_i \times q}, \quad n_i < q \tag{4.35}$$

is equal to n_i.

Proof We assume that the subspace spanned by $\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q \in \mathsf{R}^{n_i}$ is r-dimensional and $r < n_i$, and denote that subspace by U and the orthogonal complement of U by $\mathsf{U}^{\perp}$. Then each $\boldsymbol{y} \in \mathsf{R}^{n_i}$ can be described as $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{v}$ where $\boldsymbol{u} \in \mathsf{U}$ and $\boldsymbol{v} \in \mathsf{U}^{\perp}$. If $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{v}$ is a element of the dual cone of $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}$, then

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{f}_i^j \geq 0 \quad \text{for} \quad j = 1, \cdots, q. \tag{4.36}$$

That is

$$\boldsymbol{u} \cdot \boldsymbol{f}_i^j \geq 0 \quad \text{for} \quad j = 1, \cdots, q \tag{4.37}$$

because

$$\boldsymbol{v} \in \mathsf{U}^{\perp}. \tag{4.38}$$

Hence, if $(\boldsymbol{u} + \boldsymbol{v}) \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^*$, then $\boldsymbol{u} \in (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^*$. This implies

$$\begin{aligned} &(\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^* \\ &= (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^* \cap \mathsf{U} + \mathsf{U}^{\perp}. \end{aligned} \tag{4.39}$$

If the dual cone $(\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^*$ has a subspace of dimension 1, then the dual cone has 2 elements $\boldsymbol{b}$ and $-\boldsymbol{b}$ among the elements spanning the dual cone, where $\boldsymbol{b}$ is a basis of the 1-dimensional subspace. Then $\boldsymbol{b}$ satisfies

$$\boldsymbol{b} \cdot \boldsymbol{f}_i^j \geq 0 \text{ and } -\boldsymbol{b} \cdot \boldsymbol{f}_i^j \geq 0 \quad \text{for } j = 1, \cdots, q. \tag{4.40}$$

Hence, $\boldsymbol{b} \cdot \boldsymbol{f}_i^j = 0$ holds for $j = 1, \cdots, q$. This implies that the 1-dimensional subspace is involved in $\mathsf{U}^{\perp}$. If the dual cone has a subspace of dimension t and $1 < t < r$, the subspace will be shown to be involved in $\mathsf{U}^{\perp}$ in a similar way.

We defined that polyhedral vertex-convex cone does not contain a subspace of dimension t $(t \geq 1)$. Then the dual cone is polyhedral vertex-convex cone if and only if $\mathsf{U}^{\perp}$ is empty, and $\mathsf{U}^{\perp}$ is empty if and only if the rank of $\boldsymbol{F}_i$ is n_i. ∎

The rank of $\boldsymbol{F}_i$ is n_i as we mentioned above, hence by **Lemma 4.3.3** the dual cone is polyhedral vertex-convex cone. ∎

Though we can obtain the set of the solutions of the system $\boldsymbol{F}_i^t\boldsymbol{p}_i \geq \mathbf{o}$ by finite calculations from **Theorem 4.3.2**, we must examine the set because what we should obtain is the set of the solutions of the system $\boldsymbol{F}_i^t\boldsymbol{p}_i > \mathbf{o}$, not $\boldsymbol{F}_i^t\boldsymbol{p}_i \geq \mathbf{o}$. We will examine the set below.

The dual cone of $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}$ is the intersection of finitely many halfspaces

$$\bigcap_{j=1}^{q}\{\boldsymbol{x} \in \mathsf{R}^{n_i} \mid \boldsymbol{f}_i^j \cdot \boldsymbol{x} \geq 0\} \tag{4.41}$$

whose boundary hyperplanes are

$$\{\boldsymbol{x} \in \mathsf{R}^{n_i} \mid \boldsymbol{f}_i^j \cdot \boldsymbol{x} = 0\} \quad (1 \leq j \leq q). \tag{4.42}$$

Then let Q_i^j denote the subset of R^{n_i} such that

$$\{\boldsymbol{x} \in \mathsf{R}^{n_i} \mid \boldsymbol{f}_i^j \cdot \boldsymbol{x} > 0\}, \tag{4.43}$$

and S_i^j the hyperplane of R^{n_i} such that

$$\{\boldsymbol{x} \in \mathsf{R}^{n_i} \mid \boldsymbol{f}_i^j \cdot \boldsymbol{x} = 0\}. \tag{4.44}$$

Because of the reason just mentioned above, we can describe the dual cone as

$$\mathsf{D}_i = \bigcap_{j=1}^{q}(\mathsf{Q}_i^j \bigcup \mathsf{S}_i^j), \tag{4.45}$$

hence, by short calculation, we can obtain

$$\mathsf{D}_i = \{\bigcap_{j=1}^{q}\mathsf{Q}_i^j\}\bigcup_{k=2}^{q}[\{\bigcap_{S=1}^{k-1}\mathsf{Q}_i^s\}\bigcap\{\mathsf{S}_i^k \bigcap \mathsf{D}_i\}]\bigcup\{\mathsf{S}_i^1 \bigcap \mathsf{D}_i\}. \tag{4.46}$$

Therefore, we can see that the dual cone D_i is constituted of the interior $\{\bigcap_{j=1}^{q} \mathsf{Q}_i^j\}$, which is the subset of the solutions of the system $\boldsymbol{F}_i^t \boldsymbol{p}_i > \mathbf{o}$, and boundary that is union of such subset of hyperplanes that defined in (4.43). We can easily see that the interior is convex set.

From the examination above, we also see that the hyperplanes whose subset constitute the boundary play very important role when we make use of the convex cone. Hence, it would be worth while to obtain such hyperplanes.

We use the following theorem:

Farkas' Theorem[50]: If A is a finite set of vectors, then $\mathsf{A}^{**} = \mathsf{A}^{\angle}$.

Applying **Farkas' Theorem** to $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}$, which is a finite set, and taking **Lemma 4.3.2** into account, we obtain

$$\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle} = (\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle})^{**}. \tag{4.47}$$

By **Theorem 4.3.2**, the dual cone of $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle}$ is polyhedral vertex-convex cone that is described as $\{\boldsymbol{y}_{\varphi_i} \mid \varphi_i \in \mathcal{B}_i\}^{\angle}$. We will apply **Theorem 4.3.2** again. For the purpose, we number the elements of $\{\boldsymbol{y}_{\varphi_i} \mid \varphi_i \in \mathcal{B}_i\}^{\angle}$, and express them as $\{\boldsymbol{y}_i^1,\ \boldsymbol{y}_i^2,\ \cdots, \boldsymbol{y}_i^s\}$ $(s \leq 2^q)$. Let $\Omega_{\boldsymbol{y}_i}$ denote the set of all indices of $\boldsymbol{y}_i^j$, then $\Omega_{\boldsymbol{y}_i} = \{1,\ 2,\ \cdots\ s\}$, and ξ_i a subset of $\Omega_{\boldsymbol{y}_i}$ that satisfies the followings:

1. $\{\boldsymbol{y}_i^j \mid j \in \xi_i\}$ spans a $(n_i - 1)$-dimensional subspace that will be denoted by $\boldsymbol{W}_{\xi_i}$;

2. There is 1-dimensional vector $\boldsymbol{v}_{\xi_i}$ in the orthogonal complement of $\boldsymbol{W}_{\xi_i}$ $(= \boldsymbol{W}_{\xi_i}^{\perp})$ such that $\boldsymbol{v}_{\xi_i} \in (\{\boldsymbol{y}_i^1,\ \boldsymbol{y}_i^2,\ \cdots, \boldsymbol{y}_i^s\}^{\angle})^*$ and non-zero.

Let $\mathcal{C}_i$ denote the family consisting of all the set ξ_i. From **Theorem 4.3.2**, we obtain

$$\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle} = \{\boldsymbol{v}_{\xi_i} \mid \xi_i \in \mathcal{C}_i\}^{\angle}, \tag{4.48}$$

hence, we can conclude that each element spanning the convex cone $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle}$ corresponds to one of the elements of $\{\boldsymbol{v}_{\xi_i} \mid \xi_i \in \mathcal{C}_i\}$ and they would coincide if they were normalized. The hyperplanes whose subsets constitute the boundary are the set $\{\boldsymbol{W}_{\xi_i} \mid \xi_i \in \mathcal{C}_i\}$ because each of them is orthogonal complement of the element which spans the convex cone $\{\boldsymbol{f}_i^1,\ \boldsymbol{f}_i^2,\ \cdots \boldsymbol{f}_i^q\}^{\angle}$.

We obtained N polyhedral vertex-convex cones and the vectors $\boldsymbol{p}_i = [p_{mi1}\ \ p_{mi2}\ \ \cdots\ \ p_{min_i}]^t$ for $i = 1,\ 2,\ \cdots\ ,N$, each of the vectors corresponds to the cone. Each polyhedral vertex-convex cone can be obtained by finite calculations. Here, we state the main result of this section:

A set of base-parameter values is "possible" if and only if each $\boldsymbol{p}_i$ *that is determined by the set of base-parameter values exists in the domain that is determined from* $\boldsymbol{f}_i$*, and the domain is approximately the interior of the polyhedral vertex-convex cone corresponding to* $\boldsymbol{F}_i$.

4.4 Modification of Estimated Base-Parameter Values

After estimating a set of base-parameter values, we will be able to determine $\boldsymbol{p}_i$ for $i = 1, 2, \cdots, N$ as the constant vectors whose entries are real. Each constant vector $\boldsymbol{p}_i$ must be a element of the subset that is determined by $\boldsymbol{f}_i$, otherwise the inertial matrix would not always be positive definite, that would be physically impossible. Nevertheless, we may estimate such a set of base-parameter values that determine $\boldsymbol{p}_i$ which is outside of the cone. If we use such a set of base-parameter values, we can not ensure good control performance of the manipulator. Hence, after estimating a set of base-parameter values we should judge if it is "possible" or not, and modify the estimated base-parameter values for the set of them to be at least "possible" if it is judged not to be. Then in this chapter we propose one method to do it using the convex cone on computers.

4.4.1 Judgement

Let $\hat{\boldsymbol{p}}_i$ for $i = 1, 2, \cdots, N$ denote the constant vector whose entries are determined by a set of estimated base-parameter values. A simple method to judge if it is "possible" or not is to examine each $\hat{\boldsymbol{p}}_i$ if

$$\boldsymbol{v}_{\xi_i} \cdot \hat{\boldsymbol{p}}_i > 0 \tag{4.49}$$

holds for each $\xi_i \in \mathcal{C}_i$ and for $i = 1, 2, \cdots, N$.

4.4.2 A Method to Modify

If the set of estimated base-parameter values is judged not to be "possible" , there is at least one $\hat{\boldsymbol{p}_i}$ among $\hat{\boldsymbol{p}_1}$, $\hat{\boldsymbol{p}_2}$, $\cdots, \hat{\boldsymbol{p}_N}$ that does not hold $\boldsymbol{v}_{\xi_i} \cdot \hat{\boldsymbol{p}_i} > 0$ for each $\xi_i \in \mathcal{C}_i$. Then let Λ denote the set consisting of all indices: $1,\ 2,\ \cdots, N$, and Γ the subset of Λ which satisfy that for $j \in \Gamma, \boldsymbol{v}_{\xi_j} \cdot \hat{\boldsymbol{p}_j} \leq 0$ holds for some $\xi_j \in \mathcal{C}_j$. Let $\mathcal{E}_j$ denote the subset of $\mathcal{C}_j, (j \in \Gamma)$ which satisfy that for $\xi_j \in \mathcal{E}_j$, $\boldsymbol{v}_{\xi_j} \cdot \hat{\boldsymbol{p}_j} \leq 0$ holds.

We propose a method to modify the estimated base-parameter values taking the following steps.

Step1: We find the nearest point on the boundary of convex cone corresponding to $\boldsymbol{p}_i$ for $i \in \Gamma$ from the point $\hat{\boldsymbol{p}_i}$. We use "nearest" in the sense of Euclidean distance. The nearest point would be easily found because we obtained hyperplanes $\boldsymbol{W}_{\xi_i}$ $(\xi_i \in \mathcal{C}_j)$ whose subset constitute the boundary of convex cone, and it would be sufficient to examine the only hyperplanes $\boldsymbol{W}_{\xi_i}$ for $\xi_i \in \mathcal{E}_i$ because $\hat{\boldsymbol{p}_i}$ is near them outside the convex cone.

Let $\tilde{\boldsymbol{p}_i}$ $(i \in \Gamma)$ denote the nearest point on the boundary of convex cone.

Step2: We obtain a line through the point $\hat{\boldsymbol{p}_i}$ and $\tilde{\boldsymbol{p}_i}$ for $i \in \Gamma$, and take another point $\check{\boldsymbol{p}_i}$ in the interior of the convex cone and on the line.

Step3: We make a hypercube in the interior, taking its center on $\check{\boldsymbol{p}_i}$ and the length of edges as $2\alpha_i$ for each $i \in \Gamma$. In the same way we make a hypercube, taking its center on $\hat{\boldsymbol{p}_i}$ for each $i \notin \Gamma$. Thereby we obtain a inequality for each element of $\boldsymbol{p}_i$ for each $i \in \Lambda$. It is described as

$$\check{p}_{miv_i} - \alpha_i \leq p_{miv_i} \leq \check{p}_{miv_i} + \alpha_i \tag{4.50}$$

for each $i \in \Gamma$,

$$\tilde{p}_{miv_i} - \alpha_i \leq p_{miv_i} \leq \tilde{p}_{miv_i} + \alpha_i \tag{4.51}$$

for each $i \notin \Gamma$, for $1 \leq v_i \leq n_i$.

Step4: We search the system of inequalities in **Step3** for a set of base-parameter values. After searching, if we fail to find the solution, we take α_i as $\alpha_i + \varepsilon_i$, ε_i is appropriate positive value, and go back to **Step3**.

If we make the domain in each convex cone, for which a solution is searched, enlarge so as to gradually cover the interior of the convex cone, we will surely be able to obtain a solution. Because each $\acute{\boldsymbol{p}}_i$ that is determined by the set of true base-parameter values is in the interior for each $1 \leq i \leq N$, and p_{miv_i} are continuous on the continuous changes of the base-parameter values as mentioned in section 3, there is a domain in each convex cone that is the neighborhood of $\acute{\boldsymbol{p}}_i$.

It is needed to find a good algorithm for **Step4** to search for a solution, but it should be done after the structure of the manipulator is fixed because the algorithm needs the information about the explicit forms of p_{miv_i}.

4.5 Conclusion

We proposed a method to judge if a set of base-parameter values determines the inertial matrix to be positive definite for each configuration of the manipulator or not, when we approximately consider the continuous change of each joint variable of the manipulator as a finite set of discrete points. The method can be executed on computers. Using this method we can judge if a set of estimated base-parameter values is "possible" or not. If a set of base-parameter values is not "possible", it is physically impossible. Also we proposed one method to modify the estimated base-parameter values for the set of them to be at least "possible" if we judge it is not.

Though we used approximation in obtaining the subset of the solutions of each inequality in (4.8), we could make the accuracy of approximation high enough for the purpose of judgement and modification of the set of estimated base-parameter values. Obtaining a good algorithm in **Step4** in chapter 4 would be very important.

Even if a set of base-parameter values is "possible", it is not always physically possible, since being "possible" of a set of base-parameter values only ensures the inertial matrix to be positive definite for each configuration of the manipulator. Then it would be important to make it clear which set of base-parameter values is physically possible. The study in this chapter is the first step for the goal. The other properties that characterize the dynamic equations for manipulators should be taken into consideration.

Chapter 5

Concluding Remarks

Dynamics of robot manipulators has been discussed in this dissertation, focusing on inertial parameters of kinematic chains of the robot manipulators and identification of them for dynamic modeling. For the model-based control of a robot manipulator, it is very crucial to obtain an accurate dynamic model of the manipulator. The dynamic model of the manipulator consisting of rigid links is described as a set of nonlinear differential equations involving various constant parameters: kinematic parameters, link inertial parameters of its kinematic chain, and dynamic parameters of driving systems. If all the values of these parameters are known, we can determine the dynamic model. Hence, accurate values of the parameters are required to obtain an accurate dynamic model. The values of the kinematic parameters can be obtained from design data or by kinematic calibration. The most practical way to obtain the values of the link inertial parameters and driving system parameters is to make test motions of the manipulator and to estimate them from the input data and joint motion data which are taken while the manipulator is in the test motions.

However, unfortunately, it is impossible to estimate all the link inertial parameter values from the input data and the joint motion data in general since they are redundant to determine the dynamic model uniquely. Hence in **Chapter 2**, we have investigated a base parameter set which is defined to be a minimum set of inertial parameters whose values can determine the dynamic model uniquely for each of three types of manipulators. The investigation of a base parameter set would give us many insights into the structure of the dynamic equations.

The base parameters are also the parameters that can be identified independently from input data and joint motion data. We have described each element of the base parameter set in a linear combination of the link inertial parameters directly and completely in closed form, also we have given the exact number of the base parameters.

Next, it would be very important to have a good identification method to obtain the values of the base parameters for modeling. Then, in **Chapter 3**, we have experimentally examined to estimate the base parameters for an industrial manipulator applying the identification methods: step-by-step method, simultaneous method, and advanced simultaneous method. We have compared the methods about the accuracy of estimates. To evaluate the accuracy of them, we have simulated the manipulator motion using the estimates and compared the simulated trajectories with measured trajectories. We have also described in detail the contents of the work which is needed to obtain the estimates about each identification method, and compared them about the amount of labour and consuming time on a computer.

If we could obtain the true values of the parameters, no problem would happen. However we are forced to have the estimates biased more or less, and determine the dynamic models using them. Thereby it may happen that the inertial matrix of the dynamic model is not always positive definite for each configuration of the manipulator, though it is the fact that the inertial matrix is positive definite for each configuration of the manipulator. If a set of estimated base-parameter values determines such inertial matrix, it is physically impossible. Hence, in **Chapter 4** we have proposed a method to judge if a set of base-parameter values determines the inertial matrix to be positive definite for each configuration of the manipulator or not, when we approximately consider the continuous change of each joint variable of the manipulator as a finite set of discrete points. The method can be executed on computers. Using this method we can judge if a set of estimated base-parameter values is "possible" or not. We have also proposed one method to modify the estimated base-parameter values for the set of them to be at least "possible" if we judge it is not. Even if a set of base-parameter values is "possible", it is not always physically possible, since being "possible" of a set of base-parameter values only ensures the inertial matrix to be positive definite for each configuration of the manipulator. Hence the study in **Chapter 4** is the first step in the context of making it clear which set of base-parameter values is physically possible. It would be important to make it clear which set of base-parameter values is physically possible.

The results in this dissertation would have direct contribution to the identification problem of the inertial parameters for robot manipulators. Moreover, through the detailed examination of the dynamic equations we have had a fact that some link inertial parameters appear in the form of linear combinations in dynamic equations. Also we have noticed that some sets of base-parameter values for the dynamic model are physically impossible. Some other features of the dynamics of the robot manipulators have been found to be quite important by several researchers. Those would help us to better understanding of the dynamics of the robot manipulators.

Appendix A

Proof of Lemma 2.3.3, 2.3.4, 2.3.5, and 2.3.6

Since inner product of two vectors can be executed about any coordinate system, $H(i,j)$ in (2.17) can be evaluated for $1 \leq j \leq i \leq N$ as follows:

$$
\begin{aligned}
H(i,j) &= \sum_{S=i}^{N} {}^{S}\boldsymbol{z}_i^{t}\,{}^{S}\boldsymbol{J}_S\,{}^{S}\boldsymbol{z}_j \\
&+ \sum_{S=i}^{N-1} {}^{S}\boldsymbol{z}_i^{t}[2({}^{S}\boldsymbol{L}_S^{t}\,{}^{S}\boldsymbol{SR}_{S+1})\boldsymbol{E} - {}^{S}\boldsymbol{L}_S\,{}^{S}\boldsymbol{SR}_{S+1}^{t} - {}^{S}\boldsymbol{SR}_{S+1}\,{}^{S}\boldsymbol{L}_S^{t}]\ {}^{S}\boldsymbol{z}_j \qquad \text{(A.1)} \\
&+ {}^{i}\boldsymbol{z}_i^{t}[({}^{i}\boldsymbol{L}_{j,i}^{t}\,{}^{i}\boldsymbol{SR}_i)\boldsymbol{E} - {}^{i}\boldsymbol{L}_{j,i}\,{}^{i}\boldsymbol{SR}_i^{t}]\ {}^{i}\boldsymbol{z}_j.
\end{aligned}
$$

Since using (2.13) and **Lemma 2.2**, we can derive that

$$
{}^{S}\boldsymbol{SR}_{S+1} = {}^{S}\boldsymbol{SR}_S - {}^{S}\boldsymbol{R}_S = \begin{bmatrix} 0 & -\mathbf{RZB}(s) & \mathbf{RZ}(s) \end{bmatrix}^t + \boldsymbol{GA}_{1_s} \qquad \text{(A.2)}
$$

where $\boldsymbol{GA}_{1_s}$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq s+1$. We obtain by direct calculations that

$$
\begin{aligned}
&2({}^{S}\boldsymbol{L}_S^{t}\,{}^{S}\boldsymbol{SR}_{S+1})\boldsymbol{E} - {}^{S}\boldsymbol{L}_S\,{}^{S}\boldsymbol{SR}_{S+1}^{t} - {}^{S}\boldsymbol{SR}_{S+1}\,{}^{S}\boldsymbol{L}_S^{t} \\
&= \begin{bmatrix} 2[L]_S^z\mathbf{RZ}(s) & [L]_S^x\mathbf{RZB}(s) & -[L]_S^x\mathbf{RZ}(s) \\ 1[L]_S^x\mathbf{RZB}(s) & 2[L]_S^z\mathbf{RZ}(s) & [L]_S^z\mathbf{RZB}(s) \\ -[L]_S^x\mathbf{RZ}(s) & [L]_S^z\mathbf{RZB}(s) & 0 \end{bmatrix} + D_S
\end{aligned} \qquad \text{(A.3)}
$$

where $\boldsymbol{D}_S = 2({}^S\boldsymbol{L}_S^t \boldsymbol{G}\boldsymbol{A}_{1s})\boldsymbol{E} - {}^S\boldsymbol{L}_S \boldsymbol{G}\boldsymbol{A}_{1s}^t - \boldsymbol{G}\boldsymbol{A}_{1s}{}^S\boldsymbol{L}_S^t$. Denoting ${}^i\boldsymbol{L}_{j,i}$ by

$$
{}^i\boldsymbol{L}_{j,i} = \left[\ [{}^iL_{j,i}]^x \quad [{}^iL_{j,i}]^y \quad [{}^iL_{j,i}]^z\ \right]^t, \tag{A.4}
$$

it can be derived by **Lemma 2.3.2** and ${}^i\boldsymbol{z}_i = \boldsymbol{e}_3$ that

$$
\begin{aligned}
&{}^i\boldsymbol{z}_i^t[({}^i\boldsymbol{L}_{j,i}^t\,{}^i\boldsymbol{SR}_i)\boldsymbol{E} - {}^i\boldsymbol{L}_{j,i}\,{}^i\boldsymbol{SR}_i^t] \\
&= \begin{bmatrix} -\mathbf{R}_i^x[{}^iL_{j,i}]^z \\ -(\mathbf{R}_i^y - \mathbf{RZB}(i))[{}^iL_{j,i}]^z \\ \mathbf{R}_i^x[{}^iL_{j,i}]^x + (\mathbf{R}_i^y - \mathbf{RZB}(i))[{}^iL_{j,i}]^y \end{bmatrix}^t + \boldsymbol{G}\boldsymbol{A}_{2s}^t
\end{aligned} \tag{A.5}
$$

where $\boldsymbol{G}\boldsymbol{A}_{2s}$ is a vector whose entries are functions of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i+1$. Define

$$
\begin{aligned}
&\overline{\mathbf{J}}_S^x = \mathbf{J}_S^x + 2[L]_S^z\mathbf{RZ}(s), \qquad \overline{\mathbf{J}}_S^y = \mathbf{J}_S^y + 2[L]_S^z\mathbf{RZ}(s) \\
&\overline{\mathbf{J}}_S^{xy} = \mathbf{J}_S^{xy} + [L]_S^x\mathbf{RZB}(s), \quad \overline{\mathbf{J}}_S^{yz} = \mathbf{J}_S^{yz} + [L]_S^z\mathbf{RZB}(s) \\
&\overline{\mathbf{J}}_S^{xz} = \mathbf{J}_S^{xz} - [L]_S^x\mathbf{RZ}(s)
\end{aligned} \tag{A.6}
$$

and

$$
{}^S\overline{\boldsymbol{J}}_S = \begin{bmatrix} \overline{\mathbf{J}}_S^x & \overline{\mathbf{J}}_S^{xy} & \overline{\mathbf{J}}_S^{xz} \\ \overline{\mathbf{J}}_S^{xy} & \overline{\mathbf{J}}_S^y & \overline{\mathbf{J}}_S^{yz} \\ \overline{\mathbf{J}}_S^{xz} & \overline{\mathbf{J}}_S^{yz} & \mathbf{J}_S^z \end{bmatrix}. \tag{A.7}
$$

Then substituting (A.3)-(A.7), ${}^S\boldsymbol{z}_i = ({}^i\boldsymbol{A}_S)^t\boldsymbol{e}_3$, ${}^S\boldsymbol{z}_j = ({}^j\boldsymbol{A}_S)^t\boldsymbol{e}_3$, and ${}^i\boldsymbol{z}_j = {}^i\boldsymbol{A}_j\boldsymbol{e}_3$ for (A.1), we obtain

$$
\begin{aligned}
H(i,j) &= \sum_{S=i}^{N} \boldsymbol{e}_3^t\,{}^i\boldsymbol{A}_S{}^S\overline{\boldsymbol{J}}_S({}^j\boldsymbol{A}_S)^t\,\boldsymbol{e}_3 \\
&+ \begin{bmatrix} -\mathbf{R}_i^x[{}^iL_{j,i}]^z \\ -(\mathbf{R}_i^y - \mathbf{RZB}(i))[{}^iL_{j,i}]^z \\ \mathbf{R}_i^x[{}^iL_{j,i}]^x + (\mathbf{R}_i^y - \mathbf{RZB}(i))[{}^iL_{j,i}]^y \end{bmatrix}^t {}^i\boldsymbol{A}_j\boldsymbol{e}_3 + GA_3
\end{aligned} \tag{A.8}
$$

where GA_3 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i+1$.

Since $\boldsymbol{L}_{i,\,i} = 0$, it is obvious from (A.8) that

$$H(i,i) = \sum_{S=i}^{N} \boldsymbol{e}_3^t\, {}^i\boldsymbol{A}_S{}^S\overline{\boldsymbol{J}}_S({}^i\boldsymbol{A}_S)^t\, \boldsymbol{e}_3 \ + \ GA_4 \tag{A.9}$$

where GA_4 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i+1$. $({}^i\boldsymbol{A}_S)^t\boldsymbol{e}_3 = \left[\ ({}^iA_S)_{31} \quad ({}^iA_S)_{32} \quad ({}^iA_S)_{33}\ \right]^t$ and $({}^iA_S)_{31} = 0,\ ({}^iA_S)_{32} = 0$ and $({}^iA_S)_{33} = 1$ for $i \leq s \leq \beta_{c(i)}$ by **Property 2.3.1**. Thus, we can derive by direct calculations that

$$\begin{aligned}
H(i,i) \ &= \sum_{S=i}^{\beta_{c(i)}} \mathbf{J}_S^z + \sum_{d=c(i)+1}^{K} \ \sum_{S=\alpha_d}^{\beta_d} (\overline{\mathbf{J}}_S^x({}^iA_S)_{31}^2 + \overline{\mathbf{J}}_S^y({}^iA_S)_{32}^2 + \mathbf{J}_S^z({}^iA_S)_{33}^2) \\
&+2 \sum_{d=c(i)+1}^{K} \ \sum_{S=\alpha_d}^{\beta_d} (\overline{\mathbf{J}}_S^{xy}({}^iA_S)_{31}({}^iA_S)_{32} + \overline{\mathbf{J}}_S^{xz}({}^iA_S)_{31}({}^iA_S)_{33} \\
&+\overline{\mathbf{J}}_S^{yz}({}^iA_S)_{32}({}^iA_S)_{33}) \\
&+ \ GA_4.
\end{aligned} \tag{A.10}$$

Since $({}^iA_S)_{32}^2 = 1 - ({}^iA_S)_{31}^2 - ({}^iA_S)_{33}^2$ (because ${}^i\boldsymbol{A}_S$ is a rotation matrix) and $({}^iA_S)_{33}^2 = ({}^iA_{\beta_{c(s)-1}})_{32}^2$ for $c(i) < c(s)$ by **Property 2.3.1**, $({}^iA_S)_{32}^2 = -({}^iA_S)_{31}^2 + (1 - ({}^iA_{\beta_{c(s)-1}})_{32}^2) = -({}^iA_S)_{31}^2 + ({}^iA_{\beta_{c(s)-1}})_{31}^2 + ({}^iA_{\beta_{c(s)-1}})_{33}^2$. Using this relation, we can derive that

$$\begin{aligned}
\sum_{d=c(i)+1}^{K} \ \sum_{S=\alpha_d}^{\beta_d} \overline{\mathbf{J}}_S^y({}^iA_S)_{32}^2 \ &= - \sum_{d=c(i)+1}^{K} \ \sum_{S=\alpha_d}^{\beta_d} \overline{\mathbf{J}}_S^y({}^iA_S)_{31}^2 \\
&+(\sum_{S=\alpha_{c(i)+1}}^{\beta_{c(i)+1}} \overline{\mathbf{J}}_S^y)(({}^iA_{\beta_{c(i)}})_{31}^2 + ({}^iA_{\beta_{c(i)}})_{33}^2) \\
&+ \sum_{d=c(i)+1}^{K-1} (\sum_{S=\alpha_{d+1}}^{\beta_{d+1}} \overline{\mathbf{J}}_S^y)(({}^iA_{\beta_d})_{31}^2 + ({}^iA_{\beta_d})_{33}^2).
\end{aligned} \tag{A.11}$$

Since $({}^iA_{\beta_{c(i)}})_{31} = 0$ and $({}^iA_{\beta_{c(i)}})_{33} = 1$ by **Property 2.3.1**, we obtain from (A.11), (A.6), and (2.24) that

$$\sum_{S=i}^{\beta_{c(i)}} \mathbf{J}_S^z + \sum_{d=c(i)+1}^{K} \sum_{S=\alpha_d}^{\beta_d} (\overline{\mathbf{J}}_S^x ({}^i A_S)_{31}^2 + \overline{\mathbf{J}}_S^y ({}^i A_S)_{32}^2 + \mathbf{J}_S^z ({}^i A_S)_{33}^2)$$

$$= \sum_{S=i}^{\beta_{c(i)}} (\mathbf{J}_S^z + \mathbf{JYB}(s)) \tag{A.12}$$

$$+ \sum_{d=c(i)+1}^{K} \sum_{S=\alpha_d}^{\beta_d} [(\overline{\mathbf{J}}_S^x - \overline{\mathbf{J}}_S^y + \mathbf{JYB}(s))({}^i A_S)_{31}^2 + (\mathbf{J}_S^z + \mathbf{JYB}(s))({}^i A_S)_{33}^2]$$

By (A.10), (A.12) and the fact that $\overline{\mathbf{J}}_S^x - \overline{\mathbf{J}}_S^y = \mathbf{J}_S^x - \mathbf{J}_S^y$ from (A.6), **Lemma 2.3.3** is proved directly.

It is obvious from (A.8) that

$$H(i,j) = \sum_{S=i}^{N} \boldsymbol{e}_3^t\, {}^i\boldsymbol{A}_S {}^S\overline{\boldsymbol{J}}_S ({}^j\boldsymbol{A}_S)^t\, \boldsymbol{e}_3 \;+\; GA_5 \tag{A.13}$$

for $i > j$ where GA_5 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i$. We can obtain by direct calculations that

$$\begin{aligned} \boldsymbol{e}_3^t\, {}^i\boldsymbol{A}_S {}^S\overline{\boldsymbol{J}}_S ({}^j\boldsymbol{A}_S)^t\, \boldsymbol{e}_3 &= \overline{\mathbf{J}}_S^{xy}(({}^i A_S)_{32}({}^j A_S)_{31} + ({}^i A_S)_{31}({}^j A_S)_{32}) \\ &+ \overline{\mathbf{J}}_S^{xz}(({}^i A_S)_{33}({}^j A_S)_{31} + ({}^i A_S)_{31}({}^j A_S)_{33}) \\ &+ \overline{\mathbf{J}}_S^{yz}(({}^i A_S)_{33}({}^j A_S)_{32} + ({}^i A_S)_{32}({}^j A_S)_{33}) \\ &+ \overline{\mathbf{J}}_S^x ({}^i A_S)_{31}({}^j A_S)_{31} + \overline{\mathbf{J}}_S^y ({}^i A_S)_{32}({}^j A_S)_{32} \\ &+ \mathbf{J}_S^z ({}^i A_S)_{33}({}^j A_S)_{33}. \end{aligned} \tag{A.14}$$

Here, let us consider $i > \alpha_2$ and $j = \beta_{c(i)-1}$. When $\beta_{c(i)-1} < i \leq s \leq \beta_{c(i)}$, it is obvious by **Property 2.3.1** that $({}^i A_S)_{31} = 0$, $({}^i A_S)_{32} = 0$, $({}^i A_S)_{33} = 1$, $({}^{\beta_{c(i)-1}} A_S)_{31} = \sin\theta(\alpha_{c(i)}, s)$, $({}^{\beta_{c(i)-1}} A_S)_{32} = \cos\theta(\alpha_{c(i)}, s)$ and $({}^{\beta_{c(i)-1}} A_S)_{33} = 0$. Using these and (A.14), we can obtain from (A.13) that

$$\begin{aligned}
H(i,\beta_{c(i)-1}) &= \sum_{S=i}^{\beta_{c(i)}} (\overline{\mathbf{J}}_S^{xz}\sin\theta(\alpha_{c(i)},s) + \overline{\mathbf{J}}_S^{yz}\cos\theta(\alpha_{c(i)},s)) \\
&+ \sum_{d=c(i)+1}^{K}\ \sum_{S=\alpha_d}^{\beta_d} [\overline{\mathbf{J}}_S^{x}({}^{i}A_S)_{31}({}^{\beta_{c(i)-1}}A_S)_{31} + \overline{\mathbf{J}}_S^{y}({}^{i}A_S)_{32}({}^{\beta_{c(i)-1}}A_S)_{32} \\
&+\mathbf{J}_S^{z}({}^{i}A_S)_{33}({}^{\beta_{c(i)-1}}A_S)_{33}] + \ GA_6
\end{aligned} \tag{A.15}$$

where GA_6 is a function of $\boldsymbol{\theta}$ generated $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i$ and $\overline{\mathbf{J}}_t^{xy}$, $\overline{\mathbf{J}}_t^{yz}$ and $\overline{\mathbf{J}}_t^{xz}$ for $t \geq \alpha_{c(i)+1}$. Since ${}^{\beta_{c(i)-1}}\boldsymbol{A}_S = {}^{\beta_{c(i)-1}}\boldsymbol{A}_i{}^{i}\boldsymbol{A}_S$, it is derived by **Property 2.3.1** that

$$({}^{\beta_{c(i)-1}}A_S)_{3w} = \sin\theta(\alpha_{c(i)},i)({}^{i}A_S)_{1w} + \cos\theta(\alpha_{c(i)},i)({}^{i}A_S)_{2w} \tag{A.16}$$

for $w = 1,2,3$. Therefore

$$\begin{aligned}
\sum_{w=1}^{3}({}^{i}A_S)_{3w}({}^{\beta_{c(i)-1}}A_S)_{3w} &= \sin\theta(\alpha_{c(i)},i)\sum_{w=1}^{3}({}^{i}A_S)_{3w}({}^{i}A_S)_{1w} \\
&+\cos\theta(\alpha_{c(i)},i)\sum_{w=1}^{3}({}^{i}A_S)_{3w}({}^{i}A_S)_{2w} \\
&= 0
\end{aligned} \tag{A.17}$$

since $\sum_{w=1}^{3}({}^{i}A_S)_{uw}({}^{i}A_S)_{vw} = 0$ if $u \neq v$ (because ${}^{i}\boldsymbol{A}_S$ is a rotation matrix). By (A.17) and **Property 2.3.1**, we derive

$$\begin{aligned}
({}^{i}A_S)_{32}({}^{\beta_{c(i)-1}}A_S)_{32} &= -({}^{i}A_S)_{31}({}^{\beta_{c(i)-1}}A_S)_{31} - ({}^{i}A_S)_{33}({}^{\beta_{c(i)-1}}A_S)_{33} \\
&= -({}^{i}A_S)_{31}({}^{\beta_{c(i)-1}}A_S)_{31} - ({}^{i}A_{\beta_{c(s)-1}})_{32}({}^{\beta_{c(i)-1}}A_{\beta_{c(s)-1}})_{32} \\
&= -({}^{i}A_S)_{31}({}^{\beta_{c(i)-1}}A_S)_{31} + ({}^{i}A_{\beta_{c(s)-1}})_{31}({}^{\beta_{c(i)-1}}A_{\beta_{c(s)-1}})_{31} \\
&+({}^{i}A_{\beta_{c(s)-1}})_{33}({}^{\beta_{c(i)-1}}A_{\beta_{c(s)-1}})_{33}
\end{aligned} \tag{A.18}$$

for $s \geq \alpha_{c(i)+1}$. Since $({}^{i}A_{\beta_{c(s)-1}})_{32} = 0$ when $c(i) = c(s) - 1$, using (A.18), we can obtain that

$$\sum_{d=c(i)+1}^{K}\sum_{S=\alpha_d}^{\beta_d}\overline{\mathbf{J}}^y_S({}^iA_S)_{32}({}^{\beta_{c(i)}-1}A_S)_{32} = -\sum_{d=c(i)+1}^{K}\sum_{S=\alpha_d}^{\beta_d}\overline{\mathbf{J}}^y_S({}^iA_S)_{31}({}^{\beta_{c(i)}-1}A_S)_{31}$$

$$+\sum_{d=c(i)+1}^{K-1}\Big(\sum_{S=\alpha_{d+1}}^{\beta_{d+1}}\overline{\mathbf{J}}^y_S\Big)\big(({}^iA_{\beta_d})_{31}({}^{\beta_{c(i)}-1}A_{\beta_d})_{31} \tag{A.19}$$

$$+({}^iA_{\beta_d})_{33}({}^{\beta_{c(i)}-1}A_{\beta_d})_{33}\big).$$

Hence

$$\sum_{d=c(i)+1}^{K}\sum_{S=\alpha_d}^{\beta_d}[\overline{\mathbf{J}}^x_S({}^iA_S)_{31}({}^{\beta_{c(i)}-1}A_S)_{31}+\overline{\mathbf{J}}^y_S({}^iA_S)_{32}({}^{\beta_{c(i)}-1}A_S)_{32}$$

$$+\mathbf{J}^z_S({}^iA_S)_{33}({}^{\beta_{c(i)}-1}A_S)_{33}]$$

$$= \sum_{d=c(i)+1}^{K}\sum_{S=\alpha_d}^{\beta_d}[(\overline{\mathbf{J}}^x_S-\overline{\mathbf{J}}^y_S+\mathbf{JYB}(s))({}^iA_S)_{31}({}^{\beta_{c(i)}-1}A_S)_{31} \tag{A.20}$$

$$+(\mathbf{J}^z_S+\mathbf{JYB}(s))({}^iA_S)_{33}({}^{\beta_{c(i)}-1}A_S)_{33}].$$

By (A.15) and (A.20), **Lemma 2.3.4** is obviously proved.

When $\alpha_{c(i)} \le j < i \le s$, ${}^i\boldsymbol{A}_j\boldsymbol{e}_3 = \boldsymbol{e}_3$ and $({}^j\boldsymbol{A}_S)^t\boldsymbol{e}_3 = ({}^i\boldsymbol{A}_S)^t\boldsymbol{e}_3$. Thus, we can derive from (A.8) and (A.10) that

$$H(i,j) = H(i,i) + \mathbf{R}^x_i[{}^iL_{j,i}]^x + (\mathbf{R}^y_i - \mathbf{RZB}(i))[{}^iL_{j,i}]^y \ + \ GA_7 \tag{A.21}$$

where GA_7 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}^x_t$ and $\mathbf{R}^y_t - \mathbf{RZB}(t)$ for $t \ge i+1$. By (2.15), we can describe ${}^i\boldsymbol{L}_{j,i}$ as ${}^i\boldsymbol{L}_{j,i} = \sum_{S=j}^{i-1}({}^S\boldsymbol{A}_i)^t\,{}^S\boldsymbol{L}_S$. When $\alpha_{c(i)} \le j \le s < i$, we obtain by (2.3) and **Property 2.3.1** that

$$[{}^iL_{j,i}]^x = \sum_{s=j}^{i-1}[L]^x_S\cos\theta(s+1,i), \quad [{}^iL_{j,i}]^y = -\sum_{s=j}^{i-1}[L]^x_S\sin\theta(s+1,i). \tag{A.22}$$

Lemma 2.3.5 is obviously proved by (A.21) and (A.22).

It is obvious from (2.34) in the proof of **Theorem 2.3.1** that all the $\boldsymbol{g}\cdot(\boldsymbol{z}_i \times \boldsymbol{SR}_i)$ for $1 \le i \le N$ are functions of θ generated by $\mathbf{R}^x_S$ and $\mathbf{R}^y_S - \mathbf{RZB}(s)$ for $1 \le s \le N$ if $\boldsymbol{z}_i$ is not parallel to $\boldsymbol{g}$ or for $\alpha_2 \leq s \leq N$ if $\boldsymbol{z}_1$ is parallel to $\boldsymbol{g}$ (in this case $\boldsymbol{g} \cdot (\boldsymbol{z}_i \times \boldsymbol{SR}_i) = 0$ for $1 \leq i \leq \beta_1$). From (A.8), $H(i,j)$ for $1 \leq j \leq i \leq N$ can be represented by

$$H(i,j) = \sum_{S=i}^{N} \boldsymbol{e}_3^{t\,i} \boldsymbol{A}_S {}^S\overline{\boldsymbol{J}}_S ({}^j\boldsymbol{A}_S)^t \boldsymbol{e}_3 \;+\; GA_8 \tag{A.23}$$

where GA_8 is a function of $\boldsymbol{\theta}$ generated by $\mathbf{R}_t^x$ and $\mathbf{R}_t^y - \mathbf{RZB}(t)$ for $t \geq i$. Here, we define

$$\begin{bmatrix} W_{i1} & W_{i2} & W_{i3} \end{bmatrix} = \sum_{S=i}^{N} \boldsymbol{e}_3^t \,{}^i\boldsymbol{A}_S {}^S\overline{\boldsymbol{J}}_S ({}^i\boldsymbol{A}_S)^t. \tag{A.24}$$

Then, since $({}^j\boldsymbol{A}_S)^t = ({}^i\boldsymbol{A}_S)^t ({}^j\boldsymbol{A}_i)^t$,

$$\sum_{S=i}^{N} \boldsymbol{e}_3^t \,{}^i\boldsymbol{A}_S {}^S\overline{\boldsymbol{J}}_S ({}^j\boldsymbol{A}_S)^t \boldsymbol{e}_3 = \begin{bmatrix} W_{i1} & W_{i2} & W_{i3} \end{bmatrix} ({}^j\boldsymbol{A}_i)^t \boldsymbol{e}_3. \tag{A.25}$$

When $j = i$, $H(i,i) = W_{i3} \;+\; GA_4$ from (A.10) and (A.25). Since it has been shown by **Lemma 2.3.3** that $H(i,i)$ is a function of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1**, so is W_{i3} for $1 \leq i \leq N$. When $j = \beta_{c(i)-1}$,

$$H(i, \beta_{c(i)-1}) = W_{i1} \sin\theta(\alpha_{c(i)}, i) + W_{i2} \cos\theta(\alpha_{c(i)}, i) + GA_8 \tag{A.26}$$

from (A.23) and (A.25) since $({}^{\beta_{c(i)-1}}\boldsymbol{A}_i)^t \boldsymbol{e}_3 = \begin{bmatrix} \sin\theta(\alpha_{c(i)}, i) & \cos\theta(\alpha_{c(i)}, i) & 0 \end{bmatrix}^t$ by **Property 2.3.1**. Since it has been shown by **Lemma 2.3.4** that $H(i, \beta_{c(i)-1})$ is a function of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1**, so is $W_{i1} \sin\theta(\alpha_{c(i)}, i) + W_{i2} \cos\theta(\alpha_{c(i)}, i)$ for $\alpha_2 \leq i \leq N$. $\cos\theta(\alpha_{c(i)}, i)$ and $\sin\theta(\alpha_{c(i)}, i)$ are linearly independent functions, and $\theta_{\alpha_{c(i)}}, \theta_{\alpha_{c(i)+1}}, \ldots, \theta_i$ appear in neither W_{i1} nor W_{i2}. Hence, we can conclude that W_{i1} and W_{i2} for $\alpha_2 \leq i \leq N$ are functions of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1**. It is obvious from (A.23) and (A.25) that all the $H(i,j)$ for $j \leq i$ and $\alpha_2 \leq i \leq N$ are functions of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1**. In the case that $1 \leq i \leq \beta_1$, $H(i,j) = W_{i3} \;+\; GA_8$ for $j \leq i$ from (A.23) and (A.25) since $({}^j\boldsymbol{A}_i)^t \boldsymbol{e}_3 = \boldsymbol{e}_3$ for $1 \leq j \leq i \leq \beta_1$. Thus, $H(i,j)$ is also a function of $\boldsymbol{\theta}$ generated by the inertial parameters given in **Theorem 2.3.1** for $1 \leq j \leq i \leq \beta_1$. **Lemma 2.3.6** is proved. ∎

Appendix B

Proof of Lemma 2.5.2, 2.5.3, 2.5.4, 2.5.5, and 2.5.6

We begin with the proof of **Lemma 2.5.2**.

In case of $i \leq \xi - 1$, using (2.145) and (3.35) we can describe ${}^{i}\boldsymbol{SR}_{i} + {}^{i}\boldsymbol{R}_{\zeta} + {}^{i}\boldsymbol{R}_{\zeta+1}$ as

$$\sum_{s=i}^{\xi-1} {}^{i}\boldsymbol{A}_{s}{}^{s}\boldsymbol{R}_{s} + {}^{i}\boldsymbol{A}_{\xi}({}^{\xi}\boldsymbol{R}_{\xi} + {}^{\zeta+1}\boldsymbol{R}_{\zeta+1}) + {}^{i}\boldsymbol{A}_{\xi+1}({}^{\zeta+1}\boldsymbol{R}_{\xi+1} + \boldsymbol{A}_{\rho}{}^{\zeta}\boldsymbol{R}_{\zeta}) + \sum_{s=\xi+1}^{N} {}^{i}\boldsymbol{A}_{s}{}^{s}\boldsymbol{R}_{s}. \tag{B.1}$$

Then, using (2.147)-(2.149), we obtain the w-th ($w = 1, 2, 3$) entry of (B.1) as

$$\sum_{s=i}^{N}\{({}^{i}A_{s})_{w1}\bar{\mathbf{R}}_{s}^{x} + ({}^{i}A_{s})_{w2}\bar{\mathbf{R}}_{s}^{y} + ({}^{i}A_{s})_{w3}\bar{\mathbf{R}}_{s}^{z}\}. \tag{B.2}$$

We can derive from (2.144) that

$$({}^{i}A_{s})_{w3} = -\sin\alpha_{s}({}^{i}A_{s-1})_{w2} + \cos\alpha_{s}({}^{i}A_{s-1})_{w3}. \tag{B.3}$$

Applying (B.3) repeatedly and using (2.153)-(2.155), we can deform $\sum_{s=i}^{N}({}^{i}A_{s})_{w3}\bar{\mathbf{R}}_{s}^{z}$ in (B.2) as

$$\begin{aligned}\sum_{s=i}^{N}({}^{i}A_{s})_{w3}\bar{\mathbf{R}}_{s}^{z} &= -\sum_{s=i}^{N-1}({}^{i}A_{s})_{w2}\sin\alpha_{s+1}\bar{\mathbf{RZ}}(s) \\ &\quad +({}^{i}A_{i})_{w3}[\bar{\mathbf{R}}_{i}^{z} + \cos\alpha_{i+1}\bar{\mathbf{RZ}}(i)].\end{aligned} \tag{B.4}$$

Then, we can describe (B.2) as

$$\sum_{S=i}^{N}({}^{i}A_S)_{w1}\bar{\mathbf{R}}_S^x + \sum_{S=i}^{N}({}^{i}A_S)_{w2}\hat{\bar{\mathbf{R}}}_S^y + ({}^{i}A_i)_{w3}[\bar{\mathbf{R}}_i^z + \cos\alpha_{i+1}\bar{\mathbf{RZ}}(i)]. \tag{B.5}$$

Consequently, we obtain

$$ {}^{i}\boldsymbol{SR}_i + {}^{i}\boldsymbol{R}_\zeta + {}^{i}\boldsymbol{R}_{\zeta+1} = \begin{bmatrix} \bar{\mathbf{R}}_i^x \\ \hat{\bar{\mathbf{R}}}_i^y \\ \bar{\mathbf{R}}_i^z + \cos\alpha_{i+1}\bar{\mathbf{RZ}}(i) \end{bmatrix} + \boldsymbol{G}_{1i} \tag{B.6}$$

where

$$\boldsymbol{G}_{1i} = \begin{bmatrix} (G_{1i})^x \\ (G_{1i})^y \\ (G_{1i})^z \end{bmatrix} = \begin{bmatrix} \sum_{S=i+1}^{N} [({}^{i}A_S)_{11}\bar{\mathbf{R}}_S^x + ({}^{i}A_S)_{12}\hat{\bar{\mathbf{R}}}_S^y] \\ \sum_{S=i+1}^{N} [({}^{i}A_S)_{21}\bar{\mathbf{R}}_S^x + ({}^{i}A_S)_{22}\hat{\bar{\mathbf{R}}}_S^y] \\ \sum_{S=i+1}^{N} [({}^{i}A_S)_{31}\bar{\mathbf{R}}_S^x + ({}^{i}A_S)_{32}\hat{\bar{\mathbf{R}}}_S^y] \end{bmatrix}. \tag{B.7}$$

In the similar way , we can show that ${}^{i}\boldsymbol{SR}_i + {}^{i}\boldsymbol{R}_\zeta + {}^{i}\boldsymbol{R}_{\zeta+1}$ is described as shown in **Lemma 2.5.2** when $i = \xi$ or $\xi + 1 \leq i$. ∎

Next, we give the proof of **Lemma 2.5.3, 2.5.4, 2.5.5** and **2.5.6**. First of all, we rewrite (3.25) as

$$
\begin{aligned}
H_{11}(i,j) = \; & \boldsymbol{z}_i \cdot [\sum_{S=i}^{\xi-1} \boldsymbol{J}_S]\boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot [\sum_{S=i}^{\xi-1} \{2\boldsymbol{L}_S \cdot (\boldsymbol{SR}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})\boldsymbol{E} \\
& \qquad -\boldsymbol{L}_S \otimes (\boldsymbol{SR}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1}) - (\boldsymbol{SR}_{S+1} + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1}) \otimes \boldsymbol{L}_S\}]\boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot [\sum_{S=\xi}^{N} \boldsymbol{J}_S]\boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot [\sum_{S=\xi}^{N-1} \{2\boldsymbol{L}_S \cdot \boldsymbol{SR}_{S+1}\boldsymbol{E} - \boldsymbol{L}_S \otimes \boldsymbol{SR}_{S+1} - \boldsymbol{SR}_{S+1} \otimes \boldsymbol{L}_S\}]\boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot \boldsymbol{J}_\zeta \boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot [2\boldsymbol{L}_\zeta \cdot \boldsymbol{R}_{\zeta+1}\boldsymbol{E} - \boldsymbol{L}_\zeta \otimes \boldsymbol{R}_{\zeta+1} - \boldsymbol{R}_{\zeta+1} \otimes \boldsymbol{L}_\zeta]\boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot \boldsymbol{J}_{\zeta+1} \boldsymbol{z}_j \\
+ \; & \boldsymbol{z}_i \cdot [\boldsymbol{L}_{j,i} \cdot (\boldsymbol{SR}_i + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})\boldsymbol{E} - \boldsymbol{L}_{j,i} \otimes (\boldsymbol{SR}_i + \boldsymbol{R}_\zeta + \boldsymbol{R}_{\zeta+1})]\boldsymbol{z}_j.
\end{aligned}
\tag{B.8}
$$

Since the inner product of two vectors can be executed about any coordinate system, the first and second terms of (B.8) can be evaluated as follows

$$
\begin{aligned}
& {}^S\boldsymbol{z}_i^t[\sum_{S=i}^{\xi-1} {}^S\boldsymbol{J}_S]\,{}^S\boldsymbol{z}_j \\
+ \; & {}^S\boldsymbol{z}_i^t[\sum_{S=i}^{\xi-1} \{2^S\boldsymbol{L}_S^t({}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})\boldsymbol{E} \\
- \; & {}^S\boldsymbol{L}_S({}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})^t - {}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})^S\boldsymbol{L}_S^t\}]^S\boldsymbol{z}_j
\end{aligned}
\tag{B.9}
$$

From **Lemma 2.5.2** and (2.147)-(2.149), we can derive that

$$ {}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1} = \begin{bmatrix} 0 \\ -\sin\alpha_{S+1}\bar{\mathbf{RZ}}(s) \\ \cos\alpha_{S+1}\bar{\mathbf{RZ}}(s) \end{bmatrix} + \boldsymbol{G}_{1S} \tag{B.10} $$

where

$$ \boldsymbol{G}_{1S} = \begin{bmatrix} \sum_{k=S+1}^{N} [({}^S A_k)_{11}\bar{\mathbf{R}}_k^x + ({}^S A_k)_{12}\hat{\bar{\mathbf{R}}}_k^y] \\ \sum_{k=S+1}^{N} [({}^S A_k)_{21}\bar{\mathbf{R}}_k^x + ({}^S A_k)_{22}\hat{\bar{\mathbf{R}}}_k^y] \\ \sum_{k=S+1}^{N} [({}^S A_k)_{31}\bar{\mathbf{R}}_k^x + ({}^S A_k)_{32}\hat{\bar{\mathbf{R}}}_k^y] \end{bmatrix}. \tag{B.11} $$

Then we can deform the 2nd term of (B.9) as

$$ \begin{aligned} & 2{}^S\boldsymbol{L}_S^t({}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})\boldsymbol{E} \\ & -{}^S\boldsymbol{L}_S({}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})^t - {}^S\boldsymbol{SR}_{S+1} + {}^S\boldsymbol{R}_\zeta + {}^S\boldsymbol{R}_{\zeta+1})^S\boldsymbol{L}_S^t \\ & = \begin{bmatrix} 2[L_S]^z\cos\alpha_{S+1}\bar{\mathbf{RZ}}(s) & [L_S]^x\sin\alpha_{S+1}\bar{\mathbf{RZ}}(s) & -[L_S]^x\cos\alpha_{S+1}\bar{\mathbf{RZ}}(s) \\ 1[L_S]^x\sin\alpha_{S+1}\bar{\mathbf{RZ}}(s) & 2[L_S]^z\cos\alpha_{S+1}\bar{\mathbf{RZ}}(s) & [L_S]^z\sin\alpha_{S+1}\bar{\mathbf{RZ}}(s) \\ -[L_S]^x\cos\alpha_{S+1}\bar{\mathbf{RZ}}(s) & [L_S]^z\sin\alpha_{S+1}\bar{\mathbf{RZ}}(s) & 0 \end{bmatrix} + \boldsymbol{D}_S \end{aligned} \tag{B.12} $$

where

$$ \boldsymbol{D}_S = 2\,{}^S\boldsymbol{L}_S^t\boldsymbol{G}_{1S}\boldsymbol{E} - {}^S\boldsymbol{L}_S\boldsymbol{G}_{1S}^t - \boldsymbol{G}_{1S}{}^S\boldsymbol{L}_S^t. \tag{B.13} $$

Using (2.153),(2.157)-(2.161), and (B.11), we can deform (B.9) as

$$ \sum_{S=i}^{\xi-1} {}^S\boldsymbol{z}_i^t{}^S\tilde{\boldsymbol{J}}_S{}^S\boldsymbol{z}_j + \sum_{S=i}^{\xi-1} {}^S\boldsymbol{z}_i^t\boldsymbol{D}_S{}^S\boldsymbol{z}_j. \tag{B.14} $$

In the same manner, using (2.154),(2.157)-(2.161), we can deform the 3rd and 4th terms of (B.8) as

$$\sum_{S=\xi}^{N} {}^{S}\boldsymbol{z}_i^t\, {}^{S}\tilde{\boldsymbol{J}}_S\, {}^{S}\boldsymbol{z}_j + \sum_{S=\xi}^{N} {}^{S}\boldsymbol{z}_i^t \boldsymbol{D}_S {}^{S}\boldsymbol{z}_j, \tag{B.15}$$

and using (2.155)-(2.161), we can deform the 5th, 6th, and 7th terms of (B.8) as

$${}^{\zeta}\boldsymbol{z}_i^t\, {}^{\zeta}\tilde{\boldsymbol{J}}_{\zeta}\, {}^{\zeta}\boldsymbol{z}_j + {}^{\zeta}\boldsymbol{z}_i^t\, {}^{\zeta}\boldsymbol{D}_{\zeta}\, {}^{\zeta}\boldsymbol{z}_j + {}^{\zeta+1}\boldsymbol{z}_i^t\, {}^{\zeta+1}\tilde{\boldsymbol{J}}_{\zeta+1}\, {}^{\zeta+1}\boldsymbol{z}_j \tag{B.16}$$

where

$$\boldsymbol{D}_{\zeta} = 2\, {}^{\zeta}\boldsymbol{L}_{\zeta}^t \boldsymbol{G}_{1\zeta} \boldsymbol{E} - {}^{\zeta}\boldsymbol{L}_{\zeta} \boldsymbol{G}_{1\zeta}^t - \boldsymbol{G}_{1\zeta}\, {}^{\zeta}\boldsymbol{L}_{\zeta}^t, \tag{B.17}$$

$$\boldsymbol{G}_{1\zeta} = \begin{bmatrix} ({}^{\zeta}A_{\zeta+1})_{11}\bar{\mathbf{R}}_{\zeta+1}^{x} + ({}^{\zeta}A_{\zeta+1})_{12}\hat{\bar{\mathbf{R}}}_{\zeta+1}^{y} \\ ({}^{\zeta}A_{\zeta+1})_{21}\bar{\mathbf{R}}_{\zeta+1}^{x} + ({}^{\zeta}A_{\zeta+1})_{22}\hat{\bar{\mathbf{R}}}_{\zeta+1}^{y} \\ ({}^{\zeta}A_{\zeta+1})_{31}\bar{\mathbf{R}}_{\zeta+1}^{x} + ({}^{\zeta}A_{\zeta+1})_{32}\hat{\bar{\mathbf{R}}}_{\zeta+1}^{y} \end{bmatrix}. \tag{B.18}$$

Then, using (2.166)-(2.168), we can describe (B.15) together with (B.16) as

$$\sum_{S=\xi}^{N} {}^{S}\boldsymbol{z}_i^t {}^{S}\hat{\boldsymbol{J}}_S {}^{S}\boldsymbol{z}_j + \sum_{S=\xi}^{N} {}^{S}\boldsymbol{z}_i^t \boldsymbol{D}_S {}^{S}\boldsymbol{z}_j + {}^{\zeta}\boldsymbol{z}_i^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{z}_j. \tag{B.19}$$

Here, we rewrite ${}^{\zeta}\boldsymbol{z}_i^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{z}_j$ of (B.19) as ${}^{\xi}\boldsymbol{z}_i^t ({}^{\zeta}\boldsymbol{A}_{\xi}^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{A}_{\xi}) {}^{\xi}\boldsymbol{z}_j$ where

$${}^{\zeta}\boldsymbol{A}_{\xi}^t = \begin{bmatrix} \cos(\theta_{\zeta}-\theta_{\xi}) & -\sin(\theta_{\zeta}-\theta_{\xi}) & 0 \\ \sin(\theta_{\zeta}-\theta_{\xi}) & \cos(\theta_{\zeta}-\theta_{\xi}) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{B.20}$$

Thus, from (B.18), we see that ${}^{\zeta}\boldsymbol{A}_{\xi}^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{A}_{\xi}$ includes dependent variables. Using (2.113)-(3.2) we delete the dependent variables in ${}^{\zeta}\boldsymbol{A}_{\xi}^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{A}_{\xi}$,then, after lengthy calculation, we obtain that

$$\begin{aligned} & {}^{\zeta}\boldsymbol{A}_{\xi}^t \boldsymbol{D}_{\zeta} {}^{\zeta}\boldsymbol{A}_{\xi} + \boldsymbol{D}_{\xi} \\ &= \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} + 2\, {}^{\xi}\boldsymbol{L}_{\xi}^t \boldsymbol{G}_{1\xi} \boldsymbol{E} - {}^{\xi}\boldsymbol{L}_{\xi} \boldsymbol{G}_{1\xi}^t - \boldsymbol{G}_{1\xi}\, {}^{\xi}\boldsymbol{L}_{\xi}^t \end{aligned} \tag{B.21}$$

where

$$
\begin{aligned}
d_{11} &= -2(\sin\theta_{\xi+1}\cos\rho[L_\zeta]^x + \cos\theta_{\xi+1}\sin\rho[L_\zeta]^x)\mathbf{R}^y_{\zeta+1}\\
d_{12} &= \sin\theta_{\xi+1}(-[L_\xi]^x\mathbf{R}^x_{\xi+1} + \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})\\
&\quad + \cos\theta_{\xi+1}(-[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} + \cos\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})\\
d_{13} &= 0\\
d_{22} &= 2\sin\theta_{\xi+1}(-[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1}\\
&\quad +2\cos\theta_{\xi+1}([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1})\\
d_{23} &= 0\\
d_{33} &= 2\sin\theta_{\xi+1}(-[L_\xi]^x\hat{\mathbf{R}}^y_{\xi+1} + \sin\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \cos\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})\\
&\quad +2\cos\theta_{\xi+1}([L_\xi]^x\mathbf{R}^x_{\xi+1} - \cos\rho[L_\zeta]^x\mathbf{R}^x_{\zeta+1} - \sin\rho[L_\zeta]^x\mathbf{R}^y_{\zeta+1})
\end{aligned}
\tag{B.22}
$$

and

$$
\boldsymbol{G}_{1\xi} = \begin{bmatrix} \sum_{k=\xi+2}^{N} [({}^\xi A_k)_{11}\bar{\mathbf{R}}^x_k + ({}^\xi A_k)_{12}\hat{\bar{\mathbf{R}}}^y_k] \\ \sum_{k=\xi+2}^{N} [({}^\xi A_k)_{21}\bar{\mathbf{R}}^x_k + ({}^\xi A_k)_{22}\hat{\bar{\mathbf{R}}}^y_k] \\ \sum_{k=\xi+2}^{N} [({}^\xi A_k)_{31}\bar{\mathbf{R}}^x_k + ({}^\xi A_k)_{32}\hat{\bar{\mathbf{R}}}^y_k] \end{bmatrix}. \tag{B.23}
$$

In the calculation above, we used ${}^\zeta\boldsymbol{L}_\zeta = \begin{bmatrix} [L_\zeta]^x & 0 & 0 \end{bmatrix}^t$. Let $\boldsymbol{D}_\xi$ denote ${}^\zeta\boldsymbol{A}^t_\xi\boldsymbol{D}_\zeta{}^\zeta\boldsymbol{A}_\xi + \boldsymbol{D}_\xi$, then, (B.14) and (B.19) can be unified and described as

$$
\sum_{S=i}^{N} {}^S\boldsymbol{z}^t_i{}^S\hat{\boldsymbol{J}}_S{}^S\boldsymbol{z}_j + \sum_{S=i}^{N} {}^S\boldsymbol{z}^t_i\boldsymbol{D}_S{}^S\boldsymbol{z}_j. \tag{B.24}
$$

Since ${}^S\boldsymbol{z}_i = {}^S\boldsymbol{z}_j = \boldsymbol{e}_3\,(\boldsymbol{e}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^t)$ for $1 \le j \le i < s \le K$ and $\boldsymbol{e}^t_3\boldsymbol{D}_S\boldsymbol{e}_3 = 2[L_S]^x(G_{1S})^x$, then $\sum_{S=i}^{N} {}^S\boldsymbol{z}^t_i\boldsymbol{D}_S{}^S\boldsymbol{z}_j$ in (B.24) can be described as

$$2\sum_{S=i}^{K}[L_S]^x(G_{1S})^x + d_{33} + \sum_{S=max(K+1,i)}^{N} {}^S\boldsymbol{z}_i^t \boldsymbol{D}_S {}^S\boldsymbol{z}_j \tag{B.25}$$

if $i \le \xi < K$, otherwise d_{33} disappears.

Next, the last term of (B.8) will be evaluated. Denoting ${}^i\boldsymbol{L}_{j,i}$ by

$${}^i\boldsymbol{L}_{j,i} = \left[\ [{}^iL_{j,i}]^x \quad [{}^iL_{j,i}]^y \quad [{}^iL_{j,i}]^z\ \right]^t, \tag{B.26}$$

we can describe it by using **Lemma 2.5.2** as

$$\begin{aligned}
&\bar{\mathbf{R}}_i^x([{}^iL_{j,i}]^x({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{13})\\
&+\hat{\bar{\mathbf{R}}}_i^y([{}^iL_{j,i}]^y({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{23})\\
&+G
\end{aligned} \tag{B.27}$$

where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \le s \le N$.

Consequently, from above, we obtain that

$$\begin{aligned}
Hr(i,j) &= \sum_{S=i}^{N} {}^S\boldsymbol{z}_i^t {}^S\hat{\boldsymbol{J}}_S {}^S\boldsymbol{z}_j\\
&\quad +2\sum_{S=i}^{K}[L_S]^x(G_{1S})^x + d_{33} + \sum_{S=max(K+1,i)}^{N} {}^S\boldsymbol{z}_i^t \boldsymbol{D}_S {}^S\boldsymbol{z}_j\\
&\quad +\bar{\mathbf{R}}_i^x([{}^iL_{j,i}]^x({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{13})\\
&\quad +\hat{\bar{\mathbf{R}}}_i^y([{}^iL_{j,i}]^y({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{23})\\
&\quad +G
\end{aligned} \tag{B.28}$$

for $1 \le j \le i \le \xi - 1$ where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \le s \le N$.

In the same manner as was used in deriving (B.28) but more simply, we can derive that

$$
\begin{aligned}
Hr(i,j) &= \sum_{s=i}^{N} {}^s\boldsymbol{z}_i^{t\,s}\hat{\boldsymbol{J}}_S{}^s\boldsymbol{z}_j \\
&+2\sum_{s=i}^{K}[L_S]^x(G_{1S})^x + \sum_{s=max(K+1,i)}^{N} {}^s\boldsymbol{z}_i^t\boldsymbol{D}_S{}^s\boldsymbol{z}_j \\
&+\bar{\mathbf{R}}_i^x([{}^iL_{j,i}]^x({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{13}) \\
&+\hat{\bar{\mathbf{R}}}_i^y([{}^iL_{j,i}]^y({}^iA_j)_{33} - [{}^iL_{j,i}]^z({}^iA_j)_{23}) \\
&+G
\end{aligned} \tag{B.29}
$$

for $\xi+2 \le j \le i \le N$ where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $i+1 \le s \le N$.

Next, from (3.22)-(3.25) and (2.137)-(2.140) we can derive $Hr(\xi+1,j)$ for $1 \le j \le \xi-1$, $Hr(j,,\xi+1)$ for $\xi+2 \le j \le N$, $Hr(\xi+1,\xi+1), Hr(\xi,j)$ for $1 \le j \le \xi-1$, $Hr(j,\xi)$ for $\xi+2 \le j \le N$, $Hr(\xi,\xi)$, and $Hr(\xi+1,\xi)$ by lengthy and direct calculations that

$$
\begin{aligned}
Hr(\xi+1,j) &= \sum_{s=\xi+1}^{N} {}^s\boldsymbol{z}_{\xi+1}({}^s\hat{\boldsymbol{J}}_S + \boldsymbol{D}_S)^s\boldsymbol{z}_j \\
&+[\cos\theta_{\xi+1}([L_\xi]^x\mathbf{R}_{\xi+1}^x - \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x - \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y) \\
&\quad -\sin\theta_{\xi+1}([L_\xi]^x\hat{\mathbf{R}}_{\xi+1}^y - \sin\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^x + \cos\rho[L_\zeta]^x\mathbf{R}_{\zeta+1}^y)]({}^{\xi+1}A_j)_{33} \\
&+\bar{\mathbf{R}}_{\xi+1}^x[({}^{\xi+1}A_j)_{33}[{}^{\xi+1}L_{j,\xi}]^x - ({}^{\xi+1}A_j)_{13}[{}^{\xi+1}L_{j,\xi}]^z] \\
&+\hat{\bar{\mathbf{R}}}_{\xi+1}^y[({}^{\xi+1}A_j)_{33}[{}^{\xi+1}L_{j,\xi}]^y - ({}^{\xi+1}A_j)_{23}[{}^{\xi+1}L_{j,\xi}]^z] \\
&+G
\end{aligned} \tag{B.30}
$$

for $1 \le j \le \xi-1$ where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_S^x$ and $\hat{\bar{\mathbf{R}}}_S^y$ for $\xi+2 \le s \le N$. $Hr(j,\xi+1)$ for $\xi+2 \le j \le N$ and $Hr(\xi+1,\xi+1)$ can be included in (B.29), hence, (B.29) holds for $\xi+1 \le j \le i \le N$.

$$
\begin{aligned}
Hr(\xi,j) \ &= {}^{\xi}\boldsymbol{z}_{\xi}^{t}\,{}^{\xi}\hat{\boldsymbol{J}}_{\xi}{}^{\xi}\boldsymbol{z}_{j} \\
&+[\cos\theta_{\xi+1}([L_{\xi}]^{x}\mathbf{R}_{\xi+1}^{x}-\cos\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{x}-\sin\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{y}) \\
&\ -\sin\theta_{\xi+1}([L_{\xi}]^{x}\hat{\mathbf{R}}_{\xi+1}^{y}-\sin\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{x}\cos\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{y})]({}^{j}A_{\xi})_{33} \\
&+\bar{\mathbf{R}}_{\xi}^{x}([{}^{\xi}L_{j,\xi}]^{x}({}^{j}A_{\xi})_{33}-[{}^{\xi}L_{j,\xi}]^{z}({}^{j}A_{\xi})_{31}) \\
&+\hat{\bar{\mathbf{R}}}_{\xi}^{y}([{}^{\xi}L_{j,\xi}]^{y}({}^{j}A_{\xi})_{33}-[{}^{\xi}L_{j,\xi}]^{z}({}^{j}A_{\xi})_{32}) \\
&+G
\end{aligned}
\tag{B.31}
$$

for $1 \leq j \leq \xi - 1$ where G is a function of $\boldsymbol{\theta}$ generated by $\bar{\mathbf{R}}_{s}^{x}$ and $\hat{\bar{\mathbf{R}}}_{s}^{y}$ for $\xi + 2 \leq s \leq N$.

$$
\begin{aligned}
Hr(j,\xi) \ &= [L_{\xi}]^{x}(\bar{\mathbf{R}}_{j+1}^{x}+\sum_{k=j+2}^{N}\{({}^{k}A_{j+1})_{11}\bar{\mathbf{R}}_{k}^{x}+({}^{k}A_{j+1})_{12}\hat{\bar{\mathbf{R}}}_{k}^{y}\}) \\
&\quad [({}^{j+1}A_{\xi})_{11}({}^{j+1}A_{\xi+1})_{33}-({}^{j+1}A_{\xi})_{31}({}^{j+1}A_{\xi+1})_{13}] \\
&+[L_{\xi}]^{x}(\hat{\bar{\mathbf{R}}}_{j+1}^{y}+\sum_{k=j+2}^{N}\{({}^{k}A_{j+1})_{21}\bar{\mathbf{R}}_{k}^{x}+({}^{k}A_{j+1})_{22}\hat{\bar{\mathbf{R}}}_{k}^{y}\}) \\
&\quad [({}^{j+1}A_{\xi})_{21}({}^{j+1}A_{\xi+1})_{33}-({}^{j+1}A_{\xi})_{31}({}^{j+1}A_{\xi+1})_{23}]
\end{aligned}
\tag{B.32}
$$

for $\xi + 2 \leq j \leq N$.

$$
Hr(\xi,\xi) = \hat{\mathbf{J}}_{\xi}^{z}. \tag{B.33}
$$

$$
\begin{aligned}
Hr(\xi+1,\xi) \ &= \cos\theta_{\xi+1}([L_{\xi}]^{x}\mathbf{R}_{\xi+1}^{x}-\cos\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{x}-\sin\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{y}) \\
&\ -\sin\theta_{\xi+1}([L_{\xi}]^{x}\hat{\mathbf{R}}_{\xi+1}^{y}-\sin\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{x}+\cos\rho[L_{\zeta}]^{x}\mathbf{R}_{\zeta+1}^{y}) \\
&+[L_{\xi}]^{x}(\sum_{k=\xi+2}^{N}\{({}^{\xi}A_{k})_{11}\bar{\mathbf{R}}_{k}^{x}+({}^{\xi}A_{k})_{12}\hat{\bar{\mathbf{R}}}_{k}^{y}\}).
\end{aligned}
\tag{B.34}
$$

Next, we can describe $\sum_{s=i}^{N} {}^{s}\boldsymbol{z}_{i}^{t}\,{}^{s}\hat{\boldsymbol{J}}_{s}{}^{s}\boldsymbol{z}_{j}$ of (B.28)-(B.31) as

$$\sum_{S=i}^{N} {}^{S}\boldsymbol{z}_i^{t}{}^{S}\hat{\boldsymbol{J}}_S{}^{S}\boldsymbol{z}_j = \sum_{S=i}^{N}[({}^{i}A_S)_{31}({}^{j}A_S)_{31}\hat{\mathbf{J}}_S^{x} + ({}^{i}A_S)_{32}({}^{j}A_S)_{32}\hat{\mathbf{J}}_S^{y} + ({}^{i}A_S)_{33}({}^{j}A_S)_{33}\hat{\mathbf{J}}_S^{z}$$
$$+\{({}^{i}A_S)_{31}({}^{j}A_S)_{32} + ({}^{i}A_S)_{32}({}^{j}A_S)_{31}\}\hat{\mathbf{J}}_S^{xy}$$
$$+\{({}^{i}A_S)_{32}({}^{j}A_S)_{33} + ({}^{i}A_S)_{33}({}^{j}A_S)_{32}\}\hat{\mathbf{J}}_S^{yz}$$
$$+\{({}^{i}A_S)_{31}({}^{j}A_S)_{33} + ({}^{i}A_S)_{33}({}^{j}A_S)_{31}\}\hat{\mathbf{J}}_S^{xz}]. \quad \text{(B.35)}$$

Since ${}^{n}\boldsymbol{A}_S = {}^{n}\boldsymbol{A}_{S-1}{}^{S-1}\boldsymbol{A}_S$ for $j \le i \le n+1 \le s$, we obtain

$$({}^{n}A_S)_{32} = -\sin\theta_S({}^{n}A_{s-1})_{31} + \cos\theta_S\cos\alpha_S({}^{n}A_{S-1})_{32}$$
$$+\cos\theta_S\sin\alpha_S({}^{n}A_{S-1})_{33}. \quad \text{(B.36)}$$

Using (B.36), we can describe $({}^{i}A_S)_{32}({}^{j}A_S)_{32}$ of (B.35) as

$$({}^{i}A_S)_{32}({}^{j}A_S)_{32} = -({}^{i}A_S)_{31}({}^{j}A_S)_{31} + ({}^{i}A_{S-1})_{31}({}^{j}A_{S-1})_{31}$$
$$+\sin^2\alpha_S({}^{i}A_{S-1})_{31}({}^{j}A_{S-1})_{33}$$
$$+\sin\alpha_S\cos\alpha_S[({}^{i}A_{S-1})_{32}({}^{j}A_{S-1})_{33} + ({}^{i}A_{S-1})_{33}({}^{j}A_{S-1})_{32}]$$
$$+\cos^2\alpha_S({}^{i}A_{S-1})_{32}({}^{j}A_{S-1})_{32}. \quad \text{(B.37)}$$

Using (B.37) repeatedly, we can deform (B.35) as

$$\sum_{S=i}^{N} {}^{S}\boldsymbol{z}_i^{t}{}^{S}\boldsymbol{J}_S{}^{S}\boldsymbol{z}_j = \sum_{S=i}^{N}[({}^{i}A_S)_{31}({}^{j}A_S)_{31}(\hat{\mathbf{J}}_S^{x} - \hat{\mathbf{J}}_S^{y} + \sin^2\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s))$$
$$+({}^{i}A_S)_{33}({}^{j}A_S)_{33}(\hat{\mathbf{J}}_S^{z} + \sin^2\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s))$$
$$+\{({}^{i}A_S)_{31}({}^{j}A_S)_{32} + ({}^{i}A_S)_{32}({}^{j}A_S)_{31}\}\hat{\mathbf{J}}_S^{xy}$$
$$+\{({}^{i}A_S)_{32}({}^{j}A_S)_{33} + ({}^{i}A_S)_{33}({}^{j}A_S)_{32}\}(\hat{\mathbf{J}}_S^{yz} + \sin\alpha_{s+1}\cos\alpha_{s+1}\mathbf{J}\hat{\mathbf{Y}}(s))$$
$$+\{({}^{i}A_S)_{31}({}^{j}A_S)_{33} + ({}^{i}A_S)_{33}({}^{j}A_S)_{31}\}\hat{\mathbf{J}}_S^{xz}]. \quad \text{(B.38)}$$

Here, let us consider the case that $j = i$. Since $\boldsymbol{L}_{j,i} = \mathbf{o}$, we can derive from (B.28) and (B.29) that

$$\begin{aligned} Hr(i,i) &= \sum_{S=i}^{N} {}^{S}\boldsymbol{z}_i^{t\,S}\hat{\boldsymbol{J}}_S{}^{S}\boldsymbol{z}_i \\ &+2\sum_{S=i}^{K}[L_S]^x (G_{1S})^x + d_{33} + \sum_{S=max(K+1,i)}^{K} {}^{S}\boldsymbol{z}_i^t \boldsymbol{D}_S{}^{S}\boldsymbol{z}_i \end{aligned} \tag{B.39}$$

for $1 \leq i \leq \xi - 1$ or $\xi + 1 \leq i \leq N$. Since $\alpha_n = 0$ for $i + 1 \leq n \leq \delta_2(i) - 1$ and $n \neq \delta_1(i)$, we can easily derive that ${}^{s}\boldsymbol{z}_i = {}^{i}\boldsymbol{A}_S^t e_3 = \boldsymbol{e}_3$ for $i \leq s \leq \delta_1(i) - 1$ and

$${}^{s}\boldsymbol{z}_i = \begin{bmatrix} \sin\theta(\delta_1(i), s)\sin\alpha_{\delta_1(i)} & \cos\theta(\delta_1(i), s)\sin\alpha_{\delta_1(i)} & \cos\alpha_{\delta_1(i)} \end{bmatrix}^t \tag{B.40}$$

for $\delta_1(i) \leq s \leq \delta_2(i) - 1$, ${}^{\xi}\boldsymbol{z}_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^t$ for $\delta(\xi) \leq i \leq \xi - 1$, moreover $({}^{i}A_\xi)_{31}, ({}^{i}A_\xi)_{32}$, and $({}^{i}A_\xi)_{33}$ are not zero for $i \leq \delta(\xi) - 1$. Using these and (B.38), we can prove **Lemma 2.5.3** from (B.39) and (B.40).

In case that $j = \delta(i) - 1$ $(K + 1 \leq i)$, we can derive that ${}^{s}\boldsymbol{z}_i = \boldsymbol{e}_3$ and

$${}^{s}\boldsymbol{z}_{\delta(i)-1} = \begin{bmatrix} \sin\theta(\delta(i), s)\sin\alpha_{\delta(i)} & \cos\theta(\delta(i), s)\sin\alpha_{\delta(i)} & \cos\alpha_{\delta(i)} \end{bmatrix}^t \tag{B.41}$$

for $i \leq s \leq \delta_1(i) - 1$ since $\alpha_n = 0$ for $\delta(i) + 1 \leq n \leq \delta_1(i) - 1$, ${}^{\xi}\boldsymbol{z}_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^t$ for $\delta(\xi) \leq i \leq \xi - 1$, moreover $({}^{i}A_\xi)_{31}, ({}^{i}A_\xi)_{32}$, and $({}^{i}A_\xi)_{33}$ are not zero for $i \leq \delta(\xi) - 1$. Then, Using these, we can prove **Lemma 2.5.4** by direct calculations from (B.28)-(B.31).

Next, since $\alpha_n = 0$ for $2 \leq n \leq K$, we can derive that

$${}^{i}\boldsymbol{L}_{j,i} = \begin{bmatrix} [L_S]^x \cos\theta(s+1, i) & -[L_S]^x \sin\theta(s+1, i) & [L_S]^z \end{bmatrix}^t \tag{B.42}$$

for $j \leq i \leq K$. Also we obtain ${}^{s}\boldsymbol{z}_j = {}^{s}\boldsymbol{z}_i$ for $j \leq i \leq K$. Using these, we can prove **Lemma 2.5.5** from (B.28)-(B.31).

Taking it into account that ${}^{s}\boldsymbol{z}_i = \boldsymbol{e}_3$ for $1 \leq i \leq s \leq K$, we can easily show from (B.28), (B.29), and (B.38) that $Hr(i,j)$ $(j \leq i)$ for $1 \leq i \leq \xi - 1$ or $\xi + 1 \leq i \leq N$ is a function of $\boldsymbol{\theta}$ generated by some fundamental parameters in **Theorem**, and it is obvious from (B.30)-(B.34) that $Hr(\xi + 1, j)$ for $1 \leq j \leq \xi - 1$, $Hr(\xi, j)$ for $1 \leq j \leq \xi - 1$, $Hr(j, \xi)$ for $\xi + 2 \leq j \leq N$, $Hr(\xi, \xi)$, and $Hr(\xi + 1, \xi)$ are functions of $\boldsymbol{\theta}$ generated by some fundamental parameters in **Theorem 2.5.1**. Therefore, **Lemma 2.5.6** is proved . ∎

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