ORIENTED BORDISM MODULES OF S^1- AND $(Z_2)^k$-ACTIONS

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Introduction

In [2] P.E.Conner and E.E.Floyd demonstrated the effectiveness of bordism methods in the studies of group actions. Afterwards, using the bordism methods, many topologists obtained various results in the area. The central tools in these studies are the bordism modules of group actions.

Let G be a compact Lie group, and F, F' be families of subgroups of G such that $F \supset F'$. We may define the oriented bordism module $\Omega_*(G ; F,F')$, over the oriented cobordism ring Ω_*, which consists of bordism classes of (F, F')-free oriented G-manifolds. In this paper we are concerned with the module structure of $\Omega_*(G ; F, F')$. If F' is empty, then we denote this module by $\Omega_*(G ;F)$. Let F_A be the family of all subgroups of G. Then $\Omega_*(G ; F_A)$ is the bordism module of all closed oriented G-manifolds. Especially we are interested in the module structure of $\Omega_*(G ; F_A)$.

R.E.Stong [7] has shown that $\Omega_*(G ; F_A)$ is a free Ω_*-module on even dimensional generators when G is a finite p-primary abelian group for odd prime p. Recently E.R.Wheeler [8] has shown that $\Omega_*(G ; F_A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators when G is a certain finite cyclic

group, where $R_2 = Z[\frac{1}{2}]$.

We study the cases in which G is the circle group S^1 or $(Z_2)^k = Z_2 \oplus \cdots \oplus Z_2$ (k times). We obtain that both $\Omega_*(S^1 ; F_A) \otimes R_2$ and $\Omega_*((Z_2)^k ; F_A) \otimes R_2$ are free $\Omega_* \otimes R_2$-modules on even dimensional generators. In fact we prove such "freeness" theorems for more general families, as stated in Theorem 2-1-1 and Theorem 3-1-1.

Our main tools are the Conner-Floyd exact sequences and the fact that $\Omega_*(G ; F, F')$ can be interpreted as (direct sum of) singular bordism modules of adequate spaces when $F - F'$ consists of a single element H. When G is S^1 or $(Z_2)^k$, this interpretation involves a difficulty for the sake of non-orientability of normal bundles of H-stationary point sets. We overcome this difficulty by a modification of the methods due to E.Ossa [5; Lemma 1-2-5][6; Lemma 4], (see Lemmas 2-2-3,3-2-4).

Chapter 1. Preliminalies

In this chapter we give common preliminaries for S^1- and $(Z_2)^k$-actions. Throughout this chapter, G denotes a compact abelian Lie group.

1-1. Bordism of G-manifolds

A family F in G is a collection of closed subgroups of G such that if $H \in F$ and if K is a closed subgroup of H then $K \in F$.

Being given families F, F' in G with $F \supset F'$, an (F, F')-free G-manifold M is a compact differentiable manifold M with differentiable G-action such that for all $x \in M$ the isotropy groups I(x) belong to F and for all $x \in \partial M$ I(x) belong to F'. An (F, F')-free oriented G-manifold M is an (F, F')-free G-manifold M such that M is an oriented manifold and the G-action preserves the orientation of M. If F' is empty, then necessarily $\partial M = \phi$.

Being given an (F, F')-free oriented G-manifold M, we define -M to be the (F, F')-free oriented G-manifold whose ~~base~~ underlying manifold and G-action are same as M but orientation is reversed. We also define ∂M to be the (F', ϕ)-free oriented G-manifold whose G-action is the restriction of the G-action on M and orientation is given by inward normal vectors.

Two (F, F')-free oriented G-manifolds M, M' are bordant, if there are an (F', F')-free oriented G-manifold V and an (F, F)-free oriented G-manifold W such that

(1-1-1) ∂V is diffeomorphic, as oriented G-manifolds,

to the disjoint union of ∂M and $-\partial M'$, and

(1-1-2) ∂W is diffeomorphic, as oriented G-manifolds, to the manifold $M \cup V \cup -M'$ obtained by glueing the boundaries.

This relation "bordant" is an equivalence relation on the set of all (F, F')-free oriented G-manifolds. An equivalence class by this relation is called an (F, F')-free bordism class.

The set of all (F, F')-free bordism classes of (F, F')-free oriented G-manifolds forms an abelian group with the operation induced by disjoint union, and this group will be denoted by $\Omega_*(G ; F, F')$. $\Omega_n(G ; F, F')$ denotes the summand consisting of (F, F')-free bordism classes of (F, F')-free oriented G-manifolds of dimension n.

When F' is empty, $\Omega_*(G ; F, F'), \Omega_n(G ; F, F')$ are denoted by $\Omega_*(G ; F), \Omega_n(G ; F)$ in brief.

For a representative N of any element in the oriented cobordism ring Ω_* we can see N to be an oriented G-manifold by giving the trivial G-action. Therefore $\Omega_*(G ; F, F')$ is a module over Ω_* by the cartesian product.

Being given families F, F', F" in G with $F \supset F' \supset F"$, we have Ω_*-module homomorphisms

$$i_* : \Omega_*(G ; F', F") \longrightarrow \Omega_*(G ; F, F")$$

$$j_* : \Omega_*(G ; F, F") \longrightarrow \Omega_*(G ; F, F')$$

obtained by considering (F', F")-free (or (F, F")-free) bordism classes as (F, F")-free (or (F, F')-free) bordism classes. We also have an Ω_*-module homomorphism

$$\partial_* : \Omega_*(G ; F, F') \longrightarrow \Omega_*(G ; F', F'')$$

of degree -1 obtained by sending the class of M to the class of ∂M.

Then

Theorem 1-1-3. The sequence

$$\cdots \to \Omega_n(G ; F', F'') \otimes R_2 \xrightarrow{i_* \otimes 1} \Omega_n(G ; F, F'') \otimes R_2$$
$$\xrightarrow{j_* \otimes 1} \Omega_n(G ; F, F') \otimes R_2 \xrightarrow{\partial_* \otimes 1} \Omega_{n-1}(G ; F', F'') \otimes R_2 \to \cdots$$

is exact, where R_2 is the subring of the rationals given by $R_2 = Z[\frac{1}{2}]$.

Proof. The sequence obtained from the above sequence by taking away $\otimes R_2$ is a Conner-Floyd's exact sequence [3 ; (5.3)]. Since R_2 is torsion free, the above sequence is also exact. q.e.d.

1-2. Bordism of G-vector bundles

A differentiable vector bundle $E \to X$ is called a G-vector bundle, when the total space E is an oriented manifold on which G acts as a group of bundle maps preserving the orientation of the manifold E.

Let H be a closed subgroup of G. A G-vector bundle $E \to X$ is called to be of type (r, s, H), if $E \to X$ is an r-dimensional G-vector bundle over an s-dimensional compact manifold X such that for any vector $e \in E$ the isotropy group $I(e)$ is a subgroup of H and $I(e)$ is equal to H if and only if e is a zero vector.

Being given a G-vector bundle $E \to X$, we define $-(E \to X)$ to be the G-vector bundle obtained from $E \to X$ by reversing the orientation of the total space E.

Two G-vector bundles $E \to X$, $E' \to X'$ over closed manifolds of type (r, s, H) are bordant, if there is a G-vector bundle $F \to Y$ of type $(r, s+1, H)$ such that the restriction $F | \partial Y \longrightarrow \partial Y$ is isomorphic, as G-vector bundles, to the disjoint union of $E \to X$ and $-(E' \longrightarrow X')$.

This relation "bordant" is an equivalence relation on the set of G-vector bundles over closed manifolds of type (r, s, H). The set of equivalence classes of G-vector bundles over closed manifolds of type (r, s, H) forms an abelian group with the operation induced by disjoint union, and this group is denoted by $B_{r,s}(G ; H)$.

We note that for $G = S^1$ $B_{2r+1,s}(G ; H) = 0$ by orientation reason.

For a closed oriented manifold N of dimension n and a G-vector bundle $E \to X$ of type (r, s, H) we obtain a G-vector bundle $N \times E \to N \times X$ of type $(r, s+n, H)$ by a natural way. This makes the direct sum $\oplus_{s \geq 0} B_{r,s}(G ; H)$ a module over Ω_*.

Let F be a family in G and H be a maximal element in F. Let M be an $(F, F-\{H\})$-free oriented G-manifold and M_H be the set of all points $x \in M$ whose isotropy groups are equal to H. Then the normal bundle $\nu_H(M) \to M_H$ of M_H in M is a G-vector bundle over a closed manifold M_H.

Lemma 1-2-1. The correspondence $M \longmapsto \nu_H(M) \longrightarrow M_H$ induces an $\mathfrak{N}_*$-module isomorphism

$$\mathfrak{N}_n(G\ ;\ F,\ F - \{H\}) \cong \oplus_{n=r+s} B_{r,s}(G\ ;\ H).$$

The inverse isomorphism may be obtained by corresponding a G-vector bundle $E \to X$ to the associated disc bundle $D(E)$.

The proof is easy.

1-3. Homology interpretation of the singular bordism groups

For later uses we present this interpretation in the following fashion.

Theorem 1-3-1. If X is a CW-complex such that for each n $H_n(X\ ;\ Z)$ is finitely generated and has no odd torsion, then there is an $\mathfrak{N}_* \otimes R_2$-module isomorphism

$$\mathfrak{N}_*(X) \otimes R_2 \cong H_*(X\ ;\ R_2) \otimes_{R_2} (\mathfrak{N}_* \otimes R_2)$$

of degree 0.

Proof. Since $H_*(X\ ;\ Z)$ has no odd torsion, the Thom homomorphism

$$\mu : \mathfrak{N}_*(X) \longrightarrow H_*(X\ ;\ Z)$$

is epic by Conner-Floyd [2 ; Theorem 15. 2]. Hence

$$\mu \otimes 1 : \mathfrak{N}_*(X) \otimes R_2 \longrightarrow H_*(X\ ;\ Z) \otimes R_2$$

is epic. As in [2 ; Theorem 17. 1] we obtain the desired isomorphism. q.e.d.

Chapter 2. Bordism of S^1-actions

In this chapter we consider the case in which G is the circle group S^1. For any positive integer i we define a family F_i in S^1 to be the family of all closed subgroups whose orders are at most i. We also define a family F_∞ to be the family of all proper closed subgroups of S^1, i.e., $F_\infty = \bigcup_{i\geq 1} F_i$, and a family F_A to be the family of all closed subgroups of S^1, i.e., $F_A = F_\infty \cup \{S^1\}$.

2-1. The main theorem and the key lemma

Main Theorem 2-1-1. (1) Both $\Omega_*(S^1 ; F_i) \otimes R_2$ (for any $i \geq 1$) and $\Omega_*(S^1 ; F_\infty) \otimes R_2$ are free $\Omega_* \otimes R_2$-modules on odd dimensional generators.

(2) $\Omega_*(S^1 ; F_A) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators.

Key Lemma 2-1-2. (1) $\Omega_*(S^1 ; F_i, F_{i-1}) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on odd dimensional generators.

(2) $\Omega_*(S^1 ; F_A, F_\infty) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators.

The key lemma will be proved in the following sections. We may prove the main theorem by using the key lemma as follows.

First we obtain

Proposition 2-1-3. $\Omega_*(S^1 ; F_1) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on odd dimensional generators.

Proof. $\Omega_*(S^1 ; F_1)$ is the bordism group of all fixed

point free closed oriented S^1-manifolds. By corresponding a fixed point free closed oriented S^1-manifold M to a classifying map of the principal S^1-bundle $M \to M/S^1$, we obtain an Ω_*-module isomorphism

$$\Omega_*(S^1 ; F_1) \cong \Omega_*(BS^1)$$

of degree -1. Since $\Omega_*(BS^1)$ is a free Ω_*-module on even dimensional generators, the proposition follows. q.e.d.

Proposition 2-1-4. $\Omega_{even}(S^1 ; F_i) \otimes R_2 = 0$ for all i.

Proof. First $\Omega_{even}(S^1 ; F_1) \otimes R_2 = 0$ by Proposition 2-1-3.

By applying the exact sequence of Theorem 1-1-3 to the case in which $(F, F', F'') = (F_i, F_{i-1}, \phi)$ and using Lemma 2-1-2 (1), we see that the canonical homomorphism

$$\Omega_{ev}(S^1 ; F_{i-1}) \otimes R_2 \longrightarrow \Omega_{ev}(S^1 ; F_i) \otimes R_2$$

is epic. Then the proposition follows by the induction for i. q.e.d.

Lemma 2-1-5. We obtain a split short exact sequence

$$0 \to \Omega_*(S^1 ; F_{i-1}) \otimes R_2 \to \Omega_*(S^1 ; F_i) \otimes R_2 \to \Omega_*(S^1 ; F_i, F_{i-1}) \otimes R_2 \to 0$$

of $\Omega_* \otimes R_2$-modules.

Proof. Lemma 2-1-2 (1) and Proposition 2-1-4 make the exact sequence of Theorem 1-1-3 for the families $F_i \supset F_{i-1} \supset \phi$ to the above sequence. q.e.d.

From this lemma

$$\Omega_*(S^1 ; F_i)\otimes R_2 \cong \Omega_*(S^1 ; F_{i-1})\otimes R_2 \oplus \Omega_*(S^1 ; F_i, F_{i-1})\otimes R_2.$$

By Lemma 2-1-2, Proposition 2-1-3 and using the induction for i we may assert that $\Omega_*(S^1 ; F_i) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on odd dimensional generators.

Clearly

$$\Omega_*(S^1 ; F_\infty) \otimes R_2 \cong \lim_{i\to\infty} \Omega_*(S^1 ; F_i) \otimes R_2.$$

Since the image of the canonical homomorphism

$$\Omega_*(S^1 ; F_{i-1}) \otimes R_2 \longrightarrow \Omega_*(S^1 ; F_i) \otimes R_2$$

splits by Lemma 2-1-5, $\Omega_*(S^1 ; F_\infty) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on odd dimensional generators.

By Ossa [6 ; Satz 2], the canonical homomorphism

$$\Omega_*(S^1 ; F_\infty) \otimes R_2 \longrightarrow \Omega_*(S^1 ; F_A) \otimes R_2$$

is the zero homomorphism. Then the exact sequence of Theorem 1-1-3 for the families $F_A \supset F_\infty \supset \phi$ becomes a short exact sequence

$$0\to\Omega_*(S^1 ; F_A)\otimes R_2\to\Omega_*(S^1 ; F_A, F_\infty)\otimes R_2\to\Omega_*(S^1 ; F_\infty)\otimes R_2\to 0$$

of $\Omega_* \otimes R_2$-modules. Since $\Omega_*(S^1 ; F_\infty) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module, this short exact sequence is split. Therefore $\Omega_*(S^1 ; F_A) \otimes R_2$ is a direct summand of $\Omega_*(S^1 ; F_A, F_\infty) \otimes R_2$ which is a free $\Omega_* \otimes R_2$-module on even dimensional generators by Lemma 2-1-2 (2). Hence $\Omega_*(S^1 ; F_A) \otimes R_2$ is a projective $\Omega_* \otimes R_2$-module. Moreover it is a free $\Omega_* \otimes R_2$-module by

Conner-Smith [4 ; Proposition 3.2].

Thus Theorem 2-1-1 is obtained from Lemma 2-1-2.

The remaining sections in this chapter will be devoted to the proof of Lemma 2-1-2.

2-2. Oriented S^1-vector bundles

Let H be a closed subgroup of S^1, and $P(H)$ be the set of equivalence classes of (real) representations of H which do not contain a direct summand of trivial actions and on which H acts orientation preservingly. For an element $\rho \in P(H)$ we denote a representative of ρ by the same letter ρ as long as it causes no confusion.

For an element $\rho \in P(H)$, a differentiable vector bundle $E \to X$ is called to be an oriented S^1-vector bundle of type (r, s, H, ρ), if $E \to X$ is an oriented vector bundle and an S^1-vector bundle of type (r, s, H) and the H-action on any fibre of E is equivalent to ρ.

Two oriented S^1-vector bundles $E \to X$, $E' \to X'$ over closed manifolds of type (r, s, H, ρ) are bordant, if there is an oriented S^1-vector bundle $F \to Y$ of type $(r, s+1, H, \rho)$ such that the restriction $F | \partial Y \to \partial Y$ is isomorphic, as oriented S^1-vector bundles, to the disjoint union of $E \to X$ and $-(E' \to X')$, where $-(E' \to X')$ is the oriented S^1-vector bundle obtained from $E' \to X'$ by reversing only the orientation of the total space.

By this relation we may define a bordism group $B^O_{r,s}(S^1 ; H, \rho)$ of all oriented S^1-vector bundles over closed manifolds

of type (r, s, H, ρ). We note that for odd r $B^0_{r,s}(S^1 ; H, \rho) = 0$ by orientation reason. The direct sum $\oplus_{s\geq 0} B^0_{r,s}(S^1 ; H, \rho)$ is a module over Ω_* by the usual way.

We also define $B^0_{r,s}(S^1 ; H)$ to be the direct sum $\oplus_{\rho\in P(H)} B^0_{r,s}(S^1 ; H, \rho)$.

Let (G, G') denote one of (F_i, F_{i-1}) and (F_A, F_∞), and K be the subgroup which belongs to $G - G'$. Let M be a (G, G')-free oriented S^1-manifold and M_K be the set of all points $x \in M$ whose isotropy groups are equal to K. When the normal bundle $\nu_K(M) \to M_K$ of M_K in M is oriented, M is called to be a (G, G')-free oriented S^1-manifold with oriented normal bundle.

Two (G, G')-free oriented S^1-manifolds M, M' with oriented normal bundles are bordant, if there are a (G', G')-free oriented S^1-manifold V and (G, G)-free oriented S^1-manifold W satisfying the conditions (1-1-1), (1-1-2), and if the two oriented S^1-vector bundles $\nu_K(M)$, $\nu_K(M')$ are bordant by the oriented S^1-vector bundle $\nu_K(W)$. By this relation we then define a bordism group $\Omega^0_*(S^1 ; G, G')$ of all (G, G')-free oriented S^1-manifolds with oriented normal bundles. By the cartesian product $\Omega^0_*(S^1 ; G, G')$ becomes a module over Ω_*.

Then we obtain an analogue of Lemma 1-2-1.

Lemma 2-2-1. The correspondence $M \longmapsto \nu_K(M) \to M_K$ induces an Ω_*-module isomorphism

$$\Omega^0_n(S^1 ; G, G') \cong \oplus_{n=2r+s} B^0_{2r,s}(S^1 ; K).$$

The inverse isomorphism may be obtained by corresponding an

oriented S^1-vector bundle $E \to X$ to the associated disc bundle $D(E)$.

The proof is as easy as the proof of Lemma 1-2-1.

Lemma 2-2-2. (1) $\Omega^0_*(S^1 ; F_i, F_{i-1}) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on odd dimensional generators.

(2) $\Omega^0_*(S^1 ; F_A, F_\infty) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators.

This lemma will be proved in the next section 2-3.

Lemma 2-2-3. There are $\Omega_* \otimes R_2$-module homomorphisms f and g of degree 0

$$\Omega^0_*(S^1 ; G, G') \otimes R_2 \underset{g}{\overset{f}{\rightleftarrows}} \Omega_*(S^1 ; G, G') \otimes R_2$$

satisfying $f \circ g$ = identity.

This lemma will be proved in the section 2-4.

These lemmas assume ~~of~~ Lemma 2-1-2 as follows. By Lemma 2-2-3 $\Omega_*(S^1 ; G, G') \otimes R_2$ is a direct summand of $\Omega^0_*(S^1 ; G, G') \otimes R_2$. Since $\Omega^0_*(S^1 ; G, G') \otimes R_2$ is a free $\Omega_* \otimes R_2$-module by Lemma 2-2-2, $\Omega_*(S^1 ; G, G') \otimes R_2$ is a free $\Omega_* \otimes R_2$-module by Conner-Smith [4 ; Proposition 3.2]. The dimensions of generators are obtained from Lemma 2-2-2 as desired.

Thus Lemma 2-1-2 is proved.

Now the remaining subjects to prove the main theorem are to prove Lemmas 2-2-2 and 2-2-3.

2-3. The proof of Lemma 2-2-2

Let H be a closed subgroup of S^1. An element $\rho \in P(H)$ gives a homomorphism

$$\rho : H \longrightarrow SO(2r).$$

We denote the centralizer of the image of ρ in $SO(2r)$ by $C(\rho)$. And we set

$$\Delta = \{(h, \rho(h)) \mid h \in H\}.$$

This is a normal subgroup of $S^1 \times C(\rho)$.

Let $E \to X$ be an oriented S^1-vector bundle representing a class in $B^0_{2r,s}(S^1 ; H, \rho)$, and $\tilde{E} \to X$ be the principal $SO(2r)$-bundle associated to $E \to X$. By the natural way $\tilde{E}$ is given a left S^1-action and a right $SO(2r)$-action. We set

$$F = \{e \in \tilde{E} \mid h \cdot e = e \cdot \rho(h) \quad \text{for all} \quad h \in H\}.$$

The left S^1-action on $\tilde{E}$ induces a left S^1-action on F. The right $SO(2r)$-action on $\tilde{E}$ also induces a left $C(\rho)$-action defined by $\gamma \cdot e = e \cdot \gamma^{-1}$ for $\gamma \in C(\rho)$. So we have a left $S^1 \times C(\rho)$-action on F, and all isotropy groups of points in F are equal to Δ. Then we have a principal $S^1 \times C(\rho)/\Delta$-bundle

$$F \to F/(S^1 \times C(\rho)/\Delta) = X/S^1.$$

Then

Lemma 2-3-1 (Conner-Floyd [3], Ossa [5]). By corresponding $E \to X$ to $F \to X/S^1$ we obtain an Ω_*-module isomorphism

$$B^0_{2r,s}(S^1 ; H, \rho) \cong \Omega_{s+\dim H-1}(B(S^1 \times C(\rho)/\Delta)).$$

For the proof of Lemma 2-2-2, it suffices to prove the

following lemma.

Lemma 2-3-2. For any closed subgroup H of S^1 and any $\rho \in P(H)$, $\Omega_*(B(S^1 \times C(\rho)/\Delta)) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional genrators.

Proof. Considering S^1 to be the unit sphere in C^1, we let

$$\rho_j : S^1 \longrightarrow U(1)$$

be the representation of S^1 defined by

$$\rho_j(z) = (z^j), \quad z \in S^1.$$

Let

$$i : H \longrightarrow S^1, \quad \text{and} \quad \iota : U(1) \longrightarrow O(2)$$

be the natural inclusions. Then

$$\left\{ \iota\rho_j i \mid \frac{1}{\not{2}} \leq j \leq \frac{o(H)-1}{2} \right\}$$

gives a complete set of (real) irreducible representations of H if the order $o(H)$ of H is odd or H is equal to S^1.

For H of even order we let

$$\lambda : H \longrightarrow O(1)$$

be the representation of H defined by

$$\lambda(h) = (h^{o(H)/2}), \quad h \in H.$$

Then

$$\left\{ \iota\rho_j i, \lambda \mid \frac{1}{\not{2}} \leq j \leq \frac{o(H)}{2} - 1 \right\}$$

gives a complete set of (real) irreducible representations of H of even order.

From the above remarks and elementary computations we see that for any $\rho \in P(H)$ $C(\rho)$ is isomorphic to

$$U(r_1) \times \cdots \times U(r_{a-1}) \times SO(r_a)$$

for some sequence $(r_1, \cdots, r_{a-1}, r_a)$ with $\dim \rho = 2\sum_{j=1}^{a-1} r_j + r_a$. Moreover we see that $r_a = 0$ if H is of odd order or S^1.

For $1 \leqq b \leqq a$ let

$p_b : H \longrightarrow U(r_b)$ (or $SO(r_a)$ if $b = a$)

be the composition

$$H \xrightarrow{\rho} C(\rho) \cong U(r_1) \times \cdots \times U(r_{a-1}) \times SO(r_a) \xrightarrow{\text{proj}} U(r_b) \text{ (or } SO(r_a)).$$

Then p_b is extendable to S^1 such that if $b \neq a$ the image of p_b lies in the center of $U(r_b)$, and we denote this extension by the same symbol p_b. We set

$$\Delta' = \{(h, p_a(h)) \mid h \in H\}.$$

Then there is an epimorphism from $S^1 \times C(\rho)$ to $U(r_1) \times \cdots \times U(r_{a-1}) \times (S^1 \times SO(r_a)/\Delta')$ which sends $(z, x_1, \cdots, x_a)$ to $(p_1(z)^{-1}x_1, \cdots, p_{a-1}(z)^{-1}x_{a-1}, [z, x_a])$, and the kernel of this homomorphism is Δ. So we obtain an isomorphism

$$(2\text{-}3\text{-}3) \quad S^1 \times C(\rho)/\Delta \cong U(r_1) \times \cdots \times U(r_{a-1}) \times (S^1 \times SO(r_a)/\Delta').$$

(1) The case in which H is of odd order or S^1: We obtain $\Omega_* \otimes R_2$-module isomorphisms

$$\Omega_*(B(S^1 \times C(\rho)/\Delta)) \otimes R_2 \cong \Omega_*(BX) \otimes R_2$$
$$\cong H_*(BX ; R_2) \otimes_{R_2} (\Omega_* \otimes R_2),$$

where $BX = B(U(r_1) \times \cdots \times U(r_{a-1}) \times (S^1/H))$. The first isomorphism is obtained from (2-3-3), and the last one is obtained from Theorem 1-3-1. Since $H_*(BX\ ;\ R_2)$ is a free R_2-module on even dimensional generators, $\mathfrak{N}_*(B(S^1 \times C(\rho)/\Delta)) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module on even dimensional generators.

(2) The case in which H is of even order (The following method is largely due to Ossa [5],[6]): We set

$$\Delta'' = \{(1, 1), (-1, -1)\} \subset S^1 \times SO(r_a).$$

Then we obtain an isomorphism

$$(2\text{-}3\text{-}4) \quad S^1 \times SO(r_a)/\Delta' \cong S^1 \times SO(r_a)/\Delta''.$$

Let T' be a maximal torus in $S^1 \times SO(r_a)$ such that T' contains Δ''. Then $T = T'/\Delta''$ is a maximal torus in $S^1 \times SO(r_a)/\Delta''$.

$H^*(S^1 \times SO(r_a)/\Delta''\ ;\ Z)$ and $H^*((S^1 \times SO(r_a)/\Delta'')/T\ ;\ Z)$ have no odd torsion. So the canonical homomorphism

$$H^*(B(S^1 \times SO(r_a)/\Delta'')\ ;\ Z_p) \longrightarrow H^*(BT\ ;\ Z_p)$$

is injective for any odd prime p by Borel [1; Proposition 29.2]. Hence $H_*(B(S^1 \times SO(r_a)/\Delta'')\ ;\ Z)$ has no odd torsion and $H_{odd}(B(S^1 \times SO(r_a)/\Delta'')\ ;\ Z)$ is a 2-torsion group.

We obtain $\mathfrak{N}_* \otimes R_2$-module isomorphisms

$$\begin{aligned}\mathfrak{N}_*(B(S^1 \times C(\rho)/\Delta)) \otimes R_2 &\cong \mathfrak{N}_*(BY) \otimes R_2 \\ &\cong H_*(BY\ ;\ R_2) \otimes_{R_2} (\mathfrak{N}_* \otimes R_2),\end{aligned}$$

where $BY = B(U(r_1) \times \cdots \times U(r_{a-1}) \times (S^1 \times SO(r_a)/\Delta''))$. The first isomorphism is obtained from (2-3-3) and (2-3-4), and the last

one is obtained from the above argument and Theorem 1-3-1.

$H_{odd}(BY\ ;\ R_2) = 0$ since $H_{odd}(B(S^1 \times SO(r_a)/\Delta''\ ;\ Z)$ is a 2-torsion group. Then $\mathfrak{N}_*(B(S^1 \times C(\rho)/\Delta)) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module on even dimensional generators. q.e.d.

Thus Lemma 2-2-2 is completely proved.

2-4. The proof of Lemma 2-2-3

First we obtain

Lemma 2-4-1 (Ossa [5][6]). For an S^1-vector bundle $E \longrightarrow X$ (X closed) of type (r, s, H) there exists an oriented S^1-vector bundle $E^{(1)} \longrightarrow X^{(1)}$ of type (r, s, H) such that

$$2[E \longrightarrow X] = [E^{(1)} \longrightarrow X^{(1)}] \quad \text{in} \quad B_{r,s}(S^1\ ;\ H).$$

Proof. Let $\pi : X \longrightarrow X/S^1$ be the natural projection. Let Y' be a minimal 1-codimensional submanifold of X/S^1 which represents the first Stiefel-Whitney class of X/S^1, and set $Y = \pi^{-1}(Y')$. Then Y is a minimal 1-codimensional invariant closed submanifold of X such that $X - Y$ is orientable.

Let U be an invariant closed tubular neighborhood of Y in X, and set $X_1 = X - \text{int } U$. Considering U as a disc bundle over Y, $(E|Y) \oplus U$ is equivariantly diffeomorphic to $E|U$. So the antipodal involution on U induces an involution T on $E|U$, which is equivariant with the S^1-action and reverses the orientation of $E|U$.

Using the involution T we may obtain an S^1-vector bundle $W \longrightarrow V$ of type $(r, s+1, H)$ as follows. W is formed from two

copies $\{0\} \times E \times I$ and $\{1\} \times E \times I$ of $E \times I$ by identifying $(0, e, 0)$ with $(1, T(e), 0)$ for all $e \in E|U$, and V is formed from two copies $\{0\} \times X \times I$ and $\{1\} \times X \times I$ of $X \times I$ by identifying $(0, x, 0)$ with $(1, T(x), 0)$ for all $x \in U$, where I is the interval $[0, 1]$.

Let $E^{(1)} \longrightarrow X^{(1)}$ be the subbundle of $W \longrightarrow V$ defined by

$$(2\text{-}4\text{-}2) \quad E^{(1)} = \{0\} \times (E|X_1) \times \{0\} \bigcup_T \{1\} \times (E|X_1) \times \{0\}, \text{ and}$$

$$(2\text{-}4\text{-}3) \quad X^{(1)} = \{0\} \times X_1 \times \{0\} \bigcup_T \{1\} \times X_1 \times \{0\}.$$

Then $E^{(1)} \longrightarrow X^{(1)}$ is an S^1-vector bundle of type (r, s, H), and is bordant to $2(E \to X)$ by the bordism $W \longrightarrow V$.

X_1 is an orientable manifold. We orient the two copies of X_1 in (2-4-3) so that those orientations are reverse each other, then $X^{(1)}$ is oriented since Y is minimal. So $E^{(1)} \longrightarrow X^{(1)}$ is an oriented S^1-vector bundle. q.e.d.

We may construct $\Omega_* \otimes R_2$-module homomorphisms f and g

$$B^O_{r,s}(S^1 ; H) \otimes R_2 \underset{g}{\overset{f}{\rightleftarrows}} B_{r,s}(S^1 ; H) \otimes R_2$$

satisfying $f \circ g$ = identity. Then Lemma 2-2-3 follows from Lemma 1-2-1 and Lemma 2-2-1.

First we define f by $f = f' \otimes 1$ where

$$f' : B^O_{r,s}(S^1 ; H) \longrightarrow B_{r,s}(S^1 ; H)$$

is the canonical homomorphism forgetting orientation of bundles.

Next we define g as follows. Let $E \longrightarrow X$ be an S^1-vector bundle over a closed manifold X of type (r, s, H), and $E^{(1)}$

$\longrightarrow X^{(1)}$ be an oriented S^1-vector bundle constructed by Lemma 2-4-1. We must note that $E^{(1)} \longrightarrow X^{(1)}$ can not be canonically oriented.

We devide $X^{(1)}$ into the connected components,

$$X^{(1)} = A_1 \cup \cdots \cup A_n .$$

Each A_α $(1 \leqq \alpha \leqq n)$ is invariant under the S^1-action, and $E^{(1)}|A_\alpha \longrightarrow A_\alpha$ is an orientable S^1-vector bundle of type (r, s, H). Since A_α is connected, $E^{(1)}|A_\alpha \longrightarrow A_\alpha$ is given exactly two orientations. Let $\sigma(E^{(1)}|A_\alpha \longrightarrow A_\alpha)$ be the class in $B^0_{r,s}(S^1 ; H)$ which is represented by the sum of the two kinds of oriented S^1-vector bundles obtained from $E^{(1)}|A_\alpha \longrightarrow A_\alpha$. Then we define $\sigma(E^{(1)} \longrightarrow X^{(1)})$ to be the sum of $\sigma(E^{(1)}|A_\alpha \longrightarrow A_\alpha)$, $\alpha = 1, \cdots, n$. We see that

$$f'(\sigma(E^{(1)} \longrightarrow X^{(1)})) = 4[E \to X] \quad \text{in} \quad B_{r,s}(S^1 ; H).$$

So we define g by

$$g([E \to X] \otimes x) = \sigma(E^{(1)} \longrightarrow X^{(1)}) \otimes \frac{x}{4}, \quad x \in R_2 .$$

This is a well-defined homomorphism and satisfies $f \circ g =$ identity.

Chapter 3. Bordism of $(Z_2)^k$-actions

In this chapter we consider $(Z_2)^k$-actions.

We remark that

(1) any subgroup of $(Z_2)^k$ is isomorphic to $(Z_2)^a$ for some $a \leqq k$, and

(2) for any subgroup H of $(Z_2)^k$, there exists a "complement" H^c such that $H \oplus H^c$ is equal to $(Z_2)^k$.

For each H we fix one complement H^c throughout this chapter.

3-1 The main theorem and the key lemma

Main Theorem 3-1-1. For any family F in $(Z_2)^k$, $\Omega_*((Z_2)^k ; F) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators.

Let

$$H_1, H_2, \cdots, H_a$$

be a sequence of all subgroups belonging to F such that the order $o(H_i)$ of H_i is larger than or equal to the order $o(H_{i-1})$ of H_{i-1} for any $i = 2, \cdots, a$. Then the collection

$$F_i = \{ H_j \mid j \leqq i \}$$

is a family in $(Z_2)^k$ for all $i = 1,2,\cdots, a$, and $F_a = F$.

Key Lemma 3-1-2. $\Omega_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on even dimensional generators.

This key lemma will be proved in the following sections. We may prove the main theorem by using the key lemma as follows.

First we obtain

Proposition 3-1-3. $\Omega_*((Z_2)^k ; F_1) \otimes R_2$ is a free $\Omega_* \otimes R_2$-module on a 0-dimensional generator.

Proof. As the proof of Lemma 2-1-3 we obtain an Ω_*-module

isomorphism

$$\mathfrak{N}_*((Z_2)^k ; F_1) \cong \mathfrak{N}_*(B((Z_2)^k))$$

of degree 0. Since $\mathfrak{N}_*(B((Z_2)^k)) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module on a 0-dimensional generator, the proposition follows.

q.e.d.

We also obtain the following proposition and lemma by the similar ways to Proposition 2-1-4 and Lemma 2-1-5.

Proposition 3-1-4. $\mathfrak{N}_{odd}((Z_2)^k ; F_i) \otimes R_2 = 0$ for all i.

Lemma 3-1-5. We obtain a split short exact sequence

$$0 \to \mathfrak{N}_*((Z_2)^k ; F_{i-1}) \otimes R_2 \to \mathfrak{N}_*((Z_2)^k ; F_i) \otimes R_2 \to \mathfrak{N}_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2 \to 0$$

of $\mathfrak{N}_* \otimes R_2$-modules.

From this lemma we obtain an $\mathfrak{N}_* \otimes R_2$-module isomorphism

$$\mathfrak{N}_*((Z_2)^k; F_i) \otimes R_2 \cong \mathfrak{N}_*((Z_2)^k; F_{i-1}) \otimes R_2 \oplus \mathfrak{N}_*((Z_2)^k; F_i, F_{i-1}) \otimes R_2.$$

By Lemma 3-1-2, Proposition 3-1-3 and using the induction for i we may assert that $\mathfrak{N}_*((Z_2)^k ; F_i) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module on even dimensional generators for all i.

Thus Theorem 3-1-1 is obtained from Lemma 3-1-2. The following sections will be devoted to the proof of Lemma 3-1-2.

3-2. Special $(Z_2)^k$-vector bundles

Let H be a subgroup of $(Z_2)^k$, and $\{V_1, \cdots, V_q\}$ be a

complete set of non-trivial (real) irreducible representation spaces of H. We note that any V_j $(1 \leqq j \leqq q)$ is a 1-dimensional vector space and q is equal to $o(H) - 1$, $o(H)$ the order of H.

Then we obtain

Lemma 3-2-1. For any $(Z_2)^k$-vector bundle $E \to X$ of type (r, s, H) we have a canonical equivariant decomposition of E by $(Z_2)^k$-vector bundles E_j $(j = 1, \cdots, q)$

$$E \cong \bigoplus_{i=1}^{q} E_j$$

such that the H-action on any fibre of E_j is equivalent to V_j for $j = 1, \cdots, q$.

Proof. Let

$$\underline{V}_j = V_j \times X \longrightarrow X$$

be the product bundle over X. Giving the trivial H^C-action on V_j, we may regard V_j as a $(Z_2)^k$-vector space. We define a $(Z_2)^k$-action on $\underline{V}_j$ by the diagonal action.

Let $\mathrm{Hom}_H(\underline{V}_j, E)$ be the bundle of H-equivariant homomorphisms. For any element $f \in \mathrm{Hom}_H(\underline{V}_j, E)$ and $g \in (Z_2)^k$ we define $g \cdot f \in \mathrm{Hom}_H(\underline{V}_j, E)$ to be the composition

$$\underline{V}_j \xrightarrow{g^{-1}\cdot} \underline{V}_j \xrightarrow{f} E \xrightarrow{g\cdot} E.$$

This defines a $(Z_2)^k$-action on $\mathrm{Hom}_H(\underline{V}_j, E)$.

Let E_j be the $(Z_2)^k$-vector bundle $\underline{V}_j \otimes \mathrm{Hom}_H(\underline{V}_j, E)$. Then the direct sum of the canonical homomorphisms

$$E_j \longrightarrow E, \quad j = 1, \cdots, q$$

gives an equivariant isomorphism

$$\oplus_{j=1}^{q} E_j \longrightarrow E,$$

since all V_j's are 1-dimensional. q.e.d.

A $(Z_2)^k$-vector bundle $E \to X$ of type (r, s, H) is called to be a special $(Z_2)^k$-vector bundle, if in the canonical decomposition

$$E \cong \oplus_{j=1}^{q} E_j$$

each E_j is an oriented vector bundle and the H^c-action on E_j preserves the orientation of the bundle E_j.

Let V be a representation space of H which has no direct summand of trivial action, then V is isomorphic to $V_1^{r_1} \oplus \cdots \oplus V_q^{r_q}$ for some $r_1, \cdots, r_q$. For any $x \in H$ we set

$$J(x) = \{ j \mid x \text{ non-trivially acts on } V_j \ (1 \leqq j \leqq q). \},$$

and we denote the number of the elements of the set

$$J(x) \cap \{ j \mid r_j \text{ is odd } (1 \leqq j \leqq q). \}$$

by $\alpha_x(V)$.

We define $S(H)$ to be the set of equivalence classes of representation spaces V of H which satisfy that $\alpha_x(V)$ is even for all $x \in H$. We denote an equivalence class and its representative by a same letter V as long as it causes no confusion. We note that the H-action on V is orientation preserving for any $V \in S(H)$.

For $V \in S(H)$ a $(Z_2)^k$-vector bundle $E \to X$ is called to be of type (r, s, H, V), if $E \to X$ is a $(Z_2)^k$-vector bundle of type (r, s, H) and the H-action on any fibre of E is equivalent to V.

Two special $(Z_2)^k$-vector bundles $E \to X$, $E' \to X'$ over closed manifolds of type (r, s, H, V) are bordant, if there is a special $(Z_2)^k$-vector bundle $F \to Y$ of type $(r, s+1, H, V)$ such that the restriction $F|\partial Y \to \partial Y$ is isomorphic, as special $(Z_2)^k$-vector bundles, to the disjoint union of $E \to X$ and $-(E' \to X')$, where $-(E' \to X')$ is the special $(Z_2)^k$-vector bundle obtained from $E' \to X'$ by reversing only the orientation of the total space.

By this relation "bordant" we may define a bordism group $B^S_{r,s}((Z_2)^k ; H, V)$ of all special $(Z_2)^k$-vector bundles over closed manifolds of type (r, s, H, V). The direct sum $\oplus_{s \geq 0} B^S_{r,s}((Z_2)^k ; H, V)$ is a module over $\mathfrak{N}_*$ by the usual way. We also define $B^S_{r,s}((Z_2)^k ; H)$ to be the direct sum $\oplus_{V \in S(H)} B^S_{r,s}((Z_2)^k ; H, V)$.

Let M be an (F_i, F_{i-1})-free oriented $(Z_2)^k$-manifold and M_{H_i} be the set of all points $x \in M$ whose isotropy groups are H_i. When the normal bundle $\nu_{H_i}(M) \to M_{H_i}$ of M_{H_i} in M is a special $(Z_2)^k$-vector bundle, M is called to be an (F_i, F_{i-1})-free oriented $(Z_2)^k$-manifold with special normal bundle.

Two (F_i, F_{i-1})-free oriented $(Z_2)^k$-manifolds M, M' with special normal bundles are bordant, if there are an (F_{i-1}, F_{i-1})-free oriented $(Z_2)^k$-manifold V and an (F_i, F_i)-free oriented

$(Z_2)^k$-manifold W satisfying the conditions (1-1-1),(1-1-2), and if the two special $(Z_2)^k$-vector bundles $\nu_{H_i}(M)$, $\nu_{H_i}(M')$ are bordant by the special $(Z_2)^k$-vector bunlde $\nu_{H_i}(W)$. By this relation we may define a bordism group $\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1})$ of all (F_i, F_{i-1})-free oriented $(Z_2)^k$-manifolds with special normal bundles. By the cartesian product $\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1})$ becomes a module over $\mathfrak{N}_*$.

Then we obtain an analogue of Lemma 1-2-1 (or Lemma 2-2-1).

Lemma 3-2-2. The correspondence $M \longmapsto \nu_{H_i}(M) \longrightarrow M_{H_i}$ induces an $\mathfrak{N}_*$-module isomorphism

$$\mathfrak{N}^S_n((Z_2)^k ; F_i, F_{i-1}) \cong \oplus_{n=r+s} B^S_{r,s}((Z_2)^k ; H_i).$$

The inverse isomorphism may be obtained by corresponding a special $(Z_2)^k$-vector bundle $E \longrightarrow X$ to the associated disc bundle $D(E)$.

Lemma 3-2-3. $\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$ *is a free* $\mathfrak{N}_* \otimes R_2$-module.

This lemma will be proved in the next section 3-3.

Lemma 3-2-4. There are $\mathfrak{N}_* \otimes R_2$-module homomorphisms f and g of degree 0

$$\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2 \underset{g}{\overset{f}{\rightleftarrows}} \mathfrak{N}_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$$

satisfying

(1) $f \circ g$ = identity, and

(2) $f(\mathfrak{N}^S_{odd}((Z_2)^k ; F_i, F_{i-1}) \otimes R_2) = 0$.

This lemma will be proved in 3-4.

These lemmas assure ~~of~~ Lemma 3-1-2 as follows. By Lemma 3-2-4 $\mathfrak{N}_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$ is a direct summand of $\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$. Since $\mathfrak{N}^S_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module by Lemma 3-2-3, $\mathfrak{N}_*((Z_2)^k ; F_i, F_{i-1}) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module by Conner-Smith [4 ; Proposition 3.2]. The dimensions of generators are obtained from Lemma 3-2-4 (2) as desired.

Thus Lemma 3-1-2 is proved.

Now the remaining subjects to prove the main theorem are to prove Lemmas 3-2-3 and 3-2-4.

3-3. The proof of Lemma 3-2-3

Let H be a subgroup of $(Z_2)^k$ and $\{V_1, \cdots, V_q\}$ be a complete set of non-trivial irreducible representation spaces ($q = o(H) - 1$). For any element $V \in S(H)$ we set

$$SO(V) = SO(r_1) \times \cdots \times SO(r_q)$$

where $r_1, \cdots, r_q$ are defined by

$$V \cong V_1^{r_1} \oplus \cdots \oplus V_q^{r_q}.$$

Lemma 3-3-1. There is an $\mathfrak{N}_*$-module isomorphism

$$B^S_{r,s}((Z_2)^k ; H, V) \cong \mathfrak{N}_s(B(H^c \times SO(V))).$$

By Theorem 1-3-1 $\mathfrak{N}_*(B(H^c \times SO(V))) \otimes R_2$ is a free $\mathfrak{N}_* \otimes R_2$-module isomorphic to $H_*(B(H^c \times SO(V)) ; R_2) \otimes_{R_2} (\mathfrak{N}_* \otimes R_2)$.

Hence we obtain Lemma 3-2-3 from Lemmas 3-2-2, 3-3-1.

Proof of Lemma 3-3-1. Let $E \to X$ be a special $(Z_2)^k$-vector bundle of type (r, s, H, V). We obtain the canonical equivariant decomposition of E by $(Z_2)^k$-vector bundles

$$E \cong \bigoplus_{j=1}^{q} E_j$$

from Lemma 3-2-1.

H^C freely acts on X, and each E_j is an oriented vector bundle on which H^C acts orientation preservingly. Let

$$h : X/H^C \longrightarrow BH^C \quad \text{and}$$

$$h_j : X/H^C \longrightarrow BSO(r_j)$$

be classifying maps of the principal bundle $X \to X/H^C$ and the oriented vector bundles $E_j/H^C \to X/H^C$, respectively. Then h and h_j $(j = 1, \cdots, q)$ define a map

$$h \times \prod_{j=1}^{q} h_j : X/H^C \to BH^C \times \prod_{j=1}^{q} BSO(r_j) \approx B(H^C \times SO(V)).$$

By corresponding $E \to X$ to $h \times \prod h_j$ we obtain a homomorphism

$$B^S_{r,s}((Z_2)^k ; H, V) \longrightarrow \Omega_s(B(H^C \times SO(V))).$$

The inverse homomorphism is obtained as follows. Let

$$\bar{h} : M \longrightarrow B(H^C \times SO(V))$$

represent a class in $\Omega_s(B(H^C \times SO(V)))$. Then $\bar{h}$ defines maps

$$h : M \longrightarrow BH^C \quad \text{and}$$

$$h_j : M \longrightarrow BSO(r_j) \quad (j = 1, \cdots, q).$$

Let $\pi : \tilde{M} \longrightarrow M$ be the principal H^C- bundle induced by h, and $E_j \longrightarrow \tilde{M}$ be the oriented vector bundle induced by $h_j \circ \pi$.

We may define a $(Z_2)^k$-action on $\oplus_{j=1}^{q} E_j \longrightarrow \tilde{M}$ so that the bundle is a special $(Z_2)^k$-vector bundle of type (r, s, H, V) and the correspondence $\bar{h}$ to $\oplus_{j=1}^{q} E_j \longrightarrow \tilde{M}$ defines the desired inverse homomorphism. q.e.d.

3-4. The proof of Lemma 3-2-4

For the proof we need the following lemma.

Lemma 3-4-1. For a $(Z_2)^k$-vector bundle $E \longrightarrow X$ (X closed) of type (r, s, H) there exists a special $(Z_2)^k$-vector bundle $E^{(q)} \longrightarrow X^{(q)}$ of type (r, s, H) such that

$$2^q[E \to X] = [E^{(q)} \longrightarrow X^{(q)}] \quad \text{in} \quad B_{r,s}((Z_2)^k ; H),$$

where $q = o(H) - 1$.

Proof. We have the canonical decomposition

$$E \cong \oplus_{j=1}^{q} E_j$$

by Lemma 3-2-1.

For $0 \leqq m \leqq q$ let $D(m)$ be the following statement:

There exists a $(Z_2)^k$-vector bundle $E^{(m)} \longrightarrow X^{(m)}$ of type (r, s, H) such that

(1) in the canonical decomposition

$$E^{(m)} \cong \oplus_{j=1}^{q} E_j^{(m)}$$

the direct summands $E_1^{(m)}, \cdots, E_m^{(m)}$ are oriented vector bundles on which H^c acts orientation preservingly, and

(2) $2^m[E \to X] = [E^{(m)} \longrightarrow X^{(m)}]$ in $B_{r,s}((Z_2)^k ; H)$.

We may prove Lemma 3-4-1 by the induction for m. The statement $D(0)$ is trivially valid. In the following we show that the statement $D(m)$ implies the statement $D(m+1)$.

Since H^c freely acts on $X^{(m)}$, $E^{(m)}_{m+1}/H^c \longrightarrow X^{(m)}/H^c$ is also a vector bundle. Let Y' be a minimal 1-codimensional submanifold of $X^{(m)}/H^c$ which represents the first Stiefel-Whitney class of the vector bundle $E^{(m)}_{m+1}/H^c \longrightarrow X^{(m)}/H^c$, and set $Y = \pi^{-1}(Y')$ where $\pi : X^{(m)} \longrightarrow X^{(m)}/H^c$ is the natural projection. Then Y is a minimal 1-codimensional invariant closed submanifold of $X^{(m)}$ satisfying that the restricted vector bundle of $E^{(m)}_{m+1}$ on $X^{(m)} - Y$ is orientable so that H^c acts orientation preservingly.

Let U be an invariant closed tubular neighborhood of Y in $X^{(m)}$, and set $X_1 = X^{(m)} - \text{int } U$. Considering U as a disc bundle over Y, $(E^{(m)}|Y) \oplus U$ is equivariantly diffeomorphic to $E^{(m)}|U$. So the antipodal involution on U induces an involution T on $E^{(m)}|U$. As in the proof of Lemma 2-4-1, pasting two copies of $E^{(m)}|X_1$ each other by T, we may construct a new $(Z_2)^k$-vector bundle $E^{(m+1)} \longrightarrow X^{(m+1)}$ of type (r, s, H) such that

$$2[E^{(m)} \longrightarrow X^{(m)}] = [E^{(m+1)} \longrightarrow X^{(m+1)}] \quad \text{in} \quad B_{r,s}((Z_2)^k ; H).$$

Then

$$2^{m+1}[E \to X] = [E^{(m+1)} \longrightarrow X^{(m+1)}]$$

by the statement $D(m)$.

In the canonical decomposition

$$E^{(m+1)} \cong \oplus_{j=1}^{q} E^{(m+1)}_j$$

each $E_j^{(m+1)}$ is equivariantly isomorphic to the bundle

(3-4-2) $$(E_j^{(m)} | X_1) \cup_T (E_j^{(m)} | X_1).$$

For $1 \leqq j \leqq m$ we give a same orientation to the two copies of the vector bundle $E_j^{(m)} | X_1$ in (3-4-2), then $E_j^{(m+1)}$ is an oriented vector bundle on which H^c acts orientation preservingly. For $j = m + 1$ we orient the two copies of $E_j^{(m)} | X_1$ in (3-4-2) such that those orientations are reverse each other, then $E_j^{(m+1)}$ is an oriented vector bundle since Y is minimal, and H^c acts orientation preservingly on $E_j^{(m+1)}$.

Thus the statement $D(m)$ implies the statement $D(m + 1)$.

q.e.d.

We may construct $\Omega_* \otimes R_2$-module homomorphisms f and g

$$B^S_{r,s}((Z_2)^k ; H) \otimes R_2 \underset{g}{\overset{f}{\rightleftarrows}} B_{r,s}((Z_2)^k ; H) \otimes R_2$$

satisfying that

(3-4-3) $f \circ g =$ identity, and

(3-4-4) if $r + s$ is odd, f is the zero homomorphism.

Then Lemma 3-2-4 follows from Lemma 1-2-1 and Lemma 3-2-2.

First we define f by $f = f' \otimes 1$ where

$$f' : B^S_{r,s}((Z_2)^k ; H) \longrightarrow B_{r,s}((Z_2)^k ; H)$$

is the canonical homomorphism forgetting the "speciality" of bundles. Then we obtain

Lemma 3-4-5. If $r + s$ is odd, every element in $f'(B^S_{r,s}((Z_2)^k ; H))$ is of order 2. So the statement (3-4-4)

follows.

Proof. Let V be any element in $S(H)$, and

$$\lambda : B^S_{r,s}((Z_2)^k ; H, V) \cong \Omega_s(B(H^c \times SO(V)))$$

be the isomorphism obtained by Lemma 3-3-1. And let x be any element in $B^S_{r,s}((Z_2)^k ; H, V)$. If s is odd, then $\lambda(x)$ is of order 2, and $f'(x)$ is so.

If $r + s$ is odd and s is even, we may construct an element $\bar{x} \in B^S_{r,s}((Z_2)^k ; H, V)$ such that

(3-4-6) $x + \bar{x} = 0$, and

(3-4-7) $f'(x) = f'(\bar{x})$.

Then $f'(x)$ is of order 2.

$\bar{x}$ is constructed as follows. Let $E \to X$ be a representative of x, and

$$E \cong \oplus_{j=1}^{q} E_j$$

be the canonical decomposition. We define an oriented vector bundle $\bar{E}_j$ such that if $\dim E_j$ is even $\bar{E}_j$ is E_j, and if $\dim E_j$ is odd $\bar{E}_j$ is the oriented vector bundle obtained from E_j by reversing the orientation. Let $\bar{x}$ be the class represented by $\oplus_{j=1}^{q} \bar{E}_j \to X$ which is given the same orientation of the total space as E.

The number of j's with $\dim E_j$ = odd is odd since r is odd. From this fact the condition (3-4-6) follows. The condition (3-4-7) follows easily. q.e.d.

Next we define g as follows. Let $E \to X$ be a $(Z_2)^k$-vector bundle over a closed manifold X of type (r, s, H), and $E^{(q)} \to X^{(q)}$ be a special $(Z_2)^k$-vector bundle constructed by Lemma 3-4-1. Let

$$E^{(q)} \cong \oplus_{j=1}^{q} E_j^{(q)}$$

be the canonical decomposition. We must note that $E^{(q)} \to X^{(q)}$ can not be canonically "specialized", i.e., each $E_j^{(q)}$ can not be canonically oriented. We devide $X^{(q)}$ into the form

$$X^{(q)} = A_1 \cup \cdots \cup A_n$$

such that each A_α $(1 \leq \alpha \leq n)$ is invariant under the $(Z_2)^k$-action and each $A_\alpha/(Z_2)^k$ is connected. Then each $E^{(q)}|A_\alpha \to A_\alpha$ is a $(Z_2)^k$-vector bundle, and specialized to exactly 2^q kinds of special $(Z_2)^k$-vector bundles. We sum up the 2^q special $(Z_2)^k$-vector bundles obtained from $E^{(q)}|A_\alpha \to A_\alpha$, and denote the class of the sum in $B^S_{r,s}((Z_2)^k ; H)$ by

$$\sigma(E^{(q)}|A_\alpha \to A_\alpha).$$

And we set

$$\sigma(E^{(q)} \to X^{(q)}) = \sum_{\alpha=1}^{n} \sigma(E^{(q)}|A_\alpha \to A_\alpha).$$

Then we see that

$$f'(\sigma(E^{(q)} \to X^{(q)})) = 2^{2q}[E \to X].$$

So we define g by

$$g([E \to X] \otimes x) = \sigma(E^{(q)} \to X^{(q)}) \otimes x/2^{2q}, \quad x \in R_2.$$

This is a well-defined homomorphism and satisfies (3-4-3).

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