Formulation for the hydrodynamics in the 2.5 post-Newtonian approximation of general relativity

(Doctoral Thesis)

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ABSTRACT

Using the (3+1) formalism in general relativity, we perform the post-Newtonian (PN) approximation to clarify what sort of gauge condition is suitable for numerical analysis of coalescing compact binary neutron stars and gravitational waves from them. We adopt a kind of transverse gauge condition to determine the shift vector. On the other hand, for determination of the time slice, we adopt three slice conditions (conformal slice, maximal slice and harmonic slice) and discuss their properties. It is found that the conformal slice seems appropriate for analysis of gravitational waves in the wave zone and the maximal slice will be useful for describing the equilibrium configurations. Using these conditions, the PN hydrodynamic equations are obtained up through the 2.5PN order including the quadrupole gravitational radiation reaction. In particular, it is shown that we can solve the 2PN tensor potential by the method used in the Newtonian hydrodynamics. The PN approximation in the (3+1) formalism will be also useful to perform numerical simulations using various slice conditions and, as a result, to provide an initial data for the final merging phase of coalescing binary neutron stars which can be treated only by fully general relativistic simulations.

We also present a formalism to obtain equilibrium configurations of uniformly rotating fluid in the second post-Newtonian approximation. In our formalism, we need to solve 29 Poisson equations, but their source terms decrease rapidly enough at the external region of the matter (i.e., at worst $O(r^{-4})$). Hence these Poisson equations can be solved accurately as the boundary value problem using standard numerical methods. This formalism will be useful to obtain nonaxisymmetric uniformly rotating equilibrium configurations such as synchronized binary neutron stars just before merging and the Jacobi ellipsoid.

Contents

I. INTRODUCTION

Kilometer-size interferometric gravitational wave detectors, such as LIGO (Abramovici et al. 1992; Thorne 1994; Will 1994) and VIRGO (Bradaschia 1990), are now under construction aiming at direct detection of gravitational waves from relativistic astrophysical objects or the early universe. Coalescing binary neutron stars are the most promising sources of gravitational waves for such detectors. The reasons are that (1) we expect to detect the signal of coalescence of binary neutron stars about several times per year (Phinney 1991), and (2) the wave form from coalescing binaries can be predicted with a high accuracy compared with other sources (Abramovici et al. 1992; Thorne 1994; Will 1994).

In the case when the orbital separation of each star is large compared with the radius of neutron stars, i.e., in the so-called inspiraling phase, binary neutron stars are evolving in the adiabatic manner due to gravitational radiation reaction with much longer time scale than the orbital period. As for the inspiraling phase, the theoretical investigation is usually done by the point particle approach using the post-Newtonian (PN) approximation in general relativity (Lincoln and Will 1990; Blanchet 1993, 1995, 1996; Will 1994; Sasaki 1994; Tagoshi et.al. 1994a, 1994b, 1996). Since the separation is large compared with the neutron star radius, the hydrodynamic effect is so small that we can regard each star of binary as a point particle. Theoretical studies for such a phase is potentially important because by comparing the observational signal with the theoretical prediction of the signal of inspiraling binary, we may be able to know not only the various parameters of binary (Cutler et.al. 1993; Cutler and Flanagan 1994), but also the cosmological parameters (Schutz 1986; Markovic 1993; Finn 1996; Wang, Stebbins and Turner 1996).

After a long time emission of gravitational waves, the orbital separation becomes comparable to the radius of the neutron star. Then, each star of binary neutron stars begins to behave as a hydrodynamic object, not as a point particle, because they are tidally coupled each other. Recently, Lai, Rasio and Shapiro (1993, 1994) have pointed out that such a tidal coupling of binary neutron stars is very important for their evolution in the final merging phase because the tidal effect causes the instability to the circular motion of them. Also important is the general relativistic gravity because in such a phase, the orbital separation is as large as about ten times of the Schwarzschild radius of the system. Thus, we need not only a hydrodynamic treatment, but also general relativistic one in order to study the final phase of binary neutron stars.

Fully general relativistic simulation is sure to be the best method, but it is also one of the most difficult ones. Although much effort has been focused and much progress can be expected there (Nakamura 1994), it will take a long time until numerical relativistic calculations become reliable. One of the main reasons is that we do not know the behavior of the geometric variables in the strong gravitational field around coalescing binary neutron stars. Owing to this, we do not know what sort of gauge condition is useful and how to give an appropriate general relativistic initial condition for coalescing binary neutron stars.

The other reason is a technical one: In numerical relativistic simulations, gravitational waves are generated, and in the case of coalescing binary neutron stars, the wavelength is of the order of $\lambda \sim \pi R^{3/2} M^{-1/2}$, where R and M are the orbital radius and the total mass of binary, respectively. Thus, we need to cover a region $L > \lambda \propto R^{3/2}$ with numerical grids in order to perform accurate simulations. This is in contrast with the case of Newtonian and/or PN simulations, in which we only need to cover a region $\lambda > L > R$. Since the circular orbit of binary neutron stars becomes unstable at $R \leq 10M$ owing to the tidal effects (Lai, Rasio and Shapiro 1993, 1994) and/or the strong general relativistic gravity (Kidder, Will and Wiseman 1993a), we must set an initial condition of binary at $R \geq 10M$. For such a case, to perform an accurate simulation, the grid must cover a region $L > \lambda \sim 100M$ in numerical relativistic simulations. When we assume to cover each neutron star of its radius $\sim 5M$ with ~ 30 homogeneous grids (Oohara and Nakamura 1990, 1991, 1992; Shibata, Nakamura and Oohara 1992, 1993), we need to take grids of at least $\sim 500^3$, but it seems impossible to take such a large amount of mesh points even for the present power of supercomputer. At present, we had better search other methods to prepare a precise initial condition for binary neutron stars.

In the case of PN simulations, the situation is completely different because we do not have to treat gravitational waves explicitly in numerical simulations, and as the result, only need to cover a region at most $L \sim 20-30M$. In this case, it seems that $\sim 200^3$ grid numbers are enough. Furthermore, we can take into account general relativistic effects with a good accuracy: In the case of coalescing binary neutron stars, the error will be at most $\sim M/R \sim$ a few $\times$ 10% for the first PN approximation, and $\sim (M/R)^2 \sim$ several % for 2PN approximation. Hence, if we can take into account up through 2PN terms, we will be able to give a highly accurate initial condition (the error $\leq$ several %). For these reasons, we present the 2.5PN hydrodynamic equations including the 2.5PN radiation reaction potential in this thesis.

This thesis is divided into two parts (Part 1 and 2). The purpose of Part 1 is twofold: One is to establish the basic formulation of the 2.5PN hydrodynamic equation, in particular taking into account the numerical application. The other is to investigate what kind of gauge condition is appropriate for simulation of the coalescing binary neutron stars and extraction of gravitational waves from them. As for the PN hydrodynamic equation, Blanchet, Damour and Schäfer (1990) have already obtained the (1+2.5) PN formula. In their formulation, the source terms of all Poisson equations take nonvanishing values only on the matter, like in the Newtonian hydrodynamics. Although their formula is very useful for PN hydrodynamic simulations including the radiation reaction (Oohara and Nakamura 1990, 1991, 1992; Shibata, Nakamura and Oohara 1992, 1993; Ruffert, Janka and Schäfer 1996), they did not take into account 2PN terms. In their formula, they also fixed the gauge conditions to the ADM gauge, but in numerical relativity, it has not been known yet what sort of gauge condition is suitable for simulation of the coalescing binary neutron stars and estimation of gravitational waves from them. First, we develop the formalism for the hydrodynamics using the PN approximation. In particular, we use the (3+1) formalism of general relativity so that we can adopt more general class of slice conditions. Next, we present methods to obtain numerically terms at the 2PN order. We also investigate several gauge conditions using the (3+1) formalism in general relativity.

In Part 2, we consider the problem of how to construct close binary neutron stars by taking them as uniformly rotating equilibrium configurations. Here, we mention the importance of this investigation, though more detailed explanation is given in section 7. To interpret the implication of the signal of gravitational waves, we need to understand the theoretical mechanism of merging in detail. The little knowledge we have about the very last phase of BNS's is as follows: When the orbital separation of BNS's is $\lesssim 10GM/c^2$, where M is the total mass of BNS's, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation t_{GW} is much longer than the orbital period P as

$$\frac{t_{GW}}{P} \sim 15\Big(\frac{dc^2}{10GM}\Big)^{5/2}\Big(\frac{M}{4\mu}\Big), \tag{1.1}$$

where μ and d are the reduced mass and the separation of BNS's. Thus, BNS's adiabatically evolve radiating gravitational waves. However, when the orbital separation becomes $6-10GM/c^2$, they cannot maintain the circular orbit because of instabilities due to the GR gravity (Kidder, Will and Wiseman 1993a) or the tidal field (Lai, Rasio and Shapiro 1993, 1994). As a result of such instabilities, the circular orbit of BNS's changes into the plunging orbit to merge. This means that the nature of the signal of gravitational waves changes around the transition between the circular orbit and plunging one. Gravitational waves emitted at this transition region may bring us an important information about the structure of NS's because the location where the instability occurs will depend on the equation of state (EOS) of NS sensitively (Lai, Rasio and Shapiro 1993, 1994; Zhung, Centrella and McMillan 1994). Thus, it is very important to investigate the location of the innermost stable circular orbit (ISCO) of BNS's.

As mentioned above, the ISCO is determined not only by the GR effects, but also by the hydrodynamic one. We emphasize that the tidal effects depend strongly on the structure of NS. Here, NS is a GR object because of its compactness, $Gm/c^2R \sim 0.2$, where m and R are the mass and radius of NS. Thus, in order to know accurately the location of the ISCO, we need to solve the GR hydrodynamic equations in general. A strategy to search the ISCO in GR manner is as follows; since the timescale of the energy loss is much longer than the orbital period according to Eq.(1.1), we may suppose that the motion of BNS's is composed of the stationary part and the small radiation reaction part. From this physical point of view, we may consider that BNS's evolve quasi-stationally, and we can take the following procedure; first, neglecting the evolution due to gravitational radiation, equilibrium configurations are constructed, and then the radiation reaction is taken into account as a correction to the equilibrium configurations. The ISCO is determined from the point, where the dynamical instability for the equilibrium configurations occurs. Hence, in Part 2, we develop a formalism to obtain equilibrium configurations of uniformly rotating fluid in the 2PN order as a first step.

This paper is organized as follows:

Part 1 consists of sections from 2 to 6: In section 2 we present the (3+1) formalism of the Einstein equation and the equations for the PN approximation. Several slice conditions are discussed in section 3. The methods to solve the 2PN tensor potential are discussed in detail for the sake of actual numerical simulations in section 4. In section 5, the quadrupole radiation-reaction potential is calculated more easily in combination of the conformal slice (Shibata and Nakamura 1992) and the transverse gauge. It is also shown that the work done by the reaction force takes the form invariant for slice conditions under the transverse gauge. We describe the 2PN expression of the conserved quantities, such as the conserved mass, the ADM mass, the total energy and the total angular momentum in section 6.

Part 2 consists of sections from 7 to 11: In section 7, we describe the approach to construct the stationary close binary neutron stars, i.e. without gravitational radiation reaction. Moreover, we discuss the importance of this approach by comparing Wilson's approach. In section 8, we review the basic equations up to the 2PN order in order to obtain equilibrium configurations of uniformly rotating fluid in the 2PN order. In section 9, we rewrite the Poisson equation for potential functions, which are described in section 8, into useful forms in which the source terms of the Poisson equations decrease rapidly enough ($O(r^{-4})$). In section 10, we show a formulation to obtain numerically equilibrium solutions of uniformly rotating fluid in the 2PN approximation. In particular, we rewrite potentials defined in section 9 into a polynomial form in the angular velocity, Ω. Then, we transform the integrated Euler equation into the polynomial form in Ω^2 so that the convergence property in iteration procedures can be much improved. For the sake of analysis for numerical results, we describe the 2PN expression of the conserved quantities for equilibrium configurations of uniformly rotating fluid in section 11.

Section 12 is devoted to discussion and summary. In appendix A, we transform the equation of motion in the (3+1) formalism into the form in section 2. In appendix B, we describe a method to derive the logarithmic kernel. We calculate some metric variables up to the 2PN order in appendices C and D. The integration of Euler's equation is done for the uniformly rotating equilibrium configurations in appendix E. We mention tail terms in appendix F. The brief history of the PN approximation is given in appendix G.

We use the units of $c = G = 1$ in this paper. Greek and Latin indices take $0, 1, 2, 3$ and $1, 2, 3$, respectively.

Part 1

II. POST-NEWTONIAN APPROXIMATION IN THE (3+1) FORMALISM

A. (3+1) Formalism for Post-Newtonian Approximation

We consider the (3+1) formalism to perform the PN approximation. In the (3+1) formalism (Arnowitt, Deser and Misner 1962; Wald 1984; Nakamura, Oohara and Kojima 1987), the metric is split as

$$g_{\mu\nu} = \gamma_{\mu\nu} - \hat{n}_\mu \hat{n}_\nu, \tag{2.1}$$

and

$$\begin{aligned} \hat{n}_\mu &= (-\alpha,\ \mathbf{0}), \\ \hat{n}^\mu &= \Big(\frac{1}{\alpha},\ -\frac{\beta^i}{\alpha}\Big), \end{aligned} \tag{2.2}$$

where α, β^i and γ_{ij} are the lapse function, shift vector and metric on the 3D hypersurface, respectively. Then the line element is written as

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j. \tag{2.3}$$

Using the (3+1) formalism, the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.4}$$

is split into the constraint equations and the evolution equations. The formers are the so-called Hamiltonian and momentum constraints which respectively become

$$\mathrm{tr}R - K_{ij}K^{ij} + K^2 = 16\pi\rho_H, \tag{2.5}$$

$$D_i K^i{}_j - D_j K = 8\pi J_j, \tag{2.6}$$

where K_{ij}, K, tr$R$ and D_i are the extrinsic curvature, the trace part of K_{ij}, the scalar curvature of 3D hypersurface and the covariant derivative with respect of γ_{ij}. ρ_H and J_j are defined as

$$\rho_H = T_{\mu\nu}\hat{n}^\mu\hat{n}^\nu,$$
$$J_j = -T_{\mu\nu}\hat{n}^\mu\gamma^\nu{}_j. \tag{2.7}$$

Evolution equations for the spatial metric and extrinsic curvature are respectively

$$\frac{\partial}{\partial t}\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \tag{2.8}$$

$$\frac{\partial}{\partial t}K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_{il}K^l{}_j) - D_iD_j\alpha$$
$$+(D_j\beta^m)K_{mi} + (D_i\beta^m)K_{mj} + \beta^m D_m K_{ij} - 8\pi\alpha\Big(S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - S^l{}_l)\Big), \tag{2.9}$$

$$\frac{\partial}{\partial t}\gamma = 2\gamma(-\alpha K + D_i\beta^i), \tag{2.10}$$

$$\frac{\partial}{\partial t}K = \alpha(\mathrm{tr}R + K^2) - D^iD_i\alpha + \beta^jD_jK + 4\pi\alpha(S^l{}_l - 3\rho_H), \tag{2.11}$$

where R_{ij}, γ and S_{ij} are, respectively, the Ricci tensor with respect of γ_{ij}, determinant of γ_{ij} and

$$S_{ij} = T_{kl}\gamma^k{}_i\gamma^l{}_j. \tag{2.12}$$

Hereafter we use the conformal factor $\psi = \gamma^{1/12}$ instead of γ for simplicity.

To distinguish the wave part from the non-wave part (for example, Newtonian potential) in the metric, we use $\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$ instead of γ_{ij}. Then $\det(\tilde{\gamma}_{ij}) = 1$ is satisfied. We also define $\tilde{A}_{ij}$ as

$$\tilde{A}_{ij} \equiv \psi^{-4}A_{ij} \equiv \psi^{-4}\Big(K_{ij} - \frac{1}{3}\gamma_{ij}K\Big). \tag{2.13}$$

We should note that in our notation, indices of $\tilde{A}_{ij}$ are raised and lowered by $\tilde{\gamma}_{ij}$, so that the relations, $\tilde{A}^i{}_j = A^i{}_j$ and $\tilde{A}^{ij} = \psi^4 A^{ij}$, hold. Using these variables, the evolution equations (2.8-2.11) can be rewritten as follows;

$$\frac{\partial}{\partial n}\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{il}\frac{\partial\beta^l}{\partial x^j} + \tilde{\gamma}_{jl}\frac{\partial\beta^l}{\partial x^i} - \frac{2}{3}\tilde{\gamma}_{ij}\frac{\partial\beta^l}{\partial x^l}, \tag{2.14}$$

$$\begin{aligned}\frac{\partial}{\partial n}\tilde{A}_{ij} = &\frac{1}{\psi^4}\Big[\alpha\Big(R_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}R\Big) - \Big(\tilde{D}_i\tilde{D}_j\alpha - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\alpha\Big) - \frac{2}{\psi}\Big(\psi_{,i}\alpha_{,j} + \psi_{,j}\alpha_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}\psi_{,k}\alpha_{,l}\Big)\Big] \\ &+\alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) + \frac{\partial\beta^m}{\partial x^i}\tilde{A}_{mj} + \frac{\partial\beta^m}{\partial x^j}\tilde{A}_{mi} - \frac{2}{3}\frac{\partial\beta^m}{\partial x^m}\tilde{A}_{ij} - 8\pi\frac{\alpha}{\psi^4}\Big(S_{ij} - \frac{1}{3}\gamma_{ij}S^l{}_l\Big),\end{aligned} \tag{2.15}$$

$$\frac{\partial}{\partial n}\psi = \frac{\psi}{6}\Big(-\alpha K + \frac{\partial\beta^i}{\partial x^i}\Big), \tag{2.16}$$

$$\frac{\partial}{\partial n}K = \alpha\Big(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\Big) - \frac{1}{\psi^4}\tilde{\Delta}\alpha - \frac{2}{\psi^5}\tilde{\gamma}^{kl}\psi_{,k}\alpha_{,l} + 4\pi\alpha(S^i{}_i + \rho_H), \tag{2.17}$$

where $\tilde{D}_i$ and $\tilde{\Delta}$ are the covariant derivative and Laplacian with respect to $\tilde{\gamma}_{ij}$ and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial t} - \beta^i\frac{\partial}{\partial x^i}. \tag{2.18}$$

The Hamiltonian constraint equation is written as

$$\tilde{\Delta}\psi = \frac{1}{8}\mathrm{tr}\tilde{R}\psi - 2\pi\rho_H\psi^5 - \frac{\psi^5}{8}\Big(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\Big), \tag{2.19}$$

where $\mathrm{tr}\tilde{R}$ is the scalar curvature with respect to $\tilde{\gamma}_{ij}$. Here we used the following relation for the conformal transformation (For example, Wald's *General Relativity* (1984))

$$\mathrm{tr}R = \frac{1}{\psi^5}\Big(\tilde{R}\psi - 8\tilde{\Delta}\psi\Big). \tag{2.20}$$

The momentum constraint is also written as

$$\tilde{D}_j(\psi^6\tilde{A}^j{}_i) - \frac{2}{3}\psi^6\tilde{D}_iK = 8\pi\psi^6 J_i. \tag{2.21}$$

Now let us consider R_{ij} in Eq.(2.15), which is one of the main source terms of the evolution equation for $\tilde{A}_{ij}$. First we split R_{ij} into two parts as

$$R_{ij} = \tilde{R}_{ij} + R^{\psi}_{ij}, \tag{2.22}$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$ and R^{ψ}_{ij} is defined as

$$R^{\psi}_{ij} = -\frac{2}{\psi}\tilde{D}_i\tilde{D}_j\psi - \frac{2}{\psi}\tilde{\gamma}_{ij}\tilde{D}^k\tilde{D}_k\psi + \frac{6}{\psi^2}(\tilde{D}_i\psi)(\tilde{D}_j\psi) - \frac{2}{\psi^2}\tilde{\gamma}_{ij}(\tilde{D}_k\psi)(\tilde{D}^k\psi). \tag{2.23}$$

Using the property of $\det(\tilde{\gamma}_{ij}) = 1$, $\tilde{R}_{ij}$ is written as

$$\tilde{R}_{ij} = \frac{1}{2}\Big[\tilde{\gamma}^{kl}(\tilde{\gamma}_{lj,ik} + \tilde{\gamma}_{li,jk} - \tilde{\gamma}_{ij,lk}) + \tilde{\gamma}^{kl}{}_{,k}(\tilde{\gamma}_{lj,i} + \tilde{\gamma}_{li,j} - \tilde{\gamma}_{ij,l})\Big] - \tilde{\Gamma}^{l}_{kj}\tilde{\Gamma}^{k}_{li}\ , \tag{2.24}$$

where ${}_{,i}$ denotes $\partial/\partial x^i$ and $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol with respect to $\tilde{\gamma}_{ij}$. We split $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ as $\delta_{ij} + h_{ij}$ and $\delta^{ij} + f^{ij}$, where δ_{ij} denotes the flat metric, and rewrite $\tilde{R}_{ij}$ as

$$\begin{aligned}\tilde{R}_{ij} = \frac{1}{2}\Big[&-h_{ij,kk} + h_{jl,li} + h_{il,lj} + f^{kl}{}_{,k}(h_{lj,i} + h_{li,j} - h_{ij,l}) \\ &+f^{kl}(h_{kj,il} + h_{ki,jl} - h_{ij,kl})\Big] - \tilde{\Gamma}^{l}_{kj}\tilde{\Gamma}^{k}_{li}\ .\end{aligned} \tag{2.25}$$

In this paper, we consider only the linear order in h_{ij} and f_{ij} assuming $|h_{ij}|,\ |f_{ij}| \ll 1$. (As a result, $h_{ij} = -f^{ij}$.) Such an assumption is justified because in this paper, we choose a gauge condition, in which h_{ij} is a 2PN quantity (see below). This implies that we neglect higher PN effects such as the non-linear coupling between gravitational waves themselves, but does not imply that we neglect the non-linear coupling between the Newtonian potentials themselves and between gravitational waves and the Newtonian potentials. In other words, although we can not see the non-linear memory of gravitational waves (Christodoulou 1991; Wiseman and Will 1991; Thorne 1992), we can see the tail term of gravitational waves and can derive the exact quadrupole formula (see below). Here, to guarantee the wave property of $\tilde{\gamma}_{ij}$, we impose a kind of transverse gauge to h_{ij} as

$$h_{ij,j} = 0. \tag{2.26}$$

Hereafter, we call this condition merely the transverse gauge. This condition is guaranteed by β^i which satisfies

$$-\beta^k{}_{,j}\tilde{\gamma}_{ij,k} = \Big(-2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{il}\beta^l{}_{,j} + \tilde{\gamma}_{jl}\beta^l{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}\Big)_{,j}. \tag{2.27}$$

Using the above conditions, Eq. (2.25) becomes

$$\tilde{R}_{ij} = -\frac{1}{2}\Delta_{flat}h_{ij} + O(h^2), \tag{2.28}$$

where Δ_{flat} is the Laplacian with respect to δ_{ij}. Note that $\mathrm{tr}\tilde{R} = O(h^2)$ is guaranteed in the transverse gauge because the traceless property of h_{ij} holds in the linear order.

Finally, we show the equations for the perfect fluid. The energy momentum tensor for the perfect fluid is written as

$$T^{\mu\nu} = (\rho + \rho\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu}, \tag{2.29}$$

where u^μ, ρ, ε and P are the four velocity, the mass density, the specific internal energy and the pressure. Then we obtain

$$\begin{aligned} \rho_H &= (\rho + \rho\epsilon + P)(\alpha u^0)^2 - P, \\ J_i &= \alpha(\rho + \rho\epsilon + P)u^0 u_i, \\ S_{ij} &= (\rho + \rho\epsilon + P)u_i u_j + P\gamma_{ij}. \end{aligned} \tag{2.30}$$

The mass density obeys the continuity equation

$$\nabla_\mu(\rho u^\mu) = 0, \tag{2.31}$$

where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$. The explicit form is

$$\frac{\partial \rho_*}{\partial t} + \frac{\partial(\rho_* v^i)}{\partial x^i} = 0, \tag{2.32}$$

where ρ_* is the conserved density defined as

$$\rho_* = \alpha\psi^6 \rho u^0. \tag{2.33}$$

The equations of motion and the energy equation are derived from

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.34}$$

Explicit forms of them become

$$\frac{\partial S_i}{\partial t} + \frac{\partial(S_i v^j)}{\partial x^j} = -\alpha\psi^6 P_{,i} - \alpha\alpha_{,i}S^0 + S_j\beta^j{}_{,i} - \frac{1}{2S^0}S_j S_k \gamma^{jk}{}_{,i}, \tag{2.35}$$

and

$$\frac{\partial H}{\partial t}+\frac{\partial (Hv^j)}{\partial x^j}=-P\Big(\frac{\partial(\alpha\psi^6u^0)}{\partial t}+\frac{\partial(\alpha\psi^6u^0v^j)}{\partial x^j}\Big), \tag{2.36}$$

where

$$\begin{aligned}
&S_i=\alpha\psi^6(\rho+\rho\varepsilon+P)u^0u_i=\rho_*\Big(1+\varepsilon+\frac{P}{\rho}\Big)u_i(=\psi^6J_i),\\
&S^0=\alpha\psi^6(\rho+\rho\varepsilon+P)(u^0)^2\Big(=\frac{(\rho_H+P)\psi^6}{\alpha}\Big),\\
&H=\alpha\psi^6\rho\varepsilon u^0=\rho_*\varepsilon,\\
&v^i\equiv\frac{u^i}{u^0}=-\beta^i+\frac{\gamma^{ij}S_j}{S^0}.
\end{aligned} \tag{2.37}$$

Finally, we note that in the above equations, only β^i appears, and β_i does not, so that, in the subsequent section, we only consider the PN expansion of β^i, not of β_i.

B. Post-Newtonian approximation in the (3+1) formalism

Next, we consider the PN approximation of the above equations. First of all, we review the PN expansion of the variables. Each metric variable, which is relevant to the present paper, is expanded as

$$\begin{aligned}
\psi&=1+{}_{(2)}\psi+{}_{(4)}\psi+{}_{(6)}\psi+{}_{(7)}\psi+\dots,\\
\alpha&=1+{}_{(2)}\alpha+{}_{(4)}\alpha+{}_{(6)}\alpha+{}_{(7)}\alpha+\dots,\\
&=1-U+\Big(\frac{U^2}{2}+X\Big)+{}_{(6)}\alpha+{}_{(7)}\alpha+\dots,\\
\beta^i&={}_{(3)}\beta_i+{}_{(5)}\beta_i+{}_{(6)}\beta_i+{}_{(7)}\beta_i+{}_{(8)}\beta_i+\dots,\\
h_{ij}&={}_{(4)}h_{ij}+{}_{(5)}h_{ij}+\dots,\\
\tilde{A}_{ij}&={}_{(3)}\tilde{A}_{ij}+{}_{(5)}\tilde{A}_{ij}+{}_{(6)}\tilde{A}_{ij}+\dots,\\
K&={}_{(3)}K+{}_{(5)}K+{}_{(6)}K+\dots,
\end{aligned} \tag{2.38}$$

where subscripts denote the PN order(c^{-n}) and U is the Newtonian potential satisfying

$$\Delta_{flat}U=-4\pi\rho. \tag{2.39}$$

X depends on the slice condition, and in the standard PN gauge (Chandrasekhar 1965), we usually use $\Phi = -X/2$, which satisfies

$$\Delta_{flat}\Phi = -4\pi\rho\Big(v^2 + U + \frac{1}{2}\varepsilon + \frac{3}{2}\frac{P}{\rho}\Big). \tag{2.40}$$

Note that the terms relevant to the radiation reaction appear in ${}_{(7)}\psi$, ${}_{(7)}\alpha$, ${}_{(8)}\beta_i$ and ${}_{(5)}h_{ij}$, and the quadrupole formula is derived from ${}_{(7)}\alpha$ and ${}_{(5)}h_{ij}$.

The four velocity is expanded as

$$\begin{aligned}
u^0 &= 1 + \Big(\frac{1}{2}v^2 + U\Big) + \Big(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta_i v^i - X\Big) + O(c^{-6}),\\
u_0 &= -\Big[1 + \Big(\frac{1}{2}v^2 - U\Big) + \Big(\frac{3}{8}v^4 + \frac{3}{2}v^2U + \frac{1}{2}U^2 + X\Big)\Big] + O(c^{-6}),\\
u^i &= v^i\Big[1 + \Big(\frac{1}{2}v^2 + U\Big) + \Big(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta_i v^i - X\Big)\Big] + O(c^{-7}),\\
u_i &= v^i + \Big\{{}_{(3)}\beta_i + v^i\Big(\frac{1}{2}v^2 + 3U\Big)\Big\} + \Big[{}_{(5)}\beta_i + {}_{(3)}\beta_i\Big(\frac{1}{2}v^2 + 3U\Big) + {}_{(4)}h_{ij}v^j\\
&\quad + v^i\Big(\frac{3}{8}v^4 + \frac{7}{2}v^2U + 4U^2 - X + 4{}_{(4)}\psi + {}_{(3)}\beta_j v^j\Big)\Big] + \Big({}_{(6)}\beta_i + {}_{(5)}h_{ij}v^j\Big) + O(c^{-7}),
\end{aligned} \tag{2.41}$$

where $v^2 = v^i v^i$. This expansion can be done as follows: The four velocity satisfies

$$\begin{aligned}
-1 &= g_{\mu\nu}u^\mu u^\nu\\
&= \Big(g_{00} + 2g_{0i}v^i + g_{ij}v^iv^j\Big)(u^0)^2\\
&= -\Big[\Big(1 - 2U + 2U^2 + 2X\Big) - 2{}_{(3)}\beta_i v^i - v^2(1 + 2U) + O(c^{-6})\Big](u^0)^2.
\end{aligned} \tag{2.42}$$

Hence we obtain

$$u^0 = 1 + \Big(\frac{1}{2}v^2 + U\Big) + \Big(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta_i v^i - X\Big) + O(c^{-6}). \tag{2.43}$$

Next we obtain u_0 as

$$\begin{aligned}
u_0 &= g_{00}u^0 + g_{0i}u^i\\
&= -\Big[1 + \Big(\frac{1}{2}v^2 - U\Big) + \Big(\frac{3}{8}v^4 + \frac{3}{2}v^2U + \frac{1}{2}U^2 + X\Big)\Big] + O(c^{-6}).
\end{aligned} \tag{2.44}$$

Finally, we obtain u_i from

$$u_i = g_{0i}u^0 + g_{ij}u^j. \tag{2.45}$$

Here we used

$$g_{ij} = \Big[1 + 2U + \Big(\frac{3}{2}U^2 + 4_{(4)}\psi\Big)\Big]\delta_{ij} + {}_{(4)}h_{ij} + {}_{(5)}h_{ij} + O(c^{-6}). \tag{2.46}$$

The PN expansion of the relation $u^\mu u_\mu = -1$ becomes

$$\begin{aligned}(\alpha u^0)^2 &= 1 + \gamma^{ij}u_i u_j \\ &= 1 + v^2 + v^4 + 4v^2 U + 2_{(3)}\beta_i v^i + O(c^{-6}).\end{aligned} \tag{2.47}$$

Thus ρ_H, J_i and S_{ij} are respectively expanded as

$$\begin{aligned}\rho_H &= \rho\Big[1 + \Big(v^2 + \varepsilon\Big) + \Big\{v^4 + v^2\Big(4U + \varepsilon + \frac{P}{\rho}\Big) + 2_{(3)}\beta_i v^i\Big\} + O(c^{-6})\Big], \\ J_i &= \rho\Big[v^i\Big(1 + v^2 + 3U + \varepsilon + \frac{P}{\rho}\Big) + {}_{(3)}\beta_i + O(c^{-5})\Big], \\ S_{ij} &= \rho\Big[\Big(v^i v^j + \frac{P}{\rho}\delta_{ij}\Big) + \Big\{\Big(v^2 + 6U + \varepsilon + \frac{P}{\rho}\Big)v^i v^j + v^i{}_{(3)}\beta_j + v^j{}_{(3)}\beta_i + 2\frac{UP}{\rho}\delta_{ij}\Big\} \\ &\qquad + O(c^{-6})\Big], \\ S_l{}^l &= \rho\Big[v^2 + \frac{3P}{\rho} + \Big\{2_{(3)}\beta_i v^i + v^2\Big(v^2 + 4U + \varepsilon + \frac{P}{\rho}\Big)\Big\} + O(c^{-6})\Big].\end{aligned} \tag{2.48}$$

The conformal factor ψ(and α in the conformal slice) is determined by the Hamiltonian constraint. In the PN approximation, the Laplacian for the scalar is expanded as

$$\tilde{\Delta} = \Delta_{flat} - ({}_{(4)}h_{ij} + {}_{(5)}h_{ij})\partial_i\partial_j + O(c^{-6}). \tag{2.49}$$

At the lowest order, the Hamiltonian constraint becomes

$$\Delta_{flat\,(2)}\psi = -2\pi\rho. \tag{2.50}$$

Thus, ${}_{(2)}\alpha = -2_{(2)}\psi = -U$ is satisfied in this paper. At the 2PN and 3PN orders, the Hamiltonian constraint equation becomes, respectively,

$$\Delta_{flat\,(4)}\psi = -2\pi\rho\Big(v^2 + \varepsilon + \frac{5}{2}U\Big), \tag{2.51}$$

and

$$\begin{aligned}\Delta_{flat\,(6)}\psi = -2\pi\rho\Big\{v^4 + v^2\Big(\varepsilon + \frac{P}{\rho} + \frac{13}{2}U\Big) + 2_{(3)}\beta_i v^i + \frac{5}{2}\varepsilon U + \frac{5}{2}U^2 + 5_{(4)}\psi\Big\} \\ + \frac{1}{2}{}_{(4)}h_{ij}U_{,ij} - \frac{1}{8}\Big({}_{(3)}\tilde{A}_{ij\,(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\Big).\end{aligned} \tag{2.52}$$

The term relevant to the radiation reaction first appears in ${}_{(7)}\psi$ and the equation for it becomes

$$\Delta_{flat(7)}\psi = \frac{1}{2}{}_{(5)}h_{ij}U_{,ij}. \tag{2.53}$$

Hence, ${}_{(7)}\alpha$ may be also relevant to the radiation reaction and whether it may or not depends on the slice condition.

From Eq.(2.27), the relation between ${}_{(3)}\tilde{A}_{ij}$ and ${}_{(3)}\beta_i$ becomes

$$-2_{(3)}\tilde{A}_{ij} + {}_{(3)}\beta_{i,j} + {}_{(3)}\beta_{j,i} - \frac{2}{3}\delta_{ij(3)}\beta_{l,l} = 0. \tag{2.54}$$

${}_{(3)}\tilde{A}_{ij}$ must also satisfy the momentum constraint. Since ${}_{(3)}\tilde{A}_{ij}$ does not contain the transverse-traceless (TT) part and only contains the longitudinal part, it can be written as

$${}_{(3)}\tilde{A}_{ij} = {}_{(3)}W_{i,j} + {}_{(3)}W_{j,i} - \frac{2}{3}\delta_{ij(3)}W_{k,k}\ , \tag{2.55}$$

where ${}_{(3)}W_i$ is a vector on the 3D hypersurface and satisfies the momentum constraint at the first PN order as follows;

$$\Delta_{flat(3)}W_i + \frac{1}{3}{}_{(3)}W_{j,ji} - \frac{2}{3}{}_{(3)}K_{,i} = 8\pi\rho v^i. \tag{2.56}$$

From Eq.(2.54), the relation,

$${}_{(3)}\beta_i = 2_{(3)}W_i\ , \tag{2.57}$$

holds and at the first PN order, Eq.(2.16) becomes

$$3\dot{U} = -{}_{(3)}K + {}_{(3)}\beta_{l,l}\ , \tag{2.58}$$

where $\dot{U}$ denotes the derivative of U with respect to time. Thus Eq.(2.56) is rewritten as

$$\Delta_{flat(3)}\beta_i = 16\pi\rho v^i + \Big({}_{(3)}K_{,i} - \dot{U}_{,i}\Big). \tag{2.59}$$

This is the equation for the vector potential at the first PN order.

From the next order, ${}_{(n)}\beta_i$ is determined by the gauge condition, $h_{ij,j} = 0$. Making use of the momentum constraint and the 2PN order of Eq.(2.16),

$$6_{(4)}\dot{\psi} - 3_{(3)}\beta_l U_{,l} - \frac{1}{2}U(2_{(3)}K + 3\dot{U}) + {}_{(5)}K = {}_{(5)}\beta_{l,l} \ , \tag{2.60}$$

the equation for ${}_{(5)}\beta_i$ is written as

$$\begin{aligned}\Delta_{flat(5)}\beta_i &= 16\pi\rho\Big[v^i\Big(v^2 + 2U + \varepsilon + \frac{P}{\rho}\Big) + {}_{(3)}\beta_i\Big] - 8U_{,j(3)}\tilde{A}_{ij} \\ &= +_{(5)}K_{,i} - U_{(3)}K_{,i} + \frac{1}{3}U_{,i(3)}K - 2_{(4)}\dot{\psi}_{,i} + \frac{1}{2}(U\dot{U})_{,i} + ({}_{(3)}\beta_l U_{,l})_{,i} \ . \end{aligned} \tag{2.61}$$

Since J_i at the 1.5PN order vanishes, the merely geometrical equation for ${}_{(6)}\beta_i$ is given by

$$\Delta_{flat(6)}\beta_i = {}_{(6)}K_{,i} \ . \tag{2.62}$$

Then, let us consider the wave equation for h_{ij}. From Eqs.(2.14), (2.15), (2.22) and (2.28), the wave equation for h_{ij} is written as

$$\begin{aligned}\Box h_{ij} &= \Big(1 - \frac{\alpha^2}{\psi^4}\Big)\Delta_{flat}h_{ij} + \Big(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial t^2}\Big)h_{ij} \\ &\quad + \frac{2\alpha}{\psi^4}\Big[-\frac{2\alpha}{\psi}\Big(\tilde{D}_i\tilde{D}_j - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\Big)\psi + \frac{6\alpha}{\psi^2}\Big(\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}_k\psi\tilde{D}^k\psi\Big) \\ &\qquad - \Big(\tilde{D}_i\tilde{D}_j - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\Big)\alpha - \frac{2}{\psi}\Big(\tilde{D}_i\psi\tilde{D}_j\alpha + \tilde{D}_j\psi\tilde{D}_i\alpha - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\alpha\Big)\Big] \\ &\quad + 2\alpha^2\Big(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j\Big) + 2\alpha\Big(\beta^m{}_{,i}\tilde{A}_{mj} + \beta^m{}_{,j}\tilde{A}_{mi} - \frac{2}{3}\beta^m{}_{,m}\tilde{A}_{ij}\Big) \\ &\quad - 16\pi\frac{\alpha^2}{\psi^4}\Big(S_{ij} - \frac{1}{3}\gamma_{ij}S^l{}_l\Big) - \frac{\partial}{\partial n}\Big(\beta^m{}_{,i}\tilde{\gamma}_{mj} + \beta^m{}_{,j}\tilde{\gamma}_{mi} - \frac{2}{3}\beta^m{}_{,m}\tilde{\gamma}_{ij}\Big) + 2\frac{\partial\alpha}{\partial n}\tilde{A}_{ij} \\ &\equiv \tau_{ij}, \end{aligned} \tag{2.63}$$

where

$$\Box = -\frac{\partial^2}{\partial t^2} + \Delta_{flat}. \tag{2.64}$$

We should note that ${}_{(4)}\tau_{ij}$ has the TT property, i.e., ${}_{(4)}\tau_{ij,j} = 0$ and ${}_{(4)}\tau_{ii} = 0$. This is a natural consequence of the transverse gauge, $h_{ij,j} = 0$ and $h_{ii} = O(h^2)$. Thus ${}_{(4)}h_{ij}$ is determined from

$$\Delta_{flat(4)}h_{ij} = {}_{(4)}\tau_{ij}. \tag{2.65}$$

Since $O(h^2)$ turns out to be $O(c^{-8})$, it is enough to consider only the linear order of h_{ij} in the case when we perform the PN approximation up to the 3.5PN order. We can obtain ${}_{(5)}h_{ij}$ by evaluating

$${}_{(5)}h_{ij}(t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\int {}_{(4)}\tau_{ij}(t,\mathbf{y})d^3y, \tag{2.66}$$

and the quadrupole mode of gravitational waves in the wave zone is written as

$$h_{ij}^{rad} = -\frac{1}{4\pi}\lim_{|\mathbf{x}|\to\infty}\int \frac{{}_{(4)}\tau_{ij}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d^3y. \tag{2.67}$$

In section 5, we derive the quadrupole radiation-reaction metric in the near zone using Eq.(2.66).

Finally, we show the evolution equation for K. Since we adopt slice conditions which do not satisfy $K=0$ (i.e. the maximal slice condition), the evolution equation for K is necessary. The evolution equations appear at the 1PN, 2PN and 2.5PN orders which become respectively

$$\frac{\partial}{\partial t}{}_{(3)}K = 4\pi\rho\Big(2v^2+\varepsilon+2U+3\frac{P}{\rho}\Big) - \Delta_{flat}X, \tag{2.68}$$

$$\begin{aligned}\frac{\partial}{\partial t}{}_{(5)}K = {}& 4\pi\rho\Big[2v^4+v^2\Big(6U+2\varepsilon+2\frac{P}{\rho}\Big)-\Big(\varepsilon+\frac{3P}{\rho}\Big)U-4U^2+4{}_{(4)}\psi+X+4{}_{(3)}\beta_i v^i\Big]\\ &+{}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}+\frac{1}{3}{}_{(3)}K^2-{}_{(4)}h_{ij}U_{,ij}+{}_{(3)}\beta_{i(3)}K_{,i}\\ &-\frac{3}{2}UU_{,k}U_{,k}-U_{,k}X_{,k}+2U_{,k(4)}\psi_{,k}-\Delta_{flat(6)}\alpha+2U\Delta_{flat}X,\end{aligned} \tag{2.69}$$

$$\frac{\partial}{\partial t}{}_{(6)}K = -\Delta_{flat(7)}\alpha - {}_{(5)}h_{ij}U_{,ij}. \tag{2.70}$$

We note that for the PN equations of motion up to the 2.5PN order, we need ${}_{(2)}\alpha$, ${}_{(4)}\alpha$, ${}_{(6)}\alpha$, ${}_{(7)}\alpha$, ${}_{(2)}\psi$, ${}_{(4)}\psi$, ${}_{(3)}\beta_i$, ${}_{(5)}\beta_i$, ${}_{(6)}\beta_i$, ${}_{(4)}h_{ij}$, ${}_{(5)}h_{ij}$, ${}_{(3)}K$, ${}_{(5)}K$ and ${}_{(6)}K$. Therefore, if we solve the above set of the equations, we can obtain the 2.5 PN equations of motion. Up to the 2.5PN order, the hydrodynamic equations become

$$
\begin{aligned}
\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = & -\Big(1 + 2U + \frac{5}{4}U^2 + 6_{(4)}\psi + X\Big)P_{,i} \\
& +\rho_*\Big[U_{,i}\Big\{1 + \varepsilon + \frac{P}{\rho} + \frac{3}{2}v^2 - U + \frac{5}{8}v^4 + 4v^2 U \\
& \qquad + \Big(\frac{3}{2}v^2 - U\Big)\Big(\varepsilon + \frac{P}{\rho}\Big) + 3_{(3)}\beta_j v^j\Big\} \\
& \quad -X_{,i}\Big(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2}\Big) + 2v^2{}_{(4)}\psi_{,i} - {}_{(6)}\alpha_{,i} - {}_{(7)}\alpha_{,i} \\
& \quad +v^j\Big\{{}_{(3)}\beta_{j,i}\Big(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + 3U\Big) + {}_{(5)}\beta_{j,i} + {}_{(6)}\beta_{j,i}\Big\} + {}_{(3)}\beta_{j\,(3)}\beta_{j,i} \\
& \quad +\frac{1}{2}v^j v^k({}_{(4)}h_{jk,i} + {}_{(5)}h_{jk,i}) + O(c^{-8})\Big], \qquad (2.71) \\
\frac{\partial H}{\partial t} + \frac{\partial (H v^j)}{\partial x^j} = & -P\Big[v^j{}_{,j} + \frac{\partial}{\partial t}\Big(\frac{1}{2}v^2 + 3U\Big) + \frac{\partial}{\partial x^j}\Big\{\Big(\frac{1}{2}v^2 + 3U\Big)v^j\Big\} + O(c^{-5})\Big], \qquad (2.72)
\end{aligned}
$$

where we make use of relations

$$
\begin{aligned}
\alpha S^0 &= \rho_*\Big[1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + \frac{v^2}{2}\Big(\varepsilon + \frac{P}{\rho}\Big) + \frac{3}{8}v^4 + 2v^2 U + {}_{(3)}\beta_j v^j + + O(c^{-6})\Big], \\
S_i &= \rho_*\Big[v^i\Big(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + 3U\Big) + {}_{(3)}\beta_i + O(c^{-5})\Big]. \qquad (2.73)
\end{aligned}
$$

III. SLICE CONDITIONS

In this section, we perform the PN analysis using the conformal slice (Shibata and Nakamura 1992), maximal slice and harmonic slice (Bona and Masso 1992) which are often used in 3D numerical relativity. Among them, we find that the conformal slice seems most tractable and useful to estimate gravitational waves in the far zone, while the maximal slice is suitable for describing the equilibrium configurations. Hence, first of all we describe the property of the conformal slice and then mention the properties of other slices.

A. Conformal Slice

The conformal slice (Shibata and Nakamura 1992) is defined as

$$\alpha = \exp\Bigl(-2\epsilon - \frac{2}{3}\epsilon^3 - \frac{2}{5}\epsilon^5\Bigr), \tag{3.1}$$

where $\epsilon = \psi - 1$. The lapse function is expanded in terms of ϵ as

$$\alpha = 1 - 2\epsilon + 2\epsilon^2 - 2\epsilon^3 + 2\epsilon^4 + O(\epsilon^5). \tag{3.2}$$

In the conformal slice, ${}_{(n)}\alpha$ becomes

$$\begin{aligned}
{}_{(2)}\alpha &= -2{}_{(2)}\psi,\\
{}_{(4)}\alpha &= 2({}_{(2)}\psi)^2 - 2{}_{(4)}\psi,\\
{}_{(6)}\alpha &= -2({}_{(2)}\psi)^3 + 4{}_{(2)}\psi{}_{(4)}\psi - 2{}_{(6)}\psi,\\
{}_{(7)}\alpha &= -2{}_{(7)}\psi.
\end{aligned} \tag{3.3}$$

Although in the usual PN approximation we need to solve the Poisson equation for the lapse function, this slicing saves a procedure of solving it.

In the conformal slice, equations (2.15) and (2.17) are rewritten as

$$
\begin{aligned}
\frac{\partial}{\partial n}\tilde{A}_{ij} = &-\frac{1}{2}\frac{\alpha}{\psi^4}\Delta_{flat}h_{ij} + \frac{2\alpha}{\psi^4}\Big[\Big(\tilde{D}_i\tilde{D}_j\psi - \frac{\tilde{\gamma}_{ij}}{3}\tilde{D}_k\tilde{D}^k\psi\Big)\frac{\epsilon}{\psi}(1+\epsilon+\epsilon^2+\epsilon^3+\epsilon^4) \\
&\qquad -\frac{1}{\psi^2}\Big(\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\psi\Big) \\
&\qquad\qquad (3+6\epsilon+6\epsilon^2+6\epsilon^3+6\epsilon^4+12\epsilon^5+10\epsilon^6+8\epsilon^7+6\epsilon^8+4\epsilon^9+2\epsilon^{10})\Big] \\
&+\alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l{}_j) + \beta^m{}_{,i}\tilde{A}_{mj} + \beta^m{}_{,j}\tilde{A}_{mi} - \frac{2}{3}\beta^m{}_{,m}\tilde{A}_{ij} \\
&-8\pi\frac{\alpha}{\psi^4}\Big(S_{ij} - \frac{1}{3}\gamma_{ij}S^l{}_l\Big),
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\frac{\partial K}{\partial n} = &\, 2\frac{\alpha}{\psi^4}\Big[\tilde{\Delta}\psi(1+\epsilon^2+\epsilon^4) - \frac{2}{\psi^2}\tilde{D}_k\psi\tilde{D}^k\psi(3\epsilon^5+2\epsilon^6+2\epsilon^7+\epsilon^8+\epsilon^9)\Big] \\
&+\alpha\Big(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\Big) + 4\pi\alpha(S^l{}_l + \rho_H),
\end{aligned}
\tag{3.5}
$$

where we use the TT property as well as the linear approximation for h_{ij} in the above equation.

We shall consider the equation for h_{ij}. The terms in Eq.(2.63) which contain explicitly α are evaluated in the conformal slice as

$$
\begin{aligned}
\tilde{D}_i\alpha &= -2\alpha\tilde{D}_i\psi(1+\epsilon^2+\epsilon^4), \\
\tilde{\Delta}\alpha &= -2\alpha(1+\epsilon^2+\epsilon^4)\tilde{\Delta}\psi \\
&= +4\alpha(\tilde{D}_k\psi)(\tilde{D}^k\psi)(1-\epsilon+2\epsilon^2-2\epsilon^3+3\epsilon^4+2\epsilon^6+\epsilon^8), \\
\tilde{D}_i\tilde{D}_j\alpha &= -2\alpha(1+\epsilon^2+\epsilon^4)\tilde{D}_i\tilde{D}_j\psi \\
&= +4\alpha(\tilde{D}_i\psi)(\tilde{D}_j\psi)(1-\epsilon+2\epsilon^2-2\epsilon^3+3\epsilon^4+2\epsilon^6+\epsilon^8).
\end{aligned}
\tag{3.6}
$$

Then Eq.(2.63) is written as

$$
\begin{aligned}
\Box h_{ij} = &-\Big(\frac{\alpha^2}{\psi^4} - 1\Big)\Delta_{flat}h_{ij} + \Big(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial t^2}\Big)h_{ij} \\
&+\frac{4\alpha^2}{\psi^4}\Big[\Big(\tilde{D}_i\tilde{D}_j\psi - \frac{\tilde{\gamma}_{ij}}{3}\tilde{D}_k\tilde{D}^k\psi\Big)\frac{\epsilon}{\psi}(1+\epsilon+\epsilon^2+\epsilon^3+\epsilon^4) \\
&\qquad -\frac{1}{\psi^2}\Big(\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\psi\Big) \\
&\qquad\qquad (3+6\epsilon+6\epsilon^2+6\epsilon^3+6\epsilon^4+12\epsilon^5+10\epsilon^6+8\epsilon^7+6\epsilon^8+4\epsilon^9+2\epsilon^{10})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+2\alpha^2(K\tilde{A}_{ij}-2\tilde{A}_{il}\tilde{A}^l{}_j)+2\alpha\Big(\beta^m{}_{,i}\tilde{A}_{mj}+\beta^m{}_{,j}\tilde{A}_{mi}-\frac{2}{3}\beta^m{}_{,m}\tilde{A}_{ij}\Big)\\
&-16\pi\frac{\alpha^2}{\psi^4}\Big(S_{ij}-\frac{1}{3}\gamma_{ij}S^l{}_l\Big)-\frac{d}{dt}\Big(\beta^m{}_{,i}\tilde{\gamma}_{mj}+\beta^m{}_{,j}\tilde{\gamma}_{mi}-\frac{2}{3}\beta^m{}_{,m}\tilde{\gamma}_{ij}\Big)+2\frac{\partial\alpha}{\partial n}\tilde{A}_{ij}\\
\equiv\ &\tau_{ij},
\end{aligned}
\tag{3.7}
$$

where we use $\epsilon=\psi-1$ and ψ satisfies

$$
\tilde{\Delta}\psi=-2\pi\rho_H\psi^5-\frac{\psi^5}{8}\Big(\tilde{A}_{ij}\tilde{A}^{ij}-\frac{2}{3}K^2\Big). \tag{3.8}
$$

Eq.(3.7) is expanded as follows;

$$
\begin{aligned}
\Box h_{ij}&=\Big(UU_{,ij}-\frac{1}{3}\delta_{ij}U\Delta_{flat}U-3U_{,i}U_{,j}+\delta_{ij}U_{,k}U_{,k}\Big)\\
&\quad-16\pi\Big(\rho v^iv^j-\frac{1}{3}\delta_{ij}\rho v^2\Big)-\Big({}_{(3)}\dot{\beta}_{i,j}+{}_{(3)}\dot{\beta}_{j,i}-\frac{2}{3}\delta_{ij\,(3)}\dot{\beta}_{k,k}\Big)+O(c^{-6})\\
&={}_{(4)}\tau_{ij}+O(c^{-6}),
\end{aligned}
\tag{3.9}
$$

where we use $\epsilon=2U$ which holds in the PN approximation.

In the conformal slice, the asymptotic form of ϵ becomes

$$
\epsilon\sim\frac{M_{ADM}}{2r}. \tag{3.10}
$$

Thus α behaves as $1-M_{ADM}/r$ at spatial infinity. This means that in the conformal slice, the metric at spatial infinity becomes the static Schwarzschild's one. This property seems helpful for discerning the wave part from the non-wave part in the wave zone in numerical relativity.

Also, we have an advantage to derive simply the radiation reaction potential in this slice; From a relation ${}_{(7)}\alpha=-2{}_{(7)}\psi$ and Eq.(2.53), we have

$$
\Delta_{flat\,(7)}\alpha=-{}_{(5)}h_{ij}(t)U_{,ij}. \tag{3.11}
$$

Thus, the radiation reaction potential ${}_{(7)}\alpha$ is derived as

$$
{}_{(7)}\alpha=\frac{{}_{(5)}h_{ij}(t)}{4\pi}\int U_{,ij}d^3x. \tag{3.12}
$$

Finally, we comment on the following week point of the conformal slice; in the conformal slice, the evolution equation for ${}_{(3)}K$ becomes

$$
{}_{(3)}\dot{K} = 4\pi\rho\Big(v^2 + 3\frac{P}{\rho} - \frac{1}{2}U\Big). \tag{3.13}
$$

Since $\dot{K}$ does not vanish, K continues to change even in the case of a stationary spacetime. Thus, it seems inconvenient to describe equilibrium configurations of stars and binary systems in the conformal slice. To describe equiriblium configurations, we had better use the slice, such as the maximal slice, where $\dot{K} = 0$ is satisfied.

B. Maximal Slice

The maximal slice is given by

$$
K = 0, \tag{3.14}
$$

and this equation leads to the equation for α as

$$
D_k D^k \alpha = \alpha\Big(\tilde{A}_{ij}\tilde{A}^{ij} + 4\pi(E + S^l{}_l)\Big). \tag{3.15}
$$

At the first PN order, the equation becomes

$$
\Delta_{(4)}\alpha = 4\pi(2\rho v^2 + \rho\epsilon + \rho U + 3P) + U_{,i}U_{,i}, \tag{3.16}
$$

Hence we obtain

$$
\Delta_{flat} X_{MS} = 4\pi\rho\Big(2v^2 + \varepsilon + \frac{3P}{\rho} + 2U\Big), \tag{3.17}
$$

where the subscript MS denotes "maximal slice". In the case of the conformal slice, the following relation holds;

$$
X_{CS} = -2{}_{(4)}\psi, \tag{3.18}
$$

where CS similarly denotes "conformal slice". Using the above equation, we rewrite X_{MS} as

$$
X_{MS} = -2{}_{(4)}\psi + Y. \tag{3.19}
$$

Then the equation for Y becomes

$$\Delta_{flat} Y = 4\pi\Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big). \tag{3.20}$$

We should also note that by means of the virial theorem (Chandrasekhar 1969b), the integration of the source term for Y can be written as

$$\int \Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big) d^3x = \frac{1}{2}\ddot{I}_{ll}, \tag{3.21}$$

where

$$I_{ij}(t) = \int \rho x^i x^j d^3x. \tag{3.22}$$

Hence, the behavior of Y far from the matter becomes

$$Y \sim -\frac{1}{2r}\ddot{I}_{ll}. \tag{3.23}$$

In total, the behavior of α in the wave zone becomes

$$\alpha \sim 1 - \frac{1}{r}\Big(M + \frac{1}{2}\ddot{I}_{ll}\Big). \tag{3.24}$$

Therefore, contrary to the conformal slice, in the maximal slice, the spurious time-dependent term is included in α in the wave zone. Since the metric does not approach the static Schwarzschild metric even in spatial infinity, the maximal slice is inconvenient to distinguish a wave part from non-wave parts such as the Newtonian potential.

At the 2PN order, the lapse function is given by

$$\Delta_{flat\,(6)}\alpha = 4\pi\rho\Big(2v^4 + 7v^2U + 2\varepsilon v^2 - \varepsilon U + 2v^i{}_{(3)}\beta_i + \frac{U^2}{2} + X_{MS} + \frac{P}{\rho}(2v^2 - 5U)\Big). \tag{3.25}$$

In the case of the maximal slice, the equations for the shift vector are obtained by simply taking $K = 0$ in Eqs.(2.59) and (2.61). Also, it is found that the equation for ${}_{(7)}\alpha$ is the same as that in the conformal slice: The right-hand side of Eq.(3.15) has no $O(c^{-7})$ terms. Therefore Eq.(3.15) becomes

$$\Delta_{flat(7)}\alpha = -_{(5)}h_{ij}U_{,ij}. \tag{3.26}$$

Finally, we show the wave equation for h_{ij} in the maximal slice as

$$\Box h_{ij} = -2\Big(Y_{,ij} - \frac{1}{3}\delta_{ij}\Delta_{flat}Y\Big) + \Big(UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta_{flat}U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}\Big)$$
$$-16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) - \Big({}_{(3)}\dot\beta_{i,j} + {}_{(3)}\dot\beta_{j,i} - \frac{2}{3}\delta_{ij\,(3)}\dot\beta_{k,k}\Big) + O(c^{-6}). \tag{3.27}$$

C. Harmonic Slice

The condition for the harmonic slice is

$$\Box t = 0, \tag{3.28}$$

which becomes in the (3+1) terminology,

$$\dot\alpha + \alpha^2 K - \beta^i\alpha_{,i} = 0. \tag{3.29}$$

Differentiating this equation with respect to time, we obtain

$$\ddot\alpha + 2\alpha\dot\alpha K + \alpha^2\dot K - \dot\beta^i\alpha_{,i} - \beta^i\dot\alpha_{,i} = 0. \tag{3.30}$$

We use Eq.(2.17) in order to eliminate $\dot K$ in this equation. Hence the wave equation for the lapse function is derived as

$$\begin{aligned}\Box\alpha &= 4\pi\alpha^3(S^l{}_l - 3\rho_H) - \Big(\frac{\alpha^2}{\psi^4}\tilde\Delta - \Delta_{flat}\Big)\alpha - \frac{2\alpha^2}{\psi^5}\tilde D^l\psi\tilde D_l\alpha - \frac{8\alpha^3}{\psi^5}\tilde\Delta\psi\\ &\quad +2\alpha\dot\alpha K + \frac{\alpha^3}{\psi^4}\tilde R + \alpha^3K^2 + \alpha^2\beta^i\tilde D_iK - \dot\beta^i\alpha_{,i} - \beta^i\dot\alpha_{,i}\\ &\equiv \Lambda_\alpha,\end{aligned} \tag{3.31}$$

where Λ_α is expanded as follows,

$$\Lambda_\alpha = 4\pi\rho\Big[1 + \Big(v^2 + 3\frac{P}{\rho} - \frac{U}{2}\Big)\Big] + \Delta_{flat}\Big(\frac{U^2}{2} - 2_{(4)}\psi\Big) + O(c^{-6}). \tag{3.32}$$

This equation is formally solved by using the retarded Green function and the Taylor expansion. For example, we obtain the Newtonian and first PN order lapse function

$$
{}_{(2)}\alpha = -\int d^3y \frac{\rho(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} = -\frac{1}{r}\int \rho d^3y + O(r^{-2}),
$$
$$
{}_{(4)}\alpha = -\frac{1}{2}\int d^3y \ddot{\rho}|\mathbf{x}-\mathbf{y}| - \int d^3y \frac{(\rho v^2 + 3P - \frac{1}{2}\rho U)}{|\mathbf{x}-\mathbf{y}|} + \frac{1}{2}U^2 - 2{}_{(4)}\psi. \tag{3.33}
$$

Thus, at the spatial infinity, we find the following behavior

$$
{}_{(4)}\alpha + 2{}_{(4)}\psi \sim -\frac{3}{4r}\Big(\ddot{I}_{kk} - \frac{1}{3}n^k n^l \ddot{I}_{kl}\Big), \tag{3.34}
$$

where $n^i = x^i/r$. From these equations we find that at the spatial infinity the lapse function does not behave as $1 - M/r + O(r^{-2})$ unlike in the conformal slice, but behaves as $1-(M+T(t))/r + O(r^{-2})$. Thus the harmonic slice is also inconvenient to distinguish a wave part from non-wave parts.

The quadrupole radiation reaction potential takes the following rather lengthy form.

$$
{}_{(7)}\alpha = \frac{1}{480\pi}\frac{\partial^5}{\partial t^5}\int d^3y \rho|\mathbf{x}-\mathbf{y}|^4 + \frac{1}{24\pi}\frac{\partial^3}{\partial t^3}\int d^3y {}_{(4)}\Lambda_\alpha |\mathbf{x}-\mathbf{y}|^2 + \frac{1}{4\pi}\frac{\partial}{\partial t}\int d^3y {}_{(6)}\Lambda_\alpha
$$
$$
+\frac{1}{4\pi}\int d^3y \frac{{}_{(5)}h_{ij}U_{,ij}}{|\mathbf{x}-\mathbf{y}|}. \tag{3.35}
$$

This expression is similar to Chandrasekhar and Nutku's one (1969) in the harmonic gauge and indicates that the fifth time derivative of the quadrupole moment appears in the reaction force, which is not convenient to treat in numerical calculations.

IV. STRATEGY TO OBTAIN 2PN TENSOR POTENTIAL

In this section, we describe methods to solve the equation for the 2PN tensor potential ${}_{(4)}h_{ij}$. Although Eq.(2.65) is formally solved as

$$ {}_{(4)}h_{ij}(t,\mathbf{x}) = -\frac{1}{4\pi}\int \frac{{}_{(4)}\tau_{ij}(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d^3y, \tag{4.1} $$

it seems difficult to estimate this integral in practice since ${}_{(4)}\tau_{ij} \to O(r^{-3})$ for $r \to \infty$ and the integral is taken all over the space. Thus it is desirable to replace this equation by some tractable forms in numerical evaluation. In the following, we show two approaches: In section 4-A, we change Eq.(4.1) into the form in which the integration is performed only over the matter distribution like as in the Newtonian potential. In section 4-B, we propose a method to solve Eq.(2.65) as the boundary value problem.

A. Direct integration method

The explicit form of ${}_{(4)}\tau_{ij}$ is

$$ \begin{aligned} {}_{(4)}\tau_{ij} = & -2\hat{\partial}_{ij}(X + 2{}_{(4)}\psi) + U\hat{\partial}_{ij}U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k} - 16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) \\ & -\Big({}_{(3)}\dot{\beta}_{i,j} + {}_{(3)}\dot{\beta}_{j,i} - \frac{2}{3}\delta_{ij}{}_{(3)}\dot{\beta}_{k,k}\Big), \end{aligned} \tag{4.2} $$

where

$$ \hat{\partial}_{ij} \equiv \frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\Delta_{flat}. \tag{4.3} $$

Although ${}_{(4)}\tau_{ij}$ looks as if it depends on the slice condition, the independence is shown as follows. Eq.(2.59) is rewritten as

$$ \Delta_{flat}{}_{(3)}\beta_i = \Delta_{flat}p_i + {}_{(3)}K_{,i}, \tag{4.4} $$

where

$$ p_i = -4\int \frac{\rho v^i}{|\mathbf{x}-\mathbf{y}|}d^3y - \frac{1}{2}\Big(\int \dot{\rho}|\mathbf{x}-\mathbf{y}|d^3y\Big)_{,i}\ . \tag{4.5} $$

This is solved as

$$ {}_{(3)}\beta_i = p_i - \frac{1}{4\pi}\Big(\int \frac{{}_{(3)}K}{|\mathbf{x}-\mathbf{y}|}d^3y\Big)_{,i} \ . \tag{4.6} $$

From Eqs.(2.51) and (2.68), we obtain

$$ {}_{(3)}\dot{K} = -\Delta_{flat}(X + 2{}_{(4)}\psi) + 4\pi\rho\Big(v^2 + 3\frac{P}{\rho} - \frac{U}{2}\Big). \tag{4.7} $$

Combining Eq.(4.6) with Eq.(4.7), the equation for ${}_{(3)}\dot{\beta}_i$ is written as

$$ {}_{(3)}\dot{\beta}_i = \dot{p}_i - (X + 2{}_{(4)}\psi)_{,i} - \Big[\int \frac{\big(\rho v^2 + 3P - \rho U/2\big)}{|\mathbf{x}-\mathbf{y}|}d^3y\Big]_{,i} \ . \tag{4.8} $$

Using this relation, the source term, ${}_{(4)}\tau_{ij}$, is split into five parts

$$ {}_{(4)}\tau_{ij} = {}_{(4)}\tau_{ij}^{(S)} + {}_{(4)}\tau_{ij}^{(U)} + {}_{(4)}\tau_{ij}^{(C)} + {}_{(4)}\tau_{ij}^{(\rho)} + {}_{(4)}\tau_{ij}^{(V)}, \tag{4.9} $$

where we introduce

$$ \begin{aligned} {}_{(4)}\tau_{ij}^{(S)} &= -16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big), \\ {}_{(4)}\tau_{ij}^{(U)} &= UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta_{flat}U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}, \\ {}_{(4)}\tau_{ij}^{(C)} &= 4\frac{\partial}{\partial x^j}\int \frac{(\rho v^i)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y + 4\frac{\partial}{\partial x^i}\int \frac{(\rho v^j)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \frac{8}{3}\delta_{ij}\frac{\partial}{\partial x^k}\int \frac{(\rho v^k)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y, \\ {}_{(4)}\tau_{ij}^{(\rho)} &= \hat{\partial}_{ij}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|d^3y, \\ {}_{(4)}\tau_{ij}^{(V)} &= 2\hat{\partial}_{ij}\int \frac{\big(\rho v^2 + 3P - \rho U/2\big)}{|\mathbf{x}-\mathbf{y}|}d^3y. \end{aligned} \tag{4.10} $$

Thus it becomes clear that ${}_{(4)}h_{ij}$ and ${}_{(5)}h_{ij}$ as well as ${}_{(4)}\tau_{ij}$ are expressed in terms of matter variables only and thus do not depend on slicing conditions.

Then, we define $\Delta_{flat\,(4)}h_{ij}^{(S)} = {}_{(4)}\tau_{ij}^{(S)}$, $\Delta_{flat\,(4)}h_{ij}^{(U)} = {}_{(4)}\tau_{ij}^{(U)}$, $\Delta_{flat\,(4)}h_{ij}^{(C)} = {}_{(4)}\tau_{ij}^{(C)}$, $\Delta_{flat\,(4)}h_{ij}^{(\rho)} = {}_{(4)}\tau_{ij}^{(\rho)}$ and $\Delta_{flat\,(4)}h_{ij}^{(V)} = {}_{(4)}\tau_{ij}^{(V)}$, and consider each term separately. First, since ${}_{(4)}\tau_{ij}^{(S)}$ is a compact source, we immediately obtain

$$ {}_{(4)}h_{ij}^{(S)} = 4\int \frac{\big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\big)}{|\mathbf{x}-\mathbf{y}|}d^3y. \tag{4.11} $$

Second, we consider the following equation

$$\Delta_{flat} G(\mathbf{x},\mathbf{y_1},\mathbf{y_2}) = \frac{1}{|\mathbf{x}-\mathbf{y_1}||\mathbf{x}-\mathbf{y_2}|}. \tag{4.12}$$

It is possible to write ${}_{(4)}h^{(U)}_{ij}$ using integrals over the matter if this function, G, is used. Eq.(4.12) has solutions (Fock 1959; Ohta, Okamura, Kimura and Hiida 1974; See also appendix B),

$$G(\mathbf{x},\mathbf{y_1},\mathbf{y_2}) = \ln(r_1 + r_2 \pm r_{12}), \tag{4.13}$$

where

$$\begin{aligned} r_1 &= |\mathbf{x}-\mathbf{y_1}|, \\ r_2 &= |\mathbf{x}-\mathbf{y_2}|, \\ r_{12} &= |\mathbf{y_1}-\mathbf{y_2}|. \end{aligned} \tag{4.14}$$

Note that $\ln(r_1+r_2-r_{12})$ is not regular on the interval between $\mathbf{y_1}$ and $\mathbf{y_2}$, while $\ln(r_1 + r_2 + r_{12})$ is regular on the matter. Thus we use $\ln(r_1+r_2+r_{12})$ as a kernel. Using this function, $UU_{,ij}$ and $U_{,i}U_{,j}$ are rewritten as

$$\begin{aligned} UU_{,ij} &= \Big[\frac{\partial^2}{\partial x^i \partial x^j}\Big(\int \frac{\rho(\mathbf{y_1})}{|\mathbf{x}-\mathbf{y_1}|} d^3y_1\Big)\Big]\Big(\int \frac{\rho(\mathbf{y_2})}{|\mathbf{x}-\mathbf{y_2}|} d^3y_2\Big) \\ &= \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_1^j}\Big(\frac{1}{|\mathbf{x}-\mathbf{y_1}||\mathbf{x}-\mathbf{y_2}|}\Big) \\ &= \Delta_{flat} \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_1^j} \ln(r_1+r_2+r_{12}), \\ U_{,i}U_{,j} &= \Big(\frac{\partial}{\partial x^i}\int \frac{\rho(\mathbf{y_1})}{|\mathbf{x}-\mathbf{y_1}|} d^3y_1\Big)\Big(\frac{\partial}{\partial x^j}\int \frac{\rho(\mathbf{y_2})}{|\mathbf{x}-\mathbf{y_2}|} d^3y_2\Big) \\ &= \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_2^j}\Big(\frac{1}{|\mathbf{x}-\mathbf{y_1}||\mathbf{x}-\mathbf{y_2}|}\Big) \\ &= \Delta_{flat} \int d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \frac{\partial^2}{\partial y_1^i \partial y_2^j} \ln(r_1+r_2+r_{12}). \end{aligned} \tag{4.15}$$

Thus we can express ${}_{(4)}h^{(U)}_{ij}$ using the integral over the matter as

$$\begin{aligned} {}_{(4)}h^{(U)}_{ij} = \int & d^3y_1 d^3y_2 \rho(\mathbf{y_1})\rho(\mathbf{y_2}) \\ & \Big[\Big(\frac{\partial^2}{\partial y_1^i \partial y_1^j} - \frac{1}{3}\delta_{ij}\triangle_1\Big) - 3\Big(\frac{\partial^2}{\partial y_1^i \partial y_2^j} - \frac{1}{3}\delta_{ij}\triangle_{12}\Big)\Big] \ln(r_1+r_2+r_{12}), \end{aligned} \tag{4.16}$$

where we introduce

$$\triangle_1 = \frac{\partial^2}{\partial y_1^k \partial y_1^k},$$
$$\triangle_{12} = \frac{\partial^2}{\partial y_1^k \partial y_2^k}. \tag{4.17}$$

Using relations $\Delta_{flat}|\mathbf{x}-\mathbf{y}| = 2/|\mathbf{x}-\mathbf{y}|$ and $\Delta_{flat}|\mathbf{x}-\mathbf{y}|^3 = 12|\mathbf{x}-\mathbf{y}|$, ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$ are solved as

$${}_{(4)}h_{ij}^{(C)} = 2\frac{\partial}{\partial x^i}\int (\rho v^j)^{\cdot}|\mathbf{x}-\mathbf{y}|d^3y + 2\frac{\partial}{\partial x^j}\int (\rho v^i)^{\cdot}|\mathbf{x}-\mathbf{y}|d^3y + \frac{4}{3}\delta_{ij}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|d^3y, \tag{4.18}$$

$${}_{(4)}h_{ij}^{(\rho)} = \frac{1}{12}\frac{\partial^2}{\partial x^i \partial x^j}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|^3 d^3y - \frac{1}{3}\delta_{ij}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|d^3y, \tag{4.19}$$

$${}_{(4)}h_{ij}^{(V)} = \frac{\partial^2}{\partial x^i \partial x^j}\int \Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)|\mathbf{x}-\mathbf{y}|d^3y - \frac{2}{3}\delta_{ij}\int \frac{\big(\rho v^2 + 3P - \rho U/2\big)}{|\mathbf{x}-\mathbf{y}|}d^3y. \tag{4.20}$$

In total, we obtain

$${}_{(4)}h_{ij} = {}_{(4)}h_{ij}^{(S)} + {}_{(4)}h_{ij}^{(U)} + {}_{(4)}h_{ij}^{(C)} + {}_{(4)}h_{ij}^{(\rho)} + {}_{(4)}h_{ij}^{(V)}. \tag{4.21}$$

B. Treatment as a boundary value problem

The above expression for ${}_{(4)}h_{ij}$ is quite interesting because it only consists of integrals over the matter. However, in actual numerical simulations, it will take a very long time to perform the direct integration. Therefore, we also propose other strategies where Eq.(2.65) is solved as the boundary value problem. Here, we would like to emphasize that the boundary condition should be imposed at $r(=|\mathbf{x}|) \gg |\mathbf{y_1}|, |\mathbf{y_2}|$, but r does not have to be greater than λ, where λ is a typical wave length of gravitational waves. We only need to impose $r > R$ (a typical size of matter). This means that we do not need a large amount of grid numbers compared with the case of fully general relativistic simulations, in which we require $r > \lambda \gg R$.

First of all, we consider the equation

$$\Delta_{flat}\Big({}_{(4)}h_{ij}^{(S)} + {}_{(4)}h_{ij}^{(U)}\Big) = {}_{(4)}\tau_{ij}^{(S)} + {}_{(4)}\tau_{ij}^{(U)}. \tag{4.22}$$

Since its source term behaves as $O(r^{-6})$ at $r \to \infty$, this equation can be accurately solved under the boundary condition at $r > R$ as

$$\begin{aligned} {}_{(4)}h_{ij}^{(S)} + {}_{(4)}h_{ij}^{(U)} = & \frac{2}{r}\Big(\ddot{I}_{ij} - \frac{1}{3}\delta_{ij}\ddot{I}_{kk}\Big) \\ & + \frac{2}{3r^2}\Big(n^k \dddot{I}_{ijk} - \frac{1}{3}\delta_{ij} n^k \dddot{I}_{llk} + 2n^k(\dot{S}_{ikj} + \dot{S}_{jki}) - \frac{4}{3}\delta_{ij} n^k \dot{S}_{lkl}\Big) + O(r^{-3}), \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} I_{ijk} &= \int \rho x^i x^j x^k d^3x, \\ S_{ijk} &= \int \rho(v^i x^j - v^j x^i) x^k d^3x. \end{aligned} \tag{4.24}$$

Next, we consider the equations for ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$. Noting the identity,

$$\ddot{\rho} = -(\rho v^i)^{\cdot}_{,i} = (\rho v^i v^j)_{,ij} + \Delta_{flat} P - (\rho U_{,i})_{,i}\ , \tag{4.25}$$

we find the following relations;

$$\begin{aligned} & \int \ddot{\rho} |\mathbf{x}-\mathbf{y}| d^3y = -\int d^3y \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}(\rho v^i)^{\cdot}, \\ & \int \ddot{\rho} |\mathbf{x}-\mathbf{y}|^3 d^3y = 3\int d^3y \Big[\rho v^i v^j \frac{(x^i-y^i)(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|} + \Big(4P + \rho v^2 - \rho U_{,i}(x^i - y^i)\Big)|\mathbf{x}-\mathbf{y}|\Big]. \end{aligned} \tag{4.26}$$

Using Eqs.(4.26), ${}_{(4)}h_{ij}^{(C)}$, ${}_{(4)}h_{ij}^{(\rho)}$ and ${}_{(4)}h_{ij}^{(V)}$ in Eqs.(4.18 – 4.20) can be rewritten as

$${}_{(4)}h_{ij}^{(C)} = 2\int (\rho v^j)^{\cdot} \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|} d^3y + 2\int (\rho v^i)^{\cdot} \frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|} d^3y - \frac{4}{3}\delta_{ij}\int (\rho v^k)^{\cdot} \frac{x^k - y^k}{|\mathbf{x}-\mathbf{y}|} d^3y, \tag{4.27}$$

$$\begin{aligned} {}_{(4)}h_{ij}^{(\rho)} = & \frac{1}{4}\frac{\partial^2}{\partial x^i \partial x^j}\int \rho v^k v^l \frac{(x^k-y^k)(x^l-y^l)}{|\mathbf{x}-\mathbf{y}|} d^3y + \frac{1}{3}\delta_{ij}\int (\rho v^k)^{\cdot} \frac{x^k - y^k}{|\mathbf{x}-\mathbf{y}|} d^3y \\ & + \frac{1}{2}\Big\{\frac{\partial}{\partial x^i}\int P' \frac{(x^j - y^j)}{|\mathbf{x}-\mathbf{y}|} d^3y + \frac{\partial}{\partial x^j}\int P' \frac{(x^i - y^i)}{|\mathbf{x}-\mathbf{y}|} d^3y\Big\} \\ & - \frac{1}{8}\Big\{2\int \rho \frac{U_{,j}(x^i - y^i) + U_{,i}(x^j - y^j)}{|\mathbf{x}-\mathbf{y}|} d^3y \\ & \qquad + x^k \frac{\partial}{\partial x^i}\int \rho \frac{U_{,k}(x^j - y^j)}{|\mathbf{x}-\mathbf{y}|} d^3y + x^k \frac{\partial}{\partial x^j}\int \rho \frac{U_{,k}(x^i - y^i)}{|\mathbf{x}-\mathbf{y}|} d^3y\Big\}, \end{aligned} \tag{4.28}$$

and

$$
{}_{(4)}h^{(V)}_{ij} = \frac{1}{2}\Big[\frac{\partial}{\partial x^i}\int\Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)\frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|}d^3y + \frac{\partial}{\partial x^j}\int\Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)\frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}d^3y\Big]
$$

$$
-\frac{2}{3}\delta_{ij}\int\frac{\Big(\rho v^2 + 3P - \rho U/2\Big)}{|\mathbf{x}-\mathbf{y}|}d^3y, \tag{4.29}
$$

where $P' = P + \rho v^2/4 + \rho U_{,l}y^l/4$. Hence, ${}_{(4)}h^{(C)}_{ij}$, ${}_{(4)}h^{(\rho)}_{ij}$ and ${}_{(4)}h^{(V)}_{ij}$ become

$$
\begin{aligned}
{}_{(4)}h^{(C)}_{ij} &= 2\Big[\Big(x^i\int\frac{(\rho v^j)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{(\rho v^j)^{\cdot}y^i}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) + \Big(x^j\int\frac{(\rho v^i)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{(\rho v^i)^{\cdot}y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big)\Big] \\
&\quad -\frac{4}{3}\delta_{ij}\Big[x^k\int\frac{(\rho v^k)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{(\rho v^k)^{\cdot}y^k}{|\mathbf{x}-\mathbf{y}|}d^3y\Big] \\
&= 2(x^i{}_{(3)}\dot{P}^j + x^j{}_{(3)}\dot{P}^i - Q_{ij}) + \frac{4}{3}\delta_{ij}\Big(\frac{Q_{kk}}{2} - x^k{}_{(3)}\dot{P}^k\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}_{(4)}h^{(\rho)}_{ij} &= \frac{1}{4}\frac{\partial^2}{\partial x^i\partial x^j}\Big(x^kx^l\int\rho\frac{v^kv^l}{|\mathbf{x}-\mathbf{y}|}d^3y - 2x^k\int\rho\frac{v^kv^ly^l}{|\mathbf{x}-\mathbf{y}|}d^3y + \int\rho\frac{v^kv^ly^ky^l}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) \\
&\quad +\frac{1}{3}\delta_{ij}\Big(x^k\int\frac{(\rho v^k)^{\cdot}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{(\rho v^k)^{\cdot}y^k}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) \\
&\quad +\frac{1}{2}\Big(\frac{\partial}{\partial x^i}\int P'\frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|}d^3y + \frac{\partial}{\partial x^j}\int P'\frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) \\
&\quad -\frac{1}{8}\Big[2\Big(\int\rho U_{,j}\frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}d^3y + \int\rho U_{,i}\frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) \\
&\qquad +x^k\Big(\frac{\partial}{\partial x^i}\int\rho U_{,k}\frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|}d^3y + \frac{\partial}{\partial x^j}\int\rho U_{,k}\frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|}d^3y\Big)\Big] \\
&= \frac{1}{4}\frac{\partial^2}{\partial x^i\partial x^j}\Big(V^{(\rho v)}_{kl}x^kx^l - 2V^{(\rho v)}_{k}x^k + V^{(\rho v)}\Big) + \frac{1}{3}\delta_{ij}\Big(x^k{}_{(3)}\dot{P}_k - \frac{Q_{kk}}{2}\Big) \\
&\quad +\frac{1}{2}\Big\{\frac{\partial}{\partial x^i}\Big(V^{(P)}x^j - V^{(P)}_j\Big) + \frac{\partial}{\partial x^j}\Big(V^{(P)}x^i - V^{(P)}_i\Big)\Big\} \\
&\quad -\frac{1}{8}\Big\{2\Big(x^iV^{(\rho U)}_j + x^iV^{(\rho U)}_i - V^{(\rho U)}_{ij} - V^{(\rho U)}_{ji}\Big) \\
&\qquad +x^k\frac{\partial}{\partial x^i}\Big(x^jV^{(\rho U)}_k - V^{(\rho U)}_{kj}\Big) + x^k\frac{\partial}{\partial x^j}\Big(x^iV^{(\rho U)}_k - V^{(\rho U)}_{ki}\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
{}_{(4)}h^{(V)}_{ij} &= \frac{1}{2}\Big[\frac{\partial}{\partial x^i}\Big(x^j\int\frac{\rho v^2 + 3P - \frac{\rho U}{2}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{\Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big) \\
&\qquad +\frac{\partial}{\partial x^j}\Big(x^i\int\frac{\rho v^2 + 3P - \frac{\rho U}{2}}{|\mathbf{x}-\mathbf{y}|}d^3y - \int\frac{\Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big)\Big] \\
&\quad -\frac{2}{3}\delta_{ij}\int\frac{\rho v^2 + 3P - \frac{\rho U}{2}}{|\mathbf{x}-\mathbf{y}|}d^3y \\
&= \frac{1}{2}\Big(Q^{(I)}_{,j}x^i + Q^{(I)}_{,i}x^j - Q^{(I)}_{i,j} - Q^{(I)}_{j,i}\Big) + \frac{1}{3}Q^{(I)}\delta_{ij},
\end{aligned} \tag{4.30}
$$

where

$$
\begin{aligned}
&\Delta_{flat(3)}P_i = -4\pi\rho v^i, \\
&\Delta_{flat}Q_{ij} = -4\pi\Big\{x^j(\rho v^i)^{\cdot} + x^i(\rho v^j)^{\cdot}\Big\}, \\
&\Delta_{flat}Q^{(I)} = -4\pi\Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big), \\
&\Delta_{flat}Q^{(I)}_i = -4\pi\Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big)x^i, \\
&\Delta_{flat}V^{(\rho v)}_{ij} = -4\pi\rho v^i v^j, \\
&\Delta_{flat}V^{(\rho v)}_i = -4\pi\rho v^i v^j x^j, \\
&\Delta_{flat}V^{(\rho v)} = -4\pi\rho(v^j x^j)^2, \\
&\Delta_{flat}V^{(P)} = -4\pi P', \\
&\Delta_{flat}V^{(P)}_i = -4\pi P' x^i, \\
&\Delta_{flat}V^{(\rho U)}_i = -4\pi\rho U_{,i}, \\
&\Delta_{flat}V^{(\rho U)}_{ij} = -4\pi\rho U_{,i}x^j.
\end{aligned}
\tag{4.31}
$$

Therefore, ${}_{(4)}h^{(C)}_{ij}$, ${}_{(4)}h^{(\rho)}_{ij}$ and ${}_{(4)}h^{(V)}_{ij}$ can be derived from the above potentials which satisfy the Poisson equations with compact sources.

We note that instead of the above procedure, we may solve the Poisson equation for ${}_{(4)}h_{ij}$ carefully imposing the boundary condition for $r \gg R$ as

$$
\begin{aligned}
{}_{(4)}h_{ij} = &\frac{1}{r}\Big\{\frac{1}{4}I^{(2)}_{ij} + \frac{3}{4}n^k\Big(n^i I^{(2)}_{kj} + n^j I^{(2)}_{ki}\Big) \\
&-\frac{5}{8}n^i n^j I^{(2)}_{kk} + \frac{3}{8}n^i n^j n^k n^l I^{(2)}_{kl} + \frac{1}{8}\delta_{ij}I^{(2)}_{kk} - \frac{5}{8}\delta_{ij}n^k n^l I^{(2)}_{kl}\Big\} \\
+&\frac{1}{r^2}\Big[\Big\{-\frac{5}{12}n^k I^{(2)}_{ijk} - \frac{1}{24}(n^i I^{(2)}_{jkk} + n^j I^{(2)}_{ikk}) + \frac{5}{8}n^k n^l(n^i I^{(2)}_{jkl} + n^j I^{(2)}_{ikl}) \\
&-\frac{7}{8}n^i n^j n^k I^{(2)}_{kll} + \frac{5}{8}n^i n^j n^k n^l n^m I^{(2)}_{klm} + \frac{11}{24}\delta_{ij}n^k I^{(2)}_{kll} - \frac{5}{8}\delta_{ij}n^k n^l n^m I^{(2)}_{klm}\Big\} \\
&+\Big\{\frac{2}{3}n^k(\dot{S}_{ikj} + \dot{S}_{jki}) - \frac{4}{3}(n^i\dot{S}_{jkk} + n^j\dot{S}_{ikk}) \\
&+2n^k n^l(n^i\dot{S}_{jkl} + n^j\dot{S}_{ikl}) + 2n^i n^j n^k\dot{S}_{kll} + \frac{2}{3}\delta_{ij}n^k\dot{S}_{kll}\Big\}\Big] + O(r^{-3}).
\end{aligned}
\tag{4.32}
$$

It is verified that $O(r^{-1})$ and $O(r^{-2})$ parts satisfy the traceless and divergence-free conditions respectively. It should be noted that ${}_{(4)}h_{ij}$ obtained in this way becomes meaningless at the far zone because Eq.(2.65), from which ${}_{(4)}h_{ij}$ is derived, is valid only in the near zone.

V. THE RADIATION REACTION DUE TO QUADRUPOLE RADIATION

This topic has been already investigated by using some gauge conditions in previous papers (Chandrasekhar and Esposito 1970; Schäfer 1985; Blanchet, Damour and Schäfer 1990). However, if we use the combination of the conformal slice and the transverse gauge, calculations are simplified. This is why we briefly mention the derivation of the radiation reaction potential in this section.

A. conformal slice

In combination of the conformal slice and the transverse gauge, Eq.(2.66) becomes

$$\begin{aligned}
{}_{(5)}h_{ij}(t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\int\Big[&-16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) \\
&+\Big(UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta_{flat}U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}\Big)\Big]d^3y \\
&+\frac{1}{4\pi}\frac{\partial}{\partial t}\int\Big({}_{(3)}\dot{\beta}_{i,j} + {}_{(3)}\dot{\beta}_{j,i} - \frac{2}{3}\delta_{ij\,(3)}\dot{\beta}_{k,k}\Big)d^3y.
\end{aligned} \tag{5.1}$$

From a straightforward calculation, we find that the sum of the first and second lines becomes $-2\not{I}_{ij}^{(3)}$ and the third line becomes $6\not{I}_{ij}^{(3)}/5$, where $\not{I}_{ij}^{(3)} = d^3\not{I}_{ij}/dt^3$. (This calculation is replaced by a fairly simple one when we use the transverse property of ${}_{(4)}\tau_{ij}$. It is described in the appendix C.) Thus, ${}_{(5)}h_{ij}$ in the near zone becomes

$${}_{(5)}h_{ij} = -\frac{4}{5}\not{I}_{ij}^{(3)}, \tag{5.2}$$

where

$$\not{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I_{kk}. \tag{5.3}$$

Since h_{ij} has the transverse and traceless property, it is likely that ${}_{(5)}h_{ij}$ remains the same for other slices. However it is not clear whether the TT property of h_{ij} is satisfied even after the PN expansion is taken in the near zone and, as a result, whether ${}_{(5)}h_{ij}$ is independent of slicing conditions or not. The fact that slicing conditions never affect

${}_{(5)}h_{ij}$ is understood on the ground that ${}_{(4)}\tau_{ij}$ does not depend on slices, which have been shown in the section 4.

Then the Hamiltonian constraint at the 2.5PN order, Eq.(2.53), turns out to be

$$\Delta_{flat\,(7)}\psi = -\frac{2}{5}\not{I}^{(3)}_{ij}U_{,ij} = \frac{1}{5}\not{I}^{(3)}_{ij}\Delta_{flat}\chi_{,ij}\ , \tag{5.4}$$

where χ is the superpotential (Chandrasekhar 1965) and defined as

$$\chi = -\int \rho|\mathbf{x}-\mathbf{y}|d^3y, \tag{5.5}$$

which satisfies the relation $\Delta_{flat}\chi = -2U$. From this, we find ${}_{(7)}\psi$ takes the following form,

$$\begin{aligned}{}_{(7)}\psi &= -\frac{1}{5}\not{I}^{(3)}_{ij}\int \rho_{,i}\frac{(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|}d^3y \\ &= \frac{1}{5}\not{I}^{(3)}_{ij}\Big(-x^jU_{,i} + \int \frac{\rho_{,i}y^j}{|\mathbf{x}-\mathbf{y}|}d^3y\Big).\end{aligned} \tag{5.6}$$

Therefore, the lapse function at the 2.5PN order, ${}_{(7)}\alpha = -2{}_{(7)}\psi$, is derived from U and U_r, where U_r satisfies (Blanchet, Damour and Schäfer 1990)

$$\Delta U_r = -4\pi\not{I}^{(3)}_{ij}\rho_{,i}x^j. \tag{5.7}$$

Since the right-hand side of Eq.(2.70) cancels out, ${}_{(6)}K$ disappears if the ${}_{(6)}K$ does not exist on the initial hypersurface, which seems reasonable under the condition that there are no initial gravitational waves. Also, ${}_{(6)}\beta_i$ vanishes according to Eq.(2.62). Hence, the quadrupole radiation reaction metric has the same form as that derived in the case of the maximal slice (Schäfer 1985; Blanchet, Damour and Schäfer 1990).

From Eq.(2.35), the PN equation of motion becomes

$$\dot{v}^i + v^jv^i_{\ ,\ j} = -\frac{P_{,i}}{\rho} + U_{,i} + F_i^{1PN} + F_i^{2PN} + F_i^{2.5PN} + O(c^{-8}), \tag{5.8}$$

where F_i^{1PN} and F_i^{2PN} are, respectively, the 1PN and 2PN forces and conservative ones. Since the radiation reaction potentials, ${}_{(5)}h_{ij}$ and ${}_{(7)}\alpha$, are the same as those by Schäfer

(1985) and Blanchet, Damour and Schäfer (1990) in which they use the ADM gauge, the radiation reaction force per unit mass, $F_i^{2.5PN} \equiv F_i^r$, is the same as their force and

$$
\begin{aligned}
F_i^r &= -\Big(({}_{(5)}h_{ij}v^j)^{\cdot} + v^k v^j_{,k}\,{}_{(5)}h_{ij} + {}_{(7)}\alpha_{,i} \Big) \\
&= \Big[\frac{4}{5}(\text{Ɨ}^{(3)}_{ij} v^j)^{\cdot} + \frac{4}{5}\text{Ɨ}^{(3)}_{ij} v^k v^j_{,k} + \frac{2}{5}\text{Ɨ}^{(3)}_{kl} \frac{\partial}{\partial x^i} \int \rho(t,\mathbf{y}) \frac{(x^k-y^k)(x^l-y^l)}{|\mathbf{x}-\mathbf{y}|^3} d^3y \Big].
\end{aligned}
\tag{5.9}
$$

Since the work done by the force (5.9) is given by

$$
\begin{aligned}
W &\equiv \int \rho F_i^r v^i d^3x \\
&= \frac{4}{5}\frac{d}{dt}\Big(\text{Ɨ}^{(3)}_{ij} \int \rho v^i v^j d^3x \Big) - \frac{1}{5}\text{Ɨ}^{(3)}_{ij}\text{Ɨ}^{(3)ij},
\end{aligned}
\tag{5.10}
$$

we obtain the so-called quadrupole formula of the energy loss by averaging Eq.(5.10) with respect to time as

$$
\left\langle \frac{dE_N}{dt} \right\rangle = -\frac{1}{5}\left\langle \text{Ɨ}^{(3)}_{ij}\text{Ɨ}^{(3)ij} \right\rangle + O(c^{-6}). \tag{5.11}
$$

B. Radiation reaction in other slice conditions

In this subsection, we do not specify the slice condition. From Eq.(2.35), we obtain the equation of motion as

$$
\rho\Big(\dot{v}^i + v^j v^i_{,\,j}\Big) = -P_{,i} + \rho U_{,i} + F_i^{1PN} + F_i^{2PN} + F_i^{2.5PN} + O(c^{-8}). \tag{5.12}
$$

We can obtain $F_i^{2.5PN} \equiv F_i^r$ by using ${}_{(6)}\beta_i$ and ${}_{(7)}\alpha$, which are estimated from Eqs.(2.62) and (2.70) respectively, as

$$
F_i^r = \rho\Big[\frac{4}{5}(\text{Ɨ}^{(3)}_{ij} v^j)^{\cdot} + \frac{4}{5}\text{Ɨ}^{(3)}_{ij} v^k v^j_{,k} - {}_{(7)}\alpha_{,i} - {}_{(6)}\dot{\beta}^i + v^j{}_{(6)}\beta^j{}_{,i} - v^j{}_{(6)}\beta^i{}_{,j} \Big]. \tag{5.13}
$$

Here, ${}_{(7)}\alpha$ corresponds to the slice condition. From Eq.(5.13), we obtain the work done by the reaction force as

$$
\begin{aligned}
W &\equiv \int F_i^r v^i d^3x \\
&= \int d^3x \rho v^i \Big[\frac{4}{5}\Big(\text{Ɨ}^{(3)}_{ij} v^j\Big)^{\cdot} + \frac{4}{5}\text{Ɨ}^{(3)}_{ij} v^j{}_{,l} v^l \\
&\qquad - {}_{(7)}\alpha_{,i} - {}_{(6)}\dot{\beta}^i + v^j{}_{(6)}\beta^j{}_{,i} - v^j{}_{(6)}\beta^i{}_{,j} \Big].
\end{aligned}
\tag{5.14}
$$

Explicit calculations are done separately: (1) For the first term of Eq.(5.14), we obtain

$$
\begin{aligned}
\frac{4}{5}\int d^3x\rho\Big(\not{I}^{(3)}_{ij}v^j\Big)^{\cdot}v^i &= \frac{4}{5}\frac{d}{dt}\Big[\not{I}^{(3)}_{ij}\int d^3x\rho v^iv^j\Big] \\
&\quad -\frac{4}{5}\not{I}^{(3)}_{ij}\Big(\frac{1}{2}\frac{d}{dt}\int d^3x\rho v^iv^j+\frac{1}{2}\int d^3x\dot{\rho}v^iv^j\Big) \\
&= \frac{4}{5}\frac{d}{dt}\Big[\not{I}^{(3)}_{ij}\int d^3x\rho v^iv^j\Big]-\frac{1}{5}\not{I}^{(3)}_{ij}\not{I}^{(3)}_{ij} \\
&\quad -\frac{2}{5}\not{I}^{(3)}_{ij}\int d^3x\rho(x)v^k\frac{\partial}{\partial x^k}\int d^3y\frac{\rho(y)(x^i-y^i)(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|^3} \\
&\quad -\frac{2}{5}\not{I}^{(3)}_{ij}\int d^3x\dot{\rho}v^iv^j. \qquad (5.15)
\end{aligned}
$$

Here we used

$$
I^{(2)}_{ij}=2\int d^3x\rho v^iv^j-\int d^3xd^3y\frac{\rho(x)\rho(y)(x^i-y^i)(x^j-y^j)}{|\mathbf{x}-\mathbf{y}|^3}+2\delta_{ij}\int d^3xP, \qquad (5.16)
$$

which is obtained by using the continuity equation and the Euler's equation (For instance, see Candrasekhar's "Ellipsoidal Figures".).

(2) For the second term of Eq.(5.14), we obtain

$$
\begin{aligned}
\frac{4}{5}\int d^3x\rho\not{I}^{(3)}_{ij}v^j_{,l}v^lv^i &= \frac{2}{5}\not{I}^{(3)}_{ij}\int d^3x\rho v^l(v^iv^j)_{,l} \\
&= \frac{2}{5}\not{I}^{(3)}_{ij}\int d^3x\dot{\rho}v^iv^j. \qquad (5.17)
\end{aligned}
$$

(3) On the fourth term of Eq.(5.14). From Eqs.(2.62), (2.70) and (5.2), we find the relation

$$
{}_{(6)}\dot{\beta}^i=-{}_{(7)}\alpha_{,i}-\frac{2}{5}\not{I}^{(3)}_{kl}\frac{\partial}{\partial x^i}\int d^3y\rho\frac{(x^k-y^k)(x^l-y^l)}{|\mathbf{x}-\mathbf{y}|^3}. \qquad (5.18)
$$

Using Eqs.(5.14), (5.15), (5.17) and (5.18), we obtain

$$
W=\frac{4}{5}\frac{d}{dt}\Big(\not{I}^{(3)}_{ij}\int\rho v^iv^jd^3x\Big)-\frac{1}{5}\not{I}^{(3)}_{ij}\not{I}^{(3)ij}. \qquad (5.19)
$$

This expression for W does not depend on the slice condition. However, this never means that the value of W is invariant for the slice condition, since the meaning of the time derivative depends on the slice condition.

It is a matter of course that we can obtain the standard the quadrupole energy loss formula by averaging Eq.(5.19) with respect to time as

$$
\langle W\rangle=-\frac{1}{5}\Big\langle\not{I}^{(3)}_{ij}\not{I}^{(3)ij}\Big\rangle+O(c^{-6}). \qquad (5.20)
$$

VI. CONSERVED QUANTITIES

The conserved quantities are gauge-invariant so that, in general relativity, they play important roles because we are able to compare various systems described in different gauge conditions using them. From the practical view, these are also useful for checking the numerical accuracy in simulations. Thus, in this section, we show several conserved quantities in the 2PN approximation.

A. Conserved Mass And Energy

In general relativity, the volume integral of the mass density ρ does not conserve, and instead we have the following conserved mass;

$$M_* = \int \rho_* d^3x. \tag{6.1}$$

It is verified easily that M_* conserves;

$$\begin{aligned} \frac{dM_*}{dt} &= \int \frac{\partial \rho_*}{\partial t} d^3x \\ &= 0, \end{aligned} \tag{6.2}$$

where we used Eq.(2.32). In the PN approximation, ρ_* is expanded as

$$\begin{aligned} \rho_* = \rho\Big[1 &+ \Big(\frac{1}{2}v^2 + 3U\Big) \\ &+ \Big(\frac{3}{8}v^4 + \frac{7}{2}v^2 U + \frac{15}{4}U^2 + 6{}_{(4)}\psi + {}_{(3)}\beta_i v^i\Big) + {}_{(6)}\delta_* + O(c^{-7})\Big], \end{aligned} \tag{6.3}$$

where ${}_{(6)}\delta_*$ denotes the 3PN contribution to ρ_*. This term ${}_{(6)}\delta_*$ will be calculated later.

Then, we consider the ADM mass which is also the conserved quantity. Since the asymptotic behavior of the conformal factor becomes

$$\psi = 1 + \frac{M_{ADM}}{2r} + O\Big(\frac{1}{r^2}\Big), \tag{6.4}$$

the ADM mass in the PN approximation becomes

$$
\begin{aligned}
M_{ADM} &= -\frac{1}{2\pi}\int \Delta_{flat}\psi d^3x \\
&= \int d^3x\rho\Big[\Big\{1+\Big(v^2+\varepsilon+\frac{5}{2}U\Big)+\Big(v^4+\frac{13}{2}v^2U+v^2\varepsilon+\frac{P}{\rho}v^2+\frac{5}{2}U\varepsilon+\frac{5}{2}U^2 \\
&\qquad +5_{(4)}\psi+2_{(3)}\beta_i v^i\Big)\Big\}+\frac{1}{16\pi\rho}\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K^2\Big)+{}_{(6)}\delta_{ADM}+O(c^{-7})\Big],
\end{aligned} \tag{6.5}
$$

where ${}_{(6)}\delta_{ADM}$ denotes the 3PN contribution. This term ${}_{(6)}\delta_{ADM}$ will be calculated later.

Using these two conserved quantities, we can define the conserved energy as follows;

$$
\begin{aligned}
E &\equiv M_{ADM}-M_* \\
&= \int d^3x\rho\Big[\Big\{\Big(\frac{1}{2}v^2+\varepsilon-\frac{1}{2}U\Big) \\
&\qquad +\Big(\frac{5}{8}v^4+3v^2U+v^2\varepsilon+\frac{P}{\rho}v^2+\frac{5}{2}U\varepsilon-\frac{5}{4}U^2-{}_{(4)}\psi+{}_{(3)}\beta_i v^i\Big)\Big\} \\
&\qquad +\frac{1}{16\pi\rho}\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K^2\Big)+\Big({}_{(6)}\delta_{ADM}-{}_{(6)}\delta_*\Big)+O(c^{-7})\Big] \\
&\equiv E_N+E_{1PN}+E_{2PN}+\cdots.
\end{aligned} \tag{6.6}
$$

We should note that the following equation holds

$$
\int {}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}d^3x=-8\pi\int \rho v^i{}_{(3)}\beta_i d^3x+\int\Big(\frac{2}{3}{}_{(3)}K^2+2\dot{U}_{(3)}K\Big)d^3x, \tag{6.7}
$$

where we use the identities derived from Eqs.(2.58) and (2.59)

$$
\begin{aligned}
&\int {}_{(3)}\beta_{i,j(3)}\beta_{i,j}d^3x=-16\pi\int \rho v^i{}_{(3)}\beta_i d^3x+\int({}_{(3)}K^2+2\dot{U}_{(3)}K-3\dot{U}^2)d^3x, \\
&\int {}_{(3)}\beta_{i,j(3)}\beta_{j,i}d^3x=\int({}_{(3)}K^2+6\dot{U}_{(3)}K+9\dot{U}^2)d^3x.
\end{aligned} \tag{6.8}
$$

Using these relations, we obtain the Newtonian and the first PN energies as

$$
E_N=\int\rho\Big(\frac{1}{2}v^2+\varepsilon-\frac{1}{2}U\Big)d^3x, \tag{6.9}
$$

and

$$
E_{1PN}=\int d^3x\Big[\rho\Big(\frac{5}{8}v^4+\frac{5}{2}v^2U+v^2\varepsilon+\frac{P}{\rho}v^2+2U\varepsilon-\frac{5}{2}U^2+\frac{1}{2}{}_{(3)}\beta_i v^i\Big)+\frac{1}{8\pi}\dot{U}_{(3)}K\Big]. \tag{6.10}
$$

E_{1PN} can be rewritten immediately in the following form used by Chandrasekhar (1969a);

$$E_{1PN}=\int d^3x\rho\Big[\frac{5}{8}v^4+\frac{5}{2}v^2U+v^2\Big(\varepsilon+\frac{P}{\rho}\Big)+2U\varepsilon-\frac{5}{2}U^2-\frac{1}{2}v^iq_i\Big], \tag{6.11}$$

where q_i is the first PN shift vector in the standard PN gauge (i.e., ${}_{(3)}K=0$) and satisfies

$$\Delta_{flat}q_i=-16\pi\rho v^i+\dot{U}_{,i}. \tag{6.12}$$

The total energy at the 2PN order E_{2PN} is calculated from the 3PN quantities ${}_{(6)}\delta_*$ and ${}_{(6)}\delta_{ADM}$. First we consider ${}_{(6)}\delta_*$. We expand $(\alpha u^0)^2$ up to the 2PN order as

$$(\alpha u^0)^2=1+v^2+{}_{(4)}A+{}_{(6)}A+O(c^{-7}), \tag{6.13}$$

where

$$\begin{aligned}
{}_{(4)}A&=v^4+4v^2U+2v^i{}_{(3)}\beta_i,\\
{}_{(6)}A&=v^6+8v^4U+v^2\Big(4{}_{(4)}\psi-2X+\frac{15}{2}U^2\Big)\\
&\quad+{}_{(4)}h_{ij}v^iv^j+4\Big({}_{(3)}\beta_iv^i\Big)\Big(v^2+U\Big)+2{}_{(5)}\beta_iv^i+{}_{(3)}\beta_{i(3)}\beta_i.
\end{aligned} \tag{6.14}$$

Using this expansion of $(\alpha u^0)^2$, we obtain

$${}_{(6)}(\alpha u^0)=\frac{1}{16}v^6-\frac{1}{4}v^2{}_{(4)}A+\frac{1}{2}{}_{(6)}A+O(c^{-7}). \tag{6.15}$$

Hence we obtain ${}_{(6)}\delta_*$ as

$$\begin{aligned}
{}_{(6)}\delta_*&=\frac{5}{16}v^6+\frac{33}{8}v^4U+v^2\Big(5{}_{(4)}\psi+\frac{93}{8}U^2+\frac{3}{2}{}_{(3)}\beta_iv^i-X\Big)\\
&\quad+6{}_{(6)}\psi+15U{}_{(4)}\psi+\frac{5}{2}U^3+7{}_{(3)}\beta_iv^iU+\frac{1}{2}{}_{(4)}h_{ij}v^iv^j+\frac{1}{2}{}_{(3)}\beta_{i(3)}\beta_i+{}_{(5)}\beta_iv^i.
\end{aligned} \tag{6.16}$$

Hence we obtain

$${}_{(6)}M_*=\int\rho_{(6)}\delta_*d^3x. \tag{6.17}$$

Next, we consider ${}_{(6)}\delta_{ADM}$. The Hamiltonian constraint at $O(c^{-8})$ becomes

$$\begin{aligned}
&\Delta_{flat(8)}\psi-{}_{(4)}h_{ij(4)}\psi_{,ij}-\frac{1}{2}{}_{(6)}h_{ij}U_{,ij}\\
&=-\frac{1}{32}(2{}_{(4)}h_{kl,m(4)}h_{km,l}+{}_{(4)}h_{kl,m(4)}h_{kl,m})\\
&\quad-2\pi{}_{(6)}\rho_\psi-\frac{1}{4}\Big({}_{(3)}\tilde{A}_{ij(5)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K_{(5)}K\Big)-\frac{1}{16}U\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K^2\Big),
\end{aligned} \tag{6.18}$$

where we define ${}_{(6)}\rho_\psi$ as

$$
\begin{aligned}
{}_{(6)}\rho_\psi = \rho\Big[&v^6 + v^4\Big(\varepsilon + \frac{P}{\rho} + \frac{21}{2}U\Big) + v^2\Big\{\frac{13}{2}U\Big(\varepsilon + \frac{P}{\rho}\Big) + 9{}_{(4)}\psi - 2X + 20U^2\Big\} \\
&+\varepsilon\Big(5{}_{(4)}\psi + \frac{5}{2}U^2\Big) + 5{}_{(6)}\psi + 10U{}_{(4)}\psi + \frac{5}{4}U^3 \\
&+{}_{(4)}h_{ij}v^iv^j + 2{}_{(3)}\beta_iv^i\Big\{2v^2 + \varepsilon + \frac{P}{\rho} + \frac{13}{2}U\Big\} + 2{}_{(5)}\beta_iv^i + {}_{(3)}\beta_{i(3)}\beta_i\Big]. \qquad (6.19)
\end{aligned}
$$

Making use of relations ${}_{(4)}h_{ij,j} = 0$ and ${}_{(6)}h_{ij,j} = 0$, we obtain

$$
\begin{aligned}
{}_{(6)}M_{ADM} = &\int d^3x{}_{(6)}\rho_\psi \\
&+\frac{1}{8\pi}\int d^3y\Big({}_{(3)}\tilde{A}_{ij(5)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K{}_{(5)}K\Big) \\
&+\frac{1}{32\pi}\int d^3yU\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\Big), \qquad (6.20)
\end{aligned}
$$

where we assume ${}_{(6)}h_{ij} \to O(r^{-1})$ as $r \to \infty$. Although this assumption must be verified by performing the 3PN expansions which have not been done here, it seems reasonable in the asymptotically flat spacetime. From ${}_{(6)}M_{ADM}$ and ${}_{(6)}M_*$, we obtain the conserved energy at the 2PN order

$$
\begin{aligned}
E_{2PN} &= {}_{(6)}M_{ADM} - {}_{(6)}M_* \\
&= \int d^3x\rho\Big[\frac{11}{16}v^6 + v^4\Big(\varepsilon + \frac{P}{\rho} + \frac{51}{8}U\Big) \\
&\qquad +v^2\Big\{4{}_{(4)}\psi - X + \frac{13}{2}U\Big(\varepsilon + \frac{P}{\rho}\Big) + \frac{67}{8}U^2 + \frac{5}{2}{}_{(3)}\beta_iv^i\Big\} \\
&\qquad +\varepsilon\Big(5{}_{(4)}\psi + \frac{5}{2}U^2\Big) - {}_{(6)}\psi - 5U{}_{(4)}\psi - \frac{5}{4}U^3 \\
&\qquad +\frac{1}{2}{}_{(4)}h_{ij}v^iv^j + 2{}_{(3)}\beta_iv^i\Big(\varepsilon + \frac{P}{\rho} + 3U\Big) + {}_{(5)}\beta_iv^i + \frac{1}{2}{}_{(3)}\beta_{i(3)}\beta_i\Big\}\Big] \\
&\quad +\frac{1}{8\pi}\int d^3y\Big({}_{(3)}\tilde{A}_{ij(5)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K{}_{(5)}K\Big) \\
&\quad +\frac{1}{32\pi}\int d^3yU\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij} - \frac{2}{3}{}_{(3)}K^2\Big). \qquad (6.21)
\end{aligned}
$$

When we use the relation, $\int d^3x\rho_{(6)}\psi = -\frac{1}{4\pi}\int d^3xU\Delta_{(6)}\psi$, we obtain

$$
\begin{aligned}
E_{2PN} = &\int d^3x\rho\Big[\frac{11}{16}v^6 + v^4\Big(\varepsilon + \frac{P}{\rho} + \frac{47}{8}U\Big) \\
&\qquad +v^2\Big\{4{}_{(4)}\psi - X + 6U\Big(\varepsilon + \frac{P}{\rho}\Big) + \frac{41}{8}U^2 + \frac{5}{2}{}_{(3)}\beta_iv^i\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\Big(5_{(4)}\psi+\frac{5}{4}U^2\Big)-\frac{15}{2}U_{(4)}\psi-\frac{5}{2}U^3\\
&+\frac{1}{2}{}_{(4)}h_{ij}v^iv^j+2_{(3)}\beta_iv^i\Big\{\Big(\varepsilon+\frac{P}{\rho}\Big)+5U\Big\}+{}_{(5)}\beta_iv^i+\frac{1}{2}{}_{(3)}\beta_{i(3)}\beta_i\Big]\\
+\frac{1}{8\pi}\int d^3y\Big({}_{(4)}h_{ij}UU_{,ij}+{}_{(3)}\tilde{A}_{ij(5)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K_{(5)}K\Big). \qquad (6.22)
\end{aligned}
$$

Here we used Eq.(2.52) in order to eliminate ${}_{(6)}\psi$. Then, $\Big({}_{(3)}\tilde{A}_{ij(3)}\tilde{A}_{ij}-\frac{2}{3}{}_{(3)}K^2\Big)$ in Eq.(6.21) cancels that in $\Delta_{(6)}\psi$.

B. Conserved linear momentum

When we use the center of mass system as usual, the linear momentum of the system should vanish. However, it may arise from numerical errors in numerical calculations. Since it is useful for investigation of the numerical accuracy, we mention the linear momentum derived from

$$
\begin{aligned}
P_i&=\frac{1}{8\pi}\lim_{r\to\infty}\oint\Big(K_{ij}n^j-Kn^i\Big)dS\\
&=\frac{1}{8\pi}\lim_{r\to\infty}\oint\Big(\psi^4\tilde{A}_{ij}n^j-\frac{2}{3}Kn^i\Big)dS, \qquad (6.23)
\end{aligned}
$$

where the surface integrals are taken over a sphere of constant r. Since the asymptotic behavior of $\tilde{A}_{ij}$ is determined by

$$
{}_{(3)}\tilde{A}_{ij}=\frac{1}{2}\big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij(3)}\beta_{l,l}\big)+O(r^{-3}), \qquad (6.24)
$$

and

$$
{}_{(5)}\tilde{A}_{ij}=\frac{1}{2}\Big({}_{(5)}\beta_{i,j}+{}_{(5)}\beta_{j,i}-\frac{2}{3}\delta_{ij(5)}\beta_{l,l}\Big)+{}_{(5)}\tilde{A}^{TT}_{ij}+O(r^{-3}), \qquad (6.25)
$$

the leading term of the shift vector is necessary. Using the asymptotic behavior

$$
{}_{(3)}\beta_i=-\frac{7}{2}\frac{l_i}{r}-\frac{1}{2}\frac{n^in^jl_j}{r}+O(r^{-2}), \qquad (6.26)
$$

the following relation is obtained

$$
\int\Big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij(3)}\beta_{l,l}\Big)n^jdS=16\pi l_i. \qquad (6.27)
$$

Here we defined $l_i = \int \rho v^i d^3x$ and used

$$\int \frac{n^i n^j}{r^2} dS = \frac{4\pi}{3}\delta_{ij}. \tag{6.28}$$

Therefore the Newtonian linear momentum is

$$P_N{}^i = \int d^3x \rho v^i. \tag{6.29}$$

Similarly the first PN linear momentum is obtained as follows;

$$P_{1PN}{}^i = \int d^3x \rho \Big[v^i \Big(v^2 + \varepsilon + 6U + \frac{P}{\rho} \Big) + {}_{(3)}\beta_i \Big]. \tag{6.30}$$

We obtain $P_{2PN}{}^i$ by the similar procedure as

$$\begin{aligned} P_{2PN}{}^i = \int d^3x \rho v^i \Big[& 2_{(3)}\beta_i v^i + 10_{(4)}\psi + \Big(6U + v^2\Big)\Big(\varepsilon + \frac{P}{\rho}\Big) \\ & + \frac{67}{4}U^2 + 10Uv^2 + v^4 - X \Big]. \end{aligned} \tag{6.31}$$

Part 2

VII. MOTIVATION AND APPROACH

The last stage of coalescing binary neutron stars (BNS's) is one of the most promising sources for kilometer size interferometric gravitational wave detectors, LIGO (Abramovici et.al. 1992; Thorne 1994; Will 1994) and VIRGO (Bradaschia 1990). When the orbital separation of BNS's becomes ~ 700km as a result of the emission of gravitational waves, it is observed that the frequency of gravitational waves from them becomes ~ 10Hz. After then, the orbit of BNS's shrinks owing to the radiation reaction toward merging in a few minutes (Cutler et.al. 1993). In such a phase, BNS's are the strongly self-gravitating bound systems, and gravitational waves from them will have various general relativistic (GR) information. In particular, in the last few milliseconds before merging, BNS's are in a very strong GR gravitational field because the orbital separation is less than ten times of the Schwarzschild radius of the system. Thus, if we could detect the signal of gravitational waves radiated in the last few milliseconds, we would be able to observe directly the phenomena in the GR gravitational field.

To interpret the implication of the signal of gravitational waves, we need to understand the theoretical mechanism of merging in detail. The little knowledge we have about the very last phase of BNS's is as follows: When the orbital separation of BNS's is $\lesssim 10GM/c^2$, where M is the total mass of BNS's, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation t_{GW} is much longer than the orbital period P as

$$\frac{t_{GW}}{P} \sim 15\Big(\frac{dc^2}{10GM}\Big)^{5/2}\Big(\frac{M}{4\mu}\Big), \tag{7.1}$$

where μ and d are the reduced mass and the separation of BNS's. Thus, BNS's adiabatically evolve radiating gravitational waves. However, when the orbital separation becomes $6-10GM/c^2$, they cannot maintain the circular orbit because of instabilities due to the

GR gravity (Kidder, Will and Wiseman 1993a) or the tidal field (Lai, Rasio and Shapiro 1993, 1994). As a result of such instabilities, the circular orbit of BNS's changes into the plunging orbit to merge. This means that the nature of the signal of gravitational waves changes around the transition between the circular orbit and plunging one. Gravitational waves emitted at this transition region may bring us an important information about the structure of NS's because the location where the instability occurs will depend on the equation of state (EOS) of NS sensitively (Lai, Rasio and Shapiro 1993, 1994; Zhung, Centrella and McMillan 1994). Thus, it is very important to investigate the location of the innermost stable circular orbit (ISCO) of BNS's.

As mentioned above, the ISCO is determined not only by the GR effects, but also by the hydrodynamic one. We emphasize that the tidal effects depend strongly on the structure of NS. Here, NS is a GR object because of its compactness, $Gm/c^2R \sim 0.2$, where m and R are the mass and radius of NS. Thus, in order to know the location of the ISCO accurately, we need to solve the GR hydrodynamic equations in general. A strategy to search the ISCO in GR manner is as follows; since the timescale of the energy loss is much longer than the orbital period according to Eq.(7.1), we may suppose that the motion of BNS's is composed of the stationary part and the small radiation reaction part. From this physical point of view, we may consider that BNS's evolve quasi-stationally, and we can take the following procedure; first, neglecting the evolution due to gravitational radiation, equilibrium configurations are constructed, and then the radiation reaction is taken into account as a correction to the equilibrium configurations. The ISCO is determined from the point, where the dynamical instability for the equilibrium configurations occurs. It may be a grand challenge, however, to distinguish the stationary part from the nonstationary one in general relativity. As Detweiler (1994) has pointed out, a stationary solution of the Einstein equation with standing gravitational waves, which will be constructed by adding the incoming waves from infinity, may be a valuable approximation to physically realistic solutions. However, these solutions are not asymptotically flat (Detweiler

1994) because GWs contribute to the total energy of the system and the total energy of GWs inside a radius r grows linearly with r. The lack of asymptotic flatness forces us to consider only a bounded space and impose boundary conditions in the near zone. Careful consideration will be necessary to find out an appropriate boundary condition for describing the physically realistic system in the near zone.

Recently, Wilson and his collaborators (Wilson and Mathews 1995; Wilson, Mathews and Marronetti 1996) proposed a simirelativistic approximation method in order to calculate the equilibrium configuration of BNS's just before merging. In their method, they assume the line element as

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)c^2dt^2 + 2\beta_i cdtdx^i + \psi^4 dx^3, \tag{7.2}$$

i.e., three metric γ_{ij} is chosen as the conformal flat (i.e., $\gamma_{ij} = \psi^4\delta_{ij}$), and solve only the constraint equations in the Einstein equation. In their approach, they claim that they ignore only the contribution of gravitational waves, but it is not correct at all; as shown in previous post-Newtonian (PN) analyses (Schäfer 1985; Asada, Shibata and Fuatamase 1996; Asada and Shibata 1996; Rieth and Schäfer 1996), the tensor potential term exists in the three metric even if we ignore the radiation reaction of gravitational waves (i.e., $\psi^{-4}\gamma_{ij} \neq \delta_{ij}$). Since such a term appears from the second PN order in the PN approximation, the accuracy of their results is less than the 2PN order: In reality, from results by Cook, Shapiro and Teukolsky (1996) in which they obtain equilibrium configurations of the axisymmetric NS using both the Einstein equation and Wilson's method, we can see that some quantities obtained from Wilson's scheme, such as the lapse function, the three metric, the angular velocity, and so on, deviate from the exact solution by about $O((Gm/Rc^2)^2)$. This seems to indicate that their approach for the system of BNS's is valid only at the 1PN level from the PN point of view. Furthermore, the meaning of their approximation is obscure: It is not clear at all how to estimate errors due to such an approximation scheme and in which situation but the spherical symmetric system, the scheme based on the assumption of the conformal flatness is justified.

In contrast with Wilson's method, the meaning of the PN approximation is fairly clear: In the PN approximation, the metric is formally expanded with respect to c^{-1} assuming the slow motion and weak self-gravity of matter. If we will take into account the next PN order, the accuracy of approximate solutions will be improved. This means that we can estimate the order of magnitude of the error due to the ignorance of higher PN terms. Also, in the PN approximation, we can distinguish the radiation reaction terms, which begin at the 2.5PN order (Chandrasekhar and Esposito 1970), from other terms in the metric. Thus, it is possible to construct the equilibrium configuration of BNS's in the 2PN approximation without the radiation reaction terms.

We describe schematically two approaches in Tables 1(a) and 1(b). As mentioned above, in close binary of NS's, it is important to take into account GR effects on orbital motion as well as on the internal structure of each NS. As for the orbital motion, there exist two parameters; one is the PN parameter v/c and the other is the mass ratio η of the reduced mass μ to the total mass M, and both parameters are less than unity. Thus, the physical quantities such as the orbital frequency are expanded with respect to them. In Table 1(a), we show schematically various levels of approximations in terms of v/c and η. If all terms in a level are taken into account in the 2PN approximation, we mark P^2N, while W means that all terms in the marked level are taken into account in Wilson's approach. From Table 1(a), we see that the 2PN approximation can include all corrections in η up to the 2PN order in contrast with Wilson's approach. On the other hand, Wilson's approach will hold completely in the test particle limit, i.e., at $O(\eta^0)$, whereas even in this limit the 2PN approximation is not valid at higher PN orders. As for the internal structure of each NS, there also exist two small parameters; one is the compactness Gm/c^2R and the other is the deformation parameter from its spherical shape, such as an ellipticity e. In this case, the PN approximation becomes an expansion in terms of Gm/c^2R. In Table 1(b), we also show various levels of approximation in terms of these parameters. Although Wilson's approach is exact for spherical NS's, it is not valid in nonspherical cases even at the 2PN order. On the other hand, in the 2PN approximation, the spherical compact star cannot be obtained correctly in contrast with Wilson's approach. In this way, the 2PN approximation has a week point: Although it can take into account all effects up to the 2PN order, it is inferior to Wilson's approach when we take a test-particle limit, $\eta \to 0$, or we describe an exactly spherical NS. However, as shown below, the error due to the ignorance of higher PN terms in those cases is not so large .

To estimate the error due to the ignorance of the higher PN terms, let us compare the GR exact solutions with their PN approximations. First, we consider a small star of mass μ orbiting a Schwarzschild black hole of mass $m_{bh} \gg \mu$. In this case, we may consider that the small star moves on the geodesic around the Schwarzschild black hole, and the orbital angular velocity becomes (Kidder, Will and Wiseman 1993a)

$$\Omega = \sqrt{\frac{Gm_{bh}}{(\bar{r} + Gm_{bh}c^{-2})^3}}, \tag{7.3}$$

where $\bar{r}$ is the coordinate radius of the orbit in the harmonic coordinate. In the PN approximation, Eq.(7.3) becomes

$$\Omega = \sqrt{\frac{Gm_{bh}}{\bar{r}^3}}\Big\{1 - \frac{3Gm_{bh}}{2\bar{r}c^2} + \frac{15}{8}\Big(\frac{Gm_{bh}}{\bar{r}c^2}\Big)^2 + O(c^{-6})\Big\}. \tag{7.4}$$

Comparing Eq.(7.3) with Eq.(7.4), it is found that the error size of the 2PN angular velocity is $\sim 0.3\%$ at $\bar{r} = 9Gm_{bh}c^{-2}$, and $\sim 1\%$ at $\bar{r} = 6Gm_{bh}c^{-2}$. Thus, the 2PN approximation seems fairly good to describe the motion of relativistic binary stars just before coalescence. Next, we consider a spherical NS of a uniform density in order to investigate the applicability of the PN approximation for determination of the internal structure of NS's. In this model, the pressure, P, and the density, ρ =const., are related with each other (Shapiro and Teukolsky 1983):

$$\begin{aligned}\frac{P}{\rho c^2} &= \frac{(1-2Gmr_s^2/c^2R^3)^{1/2} - (1-2Gm/c^2R)^{1/2}}{3(1-2Gm/c^2R)^{1/2} - (1-2Gmr_s^2/c^2R^3)^{1/2}}\\ &= \frac{1}{2}\frac{Gm}{c^2R}\Big(1-\frac{r_s^2}{R^2}\Big) + \frac{G^2m^2}{c^4R^2}\Big(1-\frac{r_s^2}{R^2}\Big) + \frac{G^3m^3}{c^6R^3}\Big(\frac{17}{8} - \frac{19r_s^2}{8R^2} + \frac{3r_s^4}{8R^4} - \frac{r_s^6}{8R^6}\Big) + O(c^{-8}),\end{aligned} \tag{7.5}$$

where r_s is the coordinate radius in the Schwarzschild coordinate and terms of order c^{-2}, c^{-4} and c^{-6} denote Newtonian, 1PN and 2PN terms respectively. In the second line in Eq.(7.5), we expand the equation in power of Gm/c^2R regarding it as a small quantity. In Fig.1, we show the error, $1-\tilde{P}/P$, in Newtonian, 1PN and 2PN cases as a function of r_s for $R=5Gm/c^2$(solid lines) and $8Gm/c^2$(dotted lines), where $\tilde{P}$ denotes the PN approximate pressure. It is found that the discrepancy in the Newtonian treatment is very large, while in the 2PN approximation the error is less than 10%. In this way, we can estimate rigidly the typical error size in the 2PN approximation. Furthermore, the accuracy is fairly good if the NS is not extremely compact; the 2PN approximation will be fairly accurate if the radius of NS is larger than $\sim$10km. Thus, in Part 2, we develop a formalism to obtain equilibrium configurations of uniformly rotating fluid in the 2PN order as a first step.

VIII. FORMULATION

We write the line element in the following form;

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)c^2dt^2 + 2\beta_i c dt dx^i + \psi^4\tilde{\gamma}_{ij}dx^i dx^j, \tag{8.1}$$

where we define $\det(\tilde{\gamma}_{ij}) = 1$. To fix the gauge condition in the time coordinate, we use the maximal slice condition ${K_i}^i = 0$, where ${K_i}^i$ is the trace part of the extrinsic curvature, K_{ij}. As the spatial gauge condition, we adopt the transverse gauge $\tilde{\gamma}_{ij,j} = 0$ in order to remove the gauge modes from $\tilde{\gamma}_{ij}$. In this case, up to the 2 PN approximation, each metric variable is expanded as (Asada, Shibata and Futamase 1996)

$$\psi = 1 + \frac{1}{c^2}\frac{U}{2} + \frac{1}{c^4}{}_{(4)}\psi + O(c^{-6}), \tag{8.2}$$

$$\alpha = 1 - \frac{1}{c^2}U + \frac{1}{c^4}\Big(\frac{U^2}{2} + X\Big) + \frac{1}{c^6}{}_{(6)}\alpha + O(c^{-7}), \tag{8.3}$$

$$\beta^i = \frac{1}{c^3}{}_{(3)}\beta_i + \frac{1}{c^5}{}_{(5)}\beta_i + O(c^{-7}), \tag{8.4}$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \frac{1}{c^4}h_{ij} + O(c^{-5}). \tag{8.5}$$

As for the energy-momentum tensor of the Einstein equation, we consider the perfect fluid as

$$T_{\mu\nu} = \Big(\rho c^2 + \rho\varepsilon + P\Big)u_\mu u_\nu + P g_{\mu\nu}. \tag{8.6}$$

For simplicity, we assume that the matter obeys the polytropic equation of state(EOS);

$$P = (\Gamma - 1)\rho\varepsilon = K\rho^\Gamma, \tag{8.7}$$

where Γ and K are the polytropic exponent and polytropic constant, respectively. Up to the 2PN order, the four velocity is expanded as (Chandrasekhar and Nutku 1969; Asada, Shibata and Fuatamse 1996; Asada and Shibata 1996)

$$u^0 = 1 + \frac{1}{c^2}\Big(\frac{1}{2}v^2 + U\Big) + \frac{1}{c^4}\Big(\frac{3}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}U^2 + {}_{(3)}\beta_i v^i - X\Big) + O(c^{-6}),$$

$$u_0 = -\Big[1 + \frac{1}{c^2}\Big(\frac{1}{2}v^2 - U\Big) + \frac{1}{c^4}\Big(\frac{3}{8}v^4 + \frac{3}{2}v^2U + \frac{1}{2}U^2 + X\Big)\Big] + O(c^{-6}),$$

$$
\begin{aligned}
u^i &= \frac{v^i}{c}\Big[1+\frac{1}{c^2}\Big(\frac{1}{2}v^2+U\Big)+\frac{1}{c^4}\Big(\frac{3}{8}v^4+\frac{5}{2}v^2U+\frac{1}{2}U^2+{}_{(3)}\beta_i v^i-X\Big)\Big]+O(c^{-7}),\\
u_i &= \frac{v^i}{c}+\frac{1}{c^3}\Big\{{}_{(3)}\beta_i+v^i\Big(\frac{1}{2}v^2+3U\Big)\Big\}+\frac{1}{c^5}\Big\{{}_{(5)}\beta_i+{}_{(3)}\beta_i\Big(\frac{1}{2}v^2+3U\Big)+h_{ij}v^j\\
&\qquad +v^i\Big(\frac{3}{8}v^4+\frac{7}{2}v^2U+4U^2-X+4{}_{(4)}\psi+{}_{(3)}\beta_j v^j\Big)\Big\}+O(c^{-6}),
\end{aligned}
\tag{8.8}
$$

where $v^i = u^i/u^0$ and $v^2 = v^i v^i$. Since we need u^0 up to 3PN order to obtain the 2PN equations of motion, we derive it here. Using Eq.(8.8), we can calculate $(\alpha u^0)^2$ up to 3PN order as

$$
\begin{aligned}
(\alpha u^0)^2 &= 1+\psi^{-4}\tilde{\gamma}^{ij}u_i u_j\\
&= 1+\frac{v^2}{c^2}+\frac{1}{c^4}\Big(2{}_{(3)}\beta_j v^j+4Uv^2+v^4\Big)+\frac{1}{c^6}\Big\{{}_{(3)}\beta_j{}_{(3)}\beta_j+8{}_{(3)}\beta_j v^j U+h_{ij}v^i v^j\\
&\qquad +2{}_{(5)}\beta_i v^i+\Big(4{}_{(3)}\beta_j v^j+4{}_{(4)}\psi+\frac{15}{2}U^2-2X\Big)v^2+8Uv^4+v^6\Big\}+O(c^{-7}),
\end{aligned}
\tag{8.9}
$$

where we use $\tilde{\gamma}^{ij} = \delta_{ij} - c^{-4}h_{ij} + O(c^{-5})$. Thus, we obtain u^0 up to the 3PN order as

$$
\begin{aligned}
u^0 &= 1+\frac{1}{c^2}\Big(\frac{1}{2}v^2+U\Big)+\frac{1}{c^4}\Big(\frac{3}{8}v^4+\frac{5}{2}v^2U+\frac{1}{2}U^2+{}_{(3)}\beta_i v^i-X\Big)\\
&\quad +\frac{1}{c^6}\Big\{-{}_{(6)}\alpha+\frac{1}{2}\Big({}_{(3)}\beta_j{}_{(3)}\beta_j+h_{ij}v^i v^j\Big)+{}_{(5)}\beta_j v^j+5{}_{(3)}\beta_j v^j U-2UX\\
&\qquad +\Big(\frac{3}{2}{}_{(3)}\beta_j v^j+2{}_{(4)}\psi+6U^2-\frac{3}{2}X\Big)v^2+\frac{27}{8}Uv^4+\frac{5}{16}v^6\Big\}+O(c^{-7}).
\end{aligned}
\tag{8.10}
$$

Substituting PN expansions of metric and matter variables into the Einstein equation, and using the polytropic EOS, we find that the metric variables obey the following Poisson equations (Asada, Shibata and Futamase 1996);

$$\Delta U = -4\pi\rho, \tag{8.11}$$

$$\Delta X = 4\pi\rho\Big(2v^2+2U+(3\Gamma-2)\varepsilon\Big), \tag{8.12}$$

$$\Delta_{(4)}\psi = -2\pi\rho\Big(v^2+\varepsilon+\frac{5}{2}U\Big), \tag{8.13}$$

$$\Delta_{(3)}\beta_i = 16\pi\rho v^i - \dot{U}_{,i}, \tag{8.14}$$

$$
\begin{aligned}
\Delta_{(5)}\beta_i &= 16\pi\rho\Big[v^i\Big(v^2+2U+\Gamma\varepsilon\Big)+{}_{(3)}\beta_i\Big]-4U_{,j}\Big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij}{}_{(3)}\beta_{k,k}\Big)\\
&\quad -2{}_{(4)}\dot{\psi}_{,i}+\frac{1}{2}(U\dot{U})_{,i}+({}_{(3)}\beta_l U_{,l})_{,i},
\end{aligned}
\tag{8.15}
$$

$$
\begin{aligned}
\Delta h_{ij} = &\Big(UU_{,ij} - \frac{1}{3}\delta_{ij}U\Delta_{flat}U - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k}\Big) - 16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big) \\
&-\Big({}_{(3)}\dot{\beta}_{i,j} + {}_{(3)}\dot{\beta}_{j,i} - \frac{2}{3}\delta_{ij(3)}\dot{\beta}_{k,k}\Big) - 2\Big((X + 2_{(4)}\psi)_{,ij} - \frac{1}{3}\delta_{ij}\Delta(X + 2_{(4)}\psi)\Big), \qquad (8.16)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{(6)}\alpha = 4\pi\rho\Big[&2v^4 + 2v^2\Big(5U + \Gamma\varepsilon\Big) + (3\Gamma - 2)\varepsilon U + 4_{(4)}\psi + X + 4_{(3)}\beta_i v^i\Big] \\
&- h_{ij}U_{,ij} - \frac{3}{2}UU_{,l}U_{,l} + U_{,l}(2_{(4)}\psi - X)_{,l} \\
&+ \frac{1}{2}{}_{(3)}\beta_{i,j}\Big({}_{(3)}\beta_{i,j} + {}_{(3)}\beta_{j,i} - \frac{2}{3}\delta_{ij(3)}\beta_{k,k}\Big), \qquad (8.17)
\end{aligned}
$$

where Δ is the flat Laplacian, and the dot $\cdot$ denotes $\partial/\partial t$.

Equations of motion for fluid are derived from

$$
\nabla_\mu T^\mu{}_\nu = 0. \qquad (8.18)
$$

In Part 2, we consider the uniformly rotating fluid around z-axis with the angular velocity Ω, i.e.,

$$
v^i = \epsilon_{ijk}\Omega^j x^k = (-y\Omega, x\Omega, 0), \qquad (8.19)
$$

where we choose $\Omega^j = (0, 0, \Omega)$ and ϵ_{ijk} is the completely anti-symmetric unit tensor. In this case, the following relations hold;

$$
\Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q = \Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q_i = \Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\varphi}\Big)Q_{ij} = 0, \qquad (8.20)
$$

where Q, Q_i and Q_{ij} are arbitrary scalars, vectors, and tensors, respectively. Then, Eq.(8.18) can be integrated as (Lightman, Press, Price and Teukolsky 1975; See also appendix E)

$$
\int \frac{dP}{\rho c^2 + \rho\varepsilon + P} = \ln u^0 + C, \qquad (8.21)
$$

where C is a constant. For the polytropic EOS, Eq.(8.21) becomes

$$
\ln\Big[1 + \frac{\Gamma K}{c^2(\Gamma - 1)}\rho^{\Gamma-1}\Big] = \ln u^0 + C, \qquad (8.22)
$$

or

$$1+\frac{\Gamma K}{c^2(\Gamma-1)}\rho^{\Gamma-1}=u^0\exp(C). \tag{8.23}$$

Using Eq.(8.10), the 2PN approximation of Eq.(8.22) is written as

$$\begin{aligned}
H-\frac{H^2}{2c^2}+\frac{H^3}{3c^4}=&\frac{v^2}{2}+U+\frac{1}{c^2}\Big(2Uv^2+\frac{v^4}{4}-X+{}_{(3)}\beta_i v^i\Big)\\
&+\frac{1}{c^4}\Big(-{}_{(6)}\alpha+\frac{1}{2}{}_{(3)}\beta_{i(3)}\beta_i+4{}_{(3)}\beta_i v^i U-\frac{U^3}{6}+{}_{(3)}\beta_i v^i v^2+2{}_{(4)}\psi v^2\\
&+\frac{15}{4}U^2v^2+2Uv^4+\frac{1}{6}v^6-UX-v^2X+{}_{(5)}\beta_i v^i+\frac{1}{2}h_{ij}v^iv^j\Big)+C,
\end{aligned} \tag{8.24}$$

where $H=\Gamma K\rho^{\Gamma-1}/(\Gamma-1)$, $v^2=R^2\Omega^2$ and $R^2=x^2+y^2$. Here we used the Taylor expansion of $\ln(1+x)$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}. \tag{8.25}$$

Note that Eq.(8.24) can be also obtained from the 2PN Euler equation like in the first PN case (Chandrasekhar 1967). If we solve the coupled equations (8.11-17) and (8.24), we can obtain equilibrium configurations of the non-axisymmetric uniformly rotating body.

IX. DERIVATION OF THE POISSON EQUATION OF COMPACT SOURCES FOR h_{ij}, ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$

In section 2, we derive the Poisson equations for metric variables. However, the source terms in the Poisson equations for ${}_{(3)}\beta_i$, ${}_{(5)}\beta_i$, and h_{ij} fall off slowly as $r \to \infty$ because these terms behave as $O(r^{-3})$ at $r \to \infty$. These Poisson equations do not take convenient forms when we try to solve them as the boundary value problem in numerical calculation. Hence in the following, we rewrite them into other convenient forms in numerical calculation.

As for h_{ij}, first of all, we split the equation into three parts as (Asada, Shibata and Futamase 1996)

$$\Delta h_{ij}^{(U)} = U\Big(U_{,ij} - \frac{1}{3}\delta_{ij}\Delta U\Big) - 3U_{,i}U_{,j} + \delta_{ij}U_{,k}U_{,k} \equiv -4\pi S_{ij}^{(U)}, \tag{9.1}$$

$$\Delta h_{ij}^{(S)} = -16\pi\Big(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\Big), \tag{9.2}$$

$$\begin{aligned}\Delta h_{ij}^{(G)} = &-\Big({}_{(3)}\dot{\beta}_{i,j} + {}_{(3)}\dot{\beta}_{j,i} - \frac{2}{3}\delta_{ij\,(3)}\dot{\beta}_{k,k}\Big)\\ &- 2\Big((X + 2_{(4)}\psi)_{,ij} - \frac{1}{3}\delta_{ij}\Delta(X + 2_{(4)}\psi)\Big).\end{aligned} \tag{9.3}$$

The equation for $h_{ij}^{(S)}$ has a compact source, and also the source term of $h_{ij}^{(U)}$ behaves as $O(r^{-6})$ at $r \to \infty$, so that Poisson equations for them are solved easily as the boundary value problem. On the other hand, the source term of $h_{ij}^{(G)}$ behaves as $O(r^{-3})$ at $r \to \infty$, so that it seems troublesome to solve the equation for it as the boundary value problem. In order to solve the equation for $h_{ij}^{(G)}$ as the boundary value problem, we had better rewrite the equation into useful forms. As shown by Asada, Shibata and Futamase (1996), Eq.(9.3) is integrated to give

$$\begin{aligned}h_{ij}^{(G)} = &\,2\frac{\partial}{\partial x^i}\int (\rho v^j)^{\cdot}|\mathbf{x}-\mathbf{y}|d^3y + 2\frac{\partial}{\partial x^j}\int (\rho v^i)^{\cdot}|\mathbf{x}-\mathbf{y}|d^3y + \delta_{ij}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|d^3y\\ &+\frac{1}{12}\frac{\partial^2}{\partial x^i\partial x^j}\int \ddot{\rho}|\mathbf{x}-\mathbf{y}|^3 d^3y + \frac{\partial^2}{\partial x^i \partial x^j}\int \Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big)|\mathbf{x}-\mathbf{y}|d^3y\\ &-\frac{2}{3}\delta_{ij}\int \frac{\big(\rho v^2 + 3P - \rho U/2\big)}{|\mathbf{x}-\mathbf{y}|}d^3y.\end{aligned} \tag{9.4}$$

Using the relations

$$\ddot{\rho} = -(\rho v^j)^{\cdot}_{,j} + O(c^{-2}),$$

$$\dot{v}^i = 0,$$

$$v^i x^i = 0, \tag{9.5}$$

Eq.(9.4) is rewritten as

$$\begin{aligned} h_{ij}^{(G)} &= \frac{7}{4}\Big[\int (\rho v^j)^{\cdot} \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|} d^3y + \int (\rho v^i)^{\cdot} \frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|} d^3y\Big] - \delta_{ij} x^k \int \frac{(\rho v^k)^{\cdot}}{|\mathbf{x}-\mathbf{y}|} d^3y \\ &-\frac{1}{8} x^k \Big[\frac{\partial}{\partial x^i} \int (\rho v^k)^{\cdot} \frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|} d^3y + \frac{\partial}{\partial x^j} \int (\rho v^k)^{\cdot} \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|} d^3y\Big] \\ &+\frac{1}{2}\Big[\frac{\partial}{\partial x^i} \int \Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big) \frac{x^j - y^j}{|\mathbf{x}-\mathbf{y}|} d^3y + \frac{\partial}{\partial x^j} \int \Big(\rho v^2 + 3P - \frac{\rho U}{2}\Big) \frac{x^i - y^i}{|\mathbf{x}-\mathbf{y}|} d^3y\Big] \\ &\qquad - \frac{2}{3}\delta_{ij} \int \frac{\Big(\rho v^2 + 3P - \rho U/2\Big)}{|\mathbf{x}-\mathbf{y}|} d^3y. \end{aligned} \tag{9.6}$$

From Eq.(9.6), it is found that $h_{ij}^{(G)}$ is written as

$$\begin{aligned} h_{ij}^{(G)} &= \frac{7}{4}\Big(x^i{}_{(3)}\dot{P}_j + x^j{}_{(3)}\dot{P}_i - \dot{Q}_{ij}^{(T)} - \dot{Q}_{ji}^{(T)}\Big) - \delta_{ij} x^k{}_{(3)}\dot{P}_k \\ &-\frac{1}{8} x^k \Big[\frac{\partial}{\partial x^i}\Big(x^j{}_{(3)}\dot{P}_k - \dot{Q}_{kj}^{(T)}\Big) + \frac{\partial}{\partial x^j}\Big(x^i{}_{(3)}\dot{P}_k - \dot{Q}_{ki}^{(T)}\Big)\Big] \\ &+\frac{1}{2}\Big[\frac{\partial}{\partial x^i}\Big(x^j Q^{(I)} - Q_j^{(I)}\Big) + \frac{\partial}{\partial x^j}\Big(x^i Q^{(I)} - Q_i^{(I)}\Big)\Big] - \frac{2}{3}\delta_{ij} Q^{(I)}, \end{aligned} \tag{9.7}$$

where

$$\Delta_{(3)}P_i = -4\pi\rho v^i, \tag{9.8}$$

$$\Delta Q_{ij}^{(T)} = -4\pi\rho v^i x^j, \tag{9.9}$$

$$\Delta Q^{(I)} = -4\pi\Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big), \tag{9.10}$$

$$\Delta Q_i^{(I)} = -4\pi\Big(\rho v^2 + 3P - \frac{1}{2}\rho U\Big) x^i. \tag{9.11}$$

Therefore, $h_{ij}^{(G)}$ can be deduced from variables which satisfy the Poisson equations with compact sources.

The source terms in the Poisson equations for ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$ also fall off slowly. However, if we rewrite them as (Asada, Shibata and Futamase 1996; See also appendix D)

$${}_{(3)}\beta_i = -4{}_{(3)}P_i - \frac{1}{2}\Big(x^i \dot{U} - \dot{q}_i\Big), \tag{9.12}$$

$${}_{(5)}\beta_i = -4{}_{(5)}P_i - \frac{1}{2}\Big(2x^i{}_{(4)}\dot{\psi} - \dot{\eta}_i\Big), \tag{9.13}$$

where

$$\Delta q_i = -4\pi\rho x^i, \tag{9.14}$$

$$\Delta_{(5)}P_i = -4\pi\rho\Big[v^i\Big(v^2+2U+\Gamma\varepsilon\Big)+{}_{(3)}\beta_i\Big]+U_{,j}\Big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij\,(3)}\beta_{k,k}\Big)$$
$$-\frac{1}{8}(\dot{U}U)_{,i}-\frac{1}{4}({}_{(3)}\beta_l U_{,l})_{,i}, \tag{9.15}$$

$$\Delta\eta_i = -4\pi\rho\Big(v^2+\varepsilon+\frac{5}{2}U\Big)x^i, \tag{9.16}$$

then ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$ can be obtained by solving the Poisson equations in which the fall-off of the source terms is fast enough, $O(r^{-5})$, for numerical calculation. Note that, using the relation ${}_{(3)}P_i = \epsilon_{izk}q_k\Omega$ and Eq.(8.20), ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$ may be written as

$${}_{(3)}\beta_i = \Omega\Big[-4\epsilon_{izk}q_k+\frac{1}{2}\Big(x^iU_{,\varphi}-q_{i,\varphi}\Big)\Big] \equiv \Omega_{(3)}\hat{\beta}_i, \tag{9.17}$$

$${}_{(5)}\beta_i = \Omega\Big[-4_{(5)}\hat{P}_i+\frac{1}{2}\Big(2x^i{}_{(4)}\psi_{,\varphi}-\eta_{i,\varphi}\Big)\Big], \tag{9.18}$$

where

$$\Delta_{(5)}\hat{P}_i = -4\pi\rho\Big[\epsilon_{izk}x^k\Big(v^2+2U+\Gamma\varepsilon\Big)+{}_{(3)}\hat{\beta}_i\Big]+U_{,j}\Big({}_{(3)}\hat{\beta}_{i,j}+{}_{(3)}\hat{\beta}_{j,i}-\frac{2}{3}\delta_{ij\,(3)}\hat{\beta}_{k,k}\Big)$$
$$+\frac{1}{8}(UU_{,\varphi})_{,i}-\frac{1}{4}({}_{(3)}\hat{\beta}_k U_{,k})_{,i}. \tag{9.19}$$

X. DERIVATION OF BASIC EQUATIONS

In this section, we derive the basic equation which has a suitable form to construct equilibrium configurations of uniformly rotating body in numerical calculation: Although equilibrium configurations can be formally obtained by solving Eq.(8.24) as well as metric potentials, U, X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, ${}_{(3)}\beta_i$, ${}_{(5)}\beta_i$ and h_{ij}, they do not take convenient forms for numerical calculation. Thus, we here change Eq.(8.24) into other forms appropriate to obtain numerically equilibrium configurations.

In numerical calculation, the standard method to obtain equilibrium configurations is as follows (Hachisu 1986; Oohara and Nakamura 1990);

(1) We give a trial density configuration for ρ.

(2) We solve the Poisson equations.

(3) Using Eq.(8.24), we give a new density configuration.

These procedures are repeated until a sufficient convergence is achieved. Here, at (3), we need to specify unknown constants, Ω and C. In standard numerical methods (Hachisu 1986; Oohara and Nakamura 1990), these are calculated during iteration fixing densities at two points; i.e., if we put ρ_1 and ρ_2 at x_1 and x_2 into Eq.(8.24), they become two simultaneous equations for Ω and C. Hence, we can calculate them. However, the procedure is not so simple in the PN case: Ω is included in the source of the Poisson equations for the variables such as X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, η_i, ${}_{(5)}\hat{P}_i$, $h_{ij}^{(S)}$, $Q_{ij}^{(T)}$, $Q^{(I)}$ and $Q_i^{(I)}$. Thus, if we use Eq.(8.24) as it is, equations for Ω and C become implicit equations for Ω. In such a situation, the convergence to a solution is very slow. Therefore, we transform those equations into other forms in which the potentials as well as Eq.(8.24) become explicit polynomial equations in Ω.

First of all, we define q_2, q_{2i}, q_4, q_u, q_e and q_{ij} which satisfy

$$\Delta q_2 = -4\pi\rho R^2, \tag{10.1}$$

$$\Delta q_{2i} = -4\pi\rho R^2 x^i, \tag{10.2}$$

$$\Delta q_4 = -4\pi\rho R^4, \tag{10.3}$$

$$\Delta q_u = -4\pi\rho U, \tag{10.4}$$

$$\Delta q_e = -4\pi\rho\varepsilon, \tag{10.5}$$

$$\Delta q_{ij} = -4\pi\rho x^i x^j. \tag{10.6}$$

Then, X, ${}_{(4)}\psi$, $Q^{(I)}$, $Q_i^{(I)}$, η_i, ${}_{(5)}\hat{P}_i$, $Q_{ij}^{(T)}$, and $h_{ij}^{(S)}$ are written as

$$X = -2q_2\Omega^2 - 2q_u - (3\Gamma - 2)q_e, \tag{10.7}$$

$${}_{(4)}\psi = \frac{1}{2}\Big(q_2\Omega^2 + q_e + \frac{5}{2}q_u\Big), \tag{10.8}$$

$$Q^{(I)} = q_2\Omega^2 + 3(\Gamma - 1)q_e - \frac{1}{2}q_u \equiv q_2\Omega^2 + Q_0^{(I)}, \tag{10.9}$$

$$Q_i^{(I)} = q_{2i}\Omega^2 + Q_{0i}^{(I)}, \tag{10.10}$$

$$\eta_i = q_{2i}\Omega^2 + \eta_{0i}, \tag{10.11}$$

$${}_{(5)}\hat{P}_i = \epsilon_{izk}q_{2k}\Omega^2 + {}_{(5)}P_{0i}, \tag{10.12}$$

$$Q_{ij}^{(T)} = \epsilon_{izl}q_{lj}\Omega, \tag{10.13}$$

$$h_{ij}^{(S)} = 4\Omega^2\Big(\epsilon_{izk}\epsilon_{jzl}q_{kl} - \frac{1}{3}\delta_{ij}q_2\Big), \tag{10.14}$$

where $Q_{0i}^{(I)}$, η_{0i} and ${}_{(5)}P_{0i}$ satisfy

$$\Delta Q_{0i}^{(I)} = -4\pi\Big(3P - \frac{1}{2}\rho U\Big)x^i = -4\pi\rho\Big(3(\Gamma - 1)\varepsilon - \frac{1}{2}U\Big)x^i, \tag{10.15}$$

$$\Delta\eta_{0i} = -4\pi\rho\Big(\varepsilon + \frac{5}{2}U\Big)x^i, \tag{10.16}$$

$$\begin{aligned}\Delta_{(5)}P_{0i} = &-4\pi\rho\Big[\epsilon_{izk}x^k\Big(2U + \Gamma\varepsilon\Big) + {}_{(3)}\hat{\beta}_i\Big] + U_{,j}\Big({}_{(3)}\hat{\beta}_{i,j} + {}_{(3)}\hat{\beta}_{j,i} - \frac{2}{3}\delta_{ij\,(3)}\hat{\beta}_{k,k}\Big)\\ &+ \frac{1}{8}(UU_{,\varphi})_{,i} - \frac{1}{4}({}_{(3)}\hat{\beta}_k U_{,k})_{,i} \equiv -4\pi S_i^{(P)}.\end{aligned} \tag{10.17}$$

Note that ${}_{(5)}\beta_i$ and $h_{ij}^{(G)}$ are the cubic and quadratic equations in Ω, respectively, as

$$\begin{aligned}{}_{(5)}\beta_i = &\,\Omega\Big[-4{}_{(5)}P_{0i} + \frac{1}{2}\Big\{x^i\Big(q_e + \frac{5}{2}q_u\Big)_{,\varphi} - \eta_{0i,\varphi}\Big\}\Big] + \Omega^3\Big[-4\epsilon_{izk}q_{2k} + \frac{1}{2}\Big(x^i q_{2,\varphi} - q_{2i,\varphi}\Big)\Big]\\ \equiv &\,{}_{(5)}\beta_i^{(A)}\Omega + {}_{(5)}\beta_i^{(B)}\Omega^3,\end{aligned} \tag{10.18}$$

$$\begin{aligned}h_{ij}^{(G)} = &\,\frac{1}{2}\Big[\frac{\partial}{\partial x^j}\Big(x^i Q_0^{(I)} - Q_{0i}^{(I)}\Big) + \frac{\partial}{\partial x^i}\Big(x^j Q_0^{(I)} - Q_{0j}^{(I)}\Big) - \frac{4}{3}\delta_{ij}Q_0^{(I)}\Big]\\ &+\Omega^2\Big[\frac{1}{2}\Big\{\frac{\partial}{\partial x^j}\Big(x^i q_2 - q_{2i}\Big) + \frac{\partial}{\partial x^i}\Big(x^j q_2 - q_{2j}\Big) - \frac{4}{3}\delta_{ij}q_2\Big\}\end{aligned}$$

$$
\begin{aligned}
&-\frac{7}{4}\Big(x^i\epsilon_{jzk}q_{k,\varphi}+x^j\epsilon_{izk}q_{k,\varphi}-\epsilon_{izk}q_{kj,\varphi}-\epsilon_{jzk}q_{ki,\varphi}\Big)+\delta_{ij}x^k\epsilon_{kzl}q_l\\
&+\frac{1}{8}x^k\Big\{\frac{\partial}{\partial x^i}\Big(x^j\epsilon_{kzl}q_{l,\varphi}-\epsilon_{kzl}q_{lj,\varphi}\Big)+\frac{\partial}{\partial x^j}\Big(x^i\epsilon_{kzl}q_{l,\varphi}-\epsilon_{kzl}q_{li,\varphi}\Big)\Big\}\Big]\\
\equiv\ & h_{ij}^{(A)}+h_{ij}^{(B)}\Omega^2. \qquad (10.19)
\end{aligned}
$$

Finally, we write ${}_{(6)}\alpha$ as

$$
{}_{(6)}\alpha={}_{(6)}\alpha_0+{}_{(6)}\alpha_2\Omega^2-2q_4\Omega^4, \qquad (10.20)
$$

where ${}_{(6)}\alpha_0$ and ${}_{(6)}\alpha_2$ satisfy

$$
\begin{aligned}
\Delta_{(6)}\alpha_0 &= 4\pi\rho\Big[\Big(3\Gamma-2\Big)\varepsilon U-\Big(3\Gamma-4\Big)q_e+3q_u\Big]\\
&\quad-\Big(h_{ij}^{(U)}+h_{ij}^{(A)}\Big)U_{,ij}-\frac{3}{2}UU_{,l}U_{,l}+U_{,l}\frac{\partial}{\partial x^l}\Big(\frac{9}{2}q_u+(3\Gamma+1)q_e\Big)\\
&\equiv -4\pi S^{(\alpha_0)}, \qquad (10.21)\\
\Delta_{(6)}\alpha_2 &= 8\pi\rho R^2\Big(5U+\Gamma\varepsilon+2{}_{(3)}\hat{\beta}_\varphi\Big)-\Big(4\epsilon_{izk}\epsilon_{jzl}q_{kl}-\frac{4}{3}\delta_{ij}q_2+h_{ij}^{(B)}\Big)U_{,ij}+3q_{2,l}U_{,l}\\
&\qquad+\frac{1}{2}{}_{(3)}\hat{\beta}_{i,j}\Big({}_{(3)}\hat{\beta}_{i,j}+{}_{(3)}\hat{\beta}_{j,i}-\frac{2}{3}\delta_{ij}{}_{(3)}\hat{\beta}_{k,k}\Big)\\
&\equiv -4\pi S^{(\alpha_2)}. \qquad (10.22)
\end{aligned}
$$

Using the above quantities, Eq.(8.24) is rewritten as

$$
H-\frac{H^2}{2c^2}+\frac{H^3}{3c^4}=A+B\Omega^2+D\Omega^4+\frac{R^6}{6c^4}\Omega^6+C, \qquad (10.23)
$$

where

$$
\begin{aligned}
A &= U+\frac{1}{c^2}\Big(2q_u+(3\Gamma-2)q_e\Big)+\frac{1}{c^4}\Big\{-{}_{(6)}\alpha_0-\frac{U^3}{6}+U\Big(2q_u+(3\Gamma-2)q_e\Big)\Big\},\\
B &= \frac{R^2}{2}+\frac{1}{c^2}\Big(2R^2U+2q_2+{}_{(3)}\hat{\beta}_\varphi\Big)+\frac{1}{c^4}\Big\{-{}_{(6)}\alpha_2+\frac{1}{2}{}_{(3)}\hat{\beta}_i{}_{(3)}\hat{\beta}_i+4{}_{(3)}\hat{\beta}_\varphi U\\
&\quad+(3\Gamma-1)q_eR^2+\frac{9}{2}q_uR^2+\frac{15}{4}U^2R^2+2q_2U+{}_{(5)}\beta_\varphi^{(A)}+\frac{1}{2}\Big(h_{\varphi\varphi}^{(U)}+h_{\varphi\varphi}^{(A)}\Big)\Big\},\\
D &= \frac{R^4}{4c^2}+\frac{1}{c^4}\Big\{2q_4+{}_{(3)}\hat{\beta}_\varphi R^2+\frac{7}{3}q_2R^2+2UR^4+{}_{(5)}\beta_\varphi^{(B)}+\frac{1}{2}\Big(h_{\varphi\varphi}^{(B)}+4R^2q_{RR}\Big)\Big\}. \qquad (10.24)
\end{aligned}
$$

Note that in the above, we use the following relations which hold for arbitrary vector Q_i and symmetric tensor Q_{ij},

$$Q_\varphi = -yQ_x + xQ_y,$$
$$Q_{\varphi\varphi} = y^2 Q_{xx} - 2xyQ_{xy} + x^2 Q_{yy},$$
$$R^2 Q_{RR} = x^2 Q_{xx} + 2xyQ_{xy} + y^2 Q_{yy}. \quad (10.25)$$

We also note that source terms of Poisson equations for variables which appear in A, B and D do not depend on Ω explicitly. Thus, Eq.(10.23) takes the desired form for numerical calculation.

In this formalism, we need to solve 29 Poisson equations for U, q_x, q_y, q_z, ${}_{(5)}P_{0x}$, ${}_{(5)}P_{0y}$, η_{0x}, η_{0y}, $Q^{(I)}_{0x}$, $Q^{(I)}_{0y}$, $Q^{(I)}_{0z}$, q_2, q_{2x}, q_{2y}, q_{2z}, q_u, q_e, $h^{(U)}_{xx}$, $h^{(U)}_{xy}$, $h^{(U)}_{xz}$, $h^{(U)}_{yy}$, $h^{(U)}_{yz}$, q_{xx}, q_{xy}, q_{xz}, q_{yz}, ${}_{(6)}\alpha_0$, ${}_{(6)}\alpha_2$ and q_4. In Table 2, we show the list of the Poisson equations to be solved. In Table 3, we also summarize what variables are needed to calculate the metric variables U, X, ${}_{(4)}\psi$, ${}_{(6)}\alpha$, ${}_{(3)}\beta_i$, ${}_{(5)}\beta_i$, $h^{(U)}_{ij}$, $h^{(S)}_{ij}$, $h^{(A)}_{ij}$ and $h^{(B)}_{ij}$. Note that we do not need ${}_{(5)}P_{0z}$, η_{0z}, and q_{zz} because they do not appear in any equation. Also, we do not have to solve the Poisson equations for $h^{(U)}_{zz}$ and q_{yy} because they can be calculated from $h^{(U)}_{zz} = -h^{(U)}_{xx} - h^{(U)}_{yy}$ and $q_{yy} = q_2 - q_{xx}$.

In order to derive U, q_i, q_2, q_{2i}, q_4, q_e and q_{ij}, we do not need any other potential because only matter variables appear in the source terms of their Poisson equations. On the other hand, for q_u, $Q^{(I)}_{0i}$, η_{0i} and $h^{(U)}_{ij}$, we need the Newtonian potential U, and for ${}_{(5)}P_{0i}$, ${}_{(6)}\alpha_0$ and ${}_{(6)}\alpha_2$, we need the Newtonian as well as PN potentials. Thus, U, q_i, q_2, q_{2i}, q_4, q_e and q_{ij} must be solved first, and then q_u, $Q^{(I)}_{0i}$, η_{0i}, $h^{(U)}_{ij}$, ${}_{(5)}P_{0i}$ and ${}_{(6)}\alpha_2$ should be solved. ${}_{(6)}\alpha_0$ must be solved after we obtain q_u because its Poisson equation involves q_u in the source term. In Table 2, we also list potentials which are included in the source terms of the Poisson equations for other potentials.

The configuration which we are most interested in and would like to obtain is the equilibrium state for BNS's of equal mass. Hence, we show the boundary condition at $r \to \infty$ for this problem. When we consider equilibrium configurations for BNS's where the center of mass for each NS is on the x-axis, boundary conditions for potentials at $r \to \infty$ become

$$
\begin{aligned}
U &= \frac{1}{r}\int \rho dV + O(r^{-3}), & q_x &= \frac{n^x}{r^2}\int \rho x^2 dV + O(r^{-4}),\\
q_2 &= \frac{1}{r}\int \rho R^2 dV + O(r^{-3}), & q_y &= \frac{n^y}{r^2}\int \rho y^2 dV + O(r^{-4}),\\
q_e &= \frac{1}{r}\int \rho \varepsilon dV + O(r^{-3}), & q_z &= \frac{n^z}{r^2}\int \rho z^2 dV + O(r^{-4}),\\
q_u &= \frac{1}{r}\int \rho U dV + O(r^{-3}), & q_4 &= \frac{1}{r}\int \rho R^4 dV + O(r^{-3}),
\end{aligned}
\tag{10.26}
$$

$$
\begin{aligned}
{}_{(5)}P_{0x} &= \frac{n^x}{r^2}\int S^{(P)}_x x dV + \frac{n^y}{r^2}\int S^{(P)}_y y dV + O(r^{-3}),\\
{}_{(5)}P_{0y} &= \frac{n^x}{r^2}\int S^{(P)}_y x dV + \frac{n^y}{r^2}\int S^{(P)}_y y dV + O(r^{-3}),
\end{aligned}
\tag{10.27}
$$

$$
\begin{aligned}
\eta_{0x} &= \frac{n^x}{r^2}\int \rho x^2\Big(\varepsilon + \frac{5}{2}U\Big) dV + O(r^{-4}),\\
\eta_{0y} &= \frac{n^y}{r^2}\int \rho y^2\Big(\varepsilon + \frac{5}{2}U\Big) dV + O(r^{-4}),
\end{aligned}
\tag{10.28}
$$

$$
\begin{aligned}
Q^{(I)}_{0x} &= \frac{n^x}{r^2}\int \rho x^2\Big(3(\Gamma-1)\varepsilon - \frac{1}{2}U\Big) dV + O(r^{-4}), & q_{2x} &= \frac{n^x}{r^2}\int \rho R^2 x^2 dV + O(r^{-4}),\\
Q^{(I)}_{0y} &= \frac{n^y}{r^2}\int \rho y^2\Big(3(\Gamma-1)\varepsilon - \frac{1}{2}U\Big) dV + O(r^{-4}), & q_{2y} &= \frac{n^y}{r^2}\int \rho R^2 y^2 dV + O(r^{-4}),\\
Q^{(I)}_{0z} &= \frac{n^y}{r^2}\int \rho z^2\Big(3(\Gamma-1)\varepsilon - \frac{1}{2}U\Big) dV + O(r^{-4}), & q_{2z} &= \frac{n^z}{r^2}\int \rho R^2 z^2 dV + O(r^{-4}),
\end{aligned}
\tag{10.29}
$$

$$
\begin{aligned}
h^{(U)}_{xx} &= \frac{1}{r}\int S^{(U)}_{xx} dV + O(r^{-3}), & h^{(U)}_{xy} &= \frac{3n^x n^y}{r^3}\int S^{(U)}_{xy} xy dV + O(r^{-5}),\\
h^{(U)}_{yy} &= \frac{1}{r}\int S^{(U)}_{yy} dV + O(r^{-3}), & h^{(U)}_{xz} &= \frac{3n^x n^z}{r^3}\int S^{(U)}_{xz} xz dV + O(r^{-5}),\\
h^{(U)}_{yz} &= \frac{3n^y n^z}{r^3}\int S^{(U)}_{yz} yz dV + O(r^{-5}),
\end{aligned}
\tag{10.30}
$$

$$
\begin{aligned}
q_{xx} &= \frac{1}{r}\int \rho x^2 dV + O(r^{-3}), & q_{xy} &= \frac{3n^x n^y}{r^3}\int \rho x^2 y^2 dV + O(r^{-5}),\\
q_{xz} &= \frac{3n^x n^z}{r^3}\int \rho x^2 z^2 dV + O(r^{-5}), & q_{yz} &= \frac{3n^y n^z}{r^3}\int \rho y^2 z^2 dV + O(r^{-5}),
\end{aligned}
\tag{10.31}
$$

$$
{}_{(6)}\alpha_0 = \frac{1}{r}\int S^{(\alpha_0)} dV + O(r^{-3}), \qquad {}_{(6)}\alpha_2 = \frac{1}{r}\int S^{(\alpha_2)} dV + O(r^{-3}),
\tag{10.32}
$$

where $dV = d^3x$, and

$$
n^i = \frac{x^i}{r}.
\tag{10.33}
$$

We note that at $r \to \infty$, $S^{(P)}_i \to O(r^{-5})$, $S^{(U)}_{ij} \to O(r^{-6})$, $S^{(\alpha_0)} \to O(r^{-4})$ and $S^{(\alpha_2)} \to O(r^{-4})$, so that all the above integrals are well defined.

XI. CONSERVED QUANTITIES

In this section, we show the conserved quantities in the 2PN approximation because they will be useful to investigate the stability property of equilibrium solutions obtained in numerical calculations.

(1)Conserved mass (Asada, Shibata and Futamase 1996);

$$M_* \equiv \int \rho_* d^3x, \tag{11.1}$$

where

$$\begin{aligned}\rho_* &= \rho\alpha u^0\psi^6 \\ &= \rho\Big[1+\frac{1}{c^2}\Big(\frac{1}{2}v^2+3U\Big)+\frac{1}{c^4}\Big(\frac{3}{8}v^4+\frac{7}{2}v^2U+\frac{15}{4}U^2+6_{(4)}\psi+{}_{(3)}\beta_i v^i\Big)+O(c^{-6})\Big].\end{aligned} \tag{11.2}$$

Equation (11.2) may be written as

$$\rho_* = \rho\Big[1+\frac{1}{c^2}\Big(\frac{1}{2}v^2+3U\Big)+\frac{1}{c^4}\Big(\frac{3}{8}v^4+\frac{13}{2}v^2U+\frac{45}{4}U^2+3U\varepsilon+{}_{(3)}\beta_i v^i\Big)+O(c^{-6})\Big]. \tag{11.3}$$

(2)ADM mass (Wald 1984; Asada, Shibata and Futamase 1996);

$$M_{ADM} = -\frac{1}{2\pi}\int \Delta\psi d^3x \equiv \int \rho_{ADM} d^3x, \tag{11.4}$$

where

$$\begin{aligned}\rho_{ADM} = \rho\Big[1&+\frac{1}{c^2}\Big(v^2+\varepsilon+\frac{5}{2}U\Big)+\frac{1}{c^4}\Big\{v^4+\frac{13}{2}v^2U+\Gamma\varepsilon v^2+\frac{5}{2}U\varepsilon+\frac{5}{2}U^2+5_{(4)}\psi \\ &+2_{(3)}\beta_i v^i+\frac{1}{32\pi\rho}{}_{(3)}\beta_{i,j}\Big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij\,(3)}\beta_{k,k}\Big)\Big\}+O(c^{-6})\Big],\end{aligned} \tag{11.5}$$

or

$$\begin{aligned}\rho_{ADM} = \rho\Big[1&+\frac{1}{c^2}\Big(v^2+\varepsilon+\frac{5}{2}U\Big)+\frac{1}{c^4}\Big(v^4+9v^2U+\Gamma\varepsilon v^2+5U\varepsilon+\frac{35}{4}U^2+\frac{3}{2}{}_{(3)}\beta_i v^i\Big) \\ &+O(c^{-6})\Big].\end{aligned} \tag{11.6}$$

(3)Total energy, which is calculated from $M_{ADM}-M_*$ in the third PN order (Asada, Shibata and Futamase 1996);

$$E \equiv \int \rho_E d^3x, \tag{11.7}$$

where

$$\begin{aligned}
\rho_E = \rho\Big[&\Big(\frac{1}{2}v^2+\varepsilon-\frac{1}{2}U\Big)+\frac{1}{c^2}\Big(\frac{5}{8}v^4+\frac{5}{2}v^2U+\Gamma v^2\varepsilon+2U\varepsilon-\frac{5}{2}U^2+\frac{1}{2}{}_{(3)}\beta_i v^i\Big)\\
&+\frac{1}{c^4}\Big\{\frac{11}{16}v^6+v^4\Big(\Gamma\varepsilon+\frac{47}{8}U\Big)+v^2\Big(4_{(4)}\psi+6\Gamma\varepsilon U+\frac{41}{8}U^2+\frac{5}{2}{}_{(3)}\beta_i v^i-X\Big)\\
&\quad-\frac{5}{2}U^3+2\Gamma_{(3)}\beta_i v^i\varepsilon+5\varepsilon_{(4)}\psi+5U_{(3)}\beta_i v^i-\frac{15}{2}U_{(4)}\psi+\frac{5}{4}U^2\varepsilon\\
&\quad+\frac{1}{2}h_{ij}v^iv^j+\frac{1}{2}{}_{(3)}\beta_{i(3)}\beta_i\\
&\quad+\frac{U}{16\pi\rho}\Big(2h_{ij}U_{,ij}+{}_{(3)}\beta_{i,j}\Big({}_{(3)}\beta_{i,j}+{}_{(3)}\beta_{j,i}-\frac{2}{3}\delta_{ij(3)}\beta_{k,k}\Big)\Big)\Big\}+O(c^{-6})\Big].
\end{aligned} \tag{11.8}$$

It is noteworthy that terms including ${}_{(5)}\beta_i$ cancel out in total.

(4)Total linear and angular momenta: In the case ${K_i}^i=0$, these are calculated from (Wald 1984)

$$\begin{aligned}
P_i &= \frac{1}{8\pi}\lim_{r\to\infty}\oint K_{ij}n^j dS\\
&= \frac{1}{8\pi}\lim_{r\to\infty}\oint \psi^6 K_{ij}n^j dS\\
&= \frac{1}{8\pi}\int (\psi^6 {K_i}^j)_{,j}d^3x\\
&= \int\Big(J_i+\frac{1}{16\pi}\psi^4\tilde{\gamma}_{jk,i}K^{jk}\Big)\psi^6 d^3x,
\end{aligned} \tag{11.9}$$

where $J_i=(\rho c^2+\rho\varepsilon+P)\alpha u^0 u_i$. Up to the 2PN order, the second term in the last line of Eq.(11.9) becomes

$$\begin{aligned}
&\frac{1}{16\pi}\int h_{jk,i(3)}\beta_{j,k}d^3x,\\
=&\frac{1}{16\pi}\int\Big[\Big(h_{jk,i(3)}\beta_j\Big)_{,k}-h_{jk,ik(3)}\beta_j\Big]d^3x,\\
=&\frac{1}{16\pi}\lim_{r\to\infty}\oint h_{jk,i(3)}\beta_j n^k dS=0,
\end{aligned} \tag{11.10}$$

where we use $h_{jk}\to O(r^{-1})$ and ${}_{(3)}\beta_j\to O(r^{-2})$ at $r\to\infty$, and the gauge condition $h_{jk,k}=0$. Thus, in the 2PN approximation, P_i becomes

$$P_i \equiv \int p_i d^3x, \tag{11.11}$$

where

$$p_i = \rho\Big[v^i + \frac{1}{c^2}\Big\{v^i\Big(v^2 + \Gamma\varepsilon + 6U\Big) + {}_{(3)}\beta_i\Big\} + \frac{1}{c^4}\Big\{h_{ij}v^j + {}_{(5)}\beta_i + {}_{(3)}\beta_i\Big(v^2 + 6U + \Gamma\varepsilon\Big)$$
$$+v^i\Big(2_{(3)}\beta_i v^i + 10_{(4)}\psi + 6\Gamma\varepsilon U + \frac{67}{4}U^2 + \Gamma\varepsilon v^2 + 10Uv^2 + v^4 - X\Big)\Big\} + O(c^{-5})\Big]. \tag{11.12}$$

The total angular momentum J becomes

$$J = \int p_\varphi d^3x, \tag{11.13}$$

where $p_\varphi = -yp_x + xp_y$.

XII. DISCUSSION AND SUMMARY

In this thesis, we have developed the PN approximation in the (3+1) formalism of general relativity. In this formalism, it is clarified what kind of gauge condition is suitable for each problem such as how to extract the waveforms of gravitational waves and how to describe equilibrium configurations. It was found that the combination of the conformal slice and the transverse gauge is useful to separate the wave part and the non-wave part in the metric variables such as h_{ij} and ψ. We also found that, in order to describe the equilibrium configuration, the conformal slice is not useful and instead we had better use the maximal slice. Although we restricted ourselves within some gauge conditions in this thesis, we can use any gauge condition and investigate its property relatively easily in the (3+1) formalism, compared with in the standard PN approximation performed so far (Chandrasekhar et.al. 1965, 1967, 1969, 1970). We have also developed a formalism for the hydrodynamic equation accurate up to 2.5PN order. For the sake of an actual numerical simulation, we carefully consider methods to solve the various metric quantities, especially, the 2PN tensor potential ${}_{(4)}h_{ij}$. We found it possible to solve them by using standard numerical methods. Thus, the formalism developed in this thesis will be useful also in numerical calculations.

In section 3, we used several slice conditions and investigated their properties, but, as for the spatial gauge condition, we fix it to the transverse gauge for the sake of convenience. It is not clear, however, whether this is the best gauge condition in numerical relativity. In numerical relativity, the shift vector plays a very important role to reduce the coordinate shear. If we fail to choose the appropriate condition, the coordinate shear in the spatial metric will continue to grow, and as a result, the simulation will break down. The minimal distortion gauge which was proposed by Smarr and York (1978a, 1978b) is a candidate which may reduce efficiently the coordinate shear. Even if we use this gauge condition in the PN analysis, equations for ${}_{(4)}h_{ij}$, ${}_{(5)}h_{ij}$, ${}_{(3)}\beta_i$, ${}_{(5)}\beta_i$ and ${}_{(6)}\beta_i$ remain unchanged, but higher order terms of h_{ij} and β_i may slightly change. If we investigate the effects due to the difference, we may be able to give some important suggestions about the gauge condition appropriate for numerical relativity.

In Part 2, we have developed the formulation to obtain the nonaxisymmetric uniformly rotating equilibrium configurations. It is generally expected that there exists no Killing vector in the spacetime of coalescing BNS's because such a spacetime is filled with gravitational radiation which propagates to null infinity. However, we may consider coalescing BNS's as the almost stationary object from physical point of view as described in section 7. Motivated by this idea, we have developed a formalism to obtain equilibrium configurations of uniformly rotating fluid up to the 2PN order using the PN approximation. The concept of being "almost" stationary becomes clear in the framework of the PN approximation and, in particular, the stationary rotating objects can exist exactly at the 2PN order, since the energy loss due to the gravitational radiation does occur from the 2.5PN order.

Here, we would like to emphasize that from the 2PN order, the tensor part of the 3-metric, $\tilde{\gamma}_{ij}$, cannot be neglected even if we ignore gravitational waves. Recently, Wilson and Mathews (1995), Wilson, Mathews and Marronetti (1996) presented numerical equilibrium configurations of binary neutron stars using a semi-relativistic approximation, in which they assume the spatially conformal flat metric as the spatial 3-metric, i.e., $\tilde{\gamma}_{ij} = \delta_{ij}$. Thus, in their method, a 2PN term, h_{ij}, was completely neglected. However, it should be noted that this tensor potential plays an important role at the 2PN order: This is because they appear in the equations to determine equilibrium configurations as shown in previous sections and they also contribute to the total energy and angular momentum of systems. This means that if we performed the stability analysis ignoring the tensor potentials, we might reach an incorrect conclusion. If we hope to obtain a general relativistic eqiulibrium configuration of binary neutron stars with a better accuracy (say less than 1%), we should take into account the tensor part of the 3-metric. On the other hand, Bonazzola, Frieben and Gourgoulhon (1996) obtained an approximate nonaxisymmetric neutron star by perturbing a stationary axisymmetric configuration. That is to say, they do not solve the exact 3D Einstein equation. Thus, it is important to reexamine their result on the transition between configurations, which are approximately ellipsoids, by other methods including the method presented here.

In our formalism, we extract terms depending on the angular velocity Ω from the integrated Euler equation and Poisson equations for potentials, and rewrite the integrated Euler equation as an explicit equation in Ω. This reduction will improve the convergence in numerical iteration procedure. As a result, the number of Poisson equations we need to solve in each step of iteration reaches 29. However, source terms of the Poisson equations decrease rapidly enough, at worst $O(r^{-4})$, in the region far from the source, so that we can solve accurately these equations as the boundary value problem like in the case of the first PN calculations. Thus, the formalism presented here will be useful to obtain equilibrium configurations for synchronized BNS's or the Jacobi ellipsoid. These configurations will be obtained in future work.

Acknowledgments

I would like to thank Professor S. Ikeuchi and Professor M. Sasaki for useful advice and continuous encouragement. I would like to thank Professor T. Futamase and Dr. M. Shibata for useful discussion and invaluable advice. I would like to thank Professor T. Nakamura, Professor M. Sasaki and Dr. T. Tanaka for useful discussion. I would like to thank Professor Y. Kojima and Dr. K. Nakao for useful comments. I would like to thank Professor N. Gouda for his encouragement.

APPENDIX A: EQUATION OF MOTION

(1) The spatial component of the conservation law

$$T_i^{\mu}{}_{;\mu} = 0 \tag{A1}$$

is written explicitly for the perfect fluid as

$$\Big[(\rho + \rho\epsilon + P)u^\mu u_i + P\delta_i^\mu\Big]_{;\mu} = 0. \tag{A2}$$

This is equal to

$$\frac{1}{\alpha\psi^6}\Big[\alpha\psi^6(\rho + \rho\epsilon + P)u^\mu u_i\Big]_{,\mu} - (\rho + \rho\epsilon + P)u^\mu u_\nu \Gamma^\nu_{i\mu} + P_{,i} = 0. \tag{A3}$$

Hence we obtain

$$\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = -\alpha\psi^6 P_{,i} + \alpha\psi^6(\rho + \rho\epsilon + P)u^\mu u_\nu \Gamma^\nu_{i\mu}, \tag{A4}$$

where

$$S_i = \alpha\psi^6(\rho + \rho\varepsilon + P)u^0 u_i = \rho_*\Big(1 + \varepsilon + \frac{P}{\rho}\Big)u_i (= \psi^6 J_i). \tag{A5}$$

We evaluate the last term of the right hand side of Eq.$(A4)$ as

$$u^\mu u_\nu \Gamma^\nu_{i\mu} = -\alpha\alpha_{,i}(u^0)^2 + u^0 u_k \beta^k{}_{,i} - \frac{1}{2}u_k u_l \gamma^{kl}{}_{,i}. \tag{A6}$$

Thus we obtain the equation of motion as

$$\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = -\alpha\psi^6 P_{,i} - \alpha\alpha_{,i}S^0 + S_k\beta^k{}_{,i} - \frac{1}{2}\frac{S_k S_l}{S^0}\gamma^{kl}{}_{,i}, \tag{A7}$$

where

$$S^0 = \alpha\psi^6(\rho + \rho\varepsilon + P)(u^0)^2\Big(= \frac{(\rho_H + P)\psi^6}{\alpha}\Big). \tag{A8}$$

(2) Here we consider

$$u_\nu T^{\mu\nu}{}_{;\mu} = 0. \tag{A9}$$

If Eqs.($A1$) and ($A9$) are satisfied, then we obtain

$$T_0^{\mu}{}_{;\mu} = 0, \tag{A10}$$

since u^0 does not vanish. Here we used the relation

$$u_\nu T^{\mu\nu}{}_{;\mu} = T_0^{\mu}{}_{;\mu} u^0 + T_i^{\mu}{}_{;\mu} u^i. \tag{A11}$$

Therefore the set of Eqs.($A1$) and ($A9$) is equivalent to the set of Eqs.($A1$) and ($A10$), i.e. the conservation law.

We return to Eq.($A9$), which is rewritten as

$$\Big(\rho\epsilon u^\mu\Big)_{;\mu} = -P u^\mu_{;\mu}. \tag{A12}$$

Here we used the baryon number conservation $(\rho u^\mu)_{;\mu} = 0$ and $u_\nu u^\nu{}_{;\mu} = 0$. Eq.($A12$) is rewritten as

$$\Big(\sqrt{-g}\rho\epsilon u^\mu\Big)_{,\mu} = -P\Big(\sqrt{-g}u^\mu\Big)_{,\mu}. \tag{A13}$$

Hence we obtain

$$\frac{\partial H}{\partial t} + \frac{\partial (H v^j)}{\partial x^j} = -P\Big(\frac{\partial(\alpha\psi^6 u^0)}{\partial t} + \frac{\partial(\alpha\psi^6 u^0 v^j)}{\partial x^j}\Big), \tag{A14}$$

where

$$\begin{aligned} H &= \alpha\psi^6 \rho\varepsilon u^0 = \rho_*\varepsilon, \\ v^i &\equiv \frac{u^i}{u^0} = -\beta^i + \frac{\gamma^{ij}S_j}{S^0}. \end{aligned} \tag{A15}$$

APPENDIX B: DERIVATION OF LOGARITHMIC KERNEL

We describe the derivation of the logarithmic kernel which works well at the 2PN order. There are some methods to derive the logarithmic kernel (Fock 1959; Damour 1982). For instance, the direct integral may be performed. Among all, we explain the method used by Ohta et.al. (1974b) who derived the Lagrangian for many bodies by using the logarithmic kernel. In this method, the problem to find the kernel is reduced to solving the ordinary differential equation. It has not been known which strategy works well to search suitable kernels at higher PN orders (beyond 2.5PN order). Since this method is one of candidates of the strategy, we describe it by taking into account the extension for more general cases.

We begin by considering the following type of the equation

$$\Delta f = \frac{(r_a + r_b)^n}{r_a r_b}, \tag{B1}$$

where $a \neq b$ and n takes $0, 1, 2, \cdots$. Here r_a, r_b and r_{ab} are defined as

$$\begin{aligned} r_a &= |\mathbf{x} - \mathbf{y_a}|, \\ r_b &= |\mathbf{x} - \mathbf{y_b}|, \\ r_{ab} &= |\mathbf{y_a} - \mathbf{y_b}|. \end{aligned} \tag{B2}$$

Since we can take f as $f(r_a, r_b, r_{ab})$, we obtain

$$\begin{aligned} \Delta f(r_a, r_b, r_{ab}) &= \frac{\partial}{\partial x^i}\Big(\frac{\partial r_a}{\partial x^i}\frac{\partial}{\partial r_a} + \frac{\partial r_b}{\partial x^i}\frac{\partial}{\partial r_b}\Big) f(r_a, r_b, r_{ab}) \\ &= \Big[\Big(\frac{\partial^2}{\partial r_a^2} + \frac{\partial^2}{\partial r_a^2}\Big) + 2\Big(\frac{1}{r_a}\frac{\partial}{\partial r_a} + \frac{1}{r_b}\frac{\partial}{\partial r_b}\Big) + 2\frac{r_a^i r_b^i}{r_a r_b}\frac{\partial^2}{\partial r_a \partial r_b}\Big] f(r_a, r_b, r_{ab}). \end{aligned} \tag{B3}$$

We introduce a set of variables (s, t) in place of (r_a, r_b) as

$$\begin{aligned} s &= r_a + r_b, \\ t &= r_a - r_b. \end{aligned} \tag{B4}$$

Then Eq.$(B3)$ is rewritten as

$$\Delta f(r_a, r_b, r_{ab}) = \Big[4\frac{\partial^2}{\partial t^2} + 2\Big(\frac{1}{r_a} + \frac{1}{r_b}\Big)\frac{\partial}{\partial s} + \Big(\frac{1}{r_a} - \frac{1}{r_b}\Big)\frac{\partial}{\partial t} + \frac{s^2 - r_{ab}^2}{r_a r_b}\Big(\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}\Big)\Big] f(s,t). \tag{B5}$$

However, Eq.$(B1)$ suggests that f is symmetric for r_a and r_b. Hence, since f does not depend on t, f is simply written as $f(s)$. Then Eq.$(B1)$ becomes

$$\begin{aligned}\Delta f(s) &= \Big(\frac{2s}{r_a r_b}\frac{d}{ds} + \frac{s^2 - r_{ab}^2}{r_a r_b}\frac{d^2}{ds^2}\Big) f(s) \\ &= \frac{s^n}{r_a r_b}.\end{aligned} \tag{B6}$$

That is to say, the problem to find a solution of Eq.$(B1)$ reduces to solving the following ordinary differential equation

$$\Big(2s\frac{d}{ds} + (s + r_{ab})(s - r_{ab})\frac{d^2}{ds^2}\Big) f(s) = s^n. \tag{B7}$$

This equation is rewritten as

$$\frac{d}{ds}\Big((s^2 - r_{ab}^2)\frac{df}{ds}\Big) = s^n. \tag{B8}$$

This is integrated as

$$\frac{df}{ds} = \frac{1}{n+1}\frac{s^n + s^{n-1}r_{ab} + \cdots + r_{ab}^n}{s + r_{ab}} + \frac{C_1}{s^2 - r_{ab}^2}, \tag{B9}$$

where C_1 is a constant. We can take $C_1 = 0$. In the following, we consider two cases, odd n and even n separately.

(1) odd n case

Here we consider the case of odd n. In this case, we obtain

$$s^n + s^{n-1}r_{ab} + \cdots + r_{ab}^n = (s + r_{ab})(s^{n-1} + s^{n-3}r_{ab}^2 + \cdots + r_{ab}^{n-1}). \tag{B10}$$

Therefore, Eq.$(B9)$ for $C_1 = 0$ becomes

$$\frac{df}{ds} = \frac{1}{n+1}\Big(s^{n-1} + s^{n-3}r_{ab}^2 + \cdots + r_{ab}^{n-1}\Big). \tag{B11}$$

Thus we obtain

$$f = \frac{1}{n+1}\Big(\frac{1}{n}s^n + \frac{1}{n-2}s^{n-2}r_{ab}^2 + \cdots + sr_{ab}^{n-1}\Big) + C_2, \tag{B12}$$

where C_2 is a constant.

(2) even n case

Here we consider the case of even n. For even n, we obtain

$$s^n + s^{n-1}r_{ab} + \cdots + r_{ab}^n = (s + r_{ab})(s^{n-1} + s^{n-3}r_{ab}^2 + \cdots + sr_{ab}^{n-2}) + r_{ab}^n. \tag{B13}$$

Thus, Eq.($B9$) for $C_1 = 0$ becomes

$$\frac{df}{ds} = \frac{1}{n+1}\Big(s^{n-1} + s^{n-3}r_{ab}^2 + \cdots + sr_{ab}^{n-2}\Big) + \frac{1}{n+1}\frac{r_{ab}^n}{s + r_{ab}}. \tag{B14}$$

Thus we obtain

$$f = \frac{1}{n+1}\Big(\frac{1}{n}s^n + \frac{1}{n-2}s^{n-2}r_{ab}^2 + \cdots + \frac{1}{2}s^2 r_{ab}^{n-2} + r_{ab}^n \ln(s + r_{ab})\Big) + C_3, \tag{B15}$$

where C_3 is a constant.

An alternative type of the kernel is obtained as follows: By using the nonvanishing integral constant C_1, we rewrite Eq.($B9$) as

$$\frac{df}{ds} = \frac{1}{n+1}\Big(s^{n-1} + s^{n-3}r_{ab}^2 + \cdots + sr_{ab}^{n-2}\Big) + \frac{A}{s - r_{ab}} + \frac{B}{s + r_{ab}}, \tag{B16}$$

where A and B denote

$$\begin{aligned} A &= \frac{1}{2}\Big(\frac{r_{ab}^n}{n+1} + \frac{C_1}{r_{ab}}\Big), \\ B &= \frac{1}{2}\Big(\frac{r_{ab}^n}{n+1} - \frac{C_1}{r_{ab}}\Big). \end{aligned} \tag{B17}$$

Hence we obtain

$$f = \frac{1}{n+1}\Big(\frac{1}{n}s^n + \frac{1}{n-2}s^{n-2}r_{ab}^2 + \cdots + \frac{1}{2}s^2 r_{ab}^{n-2}\Big) + A\ln(s - r_{ab}) + B\ln(s + r_{ab}), \tag{B18}$$

where constants A and B satisfy

$$A + B = \frac{r_{ab}^n}{n+1}. \tag{B19}$$

At the 2PN order, we consider the case of $n = 0$, i.e.

$$\Delta f = \frac{1}{r_a r_b}. \tag{B20}$$

Eq.($B18$) becomes

$$f = A \ln(s - r_{ab}) + B \ln(s + r_{ab}), \tag{B21}$$

where $A + B = 1$. Thus, we obtain $f = \ln(s - r_{ab})$ for $A = 1$ and $f = \ln(s + r_{ab})$ for $B = 1$.

It is worthwhile to mention that this method is not necessarily useful for more general cases which may occur at higher PN orders: When the kernel does not satisfy the symmetric equation like Eq.($B1$), we cannot replace the problem of finding the kernel with that of solving the ordinary differential equation, as implied by Eq.($B5$). Furthermore, for more than three points, we cannot transform the Poisson equation for the kernel into the ordinary differential equation in general. At the 3PN or higher PN orders, the method described here may not be so useful, since contributions from many points on the matter should appear in the source terms of the Einstein equation.

APPENDIX C: CALCULATION OF ${}_{(5)}h_{ij}$

We make use of the transverse property of τ_{ij}, which is guaranteed by the transverse gauge condition, in order to obtain ${}_{(5)}h_{ij}$. Using the following identity

$$ {}_{(4)}\tau_{ij} = ({}_{(4)}\tau_{ik}x^j)_{,k}, \tag{C1} $$

Eq.(2.66) can be rewritten in the surface integral form

$$ {}_{(5)}h_{ij} = \frac{1}{4\pi}\frac{\partial}{\partial t}\int {}_{(4)}\tau_{ik}x^j n^k dS. \tag{C2} $$

Thus, we only need to estimate terms of $O(r^{-3})$ in ${}_{(4)}\tau_{ij}$, which come only from the shift vector in the conformal slice as

$$ {}_{(3)}\beta_i = \frac{1}{r^2}\Big(n^j Z_{ij} + \frac{1}{2}n^j \dot{I}_{ij} + \frac{1}{4}n^i \dot{I}_{kk} - \frac{3}{4}n^i n^j n^k \dot{I}_{jk} + \frac{n^i}{4\pi}\int {}_{(3)}K d^3x\Big) + O(r^{-3}), \tag{C3} $$

where

$$ Z_{ij}(t) = -4\int \rho v^i y^j d^3x. \tag{C4} $$

Here, note the following relations as

$$ \frac{\partial}{\partial t}Z_{ij} = -2\ddot{I}_{ij}, \tag{C5} $$

and

$$ \int {}_{(3)}\dot{K} d^3x = 2\pi \ddot{I}_{kk}. \tag{C6} $$

Therefore, the relevant terms of ${}_{(4)}\tau_{ij}$ for the surface integral become

$$ \begin{aligned} {}_{(4)}\tau_{ij} \rightarrow \quad & {}_{(3)}\dot{\beta}_{i,j} + {}_{(3)}\dot{\beta}_{j,i} - \frac{2}{3}\delta_{ij}\,{}_{(3)}\dot{\beta}_{l,l} \\ = & \frac{1}{r^3}\Big[\Big\{-3\ddot{I}_{ij} + 3\Big(\ddot{I}_{ik}n^k n^j + \ddot{I}_{jk}n^k n^i\Big) - \frac{9}{2}\ddot{I}_{kk}n^i n^j + \frac{15}{2}\ddot{I}_{kl}n^i n^j n^k n^l\Big\} \\ & -\frac{1}{3}\delta_{ij}\Big\{-\frac{15}{2}\ddot{I}_{kk} + \frac{27}{2}\ddot{I}_{kl}n^k n^l\Big\}\Big] + O(r^{-4}). \end{aligned} \tag{C7} $$

Thus we obtain h_{ij} at the 2.5PN order as

$$
\begin{aligned}
{}_{(5)}h_{ij} &= -\frac{1}{4\pi}\frac{\partial}{\partial t}\int\Big({}_{(3)}\dot{\beta}_{i,k} + {}_{(3)}\dot{\beta}_{k,i} - \frac{2}{3}\delta_{ik\,(3)}\dot{\beta}_{l,l}\Big)x^j n^k dS \\
&= -\frac{4}{5}\rlap{--}I_{ij}^{(3)}(t).
\end{aligned}
\tag{C8}
$$

This derivation seems fairly simple owing to the gauge condition. Thus, it is expected that higher order calculations, say at 3.5PN order, may become easier in this gauge condition.

APPENDIX D: CALCULATION OF ${}_{(3)}\beta_i, {}_{(5)}\beta_i$

In this appendix, we briefly comment on a method to solve the Poisson equations for ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$, i.e., Eqs.(2.59) and (2.61). Since the source terms of them have terms such as $-\dot{U}_{,i}$ and $-2{}_{(4)}\dot{\psi}_{,i}$ which behaves as $O(r^{-2})$ at $r\to\infty$, it seems that a technical problem arises in solving these equations in numerical calculation. It should be noted that in the Newtonian limit, $-\dot{U}_{,i}$ is $O(r^{-4})$, but at the 1PN order, it becomes $O(r^{-2})$ because $\int \dot{\rho} dV \neq 0$. However, this is easily overcome in a simple manner. We consider the case of the maximal slice for simplicity, but other cases may be treated similarly.

First of all, we write ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$ as,

$$
\begin{aligned}
{}_{(3)}\beta_i &= -4{}_{(3)}P_i + \frac{1}{2}\dot{\chi}_{,i},\\
{}_{(5)}\beta_i &= -4{}_{(5)}P_i + \frac{1}{2}{}_{(4)}\dot{\chi}_{,i},
\end{aligned}
\tag{D1}
$$

where χ and ${}_{(3)}P_i$ satisfy Eq.(5.5) and the first equation of Eqs.(4.31), respectively. ${}_{(5)}P_i$ and ${}_{(4)}\chi$ satisfy the following Poisson equations;

$$
\begin{aligned}
\Delta_{flat\,(5)}P_i &= -4\pi\rho\Big[v^i\Big(v^2+2U+\varepsilon+\frac{P}{\rho}\Big)+{}_{(3)}\beta_i\Big]+2U_{,j\,(3)}\tilde{A}_{ij}-\frac{1}{8}(\dot{U}U)_{,i}-\frac{1}{4}({}_{(3)}\beta_l U_{,l})_{,i}\ ,\\
\Delta_{flat\,(4)}\chi &= -4{}_{(4)}\psi.
\end{aligned}
\tag{D2}
$$

${}_{(4)}\chi$ can be written as

$$
{}_{(4)}\chi = -\int \rho_4|\mathbf{x}-\mathbf{y}|d^3y,
\tag{D3}
$$

where

$$
\rho_4 = \rho\Big(v^2+\varepsilon+\frac{5}{2}U\Big).
\tag{D4}
$$

From Eqs.(5.5) and $(D3)$, $\chi_{,i}$ and ${}_{(4)}\chi_{,i}$ become

$$
\begin{aligned}
\chi_{,i} &= -\int d^3y\rho\frac{x^i-y^i}{|\mathbf{x}-\mathbf{y}|} = -x^iU+\eta_i,\\
{}_{(4)}\chi_{,i} &= -\int d^3y\rho_4\frac{x^i-y^i}{|\mathbf{x}-\mathbf{y}|} = -2x^i{}_{(4)}\psi+{}_{(4)}\eta_i,
\end{aligned}
\tag{D5}
$$

where η_i and ${}_{(4)}\eta_i$ satisfy

$$\Delta_{flat}\eta_i = -4\pi\rho x^i,$$
$$\Delta_{flat\,(4)}\eta_i = -4\pi\rho_4 x^i. \tag{D6}$$

Hence,

$${}_{(3)}\beta_i = -4{}_{(3)}P_i - \frac{1}{2}\Big(x^i\dot{U} - \dot{\eta}_i\Big),$$
$${}_{(5)}\beta_i = -4{}_{(5)}P_i - \frac{1}{2}\Big(2x^i{}_{(4)}\dot{\psi} - {}_{(4)}\dot{\eta}_i\Big). \tag{D7}$$

Since the source terms of the Poisson equations for ${}_{(3)}P_i$, ${}_{(5)}P_i$, η_i and ${}_{(4)}\eta_i$ behaves as $O(r^{-n})$, where $n \geq 5$, at $r \to \infty$, these vector potentials can be accurately obtained by solving the Poisson equations for them under appropriate boundary conditions. It should be noted that the non-compact sources of the Poisson equation for ${}_{(5)}P_i$ may be regarded as $O(r^{-5})$ in the 2PN approximation because $\dot{U}$ is $O(r^{-3})$ in the Newtonian order. Thus, there is no difficulty to obtain ${}_{(3)}\beta_i$ and ${}_{(5)}\beta_i$.

Finally, we note that the above method is not unique prescription. For example, ${}_{(3)}\beta_i$ in the first PN approximation may be expressed as (Blanchet, Damour and Schäfer 1990)

$${}_{(3)}\beta_i = -4{}_{(3)}P_i + \frac{1}{2}\Big(({}_{(3)}P_k x^k)_{,i} - \chi_{2,i}\Big), \tag{D8}$$

where χ_2 satisfies

$$\Delta_{flat}\chi_2 = -4\pi\rho v^i x^i. \tag{D9}$$

APPENDIX E: INTEGRATED EULER'S EQUATION

Here we shall derive the integral form of the general relativistic Euler's equation as

$$\ln u^t = \int \frac{dP}{\rho + \rho\epsilon + P}. \tag{E1}$$

The procedure to derive this equation can be divided into three parts, by following Lightman, Press, Price and Teukolsky (1975).

(1)

Let us assume that there exists a timelike Killing vector ξ so that the four velocity u^μ can be expressed as $\xi^\mu/|\xi \cdot \xi|^{1/2}$. Then we obtain the four acceleration as

$$\begin{aligned} a &\equiv u^\nu \nabla_\nu u^\mu \\ &= \frac{1}{2}\nabla^\mu \ln|\xi \cdot \xi|, \end{aligned} \tag{E2}$$

where we defined $|\xi \cdot \xi| = -\xi^\mu \xi_\mu$ and used the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{E3}$$

(2)

By acting the projection operator on the conservation law, we obtain

$$\begin{aligned} P_{\alpha\mu}\nabla_\nu T^{\mu\nu} &= (\rho + \rho\epsilon + P)u^\beta \nabla_\beta u_\alpha + \left(\nabla_\alpha P + u_\alpha u^\beta \nabla_\beta P\right) \\ &= 0, \end{aligned} \tag{E4}$$

where the projection tensor $P_{\alpha\beta}$ is defined as

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \tag{E5}$$

Here it is noteworthy that the geodesic equation is not used in order to derive this result.

(3)

Here we consider the uniformly rotating object. Then we can assume that the four velocity is expressed as

$$u^\mu = u^t\Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\Big), \tag{E6}$$

where Ω is a constant. Then the timelike Killing vector is written simply as

$$\xi^\mu = \Big(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \phi}\Big). \tag{E7}$$

From $u^\mu u_\mu = -1$, u^t is written as

$$u^t = \frac{1}{|\xi\cdot\xi|^{1/2}}. \tag{E8}$$

From Eqs.($E2$) and ($E8$), we obtain

$$u^\nu\nabla_\nu u^\mu = -\nabla \ln u^t. \tag{E9}$$

Hence from Eq.($E4$), we obtain

$$\nabla^\alpha P = (\rho + \rho\epsilon + P)\nabla^\alpha \ln u^t, \tag{E10}$$

where we used

$$\xi^\alpha\nabla_\alpha P = 0. \tag{E11}$$

Thus we obtain

$$\ln u^t = \int \frac{dP}{\rho + \rho\epsilon + P}. \tag{E12}$$

APPENDIX F: TAIL

1. radiative moments including tail effects

(1) tail terms:

In the post-Minkowskian approximation, the background geometry is the Minkowski spacetime where gravitational waves at the lowest order propagate (Thorne 1980; Blanchet and Damour 1984a, 1984b, 1986). On the other hand, the gravitational waves propagate on the true light cone. It is believed that the corrections to propagation of gravitational waves can be taken into account if one performs the post-Minkowskian approximation up to higher orders. That is to say, since the true wave operator is formally expanded as

$$\Box_{true} = \Box_{flat} + G\Box_{(1)} + G^2\Box_{(2)} + \cdots, \tag{F1}$$

all one must to do is to solve iteratively in terms of G

$$\Box_{true} h_{ij} = \frac{16\pi G}{c^4} T_{ij}. \tag{F2}$$

In fact, Blanchet and Damour (1988, 1992) obtained the tail term of gravitational waves as the integral over the past history of the source. They used the complex analytic continuation in order to produce the so-called log term in the tail contribution. As a result, it is not so physically transparent what is the origin of the tail term. As suggested by Eq.$(F1)$, the tail term originates from the difference between the flat light cone and the true one which is due to the mass of the source GM/c^2 as the lowest order correction. In this appendix, we wish to clarify that the tail term originates mostly from propagation of gravitational waves on the light cone which deviates slightly from the flat light cone owing to the mass of the source. For this purpose, we transform Eq.(2.63) into the following form

$$\begin{aligned}\Big[-\Big(1+\frac{4M}{r}\Big)\frac{\partial^2}{\partial t^2}+\Delta\Big]h_{jk} &= \tau_{jk} - \frac{4M}{r}\frac{\partial^2}{\partial t^2}h_{jk} \\ &= \tilde{\tau}_{jk}.\end{aligned} \tag{F3}$$

At the lowest order, we obtain ${}_{(4)}\tilde{\tau}_{jk} = {}_{(4)}\tau_{jk}$.

(2)Green function:

Let us consider the following tensor Green function

$$\Big[-\Big(1+\frac{4M}{r}\Big)\frac{\partial^2}{\partial t^2}+\Delta\Big]G_{jk.pq}(x^\mu, y^\mu) = -\frac{1}{2}\Big(\delta_{jp}\delta_{kq}+\delta_{jq}\delta_{kp}\Big)\delta^4(x-y). \tag{F4}$$

The tensor Green function for $M=0$ i.e. the Minkowski spacetime is written in many papers (For instance, Thorne 1980). The following procedure used here is similar to that by Thorne (1980) for the Minkowski background spacetime.

The Green function satisfying Eq.$(F4)$ can be constructed by using the homogeneous solutions for the equation

$$\Box_M \Psi_{jk} = 0, \tag{F5}$$

where we defined

$$\Box_M = \Big[-\Big(1+\frac{4M}{r}\Big)\frac{\partial^2}{\partial t^2}+\Delta\Big]. \tag{F6}$$

The homogeneous solution of Eq.$(F5)$ takes a form of

$$e^{-i\omega t} f_{l'}(\rho) T^{\lambda l' l m}(\theta, \phi), \tag{F7}$$

where we defined

$$\rho = \omega r. \tag{F8}$$

Here $T^{\lambda\, l'\, l\, m}$ represents a kind of tensor harmonics (Mathews 1962; Campbell, Macek and Morgan 1977; Thorne 1980) which satisfies

$$L^2 T^{\lambda\, l'\, l\, m} = l'(l'+1) T^{\lambda\, l'\, l\, m}. \tag{F9}$$

Here L^2 is an angular momentum operator defined as

$$L^2 = -r^2 \Delta_{flat} + \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{F10}$$

The tensor spherical harmonics $T^{\lambda\, l'\, l\, m}$ is written as

$$T^{2\, l'\, l\, m} = \sum_{m'=-l'}^{l'} \sum_{m''=-2}^{2} (l'\, 2\, m'\, m'' \,|\, l\, m) Y^{l'\, m'} t^{m''}, \tag{F11}$$

$$T^{0\, l\, l\, m} = -\frac{1}{\sqrt{3}} Y^{l\, m} \delta, \tag{F12}$$

where $(l'\, 2\, m'\, m'' \,|\, l\, m)$ is a Clebsh-Gordon coefficient and $l' = l \pm (0, 1, \mathrm{or}\, 2)$. Here t^m and δ denote symmetric basis tensors. In terms of Cartesian basis vectors e_x, e_y and e_z, these tensor basis are written as

$$\begin{aligned}
t^{\pm 2} &= \frac{1}{2}(e_x \otimes e_x - e_y \otimes e_y) \pm \frac{1}{2} i (e_x \otimes e_y + e_y \otimes e_x), \\
t^{\pm 1} &= \mp \frac{1}{2}(e_x \otimes e_z + e_z \otimes e_x) - \frac{1}{2} i (e_y \otimes e_z + e_z \otimes e_y), \\
t^0 &= \sqrt{\frac{1}{6}}(-e_x \otimes e_x - e_y \otimes e_y + 2 e_z \otimes e_z), \\
\delta &= e_x \otimes e_x + e_y \otimes e_y + e_z \otimes e_z.
\end{aligned} \tag{F13}$$

There is the orthonormal property between t^m and δ

$$\begin{aligned}
t^m_{jk} t^{n\,*}_{jk} &= \delta^{mn}, \\
t^m_{jk} \delta_{jk} &= 0, \\
\delta_{jk} \delta_{jk} &= 1,
\end{aligned} \tag{F14}$$

where $*$ denotes the complex conjugate. The tensor spherical harmonics has the orthonormal property

$$\int T^{\lambda\, l\, L\, M} T^{\lambda'\, l'\, L'\, M'\,*} d\Omega = \delta_{\lambda\lambda'} \delta_{ll'} \delta_{LL'} \delta_{MM'}. \tag{F15}$$

Then the radial function $\tilde{f}_{l'}(\rho) \equiv \rho f_{l'}(\rho)$ must satisfy

$$\Big[\frac{d^2}{d\rho^2} + 1 + \frac{4M\omega}{\rho} - \frac{l'(l'+1)}{\rho^2}\Big] \tilde{f}_{l'}(\rho) = 0, \tag{F16}$$

so that Eq.$(F7)$ is a solution of Eq.$(F5)$. Thus we can obtain homogeneous solutions of Eq.$(F5)$ by choosing $\tilde{f}_{l'}(\rho)$ as a kind of spherical Coulomb functions; $u^{(\pm)}_{l'}(\rho;\gamma)$, $F_{l'}(\rho;\gamma)$

and $G_{l'}(\rho;\gamma)$ with $\gamma = -2M\omega$. Here, we adopted the following definition of the spherical Coulomb function (See Messiah's "Quantumn Mechanics")

$$
\begin{aligned}
F_l(\rho;\gamma) &= c_l e^{i\rho}\rho^{l+1} F(l+1+i\gamma|2l+2|-2i\rho), \\
u_l^{(\pm)} &= \pm 2ie^{\mp i\sigma_l} c_l e^{\pm i\rho}\rho^{l+1} W_1(l+1\pm i\gamma|2l+2|\mp 2i\rho), \\
G_l(\rho;\gamma) &= \frac{1}{2}(u_l^{(+)}e^{i\sigma_l} + u_l^{(-)}e^{-i\sigma_l}),
\end{aligned} \tag{F17}
$$

where c_l and σ_l are defined as

$$
\begin{aligned}
c_l &= 2^l e^{-\pi\gamma/2}\frac{|\Gamma(l+1+i\gamma)|}{(2l+1)!}, \\
\sigma_l &= \arg\Gamma(l+1+i\gamma).
\end{aligned} \tag{F18}
$$

Here, F and W_1 are respectively the confluent hypergeometric function and the Whittaker's function. These spherical Coulomb functions have asymptotic behavior as

for $r \to \infty$

$$
\begin{aligned}
F_l &\sim \sin\Big(\rho - \gamma\ln 2\rho - \frac{1}{2}l\pi + \sigma_l\Big), \\
G_l &\sim \cos\Big(\rho - \gamma\ln 2\rho - \frac{1}{2}l\pi + \sigma_l\Big), \\
u^{(\pm)} &\sim \exp\Big[\pm i\Big(\rho - \gamma\ln 2\rho - \frac{1}{2}l\pi\Big)\Big],
\end{aligned} \tag{F19}
$$

and for $r \to 0$

$$
\begin{aligned}
F_l &\sim c_l\rho^{l+1}, \\
G_l &\sim \frac{1}{(2l+1)c_l}\rho^{-l}.
\end{aligned} \tag{F20}
$$

Thus we can express the Green function as

$$
\begin{aligned}
G^{(\epsilon)}_{jk.pq}(x,y) = -\sum_{\lambda l' lm} e^{-i\epsilon\sigma_{l'}} \int d\omega sgn(\omega)\Big[&\Psi^{\epsilon\,\omega\,\lambda\,l'\,l\,m}_{jk}(x)\Psi^{S\,\omega\,\lambda\,l'\,l\,m\,*}_{pq}(y)\theta(r-r') \\
&+\Psi^{S\,\omega\,\lambda\,l'\,l\,m}_{jk}(x)\Psi^{\epsilon\,\omega\,\lambda\,l'\,l\,m\,*}_{pq}(y)\theta(r'-r)\Big],
\end{aligned} \tag{F21}
$$

where we defined $\Psi^{\epsilon\,\omega\,\lambda\,l'\,l\,m}_{jk}(x)$ and $\Psi^{S\,\omega\,\lambda\,l'\,l\,m}_{jk}(x)$ as

$$\Psi^{\epsilon\omega\lambda l' \, l\, m}_{jk}(x) = \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} u^{(\epsilon)}_{l'}(\rho;\gamma) T^{\lambda l' \, l\, m}_{jk},$$
$$\Psi^{S\,\omega\,\lambda l' \, l\, m}_{jk}(x) = \sqrt{\frac{|\omega|}{2\pi}} e^{-i\omega t} \rho^{-1} F_{l'}(\rho;\gamma) T^{\lambda l' \, l\, m}_{jk}. \tag{F22}$$

We use $G^{(+)}$ for the retarded Green function.

(3) waveform:

The waveform is expressed as

$$\begin{aligned} h^{TT}_{jk} &= -\Big[\int G^{(+)}_{jk.pq}(x,y)\tilde{\tau}_{pq}(y)d^4y\Big]^{TT} \\ &= \frac{1}{r}\sum_{lm}\Big[I^{<l>lm}(t,r)T^{E\,2.l\,m}_{jk} + S^{<l>lm}(t,r)T^{B\,2.l\,m}_{jk}\Big] + O(\frac{1}{r^2}), \end{aligned} \tag{F23}$$

where the transverse and traceless tensor spherical harmonics $T^{E\,2.l\,m}$ and $T^{B\,2.l\,m}$ have respectively an electric-type parity and magnetic-type parity. By using $T^{\lambda\,l'\,l\,m}$, the transverse and traceless tensor spherical harmonics $T^{E\,2.l\,m}$ and $T^{B\,2.l\,m}$ are written as

$$\begin{aligned} T^{E\,2.l\,m}_{jk} = &\sqrt{\frac{l(l-1)}{2(2l+1)(2l+3)}}T^{2\,l+2\,l\,m} \\ &+\sqrt{\frac{3(l-1)(l+2)}{(2l-1)(2l+3)}}T^{2\,l\,l\,m} \\ &+\sqrt{\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}}T^{2\,l-2\,l\,m}, \end{aligned} \tag{F24}$$

$$\begin{aligned} T^{B\,2.l\,m}_{jk} = &-i\sqrt{\frac{l-1}{2l+1}}T^{2\,l+1\,l\,m} \\ &-i\sqrt{\frac{l+2}{2l+1}}T^{2\,l-1\,l\,m}. \end{aligned} \tag{F25}$$

Here $I^{<l>\,lm}$ and $S^{<l>\,lm}$ are defined as

$$\begin{aligned} I^{<l>lm} = &-8(-i)^{l+2}\int d\omega d^4x' e^{-i\omega(t-r-2M\ln 2\omega r - t')} \\ &\times\Big[e^{-i\sigma_{l-2}}\sqrt{\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}}T^{2\,l-2\,l\,m\,*}_{pq}F_{l-2}(\omega r') \\ &-e^{-i\sigma_l}\sqrt{\frac{3(l-1)(l+2)}{2(2l-1)(2l+3)}}T^{2\,l\,l\,m\,*}_{pq}F_l(\omega r') \end{aligned}$$

$$+e^{-i\sigma_{l+2}}\sqrt{\frac{l(l-1)}{2(2l+1)(2l+3)}}T_{pq}^{2\,l+2\,l\,m\,*}F_{l+2}(\omega r')\Big]\frac{\tilde{\tau}_{pq}}{\omega r'}, \tag{F26}$$

$$\begin{aligned} S^{<l>lm} &= -8(-i)^{l+2}\int d\omega d^4x' e^{-i\omega(t-r-2M\ln 2\omega r-t')} \\ &\quad\times\Big[e^{-i\sigma_{l-1}}\sqrt{\frac{l+2}{2l+1}}T_{pq}^{2\,l-1\,l\,m\,*}F_{l-1}(\omega r') \\ &\quad\quad+e^{-i\sigma_{l+1}}\sqrt{\frac{l-1}{2l+1}}T_{pq}^{2\,l+1\,l\,m\,*}F_{l+1}(\omega r')\Big]\frac{\tilde{\tau}_{pq}}{\omega r'}. \end{aligned} \tag{F27}$$

It should be noted that $< l >$ becomes, at the Newtonian order, the l-th temporal derivative (l) (shown later), though it is not introduced to mean the l-th temporal derivative.

(4) slow-motion sources:

For slow-motion sources, we obtain the mass moment for the lowest-order source ${}_{(4)}\tau_{pq}$, up to $O(M\omega)$ as

$$\begin{aligned} I^{<l>lm} &= \frac{8}{(2l+1)!!}(-i)^{l+2}\int d\omega d^4x' e^{-i\omega(t-r-2M\ln 2\omega r-t')}e^{-i\sigma_{l-2}} \\ &\quad\times\Big[\sqrt{\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}}T_{pq}^{2\,l-2\,l\,m\,*}(1+\pi M\omega)(\omega r')^{l-2}{}_{(4)}\tau_{pq}\Big], \end{aligned} \tag{F28}$$

where we used the expansion of c_l and σ_l in $M\omega$ as

$$\begin{aligned} c_l &= 1+\pi M\omega + O(M^2\omega^2), \\ \sigma_l &= -2CM\omega - \sum_{s=1}^{l}\frac{2M\omega}{s} + O(M^2\omega^2). \end{aligned} \tag{F29}$$

On the other hand, the mass moment becomes, at the Newtonian order (Thorne 1980),

$$\begin{aligned} I_N^{(l)lm}(t-r) &= \frac{8}{(2l-3)!!}(-i)^{l+2}\sqrt{\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}} \\ &\quad\times\int d\omega d^4x' e^{-i\omega(t-r-t')}T_{pq}^{2\,l-2\,l\,m\,*}(\omega r')^{l-2}{}_{(4)}\tau_{pq} \\ &= \frac{16\pi}{(2l+1)!!}\sqrt{\frac{(l+1)(l+2)}{2(l-1)l}}\frac{d^l}{dt^l}\int \rho Y^{lm\,*}r^l d^3x. \end{aligned} \tag{F30}$$

We evaluate the following integral in Eq.$(F28)$ up to $O(M\omega)$ as

$$\int d\omega d^4x' e^{-i\omega(t-r-2M\ln 2\omega r-t')} e^{-i\sigma_{l-2}}(1+\pi M\omega)(\omega r')^{l-2} T^{2\,l-2\,l\,m\,*}_{pq} {}_{(4)}\tau_{pq}$$
$$= \int d\omega dt' e^{-i\omega(t-r-2M\ln r-t')}\omega^{l-2} \int d^3x' T^{2\,l-2\,l\,m\,*}_{pq} {}_{(4)}\tau_{pq}$$
$$\times\Big[1+\pi M\omega + 2iM\omega\Big(\ln 2\omega + \sum_{s=1}^{l-2}\frac{1}{s} + C\Big) + O(M^2\omega^2)\Big]$$
$$= \int d\omega dt' e^{-i\omega(t-r-2M\ln r-t')}\omega^{l-2} \int d^3x' T^{2\,l-2\,l\,m\,*}_{pq} {}_{(4)}\tau_{pq}$$
$$\times\Big[1-2M\omega\Big\{i\Big(\sum_{s=1}^{l-2}\frac{1}{s}+\ln 2\Big) - \frac{\pi}{2}sgn(\omega) - i(\ln|\omega| + C)\Big\} + O(M^2\omega^2)\Big]. \tag{F31}$$

It is convenient to use the following formula (Gradshteyn and Ryzhik 1980; Blanchet and Schäfer 1993) which will be proved later

$$\lambda\int_0^1 dx \ln x e^{i\lambda x} + i\int_1^\infty \frac{dx}{x}e^{i\lambda x} = -\frac{\pi}{2}sgn(\lambda) - i(\ln|\lambda| + C), \tag{F32}$$

where C is Euler's number and $sgn(\lambda)$ is a sign of λ. By using this formula, we can evaluate Eq.($F31$) further as

$$\int d\omega d^4x' e^{-i\omega(t-r-2M\ln 2\omega r-t')} e^{-i\sigma_{l-2}}(1+\pi M\omega)(\omega r')^{l-2} T^{2\,l-2\,l\,m\,*}_{pq} {}_{(4)}\tau_{pq}$$
$$= \int d\omega dt' e^{-i\omega(t-r-2M\ln r-t')}\omega^{l-2} \int d^3x' T^{2\,l-2\,l\,m\,*}_{pq} {}_{(4)}\tau_{pq}$$
$$\times\Big[1-2M\omega\Big\{i\Big(\sum_{s=1}^{l-2}\frac{1}{s}+\ln 2\Big) + \omega\int_0^1 dx\ln x e^{i\omega x} + i\int_1^\infty \frac{dx}{x}e^{i\omega x}\Big\} + O(M^2\omega^2)\Big]$$
$$= i^{l-2}\Big[I_N^{(l-2)\,lm}(u) + 2M\Big\{\Big(\sum_{s=1}^{l-2}\frac{1}{s}+\ln 2\Big)I_N^{(l-1)\,lm}(u)$$
$$+\int_0^1 dv \ln v\, I_N^{(l)\,lm}(u-v) + \int_1^\infty \frac{dv}{v} I_N^{(l-1)\,lm}(u-v)\Big\} + O(M^2\omega^2)\Big]$$
$$= i^{l-2}\Big[I_N^{(l-2)\,lm}(u) + 2M\Big\{\int_0^\infty dv I_N^{(l)\,lm}(u-v)\Big(\ln v + \sum_{s=1}^{l-2}\frac{1}{s} + \ln 2\Big)\Big\} + O(M^2\omega^2)\Big], \tag{F33}$$

where $u \equiv t - r - 2M\ln r$. It is worthwhile to mention that $\ln 2$ in Eq.($F33$) can be removed by using the degree of the freedom to translate the time coordinate. Here we assumed

$$I_N^{(l-1)\,lm}(-\infty) = 0, \tag{F34}$$

which means that the system becomes static as it goes to the past infinity. Thus $I^{<l>\,lm}$ is rewritten as

$$I^{<l>\,lm}(u) = I_N^{(l)\,lm}(u) + 2GM\int_0^\infty dv I_N^{(l+2)\,lm}(u-v)\Big(\ln v + \sum_{s=1}^{l-2}\frac{1}{s}\Big) + O(G^2M^2). \tag{F35}$$

By using the post-Minkowskian approximation, Blanchet (1995) obtained the (radiative) mass moment as

$$I^{<l>\,lm}(u) = I_N^{(l)\,lm}(u) + 2GM\int_0^\infty dv I_N^{(l+2)\,lm}(u-v)\Big(\ln v + \kappa_l\Big) + O(G^2M^2), \tag{F36}$$

where

$$\kappa_l = \sum_{s=1}^{l-2}\frac{1}{s} + \frac{2l^2+5l+4}{l(l+1)(l+2)}. \tag{F37}$$

Eq.($F35$) does not agree with Eq.($F36$). The reason for this is that in the derivation of Eq.($F35$) we do not take into account all nonlinear terms in τ_{jk}. Some of nonlinear terms in τ_{jk} take a form of $M \times I^{lm}$. Therefore, it must be important to calculate the contribution from these nonlinear terms. Thus, the present approach may not be considered as fairly simple compared with the post-Minkowskian approximation used by Blanchet (1995). However, it is noteworthy that the luminosity of gravitational waves obtained by the present approach agrees at the tail term i.e. $O(c^{-3})$ with that by the post-Minkowskian approximation.

The similar procedure can be used to calculate the (radiative) current moment. We obtain $S^{<l>\,lm}$ as

$$S^{<l>\,lm}(u) = S_N^{(l)\,lm}(u) + 2GM\int_0^\infty dv S_N^{(l+2)\,lm}(u-v)\Big(\ln v + \sum_{s=1}^{l-1}\frac{1}{s}\Big) + O(G^2M^2), \tag{F38}$$

where $S_N^{(l)\,lm}(t-r)$ is the l-th current moment at the Newtonian order defined as

$$S_N^{(l)\,lm}(t-r) = -\frac{32\pi}{(2l+1)!!}\sqrt{\frac{(l+2)(2l+1)}{2(l-1)(l+1)}}\frac{d^l}{dt^l}\int \rho\epsilon_{jpq}x^p\rho v^q Y_j^{lm\,*} r^{l-1} d^3x. \tag{F39}$$

Here ϵ_{jpq} is the Levi-Civita symbol in the 3D Euclid space. Here, Y_j^{lm} is a vector spherical harmonics defined as

$$Y^{l'\,l\,m} = \sum_{m'=-l'}^{l'}\sum_{m''=-1}^{1}(1\,l'\,m''\,m'\,|\,l\,m)Y^{l'\,m'}\xi^{m''}, \tag{F40}$$

where $l' = l \pm (1\,\mathrm{or}\,0)$. Here ξ^m is the basis vector which is written, in terms of Cartesian basis vectors e_x, e_y and e_z, as

$$\begin{aligned} \xi^0 &= e_z, \\ \xi^{\pm 1} &= \mp \frac{1}{\sqrt{2}}(e_x \pm i e_z). \end{aligned} \tag{F41}$$

On the other hand, by using the post-Minkowskian approximation, Blanchet (1995) also obtained the (radiative) current moment as

$$S^{<l>\,lm}(u) = S_N^{(l)\,lm}(u) + 2GM \int_0^\infty dv S_N^{(l+2)\,lm}(u-v)\Big(\ln v + \kappa'_l\Big) + O(G^2M^2), \tag{F42}$$

where

$$\kappa'_l = \sum_{s=1}^{l-1} \frac{1}{s} + \frac{l-1}{l(l+1)}. \tag{F43}$$

In total, it is found that the *log* term and the $\sum 1/s$ in the tail terms originate from the propagation on the slightly curved light cone determined by Eq.($F6$). In particular, the operator defined by Eq.($F6$) includes only the gravitational redshift effect. It is worthwhile to point out the following fact: Only the *log* term has a hereditary property expressed as the integral over the past history of the source, since the constant such as $\sum 1/s$ represents merely an instantaneous part after performing the integral under the assumption that the source approaches static as the past infinity.

2. an integral formula

Here, we prove the useful formula (Gradshteyn and Ryzhik 1980; Blanchet and Schäfer 1993):

$$\lambda \int_0^1 dx \ln x e^{i\lambda x} + i \int_1^\infty \frac{dx}{x} e^{i\lambda x} = -\frac{\pi}{2} sgn(\lambda) - i(\ln|\lambda| + C), \tag{F44}$$

where C is Euler's number and $sgn(\lambda)$ is a sign of λ. We evaluate separately the real and imaginary parts of the left hand side (L) of Eq.($F44$): First we evaluate the real part of the left hand side of Eq.($F44$) as

$$
\begin{aligned}
Re(L) &= \lambda \int_0^1 dx \ln x \cos \lambda x - \int_1^\infty \frac{dx}{x} \sin \lambda x \\
&= -\frac{\pi}{2} sgn(\lambda),
\end{aligned}
\tag{F45}
$$

where we used

$$
\begin{aligned}
\int_0^1 dx \ln x \cos \lambda x &= -\frac{1}{|\lambda|}\Big(\mathrm{si}(|\lambda|) + \frac{\pi}{2}\Big), \\
\int_{|\lambda|}^\infty \frac{d(|\lambda| x)}{|\lambda| x} \sin |\lambda| x &= -\mathrm{si}(|\lambda|).
\end{aligned}
\tag{F46}
$$

Here si is a sine integral function. Next the imaginary part of Eq.($F44$) is obtained as

$$
\begin{aligned}
Im(L) &= \lambda \int_0^1 dx \ln x \sin \lambda x + \int_1^\infty \frac{dx}{x} \cos \lambda x \\
&= -C - \ln |\lambda|,
\end{aligned}
\tag{F47}
$$

where we used

$$
\begin{aligned}
\lambda \int_0^1 dx \ln x \sin \lambda x &= -\Big(C + \ln |\lambda| - \mathrm{ci}(|\lambda|)\Big), \\
\int_1^\infty \frac{dx}{x} \cos \lambda x &= -\mathrm{ci}(|\lambda|).
\end{aligned}
\tag{F48}
$$

Here ci is a cosine integral function. Hence Eq.($F44$) is proved from Eqs.($F45$) and ($F47$).

APPENDIX G: BRIEF HISTORY OF POST-NEWTONIAN APPROXIMATION

Celestial mechanics in the universe is governed by general relativity. Since it is very difficult to solve exactly Einstein equation for realistic astrophysical objects, we must use some approximation schemes. The post-Newtonian approximation scheme is one of the most useful and successful scheme. Einstein, Infeld and Hoffman (EIH 1938) initiated to derive the equation of motion in general relativity using the post-Newtonian approximation. In their work, the equation of motion is derievd from the integrability condition of the field equation (Einstein equation) without solving the conservation law. They obtained the equation of motion at the so-called first post-Newtonian order. Bertotti and Plebanski(1960), Havas and Goldberg (1957, 1962) used the post-linear approximation in order to obtain the equation of motion. The theory of general relativity has a novel property, a prediction of gravitational waves! As for this issue, long outstanding controversy had been done (Ehlers, Rosenblum, Goldberg and Havas 1976; Walker and Will 1980a, 1980b; Damour 1982, 1987): Does a moving body radiate gravitatinal waves? Is the motion of the body affected by the radiation reaction? There were a lot of arguments about radiation damping or even radiation anti-damping. For the fluid, Chandrasekhar (1965, 1967, 1969, 1970) started a series of calculations on the post-Newtonian approximation up to the higher order than the 1PN order. At last, at 1970, Chandrasekhar and Esposito obtained, at the 2.5PN order, the correct formula for the energy loss, the so-called quadrupole formula. However, their calculation is rather complicated mainly because the gauge condition and the expansion scheme change on the way of the iteration of the post-Newtonian approximation scheme. Some authors (Anderson and Decanio 1975; Papapetrou and Linet 1981; Breuer and Rudolph 1981, 1982) performed a straightforward and systematic calculation up to the 2.5 PN order by using the Harmonic gauge throughout the iteration.

Observationally, the celebrated event occurred at 1974. Hulse and Taylor (1975) discovered a binary pulsar PSR1913+16. This binary pulsar, which is called Hulse-Taylor binary in honour of them, was soon realized to be the laboratory on the theory of general relativity. The gravitational waves and the radiation reaction were verified by Taylor, Fowler and McCulloch (1979) from the analysis of the secular motion of the Hulse-Taylor binary. The accuracy of the analysis has been improved year by year and the observational value of the decay rate of the orbital period agrees wonderfully with that predicted by the theory of general relativity (Will 1987; Taylor and Weisberg 1989; Damour and Taylor 1991).

Conversely, this splendid discovery has stimulated the theoretical study of gravitational waves physics. For example, many people (Ehlers, Rosenblum, Goldberg and Havas 1976; Walker and Will 1980a, 1980b; Damour 1982) reexamined the validity of derivations of (1) the quadrupole formula and (2) the equation of motion with radiation damping, both of which were obtained till those days (Landau and Lifshitz 1962; Peters and Mathews 1963; Chandrasekhar and Esposito 1970; Burke 1971; Misner, Thorne and Wheeler 1973). The equation of motion up to the 2.5PN order were derived by many people by using some techniques (Hadamard's renormalization, Riez kernel method etc.) which are necessary for treating the point particle (Damour 1982, 1987). Along the course of renormalization for the point particle, Kimura, Ohata, Hiida and Okamura (1973, 1974a, 1974b) derived the equation of motion up to the 2PN order, while Westpfahl et.al. (1979a, 1979b, 1980) obtained it. As a consequence of many arguments, it was concluded that any satisfactorily rigorous derivation had not been done. Bel et.al. (1981) obtained the equation of motion up to the 2PN order in more rigorous manner. At last, Damour and Deruelle (1981a, 1981b, 1981c) derived the equation of motion up to the 2.5 PN order in the more rigorous manner (Damour 1982, 1987). On the other hand, without using the point particle, Grishchuk and Kopejkin (1983, 1985) derived the equation of motion by using the hydrodynamical equation which was obtained by Chandrasekhar et.al. (1969, 1970) Their result agrees with that obtained by Damour and Deruelle.

In the usual post-Newtonian approximation, it is assumed that (1) the motion of bodies is slow and (2) the gravitational field is weak everywhere. However, since the compact object including neutron stars and black holes has a strong internal gravitational field, the second assumption of the everywhere weak field may not be appropriate. Thus some authors argued whether the equations of motion obtained above are valid for the compact binary system. Prior to Damour and Deruelle's work, D'Eath (1975a, 1975b) proposed a scheme to construct an equation of motion by applying the asymptotic matching method to the Schwarzschild or Kerr metric. In fact, he obtained the equation of motion up to the 1PN order, which can describe the motion of compact spinning objects. Kates (1980a, 1980b) used this asymptotic matching method to verify the validity of the quadrupole formula and the equation of motion for slow- motion compact objects. Thorne and Hartle (1985) also argued the equation of motion using the EIH method and the asymptotic matching method. By using this asymptotic matching, Mino, Tanaka and Sasaki (1997) recently obtained the covariant form of the equation of motion up to the first order of the mass ratio. Other methods to describe the motion of the compact binary have been proposed by some people: For instance, Futamase and Schutz (1985), Futamase (1985, 1987) introduced a point particle limit and Anderson (1987) tried to extend the EIH method. As for the equation of motion of bodies with higher multipole moments, some arguments have been done at the 1PN order (Brumberg and Kopejkin 1989; Damour, Soffel and Xu 1991, 1992, 1993; Damour and Vokrouhlicky 1995).

In the early 1980's, there were many arguments on the higher order calculation of the post-Newtonian approximation. Some people obtained the divergent integrals at the 3PN order (Kerlick 1980a, 1980b; Anderson et.al. 1982). This fact cast doubts on the post-Newtonian approximation scheme itself (Damour 1982, 1987). Futamase and Schutz (1983a, 1983b) proposed a new kind of scheme and argued that the divergent term forces us to use the non-analytic term such as log term. The log terms at the external region of the source are obtained explicitly as the tail term by Blanchet and Damour (1988, 1992) who used the post-Newtonian approximation and the post-Minkowskian approximation

(Blanchet and Damour 1984a, 1984b, 1986, 1988).

Here we shall return to gravitational waves. The gravitaional waves are believed to be very weak in usual astrophysical context (Thorne 1980, 1987). However, Weber's challenge to detect the gravitational waves (Weber 1959, 1960, 1967, 1969, 1980) stimulated the theoretical study of gravitational waves from some astrophysical processes, for instance orbital motion of binaries, collision of two bodies and so on. In these astrophysical situations, the motion of the source is so slow that the post-Newtonian approximation can work well to calculate the waveform from astrophysical sources. Epstein and Wagoner (1975) presented the formula for the waveform at the 1PN order. For two body systems, Wagoner and Will (1976), Turner and Will (1978) calculated the waveform at the 1PN order for circular orbits, gravitational bremsstrahlung and head-on collisions. However, there remains a serious problem in the post-Newtonian waveform formula obtained by Epstein and Wagoner. Among all, divergent terms appear in the derivation of Epstein and Wagoner's formula, though the transverse-traceless nature of the waveform makes divergent terms in the waveform cancel out as a whole at the 1PN order. This is partly because the post-Newtonian approximation makes use of the spatial hypersurface, though the gravitational waves propagate on the light cone (null hypersurfaces). This implies that it is necessary to estimate waveforms using the post-Newtonian approximation with great caution. This drawback prevented us from extending straightforwardly Epstein and Wagoner's approach to higher orders. Thorne and Kovacs developed another formalism to obtain the waveform from weak-field sources (Thorne and Kovacs 1975; Kovacs and Thorne 1977, 1978; Crowley and Thorne 1977). They used the slightly curved wave operator in place of the wave operator in the Minkowski spacetime. However, it is not clear whether their approach can be extended straightforwardly to the higher order. At 1980, Thorne reviewed the gravitational waves physics till that time. He proposed the systematic scheme of iteration in which one starts the flat spacetime and expands the Einstein equation with respect to the gravitational constant G. This scheme is now called the post-Minkowskian (PM) approximation. Blanchet and Damour developed the systematic scheme to calculate the waveform at the higher order following the proposal by Thorne (1980). In their scheme, the post-Minkowskian approximation plays a crucial role in calculating the external field (outside the source), while the post-Newtonian approximation is mainly used near and in the source.

Recently, it becomes very important to calculate the waveform from the binary up to the higher PN order, since the interferometric gravitational waves detectors under construction, such as LIGO and VIRGO, need the accurate template of the waveform in order to apply the matched filtering method to gravitational waves with small signal-to-noise (SN) ratio (Cutler et.al. 1993; Finn and Chernoff 1993; Cutler and Flanagan 1994; Apostolatos et.al. 1994; Dhurandhar and Schutz 1994; Sathyaprakash 1994). Blanchet and Damour (1988, 1992) found the tail term at the 1.5PN order compared with the lowest (Newtonian) quadrupole waveform. Blanchet have obtained the waveform from the compact binary up to the 2.5PN order (Blanchet 1993, 1995, 1996). Recently, Will and Wiseman (1996) have developed the formalism to obtain the waveform by improving the Epsein-Wagoner formalism. Their waveform agrees with that by Blanchet up to the 2PN order (Blanchet et.al. 1995). Kidder, Will and Wiseman (1993b), Kidder (1995) considered the contribution of the spinning components of binaries to the energy flux of gravitational waves. It is worthwhile to mention that it is necessary to derive the equation of motion at the 3PN order at least for the quasi-circular orbiting binary in order to evaluate the waveform at the 3PN order. Therefore, it is of great significance to construct the eqaution of motion beyond the 2.5PN order.

Christdoulou (1991) found theoretically a new phenomena of gravitational waves which is called nonlinear memory. The nonlinear memory of gravitational waves is ascribed to the nonlinear nature of the general relativistic gravity. Wiseman and Will (1991), Thorne (1992) independently argued the physical aspects and implications of the nonlinear memory of gravitational waves. Nevertheless, no one has obtained this nonlinear memory of gravitational waves by the systematic scheme of the post-Newtonian approximation.

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Figure Captions

Fig. 1 Error of the pressure in the post-Newtonian approximation for the GR compact star of uniform density as a function of the normalized areal radius(r/R). Solid and dotted lines show the case $R = 5Gm/c^2$ and $8Gm/c^2$, where R and m are the circumference raduis and the mass of star, respectively.

Table 1 (a)

Various levels of approximation in terms of PN expansions (v^2/c^2) and mass ratio ($\eta = \mu/M$; μ = reduced mass, M = total mass). We mark P^2N if all terms in that level are taken into account in the 2PN approximation, while W is marked if Wison's approach takes into account all terms in that level. The mark $-$ means that the relevant term does not exist and the levels taken into account by neither approaches are blank. We neglect secular effects due to gravitational radiation reaction in Tables 1(a) and (b). It should be noted that, at $O(\eta^0)$, Wilson's approach produces exact GR solutions, but it is not justified at the 2PN order even at $O(\eta^1)$.

PN \ η	η^0	η^1	η^2	$O(\eta^3)$
N	P^2N, W	$-$	$-$	$-$
1PN	P^2N, W	P^2N, W	$-$	$-$
2PN	P^2N, W	P^2N	P^2N	$-$
$\geq$3PN	W			

Table 1 (b)

Various levels of approximation in terms of PN expansions (Gm/c^2R) and ellipticity of a NS (e). The meanings of P^2N and W are the same as those in Table 1(a). Wilson's approach produces exact GR solutions in the case of the completely spherical star.

PN \ e	$e = 0$	$e \neq 0$
N	P^2N, W	P^2N, W
1PN	P^2N, W	P^2N, W
2PN	P^2N, W	P^2N
$\geq$3PN	W	

Table 2

List of potentials to be solved (column 1), Poisson equations for them (column 2), and other potential variables which appear in the source term of the Poisson equation (column 3). Note that i and j run x, y, z. Also, note that we do not have to solve η_{0z}, ${}_{(5)}P_{0z}$, q_{yy}, q_{zz} and $h^{(U)}_{zz}$.

Pot.	Eq.	Needed pots.	Pot.	Eq.	Needed pots.
U	(2.11)	None	q_{ij}	(4.6)	None
q_i	(3.14)	None	$Q^{(I)}_{0i}$	(4.15)	U
q_2	(4.1)	None	η_{0i}	(4.16)	U
q_{2i}	(4.2)	None	${}_{(5)}P_{0i}$	(4.17)	$U,\ q_i$
q_4	(4.3)	None	${}_{(6)}\alpha_0$	(4.21)	$U,\ q_e,\ q_u,\ h^{(U)}_{ij},\ Q^{(I)}_{0i}$
q_u	(4.4)	U	${}_{(6)}\alpha_2$	(4.22)	$U,\ q_2,\ q_i,\ q_{2i},\ q_{ij}$
q_e	(4.5)	None	$h^{(U)}_{ij}$	(3.1)	U

Table 3

Variables to be solved in order to obtain the original metric variables.

Metric	Variables to be solved	see Eq.
U	U	(2.11)
${}_{(3)}\beta_i$	$q_i,\, U$	(3.17)
X	$q_2,\ \ q_u,\ \ q_e$	(4.7)
${}_{(4)}\psi$	$q_2,\ \ q_u,\ \ q_e$	(4.8)
${}_{(5)}\beta_i^{(A)}$	${}_{(5)}P_{0i},\, \eta_{0i},\, q_u,\, q_e$	(4.18)
${}_{(5)}\beta_i^{(B)}$	$q_{2i},\, q_2$	(4.18)
${}_{(6)}\alpha$	${}_{(6)}\alpha_0,\ \ {}_{(6)}\alpha_2,\ \ q_4$	(4.20)
$h_{ij}^{(U)}$	$h_{ij}^{(U)}$	(3.1)
$h_{ij}^{(S)}$	q_{ij}, q_2	(4.14)
$h_{ij}^{(A)}$	$Q_{0i}^{(I)},\ \ q_u,\ \ q_e$	(4.19)
$h_{ij}^{(B)}$	$q_{ij},\ \ q_2,\ \ q_{2i},\ \ q_i$	(4.19)