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INSTABILITY OF THIN LIQUID SHEET

AND

EFFECT OF VISCOSITY ON LONG GRAVITY WAVES

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INSTABILITY OF THIN LIQUID SHEET

AND

EFFECT OF VISCOSITY ON LONG GRAVITY WAVES

Contents

Main parts of this thesis are based upon the following papers:

Part I

K.Matsuuchi: Modulational Instability of Nonlinear Capillary Waves on Thin Liquid Sheet. J.Phys.Soc.Japan 37 (1974) 1680.

K.Matsuuchi: Instability of Thin Liquid Sheet and its Break-Up. in preparation.

Part II

T.Kakutani and K.Matsuuchi: Effect of Viscosity on Long Gravity Waves. J.Phys.Soc.Japan 39 (1975) 237.

K.Matsuuchi: Numerical Investigations on Long Gravity Waves under the Influence of Viscosity. submitted to J.Phys. Soc.Japan.

Abstract

Perturbation methods have been developed as a powerful technique for solving differential equations which preclude their exact solutions. It sometimes occurs that the straightforward expansions of the dependent variables in powers of a small parameter have limited regions of validity and break down in certain regions. To render these expansions uniformly valid, a number of techniques called 'singular perturbation methods' have been developed in different branches of physics, engineering, and applied mathematics. In particular, to have an understanding concerning the mechanism of nonlinear wave propagations, various techniques of singular perturbation methods such as the reductive perturbation, the derivative expansion, and the Krylov-Bogoliubov-Mitropolsky methods, are developed recently.

In this thesis, some of the methods mentioned above are applied to solve problems for water waves of finite amplitude. The first part deals with capillary waves on a thin liquid sheet, and the next part is concerned with long gravity waves on a viscous water layer.

Part I. Instability of Thin Liquid Sheet

It is well known that there are two kinds of capillary waves on a thin liquid sheet; one is symmetrical wave in which the displacements of opposite surfaces are in opposite directions, and the other is antisymmetrical wave in which the displacements are in the same direction. In the first half, we

derive a simple equation for weakly nonlinear waves called the nonlinear Schrödinger equation. Thanks to the well-known properties of this equation, it is found that these two kinds of waves of constant amplitude are always unstable to small disturbances and the growth rate for the symmetrical wave is larger than that for the antisymmetrical one, and it is expected that such an instability may lead to the break-up of the liquid sheet. However, since the above analysis is based on the assumption that the amplitude is small compared with the thickness of the sheet, the result thus obtained is far from satisfactory explanation for the break-up.

The next half deals with symmetrical waves of arbitrary amplitude but long wavelength in comparison with the sheet thickness. These waves are governed by a simple set of two differential equations for the surface displacement and the fluid velocity component in the direction of the wave propagation. These equations allow only periodic solutions as steady solutions which agree, in a certain limit, with the solutions obtained in the analysis for the nonlinear Schrödinger equation. Several properties of physical importance such as integration invariants of mass, momentum, and energy, the dependence of amplitude on wave frequency, and so on, are also studied analytically. According to the results obtained in the first half, it is suggested that these steady solutions are unstable. To confirm this conjecture, initial value problems to the set of equations are solved numerically. As initial values of these calculations, two kinds of small disturbances superposed on the steady solution are taken. One is a second harmonic disturbance to the steady solution, and the other is a subharmonic one. Within the calculations the steady solution seems to be stable to the higher harmonic disturbance. However, it is remarkably unstable to the subharmonic disturbance and the solution bursts at a finite time. It is shown that the wave energy distributed almost uniformly in the initial stage is concentrated rapidly near the minimum trough which becomes deeper and deeper, and thus the opposite surfaces are getting closer to each other. We cannot, however, perform the calculation until the two surfaces meet together, because of a violent change of the fluid velocity. Nevertheless the results thus obtained may be sufficient to confirm that such a 'burst instability' leads to the break-up of the liquid sheet.

Part II. Effect of Viscosity on Long Gravity Waves

In the first half of Part II, analytical investigations of the effect of viscosity on long gravity waves are made. First, the dispersion relation for infinitesimal waves is examined. It is, then, found that the dispersion relation consists of two distinct parts, geometrical and viscous dispersions. The former arises from the geometrical configuration and the latter from the effect of viscosity. Next, our attention is directed towards the waves of finite amplitude. The reductive perturbation method combined with the usual boundary layer theory reveals that the inviscid Korteweg-de Vries equation is not affected by the viscosity if $O(\alpha^{-5}) < R$, where R is the Reynolds number and α ($\ll 1$) the wavenumber. For $O(\alpha^{-1}) < R \leq O(\alpha^{-5})$, the effect of viscosity modifies the Korteweg-de Vries equation and yields new types of equation. On the other hand, for $R < O(\alpha^{-1})$, the complex phase velocity becomes purely imaginary and the free surface is found to be governed by a nonlinear diffusion equation which was first derived by Nakaya.

In the next half, initial value problems to the equation obtained in the first half are solved numerically. The results are compared with the experiments made by Zabusky and Galvin. It is found that the solution obtained numerically agrees with their experiment with respect to the damping of solitary waves, while their phases do not coincide with each other. By expanding the free surface elevation into Fourier series, each component is investigated separately. The temporal variations of the Fourier components together with those of the wave energy are computed. From these computations it is clarified that when the geometrical dispersion dominates over the viscous one, the wave energy decreases almost linearly with time, while in the opposite case viscosity damps it exponentially.

PART I. Instability of Thin Liquid Sheet

§1. Introduction

Since the 1950's much attention has been paid to the mechanism of drop formation resulting from the break-up of liquid sheet because of their increasing application in combustion, chemical and agricultural engineering. Disintegration of a large bulk of liquid into small drops with larger surface per unit volume is of practical imporatnce in most applications. The atomization of two-dimensional sheets has particularly received increasing attention because of the geometrical simplicity which provides a convenient model for theoretical study. In order to obtain a greater understanding of the processes involved, much attempt has been made towards analysing the hydrodynamics of flow, establishing the basic mechanism of drop formation, and determining the resulting drop-size. However, very little is obtained concerning the knowledge of the drop formation.

There is no doubt that a certain kind of instability gives rise to the break-up of the liquid sheet, and one believes that unless the instability may occur any sheet cannot disintegrate into drops. Squire[1)] has studied the linear instability which arises in a moving sheet due to the reaction of the surrounding air. This instability is the only one example which has been studied so far. He obtained the dispersion relation for disturbances with the (nondimensional) wavenumber k and the complex frequency ω ,

$$\frac{\omega}{kU_0} = \frac{1 \pm X^{\frac{1}{2}}\left[-\gamma_\rho + W^{-2}(1+\gamma_\rho X)\right]^{\frac{1}{2}}}{1 + \gamma_\rho X}, \tag{1.1}$$

where U_0 is the sheet velocity and $X = \coth k$.[1,2] The above relation is satisfied for antisymmetrical waves, and similar relation for symmetrical waves is obtained if we replace X by Y (=$\tanh k$). The ratio γ_ρ of the density of the surrounding air to that of the sheet is in general very small. Profiles of the symmetrical and antisymmetrical waves are sketched in Fig.1. Another nondimensional parameter W

(a) (b)

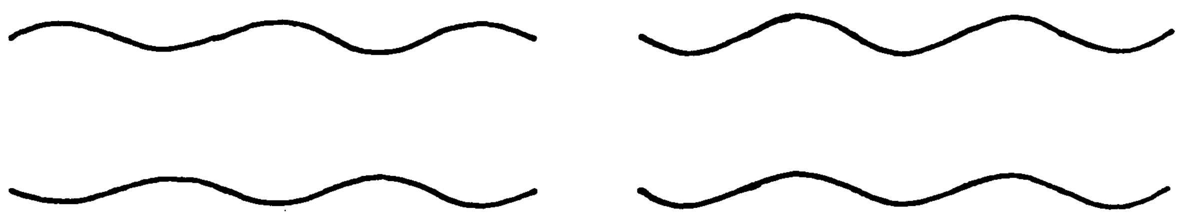

Fig.1. Sketch of (a) symmetrical, and (b) antisymmetrical waves

is the Weber number defined by

$$W = \frac{U_0}{\sqrt{\frac{2T}{\rho l}}}$$

where l is the thickness of the sheet, T the surface tension and ρ the density of the sheet. It is easy to see from eq.(1.1) that when the surface tension force is negligible, both symmetrical and antisymmetrical waves are always unstable. This fact means that the effect of the surface tension suppresses the instability, which is essentially equivalent to the so-called Kelvin-Helmholtz instability. Another important reduction from eq.(1.1) is the fact that the exponential growth rate ω_i , imaginary part of ω , for the antisymmetrical waves is much larger than that for the symmetrical ones. The analyses for Squire's instability have been studied extensively by including the effects of nonlinearity[3,4] and viscosity.[5] Nevertheless our understanding about the basic mechanism of drop formation is still far from satisfactory.

The instability which we intend to show in this thesis is very different from Squire's instability of a moving sheet. In our instability the surface tension force itself and nonlinearity play an important role.

Recently, much attention has been paid to weakly nonlinear waves in dispersive media. It was shown, in particular, that the amplitude modulation of nonlinear dispersive waves is governed by the nonlinear Schrödinger equation, by using various techniques of singular perturbation such as the reductive perturbation,[6-8] the Krylov-Bogoliubov-Mitropolsky,[9] and the derivative expansion methods.[10-13] The outstanding feature of the solution to the nonlinear Schrödinger equation is that a periodic wave train of constant amplitude such as the Stokes wave becomes unstable under certain conditions. For example, Hasimoto and Ono[10] showed that the Stokes wave is modulationally unstable when $k_0 H_c > 1.363$ where k_0 is the wavenumber and H_0 the depth of water. This criterion agrees with that of Benjamin[14] and of Whitham.[15] The main purpose of the first half of Part I is to apply the above singular perturbation method to the present problem and to show a possibility that the modulational instability plays an important role in the break-up of the sheet.

It is then found that the two kinds of waves, symmetrical and antisymmetrical waves, are always unstable to infinitesimal disturbances, and the maximum growth rate for the symmetrical waves is greater than that for the antisymmetrical ones. This tendency is contrary to Squire's instability of a moving sheet. Particularly for the symmetrical waves we discuss how important role such an instability plays in the break-up of the sheet. However, since our analysis is based on the assumption that the ratio of the amplitude to the thickness is small (but finite), we cannot confirm that such an instability certainly leads to the break-up.

The next half of Part I is devoted to obtain a certain evidence that our instability can cause the break-up of the sheet. To do so, we discard the assumption that the nonlinearity is weak, i.e., the ratio of the amplitude to the thickness of the sheet is small. In place of smallness of the amplitude we impose the asssumption called the long wave approximation, which states that the ratio of the thickness to the wavelength is small. Confining our analysis to the symmetrical waves, we derive a simple set of equations which governs the surface elevation and the velocity potential, and some important properties about nonlinear wave trains governed by these equations are first examined. Next, solving the equations numerically we show that the wave trains of constant amplitude are unstable and because of this instability the break-up of the sheet occurs certainly.

§2. Formulation of the Problem

Let us consider a two-dimensional sheet of liquid of density ρ, surface tension T and uniform thickness ℓ. It is assumed that the liquid sheet is sufficiently thin so that the gravity force can be negligible in comparison with the surface tension force. This condition is written as $\ell \ll (T/\rho g)^{\frac{1}{2}}$ where g is the acceleration due to gravity. If the motion is generated from rest, the flow may be irrotational, so that the wave propagation is governed by the two-dimensional Laplace equation for the velocity potential $\phi(x, y, t)$ in the Cartesian coordinates:

$$\phi_{xx} + \phi_{yy} = 0, \tag{2.1}$$

where t is the time, x measures horizontally to the right and y vertically upwards. The boundary conditions at the free surfaces are given by

$$\left.\begin{aligned} &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) = \eta_{xx}/(1+\eta_x^2)^{\frac{3}{2}}, && (2.2)\\ &\phi_y = \eta_t + \phi_x \eta_x, && (2.3) \end{aligned}\right\} \text{ at } y = \eta(x, t),$$

and

$$\left.\begin{aligned} &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) = -\eta'_{xx}/(1+\eta_x'^2)^{\frac{3}{2}}, && (2.4)\\ &\phi_y = \eta'_t + \phi_x \eta'_x, && (2.5) \end{aligned}\right\} \text{ at } y = \eta'(x, t),$$

where $\eta(x,t)$ and $\eta'(x,t)$ are, respectively, the displacements of the upper and the lower surfaces from the central plane. The subscripts x, y, and t denote, respectively, partial differentiation with respect to x, y, and t. All quantities have been normalized by means of the half thickness $\ell/2$ and the phase velocity $(2T/\rho\ell)^{\frac{1}{2}}$ of the antisymmetrical wave*.

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* For water sheet surrounded by air, unit of linear dimension and that of time are usually 5×10^{-3} cm and 4×10^{-5} sec, respectively, when $\ell=10^{-2}$ cm.

§3. Analysis for Weakly Nonlinear Waves with Moderate Wavenumbers

It is assumed that nonlinearity of the waves is so small that the modulation occurs slowly with respect to both space and time. In describing these waves, it is convenient to introduce new variables of multiple scales:[11]

$$x_0 = x, \qquad x_1 = \epsilon x, \qquad x_2 = \epsilon^2 x; \tag{3.1}$$

$$t_0 = t, \qquad t_1 = \epsilon t, \qquad t_2 = \epsilon^2 t; \tag{3.2}$$

where ϵ is a measure of smallness of the amplitude. Accordingly, the derivative operators $\partial/\partial x$ and $\partial/\partial t$ are expanded as

$$\frac{\partial}{\partial x} = \sum_{n=0}^{2} \epsilon^n \frac{\partial}{\partial x_n}, \tag{3.3}$$

and

$$\frac{\partial}{\partial t} = \sum_{n=0}^{2} \epsilon^n \frac{\partial}{\partial t_n}. \tag{3.4}$$

Similarly, $\phi(x, y, t)$, $\eta(x, t)$, and $\eta'(x, t)$ are expanded into power series in ϵ :

$$\phi = \sum_{n=1} \epsilon^n \phi_n (x_0, x_1, x_2, y, t_0, t_1, t_2), \tag{3.5}$$

$$\eta = 1 + \sum_{n=1} \epsilon^n \eta_n (x_0, x_1, x_2, t_0, t_1, t_2), \tag{3.6}$$

and

$$\eta' = -1 + \sum_{n=1} \epsilon^n \eta_n' (x_0, x_1, x_2, t_0, t_1, t_2). \tag{3.7}$$

Substituting (3.5), (3.6), and (3.7) into (2.1)-(2.5) and arranging in powers of ϵ, we get

$O(\epsilon)$:

$$L_0[\phi_1]=0, \tag{3.8}$$

$$\left.\begin{aligned} L_1[\phi_1,\eta_1]&=0,\\ L_2[\phi_1,\eta_1]&=0, \end{aligned}\right\} \quad \text{at} \quad y=1, \tag{3.9}$$

$$\left.\begin{aligned} L_1[\phi_1,-\eta_1']&=0,\\ L_2[\phi_1,\eta_1']&=0, \end{aligned}\right\} \quad \text{at} \quad y=-1, \tag{3.10}$$

$O(\epsilon^2)$:

$$L_0[\phi_2]=-N_0^{(2)}[\phi_1], \tag{3.11}$$

$$\left.\begin{aligned} L_1[\phi_2,\eta_2]&=-N_1^{(2)}[\phi_1,\eta_1]+M_1^{(2)}[\eta_1],\\ L_2[\phi_2,\eta_2]&=-N_2^{(2)}[\phi_1,\eta_1], \end{aligned}\right\} \quad \text{at} \quad y=1, \tag{3.12}$$

$$\left.\begin{aligned} L_1[\phi_2,-\eta_2']&=-N_1^{(2)}[\phi_1,\eta_1']+M_1^{(2)}[-\eta_1'],\\ L_2[\phi_2,\eta_2']&=-N_2^{(2)}[\phi_1,\eta_1'], \end{aligned}\right\} \quad \text{at} \quad y=-1, \tag{3.13}$$

$O(\epsilon^3)$:

$$L_0[\phi_3]=-N_0^{(3)}[\phi_1,\phi_2], \tag{3.14}$$

$$\left.\begin{aligned} L_1[\phi_3,\eta_3]&=-N_1^{(3)}[\phi_1,\phi_2,\eta_1,\eta_2]+M_1^{(3)}[\eta_1,\eta_2],\\ L_2[\phi_3,\eta_3]&=-N_2^{(3)}[\phi_1,\phi_2,\eta_1,\eta_2], \end{aligned}\right\} \quad \text{at} \quad y=1, \tag{3.15}$$

$$\left.\begin{aligned} &L_1[\phi_3, -\eta_3'] = -N_1^{(3)}[\phi_1, \phi_2, \eta_1', \eta_2'] + M_1^{(3)}[-\eta_1', -\eta_2'] ,\\ &L_2[\phi_3, \eta_3'] = -N_2^{(3)}[\phi_1, \phi_2, \eta_1', \eta_2'], \end{aligned}\right\} \text{at } y=-1, \qquad (3.16)$$

where

$$L_0[\phi_i] \equiv \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right)\phi_i , \quad L_1[\phi_i, \eta_i] \equiv \frac{\partial \phi_i}{\partial t_0} - \frac{\partial^2 \eta_i}{\partial x_0^2},$$

$$L_2[\phi_i, \eta_i] \equiv \frac{\partial \phi_i}{\partial y} - \frac{\partial \eta_i}{\partial t_0}, \qquad (i = 1,2,3),$$

$$N_0^{(2)}[\phi_1] \equiv 2\frac{\partial^2 \phi_1}{\partial x_0 \partial x_1},$$

$$N_1^{(2)}[\phi_1, \eta_1] \equiv \frac{\partial \phi_1}{\partial t_1} + \frac{\partial^2 \phi_1}{\partial t_0 \partial y}\eta_1 + \frac{1}{2}\left\{\left(\frac{\partial \phi_1}{\partial x_0}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2\right\},$$

$$M_1^{(2)}[\eta_1] \equiv 2\frac{\partial^2 \eta_1}{\partial x_0 \partial x_1},$$

$$N_2^{(2)}[\phi_1, \eta_1] \equiv \frac{\partial^2 \phi_1}{\partial y^2}\eta_1 - \frac{\partial \eta_1}{\partial t_1} - \frac{\partial \phi_1}{\partial x_0}\frac{\partial \eta_1}{\partial x_0},$$

$$N_0^{(3)}[\phi_1, \phi_2] \equiv 2\frac{\partial^2 \phi_2}{\partial x_0 \partial x_1} + 2\frac{\partial^2 \phi_1}{\partial x_0 \partial x_2} + \frac{\partial^2 \phi_1}{\partial x_1^2},$$

$$N_1^{(3)}[\phi_1, \phi_2, \eta_1, \eta_2] \equiv \frac{\partial \phi_2}{\partial t_1} + \frac{\partial \phi_1}{\partial t_2} + \frac{\partial^2 \phi_1}{\partial t_0 \partial y}\eta_2 + \frac{\partial^2 \phi_2}{\partial t_0 \partial y}\eta_1 + \frac{\partial^2 \phi_1}{\partial t_1 \partial y}\eta_1$$

$$+ \frac{1}{2}\frac{\partial^3 \phi_1}{\partial t_0 \partial y^2}\eta_1^2 + \frac{\partial \phi_1}{\partial x_0}\left(\frac{\partial \phi_2}{\partial x_0} + \frac{\partial \phi_1}{\partial x_1} + \frac{\partial^2 \phi_1}{\partial x_0 \partial y}\eta_1\right) + \frac{\partial \phi_1}{\partial y}\left(\frac{\partial \phi_2}{\partial y} + \frac{\partial^2 \phi_1}{\partial y^2}\eta_1\right).$$

$$M_1^{(3)}[\eta_1, \eta_2] \equiv 2\frac{\partial^2 \eta_2}{\partial x_0 \partial x_1} + 2\frac{\partial^2 \eta_1}{\partial x_0 \partial x_2} + \frac{\partial^2 \eta_1}{\partial x_1^2} - \frac{3}{2}\frac{\partial^2 \eta_1}{\partial x_0^2}\left(\frac{\partial \eta_1}{\partial x_0}\right)^2,$$

$$N_2^{(3)}[\phi_1, \phi_2, \eta_1, \eta_2] \equiv \frac{\partial^2 \phi_2}{\partial y^2}\eta_1 + \frac{\partial^2 \phi_1}{\partial y^2}\eta_2 + \frac{1}{2}\frac{\partial^3 \phi_1}{\partial y^3}\eta_1^2 - \frac{\partial \eta_2}{\partial t_1} - \frac{\partial \eta_1}{\partial t_2}$$

$$-\left(\frac{\partial \eta_2}{\partial x_0} + \frac{\partial \eta_1}{\partial x_1}\right)\frac{\partial \phi_1}{\partial x_0} - \frac{\partial \eta_1}{\partial x_0}\left(\frac{\partial \phi_2}{\partial x_0} + \frac{\partial \phi_1}{\partial x_1} + \eta_1\frac{\partial^2 \phi_1}{\partial x_0 \partial y}\right).$$

Let us consider the lowest order problem in ϵ. Laplace equation (3.8) together with the boundary conditions (3.9) and (3.10) permits two kinds of progressive wave solution, which represent symmetrical and antisymmetrical waves. For the symmetrical wave, the solution is given by

$$\phi_1 = -\frac{ik^2}{\omega}\frac{\cosh ky}{\cosh k}Ae^{i\theta} + C.C. + B_1^{(s)}, \tag{3.17}$$

$$\eta_1 = -\eta_1' = Ae^{i\theta} + C.C. , \tag{3.18}$$

where $\theta = kx_0 - \omega t_0$, k and ω being, respectively, the wavenumber and the frequency of the infinitesimal wave, provided that the dispersion relation

$$\frac{\omega^2}{k^2} = k \tanh k \tag{3.19}$$

is satisfied. The complex amplitude A and an additive real function $B_1^{(s)}$ may depend on the higher scales x_1, x_2, t_1, and t_2, and C.C. stands for the complex conjugate to the preceding expression. Similarly, the solution for the antisymmetrical wave is obtained as

$$\phi_1 = -\frac{ik^2}{\omega}\frac{\sinh ky}{\sinh k}Ae^{i\theta} + C.C. + B_1^{(a)}, \tag{3.20}$$

$$\eta_1 = \eta_1' = Ae^{i\theta} + C.C., \tag{3.21}$$

with the dispersion relation:

$$\frac{\omega^2}{k^2} = k \coth k, \tag{3.22}$$

where an additive real function $B_1^{(a)}$ may depend on higher scales. The linear dispersion relations (3.19) and (3.22) have already been obtained by Taylor[16].

3.1 Antisymmetrical wave

Let us now proceed to the second order problem for antisymmetrical waves. Substituting the first order solution into the second order equations (3.11)-(3.13), and solving eq.(3.11) under the boundary conditions (3.12) and (3.13), we get

$$\phi_2 = \frac{3i\omega}{4} \frac{\cosh 2ky}{\cosh^2 k} A^2 e^{2i\theta} - \frac{k}{2\omega \sinh k}(2ky\cosh ky - \sinh ky)$$

$$\times \frac{\partial A}{\partial x_1} e^{i\theta} + \text{c.c.} + B_2^{(a)}, \tag{3.23}$$

where $B_2^{(a)}$ is a homogeneous solution of eq.(3.11) depending on the higher order scales*. The surface displacements of the second order, η_2 and η_2', are given by the integra-

* The other homogeneous solution $B(x_1, x_2, t_1, t_2)e^{i\theta}$ has been dropped according to the idea of Bogoliubov and Mitropolsky[17], and that of Inoue and Matsumoto[12].

tion of the second equation of the boundary conditions (2.12) and that of the boundary conditions (2.13) with respect to t_0 as follows:

$$\eta_2 = -\frac{k^4}{2\omega^2}A^2e^{2i\theta} - \frac{ik}{\omega}\left(\frac{\omega}{2k^2}+\frac{k^2}{\omega}\right)\frac{\partial A}{\partial x_1}e^{i\theta}$$

$$-\frac{i}{\omega}\frac{\partial A}{\partial t_1}e^{i\theta} + c.c. + C_2^{(a)}, \tag{3.24}$$

$$\eta_2' = \frac{k^4}{2\omega^2}A^2e^{2i\theta} - \frac{ik}{\omega}\left(\frac{\omega}{2k^2}+\frac{k^2}{\omega}\right)\frac{\partial A}{\partial x_1}e^{i\theta}$$

$$-\frac{i}{\omega}\frac{\partial A}{\partial t_1}e^{i\theta} + c.c. + C_2'^{(a)}, \tag{3.25}$$

where $C_2^{(a)}$ and $C_2'^{(a)}$ are real functions to be determined in the higher order. Substituting the above solution ϕ_2 and η_2 into the first equation of the boundary conditions (3.12) or ϕ_2 and η_2' into the first equation of the boundary conditions (3.13) and arranging in powers of $e^{i\theta}$, we have from the coefficient of $e^{i\theta}$

$$\frac{\partial A}{\partial t_1} + C_g\frac{\partial A}{\partial x_1} = 0, \tag{3.26}$$

and from the 'constant' term

$$\frac{\partial B_1^{(a)}}{\partial t_1} = \frac{\omega^2(X^2-1)}{X^2}|A|^2, \tag{3.27}$$

where $|A|$ denotes the modulus of A and $X = \coth k$.

The group velocity C_g of the infinitesimal wave is given by

$$C_g = \frac{d\omega}{dk} = \frac{\omega}{2kX}\left[3X + k(1-X^2)\right]. \tag{3.28}$$

Equation (3.26) implies that the variation of the amplitude is transmitted with the group velocity, i.e., there is no temporal change of the amplitude in a frame of reference moving with the group velocity. According to eq.(3.27), an induced current $\partial B_1^{(a)}/\partial x_1$ of the order of ϵ^2 appears in the sheet due to the nonlinear interaction, which is represented by

$$\frac{\partial B_1^{(a)}}{\partial x_1} = \frac{2\omega k(1-X^2)}{X\left[3X+k(1-X^2)\right]}|A|^2, \tag{3.29}$$

where an arbitrary function resulting from the integration with respect to t_1 has been set equal to zero.*

The third order problem (3.14)-(3.16) can be dealt with by a similar procedure to those for the second order problem. After tedious but straightforward manipulations, we obtain from the coefficient of $e^{i\theta}$ the following equation for the upper surface

$$i\left(\frac{\partial A}{\partial t_2} + C_g\frac{\partial A}{\partial x_2}\right) + \frac{1}{2}\frac{dC_g}{dk}\frac{\partial^2 A}{\partial x_1^2} = s|A|^2A + rA, \tag{3.30}$$

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* In general, this restriction may be severe. However, this function has no important effect on the modulational instability. In this sense, the term has been dropped.

where

$$s = \frac{\omega k^2 (6X^2 - 7)}{4X^2},$$

$$r = \frac{\omega k (1 - X^2)}{2X} C_2^{(a)} + k \frac{\partial B_1^{(a)}}{\partial x_1},$$

and for the lower surface

$$i\left(\frac{\partial A}{\partial t_2} + C_g \frac{\partial A}{\partial x_2}\right) + \frac{1}{2}\frac{dC_g}{dk}\frac{\partial^2 A}{\partial x_1^2} = s|A|^2 A + r' A, \qquad (3.31)$$

where

$$r' = -\frac{\omega k (1 - X^2)}{2X} C_2^{\prime(a)} + k \frac{\partial B_1^{(a)}}{\partial x}.$$

The induced surface elevations $C_2^{(a)}$ and $C_2^{\prime(a)}$ must be determined by the condition that η_3 and η_3' are secular term free, from which we obtain the relations

$$\frac{\partial^2 B_1^{(a)}}{\partial x_1^2} + \frac{\partial C_2^{(a)}}{\partial t_1} + \frac{2k^3}{\omega}\frac{\partial |A|^2}{\partial x_1} = 0, \qquad (3.32)$$

$$\frac{\partial^2 B_1^{(a)}}{\partial x_1^2} - \frac{\partial C_2^{\prime(a)}}{\partial t_1} + \frac{2k^3}{\omega}\frac{\partial |A|^2}{\partial x_1} = 0. \qquad (3.33)$$

Substituting eq.(3.29) into (3.32) and (3.33) and remembering the first order relation (3.26), we can express the second order surface elevations $C_2^{(a)}$ and $C_2^{\prime(a)}$ as follows:

$$C_2^{(a)} = \frac{4k[3X + 2k(1 - X^2)]}{[3X + k(1 - X^2)]^2} |A|^2, \qquad (3.34)$$

$$C_2^{\prime(a)} = -\frac{4k[3X+2k(1-X^2)]}{[3X+k(1-X^2)]^2}|A|^2 . \tag{3.35}$$

Now that $C_2^{(a)}$ and $C_2^{\prime(a)}$ have been expressed in terms of A, we have from eq.(3.30) or eq.(3.31) the following closed equation for A :

$$i\left(\frac{\partial A}{\partial t_2} + C_g\frac{\partial A}{\partial x_2}\right) + \frac{1}{2}\frac{dC_g}{dk}\frac{\partial^2 A}{\partial x_1^2} = g_a|A|^2A . \tag{3.36}$$

Introducing the new variables defined as[6-8]

$$\xi = \epsilon(x - C_g t), \qquad \tau = \epsilon^2 t,$$

we have the final equation for the amplitude A :

$$i\frac{\partial A}{\partial \tau} + P_a\frac{\partial^2 A}{\partial \xi^2} = g_a|A|^2A , \tag{3.37}$$

where

$$P_a = \frac{1}{2}\frac{dC_g}{dk} = \frac{\omega[3k^2X^4 - 6kX^3 + (3-2k^2)X^2 + 6kX - k^2]}{8k^2X^2} , \tag{3.38}$$

$$g_a = \frac{\omega k^2[6k^2X^6 - 12kX^5 + (6-19k^2)X^4 + 30kX^3 - 5(3-4k^2)X^2 - 18kX - 7k^2]}{4X^2[3X+k(1-X^2)]^2} . \tag{3.39}$$

Equation (3.37) is called the nonlinear Schrödinger equation, which has already been obtained in the studies of various nonlinear dispersive systems.

The sign of the coefficient P_a and that of the coefficient g_a are always positive and always negative, respectively. The coefficient P_a and g_a are shown, respectively, in Fig.2 and 3 as functions of the wavenumber k .

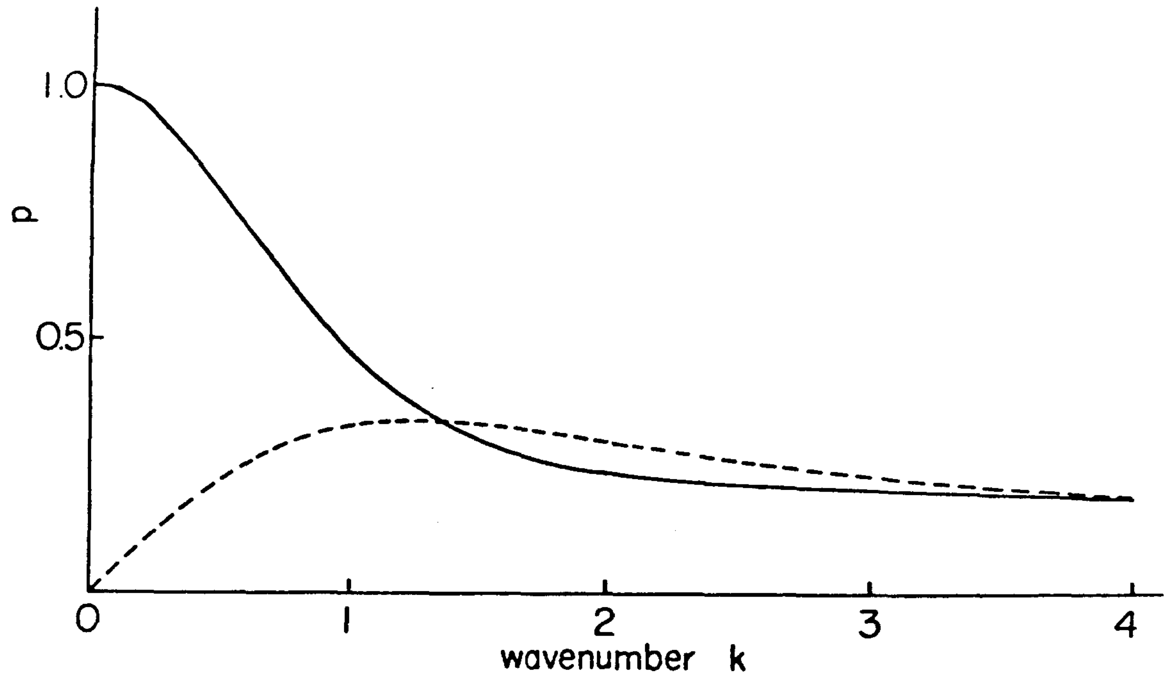

Fig.2. Variation of ρ versus wavenumber k : ———, symmetrical wave; – – – – –, antisymmetrical wave.

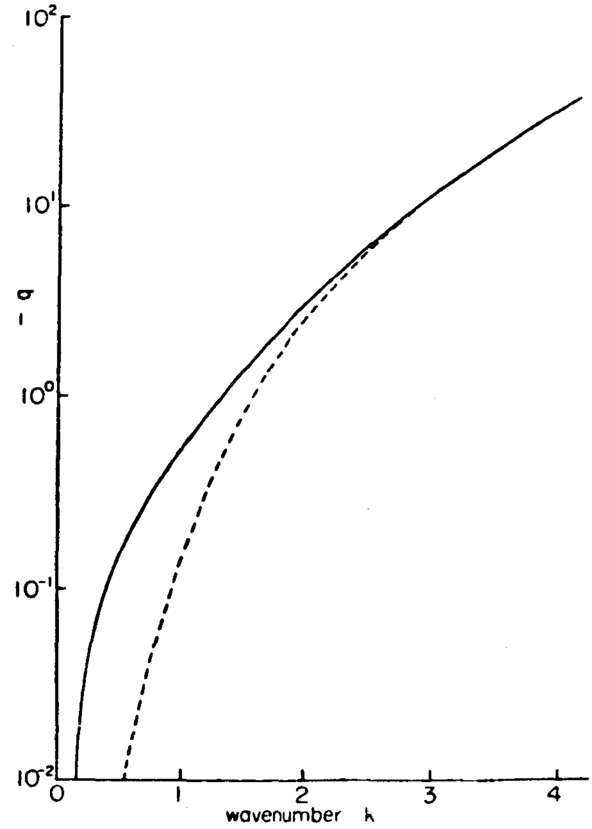

Fig.3. Variation of - ℓ versus wavenumber k : ———, symmetrical wave; – – – – –, antisymmetrical wave.

3.2 Symmetrical wave

In a similar way, the nonlinear Schrödinger equation for the symmetrical wave is obtained as

$$i\frac{\partial A}{\partial \tau} + p_s \frac{\partial^2 A}{\partial \xi^2} = q_s |A|^2 A, \tag{3.40}$$

where

$$p_s = \frac{1}{2}\frac{dC_g}{dk} = \frac{\omega[3k^2Y^4 - 6kY^3 + (3-2k^2)Y^2 + 6kY - k^2]}{8k^2Y^2}, \tag{3.41}$$

$$q_s = \frac{\omega k^2[6k^2Y^6 - 12kY^5 + (6-19k^2)Y^4 + 30kY^3 - 5(3-4k^2)Y^2 - 18kY - 7k^2]}{4Y^2[3Y + k(1-Y^2)]^2}, \tag{3.42}$$

with $Y = \tanh k$.

Inspection of eqs.(3.41) and (3.42) shows that $p_s > 0$ and $q_s < 0$ for all wavenumbers (see Figs.2 and 3). The induced current $\partial B_1^{(s)}/\partial x_1$ and the induced surface elevation $C_2^{(s)}$ due to nonlinear interaction are given as

$$\frac{\partial B_1^{(s)}}{\partial x_1} = \frac{2\omega k(1-Y^2)}{Y[3Y + k(1-Y^2)]}|A|^2, \tag{3.43}$$

$$C_2^{(s)} = \frac{4k[3Y + 2k(1-Y^2)]}{[3Y + k(1-Y^2)]^2}|A|^2. \tag{3.44}$$

It should be noted that the induced quantities are proportional to the squared modulus of amplitude.

The present symmetrical wave may also be regarded as the nonlinear capillary wave on a water layer with a rigid boundary. For example, the existing work[18,19] on capillary waves on an infinitely deep water can easily be recovered as a special case of our results. In fact, if we put $A = A_0 \exp(-i\Omega\tau)$ where A_0 and Ω are real constants and equate γ to unity, we obtain the displacements as

$$\eta = -\eta' = 1 + \frac{k}{3}\epsilon^2 a_0^2 + \epsilon a_0 \cos\zeta - \frac{k}{4}\epsilon^2 a_0^2 \cos 2\zeta, \tag{3.45}$$

where $a_0 = 2A_0$ and $\zeta = kx - (\omega - k^2\epsilon^2 a_0^2/16)t$ Equation (3.45) describes a periodic wave train moving with the phase velocity

$$C = k^{\frac{1}{2}}\left(1 - \frac{k^2}{16}\epsilon^2 a_0^2\right). \tag{3.46}$$

The results (3.45) and (3.46) coincide with those obtained by Crapper[18] and those by Pierson and Fife[19] up to the second order in ϵ, except for the trivial nonperiodic terms.

3.3 Modulational instability

In order to have an understanding concerning the mechanism of the break-up, we study solutions of the nonlinear Schrödinger equation. Equations (3.37) and (3.40) have the following type of solutions expressed as

$$A = A_c \exp i(f_1 \xi - f_2), \tag{3.47}$$

where A_c is a complex constant, f_1 and f_2 are real functions of τ. They satisfy the relation

$$\frac{d}{d\tau}(f_1 \xi - f_2) + p f_1^2 = -q |A_c|^2, \tag{3.48}$$

where the subscripts a and s have been dropped. This solution denotes a progressive wave train of constant amplitude[10]. In particular, putting $f_1 = K$ and $f_2 = \Omega \tau$, we obtain a steady periodic wave train with constant phase velocity. Further, if $K = 0$, we get a progressive wave train with argument $\zeta = kx - \bar{\omega} t$, where

$$\bar{\omega}(k, \epsilon |A_c|) = \omega + \epsilon^2 |A_c|^2 q, \tag{3.49}$$

which is called 'amplitude dispersion'[20]. Crapper's solution[18] and the Stokes wave belong to this simple type.

The stability of this solution (3.47) has already been studied by Taniuti and Washimi[6] and also by Hasimoto and Ono[10]. According to their results, it becomes unstable to small disturbances of a certain kind when $pq < 0$. Therefore, nonlinear capillary waves of constant amplitude on a thin liquid sheet are always unstable, because $P\,\gamma$ is always negative as has already been remarked (see also Figs.2 and 3). The maximum growth rate δ_{max} is given by

$$\delta_{max} = |\gamma|\,|A_0|^2 \qquad \text{for} \quad \hat{k} = \sqrt{-\frac{\gamma}{P}}\,|A_0|, \tag{3.50}$$

for the disturbed wave

$$A = (A_0 + \hat{\epsilon}\hat{\Phi})\exp i(f_1\xi - f_2 + \hat{\epsilon}\hat{\varphi}), \tag{3.51}$$

in which $\hat{k}$ is the wavenumber of the disturbance and $\hat{\epsilon}$ is a small parameter, $\hat{\Phi}$ and $\hat{\varphi}$ are real functions of ξ and τ .

In order to know the degree of instability, we examine the ratio of the maximum growth rate for the symmetrical wave to that for the antisymmetrical one. It is found that the ratio is always larger than unity and it decreases monotonically as the wavenumber k decreases (see Fig.4). This implies that the instability of the symmetrical waves may lead to the break-up of the sheet rather than that of the antisymmetrical ones, which is contrary to Squire's instability[1] of a moving liquid sheet. Since the induced surface elevation given by eq.(3.44) is proportional to the square of the amplitude and the sign of the coefficient is always positive, the exponential growth of the amplitude leads to the exponential increase of the thickness of the sheet. On the other hand, according to eq.(3.50), the modulated amplitude

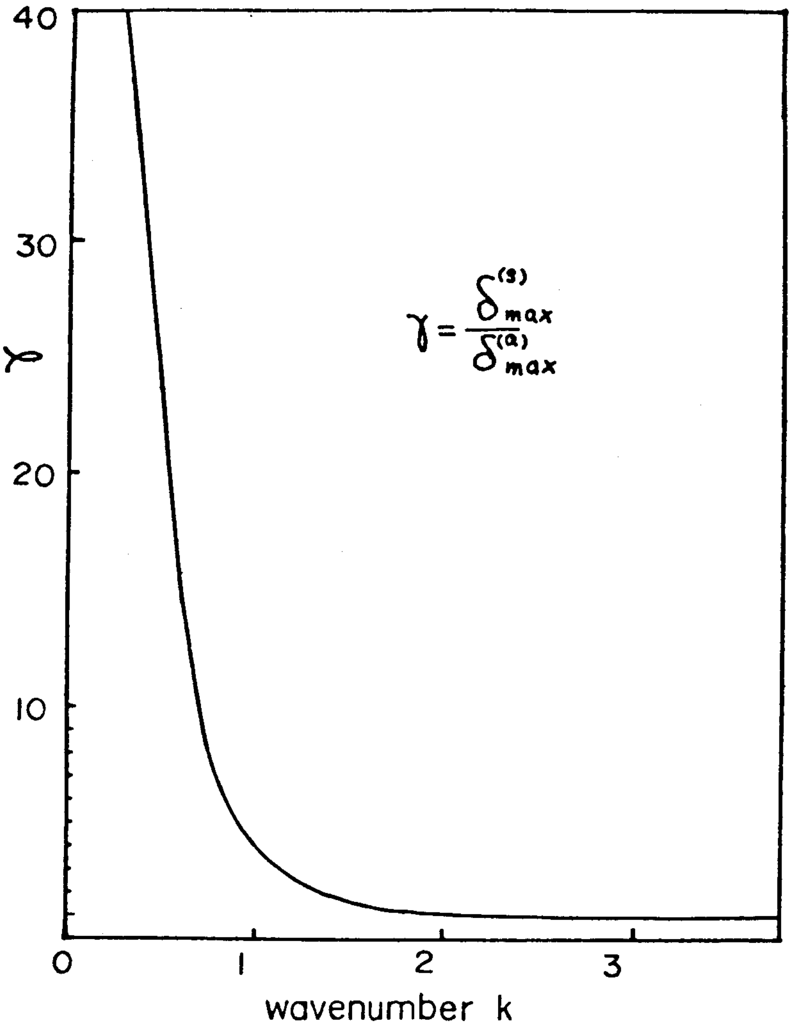

Fig.5. Variation of the ratio $\gamma = \delta^{(s)}_{max} / \delta^{(a)}_{max}$ as a function of the wavenumber k. The superscripts (s) and (a) denote the symmetrical and antisymmetrical waves, respectively.

A has a periodicity with the wavelength

$$\hat{\lambda} = \frac{2\pi}{\epsilon |A_0|} \left(- \frac{P_s}{g_s} \right)^{\frac{1}{2}}. \tag{3.52}$$

The increase in the amplitude gives rise to the increase of the thickness and vice versa, which is shown schematically in Fig.5. Hence, it may be concluded that this modulational

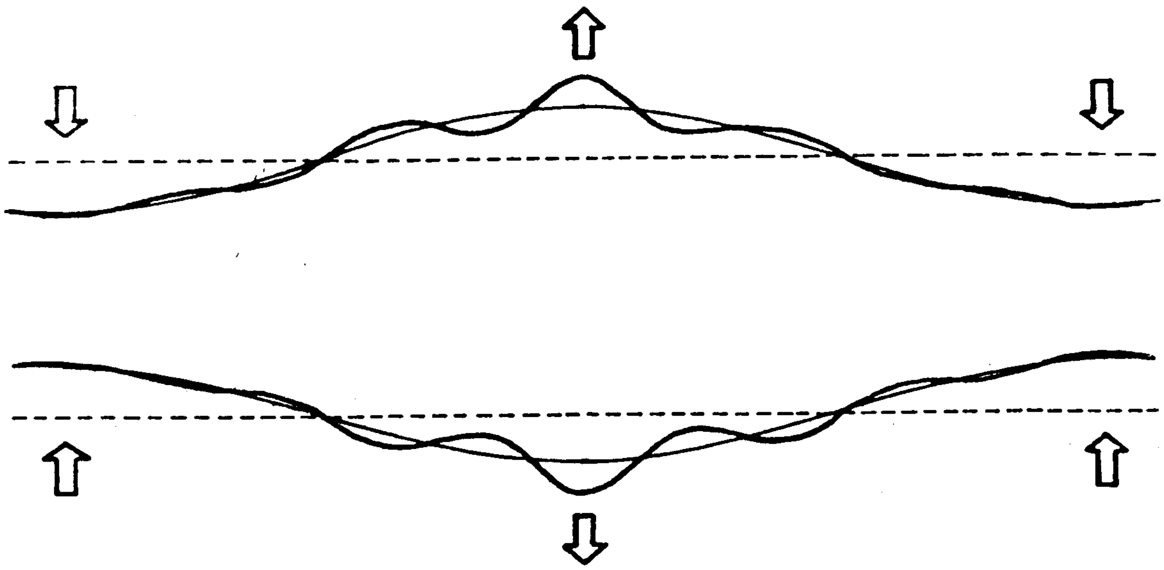

Fig.5. Sketch of time evolution of the free surfaces. The arrows indicate the direction of the variation of the free surface. _ _ _ _ _ _ _, initial state of the free surfaces with uniform thickness. __________, modulated state.

instability plays an important role in the break-up of sheet, although we cannot answer the question of what happens when the disturbance grows so large that the conditions ϵ , $\hat{\epsilon} \ll 1$ are no longer valid.

In the next sections, we will investigate successively the mechanism of the break-up based on the knowledge obtained in this section.

§4. Analysis for Long Waves of Large Amplitude

We have discussed the instability of capillary waves on a thin liquid sheet under the assumption that the dimensional amplitude a is small compared with the half thickness of the sheet, i.e.,

$$\epsilon \equiv \frac{a}{\frac{\ell}{2}} \ll 1 .$$

Because of this assumption we could merely show the existence of the instability but not explain the break-up of the sheet. The above condition is, therefore, not assumed here, but for simplicity another one called the long wave approximation is imposed. This implies that

$$\mu \equiv \left(\frac{\pi \ell}{L}\right)^2 \ll 1 ,$$

where L is the wavelength of the wave.

Our aim in this section is to show a possibility of the break-up of the sheet. First, confining our analysis to the symmetrical wave with arbitrary amplitude but long wavelength, we derive a simple set of equations governing the wave motion and investigate various properties of physical importance concerning the wave train described by the equations. Secondly, by solving the equations numerically we confirm the fact that the instability, predicted in the preceding section, proceeds until the break-up occurs.

4.1 Derivation of a simple set of equations

We write down again the equation of motion and the boundary conditions shown in §2 only for the symmetrical wave. These are

$$\phi_{xx} + \phi_{yy} = 0, \tag{4.1}$$

$$\left.\begin{aligned} &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + G^2\eta = \eta_{xx}/(1+\eta_x^2)^{\frac{3}{2}}, && (4.2)\\ &\phi_y = \eta_t + \phi_x \eta_x, && (4.3) \end{aligned}\right\} \text{ at } y = \eta(x,t),$$

$$\phi_y = 0, \qquad \text{at } y = 0, \tag{4.4}$$

where $G^2 = (\rho g/T)(l/2)^2$ and eq.(4.4) is the symmetrical condition for the upper and lower surfaces. If the dimensional wavelength is short compared with the length* L_c defined as

$$L_c^2 = \frac{T}{\rho g},$$

the last term on the left-hand side in eq.(4.2), which is due to the action of gravity force, is negligible. It should be noted that, in view of the symmetry, the set of eqs.(4.1)-(4.4) can also describe the wave motions on a water layer on

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* For pure water L_c =0.27cm, mercury 0.19cm, ethyl alchol 0.16cm and glycerine 0.23cm, when these are surrounded by air. All the values resemble each other in magnitude.

a rigid boundary.

The dispersion relation for the symmetrical wave is given as (see eq.(3.19))

$$\frac{\omega^2}{k^2} = k \tanh k \; .$$

Here, if we suppose that the wavelength is large, i.e., $k \ll 1$, the above relation can be rewritten approximately as

$$\frac{\omega^2}{k^2} = k^2.$$

This relation implies that when the space scale is set equal to $\mu^{-\frac{1}{2}}$, the time scale is μ^{-1} in magnitude, where μ stands for the squared wavenumber, i.e., $\mu \equiv k^2$. Hence it is convenient to introduce the following variables:

$$\xi = \mu^{\frac{1}{2}} x \, , \qquad \tau = \mu t. \tag{4.5}$$

We now suppose that all the quantities are functions of ξ and τ and vary at the same rate in both space and time. In terms of (ξ, y, τ) the set of eqs.(4.1)-(4.4) is written as

$$\mu \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{4.6}$$

$$\left.\begin{aligned} &\mu \frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\mu\left(\frac{\partial \phi}{\partial \xi}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] = \mu \frac{\partial^2 \eta}{\partial \xi^2} \Big/ \left[1 + \mu\left(\frac{\partial \eta}{\partial \xi}\right)^2\right]^{\frac{3}{2}}, &(4.7)\\ &\frac{\partial \phi}{\partial y} = \mu \frac{\partial \eta}{\partial \tau} + \mu \frac{\partial \phi}{\partial \xi}\frac{\partial \eta}{\partial \xi}, &(4.8)\end{aligned}\right\} \text{at } y = \eta(\xi, \tau),$$

$$\frac{\partial \phi}{\partial y} = 0, \qquad \text{at } y = 0, \tag{4.9}$$

Further, we expand $\phi(\xi, y, \tau; \mu)$ and $\eta(\xi, \tau; \mu)$ into power series in μ :

$$\left.\begin{aligned} \phi(\xi, y, \tau; \mu) &= \sum_{n=0}^{\infty} \mu^n \phi_n(\xi, y, \tau), \\ \eta(\xi, \tau; \mu) &= \sum_{n=0}^{\infty} \mu^n \eta_n(\xi, \tau). \end{aligned}\right\} \tag{4.10}$$

It should be noted that the zeroth order perturbations are taken into account; otherwise the system of equations may not lead to interesting nonlinear equations. Similar situations to this appear in nonlinear Alfvén waves[21] and long gravity waves strongly influenced by viscosity (see §4 of Part II). Substituting eq.(4.10) into eqs.(4.6)-(4.9) and noting that $\phi(\xi, y, \tau; \mu)$ at the free surface can be expressed as

$$\phi(\xi, y=\eta, \tau) = \phi\big|_{y=\eta_0} + \frac{\partial \phi}{\partial y}\Big|_{y=\eta_0} \left(\sum_{n=1}^{\infty} \mu^n \eta_n(\xi, \tau) \right) + \dots,$$

we can obtain a sequence of equations for each order of μ .

The first order of eq.(4.6) gives

$$\frac{\partial^2 \phi_0}{\partial y^2} = 0,$$

from which we have, with the boundary conditions (4.7)-(4.9)

$$\phi_0 = \phi_0(\xi, \tau). \tag{4.11}$$

From the second order of eq.(4.6) we have

$$\frac{\partial^2\phi_0}{\partial\xi^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 .$$

Integrating the above equation with respect to y , we obtain

$$\frac{\partial\phi_1}{\partial y} = - \frac{\partial^2\phi_0}{\partial\xi^2} y + C(\xi,\tau) ,$$

where $C(\xi,\tau)$ is an integration constant. The second order of eq.(4.9) requires $C=0$, whence we have

$$\frac{\partial\phi_1}{\partial y} = - \frac{\partial^2\phi_0}{\partial\xi^2} y . \tag{4.12}$$

Substitution of eqs.(4.11) and (4.12) into the second order of eq.(4.8) gives

$$\frac{\partial\eta_0}{\partial\tau} + \frac{\partial}{\partial\xi}\left(\frac{\partial\phi_0}{\partial\xi}\eta_0\right) = 0, \tag{4.13}$$

and eq.(4.7) becomes

$$\frac{\partial\phi_0}{\partial\tau} + \frac{1}{2}\left(\frac{\partial\phi_0}{\partial\xi}\right)^2 = \frac{\partial^2\eta_0}{\partial\xi^2} , \tag{4.14}$$

Accordingly, our problem is reduced to solving the set of eqs.(4.13) and (4.14).

Next, we try to arrange these two equations into a single one. We assume for a moment the amplitude to be small, although eqs.(4.13) and (4.14) are valid for long waves of arbitrary amplitude. Thus we write

$$\eta = 1 + \epsilon\hat{\eta} , \qquad u = \epsilon\hat{u} , \qquad \epsilon \ll 1 ,$$

where $u = \partial\phi/\partial\xi$ and ϵ is a measure of smallness of the amplitude (see the definition which appeared in the first page in this sub-section). Here we have omitted the subscript 0 . Substituting the above expressions into eqs. (4.13) and (4.14), neglecting the terms of higher orders in magnitude than ϵ^2 , and eliminating the velocity u , we obtain the linear equation

$$\frac{\partial^2 \hat{\eta}}{\partial \tau^2} + \frac{\partial^4 \hat{\eta}}{\partial \xi^4} = 0, \tag{4.15}$$

which was first derived by Taylor[16]. This is the same form as that governing the transverse oscillations of an elastic beam*.

Even for nonlinear case a single equation can be derived by introducing the function ψ defined by

$$\eta = \frac{\partial \psi}{\partial \xi}, \qquad u\eta = -\frac{\partial \psi}{\partial \tau},$$

which satisfy eq.(4.13) identically. Equation (4.14) is written in terms of ψ , as

$$\frac{\partial^2 \psi}{\partial \tau^2}\left(\frac{\partial \psi}{\partial \xi}\right)^2 - 2\frac{\partial^2 \psi}{\partial \xi \partial \tau}\frac{\partial \psi}{\partial \tau}\frac{\partial \psi}{\partial \xi} + \frac{\partial^2 \psi}{\partial \xi^2}\left(\frac{\partial \psi}{\partial \tau}\right)^2 + \left(\frac{\partial \psi}{\partial \xi}\right)^3\frac{\partial^4 \psi}{\partial \xi^4} = 0. \tag{4.16}$$

As this form is rather complex and moreover makes the physical meaning ambiguous, we will not intend to consider farther about this equation in this paper.

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* For a derivation of this equation and its solution, see Fourier transforms by I.N.Sneddon[22].

In concluding this sub-section, it is important to write eqs.(4.13) and (4.14) in alternative forms. Differentiating eq.(4.14) with respect to ξ, setting $u = \partial\phi/\partial\xi$, and introducing new variables $\tilde{x}$, $\tilde{t}$, and $\tilde{\eta}$, we write eqs.(4.13) and (4.14) in the form useful for interpretation, that is,

$$\frac{\partial\tilde{\eta}}{\partial\tilde{t}} + \frac{\partial}{\partial\tilde{x}}(u\tilde{\eta}) = 0, \tag{4.17}$$

$$\frac{\partial u}{\partial\tilde{t}} + u\frac{\partial u}{\partial\tilde{x}} = \delta\frac{\partial^3\tilde{\eta}}{\partial\tilde{x}^3}, \tag{4.18}$$

where the variables $\tilde{x}$, $\tilde{t}$, and $\tilde{\eta}$ are defined as

$$\tilde{x} = \frac{x^*}{L/2} = \frac{\pi\ell^2}{L^2}\xi, \quad \tilde{t} = \frac{t^*}{\pi\frac{\ell}{2}/\sqrt{2T/\rho\ell}} = \frac{\pi\ell^2}{L^2}\tau, \quad \tilde{\eta} = \frac{\eta^*}{2a} = \frac{\ell}{4a}\eta.$$

The asterisk indicates dimensional values of the variables. The nondimensional parameter δ appearing in eq.(4.18) is expressed as

$$\delta = \frac{4\pi^2 a\ell^3}{L^4},$$

which may be related to Ursell parameter[23] for long gravity waves (see Part II). Making the following transformations:

$$u \longrightarrow \sqrt{\delta}\,u, \qquad t \longrightarrow \frac{t}{\sqrt{\delta}}, \tag{4.19}$$

we have the same form as eqs.(4.13) and (4.14) expressed in terms of u . Hence it is sufficient to treat eqs.(4.13) and (4.14) in place of eqs.(4.17) and (4.18) without loss of generality.

4.2 Steady solutions

We seek a solution of eqs.(4.13) and (4.14) in the following form:

$$\eta = \eta(\zeta), \qquad \phi = \phi(\zeta), \qquad \text{with } \zeta = \xi - C\tau,$$

which represent a steady travelling wave with velocity C . Substituting the above expressions into eqs.(4.13) and (4.14), we obtain

$$-C\eta' + (\phi'\eta)' = 0, \tag{4.20}$$

$$-C\phi' + \frac{1}{2}(\phi')^2 = \eta'', \tag{4.21}$$

where the prime denotes the differentiation with respect to ζ . Integration with respect to ζ gives

$$\phi' = \frac{A + C\eta}{\eta},$$

where A is an integration constant. With use of this relation, we have from eq.(4.20),

$$\eta^2\eta'' = \frac{1}{2}(A^2 - C^2\eta^2). \tag{4.22}$$

Finally, we obtain after integration

$$\frac{1}{2}\left(\frac{d\eta}{d\zeta}\right)^2 = -\frac{1}{2}\left(\frac{A^2 + C^2\eta^2}{\eta}\right) + B, \tag{4.23}$$

in which B is another integration constant. The right-hand side must be positive, which requires that $C^2A^2 \leq B^2$. In place of A and B we introduce new constants C_1 and C_2 satisfying the relations:

$$C^2 C_1 C_2 = A^2, {}^{*} \qquad C^2(C_1 + C_2) = 2B,$$

where we can set $C_1 \leq C_2$ without loss of generality. Thus eq.(4.23) can be rewritten as

$$\left(\frac{d\eta}{d\zeta}\right)^2 = C^2 \frac{(\eta - C_1)(C_2 - \eta)}{\eta}. \tag{4.24}$$

Solutions of this equation can be written in terms of the elliptic integral (see Byrd and Friedman,[24] p.79), although we do not here write down it explicitly. It follows from this equation that the surface displacement η varies within a region expressed as

$$0 \leq C_1 \leq \eta \leq C_2,$$

and there are wave trains whose wavelength λ is given by

$$\lambda = \frac{2}{|C|}\int_{C_1}^{C_2} \sqrt{\frac{\eta}{(\eta - C_1)(C_2 - \eta)}}\, d\eta,$$

---

* We must choose $A = -C\sqrt{C_1C_2}$ in order that the velocity φ should become small quantity in the weak-nonlinear limit.

which is written simply as

$$\lambda = \frac{4}{|C|}\sqrt{C_2}\, E(\bar{a}), \text{ where } \bar{a}^2 = (C_2 - C_1)/C_2 , \tag{4.25}$$

where $E(\bar{a})$ is the complete elliptic integral of the second kind (see also Byrd and Friedman[24], p.79). These wave trains are shown in Fig.6.

The wave velocity C is equivalent to $C = \bar{\omega}/k$, $\bar{\omega}$ being the frequency, from which we have, with use of $\lambda = 2\pi/k$,

$$\bar{\omega} = \frac{2}{\pi}\sqrt{C_2}\, k^2 E(\bar{a}) . \tag{4.26}$$

Wave trains of constant amplitude can exist, only when such a relation as eq.(4.26) is satisfied among the frequency, amplitude, and wavenumber. It is found that in the present case the nonlinearity leads to decreasing of the wave velocity contrary to the case of gravity waves (see Lamb[25], p. 417). As we have already mentioned (see §3), these dependence of nonlinearity upon the frequency is called the 'amplitude dispersion'[20].

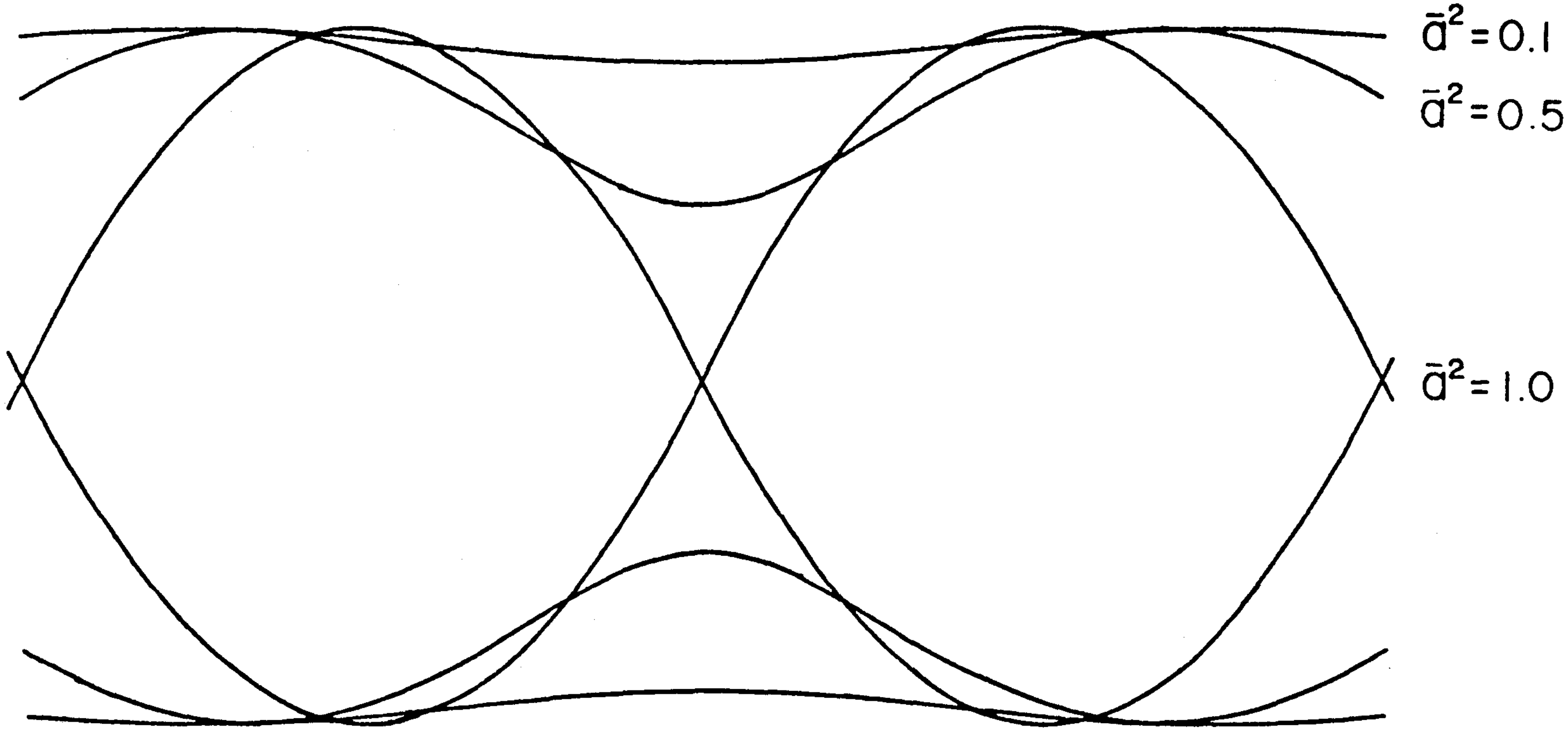

Fig.6. Steady solutions. Solutions obtained by solving eq.(4.24) numerically are shown as functions of ζ for three different values of $\bar{a}^2$. Note that η and ζ are normalized by means of C_2 and $2\sqrt{C_2}/|C|$, respectively. For the three values of $\bar{a}^2$, i.e., 0.1, 0.5, and 1.0, the normalized wavelength is equal to 3.1, 2.7, and 2.0, respectively.

4.3 Comparison between plane wave solution of the nonlinear Schrödinger equation and steady solution for long waves

We seek a plane wave solution expressed as $A=(a_c/2)\exp(-i\Omega\tau)$ in eq.(3.40), where a_c and Ω are real constants as before. This wave profile becomes

$$\eta = 1+\epsilon a_c \cos\zeta - \frac{k^4}{4\omega^2}\epsilon^2 a_c^2 \cos 2\zeta + \epsilon^2 C_2, \quad (4.27)$$

where

$$\zeta = kx - \bar{\omega} t,$$

$$C_2 = \frac{k[3\gamma+2k(1-\gamma^2)]}{[3\gamma+k(1-\gamma^2)]^2} a_c^2 .$$

The frequency $\bar{\omega}$ for the nonlinear wave train is expressed as

$$\bar{\omega}(k, \epsilon a_c) = \omega + \frac{\delta}{4}\epsilon^2 a_c^2, \quad (4.28)$$

where ω is the frequency for the linear wave given by eq.(3.19), and δ is the function of k (see §3.3). Here we assume that the wavelength is long enough to be able to set $\gamma = k$, then from eqs.(4.27) and (4.28) we have

$$\eta = 1+\epsilon a_c \cos\zeta - \frac{\epsilon^2 a_c^2}{4}\cos 2\zeta + \frac{5}{16}\epsilon^2 a_c^2, \quad (4.29)$$

$$\bar{\omega} = \omega - \frac{5}{32}k^2\epsilon^2 a_c^2. \quad (4.30)$$

Returning to the results established in §4.2, we consider a wave train of small amplitude, in which we can expand the elliptic integral in eq.(4.26) into power series in $\bar{a}^2$. According to the formula given by Byrd and Friedman[24) (see p.298), when $\bar{a}^2 \ll 1$, eq.(4.26) can be approximated by

$$\bar{\omega} \doteqdot k^2 \sqrt{C_2}\left[1 - \frac{1}{4}\frac{C_2 - C_1}{C_2} - \frac{3}{64}\frac{(C_2 - C_1)^2}{C_2^2} + O(\bar{a}^6)\right].$$

When the constants C_1 and C_2 are determined so as to satisfy the relations

$$C_1 = 1 - \epsilon a - \frac{3}{16}\epsilon^2 a_0^2,$$

$$C_2 = 1 + \epsilon a - \frac{3}{16}\epsilon^2 a_0^2,$$

we have the same dispersion relation as (4.30) and

$$\eta = 1 + \epsilon a_0 \cos\zeta - \frac{\epsilon^2 a_0^2}{4}\cos 2\zeta + \frac{1}{16}\epsilon^2 a_0^2, \qquad (4.32)$$

from eq.(4.24). This equation coincides with eq.(4.29) except the trivial constant term, which results from a special choice of the integration constant (see the marginal note on p. 15).

Therefore, it is concluded that the present wave solution with long wavelength but arbitrary amplitude can be reduced, in its weak-nonlinear limit, to the solution of the nonlinear Schrödinger equation.

4.4 Some interesting properties of periodic wave trains

We consider periodic but not necessarily steady wave trains governed by eqs.(4.13) and (4.14). Assuming two functions η and u to be periodic*, we have directly from eq.(4.13)

$$\frac{d}{d\tau}\int_L \eta \, d\xi = 0, \qquad (4.33)$$

where the integration is taken over one wavelength. This relation states that the mass

$$M[\eta] = \int_L \eta \, d\xi, \qquad (4.34)$$

is time-invariant quantity. Another invariant of physical importance is the momentum given by

$$N[\eta, u] = \int_L u\eta \, d\xi, \qquad (4.35)$$

where u is the velocity in ξ-direction as before. Multiplying η by eq.(4.14) after differentiation with respect ξ, we obtain

$$\int_L (\eta u_\tau + \eta u u_\xi) \, d\xi = 0, \qquad (4.36)$$

while from eq.(4.13) multiplied by u, we get

---

* It is easy to see from eq.(4.14) that the velocity potential ϕ cannot be assumed to be periodic.

$$\int_L [u\eta_\tau + u(u\eta)_\xi]\, d\xi = 0.$$

The second term in the latter is integrated by parts, and after summation of the two integral expressions the expected relation

$$\frac{d}{d\tau}\int_L u\eta\, d\xi = 0, \tag{4.36}$$

can be obtained.

Next, we derive some expressions concerning wave energy. The energy consists of two parts; one is the kinetic energy and the other the surface energy. Expression for the kinetic energy, say K, is known to be

$$\frac{1}{2}\int_L \eta u^2\, d\xi.$$

On the other hand, the surface energy is proportional to the surface area, so that for the present two-dimensional wave, it is given by

$$\int_L \sqrt{1 + \mu\eta_\xi^2}\; d\xi,$$

which can be approximated as

$$\int_L d\xi + \frac{1}{2}\mu\int_L \eta_\xi^2\, d\xi.$$

The second term corresponds to the energy due to the wave.

Hence, in what follows we call it simply the surface energy and denote it by symbol U, i.e.,

$$U = \frac{1}{2}\int_L \eta_\xi^2 \, d\xi ,$$

Some straightforward manipulations lead to an expression for another important invariant expressed as

$$\frac{d}{d\tau}\int_L \left(\frac{1}{2}\eta u^2 + \frac{1}{2}\eta_\xi^2\right) d\xi = 0 , \tag{4.37}$$

which implies the conservation of the total energy:

$$E[u, \eta, \eta_\xi] = \int_L \left(\frac{1}{2}\eta u^2 + \frac{1}{2}\eta_\xi^2\right) d\xi . \tag{4.38}$$

4.5 Variational principle

It sometimes occurs that when use is made of the variational principle in analyses for wave problems, some interesting and useful results can be easily attained.

Steadily travelling waves become stationary if we move with the same velocity as that of these waves, we examine the following variational principle. For a fixed mass, we virtually change the wave form while both the ends are fixed*, and we seek solutions for which the quantity

$$L = K - U,$$

becomes stationary. This is written as

$$\delta L = 0 \qquad \text{for } M \text{ fixed},$$

where δL is the first variation of the nonlinear functional $L[u, \eta, \eta_\xi]$. Here, it should be noted that the kinetic energy K is measured in the moving frame. We denote the velocity relative to the frame by u_c. Noting that for long waves the product of u_c and η may be constant, say A, we have the Euler-Lagrange equation,

$$\frac{d}{d\xi}\frac{\partial L'}{\partial \eta_\xi} = \frac{\partial L'}{\partial \eta} + \lambda, \tag{4.39}$$

where L' is the integrand of the nonlinear functional L and λ the Lagrangian multiplier. This is expressed in the

* It may be sufficient to consider only one wavelength.

form

$$\eta_{\xi\xi} = \frac{1}{2}\left(\frac{A^2}{\eta^2} - 2\lambda\right), \tag{4.40}$$

If the multiplier is chosen to be $C^2/2$, this equation is just the equation governing the steady solution (see eq. (4.22)). Similar fact was noticed by Boussinesq for solitary wave in a shallow water and utilized by Benjamin[26] to demonstrate its stability.

The following analysis is concerned with the wave energy per unit length in ξ-direction. Denoting the two kinds of energy densities by K' and U', the Lagrangian L' is written as

$$L' = K' - U'.$$

Expressions for the two functions K' and U' are given by the integrands of K and U, respectively. Differentiation with respect to ξ gives

$$\frac{dL'}{d\xi} = \frac{\partial L'}{\partial \eta}\frac{d\eta}{d\xi} + \frac{\partial L'}{\partial \eta_\xi}\frac{d\eta_\xi}{d\xi},$$

since the function L' does not include the coordinate ξ explicitly. We can arrange it in the form

$$\frac{d}{d\xi}\left(L' - \eta_\xi \frac{\partial L'}{\partial \eta_\xi} + \frac{C^2}{2}\eta\right) = 0,$$

where we have utilized the relation (4.39). This states that the function

$$L' - \eta_\xi \frac{\partial L'}{\partial \eta_\xi} + \frac{c^2}{2}\eta$$

does not depend on ξ . When the relations obtained in §4.2 (see the expression for ϕ' and the footnote on p.33) are utilized, the above expression can be written in a simple form

$$E' + \frac{c^2}{2}\eta = \frac{c^2}{2}(c_1 + c_2), \tag{4.41}$$

where E' is the energy density defined as

$$E' = \frac{1}{2}\eta u_c^2 + \frac{1}{2}\eta_\xi^2. \tag{4.42}$$

This relation is also written in terms of u in place of u_c as

$$\frac{1}{2}\eta u^2 + \frac{1}{2}\eta_\xi^2 = \frac{c^2}{2}(c_1^{\frac{1}{2}} - c_2^{\frac{1}{2}})^2, \tag{4.43}$$

which implies that the energy density is uniform with respect to ξ . It is found from eq.(4.41) that the local energy per unit length has its maximum at $\eta = c_1$, and minimum at $\eta = c_2$, i.e.,

$$E'_{max} = \frac{c^2}{2}c_2, \qquad E'_{min} = \frac{c^2}{2}c_1.$$

These are proportional to the squared velocity c^2 , hence for fixed c_1 and c_2 these become larger for the shorter waves (see (4.26)).

4.6 Numerical investigations

It is suggested from the results obtained in §3.3 that the steady wave solution represented by eq.(4.24) may be unstable to small disturbances with larger wavelength. Therefore, we solve initial value problems numerically to eqs. (4.17) and (4.18) with $\delta=1$ (or eqs.(4.13) and (4.14) expressed in terms of u), choosing the following two cases as their initial values,

Case (I): $SS(\xi;c_1,c_2,\lambda)-0.05\cos 2\pi\xi$,

Case (II): $SS(\xi;c_1,c_2,\lambda)-0.01\cos \frac{\pi}{3}\xi$,

where $SS(\xi;c_1,c_2,\lambda)$ stands for the steady solution with the crest-to-trough amplitude, c_2-c_1, and the wavelength λ (see §4.2). Here we take $c_1=0.5$, $c_2=1.5$, $\lambda=2$, and the remaining arbitrariness arising from the integration constant is determined so as to make SS minimum at $\xi=0$. Numerical calculations are carried out under periodic boundary conditions with periods λ and 3λ , respectively, for Case (I) and for Case (II), by use of the finite difference method given in Appendix. The amplitude of the disturbance for Case (I) is 10% compared with that of the steady solution, and for Case (II) this ratio is very small (merely 2%).

First, we discuss for Case (I). In Fig.7 the temporal variations of the maximum surface elevation η_{max} and the minimum one η_{min} are shown. From this figure, it may be concluded that the small disturbance, which corresponds to the

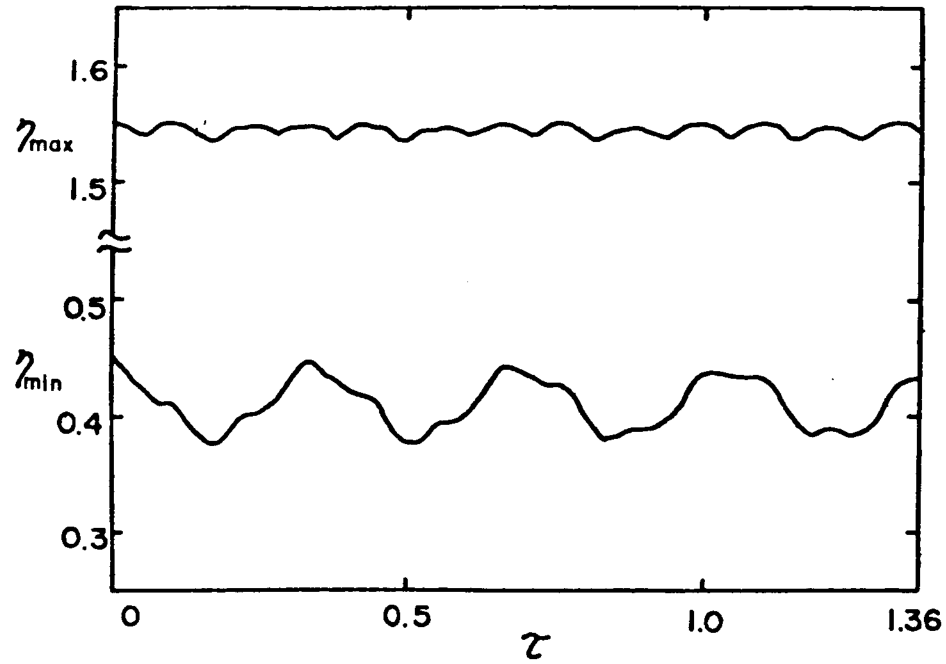

Fig.7. Temporal variations of η_{max} and η_{min} for Case (I).

second harmonic with respect to the steady solution, does not give rise to instabilities.

On the other hand, for Case (II) the variations of the free surface η and the velocity u within the time interval $0 \leq \tau \leq 4.4$ are shown in Figs.8 and 9, respectively. As has already been suggested, disturbances with larger wavelength than that of the steady solution grows with time. In fact these two figures show the instability. We cannot carry out this computation just up to the time of break-up, since the errors originating from the steepeness of the two quantities, particularly from that of the velocity (see Fig.9), become larger. However, it is highly plausible from these figures that such an instability proceeds until both surfaces

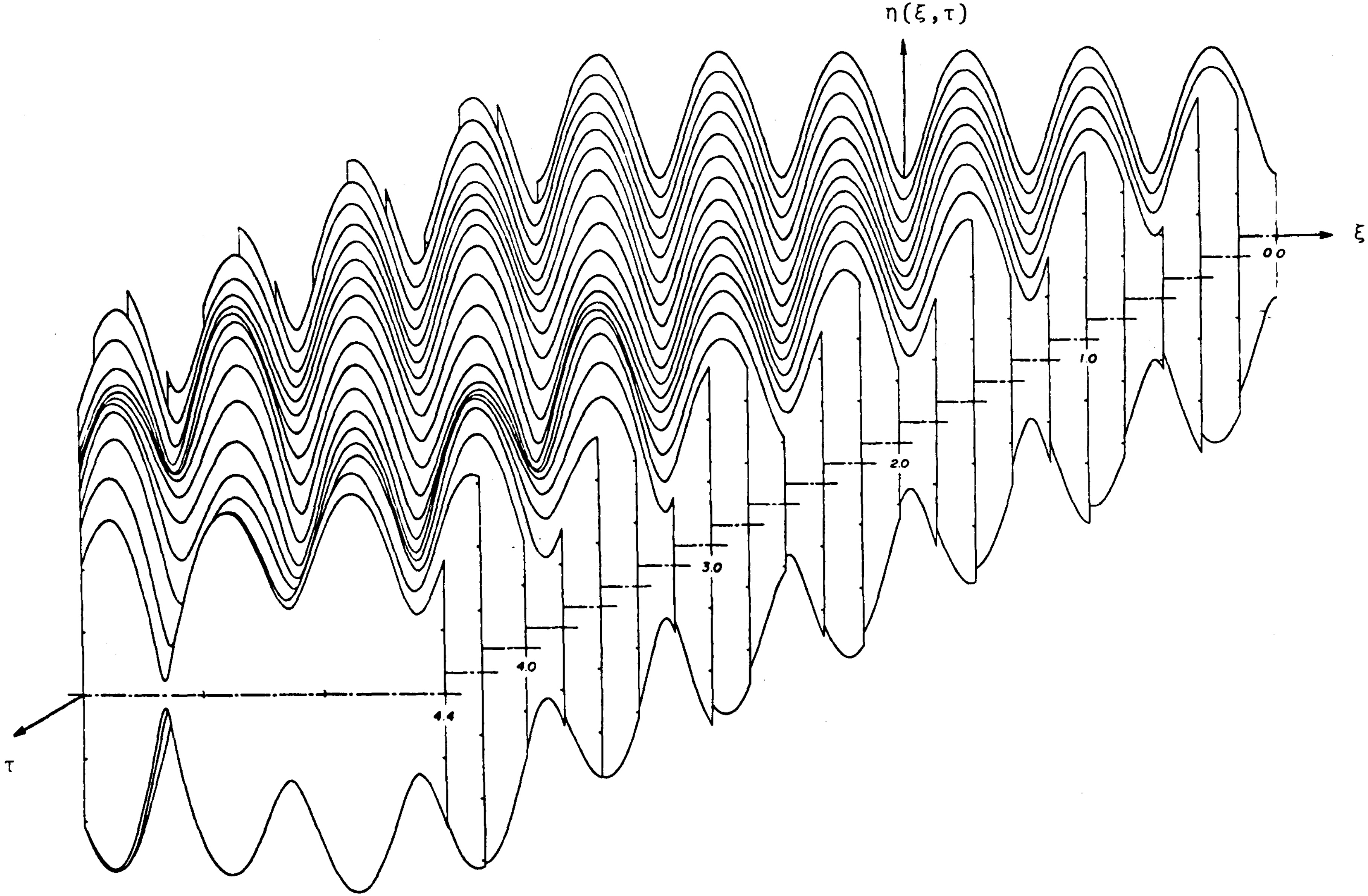

Fig.8. Variations of free surface η for Case (II). In the initial stage the minimum trough is placed at ξ =0. The wave profiles are drawn over two wavelengths of the disturbance.

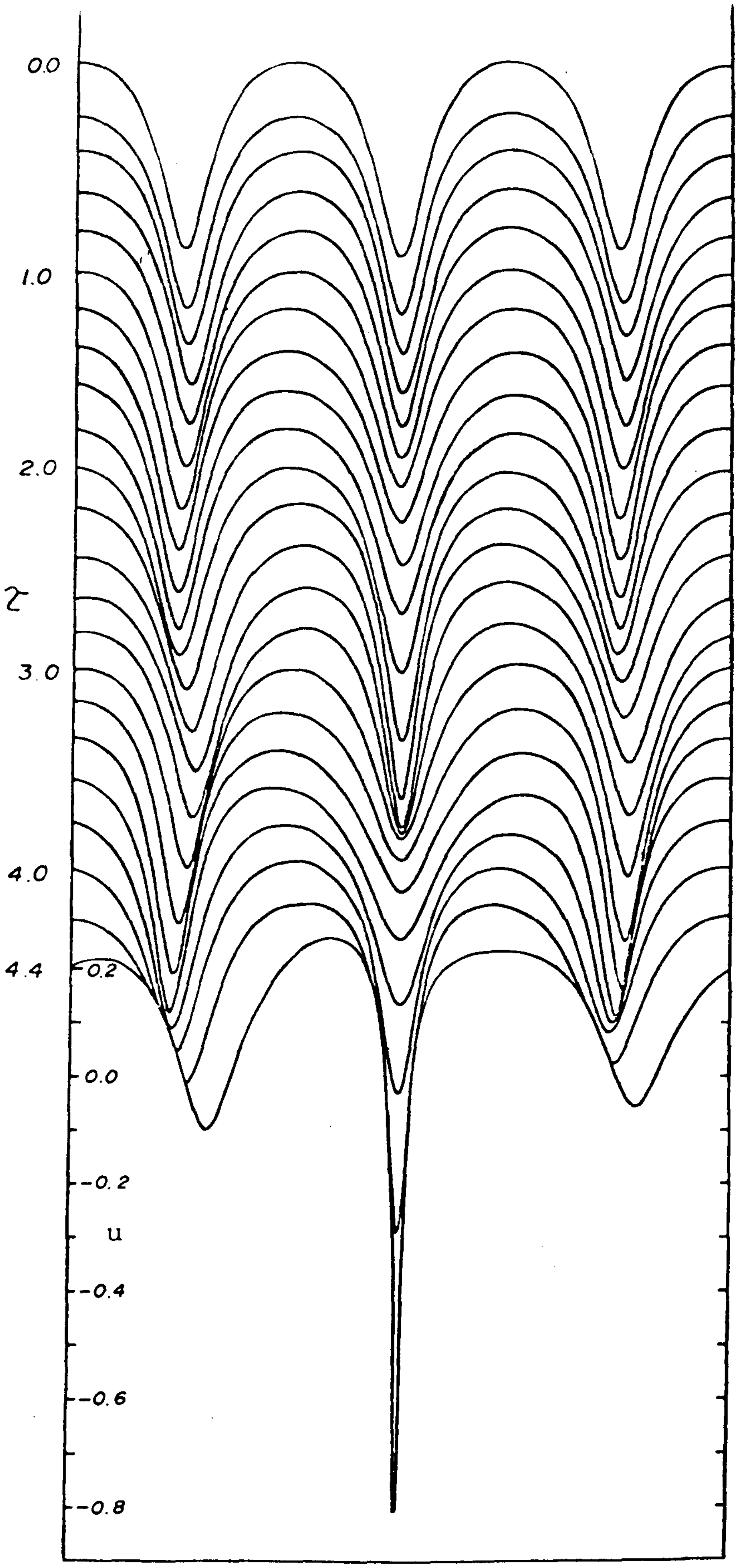

Fig.9. Variation of velocity u for Case (II). Phases are properly shifted so that the position of the minimum velocity, which is initially placed at the same position as that corresponding to the minimum trough, may be situated in the center.

meet together in the center plane. In this example, the minimum trough (initially placed at $\xi=0$) becomes smaller (but not monotonically), and hence this trough is important to have an understanding of the mechanism of the break-up. The wave energy distributed almost unifirmly in the initial stage is concentrated in a narrow region near the minimum trough (see eq.(4.43)). This tendency is more remarkable for the kinetic energy, which is easily seen from Fig.9.* This concentration becomes more noticeable and at last our solution may burst at a finite time.

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* In this respect, it should be noted that the position of the minimum velocity is placed at the same point as that of the minimum trough.

§5. Conclusions

In the first half, we derive a simple equation called nonlinear Schrodinger equation, and we show the existence of the modulational instability for weakly nonlinear waves. It is, then, conjectured that the modulational instability for two kinds of waves, especially for the symmetrical waves, may play an important role in the phenomenon of the break-up of the sheet.

In the next half, to confirm the above conjecture we analyse the symmetrical waves without the assumption that the amplitude is small compared with the thickness of the sheet. Numerical computation shows that the steadily propagating waves are unstable to small disturbance with larger wavelength than that of the steady wave solution. Such an instability proceeds until the solution bursts suddenly at a finite time. This 'burst instability' gives us an almost certain evidence that the break-up does occur. Our calculation was carried out for a special case, in which the wavelength was three times as large as that of the steady wave solution, and thus it remains to determine the optimum wavelength of small disturbance for a given steady wave solution. If the determination could be carried out, then we have a possibility that we may determine the drop-size arising from the break-up and finally find a method of controlling it.

Appendix: Numerical Algorithm with Finite difference Method

In solving the initial value problems to eqs.(4.13) and (4.14) expressed in terms of u , we use the following leap-frog explicit scheme,

$$\frac{1}{2\Delta'}[u(i,j+1)-u(i,j-1)]+\frac{1}{6\Delta}[u(i-1,j)+u(i,j)+u(i+1,j)]$$

$$\times[u(i+1,j)-u(i-1,j)]=\frac{1}{2\Delta^3}[s(i+2,j)-2s(i+1,j)+2s(i-1,j)$$

$$-s(i-2,j)], \qquad \text{(A.1)}$$

$$\frac{1}{2\Delta'}[s(i,j+1)-s(i,j-1)]+\frac{1}{6\Delta}[s(i-1,j)+s(i,j)+s(i+1,j)]$$

$$\times[u(i+1,j)-u(i-1,j)]+\frac{1}{6\Delta}[u(i-1,j)+u(i,j)+u(i+1,j)]$$

$$\times[s(i+1,j)-s(i-1,j)]=0, \qquad \text{(A.2)}$$

where $s(i,j)$ and $u(i,j)$ imply $\eta(i\Delta,j\Delta')$ and $u(i\Delta,j\Delta')$, respectively, Δ and Δ' being the lattice spacing and the time step. Both eqs.(A.1) and (A.2) have the truncation errors of order Δ^2 and Δ'^2. For the initial step we use the forward time difference scheme,

$$\frac{1}{\Delta'}[u(i,j+1)-u(i,j)]+\frac{1}{3}[u(i-1,j)+u(i,j)+u(i+1,j)]$$

$$\times D_0u(i,j)=D_+D_0D_-s(i,j),$$

$$\frac{1}{\Delta'}[s(i,j+1)-s(i,j)]+\frac{1}{3}[s(i-1,j)+s(i,j)+s(i+1,j)]$$

$$\times D_0 u(i,j) + \frac{1}{3}[u(i-1,j) + u(i,j) + u(i+1,j)] D_0 s(i,j) = 0 \ ,$$

where

$$D_+ f(i,j) = \frac{1}{\Delta}[f(i+1,j) - f(i,j)],$$

$$D_- f(i,j) = \frac{1}{\Delta}[f(i,j) - f(i-1,j)],$$

$$D_0 f(i,j) = \frac{1}{2\Delta}[f(i+1,j) - f(i-1,j)].$$

We examine the stability of the above scheme on the basis of the von Neumann theory.[27] For the application of the theory, we linearize eqs.(A.1) and (A.2), hence

$$\frac{1}{2\Delta'}[u(i,j+1) - u(i,j-1)] + \bar{u} D_0 u(i,j) = D_+ D_0 D_- s(i,j) \ , \tag{A.3}$$

$$\frac{1}{2\Delta'}[s(i,j+1) - s(i,j-1)] + \bar{s} D_0 u(i,j) + \bar{u} D_0 s(i,j) = 0 \ , \tag{A.4}$$

where $\bar{u}$ and $\bar{s}$ are regarded as constants. Equations (A.3) and (A.4) can be rewritten by usual steps as

$$\frac{1}{2\Delta'}[\hat{u}(\xi,j+1) - \hat{u}(\xi,j-1)] + \frac{\bar{u}}{2\Delta} 2i\sin\Delta\xi \, \hat{u}(\xi,j)$$
$$= \frac{1}{2\Delta^3} 4i\sin\Delta\xi (\cos\Delta\xi - 1) \hat{s}(\xi,j) \ , \tag{A.5}$$

$$\frac{1}{2\Delta'}[\hat{s}(\xi,j+1) - \hat{s}(\xi,j-1)] + \frac{\bar{s}}{2\Delta} 2i\sin\Delta\xi \, \hat{u}(\xi,j)$$
$$+ \frac{\bar{u}}{2\Delta} 2i\sin\Delta\xi \, \hat{s}(\xi,j) = 0, \tag{A.6}$$

where we have introduced the Fourier transforms,

$$\hat{s}(\hat{\xi},\tau)=\frac{1}{2\pi}\int s(\xi,\tau)\,e^{-i\hat{\xi}\xi}\,d\xi\,,$$

$$\hat{u}(\hat{\xi},\tau)=\frac{1}{2\pi}\int u(\xi,\tau)\,e^{-i\hat{\xi}\xi}\,d\xi\,.$$

Replacing $\hat{s}'(j-1)$ and $\hat{u}(j-1)$ by $\hat{s}'(j)$ and $\hat{u}'(j)$, we can arrange in the form,

$$\mathcal{U}(j+1,\hat{\xi})=\mathcal{Q}(\hat{\xi}\Delta,\Delta)\,\mathcal{U}(j,\hat{\xi}),$$

where $\mathcal{U}$ is the column vector with four components, and $\mathcal{Q}$ is the 4×4 matrix defined as

$$\mathcal{U}(j+1,\hat{\xi})=\begin{pmatrix}\hat{s}(j+1,\hat{\xi})\\ \hat{s}'(j+1,\hat{\xi})\\ \hat{u}(j+1,\hat{\xi})\\ \hat{u}'(j+1,\hat{\xi})\end{pmatrix},$$

$$\mathcal{Q}=\begin{pmatrix}-\frac{\Delta'}{\Delta}\bar{u}\,2i\sin\Delta\hat{\xi} & 1 & -\frac{\Delta'}{\Delta}\bar{s}\,2i\sin\Delta\hat{\xi} & 0\\ 1 & 0 & 0 & 0\\ \frac{\Delta'}{\Delta^3}4i\sin\Delta\hat{\xi}\times(\cos\Delta\hat{\xi}-1) & 0 & -\frac{\Delta'}{\Delta}\bar{u}\,2i\sin\Delta\hat{\xi} & 1\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

The eigenvalues λ for the matrix $\mathcal{Q}$ are given as

$$\lambda = -\frac{1}{2}\left[i(\bar{U}C+B) \pm \sqrt{4-(\bar{U}C+B)^2}\right], \quad -\frac{1}{2}\left[i(\bar{U}C-B) \pm \sqrt{4-(\bar{U}C-B)^2}\right],$$

where

$$A = 4i\,\frac{\Delta'}{\Delta^3}\sin\Delta\hat{\xi}\,(1-\cos\Delta\hat{\xi}),$$

$$B^2 = -2i\,A\,\bar{s}\,\frac{\Delta'}{\Delta}\sin\Delta\hat{\xi},$$

$$C = 2\,\frac{\Delta'}{\Delta}\sin\Delta\hat{\xi}.$$

Noting that B and C are real, it follows that when the relation

$$4 - X^2 \geq 0, \quad \text{where} \quad X = \bar{U}C \pm B,$$

is satisfied, the modulus of the eigenvalue λ is always equal to unity, and the linearized scheme is thus stable.[27)] We have an inequality:

$$|X| = \left|2\bar{u}\frac{\Delta'}{\Delta}\sin\Delta\hat{\xi} \pm \left[4\bar{s}\frac{\Delta'}{\Delta^3}\sin\Delta\hat{\xi}\,(1-\cos\Delta\hat{\xi}) \times 2\,\frac{\Delta'}{\Delta}\sin\Delta\hat{\xi}\right]^{\frac{1}{2}}\right|$$

$$\leq 2\,\frac{\Delta'}{\Delta}|\bar{u}| + 2\sqrt{2\bar{s}}\,\frac{\Delta'}{\Delta^2}\max\left|\sin\Delta\hat{\xi}\,(1-\cos\Delta\hat{\xi})^{\frac{1}{2}}\right|,$$

from which it is obvious that when an inequality

$$|\bar{u}| \frac{\Delta'}{\Delta} + \frac{8}{3\sqrt{3}} \sqrt{5} \frac{\Delta'}{\Delta^2} \leq 1 ,$$

is satisfied, our difference scheme is stable. In deriving this we have utilized the relation:

$$\max |\sin \Delta \hat{\xi} (1 - \cos \Delta \hat{\xi})^{\frac{1}{2}}| = \frac{4}{3}\sqrt{\frac{2}{3}} .$$

List of Symbols

Dimensional quantities

a : amplitude

g : acceleration due to gravity

ℓ : thickness of sheet

L : wavelength

L_c^2 : $T/\rho g$

T : surface tension per unit length

ρ : density of sheet

Nondimensional quantities

A : complex amplitude

A_0 : one half of real amplitude

a_0 : real amplitude

$\bar{a}^2$: crest-to-trough amplitude normalized by means of trough $(=(C_2 - C_1)/C_2)$

B_1, B_2 : induced potential due to nonlinear interaction

C_1 : height of trough

C_2 : height of crest

C_2' : induced surface elevation due to nonlinear interaction

C : phase velocity

C_g : group velocity $(d\omega/dk)$

$E[u, \eta, \eta_\xi]$: energy contained per one wavelength

E' : energy density $(=K' + U')$

$E(\bar{a})$: complete elliptic integral of the second kind with modulus $\bar{a}$

$f(i,j)$: $u(i,j)$ or $s(i,j)$

G_T^2 : ratio of gravity force to surface tension force $(=\rho g \ell^2/4T)$

K : kinetic energy contained in one wavelength

K' : kinetic energy per unit length

k : wavenumber

$L[u,\eta,\eta_\xi]$: $K-U$

L' : integrand of nonlinear functional L

$M[\eta]$: mass contained in one wavelength

$N[\eta,u]$: momentum contained in one wavelength

p : coefficient of dispersion term on the nonlinear Schrödinger equation

q : coefficient of nonlinear term on the nonlinear Schrödinger equation

t : time

u : velocity in ξ-direction

U : surface energy contained in one wavelength

U' : surface energy per unit length

X : coth k

Y : tanh k

γ : $\delta_{max}^{(s)}/\delta_{max}^{(a)}$

δ_{max} : maximum amplification rate

δ : $(4\pi^2 a\ell^3)/L^4$

ϵ : a measure of smallness of amplitude $(=a/(\ell/2))$, band-width for quasi-monochromatic wave

η : upper surface elevation

η': lower surface elevation

θ : phase for linear wave $(=kx_c-\omega t_c)$

λ : wavelength

μ : a measure of smallness of wavenumber $(=(\pi\ell/L)^2)$

ξ : $\epsilon(x - C_g t)$ for modulated wave, or $\mu^{\frac{1}{2}}x$ for long wave

τ : $\epsilon^2 t$ for modulated wave, or μt for long wave

ϕ : velocity potential

ω : wave frequency for linear wave

$\bar{\omega}$: wave frequency for nonlinear wave

Ω : frequency shift due to nonlinearity

superscript

(s),s: symmetrical wave

(a),a: antisymmetrical wave

* : dimensional quantities

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PART II. Effect of Viscosity on Long Gravity Waves

§1. Introduction

The effect of viscosity on long gravity waves has been investigated by a number of authors since the time of Stokes (see, for example, Lamb[1]). Most attention, however, has been confined to infinitesimal waves on the basis of linearized theory, which is surveyed by Wehausen and Laitone.[2] On the other hand, the inviscid potential theory enables us to handle finite amplitude waves, and there is also a long history of the inviscid nonlinear gravity waves (see Lamb[1] and Stoker[3]). Amongst them Korteweg and de Vries[4] derived a simple equation for the free surface, called today after their names, at the end of the last century.

In the first half of Part II, we attempt to derive a simple equation for weakly nonlinear long gravity waves on a viscous fluid layer. In order to see the effect of viscosity, we first obtain the linear dispersion relation and express the complex phase velocity c as a function of the wavenumber α and the Reynolds number R . It is found that, when $|\alpha R c| \gg 1$, the wave dispersion consists of two different parts; one is due to the geometrical configuration and the other due to the effect of viscosity. In particular, for long waves ($\alpha \ll 1$), the geometrical dispersion (inviscid dispersion) dominates over, balances with, and is dominated by the viscous dispersion according as $O(\alpha^{-5}) < R$, $R = O(\alpha^{-5})$, and $O(\alpha^{-7}) < R < O(\alpha^{-5})$. We then apply the reductive perturbation method[5] combined with the usual boundary layer theory to waves with small but finite amplitude. It is found that the inviscid Korteweg-de Vries equation is not affected by the viscosity for $O(\alpha^{-5}) < R$ but it is modified by the viscous dispersion for $O(\alpha^{-1}) < R \leq O(\alpha^{-5})$ and new types of equation are derived for the disturbed free surface. Existence of the steady solutions to the above new equations is examined but it is found that the effect of viscosity always damps the wave energy and there exist no steady solutions including shock-like solution. On the other hand, for $|\alpha R C| \ll 1$ and $\alpha \ll 1$, the complex phase velocity becomes purely imaginary and there exists no wave motion. A modified reductive perturbation method leads to a nonlinear diffusion equation for the free surface, which was first obtained by Nakaya.[6]

Recently, Zabusky and Galvin[7] made laboratory experiments on gravity waves and compared their results with the solutions of the Korteweg-de Vries (K-dV) equation. They concluded that the number of emergent solitary waves and their phases based on the K-dV equation agree quantitatively with those obtained experimentally. However, the amplitude disagrees somewhat, and they supposed that this might be due to the viscous dissipation. Therefore, the next half is devoted to the study of the effect of viscosity in more detail, confining our investigations to the case of weak viscosity ($|\alpha R C| \gg 1$).

Our aim is to answer the following questions:

(1) Can the modified Korteweg-de Vries equation obtained in

the first half explain the various properties of gravity waves obtained experimentally by Zabusky and Galvin?

(2) Can we clarify the mechanism of wave motions, in which there are three competing effects, i.e., nonlinearity, geometrical dispersion and viscous one?

For this purpose, we solve initial value problems for two cases; one is sinusoidal wave and the other solitary wave in their initial wave forms. The former is the most realizable one in experiments so that this enables us partially to attain our aim (1) mentioned above. The latter belongs to simpler and more fundamental one (from an analytical point of view). Furthermore, expanding the displacement of the free surface elevation into Fourier series, we investigate the three competing effects upon the wave motions in detail. Numerical algorithm used in these computations is summarized in Appendix C.

§2. Basic Equations

The continuity equation and the Navier-Stokes equation for two-dimensional flow of an incompressible viscous fluid under the action of gravity may be written in nondimensional form as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - 1 + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2.3}$$

where (u , v) are, respectively, the velocity components along (x , y) directions. The Cartesian coordinate x is measured horizontally along the bottom and y vertically upward, and t is the time. We have used the undisturbed depth H as the characteristic length and the wave velocity (gH)$^{\frac{1}{2}}$ in the inviscid shallow water limit as the characteristic speed, where g is the acceleration due to gravity. The pressure p has been normalized by $\rho g H$, ρ being the density of the fluid, and the Reynolds number R is defined as $R = (gH)^{\frac{1}{2}} H / \nu$, in which ν is the kinematic viscosity. Since we are concerned with long waves, we neglect the effect of surface tension.

The boundary conditions relevant to the present problem are

$$u = v = 0 \quad \text{at} \quad y = 0 , \tag{2.4}$$

$$v = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} \qquad \text{at} \quad y = h(x,t), \tag{2.5}$$

$$\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left[1 - \left(\frac{\partial h}{\partial x}\right)^2\right] + 2\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\frac{\partial h}{\partial x} = 0 \quad \text{at}\ y = h(x,t), \tag{2.6}$$

$$(p_A - p)\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right] + \frac{2}{R}\left[\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{\partial u}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial h}{\partial x}\right] = 0$$

$$\text{at} \quad y = h(x,t), \tag{2.7}$$

where $y = h(x,t)$ represents the disturbed free surface. The conditions (2.4) at the bottom are obvious, while eq.(2.5) is the kinematical condition at the free surface. The last two conditions represent the stress continuity at the free surface, i.e., eq.(2.6) and (2.7) imply, respectively, that the tangential stress at the free surface is equal to zero and that the normal stress at the free surface is equal to the atmospheric pressure p_A .

The undisturbed steady state is given by

$$\left.\begin{aligned} & u = 0, \quad v = 0, \quad h = 1, \\ & p = 1 - y + p_A . \end{aligned}\right\} \tag{2.8}$$

§3. Linear Theory

Let us first consider the effect of viscosity on gravity waves of infinitesimal amplitude. Substituting

$$\left.\begin{aligned} &u = u', \quad v = v', \quad h = 1 + h', \\ &p = 1 - y + p_A + p', \end{aligned}\right\} \tag{3.1}$$

into eqs.(2.1)-(2.3) and the boundary conditions (2.4)-(2.7) and linearizing them with respect to the small quantities with the prime, we obtain the linear equation for v' :

$$\frac{\partial^3 v'}{\partial t \partial x^2} + \frac{\partial^3 v'}{\partial t \partial y^2} = \frac{1}{R}\left(\frac{\partial^4 v'}{\partial x^4} + 2\frac{\partial^4 v'}{\partial x^2 \partial y^2} + \frac{\partial^4 v'}{\partial y^4}\right), \tag{3.2}$$

together with the boundary conditions:

$$\left.\begin{aligned} &v' = \frac{\partial v'}{\partial y} = 0 \qquad \text{at} \quad y = 0, \\ &\frac{\partial^2 v'}{\partial x^2} - \frac{\partial^2 v'}{\partial y^2} = 0 \qquad \text{at} \quad y = 1, \\ &\frac{\partial^4 v'}{\partial t \partial y^3} + 3\frac{\partial^4 v'}{\partial t \partial y \partial x^2} - R\left(\frac{\partial^3 v'}{\partial y \partial t^2} - \frac{\partial^2 v'}{\partial x^2}\right) = 0 \\ &\hspace{18em} \text{at } y = 1, \end{aligned}\right\} \tag{3.3}$$

after elimination of u', h', and p'.

As this equation is linear and does not contain the coordinate x explicitly, the function $v'(x, y, t)$ can be expanded into a Fourier series with respect to the coordinate, and the equation can be written for every Fourier component separately. This means that it is sufficient in this problem to consider solutions of the form

$$v = \hat{v}(y)\exp[i\alpha(x-ct)], \tag{3.4}$$

where α is the wavenumber (assumed to be real positive) and c the phase velocity. Substituting this expression into eq.(3.2), we have

$$\hat{v}^{(IV)} - \alpha(2\alpha - iRc)\hat{v}'' + \alpha^3(\alpha - iRc)\hat{v} = 0, \tag{3.5}$$

in which the prime denotes the differentiation with respect to y. The general solution of this equation is given by

$$\hat{v}(y) = A\cosh\alpha y + B\sinh\alpha y + C\cosh\beta y + D\sinh\beta y, \tag{3.6}$$

where $\beta^2 = \alpha(\alpha - iRc)$ and A, B, C, and D are arbitrary constants. This solution must satisfy the boundary conditions:

$$\hat{v}(0) = \hat{v}'(0) = 0, \tag{3.7}$$

$$\alpha^2\hat{v}(1) + \hat{v}''(1) = 0, \tag{3.8}$$

$$-i\alpha c\hat{v}'''(1) + (3i\alpha^3 c + \alpha^2 c^2 R)\hat{v}'(1) - \alpha^2 R\hat{v}(1) = 0. \tag{3.9}$$

These conditions are homogeneous, so that we have the dispersion relation:

$$\begin{aligned} &4\alpha^2\beta(\alpha^2+\beta^2) + 4\alpha^3\beta(\beta\sinh\alpha\sinh\beta - \alpha\cosh\alpha\cosh\beta) \\ &- (\alpha^2+\beta^2)^2(\beta\cosh\alpha\cosh\beta - \alpha\sinh\alpha\sinh\beta) \\ &- \alpha R^2(\beta\sinh\alpha\cosh\beta - \alpha\sinh\beta\cosh\alpha) = 0, \end{aligned} \tag{3.10}$$

requiring that the complex amplitude $\tilde{v}(y)$ should be non-trivial.

In order to clarify how the viscosity affects the complex phase velocity, we try to express C as an explicit function of α and R. In general this procedure may be very difficult, but for the following two cases we can obtain approximate expresssons; one is the case (a) in which $|\alpha R C| \gg 1$ and the other the case (b) where $|\alpha R C| \ll 1$*.

We first consider the case (a) in which we may set $\beta^3/\cosh\beta \approx 0$ and $\tanh\beta \approx 1$, where we have chosen the branch of β satisfying $\mathcal{R}e(\beta) > 0$. Assuming then that α is of order of unity and R is much larger than unity, we have

$$C = (\tanh\alpha/\alpha)^{\frac{1}{2}} + \exp(5\pi i/4)\alpha^{\frac{1}{4}}R^{-\frac{1}{2}}/(2\cosh^{\frac{5}{4}}\alpha\,\sinh^{\frac{3}{4}}\alpha) + O(R^{-1}), \quad (3.11)$$

where we have discarded the waves propagating to the negative x-direction. The first term in this expression coincides exactly with the phase velocity obtained in the inviscid theory[1]. Therefore we may call the dispersion due to the first term 'geometrical dispersion'. The second term is the lowest order viscous correction whose real part represents the pure dispersion while imaginary part represents the dissipation.

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* It should be noted that α may take any order of magnitude for the case (a) but for the case (b) α must be much smaller than unity in addition to $|\alpha R C| \ll 1$. Since, however, we are concerned with long waves, it is sufficient to be able to obtain approximate expressions of C for $\alpha \ll 1$ in both the cases (a) and (b).

Thus the effect of viscosity is not only dissipative but also dispersive. In this sense we may call the complex dispersion due to the second term 'viscous dispersion'. It is easily seen that eq.(3.11) is still valid even for values of smaller than unity so far as $\alpha R \gg 1$.* When α becomes much smaller than unity, the first term in eq.(3.11) varies as $1-\alpha^2/6$, while the second term as $\exp(5\pi i/4)(\alpha R)^{-\frac{1}{2}}/2$. Therefore the two different types of dispersion may balance with each other when $\alpha = O(R^{-\frac{1}{5}})$. For values of α smaller than this the viscous dispersion dominates over the geometrical one.

Next we consider the case (b). As was already remarked, we assume $\alpha \ll 1$ in addition to $|\alpha R c| \ll 1$. Then the hyperbolic functions of α or β contained in eq. (3.10) may be expanded into Taylor series of α or β. Assuming R to be of order unity and arranging eq.(3.10) in powers of α, we obtain

$$c = -\frac{i\alpha R}{3} + \frac{3i\alpha^3 R}{5}\left(1-\frac{2}{27}R^2\right) + O(\alpha^5), \tag{3.12}$$

which represents, to this order of approximation, the viscous dissipation only. It may easily be verified that the above

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* On the other hand, for $\alpha \gg 1$, the dispersion relation is also derived from eq.(3.10) and becomes

$$c = \alpha^{-\frac{1}{2}} - 2i\alpha R^{-1}.$$

This was found by Stokes for an infinitely deep water (see Lamb[1]).

relation (3.12) is valid not only for $R = O(1)$ but also for $R < O(\alpha^{-1})$. Thus in such a relatively low Reynolds number (highly viscous) case, the disturbance cannot propagate as a wave but may be diffused out.

It is useful to summarize the above results for long waves ($\alpha << 1$) and classify the dispersion relation according to the order of magnitude of the Reynolds number.

Case (a): high Reynolds number case

(i) geometrical dispersion dominant

$$c = 1 - \frac{\alpha^2}{6} + \ldots \qquad \text{for} \quad O(\alpha^{-5}) < R, \tag{3.13a}$$

(ii) balance between geometrical and viscous dispersions

$$c = 1 - \frac{\alpha^2}{6} + \frac{1}{2}\exp(5\pi i/4)(\alpha R)^{-\frac{1}{2}} + \ldots \quad \text{for} \quad R = O(\alpha^{-5}), \tag{3.13b}$$

(iii) viscous dispersion dominant

$$c = 1 + \frac{1}{2}\exp(5\pi i/4)(\alpha R)^{-\frac{1}{2}} + \ldots \text{for } O(\alpha^{-1}) < R < O(\alpha^{-5}), \tag{3.13c}$$

Case (b): low Reynolds number case, viscous dissipation dominant

$$c = -\frac{1}{3} i\alpha R + \ldots \qquad \text{for} \quad R < O(\alpha^{-1}), \tag{3.13d}$$

where the dots stand for the higher order quantities than the preceding expressions. In the region intermediate between the cases (a) and (b), i.e., when $R = O(\alpha^{-1})$, we cannot find any series expansion such as (3.13a)-(3.13d). For a water layer, the above classification is given in Appendix A

§4. Nonlinear Theory

On the basis of the linear dispersion relation obtained in the preceding section, we now consider the effect of viscosity on long gravity waves with small but finite amplitude. Our main purpose is to derive a simple equation for the surface elevation under the assumption of weak nonlinearity and long wavelength.

According to the general theory of an oscillating boundary layer, the thickness δ of a boundary layer near the bottom may be evaluated as $|\alpha R c|^{-\frac{1}{2}}$. This fact suggests us an application of the boundary layer approximation for the case (a) in which $|\alpha R c| \gg 1$. On the other hand, for the case (b), where $|\alpha R c| \ll 1$, the effect of viscosity prevails in the whole flow field. Hence we divide our problem into cases (a) and (b).

4.1 Case (a): high Reynolds number case ($|\alpha R c| \gg 1$)

It is expected in general that the effect of viscosity is dominant in two boundary layers close to the free surface and to the bottom. According to Longuet-Higgins[8], however, the ratio of the energy dissipation near the free surface to that near the bottom is of order $|c\alpha^2/R|$. Therefore the boundary layer near the free surface may be negligible in the present long wave approximation ($\alpha \ll 1$) for high Reynolds number ($R \gg 1$). In fact, as will be seen later, the boundary conditions at the free surface (2.5) -(2.7) may be replaced by the inviscid ones so far as the present order of approximation is concerned.

We shall therefore divide the flow field into two regions; one is outer region beyond the boundary layer and the other inner region near the bottom. We then match the outer solution in the limit $y \to 0$ to the inner one in the limit $\eta \to \infty$, where η is the stretched inner variable defined as $\eta = y/\delta$. Such a procedure is familiar in the conventional matched asymptotic method[9,10] and may yield a uniformly valid solution in the whole flow field.

Let us first consider the outer problem. As a typical example, we shall here take up the case (aii) in which $\chi = O(R^{-\frac{1}{5}})$. The other two cases (ai) and (aii) may be dealt with in a similar manner if we notice the relative order of magnitude of χ and R .

As is known in the inviscid theory[11], the dispersion relation (3.13b) suggests us the following coordinate-transformation:

$$\xi = \epsilon^{\frac{1}{2}}(x - t), \quad \tau = \epsilon^{\frac{3}{2}} t, \quad y = y . \tag{4.1}$$

where ϵ is a small parameter measuring the weakness of the dispersion. This means that we have assumed $\epsilon = O(\chi^2) = O(R^{-\frac{2}{5}})$. In terms of (ξ, y, τ), the basic system of equations (2.1)-(2.3) may be written as

$$\frac{\partial u^{(o)}}{\partial \xi} + \frac{\partial v'^{(o)}}{\partial y} = 0, \tag{4.2}$$

$$\epsilon \frac{\partial u^{(o)}}{\partial \tau} + (u^{(o)}-1)\frac{\partial u^{(o)}}{\partial \xi} + v'^{(o)}\frac{\partial u^{(o)}}{\partial y} = -\frac{\partial p^{(o)}}{\partial \xi} + \frac{\epsilon^2}{R^*}\left(\epsilon \frac{\partial^2 u^{(o)}}{\partial \xi^2} + \frac{\partial^2 u^{(o)}}{\partial y^2}\right), \tag{4.3}$$

$$\epsilon \frac{\partial v'^{(o)}}{\partial \tau} + \epsilon(u^{(o)}-1)\frac{\partial v'^{(o)}}{\partial \xi} + \epsilon v'^{(o)}\frac{\partial v'^{(o)}}{\partial y} = -\frac{\partial p^{(o)}}{\partial y} - 1 + \frac{\epsilon^3}{R^*}\left(\epsilon \frac{\partial^2 v'^{(o)}}{\partial \xi^2} + \frac{\partial^2 v'^{(o)}}{\partial y^2}\right), \tag{4.4}$$

where we have set $R = \epsilon^{-\frac{5}{2}} R^*$ and $v^{(o)} = \epsilon^{\frac{1}{2}} v'^{(o)}$, the latter is the same transformation as that used in the inviscid theory.[11] We have used superscript (o) to specify the outer quantities. The boundary conditions at the free surface (2.5)-(2.7) may take the form:

$$v'^{(o)} = -\frac{\partial h}{\partial \xi} + \epsilon \frac{\partial h}{\partial \tau} + u^{(o)}\frac{\partial h}{\partial \xi}, \tag{4.5}$$

$$\left(\frac{\partial u^{(o)}}{\partial y} + \epsilon \frac{\partial v'^{(o)}}{\partial \xi}\right)\left[1 - \epsilon\left(\frac{\partial h}{\partial \xi}\right)^2\right] + 2\epsilon\left(\frac{\partial v'^{(o)}}{\partial y} - \frac{\partial u^{(o)}}{\partial \xi}\right)\frac{\partial h}{\partial \xi} = 0, \tag{4.6}$$

$$(p_A - p^{(o)})\left[1 + \epsilon\left(\frac{\partial h}{\partial \xi}\right)^2\right] + \frac{2\epsilon^3}{R^*}\left[\frac{\partial v'^{(o)}}{\partial y} + \epsilon \frac{\partial u^{(o)}}{\partial \xi}\left(\frac{\partial h}{\partial \xi}\right)^2 - \frac{\partial u^{(o)}}{\partial y}\frac{\partial h}{\partial \xi} - \epsilon \frac{\partial v'^{(o)}}{\partial \xi}\frac{\partial h}{\partial \xi}\right] = 0, \tag{4.7}$$

at $y = h(\xi, \tau)$. The matching conditions may be expressed as

$$\lim_{y \to 0} \begin{pmatrix} u^{(o)}(\xi, y, \tau) \\ v^{(o)}(\xi, y, \tau) \\ p^{(o)}(\xi, y, \tau) \end{pmatrix} = \lim_{\eta \to \infty} \begin{pmatrix} u^{(i)}(\xi, \eta, \tau) \\ v^{(i)}(\xi, \eta, \tau) \\ p^{(i)}(\xi, \eta, \tau) \end{pmatrix}, \tag{4.8}$$

where the superscript (i) denotes the inner quantities.

Since we consider weakly nonlinear waves, we expand $u^{(o)}$, $v^{(o)}$, $p^{(o)}$, and h in powers of ϵ :

$$u^{(o)}(\xi, y, \tau) = \sum_{n=1}^{\infty} \epsilon^{n} u_n^{(o)}(\xi, y, \tau), \tag{4.9}$$

$$v^{(o)}(\xi, y, \tau) = \sum_{n=1}^{\infty} \epsilon^{n} v_n^{(o)}(\xi, y, \tau), \tag{4.10}$$

$$p^{(o)}(\xi, y, \tau) = 1 - y + p_A + \sum_{n=1}^{\infty} \epsilon^{n} p_n^{(o)}(\xi, y, \tau), \tag{4.11}$$

$$h(\xi, \tau) = 1 + \sum_{n=1}^{\infty} \epsilon^{n} h_n(\xi, \tau). \tag{4.12}$$

Hence the small parameter ϵ may also be regarded as a measure of weakness of nonlinearity, which means that the Ursell parameter[12] has been chosen to be of order of unity.

Substituting the above expressions (4.9)-(4.12) into eqs.(4.2)-(4.4) and the boundary conditions (4.5)-(4.7), and arranging them in powers of ϵ , we have a sequence of equations and the boundary conditions for each order of ϵ .

From the first order of eq.(4.4) together with the first order of the boundary condition (4.7), we obtain

$$p_1^{(o)} = h_1(\xi, \tau). \tag{4.13}$$

The first order of eq.(4.3) gives

$$\frac{\partial u_1^{(o)}}{\partial \xi} = \frac{\partial p_1^{(o)}}{\partial \xi}, \tag{4.14}$$

which together with (4.13) leads to the solution

$$u_1^{(0)} = h_1(\xi,\tau) + a(y,\tau) . \tag{4.15}$$

where $a(y,\tau)$ is an arbitrary function of (y,τ) resulting from the integration. Since we assume that there is no surface elevation when there is no flow, we may set $a(y,\tau)$ equal to zero. Mathematically speaking, this is valid if the condition $u_1^{(0)} = h_1$ holds somewhere in the ξ space. This was also postulated by Gardner and Morikawa in their inviscid theory.[11] From the first order of eq.(4.2) together with the first order of the kinematical condition (4.5). we obtain

$$v_1^{(0)} = -\frac{\partial h_1}{\partial \xi} y . \tag{4.16}$$

The second order of eq.(4.4) and the second order of the boundary condition (4.7) give the second order pressure as

$$p_2^{(0)} = \frac{1}{2}\frac{\partial^2 h_1}{\partial \xi^2}(1-y^2) + h_2 . \tag{4.17}$$

Eliminating $u_2^{(0)}$ from the second order of eqs.(4.2) and (4.3), and substituting the expressions $u_1^{(0)}$, $v_1^{(0)}$, and $p_2^{(0)}$ obtained above, we have

$$v_2^{(0)} = -\left(\frac{\partial h_1}{\partial \tau} + h_1\frac{\partial h_1}{\partial \xi}\right)y + \frac{1}{2}\frac{\partial^3 h_1}{\partial \xi^3}\left(\frac{y^3}{3} - y\right) - \frac{\partial h_2}{\partial \xi}y + b(\xi,\tau), \tag{4.18}$$

where $b(\xi,\tau)$ is an arbitrary function of (ξ,τ) resulting from the integration with respect to y. Setting y equal to unity and substituting $v_2^{(o)}(\xi, 1, \tau)$ into the second order of the kinematical condition (4.5), we have

$$\frac{\partial h_1}{\partial \tau} + \frac{3}{2} h_1 \frac{\partial h_1}{\partial \xi} + \frac{1}{6} \frac{\partial^3 h_1}{\partial \xi^3} = \frac{1}{2} b(\xi, \tau) . \tag{4.19}$$

The arbitrary function $b(\xi, \tau)$ will be determined by the matching condition for v (see (4.8)) after the inner solution is obtained. It should be noted that, up to the present order of approximation, the viscous term in eqs.(4.3) and (4.4) have no important effect and that the condition (4.6) is automatically satisfied. The condition (4.7) is essentially equivalent to the inviscid one, i.e., $p_1 - p^{(i)} = 0$ at $y = h_1(\xi, \tau)$.

We now proceed to the inner problem. Since $\alpha = O(\epsilon^{\frac{1}{2}})$, $R = O(\epsilon^{-\frac{5}{2}})$, and $|C| = O(1)$, we have $\delta = O((\alpha R |C|)^{-\frac{1}{2}}) = O(\epsilon)$. Therefore, in this case, the thickness of the boundary layer is just of order ϵ. Following the conventional boundary layer theory, we introduce the coordinate-transformation

$$\xi = \epsilon^{\frac{1}{2}}(x - t), \quad \tau = \epsilon^{\frac{3}{2}} t, \quad \eta = y/\epsilon , \tag{4.20}$$

where ξ and τ are identical with those in the outer problem. On the other hand, the vertical component of the velocity should be transformed as $v^{(i)} = \epsilon(\epsilon^{\frac{1}{2}} V^{(i)})$, where the ϵ outside the parentheses corresponds to the usual stretching in the boundary layer approximation, while the $\epsilon^{\frac{1}{2}}$ in the parentheses plays the same role as that in the outer problem. Thus we have $v'^{(o)} = \epsilon v'^{(i)}$.

We can now rewrite the basic system of equations in terms of (ξ, η, τ):

$$\frac{\partial u^{(i)}}{\partial \xi} + \frac{\partial v'^{(i)}}{\partial \eta} = 0, \tag{4.21}$$

$$\epsilon \frac{\partial u^{(i)}}{\partial \tau} + (u^{(i)} - 1)\frac{\partial u^{(i)}}{\partial \xi} + v'^{(i)}\frac{\partial u^{(i)}}{\partial \eta} = -\frac{\partial p^{(i)}}{\partial \xi} + \frac{\epsilon^2}{R^*}\left(\epsilon \frac{\partial^2 u^{(i)}}{\partial \xi^2} + \frac{1}{\epsilon^2}\frac{\partial^2 u^{(i)}}{\partial \eta^2}\right), \tag{4.22}$$

$$\epsilon^3 \frac{\partial v'^{(i)}}{\partial \tau} + \epsilon^2 (u^{(i)} - 1)\frac{\partial v'^{(i)}}{\partial \xi} + \epsilon^2 v'^{(i)}\frac{\partial v'^{(i)}}{\partial \eta} = -\frac{1}{\epsilon}\frac{\partial p^{(i)}}{\partial \eta} - 1 + \frac{\epsilon^4}{R^*}\left(\epsilon \frac{\partial^2 v'^{(i)}}{\partial \xi^2} + \frac{1}{\epsilon^2}\frac{\partial^2 v'^{(i)}}{\partial \eta^2}\right). \tag{4.23}$$

Expanding the inner quantities in powers of ϵ as

$$u^{(i)}(\xi, \eta, \tau) = \sum_{n=1}^{\infty} \epsilon^n u_n^{(i)}(\xi, \eta, \tau), \tag{4.24}$$

$$\epsilon v'^{(i)}(\xi, \eta, \tau) = \sum_{n=1}^{\infty} \epsilon^n v_n^{(i)}(\xi, \eta, \tau), \tag{4.25}$$

$$p^{(i)}(\xi, \eta, \tau) = 1 - \epsilon\eta + p_A + \sum_{n=1}^{\infty} p_n^{(i)}(\xi, \eta, \tau), \tag{4.26}$$

and substituting them into eqs.(4.21)-(4.23), we have a sequence of equations to be solved. On the other hand, the boundary conditions at the bottom (2.4) should also be arranged in powers of ϵ. From the first order of eq.(4.21) and the boundary condition: $v_1^{(i)} = 0$ at $\eta = 0$, it follows that

$$\zeta_1^{(i)} = 0 . \tag{4.27}$$

The first order of eq.(4.23) together with the matching condition for p (see (4.8)) gives

$$p_1^{(i)} = p_1^{(o)} = h_1(\xi, \tau), \tag{4.28}$$

where eq.(4.13) has been used. Introducing $\zeta_1^{(i)}$ and $p_1^{(i)}$ thus obtained into the first order of eq.(4.22), we obtain

$$\frac{\partial^2 u_1^{(i)}}{\partial \eta^2} + R^* \frac{\partial u_1^{(i)}}{\partial \xi} = R^* \frac{\partial h_1}{\partial \xi} , \tag{4.29}$$

where $u_1^{(i)}$ must satisfy

$$\left.\begin{aligned} u_1^{(i)} &= 0 \qquad \text{at} \qquad \eta = 0 , \\ \lim_{\eta \to \infty} u_1^{(i)} &= \lim_{y \to 0} u_1^{(o)} = h_1(\xi, \tau) . \end{aligned}\right\} \tag{4.30}$$

The solution of eq.(4.29) subject to the conditions (4.30) is given in Appendix B. Inserting this solution into the second order of eq.(4.21) and integrating it with respect to η, we have the following expression (see Appendix B):

$$\lim_{\eta \to \infty} \zeta_2^{(i)} = \lim_{\eta \to \infty} \left(- \frac{\partial h_1}{\partial \xi} \eta \right) + \frac{1}{2(\pi R^*)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \, \frac{1 - \mathrm{sgn}(\xi - \xi')}{|\xi - \xi'|^{\frac{1}{2}}} \, d\xi' , \tag{4.31}$$

from which, using the matching condition for ζ (see (4.8)), we can determine the unknown function $b(\xi, \tau)$ as

$$b(\xi, \tau) = \frac{1}{2(\pi R^*)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{1 - sgn(\xi - \xi')}{|\xi - \xi'|^{\frac{1}{2}}} d\xi' .$$

Inserting this expression $b(\xi, \tau)$ into eq.(4.19), we finally obtain the desired equation for h_1 :

$$\frac{\partial h_1}{\partial \tau} + \frac{3}{2} h_1 \frac{\partial h_1}{\partial \xi} + \frac{1}{6} \frac{\partial^3 h_1}{\partial \xi^3} =$$

$$\frac{1}{4(\pi R^*)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{d\xi'}{|\xi - \xi'|^{\frac{1}{2}}} - \frac{1}{4(\pi R^*)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{sgn(\xi - \xi')}{|\xi - \xi'|^{\frac{1}{2}}} d\xi' . \tag{4.33}$$

The first and the second terms on the right-hand side of eq. (4.33) are, respectively, due to the purely dispersive and dissipative parts of the complex viscous dispersion. Ott and Sudan have also obtained formally* an equation similar to eq.(4.33) with the right-hand side as

$$-\alpha_i \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{sgn(\xi' - \xi)}{|\xi' - \xi|^{\frac{1}{2}}} d\xi' , \tag{4.34}$$

* The linear term in eq.(4.33) can be obtained by using the Fourier transform and the convolution theorem provided that we extend the linear dispersion relation to allow negative values of α . However the inclusion of the nonlinear term is rather arbitrary and a quite formal procedure.

with $\alpha_3 > 0$. If α_3 were negative or ξ and ξ' in the sgn function were interchanged, their equation is partially equivalent to our equation (4.33) in the sense that their equation takes into account the dissipative part only out of the complex viscous dispersion. Needless to say both the purely dispersive and dissipative parts are the same order of magnitude.

So far we have confined ourselves to the case (aii). It may be shown, however, that a similar analysis to that given above leads to

$$\frac{\partial h_1}{\partial \tau} + \frac{3}{2} h_1 \frac{\partial h_1}{\partial \xi} + \frac{1}{6} \frac{\partial^3 h_1}{\partial \xi^3} = 0 \qquad \text{for case (ai)}, \tag{4.35}$$

$$\frac{\partial h_1}{\partial \tau} + \frac{3}{2} h_1 \frac{\partial h_1}{\partial \xi} = \frac{1}{4(\pi R^*)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{1 - \mathrm{sgn}(\xi - \xi')}{|\xi - \xi'|} d\xi' \qquad \text{for case (aiii)}, \tag{4.36}$$

provided that we introduce proper coordinate-transformation and asymptotic expansions depending upon the relative order of magnitude of α and R. Hence ξ and τ in the above eqs.(4.35) and (4.36) are not necessarily identical with those defined in (4.1) or (4.20).

4.2 Case (b): low Reynolds number case ($|\alpha R c| \ll 1$)

Since the effect of viscosity prevails in the whole fluid layer, the concept of boundary layer cannot be applied to this case. By virtue of the dispersion relation (3.13d), it is natural to introduce the following new variables

$$\xi = \epsilon x, \quad \tau = \epsilon^2 t, \quad y = y, \tag{4.37}$$

where we have assumed that α is of order ϵ and R of order unity. In terms of the new variables (ξ, y, τ) we can rewrite eqs.(2.1)-(2.3) and the boundary conditions (2.4) -(2.7), which we call eqs.(2.1)'-(2.3)' and the boundary conditions (2.4)'-(2.7)' although we do not write down them explicitly.

We now expand the field quantities u, v, p, and h as power series of ϵ :

$$\left.\begin{aligned}
u(\xi, y, \tau) &= \sum_{n=0}^{\infty} \epsilon^n u_n(\xi, y, \tau), \\
v(\xi, y, \tau) &= \sum_{n=0}^{\infty} \epsilon^n v_n(\xi, y, \tau), \\
p(\xi, y, \tau) &= (1 - y + P_A) + \sum_{n=0}^{\infty} \epsilon^n P_n(\xi, y, \tau), \\
h(\xi, \tau) &= 1 + \sum_{n=0}^{\infty} \epsilon^n h_n(\xi, \tau).
\end{aligned}\right\} \tag{4.38}$$

The essential difference between the above expansions and those employed in the preceding sub-section is that we have taken the 'zeroth' order perturbation into account in the above expansions. This means that in such a highly viscous case as in the present case the weaker nonlinearity considered in the case (a) cannot balance the dominant viscous dissipation. A similar situation to this is also encountered for nonlinear Alfvén waves.[14] Introducing (4.38) into eqs.(2.1)'-(2.3)' and the boundary conditions (2.4)'-(2.7)' and remembering that any field quantity $Q(\xi, y, \tau)$ at the free surface can be expressed as

$$Q(\xi, y=h, \tau) = Q\Big|_{y=1+h_0} + \frac{\partial Q}{\partial y}\Big|_{y=1+h_0}\Big(\sum_{n=1}^{\infty}\epsilon^n h_n(\xi,\tau)\Big) + \dots, \tag{4.39}$$

we can obtain a sequence of equations and the boundary conditions for each order of ϵ .

Since the procedure of calculation is similar to those given in §4.1, we only summarize here the main results:

$$v_0 = v_1 = 0, \quad v_2 = \Big[\frac{1}{2}R(1+h_0)y^2 - \frac{1}{6}Ry^3\Big]\frac{\partial^2 h_0}{\partial \xi^2} + \frac{1}{2}Ry^2\Big(\frac{\partial h_0}{\partial \xi}\Big)^2,$$

$$u_0 = 0, \quad u_1 = R\frac{\partial h_0}{\partial \xi}\Big[\frac{1}{2}y^2 - y(1+h_0)\Big]. \tag{4.40}$$

$$p_0 = h_0,$$

and h_0 satisfies the following nonlinear diffusion equation:

$$\frac{\partial h_0}{\partial \tau} = \frac{R}{3}\frac{\partial}{\partial \xi}\Big[(1+h_0)^3\frac{\partial h_0}{\partial \xi}\Big]. \tag{4.41}$$

It may be shown that the same type of equation is also derived for $R < O(\alpha^{-1})$, if we introduce proper coordinate-transformation and asymptotic expansions depending upon the relative order of magnitude of α and R.

Quite recently the same equation as eq.(4.41) was obtained by Nakaya[6] together with its similarity solutions. It is easily seen that Nakaya's nondimensional space coordinate x^* and y^* defined in his eq.(2.16) are equivalent, respectively, to our ξ and y defined in (4.37) but his time t^* is equivalent to our $Rt/2$. It seems, therfore, quite surprising that the same equation is obtained for different time stages. It is shown, however, that there is some confusion in his analysis concerning the order of t^*. In order that his analysis is consistent, his t^* should be multiplied by ϵ^2 (or by α^2 in his notation).

§5. Some Simple Properties of Eqs.(4.33) and (4.36)

Let us now consider whether or not the resultant equations obtained in the preceding section have steady solutions. It is well-known that the Korteweg-de Vries equation (4.35) has the cnoidal and solitary wave solutions. In order to see the existence of steady solutions to eq.(4.33), we examine the time evolution of the wave energy.

Assuming that

$$h_1 \quad \text{and} \quad \frac{\partial h_1}{\partial \xi} \longrightarrow 0 \qquad \text{as } \xi \longrightarrow \pm\infty , \tag{5.1}$$

and multiplying eq.(4.33) by h_1 and integrating it with respect to ξ from $-\infty$ to ∞, we obtain

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} h_1^2 d\xi = \frac{1}{2(\pi R^*)^{\frac{1}{2}}}\int_{-\infty}^{\infty} h_1 \int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{1-\operatorname{sgn}(\xi-\xi')}{|\xi-\xi'|^{\frac{1}{2}}} d\xi' d\xi . \tag{5.2}$$

By virtue of the convolution theorem, we have

$$\int_{-\infty}^{\infty} \frac{\partial h_1}{\partial \xi'} \frac{1-\operatorname{sgn}(\xi-\xi')}{|\xi-\xi'|^{\frac{1}{2}}} d\xi' = -\int_{-\infty}^{\infty} \hat{h}_1(k,\tau)|2\pi k|^{\frac{1}{2}}(1-i\operatorname{sgn} k)e^{ik\xi}dk$$

where $\hat{h}_1(k,\tau)$ is the Fourier transform of $h_1(\xi,\tau)$. Hence eq.(5.2) can be rewritten as

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} h_1^{2} d\xi = -\frac{2^{\frac{3}{2}}\pi}{R^{*\frac{1}{2}}}\int_{0}^{\infty} |k|^{\frac{1}{2}}\hat{h}_1(k,\tau)\hat{h}_1(-k,\tau)dk, \tag{5.3}$$

where it should be noted that the contribution from the purely dispersive part does vanish in the above integration. Since $\hat{h}_1(-k;\tau)$ is the complex conjugate to $\hat{h}_1(k,\tau)$, the integrand on the right-hand side is positive definite (note that $h_1(\xi,\tau)$ is real). Thus the total energy decreases monotonically with time and there exists no steady solution to eq.(4.33) satisfying the condition (5.1). For example, following Ott-Sudan[13,15] we can easily show that the amplitude $\mathcal{S}$ of the initially given Korteweg-de Vries soliton:

$$\mathcal{S}_0 \operatorname{sech}^2[(\xi - \tfrac{1}{2}\mathcal{S}_0\tau)/(\tfrac{4}{3\mathcal{S}_0})^{\frac{1}{2}}] \tag{5.4}$$

decreases with time as

$$\mathcal{S}/\mathcal{S}_0 = (1 + m\tau^*)^{-4}, \tag{5.5}$$

where $\tau^* = R^{*-\frac{1}{2}}\tau$ and

$$m = -\frac{1}{8}\left(\frac{3\mathcal{S}_0}{4\pi^2}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sech}^2 z \operatorname{sech}^2 z' \sinh z' \, \frac{\operatorname{sgn}(z-z')}{|z-z'|^{\frac{1}{2}}}\, dz'\, dz$$

$$> 0.^{*} \tag{5.6}$$

In the above analysis we have used two time scales τ and τ^* and assumed that R^{*-1} is much smaller than unity but large enough compared with ϵ so as not to invalidate the asymptotic scheme developed in the preced-

---

* It is easily shown that this expression of m is proportional to $\int_{-\infty}^{\infty} |k|^{\frac{1}{2}} |\hat{g}(k)|^2 dk$ and that the constant of proportionality is positive, where $\hat{g}(k)$ is the Fourier transform of $g(z) \equiv \operatorname{sech}^2 z$.

ing section. Due to a trivial miscalculation by Ott-Sudan[13] the value of m given in (5.6) differs from their value by factor $-\frac{1}{2}$.

On the other hand, Pfirsch and Sudan[16] have obtained a necessary condition for the existence of shock-like solutions of the Korteweg-de Vries equation with dissipation. According to them the necessary condition is given by

$$\lim_{\alpha \to 0} \frac{\gamma(\alpha)}{|\alpha|} \longrightarrow 0 , \tag{5.7}$$

where $\gamma(\alpha)$ is the linear damping rate in α space, and is proportional to $\alpha^{\frac{1}{2}}$ in the present case (see (3.13b)). It is easily shown that the presence of the purely dispersive part of the viscous dispersion does not alter the Pfirsch-Sudan's criterion. Thus we may conclude that eq. (4.33) has no steady shock-like solution for which $\phi_1(\xi = -\infty) \neq \phi_1(\xi = \infty)$.

The same conclusions obtained for eq.(4.33) may also be applied to eq.(4.36).

§6. Numerical Integration of the Modified K-dV Equation

In the preceding section, we have analytically examined some simple properties of long gravity waves under the influence of viscosity. In this section, we investigate the effect of viscosity in more detail by solving the modified K-dV equation numerically. Our aim is to answer the following questions:

(1) Can the modified K-dV equation explain the various properties of gravity waves already known in experiments?

(2) Can we clarify the mechanism of the wave motions, in which there are three competing effects, i.e., nonlinearity, geometrical dispersion and viscous one?

For this purpose, we solve initial value problems for two cases; one is sinusoidal wave, and the other solitary wave in their initial wave forms. The former is the most realizable one in experiments so that this enables us partially to attain our aim (1) mentioned above. The latter belongs to simpler and more fundamental one (from an analytical point of view). Furthermore, expanding the displacement of the free surface into a Fourier series, we investigate the three competing effects upon the wave motions.

Numerical algorithm used in these computations is summarized in Appendix C.

6.1 Correspondence between experiments and the modified K-dV equation

Recently, Zabusky and Galvin[7] made laboratory experiments on gravity waves in a shallow-water tank driven by an oscillating piston which gives wave forms similar to a sinusoidal function. They compared these results with numerical solutions of the K-dV equation. Their work showed that at a downstream location, the number of crests and troughs and their phases agree fairly well with the numerical solutions, while the crest-to-trough amplitude disagrees somewhat.

Present sub-section is devoted to discussing how accurately the modified K-dV equation obtained in §4 can explain the results of the experiments made by Zabusky and Galvin. To do so, we compare the solutions of the K-dV equation with those of the modified K-dV equation taking account of the viscous dispersion, and also investigate the behaviour of the Fourier components.

Main parameters used in their experiments are summarized in Table I. In order to compare their experiments with the solutions of our modified K-dV equation, we rewrite the equations* in more convenient form (see ref.7) as

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \tilde{h}\frac{\partial \tilde{h}}{\partial \tilde{x}} + \alpha_1 \frac{\partial^3 \tilde{h}}{\partial \tilde{x}^3} = \alpha_2 \int_{\tilde{x}}^{\infty} \frac{\partial \tilde{h}}{\partial \tilde{x}'} \frac{d\tilde{x}'}{\sqrt{\tilde{x}' - \tilde{x}}} , \qquad (6.1)$$

where

$$\tilde{t} = \frac{t^*}{H/\sqrt{gH}} = \left(\frac{H}{A}\right)^{\frac{3}{2}} \frac{3A}{L}\, \tau ,$$

---

* It is easy to see that eqs.(4.33),(4.35) and (4.36) in §4 are unified into a single form represented by eq.(6.1).

$$\tilde{x} = \frac{x^* - \sqrt{gH}\, t^*}{L/2} = \left(\frac{H}{A}\right)^{\frac{1}{2}} \frac{2H}{L}\, \xi ,$$

$$\tilde{h} = \frac{h^*}{A} \doteq h_1 .$$

The asterisk indicates dimensional value of the variables. Two nondimensional parameters α_1 and α_2 appeared in the above equation represent, respectively, a measure of the geometrical and viscous dispersions relative to the non-linearity, and they are defined as

$$\alpha_1 = \frac{4}{9} U_r^{-1} = \frac{4}{9} \frac{H^3}{AL^2} , \quad \alpha_2 = \frac{1}{3} \left(\frac{\nu^2 L^2}{4\pi^2 A^4 g H} \right)^{\frac{1}{4}} , \qquad (6.2)$$

where U_r is known as the Ursell parameter.[12] In Table I, are listed their values corresponding to Zabusky-Galvin's experiments.[7]

Table I. Parameters of Zabusky-Galvin's experiments. The last parameter α_2 was calculated for pure water at 20°C.

Case	1	2	3
Depth H (ft)	0.493	0.242	0.242
Wavelength L (ft)	7.92	8.38	9.46
Amplitude A (ft)	0.0418	0.0192	0.0243
α_1	0.02025	0.004672	0.002896
α_2	0.01475	0.03945	0.03312

It is seen from Table I that the viscous dispersion dominates or at most balances with the geometrical one in magnitude of the parameters. As will be shown later, however, the ratio of these parameters itself is not so effective measure of the two dispersions as it appears, because the geometrical dispersion has a local effect while the viscous one global.

We solve an initial value problem to eq.(6.1) under the following initial condition:

$$h = \cos \pi x ,$$

where and hereafter we suppress the tilde for simplicity. We set here each parameter as follows:

Case A_1 : $\alpha_1 = 500/64^3 \doteqdot 0.00191, \quad \alpha_2 = 10\alpha_1,$

Case A_2 : $\alpha_1 = 500/64^3, \quad \alpha_2 = 1.0 \;(\doteqdot 500\alpha_1).$

The magnitude of the factors in Case A_1 is similar to that in Case 3 of Zabusky-Galvin's experiments. Main constants used and obtained in these computations are summarized in Appendix D.

6.1.1 <u>Wave forms for K-dV and modified K-dV equation</u>

Wave forms calculated numerically by using the method given in Appendix C are shown in Fig.1 for Case A_1 and also in Fig.2 for Case A_2.

We first compare the results of Case A_1 with those of

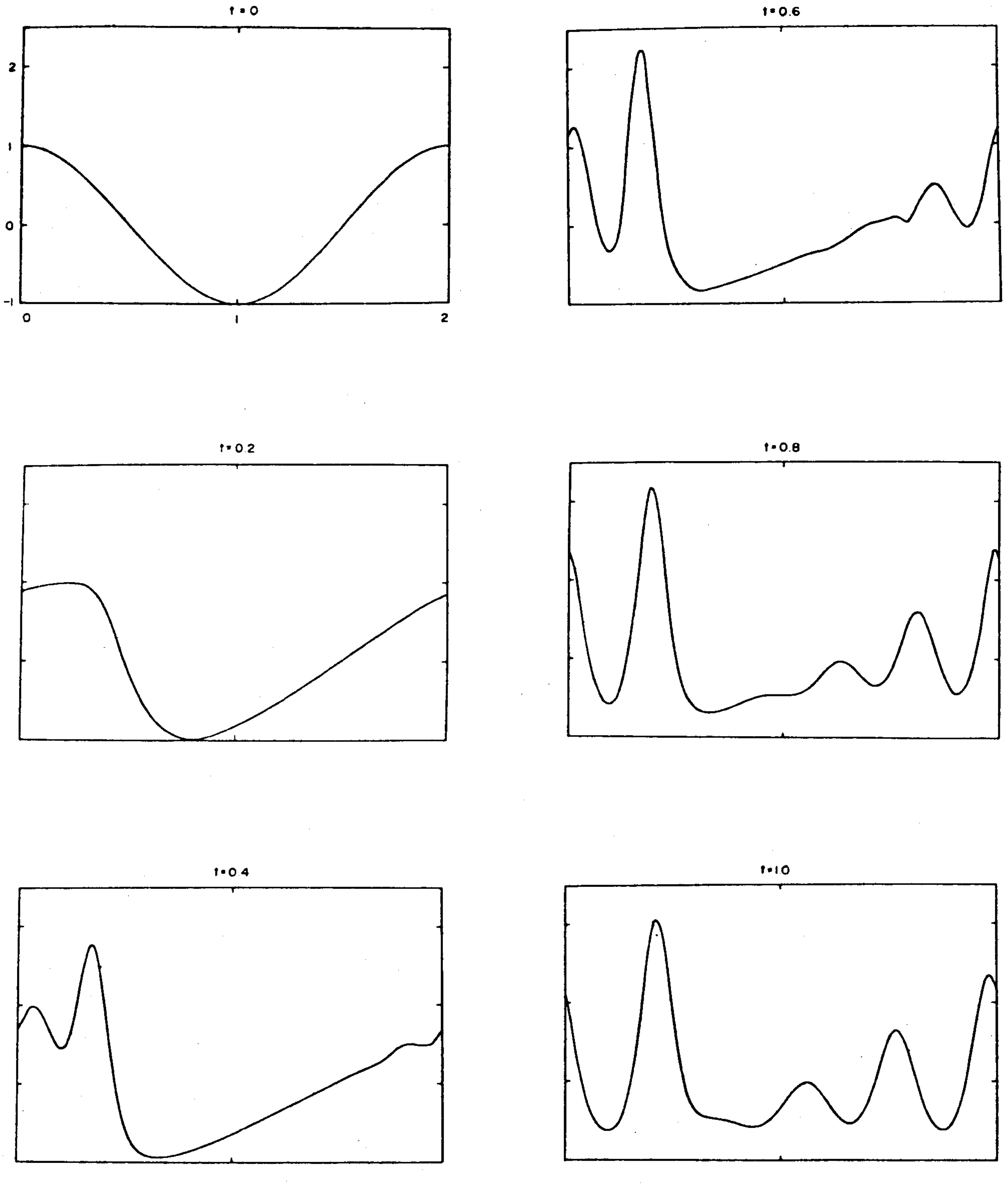

Fig.1. A series of the wave forms for Case A_1 .

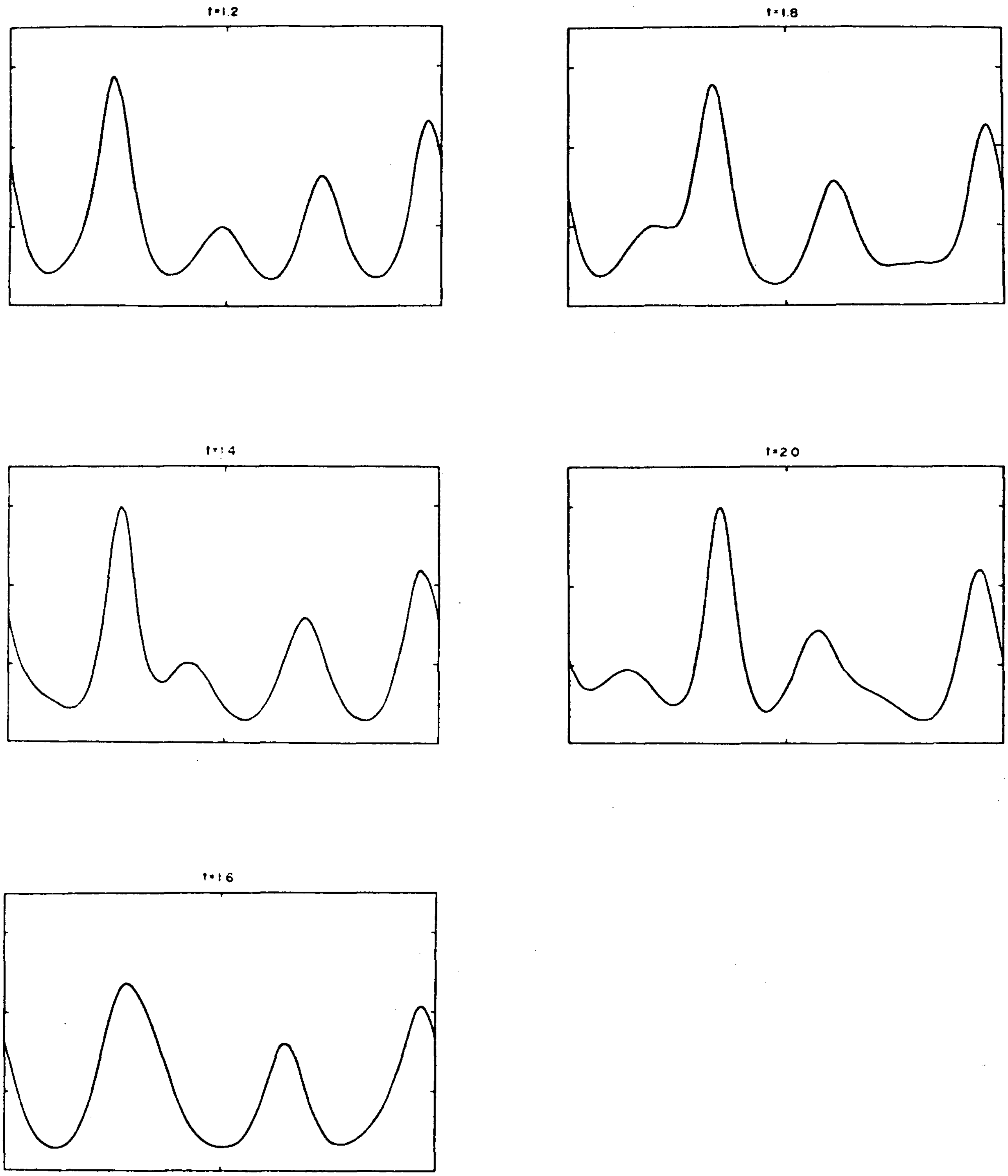

Fig.1. A series of the wave forms for Case A_1 (continued from the preceding page).

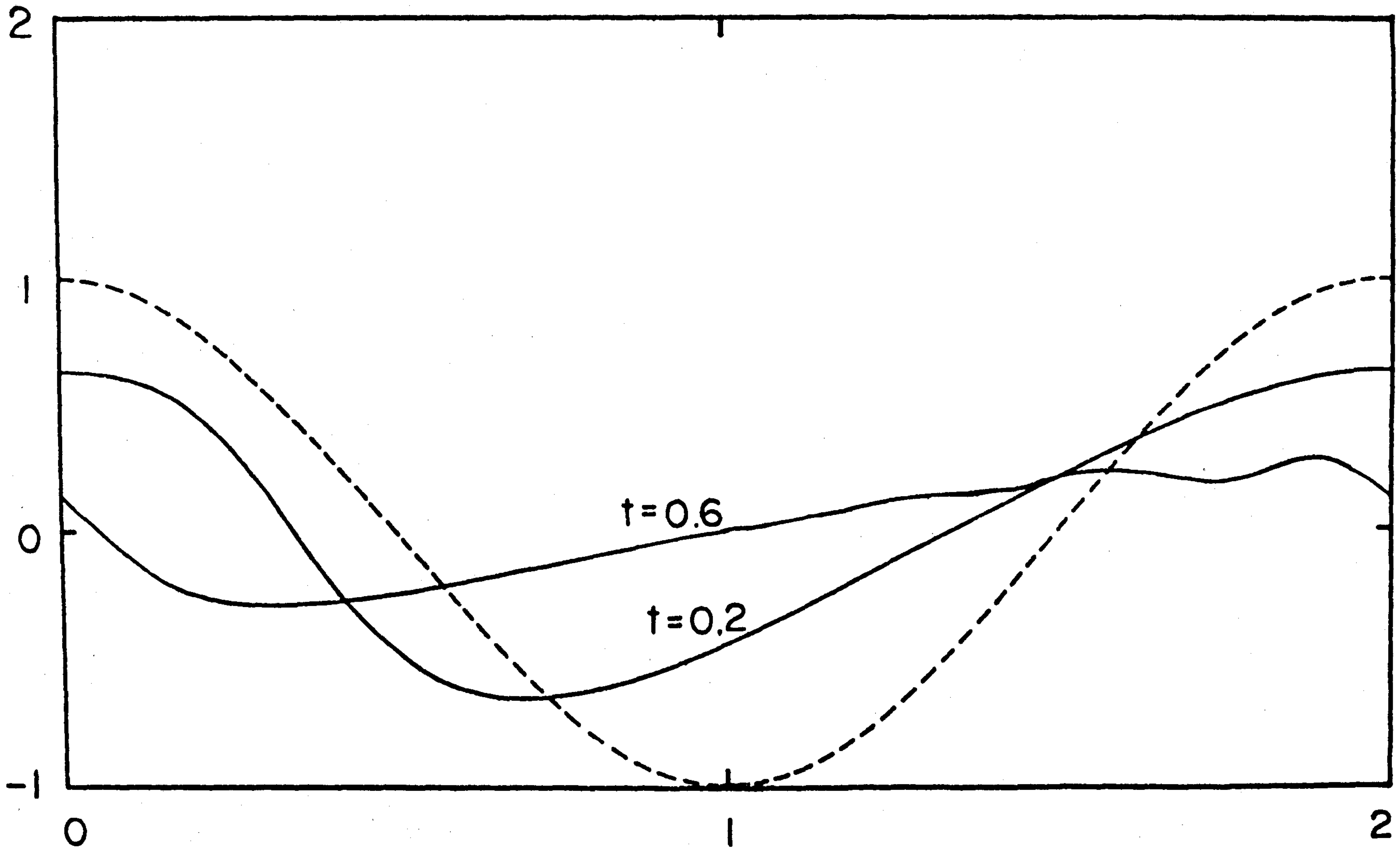

Fig.2. Wave forms for Case A_2 at two different times.

the K-dV equation. The differences between the wave forms of the K-dV equation and those of Case A_1 of the modified K-dV equation are shown in Figs. 3 and 4 at time t =1.0 and 2.0, respectively. From these figures, it is found that the two wave forms are not so different from each other under such a weak influence of viscosity. Nevertheless one can observe a slight damping of wave amplitude and a phase shift in the present Case A_1 , which are due to the viscous dispersion.

According to linear theory, real part of the viscous dispersion gives a phase shift if we assume the initial wave form remains unchanged. This phase shift $\delta \hat{c}$ is represented as (see eq.(3.13b) or (3.13c))

$$\delta \hat{c} = \gamma_2 \sqrt{\frac{\pi}{2}}\, \hat{\kappa}^{-\frac{1}{2}}\, t , \tag{6.3}$$

where $\hat{\kappa}$ is a wavenumber in the present frame of reference. Setting $\hat{\kappa}$ equal to π , we have

$$\delta c = 0.0134 , \qquad 0.0268 ,$$

at time t =1.0 and 2.0, respectively.

On the other hand, numerical calculations give the positions of crests, which are listed in Table II for Case A_1 and K-dV equation, from which it is concluded that the real phase shift calculated numerically is, in general, larger than that predicted by the linear theory and the larger crest gives the larger shift in the phase except for P_4 at t =1.0. The lat-

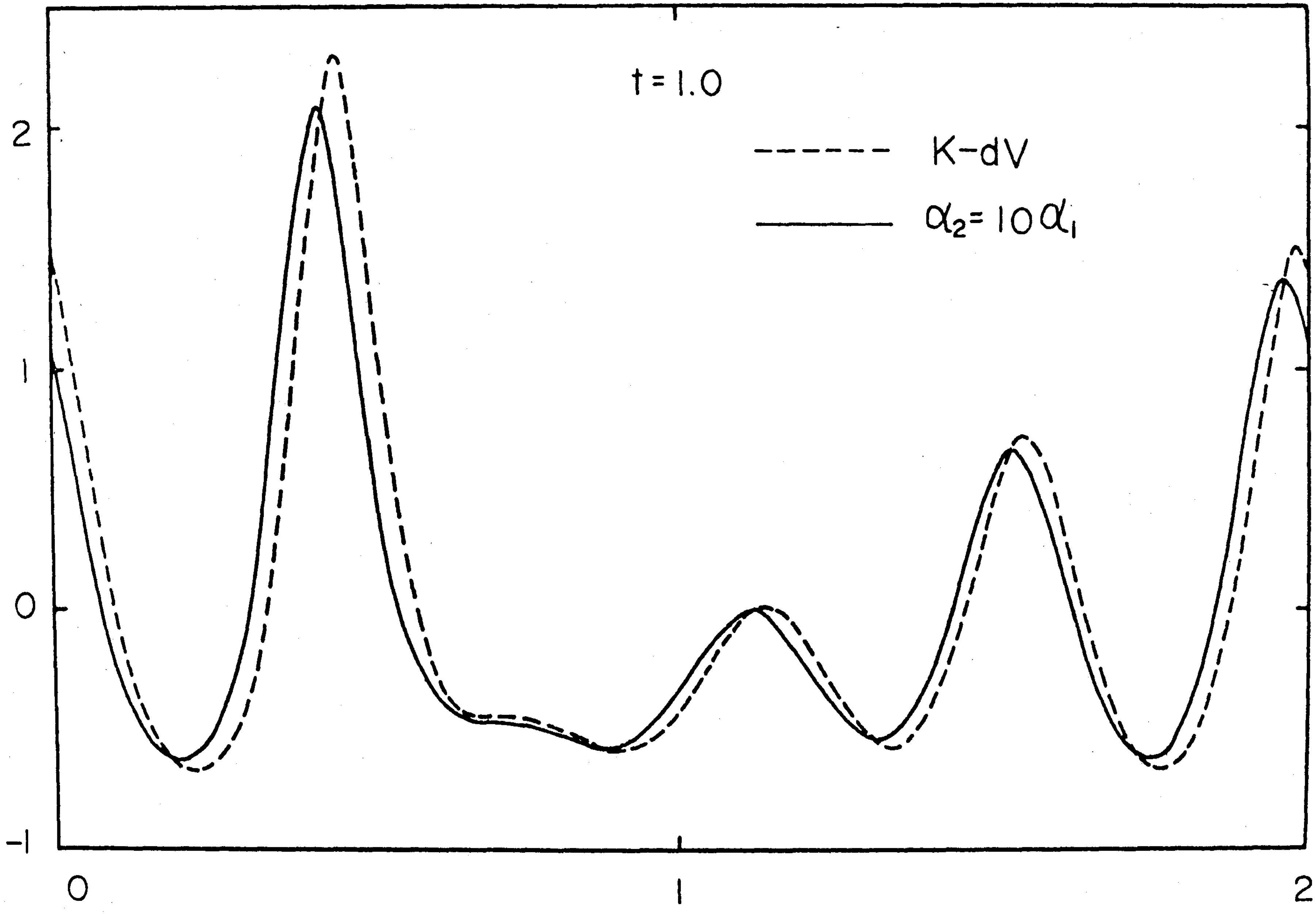

Fig.3. Wave forms at t =1.0 for Case A_1 and K-dV solution.

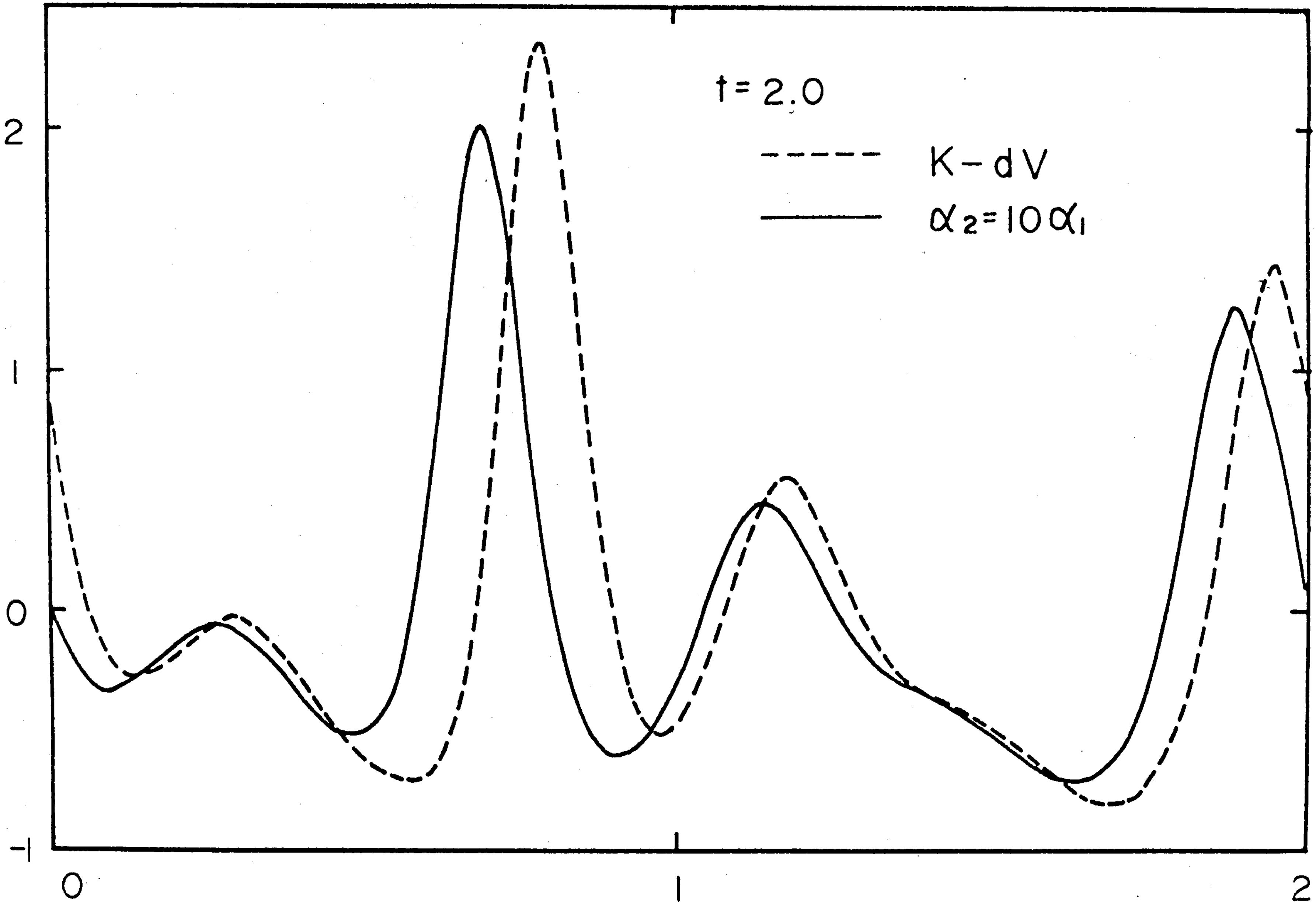

Fig.4. Wave forms at t =2.0 for Case A_1 and K-dV solution

ter can be explained by the fact that the amplitude dispersion* plays an important role and hence the damped solitary waves move with the slower speed. As already shown in Figs. 3 and 4 (see also Table III in the next paragraph), the waves with larger amplitude are damped more significantly.

Table II. Positions of crests for Case A_1 and K-dV equation. The subscripts indicate the number of the crests designated in order of magnitude. The n-th crest is shortly written as P_n .

	t =1.0			t =2.0		
	K-dV	eq.(6.1)	$\delta\theta$	K-dV	eq.(6.1)	$\delta\theta$
P_1	0.453	0.421	0.032	0.789	0.687	0.102
P_2	1.984	1.968	0.016	1.945	1.890	0.055
P_3	1.546	1.531	0.015	1.171	1.140	0.031
P_4	1.132	1.109	0.023	0.289	0.265	0.024

Next we consider Case A_2 , in which there exists more dominant contribution of the viscous dispersion. In Fig. 2 is shown the variation of wave forms at two different time stages. Peculiar properties of the K-dV equation such as soliton formation almost disappear in such a highly viscous case. Initial wave form steepens due to nonlinearity, which

* This means that the solitary waves propagate with their proper speeds proportional to their amplitude (see §6.2).

is similar to case A_1, but the maximum amplitude decreases so rapidly that the geometrical dispersion has no important effect and moreover its position cannot proceed to the positive x-direction.

6.1.2 Comparison of Zabusky-Galvin's experiments and the solution of the modified K-dV equation

Zabusky and Galvin[7] examined their experiments in detail for Case 3 and the corresponding K-dV solution. As pointed out before, their Case 3 is closely related to our Case A_1. Let us compare the wave forms calculated numerically for Case A_1 with those obtained experimentally by Zabusky and Galvin. For this purpose, the heights of solitary waves measured from mean water level are shown in Table III. It

Table III. Comparisons of heights of solitary waves: h^E:experiments, h^K :K-dV h^M :modified K-dV solution.

	h^E	h^K	h^K-h^E	h^M	h^K	h^K-h^M
P_1	1.88	2.10	0.22	2.21	2.36	0.15
P_2	0.792	0.873	0.081	1.35	1.43	0.08
P_3	0.067	0.097	0.030	0.54	0.57	0.03

should be, however, that this correspondence is somewhat incomplete, because the heights of the K-dV solution corresponding to the experiment are the values at t=0.675 while the heights of the modified K-dV solution and the corresponding K-dV solution are the values at t =0.680. Needless to say, there are slight differences between the two dispersion parameters in Case 3 and those in Case A_1, and also between their initial wave forms. Therefore we cannot compare the two results completely. Nevertheless it can be pointed out from Table III that there exists a similar tendency concerning the damping of the solitary waves, so that our modified K-dV equation can describe almost accurately the long wave motions under the influence of weak dissipation so far as the damping is concerned.

We now summarize briefly the results obtained above. The wave forms with weak viscosity (Case A_1) are not so different from those with no dissipation (K-dV), while the number of emergent solitary waves coincides exactly with each other. This agreement between the modified K-dV and the K-dV solutions corresponds accurately with that between experiments and the K-dV solution. Moreover, the damping of solitary waves obtained by numerical computations for Case A_1 is well compared with that found experimentally.

As a result, it may be concluded that our modified K-dV equation can describe the observed wave behaviours except the fact that the phase shift obtained by the calculations is not confirmed by their experiments. In this respect, it should be noted that the phase shift is always caused by the viscosity and is of the same order in magnitude as the viscous damping. It is therefore hoped that careful experiments concerning the phase shift will be carried out. Then the results may be compared quantitatively with our modified K-dV solution.

6.1.3 Temporal changes of energy spectrum

The temporal changes of the energy spectrum for Cases A_1, A_2 and K-dV solution are shown in Fig.5. The ordinate measures the components of the energy spectrum for the n-th Fourier component defined by

$$E_n = a_n^2 + b_n^2, \qquad (n=1,2,3,\ldots), \qquad (6.4)$$

where a_n and b_n satisfy the relation

$$h = \frac{a_0}{2} + \sum_{n=1} (a_n \cos n\pi x + b_n \sin n\pi x).$$

Both spectra for K-dV solution and for Case A_1 resemble each other in shape. On the other hand, for Case A_2, E_1 varies linearly with time in log-scale. The second component E_2 grows up to a certain equilibrium value, and then it decreases with time at a rate similar to that of E_1. This tendency applies to E_3 and the higher order components. On the other hand, we show the energy spectra including up to the 11-th component of several time stages in Fig.6 for Case A_1 and also in Fig.7 for Case A_2.

We now consider the changes of the wave energy:

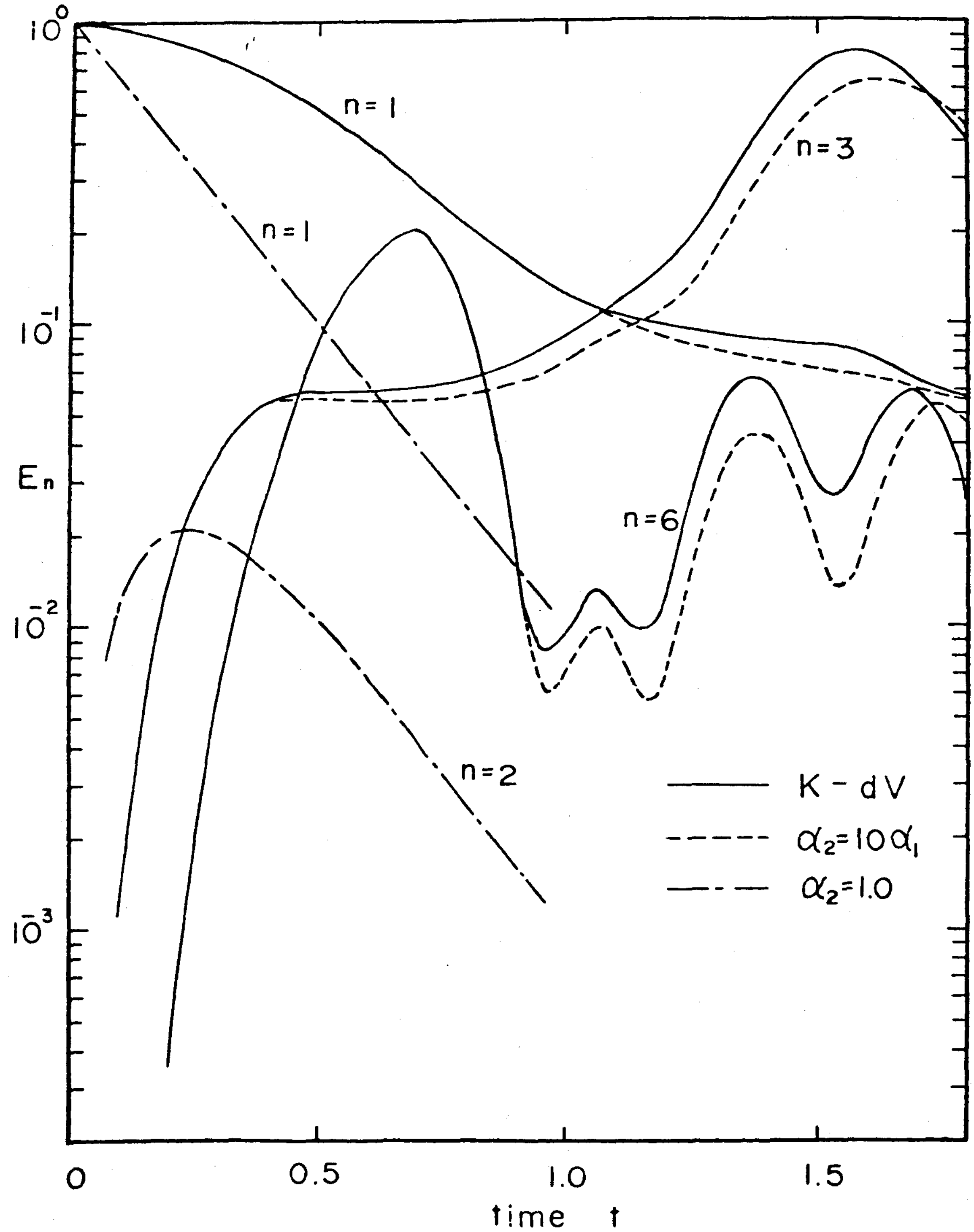

Fig.5. Temporal changes of energy spectrum for each Fourier component.

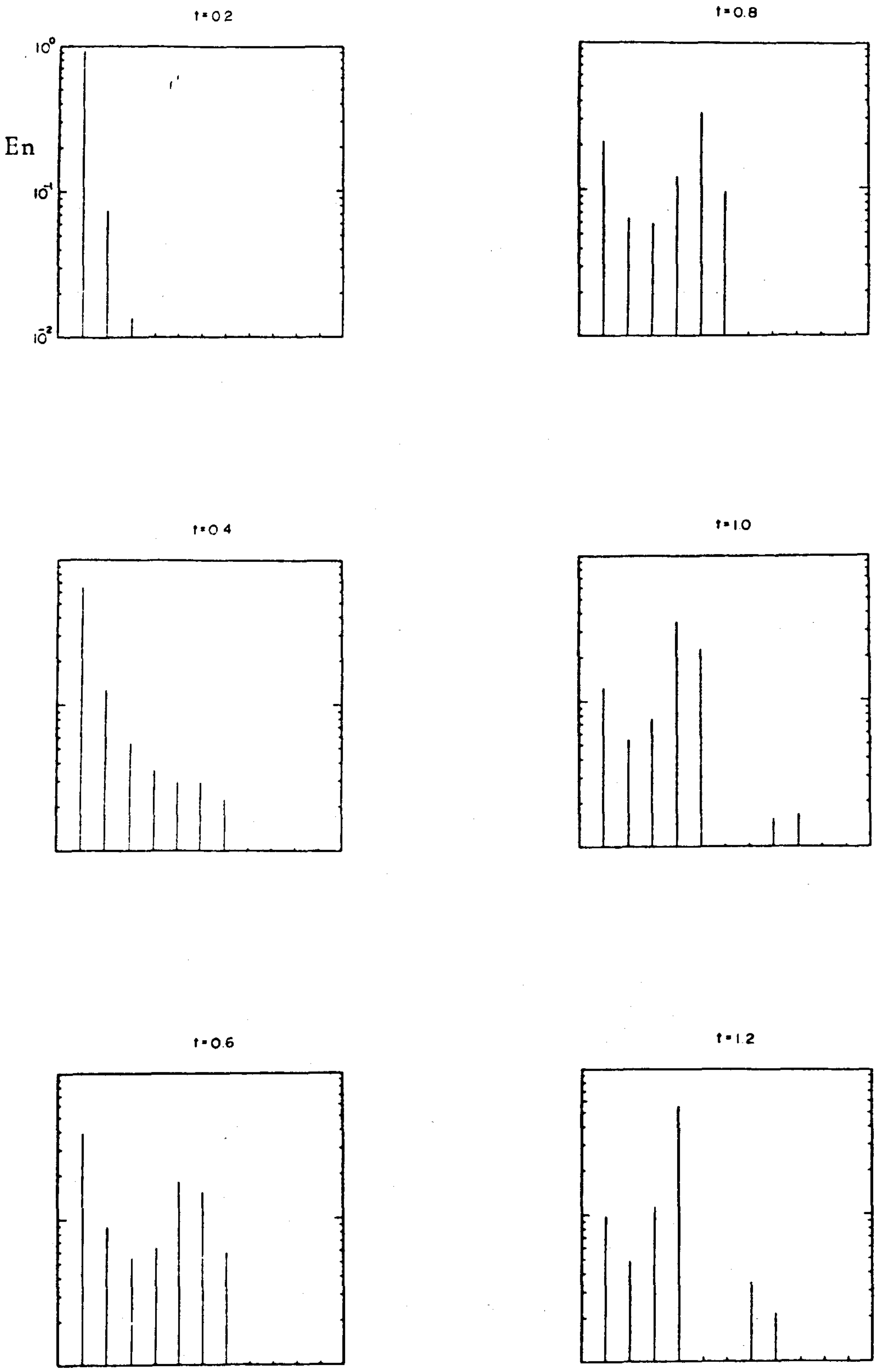

Fig.6. Energy spectra for Case A_1 .

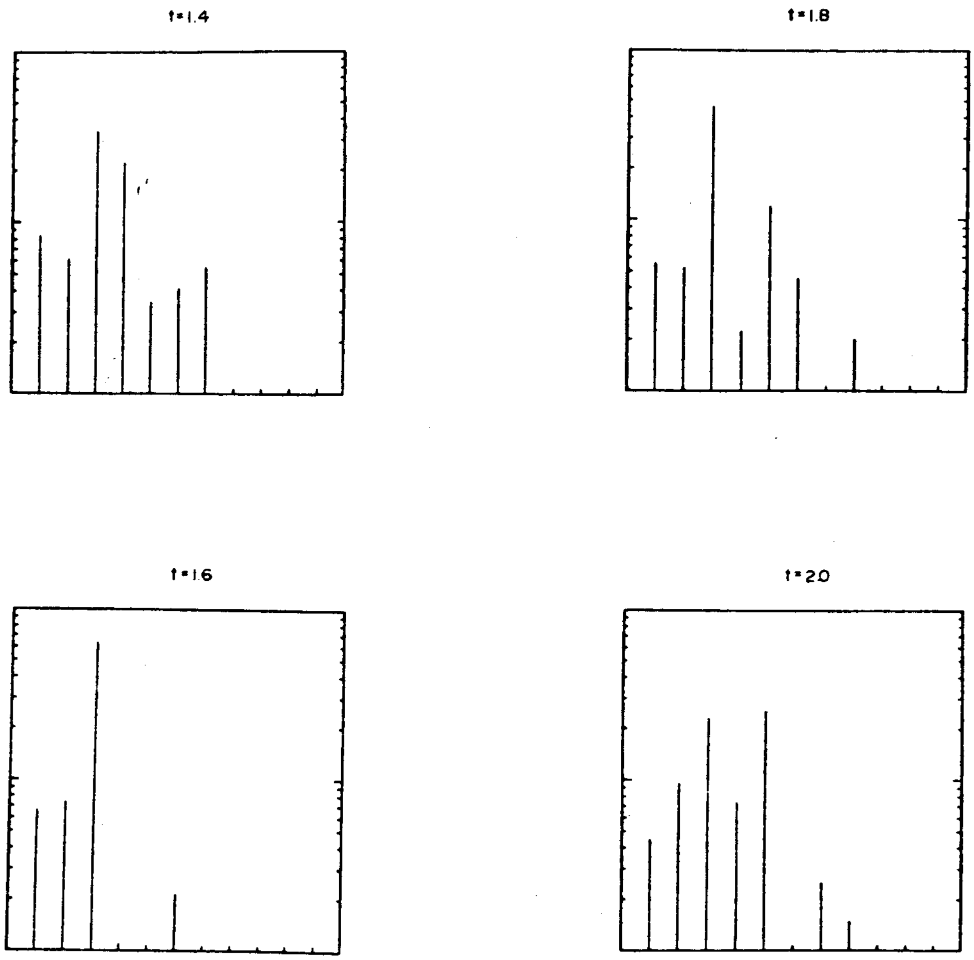

Fig.6. Energy spectra for Case A_1 .

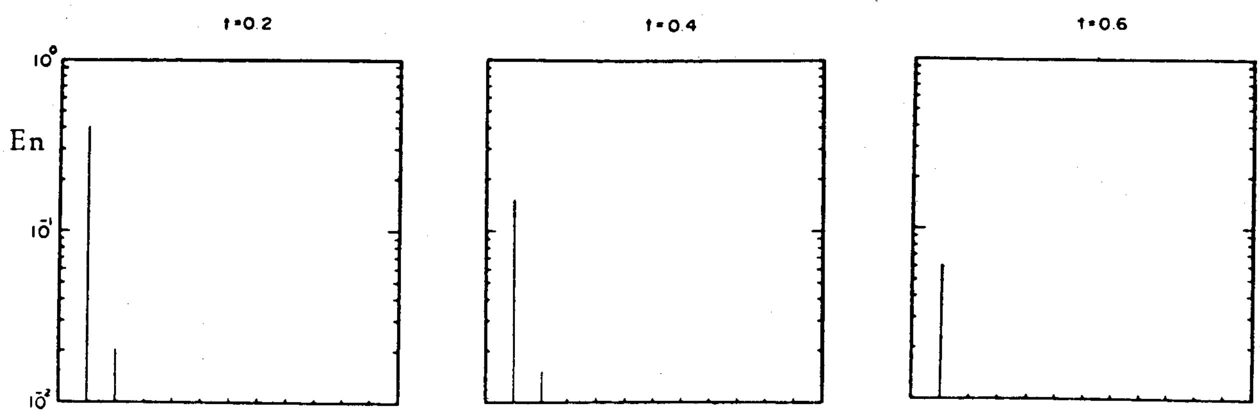

Fig.7. Energy spectra for Case A_2 .

$$E = \sum_{n=1} E_n \text{ .}^{*}$$

In Case A_1 it damps almost linearly with time, which is shown in Fig.8. On the other hand, for Case A_2 we find the exponential decay represented as

$$E = e^{-\gamma_E t},$$

where the value of γ_E is nearly equal to 2, since the wave energy may be nearly equal to the first component E_1. As this value is about one half of that predicted by the linear theory (γ_E=4.4, see eq.(3.13b) or (3.13c)), we may conclude that nonlinearity gives rise to negative influence against the damping.

6.2 Damping of solitary wave

By using the transformations

$$h \rightarrow \alpha_1^{\frac{1}{5}} h, \quad x \rightarrow \alpha_1^{\frac{2}{5}} x, \quad t \rightarrow \alpha_1^{\frac{1}{5}} t,$$

eq.(6.1) can be rewritten as

$$\frac{\partial h}{\partial t} + h\frac{\partial h}{\partial x} + \frac{\partial^3 h}{\partial x^3} = \alpha_2 \int_x^{\infty} \frac{\partial h}{\partial x'} \frac{dx'}{\sqrt{x'-x}} . \tag{6.5}$$

Setting α_2 =0 in the above equation, we recover the K-dV equation and have the solitary wave solution represented

* This is nearly equal to the sum of the potential and kinetic energies per unit wavelength.

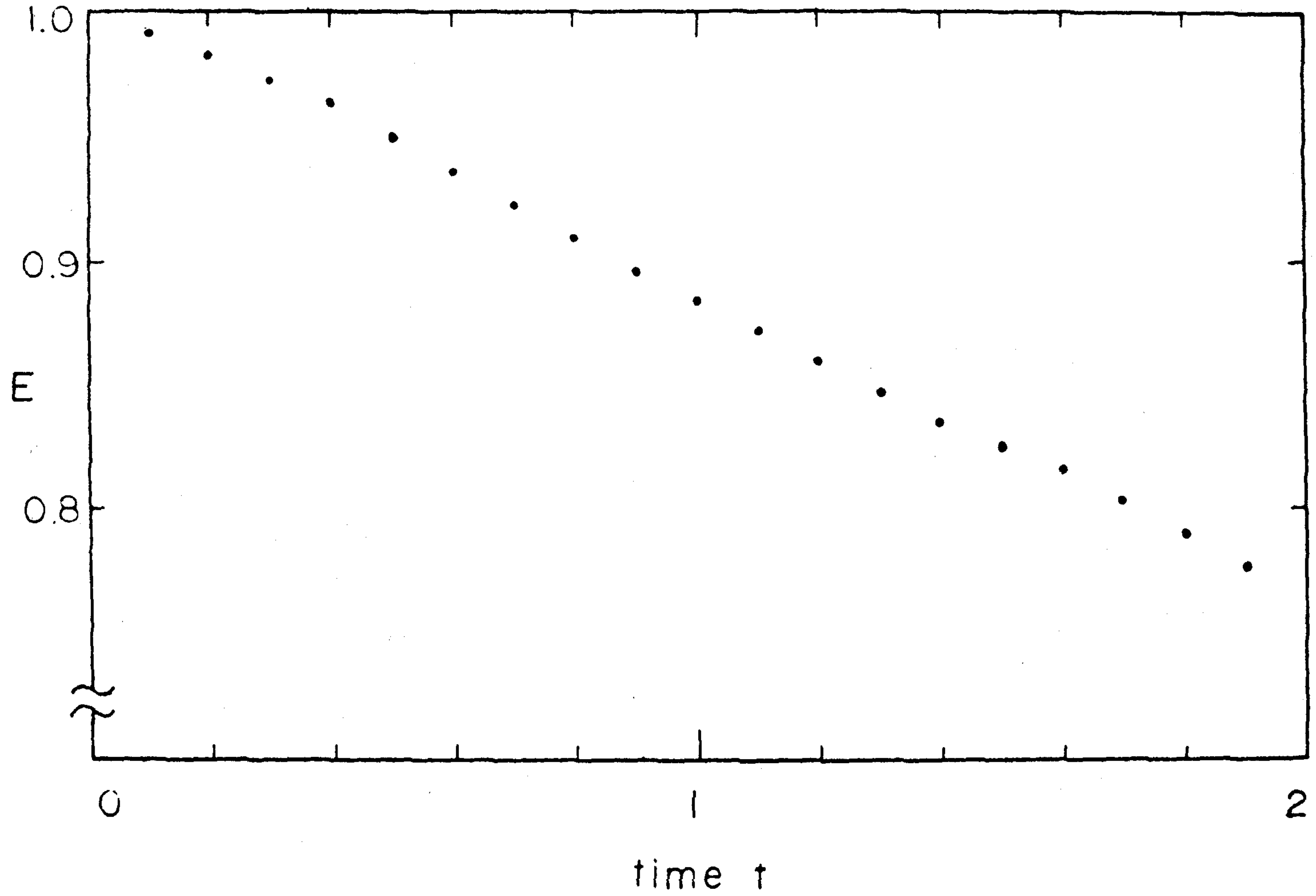

Fig.8. Damping of total energy for Case A_1 .

as

$$h = S \operatorname{sech}^2 \sqrt{\frac{S}{12}} \left(x - \frac{S}{3} t \right).$$

A solitary wave whose amplitude S is equal to 24 has the width $D = 1/\sqrt{2}$ and propagates with speed $V = 8$.

We solve the initial value problems to eq.(6.5) for the following two cases,

Case B_1 : $\alpha_2 = 0.1$,

Case B_2 : $\alpha_2 = 10$,

choosing the solitary wave placed at $x = 6.4$ as its initial value. In the numerical calculations, we replaced the region extended from $x = -\infty$ to $x = \infty$ by a periodic boundary condition. Several constants used in these calculations are summarized in Appendix D. The separation distance was taken to be 12.8 ($\fallingdotseq 18D$). In this configuration, each solitary wave is well-separated geometrically, but the effect due to the presence of the others may not be weak because of the non-locality of the viscous effect as shown in the right-hand side of eq.(6.5). However, we carry out the computations under the periodic boundary condition mainly because of their simplicity in numerical algorithm.

The wave forms thus obtained for Cases B_1 and B_2 are given in Figs.9 and 10, respectively. As we have already shown in §5, under the influence of weak dissipation, the

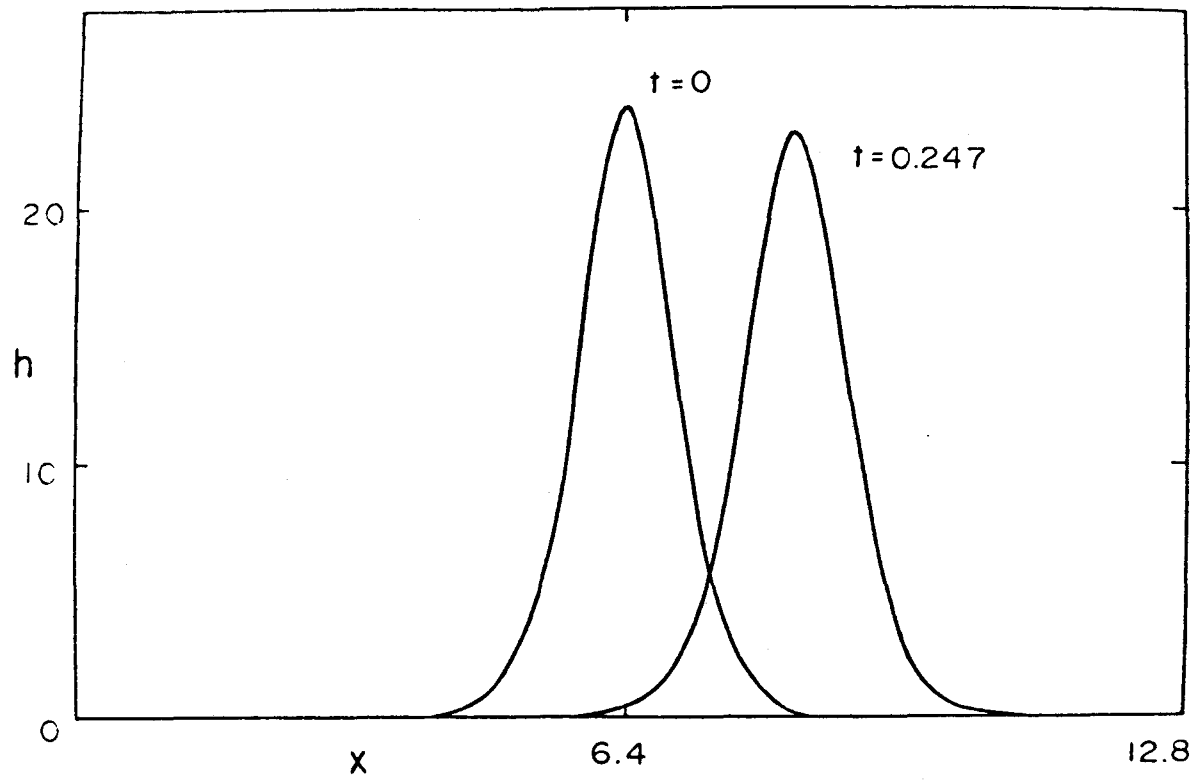

Fig.9. Wave forms for Case B_1 .

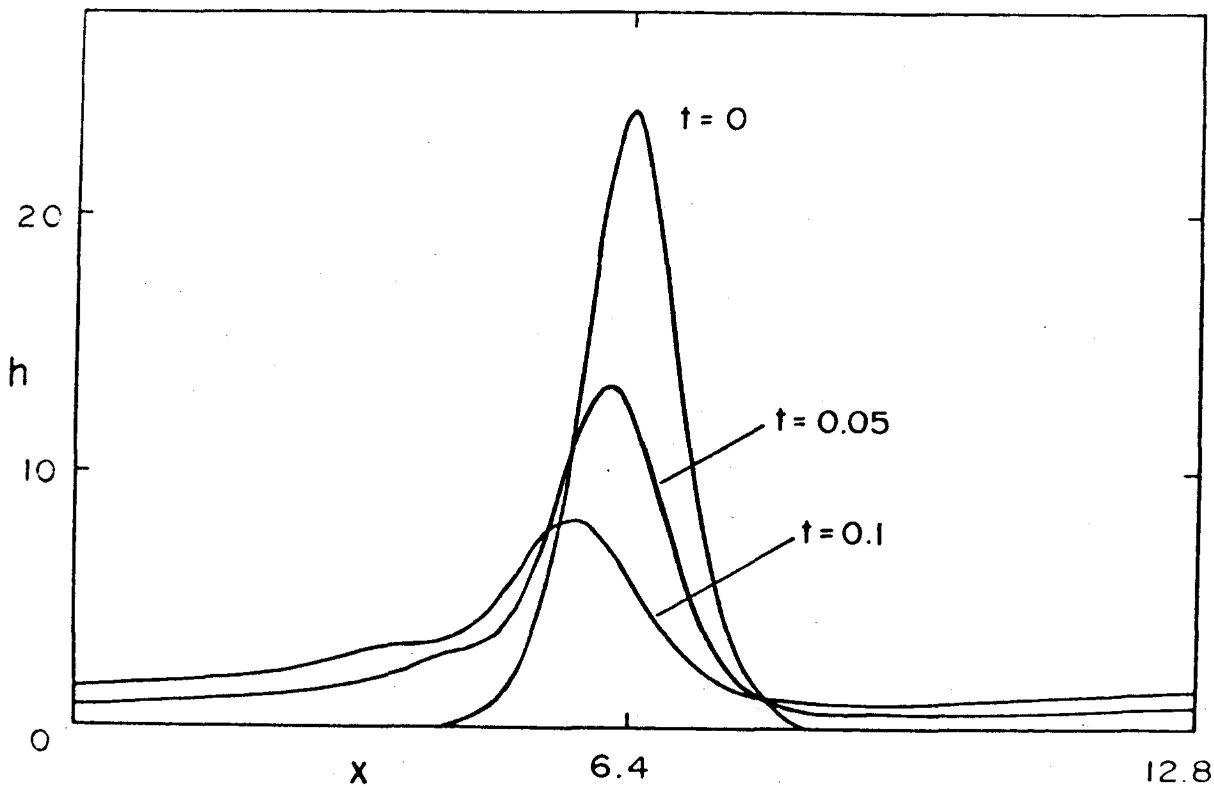

Fig.10. Wave forms for Case B_2 .

amplitude S decreases with time as

$$S(t)/S(0)=(1+m\alpha_2 t)^{-4}, \qquad (6.6)$$

if we note that eq.(4.33) is replaced by eq.(6.5)(cf. eq.(5.5)), where

$$m=\frac{1}{8}\left(\frac{S(0)}{12}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sech}^2 z\,\operatorname{sech}^3 z'\sinh z'\frac{\operatorname{sgn}(z'-z)}{|z'-z|^{\frac{1}{2}}}dz'dz\,.$$

Setting S =24 for the present problem, we have m =0.190. When $m\alpha_2 t\ll 1$, we may approximate eq.(6.6) for Case B_1 as

$$S(t)/S(0)\fallingdotseq 1-4m\alpha_2 t,$$

which shows the linear damping of the amplitude. On the other hand, by solving the initial value problem, it is found that the amplitude also decreases almost linearly with time and has the value S =23.0 at t=0.247. However, the linear damping rate thus obtained (γ_S=0.17) does not agree with the value given by the above theory (γ_S=4$\cdot m\alpha_2$ =0.076). This disagreement arises mainly from the assumption made in §5 that solitary wave preserves its symmetry around the central axis. On the other hand, the function expressed in the right-hand side in eq.(6.5) has initially a negative sharp peak near the axis and is roughly positive in the left half to the center and negative in the right half to it (see Fig.11). Such an ansymmetry will appear more remarkably in Case B_2 (see Fig.

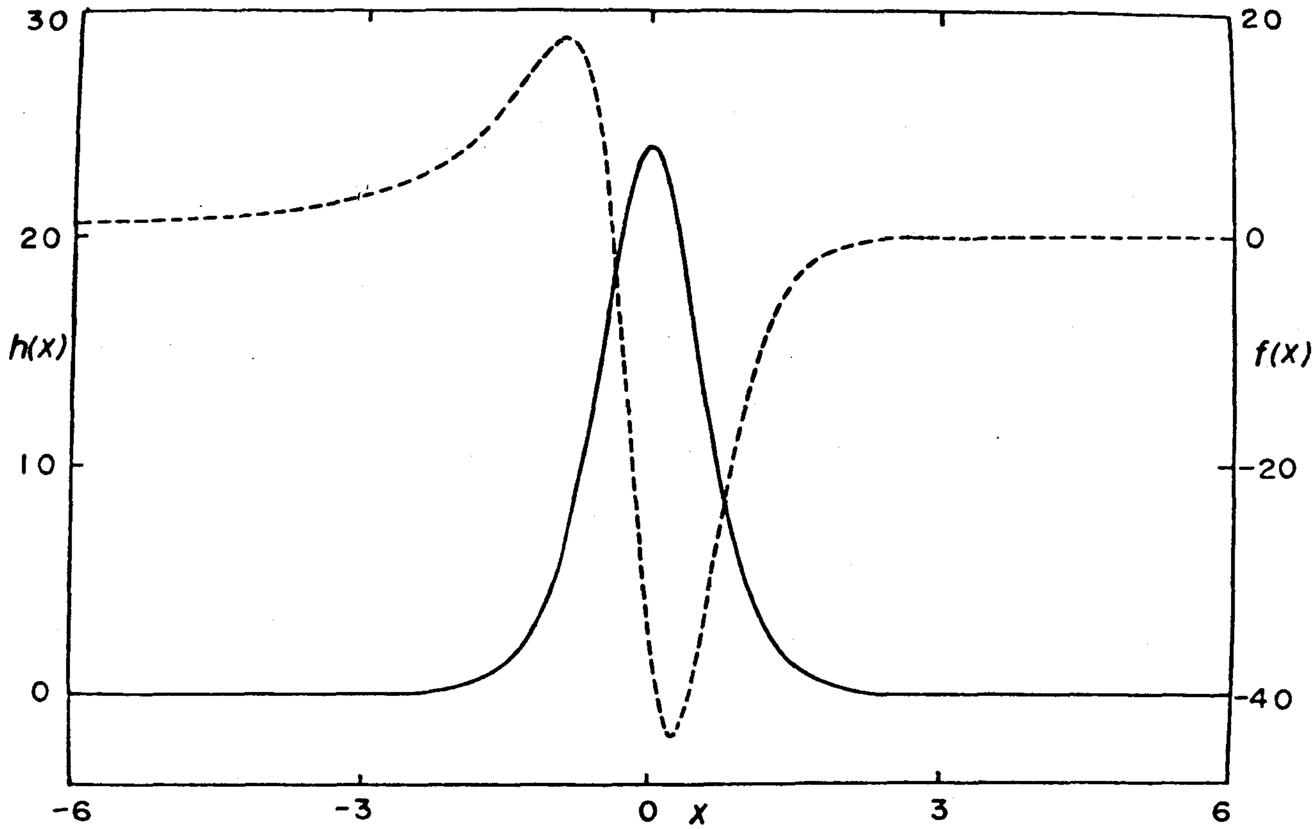

Fig.11. Function expressed initially in the right-hand side of eq.(6.5). Function $f(x)$ is the right-hand side of eq. (6.5), so that

$$f(x)=\int_{-\infty}^{x}\frac{\partial h}{\partial x'}\frac{dx'}{\sqrt{x'-x}}\ , \quad \text{where } h(x)=24\,\mathrm{sech}^2\sqrt{2}\,x.$$

10).

Next we consider the phase shift for Case B_1. The position of the solitary wave after 0.247 time unit is computed as $x=6.4+1.90$, while the K-dV equation gives the position to be $x=6.4+1.976$. Thus we have the phase shift $\delta\theta=0.076$, which is twice as large as that predicted by the linear theory ($\delta\theta=0.039$, see eq.(6.3)). Such a tendency is also found in the case of the sinusoidal function discussed in §6.1.2, and this can be explained by taking account of the amplitude dispersion.

In Case B_2' subject to the stronger influence of viscosity the solitary wave does not propagate downstream and decreases rapidly (see Fig.10). The fact that the left to the peak is fat and the right thin originates from the ansymmetry of the right-hand of eq.(6.1) pointed out above (see Fig.11).

6.3 Conclusion

First, it was clarified that in addition to the nondimensional parameter α_1 already known in the inviscid shallow water theory, under the influence of viscosity another nondimensional parameter α_2 plays an essential role in long gravity waves.

Secondly, we obtained the results that the solution of our modified K-dV equation agrees with Zabusky-Galvin's experiment with respect to the damping of solitary waves, while it produces a new disagreement in their phases.

Lastly, concerning the damping of wave energy it is found that when the geometrical dispersion dominates over the viscous one, wave energy decreases almost linearly with time, while in the opposite case viscosity damps it exponentially.

Appendix A: Classification of the Dispersion Relations for Water Waves

We inquire whether there are cases in which the viscous dispersion plays an important role in a usual water layer. In order to see this, we now investigate the relations between the depth H (measured in cm) and the dimensional wavelength L (in cm). For a usual water layer at 20°C, the wavelength vs the depth for $\alpha=(9/2)^{\frac{1}{5}}R^{-\frac{1}{5}}$ and $\alpha=R^{-1}$ is drawn by solid lines in Fig.A. It is seen from this figure that, for example, when $L=10^2$ cm, the geometrical dispersion dominates over the viscous dispersion for a layer deeper than 3 cm and balances with that for a layer 3 cm in deep. In a layer shallower than 3 cm the viscous dispersion makes a dominant contribution, and thus we cannot discard the effect of viscosity. For a water layer shallower than or as shallow as 0.1 cm, any wave cannot propagate and diturbances are diffused out.

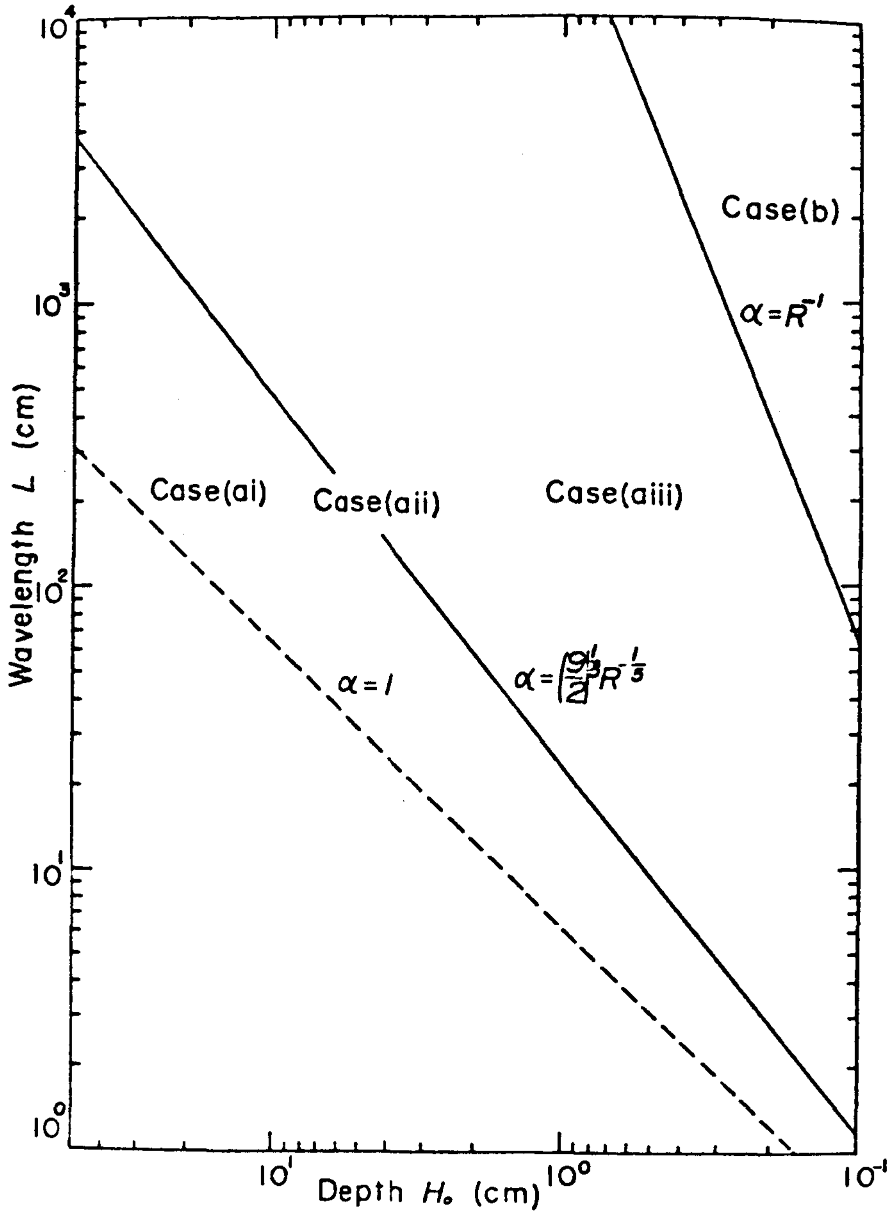

Fig.A. Classification of the dispersion relations for water waves. Regions corresponding to the Cases (ai)-(aiii) and (b) are shown for a water layer at 20° C. The wavelength L (measured in cm) is expressed as a function of the depth H, (also in cm). Two solid lines show the relations $\alpha = (\frac{9}{2})^{\frac{1}{5}} R^{-\frac{1}{5}}$ and $\alpha = R^{-1}$. The dashed line shows $\alpha = 1$ above which the long wave approximation is valid.

Appendix B: Evaluation of $u_1^{(i)}$ and $v_1^{(i)}$

Equation (4.29) and the boundary conditions (4.30) may be rewritten as

$$\frac{\partial^2 f}{\partial \eta^2} + R^* \frac{\partial f}{\partial \xi} = 0, \tag{B.1}$$

$$\left.\begin{aligned} f &= -h_1 \quad \text{at} \quad \eta = 0, \\ \lim_{\eta \to \infty} f &= 0, \end{aligned}\right\} \tag{B.2}$$

where $f(\xi, \eta) = u_1^{(i)} - h_1$. This equation can be solved by using the Fourier transform. Introducing

$$\left.\begin{aligned} f(\xi, \eta) &= \int_{-\infty}^{\infty} \hat{f}(k, \eta)\, e^{ik\xi}\, dk, \\ \text{where} \qquad & \\ \hat{f}(k, \eta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi, \eta)\, e^{-ik\xi}\, d\xi, \end{aligned}\right\} \tag{B.3}$$

into eq. (B.1) and solving the second order equation for $\hat{f}$, we obtain the general solution

$$\hat{f} = C_1 e^{-\sigma\eta} + C_2 e^{\sigma\eta}, \tag{B.4}$$

where

$$\sigma = (R^* k \,\mathrm{sgn}\, k)^{\frac{1}{2}} \exp\left(-\frac{\pi i}{4} \mathrm{sgn}\, k\right). \tag{B.5}$$ [17]

The integration constants C_1 and C_2 must be determined so as to satisfy the boundary condition (B.2). Hence we have

$$u_1^{(1)} = h_1 - \int_{-\infty}^{\infty} \hat{h}_1 e^{-\sigma\eta} e^{ik\xi} dk, \tag{B.6}$$

where $\hat{h}_1$ is the Fourier transform of h_1.

Substituting $u_1^{(1)}$ thus obtained into the second order of the continuity equation (4.21) and integrating it with respect to η, we find that

$$v_2^{(1)} = -\frac{\partial h_1}{\partial \xi}\eta + \int_{-\infty}^{\infty} \frac{ik}{\sigma}(1-e^{-\sigma\eta})\hat{h}_1 e^{ik\xi} dk, \tag{B.7}$$

where the integration constant has been determined by the condition:

$$v_2^{(1)} = 0 \quad \text{at} \quad \eta = 0 .$$

The second term on the right-hand side of eq.(B.7) becomes, in the limit $\eta \to \infty$,

$$\frac{1}{(2R^*)^{\frac{1}{2}}}\int_{-\infty}^{\infty}(-1+i\,\mathrm{sgn}\,k)|k|^{\frac{1}{2}}\hat{h}_1 e^{ik\xi} dk, \tag{B.8}$$

where the explicit form of σ given by (B.5) has been substituted. Applying the convolution theorem we can rewrite (B.8) as

$$\left.\begin{aligned} &\frac{1}{2(\pi R^*)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\partial h_1}{\partial \xi'}\,\frac{1-\mathrm{sgn}(\xi-\xi')}{|\xi-\xi'|^{\frac{1}{2}}}\,d\xi', \\ &\frac{1}{(\pi R^*)^{\frac{1}{2}}}\int_{\xi}^{\infty}\frac{\partial h_1}{\partial \xi'}\,\frac{d\xi'}{(\xi'-\xi)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{B.9}$$

Appendix C: Finite Difference Method Combined with FFT

In order to solve the initial value problems for eq.(6.1) or (6.5), we replace the left-hand side in each equation by a right-to-left sweeping iterative algorithm validated by Zabusky[18] and the right-hand side is estimated by use of the fast Fourier transform (shortly, FFT). This algorithm is given as

$$\frac{\alpha_1}{\Delta^3}\left[u(i+2,j+1)-3u(i+1,j)+3u(i,j+1)-u(i-1,j)\right]$$
$$+\frac{1}{8\Delta}\left[u(i+2,j+1)+u(i-1,j)\right]\left[u(i+2,j+1)+u(i+1,j)\right.$$
$$\left.-u(i,j+1)-u(i-1,j)\right]=\alpha_2 f(i,j), \qquad \text{(C.1)}$$

where

$$u(i,j)=h(i\Delta,j\Delta'),$$
$$f(i,j)=\pi\sum_{n=1}^{M-1}\left(\frac{n}{2\ell}\right)^{\frac{1}{2}}\left[(b_n(j\Delta')-a_n(j\Delta'))\cos\frac{n\pi}{\ell}i\Delta\right.$$
$$\left.-(b_n(j\Delta')+a_n(j\Delta'))\sin\frac{n\pi}{\ell}i\Delta\right],$$

Δ and Δ' being, respectively, the lattice spacing and the time step, which satisfy the relation

$$\Delta'=\Delta^3/4\alpha_1.$$

Equation (C.1) can be arranged in the form

$$u(i,j+1)=u(i+1,j)-\left[u(i+2,j+1)-u(i-1,j)\right]\frac{1+s}{3-s}$$
$$+\frac{\alpha_2}{\alpha_1}\,\frac{\Delta^3 f(i,j)}{3-s}, \qquad \text{(C.2)}$$

where

$$S = \frac{\Delta^2}{8x_1}\left[u(i+2,j+1)+u(i-1,j)\right].$$

All the calculations are carried out for periodic boundary conditions with period 2ℓ. In the first place, $u(N+2,j+1)$ is evaluated by means of the explicit finite difference method, and then $u(N,j+1)$ is calculated from eq.(C.2) by using the values of $u(N-1,j)$, $u(N+1,j)$, and $f(N,j)$, where $N=2\ell/\Delta$. We sweep to the left until we compute $u(2,j+1)$ and we set $u(N+2,j+1)=u(2,j+1)$ because of periodicity. We repeat this process until two successive sweeps differ by a small amount given by

$$|u(N+2,j+1)-u(2,j+1)|<\epsilon',$$

where we take $\epsilon' = 10^{-7}$ for the NEAC 2200 whose single precision arithmetic carries ten significant figures.

According to Zabusky,[18] the discretization errors contained in the left-hand side of eq.(C.1) are of order $(\Delta/D)^2$ and $x_1 S(\Delta/D)^2$ where S and D are the amplitude and width of the solitary wave and satisfy $D = (12x_1/S)^{\frac{1}{2}}$. For example, for Cases B_1 and B_2, these errors $(\Delta/D)^2$ and $x_1 S(\Delta/D)^2$ can be estimated as 2×10^{-2} and 4.8×10^{-1}, respectively, which are the same order as those of Zabusky's computations.

Appendix D: Summary of Constants Used in Numerical Calculations

We summarize briefly the constants used in the computations. These calculations are performed in the time interval $0 \leq t \leq t_E$ for K-dV equation, Cases A_1, A_2, B_1, and B_2.

Table IV. Summary of constants. Last two quantities $a_{0,E}^2$ and $E_{M-1,E}$ imply, respectively, a_0^2 and E_{M-1} at time $t = t_E$. It should be noted that the former is a measure of errors accumulated during the calculations. If there were no approximation, this value would be equal to zero exactly.

	x_1	x_2	N	M	$2l$	t_E	$a_{0,E}^2$	$E_{M-1,E}$
K-dV	$500/64^3$	0	256		2	2.4	2×10^{-11}	
Case A_1	$500/64^3$	$10\,x_1$	128	32	2	2.4	5×10^{-9}	5×10^{-12}
Case A_2	$500/64^3$	1	64	32	2	1.0	2×10^{-7}	2×10^{-11}
Case B_1	1	0.1	128	32	12.8	0.247		7×10^{-11}
Case B_2	1	10	128	32	12.8	0.1		8×10^{-4}

List of Symbols

Dimensional quantities

A : amplitude

g : acceleration due to gravity

H : depth of fluid layer

L : wavelength

ν : kinetic viscosity

ρ : density

Nondimensional quantities

a_n : coefficient of $\cos n\pi x$ in Fourier series

b_n : coefficient of $\sin n\pi x$ in Fourier series

c : complex phase velocity

D : width of solitary wave $(=12\chi/\delta)$

E_n : energy spectrum for the n-th Fourier component $(=a_n^2+b_n^2)$

E : wave energy which is nearly equal to the sum of the potential and kinetic energies $(=\sum_{n=1} E_n)$

$f(x)$: function expressed on the right-hand side of eq.(6.1) or eq.(6.5)

$f(\xi,\eta)$: $u_1^{(i)}-h_1$

h : surface elevation

h': surface elevation for linear wave

2ℓ : period of numerical computation with periodic boundary condition

N : number of lattice spacing

m : damping factor for solitary wave

p : pressure

p_A : atmospheric pressure

p' : perturbed pressure for linear wave

P_n : n-th crest designated in order of magnitude

$Q(\xi, y, \tau)$: field quantities (u, v, and p)

R : Reynolds number

R^* : Reynolds number stretched into quantity of order unity

$Re(\beta)$: real part of β

sgn(ξ): signum function which is equal to unity for positive ξ and minus unity for negative ξ

S : amplitude of solitary wave

S_0 : initial amplitude of solitary wave

t : time

(u, v): velocity components corresponding to coordinates (x, y)

(u', v'): perturbed fluid velocity for linear wave

U_r : Ursell parameter ($=AL^2/H^3$)

V : stretched velocity

α : wavenumber

α_1 : $(4/9)(H^3/AL^2)$

α_2 : $(1/3)(\nu^2L^2/4\pi^2A^4gH)^{\frac{1}{4}}$

β^2 : $\alpha(\alpha - iRc)$

$\gamma(\alpha)$: linear damping rate

γ_E : damping rate for wave energy

γ_S : damping rate for amplitude of solitary wave

δ : thickness of boundary layer

$\delta\theta$: phase shift

Δ : lattice spacing in computation with finite difference method ($=2\ell/N$)

Δ' : time step ($=\Delta^3/4\alpha_1$)

ϵ : a measure of smallness of amplitude or that of wave-number

η : stretched coordinate in y-direction ($=y/\delta$)

ξ : stretched coordinate in x-direction

τ : stretched time

superscript

(o): outer quantity

(i): inner quantity

$\wedge$: Fourier transform

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