Convexity properties of generalized moment maps

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Abstract. In this paper, we consider generalized moment maps for Hamiltonian actions on H-twisted generalized complex manifolds introduced by Lin and Tolman [**15**]. The main purpose of this paper is to show convexity and connectedness properties for generalized moment maps. We study Hamiltonian torus actions on compact H-twisted generalized complex manifolds and prove that all components of the generalized moment map are Bott-Morse functions. Based on this, we shall show that the generalized moment maps have a convex image and connected fibers. Furthermore, by applying the arguments of Lerman, Meinrenken, Tolman, and Woodward [**13**] we extend our results to the case of Hamiltonian actions of general compact Lie groups on H-twisted generalized complex orbifolds.

1. Introduction

The notion of H-twisted generalized complex structures was introduced by Hitchin [**9**] inspired by physical motivations. It provides us with a unifying framework for both complex and symplectic geometry and with a useful geometric language for understanding some recent development in string theory. The associated notion of H-twisted generalized Kähler structures was introduced by Gualtieri [**8**], showing that this notion is essentially equivalent to that of bihermitian structures. This equivalence was first observed by physicists in their study [**5**] of a super-symmetric nonlinear sigma model.

For Hamiltonian group actions on manifolds, moment maps are a very useful tool in geometry. In generalized complex geometry, Lin and Tolman studied the notions of Hamiltonian actions and generalized moment maps for actions of compact Lie groups on H-twisted generalized complex manifolds [**15**], and established a reduction theorem. In the present paper we study the convexity properties of generalized moment maps for Hamiltonian actions. Both convexity and connectedness for moment maps in symplectic geometry were studied by Atiyah [**1**] and Guillemin-Sternberg [**7**] in the case of torus actions on compact symplectic manifolds. We here consider Hamiltonian torus actions on compact connected H-twisted generalized complex manifolds and prove such convexity and connectedness for generalized moment maps (cf. Sections 2 and 3).

THEOREM A. *Let an m-dimensional torus T^m act on a compact connected H-twisted generalized complex manifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{t}^*)$. Then*

1. *the levels of μ are connected,*
2. *the image of μ is convex, and*

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3. *the fixed points of the action form a finite union of connected generalized complex submanifolds* $C_1, \cdots, C_N$*:*

$$\mathrm{Fix}(T^m) = \bigcup_{i=1}^{N} C_i.$$

On each component the generalized moment map μ *attains a constant:* $\mu(C_i) = \{a_i\}$*, and the image of* μ *is the convex hull of the images* $a_1, \cdots, a_N$ *of the fixed points, that is,*

$$\mu(M) = \left\{ \sum_{i=1}^{N} \lambda_i a_i \mid \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

For the proof of the theorem, we need to show that all components of the generalized moment map are Bott-Morse functions, that is, the function $\mu^\xi : M \to \mathbb{R}$ is a Bott-Morse function for all $\xi \in \mathfrak{t}$ (cf. Proposition 3.9). This is crucial in the proof, and is obtained by the maximum principle for pseudoholomorphic functions on almost complex manifolds.

In the latter part of this paper, we shall extend our results to the case of general compact Lie group actions on H-twisted generalized complex orbifolds under the assumption of *weak nondegeneracy* (cf. Definition 4.1) for generalized moment maps, where weak nondegeneracy is always the case for compact orbifolds. Recall that the non-abelian convexity theorem in symplectic geometry was proved by Kirwan [**11**] and Lerman-Meinrenken-Tolman-Woodward [**13**]. A subset Δ of a vector space V is *polyhedral* if it is an intersection of finitely many closed half-spaces, and is *locally polyhedral* if for each point $p \in \Delta$ there exist a neighborhood U of p in V and a polyhedral set P in V such that $U \cap \Delta = U \cap P$. Then we obtain:

Theorem B. *Let* $(M, \mathcal{J})$ *be a connected* H*-twisted generalized complex orbifold with a Hamiltonian action of a compact connected Lie group* G*, a proper generalized moment map* $\mu : M \to \mathfrak{g}^*$*, and a moment one form* $\alpha \in \Omega^1(M; \mathfrak{g}^*)$*. Suppose that the generalized moment map* μ *has weak nondegeneracy.*

1. *Let* $\mathfrak{t}^*_+$ *be a closed Weyl chamber for the Lie group considered as a subset of* $\mathfrak{g}^*$*. The moment set* $\Delta = \mu(M) \cap \mathfrak{t}^*_+$ *is a convex locally polyhedral set. In particular, if* M *is compact then* Δ *is a convex polytope.*
2. *The levels of* μ *are connected.*

Let us explain the real meaning of the convexity property for generalized moment maps. In general, an H-twisted generalized complex structure is of an intermediate type, i.e., it is neither a complex structure nor a symplectic structure. Then the manifold is locally fibered over a complex base space such that symplectic structures appear in the fiber directions. The generalized moment map is thought of as a "relative version" of the ordinary moment map. Now our theorems on generalized moment maps show not only the convexity of the image of each fiber but also the convexity of the global image of the generalized moment maps (cf. Section 4.4).

This paper is organized as follows. In section 2 we briefly review the theory of generalized complex structures and generalized Kähler structures. Furthermore we introduce generalized complex submanifolds of H-twisted generalized complex manifolds in the sense of Ben-Bassat and Boyarchenko [**2**]. In section 3 we consider the notion of generalized moment maps [**15**] for Hamiltonian actions on H-twisted generalized complex manifolds and prove that all components of the generalized moment map are Bott-Morse functions. After that we shall give a proof of Theorem A. Finally in the last section, we give a proof of Theorem B. Our proof follows that of the non-abelian convexity and connectedness properties in symplectic geometry by Lerman-Meinrenken-Tolman-Woodward in [**13**]. To complete our proof, we need an additional constraint "weak nondegeneracy" for generalized moment maps and a generalized complex geometry analogue of the cross-section theorem in [**13**] (cf. Theorem 4.9).

2. Generalized complex structures

We recall the basic theory of generalized complex structures; see [**8**] for the details. Throughout this paper, we assume that all manifolds and orbifolds are connected.

2.1. Generalized complex structures

Given a closed differential 3-form H on an n-dimensional manifold M, we define the H-twisted Courant bracket of sections of the direct sum $TM \oplus T^*M$ of the tangent and cotangent bundles by

$$[X+\alpha, Y+\beta]_H = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d\left(\beta(X) - \alpha(Y)\right) + i_Y i_X H,$$

where $\mathcal{L}_X$ denotes the Lie derivative along a vector field X. The vector bundle $TM \oplus T^*M$ is also endowed with a natural inner product of signature (n,n):

$$\langle X+\alpha, Y+\beta\rangle = \frac{1}{2}(\beta(X) + \alpha(Y)).$$

Definition 2.1. Let M be a manifold and H be a closed 3-form on M. A generalized almost complex structure on M is a complex structure $\mathcal{J}$ on the bundle $TM \oplus T^*M$ which preserves the natural inner product. If sections of the $\sqrt{-1}$-eigenspace L of $\mathcal{J}$ is closed under the H-twisted Courant bracket, then $\mathcal{J}$ is called an H-twisted generalized complex structure of M. If $H = 0$, we call it simply a generalized complex structure.

An H-twisted generalized complex structure $\mathcal{J}$ can be fully described in terms of its $\sqrt{-1}$-eigenbundle L, which is a maximal isotropic subbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$ satisfying $L \cap \bar{L} = \{0\}$ and to be closed under the H-twisted Courant bracket. For the natural projection $\pi : (TM \oplus T^*M) \otimes \mathbb{C} \to TM \otimes \mathbb{C}$, the codimension of $\pi(L)$ in $TM \otimes \mathbb{C}$ is called the type of $\mathcal{J}$ and written by $\mathrm{type}(\mathcal{J})$.

Example 2.2 (Complex structures (type n)). Let J be a complex structure

on an n-dimensional complex manifold M. Consider the endomorphism

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

where the matrix is written with respect to the direct sum $TM \oplus T^*M$. Then $\mathcal{J}_J$ is a generalized complex structure of type n on M; the $\sqrt{-1}$-eigenspace of $\mathcal{J}_J$ is $L_J = T_{1,0}M \oplus T^{0,1}M$, where $T_{1,0}M$ is the $\sqrt{-1}$-eigenspace of J.

Example 2.3 (Symplectic structures (type 0)). Let M be a symplectic manifold with a symplectic structure ω, viewed as a skew-symmetric isomorphism $\omega : TM \to T^*M$ via the interior product $X \mapsto i_X\omega$. Consider the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

Then $\mathcal{J}_\omega$ is a generalized complex structure on M of type 0; the $\sqrt{-1}$-eigenspace of $\mathcal{J}_\omega$ is given by

$$L_\omega = \{X - \sqrt{-1}i_X\omega \mid X \in TM \otimes \mathbb{C}\}.$$

Example 2.4 (B-field shift). Let $(M, \mathcal{J})$ be an H-twisted generalized complex manifold and $B \in \Omega^2(M)$ be a closed 2-form on M. Then the endomorphism

$$\mathcal{J}_B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

is also an H-twisted generalized complex structure. It is called the B-field shift of $\mathcal{J}$. The type of $\mathcal{J}_B$ coincides with that of $\mathcal{J}$. Indeed, the $\sqrt{-1}$ eigenspace L_B of $\mathcal{J}_B$ can be written as

$$L_B = \{X + \alpha + i_X B \mid X + \alpha \in L\},$$

where L is the $\sqrt{-1}$ eigenspace of $\mathcal{J}$.

Next we describe the notions of H-twisted generalized complex structures from the view point of differential forms. For the details, see [**8**]. Let $(M, \mathcal{J})$ be a $2n$-dimensional H-twisted generalized complex manifold with its $\sqrt{-1}$-eigenspace L. Recall that the exterior algebra $\wedge^\bullet T^*M$ carries a natural spin representation for the metric bundle $TM \oplus T^*M$; the Clifford action of $X + \alpha \in TM \oplus T^*M$ on $\varphi \in \wedge^\bullet T^*M$ is given by

$$(X + \alpha) \cdot \varphi = i_X\varphi + \alpha \wedge \varphi.$$

The annihilator K of L by the spin representation forms a complex line subbundle of the

complex spinors $\wedge^\bullet T^*M \otimes \mathbb{C}$. We call K the canonical line bundle of $\mathcal{J}$:

$$K = \{\varphi \in \wedge^\bullet T^*M \otimes \mathbb{C} \mid (X+\alpha)\cdot\varphi = 0 \ (\forall X+\alpha \in L)\}.$$

Then the $\sqrt{-1}$-eigenspace L can also be viewed as an annihilator of K.

Conversely, for a complex spinor $\varphi \in \wedge^\bullet T^*M \otimes \mathbb{C}$, consider L_φ the annihilator of φ:

$$L_\varphi = \{X+\alpha \in (TM \oplus T^*M)\otimes\mathbb{C} \mid (X+\alpha)\cdot\varphi = 0\}.$$

Then the subspace $L_\varphi \subset (TM\oplus T^*M)\otimes\mathbb{C}$ is always isotropic. If L_φ is maximal isotropic, φ is called a complex pure spinor. A necessary and sufficient condition that φ is pure can be described as follows. We call a complex differential k-form Ω to be decomposable if it has the algebraic form $\Omega = \theta^1 \wedge \cdots \wedge \theta^k$ at each point, where $\theta^1, \cdots, \theta^k$ are linearly independent complex 1-forms. Then the spinor φ is pure if and only if it can be written locally as

$$\varphi = e^{B+\sqrt{-1}\omega} \wedge \Omega,$$

where B and ω are real 2-forms and Ω is a complex decomposable k-form. The condition $L_\varphi \cap \bar{L}_\varphi = \{0\}$ is equivalent to an additional constraint on φ:

$$\omega^{2(n-k)} \wedge \Omega \wedge \bar{\Omega} \neq 0.$$

A complex pure spinor φ which satisfies the condition above is said to be nondegenerate. If a complex differential form $\varphi \in \Omega^\bullet \otimes \mathbb{C}$ is a nondegenerate complex pure spinor at every point on M, then we have $(TM \oplus T^*M)\otimes\mathbb{C} = L_\varphi \oplus \bar{L}_\varphi$, and L_φ defines a generalized almost complex structure on M. For each point, the integer k defined above coincides with the type of the generalized almost complex structure. The canonical line bundle is generated by the complex pure spinor φ.

Finally as shown in [**8**], the involutivity of L_φ under the Courant bracket is equivalent to the condition that there exist a section $X+\alpha$ of $(TM \oplus T^*M)\otimes\mathbb{C}$ such that

$$d\varphi + H \wedge \varphi = (X+\alpha)\cdot\varphi.$$

2.2. Generalized Kähler structures

We briefly review the notion of H-twisted generalized Kähler structures.

Definition 2.5. Let M be a manifold and H a closed 3-form on M. An H-twisted generalized Kähler structure on M is a pair of commuting H-twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ so that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric, that is, $\mathcal{G}^2 = \mathrm{id}$, $\mathcal{G}$ preserves the natural inner product and $\mathcal{G}(X+\alpha, X+\alpha) := \langle \mathcal{G}(X+\alpha), X+\alpha\rangle > 0$ for all non-zero $X+\alpha \in TM \oplus T^*M$.

Example 2.6. Let (M, g, J) be a Kähler manifold and $\omega = gJ$ be the Kähler form.

As seen in the examples above, J and ω induce generalized complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ respectively. Moreover, we see that $\mathcal{J}_J$ commutes with $\mathcal{J}_\omega$, and

$$\mathcal{G} = -\mathcal{J}_J\mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

is a positive definite metric on $TM \oplus T^*M$. Hence $(\mathcal{J}_J, \mathcal{J}_\omega)$ is a generalized Kähler structure on M.

EXAMPLE 2.7. Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an H-twisted generalized Kähler structure, and B be a closed 2-form on M. Then $((\mathcal{J}_1)_B, (\mathcal{J}_2)_B)$ is also an H-twisted generalized Kähler structure. It is called the B-field shift of $(\mathcal{J}_1, \mathcal{J}_2)$.

In [**8**], a characterization of H-twisted generalized Kähler structures was given in terms of Hermitian geometry, which is represented below.

THEOREM 2.8 (M. GUALTIERI, [**8**]). *For each H-twisted generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, there exists a uniquely determined 2-form b, a Riemannian metric g and two orthogonal complex structures $J_\pm$ such that*

$$\mathcal{J}_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

where $\omega_\pm = gJ_\pm$ with the condition

$$d_-^c\omega_- = -d_+^c\omega_+ = H + db. \tag{1}$$

Conversely, any quadruple $(g, b, J_\pm)$ satisfying the condition (1) defines an H-twisted generalized Kähler structure.

Not every H-twisted generalized complex manifold admits an H-twisted generalized Kähler structure. However, the following lemma claims that every H-twisted generalized complex manifold always admits a "generalized almost Kähler structure". This is a generalized complex geometry analogue of the fact that a symplectic manifold admits an almost complex structure which is compatible with the symplectic structure.

LEMMA 2.9. *Let $(M, \mathcal{J})$ be an H-twisted generalized complex manifold. Then there exists a generalized almost complex structure $\mathcal{J}'$ which is compatible with $\mathcal{J}$, that is, $\mathcal{J}'$ is a generalized almost complex structure which commutes with $\mathcal{J}$, and $\mathcal{G} = -\mathcal{J}\mathcal{J}'$ is a positive definite metric.*

Proof. Choose a Riemannian metric g on M and put

$$\tilde{\mathcal{G}} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Then $\tilde{\mathcal{G}}$ is a positive definite metric on $TM \oplus T^*M$. Define a symplectic structure $\mathcal{W}$ on $TM \oplus T^*M$ by

$$\mathcal{W}(X+\alpha, Y+\beta) = \langle \mathcal{J}(X+\alpha), Y+\beta \rangle.$$

Since $\tilde{\mathcal{G}}$ and $\mathcal{W}$ are nondegenerate, there exists an endomorphism $\mathcal{A}$ on $TM \oplus T^*M$ which satisfies

$$\mathcal{W}(X+\alpha, Y+\beta) = \tilde{\mathcal{G}}(\mathcal{A}(X+\alpha), Y+\beta)$$

for all $X+\alpha, Y+\beta \in TM \oplus T^*M$. The endomorphism $\mathcal{A}$ is skew-symmetric with respect to the positive definite metric $\tilde{\mathcal{G}}$ because $\mathcal{W} = \tilde{\mathcal{G}}\mathcal{A}$ is an alternating 2-form on $TM \oplus T^*M$. Let $\mathcal{A}^*$ be the adjoint operator of $\mathcal{A}$ with respect to $\tilde{\mathcal{G}}$. Since $\mathcal{A}$ is invertible, $\mathcal{A}\mathcal{A}^* = -\mathcal{A}^2$ is symmetric and positive, that is, $(\mathcal{A}\mathcal{A}^*)^* = \mathcal{A}\mathcal{A}^*$ and

$$\tilde{\mathcal{G}}(\mathcal{A}\mathcal{A}^*(X+\alpha), X+\alpha) > 0$$

for all non-zero $X+\alpha \in TM \oplus T^*M$. Hence we can define $\sqrt{\mathcal{A}\mathcal{A}^*}$ the square root of $\mathcal{A}\mathcal{A}^*$. Here $\sqrt{\mathcal{A}\mathcal{A}^*}$ is also symmetric and positive definite.

Let $\mathcal{J}'$ be an endomorphism on $TM \oplus T^*M$ defined by

$$\mathcal{J}' = (\sqrt{\mathcal{A}\mathcal{A}^*})^{-1}\mathcal{A}.$$

Since $\mathcal{A}$ commutes with $\sqrt{\mathcal{A}\mathcal{A}^*}$, $\mathcal{J}'$ commutes with both $\mathcal{A}$ and $\sqrt{\mathcal{A}\mathcal{A}^*}$. Hence we obtain $(\mathcal{J}')^2 = -\mathrm{id}$. By the definition of $\mathcal{A}$, we have $\mathcal{A}\mathcal{J} = -\mathcal{J}\mathcal{A}^{-1}$ and hence $\mathcal{J}$ commutes with $\sqrt{\mathcal{A}\mathcal{A}^*}$. In particular, we see that $\mathcal{J}'$ commutes with $\mathcal{J}$. Moreover, since $\mathcal{J}'$ is orthogonal with respect to $\tilde{\mathcal{G}}$, we can check easily that $\mathcal{J}'$ is orthogonal with respect to the natural inner product on $TM \oplus T^*M$. Hence $\mathcal{J}'$ is a generalized almost complex structure on M which commutes with $\mathcal{J}$. Finally $\mathcal{G} := -\mathcal{J}\mathcal{J}'$ is a positive definite metric on $TM \oplus T^*M$ since $\mathcal{G} = \tilde{\mathcal{G}}\sqrt{\mathcal{A}\mathcal{A}^*}$. This completes the proof. □

If $\mathcal{J}'$ is a generalized almost complex structure which is compatible with an H-twisted generalized complex structure $\mathcal{J}$, then we can apply the argument of Gualtieri in [**8**] and construct a 2-form b, a Riemannian metric g and two orthogonal almost complex structures $J_\pm$ which satisfy the equation

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} J_+ + J_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(J_+^* + J_-^*) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}. \tag{2}$$

In general, J_+ and J_- may not be integrable.

2.3. Generalized complex submanifolds

Here we introduce the notion of generalized complex submanifolds in the sense of Ben-Bassat and Boyarchenko in [**2**]. Let $i : S \to M$ be a submanifold of an H-twisted

generalized complex manifold $(M, \mathcal{J})$. For each $p \in S$, define a subspace $(L_S)_p \subset (T_pS \oplus T_p^*S) \otimes \mathbb{C}$ by

$$(L_S)_p = \{X + i^*\alpha \in (T_pS \oplus T_p^*S) \otimes \mathbb{C} \mid X + \alpha \in L_p\},$$

where L is the $\sqrt{-1}$-eigenspace of $\mathcal{J}$. Clearly $(L_S)_p$ is an isotropic subspace of $(T_pS \oplus T_p^*S) \otimes \mathbb{C}$. Furthermore it is easy to see that $\dim_{\mathbb{C}}(L_S)_p = \dim S$ and hence $(L_S)_p$ is a maximal isotropic subspace. However, the distribution $L_S := \cup_{p \in S}(L_S)_p$ may not be a subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$ in general. We refer the reader to [**4**] for a detailed discussion in the case of submanifolds of Dirac manifolds. In particular, Courant's arguments can be easily adapted to give a necessary condition under which L_S is a subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$ (cf. [**4**], Theorem 3.1.1), and to prove that if L_S is a subbundle and L is integrable, then so is L_S (cf. [**4**], Corollary 3.1.4).

Definition 2.10 (Ben-Bassat, Boyarchenko, [**2**]). We say that S is a generalized complex submanifold of M if L_S is a subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$ and satisfies that $L_S \cap \bar{L}_S = \{0\}$.

If $i : S \to M$ is a generalized complex submanifold of an H-twisted generalized complex manifold $(M, \mathcal{J})$, then L_S gives an i^*H-twisted generalized complex structure on S.

Example 2.11. Let S be a complex submanifold of a complex manifold (M, J). Note that S has a natural complex structure induced by J. Then we have

$$L_S = T_{1,0}S \oplus T^{0,1}S,$$

which is of course a subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$ and satisfies $L_S \cap \bar{L}_S = \{0\}$. Hence S is a generalized complex submanifold of $(M, \mathcal{J}_J)$. The induced generalized complex structure of S is the natural generalized complex structure which is induced by the complex structure of S.

Example 2.12. Let $i : S \to M$ be a symplectic submanifold of a symplectic manifold (M, ω). Then for the generalized complex structure $\mathcal{J}_\omega$ induced by the symplectic structure ω, we have

$$L_S = \{X - \sqrt{-1}i_X(i^*\omega) \mid X \in TS \otimes \mathbb{C}\},$$

which coincides with the $\sqrt{-1}$-eigenspace of the generalized complex structure $\mathcal{J}_{i^*\omega}$ induced by the symplectic structure $i^*\omega$. In particular S is a generalized complex submanifold of $(M, \mathcal{J}_\omega)$.

Example 2.13. Let S be a Lagrangian submanifold of a symplectic manifold (M, ω). Then we can see easily that $L_S = TS \otimes \mathbb{C}$ and hence L_S is a maximal isotropic subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$. However, since it is clear that $L_S \cap \bar{L}_S = TS \otimes \mathbb{C} \neq \{0\}$, the submanifold S is not a generalized complex submanifold.

In general, it may not be easy to determine if a given submanifold is a generalized complex submanifold. Here we give a simple sufficient condition.

PROPOSITION 2.14. *Let $(M, \mathcal{J})$ be an H-twisted generalized complex manifold, and $\mathcal{J}'$ be a generalized almost complex structure of M which is compatible with $\mathcal{J}$. We denote $(g, b, J_\pm)$ the corresponding quadruple. If $i : S \to M$ is an almost complex submanifold of M with respect to both J_+ and J_-, then S is a generalized complex submanifold of $(M, \mathcal{J})$.*

Proof. Let $T^{\pm}_{1,0}M \subset TM \otimes \mathbb{C}$ be the $\sqrt{-1}$-eigenspace of $J_\pm$. Then we can check easily that L is given by

$$L = \{X + (b+g)(X) \mid X \in T^+_{1,0}M\} \oplus \{Y + (b-g)(Y) \mid Y \in T^-_{1,0}M\}.$$

Hence for a given almost complex submanifold S, the subspace L_S can be written as

$$L_S = \{X + i^*(b+g)(X) \mid X \in T^+_{1,0}S\} \oplus \{Y + i^*(b-g)(Y) \mid Y \in T^-_{1,0}S\},$$

where $T^{\pm}_{1,0}S$ is the $\sqrt{-1}$-eigenspace of the restriction of $J_\pm$ to S. In particular, L_S is a subbundle of $(TS \oplus T^*S) \otimes \mathbb{C}$. In addition if $(X+i^*(b+g)(X))+(Y+i^*(b-g)(Y)) \in L_S \cap \bar{L}_S$ for $X \in T^+_{1,0}S$ and $Y \in T^-_{1,0}S$, then we see that $(X + (b+g)(X)) + (Y + (b-g)(Y)) \in L \cap \bar{L} = \{0\}$. Thus we obtain $X = Y = 0$, and hence $L_S \cap \bar{L}_S = \{0\}$. This proves the proposition. □

REMARK 2.15. If M is an orbifold and H is a closed 3-form on M, we can define the notions of H-twisted generalized complex structures of M in usual way. The detailed description is as follows. A definition of orbifolds can be seen in [**18**] for example. Let M be an orbifold and $(V_i, G_i, \pi_i)_{i \in I}$ be a local uniformizing system of M. A generalized almost complex structure $\mathcal{J}$ of M is a family of endomorphisms $\{\mathcal{J}_i : TV_i \oplus T^*V_i \to TV_i \oplus T^*V_i\}_{i \in I}$ such that $\mathcal{J}_i$ is a generalized almost complex structure on V_i for each $i \in I$ and they are equivariant under the local group actions and compatible with respect to the injections. If each $\mathcal{J}_i$ is integrable with respect to H-twisted Courant brackets, then $\mathcal{J}$ is called to be integrable and we call it a H-twisted generalized complex structure of an orbifold M.

In the case that $(M, \mathcal{J})$ is an H-twisted generalized complex orbifold, we can describe the same notions in section 2, and the assertions in section 2 still hold in the language of orbifolds.

3. Hamiltonian actions and generalized moment maps

3.1. Hamiltonian actions on H-twisted generalized complex manifolds

In this section we recall the definition of Hamiltonian actions on H-twisted generalized complex manifolds given by Lin and Tolman in [**15**].

DEFINITION 3.1 (Y. LIN AND S. TOLMAN, [**15**]). Let a compact Lie group G with its Lie algebra $\mathfrak{g}$ act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ preserving

$\mathcal{J}$, where $H \in \Omega^3(M)$ is a G-invariant closed 3-form. The action of G is said to be Hamiltonian if there exists a G-equivariant smooth function $\mu : M \to \mathfrak{g}^*$, called the generalized moment map, and a $\mathfrak{g}^*$-valued one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$, called the moment one form, such that

1. $\xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi)$ lies in L for all $\xi \in \mathfrak{g}$, where ξ_M denotes the induced vector field on M, and
2. $i_{\xi_M} H = d\alpha^\xi$ for all $\xi \in \mathfrak{g}$.

Since L is an isotropic subbundle, we have

$$\left\langle \xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi),\ \xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi) \right\rangle = 0$$

and hence $\iota_{\xi_M}\alpha^\xi = \iota_{\xi_M} d\mu^\xi = 0$ for each $\xi \in \mathfrak{g}$.

Example 3.2. Let a compact Lie group G act on a symplectic manifold (M, ω) preserving the symplectic structure ω, and $\mu : M \to \mathfrak{g}^*$ be an usual moment map, that is, μ is G-equivariant and $i_{\xi_M}\omega = d\mu^\xi$ for all $\xi \in \mathfrak{g}$. Then G also preserves $\mathcal{J}_\omega$, μ is also a generalized moment map, and $\alpha = 0$ is a moment one form for this action. Hence the G-action on $(M, \mathcal{J}_\omega)$ is Hamiltonian.

Example 3.3. Let (M, J) be a complex manifold and G act on $(M, \mathcal{J}_J)$ in a Hamiltonian way. Then G also preserves the original complex structure J. Since $L_J = T_{1,0}M \oplus T^{0,1}M$ and $\xi_M \in \pi(L_J)$, we have $\xi_M = 0$ for all $\xi \in \mathfrak{g}$. Thus if G is connected, the G-action on M must be trivial.

Example 3.4. Let G act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ with a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$. If $B \in \Omega^2(M)^G$ is closed, then G acts on M preserving the B-field shift of $\mathcal{J}$ with generalized moment map μ and moment one form α', where $(\alpha')^\xi = \alpha^\xi + i_{\xi_M} B$ for all $\xi \in \mathfrak{g}$.

By the definition, we can treat the notion of generalized moment maps as a generalization of the notion of moment maps in symplectic geometry. Generalized moment maps are studied by Lin and Tolman in [**15**]. In their paper, they showed that a reduction theorem for Hamiltonian actions of compact Lie groups on H-twisted generalized complex manifold holds. We shall use this fact later. Note that since $i_{\xi_M} H = d\alpha^\xi$ and $\iota_{\xi_M}\alpha^\xi = 0$ for each $\xi \in \mathfrak{g}$, we can see $H + \alpha$ as an equivariantly closed form.

Lemma 3.5 (Y. Lin, S. Tolman, [**15**]). *Let a compact Lie group G act freely on a manifold M. Let H be an invariant closed 3-form and α be an equivariant mapping from $\mathfrak{g}$ to $\Omega^1(M)$. Fix a connection $\theta \in \Omega(M; \mathfrak{g})$. Then if $H + \alpha$ is equivariantly closed, there exists a natural form $\Gamma \in \Omega^2(M)^G$ such that $i_{\xi_M}\Gamma = \alpha^\xi$ for all $\xi \in \mathfrak{g}$. In particular, $H + \alpha + d_G\Gamma \in \Omega^3(M) \subset \Omega_G(M)$, where $\Omega_G(M)$ is the set of equivariant differential forms of M and d_G denotes the equivariant exterior differential, is closed and basic and so descends to a closed form $\tilde{H} \in \Omega^3(M/G)$ such that the cohomology class $[\tilde{H}]$ is the image of $[H + \alpha]$ under the Kirwan map.*

THEOREM 3.6 (Y. LIN, S. TOLMAN, [**15**]). *Let a compact Lie group G act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$. Let $\mathcal{O}_a$ be a coadjoint orbit through $a \in \mathfrak{g}^*$ so that G acts freely on $\mu^{-1}(\mathcal{O}_a)$. Given a connection on $\mu^{-1}(\mathcal{O}_a)$, the twisted generalized complex quotient $M_a = \mu^{-1}(\mathcal{O}_a)/G$ inherits an $\tilde{H}$-twisted generalized complex structure $\tilde{\mathcal{J}}$, where $\tilde{H}$ is defined as in Lemma 3.5. Up to B-field shift, $\tilde{\mathcal{J}}$ is independent of the choice of connection. Finally, for each $p \in \mathcal{O}_a$,*

$$\mathrm{type}(\tilde{\mathcal{J}})_{[p]} = \mathrm{type}(\mathcal{J})_p.$$

In the case that $(M, \mathcal{J})$ is an H-twisted generalized complex orbifold, we can define the notion of Hamiltonian actions of a compact Lie group on $(M, \mathcal{J})$ in usual way. In this case, the reduction theorem still holds in the language of orbifolds. The detailed statement is as follows. Let a compact Lie group G act on an H-twisted generalized complex orbifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$. For a coadjoint orbit $\mathcal{O}_a$ through $a \in \mathfrak{g}^*$, suppose that the G-action on $\mu^{-1}(\mathcal{O}_a)$ is locally free. Given a connection on $\mu^{-1}(\mathcal{O}_a)$, the twisted generalized complex quotient $M_a = \mu^{-1}(\mathcal{O}_a)/G$ is an orbifold and inherits an $\tilde{H}$-twisted generalized complex structure $\tilde{\mathcal{J}}$, where $\tilde{H}$ is defined as in Lemma 3.5. Up to B-field shift, $\tilde{\mathcal{J}}$ is independent of the choice of connection and the type is preserved.

Before we begin a proof of Theorem A, we shall prove a remarkable fact of generalized moment maps. At first we prove the following lemmata.

LEMMA 3.7. *Let a compact Lie group G act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ preserving $\mathcal{J}$. Then there exists a G-invariant generalized almost complex structure which is compatible with $\mathcal{J}$.*

Proof. Choose a G-invariant Riemannian metric g on M and put

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Then $\mathcal{G}$ is a G-invariant positive definite metric on $TM \oplus T^*M$. Let $\mathcal{A}$ be an endomorphism on $TM \oplus T^*M$ defined by $\mathcal{A} = \mathcal{G}^{-1}\mathcal{J}$. Since $\mathcal{G}$ and $\mathcal{J}$ are G-invariant, $\mathcal{A}$ is also G-invariant. Now if we define

$$\mathcal{J}' = (\sqrt{\mathcal{A}\mathcal{A}^*})^{-1}\mathcal{A},$$

then $\mathcal{J}'$ is a generalized almost complex structure on M which is compatible with $\mathcal{J}$. Furthermore since $\mathcal{A}$ is G-invariant, $\mathcal{J}'$ is also G-invariant. This completes the proof. □

LEMMA 3.8. *Let an m-dimensional torus T^m act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map μ and a moment one form α. Then for an arbitrary subtorus $G \subset T^m$ the fixed point set of*

G-action

$$\mathrm{Fix}(G) = \{p \in M \mid \theta \cdot p = p \ (\forall \theta \in G)\}$$

is a generalized complex submanifold of $(M, \mathcal{J})$.

Proof. Choose a G-invariant generalized almost complex structure $\mathcal{J}'$ which is compatible with $\mathcal{J}$. Then there exists a Riemannian metric g, a 2-form b, and two orthogonal almost complex structures $J_\pm$ which satisfies the equation (2). Since $\mathcal{J}$ and $\mathcal{J}'$ are G-invariant, g and $J_\pm$ are also G-invariant. For each $p \in \mathrm{Fix}(G)$ and $\theta \in G$, the differential of the action of θ at p

$$(\theta_*)_p : T_pM \to T_pM$$

preserves the almost complex structures $J_\pm$. In addition, since G-action preserves the metric g, the exponential mapping $\exp_p : T_pM \to M$ with respect to the metric g is equivariant, that is,

$$\exp_p((\theta_*)_p X) = \theta \cdot \exp_p X$$

for any $\theta \in G$ and $X \in T_pM$. This concludes that the fixed point of the action of θ near p corresponds to the fixed point of $(\theta_*)_p$ on T_pM by the exponential mapping, that is,

$$T_p\mathrm{Fix}(G) = \bigcap_{\theta \in G} \ker(1 - (\theta_*)_p).$$

Since $J_\pm$ commutes with the endomorphism $(\theta_*)_p$, the eigenspace with eigenvalue 1 of $(\theta_*)_p$ is invariant under $J_\pm$, and hence an almost complex subspace. In particular we see that $\mathrm{Fix}(G)$ is a generalized complex submanifold of $(M, \mathcal{J})$ by applying Proposition 2.14. □

Now we consider a Hamiltonian T^m-action on a compact H-twisted generalized complex manifold $(M, \mathcal{J})$ with a generalized moment map $\mu : M \to \mathfrak{g}$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$, and examine the functions $\mu^\xi : M \to \mathbb{R}$ for all $\xi \in \mathfrak{g}$. The following proposition shows that these are Bott-Morse functions with even indices and coindices. This is crucial to prove the connectedness of fibers of the generalized moment map. In our proof, the maximum principle for pseudoholomorphic functions on almost complex manifolds plays a central role. The maximum principle for pseudoholomorphic functions on almost complex manifolds is provided by the work of Boothby-Kobayashi-Wang in [**3**].

Proposition 3.9. *Let an m-dimensional torus T^m act on a compact H-twisted generalized complex manifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{t}^*)$. Then μ^ξ is a Bott-Morse function with even index and coindex for all $\xi \in \mathfrak{t}$.*

Proof. For each $\xi \in \mathfrak{t}$, we denote T^ξ the subtorus of T^m generated by ξ. First we shall prove that the critical set

$$\mathrm{Crit}(\mu^\xi) = \{p \in M \mid (d\mu^\xi)_p = 0\}$$

coincides with the fixed point set of T^ξ-action $\mathrm{Fix}(T^\xi)$. Choose a T^m-invariant generalized almost complex structure $\mathcal{J}'$ which is compatible with $\mathcal{J}$. Then $\mathcal{J}$ can be written in the form of the equation (2) for the corresponding quadruple $(g, b, J_\pm)$. Note that the metric g and orthogonal almost complex structures $J_\pm$ are all T^m-invariant.

Since $\xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi) \in L$ by the definition of Hamiltonian actions, $(d\mu^\xi)_p = 0$ implies $p \in \mathrm{Fix}(T^\xi)$. In particular we obtain $\mathrm{Crit}(\mu^\xi) \subset \mathrm{Fix}(T^\xi)$. On the other hand, since $\mathrm{Fix}(T^\xi) = \{p \in M \mid (\xi_M)_p = 0\}$, we see that $\alpha^\xi + \sqrt{-1}d\mu^\xi \in \bar{L}$ on $\mathrm{Fix}(T^\xi)$. Hence there exists a complex vector field X on M which satisfies that $\alpha^\xi + \sqrt{-1}d\mu^\xi = g(X)$ and $X \in T^+_{0,1}M \cap T^-_{0,1}M$ on $\mathrm{Fix}(T^\xi)$ because the $\sqrt{-1}$-eigenspace L can be written by

$$L = \{X + (b+g)(X) \mid X \in T^+_{1,0}M\} \oplus \{Y + (b-g)(Y) \mid Y \in T^-_{1,0}M\}.$$

Since the almost complex structures $J_\pm$ are orthogonal with respect to the metric g, we see that $\alpha^\xi + \sqrt{-1}d\mu^\xi$ is a holomorphic 1-form on $\mathrm{Fix}(T^\xi)$. Moreover, since α^ξ is a closed 1-form on $\mathrm{Fix}(T^\xi)$, we can view the function μ^ξ locally as an imaginary part of a pseudoholomorphic function on an almost complex manifold $(\mathrm{Fix}(T^\xi), J_\pm)$. By applying the maximum principle and compactness of $\mathrm{Fix}(T^m)$, we see that μ^ξ is constant on each connected component of $\mathrm{Fix}(T^\xi)$. Moreover the gradient of μ^ξ with respect to the metric g is tangent to $\mathrm{Fix}(T^\xi)$ because g and μ^ξ are T^ξ-invariant. This shows that $\mathrm{Fix}(T^\xi) \subset \mathrm{Crit}(\mu^\xi)$, and hence we obtain $\mathrm{Crit}(\mu^\xi) = \mathrm{Fix}(T^\xi)$. In particular, $\mathrm{Crit}(\mu^\xi)$ is a generalized complex submanifold of M.

To prove that the function μ^ξ is a Bott-Morse function, we shall calculate the Hessian $\nabla^2\mu^\xi$ of μ^ξ on $\mathrm{Crit}(\mu^\xi)$. Since $\xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi) \in L$ for each $\xi \in \mathfrak{g}$, we have

$$\mathcal{J}(\xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi)) = \sqrt{-1}(\xi_M - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi)). \tag{3}$$

In addition, by using the equation (2) for the H-twisted generalized complex structure $\mathcal{J}$, for the natural projection $\pi : (TM \oplus T^*M) \otimes \mathbb{C} \to TM \otimes \mathbb{C}$ we obtain the following equation;

$$\begin{aligned}\pi(\mathcal{J}(\xi_M - \sqrt{-1}(d\mu + \sqrt{-1}\alpha^\xi))) \\ = \frac{1}{2}\left((J_+ + J_-)(\xi_M) - (\omega_+^{-1} - \omega_+^{-1})(-b(\xi_M) - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi))\right).\end{aligned} \tag{4}$$

By combining the equations (3) and (4), we see that the induced vector field ξ_M can be written as

$$\xi_M = \frac{1}{2}\Big(\omega_+^{-1}(d\mu^\xi) - \omega_-^{-1}(d\mu^\xi)\Big). \tag{5}$$

Let ∇ be the Riemannian connection with respect to the metric g. Then by an easy calculation we have the following equality for $\xi^{\pm}_M := \omega_{\pm}^{-1}(d\mu^\xi) = -J_{\pm}g^{-1}(d\mu^\xi)$;

$$g(\nabla^2\mu^\xi(Y), Z) = g((\nabla_Y J_{\pm})\xi^{\pm}_M, Z) + g(J_{\pm}(\nabla_Y \xi^{\pm}_M), Z). \tag{6}$$

Since the vector field $\xi^{\pm}_M$ vanishes on $\mathrm{Crit}(\mu^\xi)$, the equation (6) shows that

$$(\nabla^2\mu^\xi)_p(Y_p) = J_{\pm}(\nabla_{Y_p}\xi^{\pm}_M) \tag{7}$$

for each $p \in \mathrm{Crit}(\mu^\xi)$ and $Y_p \in T_pM$. Let $(L_\xi)_p$ be an endomorphism on T_pM defined by $(L_\xi)_p(Y) := [\xi_M, Y]_p = -\nabla_{Y_p}\xi_M$. Then by the equation (5) and (7), we see that $(L_\xi)_p$ can be written as

$$(L_\xi)_p = -\frac{1}{2}(J_+ - J_-)(\nabla^2\mu^\xi)_p. \tag{8}$$

Now we shall prove $T_p\mathrm{Crit}(\mu^\xi) = \ker(\nabla^2\mu^\xi)_p$. Since each connected component $\mathrm{Crit}(\mu^\xi)$ is a submanifold of M, it is clear that $T_p\mathrm{Crit}(\mu^\xi) \subset \ker(\nabla^2\mu^\xi)_p$. Therefore we may only show that $\ker(\nabla^2\mu^\xi)_p \subset T_p\mathrm{Crit}(\mu^\xi)$. At first we have $\ker(\nabla^2\mu^\xi)_p \subset \ker(L_\xi)_p$ by the equation (8). If we identify $(L_\xi)_p$ with a vector field on T_pM, the one parameter family of diffeomorphisms $\{(\exp t\xi_*)_p\}_{t\in\mathbb{R}}$ on T_pM coincides with $\{\exp t(L_\xi)_p\}_{t\in\mathbb{R}}$. Hence $\ker(L_\xi)_p$ coincides with the fixed point set of $\{(\exp t\xi_*)_p\}_{t\in\mathbb{R}}$. Therefore we have

$$\ker(\nabla^2\mu^\xi)_p \subset T_p\mathrm{Crit}(\mu^\xi),$$

and this shows that $T_p\mathrm{Crit}(\mu^\xi) = \ker(\nabla^2\mu^\xi)_p$. In particular, we see that μ^ξ is a Bott-Morse function.

Finally, we shall show that the function μ^ξ has even index and coindex. By equation (7), we see that

$$g((\nabla^2\mu^\xi)_p(J_{\pm}Y), Z) = g((\nabla^2\mu^\xi)_pZ, J_{\pm}Y) = g(J_{\pm}(\nabla_Z\xi^{\pm}_M), J_{\pm}Y) = g(\nabla_Z\xi^{\pm}_M, Y)$$

for each $p \in \mathrm{Crit}(\mu^\xi)$ and $Y, Z \in T_pM$. Since $\xi_M = \frac{1}{2}(\xi^+_M - \xi^-_M)$ and ξ_M is a Killing vector field, we obtain

$$\begin{aligned} g((\nabla^2\mu^\xi)_p(J_+ - J_-)(Y), Z) &= g(\nabla_Z(\xi^+_M - \xi^-_M), Y) = 2g(\nabla_Z\xi_M, Y) \\ &= -2g(\nabla_Y\xi_M, Z) = -g(\nabla_Y(\xi^+_M - \xi^-_M), Z) \\ &= g((J_+ - J_-)(\nabla^2\mu^\xi)_p(Y), Z). \end{aligned}$$

Hence we see that $(\nabla^2\mu^\xi)_p$ commutes with $J_+ - J_-$ for all $p \in \mathrm{Crit}(\mu^\xi)$. Now we define a differential 2-form by $g(J_+ - J_-)$. Then since g is positive definite and $J_+ - J_-$ commutes with $(\nabla^2\mu^\xi)_p$, $J_+ - J_-$ preserves each eigenspace of $(\nabla^2\mu^\xi)_p$ and hence $g(J_+ - J_-)$ is

nondegenerate on each non-zero eigenspace of $(\nabla^2\mu^\xi)_p$. Thus each non-zero eigenspace of $(\nabla^2\mu^\xi)_p$ is even dimensional, in particular the index and coindex of the critical manifold are even. □

REMARK 3.10. The compactness assumption here is essential. If M is noncompact, then a generalized moment map can not be seen a Bott-Morse function in general. Indeed, if we consider a trivial torus action on a complex manifold (M, J), then the imaginary part of an arbitrary holomorphic function is a generalized moment map for this action. (See also Remark 4.12.)

3.2. A proof of Theorem A

In this section we shall prove Theorem A. This proof involves induction over $m = \dim T^m$. Consider the statements:

A_m : "the level sets of μ are connected, for any T^m-action", and

B_m : "the image of μ is convex, for any T^m-action".

At first we see that A_1 holds by Proposition 3.9 and the fact that level sets of a Bott-Morse function on a connected compact manifold are connected if the critical manifolds have index and coindex $\neq 1$ (see [**17**] for example). The claim B_1 holds clearly because in $\mathbb{R}$ connectedness is equivalent to convexity.

Now we prove $A_{m-1} \Rightarrow B_m$. Choose a matrix $A \in \mathbb{Z}^m \otimes \mathbb{Z}^{m-1}$ of maximal rank. If we identify A with a linear mapping $A : \mathbb{R}^{m-1} \to \mathbb{R}^m$ and T^m with $\mathbb{R}^m/\mathbb{Z}^m$, then A induces an action of T^{m-1} on M by

$$\theta : p \mapsto (A\theta) \cdot p,$$

for $\theta \in T^{m-1}$ and $p \in M$. The T^{m-1}-action is a Hamiltonian action with a generalized moment map $\mu_A(p) := A^t\mu(p)$ and a moment one form $\alpha_A^\xi := \alpha^{A\xi}$, where A^t denotes the transpose of A.

Given any $a \in \mu_A(M)$ and $p_0 \in \mu_A^{-1}(a)$, we have the fiber of μ_A by

$$\mu_A^{-1}(a) = \{p \in M \mid \mu(p) - \mu(p_0) \in \ker A^t\}.$$

By the assumption A_{m-1}, $\mu_A^{-1}(a)$ is connected. Therefore, for each $p_0, p_1 \in \mu_A^{-1}(a)$, if we connect p_0 to p_1 by a continuous path p_t in $\mu_A^{-1}(a)$ we obtain a path $\mu(p_t) - \mu(p_0)$ in $\ker A^t$. Since A^t is surjective, $\ker A^t$ is 1-dimensional. Hence $\mu(p_t)$ must go through any convex combination of $\mu(p_0)$ and $\mu(p_1)$, which shows that any point on the line segment from $\mu(p_0)$ to $\mu(p_1)$ must be in $\mu(M)$.

Any $p_0, p_1 \in M$ with $\mu(p_0) \neq \mu(p_1)$ can be approximated arbitrarily closely by points p_0' and p_1' with $\mu(p_1') - \mu(p_0') \in \ker A^t$ for some matrix $A \in \mathbb{Z}^m \otimes \mathbb{Z}^{m-1}$ of maximal rank. By the argument above, we see that the line segment from $\mu(p_0')$ to $\mu(p_1')$ must be in $\mu(M)$. By taking limits $p_0' \to p_0$, and $p_1' \to p_1$ we can conclude that $\mu(M)$ is convex.

Next we prove $A_{m-1} \Rightarrow A_m$. By identifying $\mathfrak{t}$ with $\mathbb{R}^m$, we can express the generalized moment map by $\mu = (\mu_1, \cdots, \mu_m)$. We call the generalized moment map μ to be effective if the 1-forms $d\mu_1, \cdots, d\mu_m$ are linearly independent. Note that $p \in M$ is a regular point of μ if and only if $(d\mu_1)_p, \cdots, (d\mu_m)_p$ are linearly independent. If the generalized moment map μ is not effective, the action reduces to a Hamiltonian action of an $(m-1)$-dimensional subtorus. Indeed, If μ is not effective, there exists $c = (c_1, \cdots, c_m) \in \mathbb{R}^m \setminus \{0\}$ such that $\sum_{i=1}^m c_i d\mu_i = 0$. Hence if we denote the canonical basis of $\mathfrak{t} \cong \mathbb{R}^m$ by $\xi_1, \cdots, \xi_n$, then we have

$$\sum_{i=1}^m c_i \Big((\xi_i)_M + \alpha_i \Big) = \sum_{i=1}^m c_i \Big((\xi_i)_M - \sqrt{-1}(d\mu_i + \sqrt{-1}\alpha_i) \Big) \in L,$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$. Since $\sum_{i=1}^m c_i \left((\xi_i)_M + \alpha_i \right)$ is real and $L \cap \bar{L} = \{0\}$, we obtain $\sum_{i=1}^m c_i (\xi_i)_M = \sum_{i=1}^m c_i \alpha_i = 0$. Now consider a vector $\xi = \sum_{i=1}^m c_i \xi_i \in \mathfrak{t}$. By the same argument in the earlier part of the proof of Proposition 3.9, we see that $\mathrm{Crit}(\mu^\xi) = \mathrm{Fix}(T^\xi)$ and hence the function μ^ξ is constant along M because $\xi_M = 0$. For the simplicity, we may assume $\xi_1, \cdots, \xi_{m-1}, \xi$ are linearly independent. Then the T^{m-1}-action generated by $\xi_1, \cdots, \xi_{m-1}$ is a Hamiltonian action with a generalized moment map $\mu' = (\mu_1, \cdots, \mu_{m-1})$ and a moment one form $\alpha' = (\alpha_1, \cdots, \alpha_{m-1})$. Hence in this case the connectedness of fibers of μ follows from that of the reduced generalized moment map μ'. Hence we may assume that μ is effective. Then for each $\xi \in \mathfrak{t} \setminus \{0\}$, μ^ξ is not a constant function, and the critical manifold $\mathrm{Crit}(\mu^\xi)$ is an even dimensional proper submanifold. Now consider the union of critical manifolds

$$C = \cup_{\eta \in \mathfrak{t} \setminus \{0\}} \mathrm{Crit}(\mu^\eta).$$

We claim that the union C is indeed a countable union of even dimensional proper submanifolds. To see this, recall that the critical points of μ^η are the fixed points of the action of the subtorus $T^\eta \subset T^m$ and form an even dimensional proper submanifold. Since the fixed point set decreases as the torus increases it suffices to consider 1-dimensional subtorus or, equivalently, integer vectors η. This shows the assertion about C. In particular, $M \setminus C$ is a dense subset of M. In addition, since the condition $p \in M \setminus C$ is equivalent to the condition that $(d\mu_1)_1, \cdots, (d\mu_m)_p$ are linearly independent, we obtain $M \setminus C$ is open dense subset of M.

LEMMA 3.11. *The set of regular values of μ in $\mu(M)$ is a dense subset of $\mu(M)$.*

Proof. For each $a = \mu(p) \in \mu(M)$, there exists a sequence $\{p_i\}_{i=1}^\infty \subset M \setminus C$ which satisfies that $\lim_{i \to \infty} p_i = p$. Since p_i is a regular point of μ, $\mu(M)$ contains a neighborhood of $\mu(p_i)$ by implicit function theorem. Moreover there exists a regular value $a_i \in \mathfrak{t}^*$ which is sufficiently close to $\mu(p_i)$ and $\mu^{-1}(a_i) \neq \phi$ by Sard's theorem. Hence the sequence $\{a_i\}_{i=1}^\infty$ approximates a. □

By a similar argument, we see that the set of $a = (a_1, \cdots, a_m) \in \mathfrak{t}^*$ that $(a_1, \cdots, a_{m-1})$ is a regular value of $(\mu_1, \cdots, \mu_{m-1})$ in $\mu(M)$ is also a dense subset of $\mu(M)$. Hence, by continuity, to prove that $\mu^{-1}(a)$ is connected for every $a =$

$(a_1, \cdots, a_m) \in \mathfrak{t}^*$, it suffics to prove that $\mu^{-1}(a)$ is connected whenever $(a_1, \cdots, a_{m-1})$ is a regular value for the reduced generalized moment map $(\mu_1, \cdots, \mu_{m-1})$. By the induction hypothesis, the submanifold

$$Q = \cap_{i=1}^{m-1} \mu_i^{-1}(a_i)$$

is connected for a regular value $(a_1, \cdots, a_{m-1})$ of $(\mu_1, \cdots, \mu_{m-1})$. To complete the proof, we need the following lemma.

LEMMA 3.12. *If $(a_1, \cdots, a_{m-1})$ is a regular value for $(\mu_1, \cdots, \mu_{m-1})$, the function $\mu_m : Q \to \mathbb{R}$ is a Bott-Morse function of even index and coindex.*

Proof. By the hypothesis, Q is a $2n - (m-1)$ dimensional connected submanifold of M. For each $p \in Q$, the subspace W of the cotangent space T_p^*M generated by $(d\mu_1)_p, \cdots, (d\mu_{m-1})_p$ is $(m-1)$ dimensional because p is regular. Therefore the tangent space T_pQ of Q coincides with the annihilator of W;

$$T_pQ = \{X \in T_pM \mid \alpha(X) = 0 \ (\forall \alpha \in W)\}.$$

Hence we see that $p \in Q$ is a critical point of $\mu_m : Q \to \mathbb{R}$ if and only if there exists real numbers $c_1, \cdots, c_{m-1}$ such that

$$\sum_{i=1}^{m-1} c_i (d\mu_i)_p + (d\mu_m)_p = 0.$$

This means that p is a critical point of the function $\mu^\xi : M \to \mathbb{R}$, where $\xi = (c_1, \cdots, c_{m-1}, 1) \in \mathfrak{t}$. By Proposition 3.9, μ^ξ is a Bott-Morse function with even index and coindex. Furthermore, by Lemma 3.8 and the fact $\mathrm{Crit}(\mu^\xi) = \mathrm{Fix}(T^\xi)$, the critical set $\mathrm{Crit}(\mu^\xi)$ is a finite union of generalized complex submanifolds. Now we shall prove the critical manifold $\mathrm{Crit}(\mu^\xi)$ intersects Q transversally at p, that is,

$$T_pM = T_p\mathrm{Crit}(\mu^\xi) + T_pQ.$$

For a subspace $S \subset T_pM$, we denote by $S^0 \subset T_p^*M$ the annihilator of S;

$$S^0 = \{\alpha \in T_p^*M \mid \alpha(X) = 0 \ (\forall X \in S)\}.$$

Then since $(T_pQ)^0 = W$, we obtain

$$(T_p\mathrm{Crit}(\mu^\xi) + T_pQ)^0 = (T_p\mathrm{Crit}(\mu^\xi))^0 \cap (T_pQ)^0 = (T_p\mathrm{Crit}(\mu^\xi))^0 \cap W.$$

Hence the critical manifold $\mathrm{Crit}(\mu^\xi)$ intersects Q transversally at p if and only if $(T_p\mathrm{Crit}(\mu^\xi))^0 \cap W = \{0\}$. Thus we may only show that the differentials $(d\mu_1)_p, \cdots, (d\mu_{m-1})_p$ remain linearly independent when restricted to the subspace

$T_p\mathrm{Crit}(\mu^\xi)$. Consider the vector fields $\xi_1^+, \cdots, \xi_{m-1}^+$ on M defined by

$$d\mu_i = \omega_+(\xi_i^+), \quad i = 1, \cdots, m-1.$$

Since $\omega_+ = gJ_+$, the vector field ξ_i^+ can be written as $\xi_i^+ = -J_+g^{-1}(d\mu_i)$. The T^m-invariance of the function μ_i implies

$$(\theta_*)_p g^{-1}(d\mu_i)_p = g^{-1}((\theta^{-1})^* d\mu_i)_p = g^{-1}(d((\theta^{-1})^*\mu_i))_p = g^{-1}(d\mu_i)_p$$

for each $\theta \in T^\xi$. In particular, we see that the vector field $g^{-1}(d\mu_i)$ is tangent to $\mathrm{Crit}(\mu^\xi)$ because $T_p\mathrm{Crit}(\mu^\xi) = T_p\mathrm{Fix}(T^\xi) = \bigcap_{\theta\in T^\xi} \ker(1 - (\theta_*)_p)$. Moreover, since the critical manifold $\mathrm{Crit}(\mu^\xi)$ is an almost complex submanifold of (M, J_+), the vector filed $\xi_i^+ = -J_+g^{-1}(d\mu_i)$ is also tangent to $\mathrm{Crit}(\mu^\xi)$. On the other hand, $(\xi_1^+)_p, \cdots, (\xi_{m-1}^+)_p$ are linearly independent on T_pM because p is regular. Hence they are also linearly independent on $T_p\mathrm{Crit}(\mu^\xi)$. Since the 2-form ω_+ is still nondegenerate when it is restricted to $\mathrm{Crit}(\mu^\xi)$, the 1-forms $(d\mu_1)_p, \cdots, (d\mu_{m-1})_p$ are linearly independent on $T_p^*\mathrm{Crit}(\mu^\xi)$ and hence $\mathrm{Crit}(\mu^\xi)$ intersects Q transversally as claimed. In particular, the critical set $\mathrm{Crit}(\mu_m|_Q)$ of $\mu_m : Q \to \mathbb{R}$ is a finite union of submanifolds of Q because $\mathrm{Crit}(\mu_m|_Q) = \mathrm{Crit}(\mu^\xi) \cap Q$.

For each $X \in T_pM$ which is orthogonal to $T_p\mathrm{Crit}(\mu^\xi)$, we have

$$(d\mu_i)_p(X) = g_p(g^{-1}(d\mu_i), X) = 0$$

for $i = 1, \cdots, m-1$. This implies that the orthogonal complement $(T_p\mathrm{Crit}(\mu^\xi))^\perp$ of the subspace $T_p\mathrm{Crit}(\mu^\xi)$ is contained in T_pQ. Hence the Hessian of μ^ξ at p is nondegenerate on $T_pQ \cap (T_p\mathrm{Crit}(\mu^\xi))^\perp = (T_p\mathrm{Crit}(\mu^\xi))^\perp$ with even index and coindex. In other words, $\mathrm{Crit}(\mu^\xi) \cap Q$ is the critical manifold of $\mu^\xi|_Q$ of even index and coindex. The same holds for $\mu_m|_Q$ since it only differs from μ^ξ by the constant $\sum_{i=1}^{m-1} c_i a_i$. Thus we have proved that the function $\mu_m : Q \to \mathbb{R}$ is a Bott-Morse function with even index and coindex. □

By applying Lemma 3.12, if $(a_1, \cdots, a_{m-1})$ is a regular value for $(\mu_1, \cdots, \mu_{m-1})$, then the level set $\mu_m^{-1}(a_m) \cap Q = \mu^{-1}(a)$ is connected. This shows that $A_{m-1} \Rightarrow A_m$.

Finally, we shall prove the third claim, that is, the image of the generalized moment map μ is the convex hull of the images of the fixed points of the action. By Lemma 3.8, the fixed point set $\mathrm{Fix}(T^m)$ of the action decomposes into finitely many even dimensional connected submanifolds $C_1, \cdots, C_N$ of M. The generalized moment map μ is constant on each of these sets because $C_i \subset \mathrm{Crit}(\mu^\xi)$ for $i = 1, \cdots, N$ and any $\xi \in \mathfrak{t}$. Hence there exists $a_1, \cdots, a_N \in \mathfrak{t}^*$ such that

$$\mu(C_i) = \{a_i\}, \quad i = 1, \cdots, N.$$

By what we have proved so far the convex hull of the points $a_1, \cdots, a_N$ is contained in $\mu(M)$. Conversely, let $a \in \mathfrak{t}^*$ be a point which is not in the convex hull of $a_1, \cdots, a_N$.

Then there exists a vector $\xi \in \mathfrak{t}$ with rationally independent components such that

$$a_i(\xi) < a(\xi), \quad i = 1, \cdots, N.$$

Since the components of ξ are rationally independent, we have $\mathrm{Crit}(\mu^\xi) = \mathrm{Fix}(T^m)$. Hence the function $\mu^\xi : M \to \mathbb{R}$ attains its maximum on one of the sets $C_1, \cdots, C_N$. This implies

$$\sup_{p \in M} \mu^\xi(p) < a(\xi),$$

and hence $a \notin \mu(M)$. This shows that $\mu(M)$ is the convex hull of the points $a_1, \cdots, a_N$ and Theorem A is proved.

Remark 3.13. By applying the same arguments of our proof and Theorem 5.1 in [**14**], Theorem A still holds in the case that M is a compact connected H-twisted generalized complex orbifold. In this case, all connected components $C_1, \cdots, C_N$ of the critical set are connected generalized complex suborbifolds.

4. Non-abelian convexity and connectedness properties

The purpose of this section is to give a proof of Theorem B. Our proof is a simple generalization of the argument of Lerman, Meinrenken, Tolman and Woodward in [**13**] to generalized complex geometry.

4.1. Weak nondegeneracy of generalized moment maps

In this subsection, we introduce an additional property "weak nondegeneracy" for generalized moment maps, which always holds for compact cases.

Definition 4.1. We say that a generalized moment map $\mu : M \to \mathfrak{g}^*$ has *weak nondegeneracy* if the following equality holds for all $\xi \in \mathfrak{g}$;

$$\mathrm{Crit}(\mu^\xi) = \mathrm{Fix}(T^\xi).$$

Example 4.2. Let a compact Lie group G act on a symplectic manifold (M, ω) in a Hamiltonian way with a moment map $\mu : M \to \mathfrak{g}^*$. Then the G-action on $(M, \mathcal{J}_\omega)$ is Hamiltonian with a generalized moment map μ and a moment one form $\alpha = 0$. In this case, the generalized moment map μ has weak nondegeneracy. Indeed, since $d\mu^\xi = \iota_{\xi_M}\omega$ for each $\xi \in \mathfrak{g}$, it follows that $\xi_M = 0$ if and only if $d\mu^\xi = 0$.

Example 4.3. Consider the trivial action of a compact torus T^m on a complex manifold (M, J). Then by identifying the Lie algebra $\mathfrak{t}$ with $\mathbb{R}^m$, each holomorphic map $h = (h_1, \cdots, h_m) : M \to \mathbb{C}^m$ defines a generalized moment map $\mu = \mathrm{Im}\ h$ and a moment one form $\alpha = d(\mathrm{Re}\ h) = (d(\mathrm{Re}\ h_1), \cdots, d(\mathrm{Re}\ h_m))$ for the T^m-action, where Re h (resp. Im h) denotes the real part (resp. the imaginary part) of h. In this case, μ has weak nondegeneracy if and only if h is locally constant, because ξ_M reduces to 0 for all $\xi \in \mathfrak{t}$.

By the former part of the proof of Proposition 3.9, we see that a generalized moment map for compact manifolds always has weak nondegeneracy. Moreover, the latter part of the proof of Proposition 3.9 tells us that, for noncompact manifolds, a generalized moment map having weak nondegeneracy is nondegenerate in the sense of abstract moment maps in Ginzburg-Guillemin-Karshon [**6**].

REMARK 4.4. Let a compact Lie group G act on an H-twisted generalized complex orbifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$. If μ has weak nondegeneracy, as in the case of symplectic orbifolds, the image of the differential of the generalized moment map at a point $p \in M$ is the annihilator of the corresponding isotropy Lie algebra $\mathfrak{g}_p$. In particular, we see that the following conditions are equivalent:

1. $p \in M$ is a regular point of μ.
2. $\mathfrak{g}_p = \{0\}$.
3. The G-action at p is locally free.

We prove the assertion here. Let $\tilde{T}_pM$ denote the uniformized tangent space of M at p. For each $\xi \in \mathfrak{g}$, weak nondegeneracy condition of the generalized moment map implies that $(d\mu^\xi)_p = 0$ if and only if $(\xi_M)_p = 0$. Since

$$(\mu_*)_p(X)(\xi) = (d\mu^\xi)_p(X) \tag{9}$$

for each $X \in \tilde{T}_pM$, we have $(\mu_*)_p(X)(\xi) = 0$ for all $\xi \in \mathfrak{g}_p$. This shows that the image of $(\mu_*)_p$ is contained in the annihilator $(\mathfrak{g}_p)^0$. On the other hand, the equation (9) implies that $X \in \ker(\mu_*)_p$ if and only if $(d\mu^\xi)_p(X) = 0$ for all $\xi \in \mathfrak{g}$. Hence we obtain the equation $\ker(\mu_*)_p = (D\mu)_p^0$, where $(D\mu)_p$ is the subspace of $\tilde{T}_p^*M$ generated by the differentials $(d\mu^\xi)_p$ for all $\xi \in \mathfrak{g}$ and $(D\mu)_p^0 \subset T_pM$ is its annihilator. In addition, since $\dim(D\mu)_p = \dim \mathfrak{g} - \dim \mathfrak{g}_p$ by weak nondegeneracy condition, we have $\dim \ker(\mu_*)_p = \dim M - (\dim \mathfrak{g} - \dim \mathfrak{g}_p)$. Hence we have

$$\dim(\mu_*)_p(\tilde{T}_pM) = \dim \mathfrak{g} - \dim \mathfrak{g}_p = \dim(\mathfrak{g}_p)^0$$

and so $(\mu_*)_p(\tilde{T}_pM) = (\mathfrak{g}_p)^0$. This shows the assertion. In particular, the generalized moment map has constant rank on the principal stratum M_{prin}, an open dense subset of M defined to be the intersection of the set of the points of principal orbit type with the set of smooth points of M. (See [**6**] for the definition of the principal orbit type.)

4.2. Generalized complex cuts

In view of symplectic geometry, we introduce the notion of generalized complex cutting. Let $(M, \mathcal{J})$ be an H-twisted generalized complex orbifold which admits a Hamiltonian circle action with a generalized moment map $\mu : M \to \mathbb{R}$ and a moment one form $\alpha \in \Omega^1(M)$. We assume that the generalized moment map μ has weak nondegeneracy.

For a regular value $\varepsilon \in \mathbb{R}$ of the generalized moment map, consider the disjoint union

$$M_{[\varepsilon,+\infty)} = \mu^{-1}((\varepsilon,+\infty)) \cup M_\varepsilon$$

obtained from the orbifold with boundary $\mu^{-1}([\varepsilon,+\infty))$ by collapsing the boundary under the circle action. Then the disjoint union $M_{[\varepsilon,+\infty)}$ admits a natural structure of a twisted generalized complex orbifold. To see this, consider the product $M \times \mathbb{C}$ of the orbifold with a complex plane. It has a natural product H-twisted generalized complex structure:

$$\mathcal{J}_{M\times\mathbb{C}} = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}_\omega \end{pmatrix},$$

where $\mathcal{J}_\omega$ is the natural generalized complex structure on $\mathbb{C}$ induced by the standard symplectic structure $\omega = (\sqrt{-1}/2)dz \wedge d\bar{z}$. The function $\nu : M \times \mathbb{C} \to \mathbb{R}$ given by $\nu(p,z) = \mu(p) - (1/2)|z|^2$ is a generalized moment map for the diagonal action of the circle, and the pull back of the moment one form α by the natural projection from $M \times \mathbb{C}$ to M is a moment one form. Since μ has weak nondegeneracy, so does ν. The point $\varepsilon \in \mathbb{R}$ is a regular value of ν if and only if it is a regular value of μ. Moreover, the map

$$\{p \in M \mid \mu(p) \geq \varepsilon\} \to \nu^{-1}(\varepsilon),\ p \mapsto (p, \sqrt{\mu(p)-\varepsilon})$$

induces a homeomorphism from $M_{[\varepsilon,+\infty)}$ to the reduced space $\nu^{-1}(\varepsilon)/S^1$. By weak nondegeneracy of the generalized moment map ν, we see that the reduced space admits a natural $\tilde{H}$-twisted generalized complex structure. In particular, $M_{[\varepsilon,+\infty)}$ also admits a twisted generalized complex structure which is induced by the $\tilde{H}$-twisted generalized complex structure on the orbifold $\nu^{-1}(\varepsilon)/S^1$.

Definition 4.5. We call the twisted generalized complex orbifold $M_{[\varepsilon,+\infty)}$ the generalized complex cut of M with respect to the ray $[\varepsilon,+\infty)$.

The construction can be generalized to general torus actions as follows. Consider a Hamiltonian action of an m-dimensional torus T^m on an H-twisted generalized complex orbifold $(M,\mathcal{J})$ with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M;\mathfrak{t}^*)$. We assume that the generalized moment map μ has weak nondegeneracy. Let $l \subset \mathfrak{t}$ denote the integral lattice. Choose N vectors $v_j \in l,\ j = 1,\cdots,N$. The endomorphism

$$\mathcal{J}_{M\times\mathbb{C}^N} = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}_\omega \end{pmatrix}$$

is an H-twisted generalized complex structure on an orbifold $M \times \mathbb{C}^N$, where $\mathcal{J}_\omega$ is the natural generalized complex structure on $\mathbb{C}^N$ induced by the standard symplectic

structure $\omega = (\sqrt{-1}/2)\sum_{i=1}^{N} dz^i \wedge d\bar{z}^i$. The map $\nu : M \times \mathbb{C}^N \to \mathbb{R}^N$ with j-th component

$$\nu_j(p,z) = \langle \mu(p), v_j \rangle - \frac{1}{2}|z_j|^2$$

is a generalized moment map for the action of T^N on $M \times \mathbb{C}^N$ induced by the Lie algebra homomorphism $\mathbb{R}^N \to \mathfrak{t},\ e_j \mapsto v_j$, where $\{e_1, \cdots, e_N\}$ is the standard basis of $\mathbb{R}^N$. The $\mathbb{R}^N$-valued 1-form β with j-th component $\beta_j(p,z) = \langle \alpha(p), v_j \rangle$ is a moment one form. Because of weak nondegeneracy of μ, the generalized moment map ν also has weak nondegeneracy. For each $b = (b_1, \cdots, b_N) \in \mathbb{R}^N$, we define a convex rational polyhedral set

$$P = \{x \in \mathfrak{t}^* \mid \langle x, v_j \rangle \geq b_j,\ j = 1, \cdots, N\}.$$

The generalized complex cut of M with respect to a rational polyhedral set P is the reduction of $M \times \mathbb{C}^N$ at b. We denote it by M_P. If b is a regular value of ν, then M_P is a twisted generalized complex orbifold by Remark 4.4. Note that regular values are generic by Sard's theorem. Furthermore if P is a compact polytope, then the fact that P is generic implies that P is simple, that is, the number of codimension one faces meeting at a given vertex is the same as the dimension of P.

A topological description of the cut space is given by the following result. This is a generalization of Proposition 2.4 in [**13**] to generalized complex geometry and we can apply their proof of the theorem by replacing moment maps with generalized moment maps.

PROPOSITION 4.6. *Let an m-dimensional torus T^m act on an H-twisted generalized complex orbifold $(M, \mathcal{J})$ effectively and in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{t}^*)$. Suppose that the generalized moment map μ has weak nondegeneracy. Consider a generic rational polyhedral set $P \subset \mathfrak{t}^*$ and the set of all open faces $\mathcal{F}_P$. Then the topological space $\tilde{M}_P$ defined by*

$$\tilde{M}_P = \bigcup_{F \in \mathcal{F}_P} \mu^{-1}(F)/T_F,$$

where $T_F \subset T^m$ is the subtorus of T^m perpendicular to F, coincides with the generalized complex cut of M with respect to P. In particular, $\tilde{M}_P$ is an H-twisted generalized complex orbifold with a natural Hamiltonian action of the torus T^m. Moreover, the map $\mu_P : \tilde{M}_P \to \mathfrak{t}^*$ induced by the restriction $\mu|_{\mu^{-1}(P)}$ is a generalized moment map, and the descending of the restriction $\alpha|_{\mu^{-1}(P)}$ of the moment one form is a moment one form for this action. Consequently,

1. *the cut space $\tilde{M}_P$ is connected if and only if $\mu^{-1}(P)$ is connected;*
2. *the fibers of μ_P are connected if and only if fibers of $\mu|_{\mu^{-1}(P)}$ are connected;*
3. *$\tilde{M}_P$ is compact if and only if $\mu^{-1}(P)$ is compact.*

Using the technique of generalized complex cuts, we can extend Theorem A to

the case that M is a noncompact orbifold and the generalized moment map has weak nondegeneracy. The proof is the same with the proof of Theorem 4.3 in [**13**], except one must use the generalized complex cuts.

THEOREM 4.7. *Let an m-dimensional torus T^m act on a connected H-twisted generalized complex orbifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{t}^*)$. If μ is proper as a map into a convex open set $U \subset \mathfrak{t}^*$ and has weak nondegeneracy, then*

1. *the image of μ is convex,*
2. *each fiber of μ is connected, and*
3. *if for every compact set $K \subset \mathfrak{t}^*$, the list of isotropy algebras for the T^m-action on $\mu^{-1}(K)$ is finite, then the image $\mu(M)$ is the intersection of U with a rational locally polyhedral set.*

4.3. The cross-section theorem

Here we recall the notion of *slices* for group actions and prove a generalized complex geometry analogue of the cross-section theorem in symplectic geometry.

DEFINITION 4.8. Suppose that a group G acts on an orbifold M. Given $p \in M$ with isotropy group G_p, a suborbifold $U \subset M$ containing p is called a slice at p if U is G_p-invariant, $G \cdot U$ is a neighborhood of p, and the map

$$G \times_{G_p} U \to G \cdot U, \ [a, u] \mapsto a \cdot u$$

is an isomorphism.

Consider the coadjoint action of a connected compact Lie group G on $\mathfrak{g}^*$. For each $x \in \mathfrak{g}^*$, there is a unique largest open subset $U_x \subset \mathfrak{g}_x^* \subset \mathfrak{g}^*$ which is a slice at x. We call U_x the natural slice at x for the coadjoint action. A detailed construction can be seen in [**13**].

THEOREM 4.9 (CROSS-SECTION). *Let a compact connected Lie group G act on an H-twisted generalized complex orbifold $(M, \mathcal{J})$ in a Hamiltonian way with a generalized moment map $\mu : M \to \mathfrak{g}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{g}^*)$. Consider the natural slice U at $x \in \mathfrak{g}^*$ for the coadjoint action. Then the cross-section $R = \mu^{-1}(U)$ is a G_x-invariant generalized complex suborbifold of M, where G_x is the isotropy group of x. Furthermore the G_x-action on R is Hamiltonian with a generalized moment map $\mu_R := \mu|_R$ and a moment one form $\alpha|_R$, the restriction of α to R.*

We shall give a proof of Theorem 4.9 below. First note that since the slice U is G_x-invariant and the generalized moment map μ is equivariant, the cross-section $R = \mu^{-1}(U)$ is also G_x-invariant. By definition of the slice, coadjoint orbits intersect U transversally. Since the generalized moment map is equivariant, it is transversal to U as well. Hence the cross-section is a suborbifold of M. We need to show that the cross-section R is a generalized complex suborbifold of M. We shall show that $(L_R)_r \subset (\tilde{T}_r R \oplus \tilde{T}_r^* R) \otimes \mathbb{C}$ defines an i^*H-twisted generalized complex structure of R, where $\tilde{T}_r R$

is the uniformized tangent space of R at $r \in R$. Then we can see easily that R is a generalized complex suborbifold. Consider a local representative φ of L. If the pull back $(i^*\varphi)_r$ is a nondegenerate complex pure spinor, then it is a local representative of L_R and hence L_R defines an i^*H-twisted generalized complex structure. Hence we may only show that the pull back $(i^*\varphi)_r$ is a nondegenerate complex pure spinor of R below.

Since φ_r is a nondegenerate complex pure spinor, there exists a decomposable complex k-form $\Omega \in \wedge^k \tilde{T}_r^* M \otimes \mathbb{C}$ and a complex 2-form $B + \sqrt{-1}\omega \in \wedge^2 \tilde{T}_r^* M \otimes \mathbb{C}$ such that

$$\varphi_r = \exp(B + \sqrt{-1}\omega) \wedge \Omega.$$

The 2-form ω is nondegenerate on the $2(n-k)$-dimensional subspace

$$S_r = \{X \in \tilde{T}_r M \mid i_X(\Omega \wedge \bar{\Omega}) = 0\}.$$

Moreover, we claim that it satisfies that for each $\xi \in \mathfrak{g}$, $i_{(\xi_M)_r}\omega = (d\mu^\xi)_r$ on S_r. Indeed, since $\iota_{\xi_M}\varphi_r - \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi) \wedge \varphi_r = 0$ by the definition of generalized moment maps, we have $\iota_{\xi_M}\Omega = 0$ and hence

$$\iota_{\xi_M}(B + \sqrt{-1}\omega) \wedge \Omega = \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi) \wedge \Omega.$$

If we write $\Omega = \theta^1 \wedge \cdots \wedge \theta^k$ by some 1-forms $\theta^1, \cdots, \theta^k \in \tilde{T}_r^* M \otimes \mathbb{C}$, the vectors $\theta^1, \cdots, \theta^k, \bar{\theta^1}, \cdots, \bar{\theta^k}$ are linearly independent because the complex pure spinor $\varphi_r = \exp(B + \sqrt{-1}\omega) \wedge \Omega$ is nondegenerate. This implies that $\iota_X \Omega = 0$ and

$$\iota_{\xi_M}(B + \sqrt{-1}\omega)(X) \wedge \Omega = \sqrt{-1}(d\mu^\xi + \sqrt{-1}\alpha^\xi)(X) \wedge \Omega$$

for each $X \in S_r$. Hence we obtain $i_{(\xi_M)_r}\omega(X) = (d\mu^\xi)_r(X)$ for each $\xi \in \mathfrak{g}$ and $X \in S_r$. This shows the claim.

Consider the complex form on $\wedge^\bullet \tilde{T}_r^* R \otimes \mathbb{C}$ defined by

$$(i^*\varphi)_r = \exp(i^*B + \sqrt{-1}i^*\omega) \wedge i^*\Omega.$$

To prove that $(i^*\varphi)_r$ is a nondegenerate complex pure spinor, we need to show the following statements:

1. $i^*\Omega \wedge i^*\bar{\Omega} \neq 0$, in particular $i^*\Omega \neq 0$.
2. $i^*\omega$ is nondegenerate on the subspace $S_r(R) = \{X \in \tilde{T}_r R \mid i_X(i^*\Omega \wedge i^*\bar{\Omega}) = 0\}$.

We first show the claim 1. For the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{g}_x$ denotes the Lie algebra of the stabilizer of $x \in \mathfrak{g}^*$. Then there exists a G_x-invariant subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{m}$. For $y = \mu(r)$, the uniformized tangent space $\tilde{T}_y U$ is just the annihilator of $\mathfrak{m}$. Consider the subspace $\mathfrak{m}_M(r) = \{(\xi_M)_r \mid \xi \in \mathfrak{m}\}$. Note that $\mathfrak{m}_M(r) \subset S_r$ and $\dim \mathfrak{m}_M(r) = \dim \mathfrak{m}$. Now we show the following lemmata.

LEMMA 4.10. *The subspace $\mathfrak{m}_M(r)$ is a symplectic vector space with respect to the 2-form ω and is perpendicular to $S_r \cap \tilde{T}_r R$.*

Proof. First observe that for $\xi \in \mathfrak{m}$ and $X \in S_r \cap \tilde{T}_r R$,

$$\omega((\xi_M)_r, X) = ((\mu_*)_r(X))(\xi) = 0$$

since $(\mu_*)_r(X) \in T_y U = \mathfrak{m}^\circ$. Hence $\mathfrak{m}_M(r)$ is perpendicular to $S_r \cap \tilde{T}_r R$ with respect to the 2-form ω.

Now we show that the subspace $\mathfrak{m}_M(r)$ is a symplectic vector space. Since for $\xi, \eta \in \mathfrak{m}$,

$$\omega((\xi_M)_r, (\eta_M)_r) = ((\mu_*)_r(\eta_M))(\xi) = (\mathrm{ad}^*(\eta)\mu(r))(\xi) = -y([\xi, \eta]),$$

$\mathfrak{m}_M(r)$ is symplectic if and only if $\mathrm{ad}^*(\mathfrak{m})y$ is a symplectic subspace of the tangent space $T_y(G \cdot y)$ of the coadjoint orbit $G \cdot y$. Since $G_x \cdot y \subset U$ and since $\mathfrak{m} = (T_y U)^\circ$, for each $\xi \in \mathfrak{m}$ and $\eta \in \mathfrak{g}_x$ we have

$$y([\xi, \eta]) = \mathrm{ad}^*(\eta)(y)\xi = 0,$$

that is, $T_y(G_x \cdot y)$ and $\mathrm{ad}^*(\mathfrak{m})y$ are symplectically perpendicular in $T_y(G \cdot y)$. Hence it remains to show that the orbit $G_x \cdot y$ is a symplectic submanifold of the coadjoint orbit $G \cdot y$ because $T_y(G \cdot y) = T_y(G_x \cdot y) \oplus \mathrm{ad}^*(\mathfrak{m})y$. Since the natural projection $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{g}_x^*$ is G_x-equivariant, we have $\mathrm{pr}(G_x \cdot y) = G_x \cdot \mathrm{pr}(y)$. By the definition of the symplectic forms on a coadjoint orbit the restriction of the symplectic form of $G \cdot y$ to $G_x \cdot y$ is just the pull-back by pr of the symplectic form of the G_x coadjoint orbit $G_x \cdot \mathrm{pr}(y)$. Hence $G_x \cdot y$ is a symplectic submanifold of $G \cdot y$, and this proves the lemma. □

LEMMA 4.11. *The uniformized tangent space $\tilde{T}_r M$ can be decomposed into the following direct sum:*

$$\tilde{T}_r M = \tilde{T}_r R \oplus \mathfrak{m}_M(r).$$

Proof. If $X \in \tilde{T}_r R \cap \mathfrak{m}_M(r)$, then X is perpendicular to $\mathfrak{m}_M(r)$ with respect to ω by Lemma 4.10. Since ω is nondegenerate on $\mathfrak{m}_M(r)$, we have $X = 0$ and hence $\tilde{T}_r R \cap \mathfrak{m}_M(r) = \{0\}$. Furthermore, since $\dim R = \dim M - \dim \mathfrak{m}$, we see that $\dim \tilde{T}_r M = \dim \tilde{T}_r R + \dim \mathfrak{m}_M(r)$, and obtain the decomposition $\tilde{T}_r M = \tilde{T}_r R \oplus \mathfrak{m}_M(r)$. □

The decomposition induces the decomposition of S_r;

$$S_r = (S_r \cap \tilde{T}_r R) \oplus \mathfrak{m}_M(r),$$

because $\mathfrak{m}_M(r)$ is contained in S_r. Hence we have the dimension

$$\dim S_r \cap \tilde{T}_r R = \dim \tilde{T}_r R - 2k.$$

This shows that we can choose a basis of $\tilde{T}_r M \otimes \mathbb{C}$;

$$e_1, \cdots, e_a, u_1, \cdots, u_{2k}, v_1, \cdots, v_{2(n-k)-a},$$

where $a = \dim S_r \cap \tilde{T}_r R$, $\{e_1, \cdots, e_a\}$ is a basis of $S_r \cap \tilde{T}_r R$, $\{e_1, \cdots, e_a,\ u_1, \cdots, u_{2k}\}$ is a basis of $\tilde{T}_r R$ and $\{v_1, \cdots, v_{2(n-k)-a}\}$ is a basis of $\mathfrak{m}_M(r)$. Since $e_i, v_j \in S_r$, we have $i_{e_i}(\Omega \wedge \bar{\Omega}) = i_{v_j}(\Omega \wedge \bar{\Omega}) = 0$. Hence we see that $\Omega \wedge \bar{\Omega}(u_1, \cdots, u_{2k}) \neq 0$ because $\Omega \wedge \bar{\Omega} \neq 0$ on $\tilde{T}_r M$. This shows that $i^*\Omega \wedge i^*\bar{\Omega} \neq 0$, and hence we have proved the claim 1.

Now we prove the claim 2. We can check easily that $S_r \cap \tilde{T}_r R \subset S_r(R)$. Since $i^*\Omega \wedge i^*\bar{\Omega} \neq 0$, we have

$$\dim(S_r \cap \tilde{T}_r R) = \dim S_r(R) = \dim R - 2k.$$

Hence we obtain the equation $S_r \cap \tilde{T}_r R = S_r(R)$. Now take a vector $X \in S_r(R)$ which is perpendicular to $S_r(R)$ with respect to ω, that is, $\omega(X, Y) = 0$ for any $Y \in S_r(R)$. Then since $\omega(X, (\xi_M)_r) = 0$ for any $\xi \in \mathfrak{m}$, we see that $\omega(X, Y) = 0$ for any $Y \in S_r$. Since ω is nondegenerate on S_r, we have $X = 0$ and hence ω is also nondegenerate on $S_r(R)$. This proves the claim 2.

By the claims 1 and 2, we see that $(i^*\varphi)_r$ is a nondegenerate complex pure spinor and that R is a generalized complex suborbifold of M. Finally, it is clear that the G_x-action on R preserves the induced i^*H-twisted generalized complex structure and is Hamiltonian with a generalized moment map $\mu_R = \mu|_R$ and the moment one form $i^*\alpha$. This completes the proof of Theorem 4.9.

4.4. A proof of Theorem B

By Remark 4.4 and Theorem 4.9, we can extend Theorem 3.1 in Lerman-Meinrenken-Tolman-Woodward [**13**] to generalized complex geometry, and we see that there is a unique open face σ of the Weyl chamber $\mathfrak{t}^*_+$ such that

1. $\mu(M) \cap \sigma$ is dense in $\mu(M) \cap \mathfrak{t}^*_+$,
2. the preimage $Y = \mu^{-1}(\sigma)$ is a connected T-invariant generalized complex suborbifold of M, and the restriction $\mu_Y = \mu|_Y$ and the pull back of α to Y are a generalized moment map and a moment one form for the action of the maximal torus T, and
3. the set $G \cdot Y$ is dense in M.

(See the proof of Theorem 3.1 in [**13**].) Since μ is proper, the restriction $\mu_Y : Y \to \mathfrak{t}^*$ is proper as a map into the open convex set σ. By Theorem 4.7, the image $\mu(Y)$ is convex and is the intersection of σ with a locally polyhedral set P, that is, $\mu(Y) = \sigma \cap P$. Therefore we have $\mu(M) \cap \mathfrak{t}^*_+ = \overline{\mu(Y)}$. Since the closure of a convex set is also convex, the moment set $\mu(M) \cap \mathfrak{t}^*_+$ is convex. Moreover, since $\mu(M) \cap \mathfrak{t}^*_+ = \overline{\sigma \cap P} = \bar{\sigma} \cap P$, $\mu(M) \cap \mathfrak{t}^*_+$ is a locally polyhedral set. Thus we have proved the first assertion. Now we shall show that the fiber $\mu^{-1}(x)$ is connected for all $x \in \mathfrak{g}^*$. We may assume $x \in \mathfrak{t}^*_+$. Since the fiber of $\mu|_Y$ is connected, the fiber of the restriction $\mu|_{G \cdot Y}$ is also connected. Observe that since $\mu^{-1}(G \cdot x)/G = \mu^{-1}(x)/G_x$ and the groups G and G_x are connected, the connectedness of $\mu^{-1}(x)$ is equivalent to that of $\mu^{-1}(G \cdot x)$. To prove the connectedness

of $\mu^{-1}(G \cdot x)$, it is suffices to show that for any convex open neighborhood B of x in $\mathfrak{t}_+^*$, the closure of the open set $\mu^{-1}(G \cdot (B \cap \mathfrak{t}_+^*))$ is connected. By the condition 3 of the open face σ, the intersection $\mu^{-1}(G \cdot (B \cap \mathfrak{t}_+^*)) \cap G \cdot Y = G \cdot \mu^{-1}(B \cap \sigma)$ is dense in $\mu^{-1}(G \cdot (B \cap \mathfrak{t}_+^*))$ and hence also dense in its closure. Since $B \cap \sigma \cap \mu(M)$ is convex and $\mu^{-1}(y)$ is connected for each $y \in \sigma$, the set $G \cdot \mu^{-1}(B \cap \sigma)$ is connected and therefore its closure is also connected. This completes the proof of Theorem B.

Remark 4.12. In noncompact cases, the assumption of weak nondegeneracy is essential for the convexity property. For instance, consider the trivial action of 3-dimensional compact torus $G = T^3$ on a complex manifold $M = \mathbb{C}$ with the standard complex structure J and a holomorphic map $h : M \to \mathbb{C}^3$ defined by

$$h(z) = (\sqrt{-1}z, z, z^2).$$

Then by identifying the Lie algebra $\mathfrak{t}$ with $\mathbb{R}^3$, we see that the action is a Hamiltonian action on a generalized complex manifold $(M, \mathcal{J}_J)$ with a generalized moment map

$$\mu = \text{Im } h = (x, y, 2xy)$$

and a moment one form

$$\alpha = d(\text{Re } h) = (-dy, dx, 2xdx - 2ydy),$$

where $z = x + \sqrt{-1}y$. Since the natural identification $\text{id} : M \to \mathbb{R}^2$ is proper, the generalized moment map $\mu : M \to \mathbb{R}^3$ is also proper. In addition, μ does not have weak nondegeneracy because for $\xi = (1, 0, 0) \in \mathfrak{t}$, we have $d\mu^\xi = dx$ and hence $\text{Crit}(\mu^\xi) = \phi$. (Note that $\text{Fix}(T^\xi) = M$ for all $\xi \in \mathfrak{t}$ since the G-action is trivial.) In this case, the convexity property of the generalized moment map does not hold. Indeed, the image of the generalized moment map μ is just the graph of the function of two variables $f(x, y) = 2xy$.

4.5. Concluding remarks

A concept of generalized complex structures arises naturally when we consider a deformation of symplectic structures. Then we can consider a Hamiltonian action on a generalized complex manifold as a family of Hamiltonian actions of symplectic manifolds. We shall give a simple example below.

Let $\mathbb{CP}^2$ be a 2-dimensional complex projective space with the homogeneous coordinates $[z_0 : z_1 : z_2]$, and $\omega_{\text{F.S.}}$ the Fubini-Study metric on $\mathbb{CP}^2$. For each $w = (w_1, w_2) \in \mathbb{C}^* \times \mathbb{C}^*$ we define a projective transformation $T_w \in \text{PGL}(3, \mathbb{C})$ by

$$T_w([z_0 : z_1 : z_2]) = [z_0 : |w_1|z_1 : |w_2|z_2].$$

Then we have a deformation of the Fubini-Study metric $T_w^*\omega_{\text{F.S.}}$. Consider the T^2-action

on $\mathbb{CP}^2$ defined by

$$(\theta_1, \theta_2) \cdot [z_0 : z_1 : z_2] = [z_0 : \theta_1 z_1 : \theta_2 z_2]$$

for all $(\theta_1, \theta_2) \in T^2$. Since the transformation T_w commutes with the T^2-action, the action on a symplectic manifold $(\mathbb{CP}^2, T_w^* \omega_{\text{F.S.}})$ is Hamiltonian with a moment map

$$\mu_w([z_0 : z_1 : z_2]) = -\frac{1}{2|z|^2}(|w_1| \cdot |z_1|^2, |w_2| \cdot |z_2|^2).$$

By symplectic convexity theorem, we see that the image Δ_w of the moment map μ_w is the convex hull of $\{(0,0), (-|w_1|/2, 0), (0, -|w_2|/2)\}$, which is of course a compact polytope.

Here we have obtained a family of Hamiltonian actions on symplectic manifolds. By considering a generalized complex structure, we can treat them at once. Consider the product $M = (\mathbb{C}^*)^2 \times \mathbb{CP}^2$ of an algebraic torus with a projective space. Since the 2-form $T_w^* \omega_{\text{F.S.}}$ is a symplectic form of $\mathbb{CP}^2$ for each $w \in (\mathbb{C}^*)^2$, we can define a complex pure spinor φ on M by

$$\varphi = dw_1 \wedge dw_2 \wedge \exp \sqrt{-1} T_w^* \omega_{\text{F.S.}}.$$

Furthermore, since the complex pure spinor φ is nondegenerate, it defines a generalized complex structure $\mathcal{J}_\varphi$ on M.

Now consider a T^2-action on a generalized complex manifold $(M, \mathcal{J}_\varphi)$ defined by lifting the T^2-action on $\mathbb{CP}^2$ to M;

$$(\theta_1, \theta_2) \cdot (w, [z_0 : z_1 : z_2]) = (w, [z_0 : \theta_1 z_1 : \theta_2 z_2]),$$

for each $(\theta_1, \theta_2) \in T^2$. The T^2-action on $(M, \mathcal{J}_\varphi)$ is Hamiltonian with a generalized moment map

$$\mu(w, [z_0 : z_1 : z_2]) = \mu_w([z_0 : z_1 : z_2])$$

and a moment one form $\alpha = 0$. The image Δ of the generalized moment map μ is a convex polyhedral set,

$$\Delta = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0,\ y \leq 0\}.$$

When we restrict the T^2-action to the fiber $M_w = \{w\} \times \mathbb{CP}^2 \cong \mathbb{CP}^2$, the action on M_w is equivalent to the Hamiltonian T^2-action on $\mathbb{CP}^2$ and the generalized moment map μ restricted to M_w coincides with the moment map μ_w. This shows that we can think of the Hamiltonian T^2-action on a generalized complex manifold $(M, \mathcal{J}_\varphi)$ as a family of Hamiltonian T^2-actions on symplectic manifolds $(\mathbb{CP}^2, T_w^* \omega_{\text{F.S.}})$. Then the image Δ of

the generalized moment map μ coincides with the union of Δ_w;

$$\Delta = \cup_{w\in(\mathbb{C}^*)^2}\Delta_w.$$

Here we see that not only each Δ_w is convex, but the union Δ is also convex. Note that Δ is not compact although Δ_w is a compact polytope for each $w \in (\mathbb{C}^*)^2$.

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