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Studies On Unsteady Flows Through Cascades

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by

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Preface

In modern turbomachinery such as axial flow compressors or turbines, thin airfoils of high aspect ratio are often used owing to the demand of high speed performance and minimization of engine weight. High speed operation lowers the critical flutter frequency and the adoption of thin and high aspect ratio airfoils lowers the natural frequency of the blades. These tendencies make the cascade blades easy to oscillate. Along with the growth of the aircraft in its size and speed the frequent service has made the noise problems around airport a serious social problem. From the viewpoint of engineering these problems can be seen as the unsteady flow problems of airfoil cascades, and intensive researches have been reported on this problem nowadays.

J.S.M.E. founded the Research and Study Division of Unsteady Cascade Problems composed of the authorities of the problems in Japan and The Report of Research and Study on Unsteady Cascade Problems was published on March 1976, in which broad and precise reviews of many recent papers are contained.

Most of the unsteady cascade theories are two dimensional inviscid and linear theories and can be classified into the following three classes from the method applied.

(1) Vortex formulation

A lift in uniform flow corresponds to a vortex and in case the lift fluctuates, free vortices are shed so as to maintain the circulation of the entire flow region constant. The flow field can be represented by a bound vortex at the application point of the external force and vortices which flow downstream with the mainstream velocity.

(2) Acceleration potential formulation

The fluctuating lifts generate fluctuating doublets in the pressure field. Since the pressure is continuous across the shed-off vortex sheets the pressure field can be represented by the bound doublets only at the point of force application. This makes the analysis simple compared to the vortex formulation.

(3) Actuator disk formulation

By representing the cascade by an actuator disk on the assumption of infinitestimal chord length and blade spacing, the flows upstream and downstream of it can be easily found. The flows are coupled together by appropriate conditions in which the finiteness of the chord and the blade spacing can be taken into account approximately.

Most of the theories adopting the formulations (1) or (2) are solved by singular point procedure. In spite of the virtue of the simplicity, the formulation (3) is not in frequent use nowadays with the development of the finite pitch theories by the formulation (1) or (2). Unsteady theories considering the viscous effects are few. One group of them are isolated airfoil theories of boundary layer approach, having much interests on the trailing edge problems. Those analyses are very complicated in its nature and are applicable to restricted flow conditions. Another group is numerical ones owing much to the developments of computer.

The first chapter of this report shows the analytical method of unsteady forces on cascades by applying the acceleration potential method combined with conformal mapping method. The fluid is assumed to be incompressible and inviscid. Transient flows are treated as well as many types of oscillating flows.

In the second chapter the effects of the fluid viscosity are taken into account. Rigorous elementary solutions of linearized Navier-Stokes equation are given on the assumption of small amplitude of oscillation. Firstly they are applied to isolated oscillating airfoil and the effects of the viscosity on unsteady forces are elucidated. Secondly they are applied to cascade. The dissipating sinusoidal gusts are also given to satisfy the basic equations, and the lift and the drag response of the cascade blades to the gusts are given as well as the unsteady forces on the oscillating blades.

In the last chapter, the viscosity and the compressibility are considered simultaneously. Firstly, an actuator disk theory is developed for viscous compressible fluid, in which the three dimensionality effects are also taken into account. Secondly, a finite pitch theory is given on the basis of the actuator disk theory. The singularities of the elementary solutions are analytically shown to agree with those for inviscid or incompressible limits given in the preceding chapters.

The results of these three chapters agree with each other in spite of the different methods applied. All of the numerical caluculations are made to get the unsteady forces on the blades since the entire flow fields can be easily calculated after getting the force distributions.

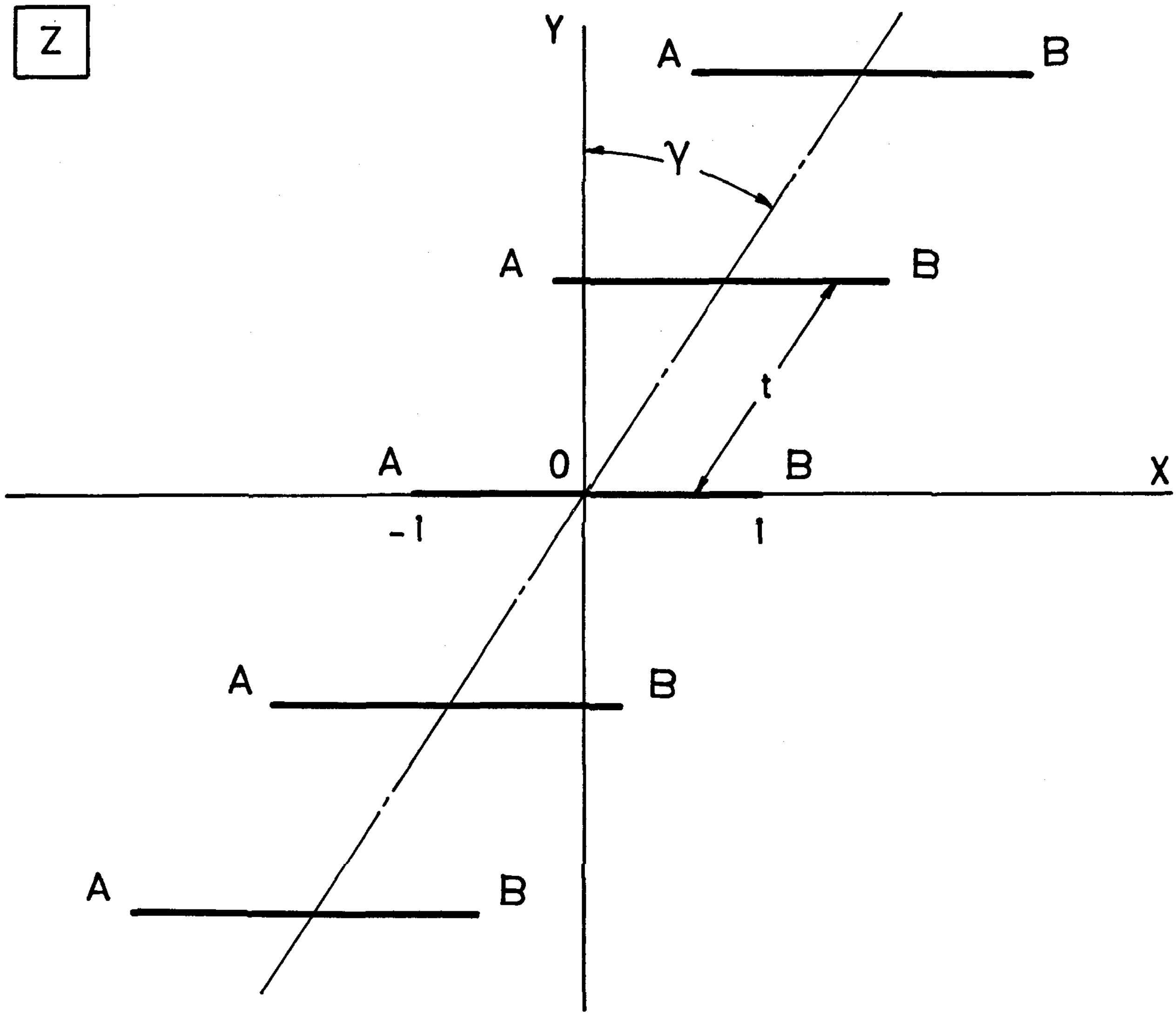

Fig.0 Cascade geometry

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Chapter 1 Inviscid Flow

1.1 Sinusoidal gusts of equal phase

1.1.1 Introduction

The blades of a turbomaschine are subjected to unsteady forces arising from the relative motion of adjacent blade rows. These unsteady forces have important effects on fluttering of blades, noise generation, fatigue failures of blades and, in case of hydraulic maschines, on the cavitation characteristics.

Karman & Sears [1] first suggested to use the prevailing vortex theory for the analysis of the unsteady flow around airfoils. Sears [2] investigated the unsteady forces on a single airfoil in a sinusoidal travelling gust in which the velocity purterbation is normal to the undisturbed flow. Utilizing the results of this work, Kemp & Sears [3] analyzed the potential flow effects of adjacent blade rows, and the lift fluctuation of the blades due to the viscous wakes shed from an upstream blade row. [4] The results of these analyses are valid only for cascades with low solidity because they are based on the single airfoil theory. Whitehead [5] studied the fluctuating forces on a blade in cascade when the blades are in a sinusoidal gust, together with the lift fluctuation on the oscillating blades. Ohashi [6] analyzed the lift fluctuation on the blades of a non-staggered cascade, using vortex theory in combination with a conformal mapping method. For staggered cascades he showed the way to apply his results for non-staggered cascades considering the ratio of circulation around a blade of staggered cascade to that around a blade of non-staggered cascade. Schorr & Reddy [7] treated the flow of sinusoidal gusts through cascades and showed numerical results solving approximately the integral equations for the determination of vortex distribution.

All the analyses mentioned above treated the velocity fluctuation normal to the relative flow direction. Horlock [8] analyzed the fluctuating lift on a single airfoil with an angle of attack due to a longitudinal (i.e., parallel to the undisturbed flow) gust. Applying his results to a viscous wake interaction problem, he found that those two fluctuating lifts produced by gusts parallel to and normal to the relative flow direction are, for the most part, opposite in sign and tend to cancel each other, then he presented the designing method to reduce the lift fluctuation to the minimum. Nauman & Yeh [9] considered the effect of camber on lift fluctuation arising from longitudinal velocity fluctuation.

This chapter presents an analysis of fluctuating lift on the flat plate airfoils in cascade produced by gusts parallel to and normal to the undisturbed flow direction, introducing a conformal mapping method for acceleration potential which makes the analysis clearer than the conventional analysis of the velocity field as was shown by M.A.Biot[12] for the case of single oscillating airfoil. The analysis in this section is confined to the case, in which the phase angle of the velocity fluctuation is the same for all the blades. In actual machines the blade numbers of stator and rotor are usually different, but the results of this section will be useful as an approximation, because the difference of blade numbers between stator and rotor is usually small in actual machines.

1.1.2 Fundermental equations

In the Euler's equations (1),(2), we assume the fluid is incompressible.

$$\frac{\partial v_x}{\partial \tau} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = a_x \tag{1}$$

$$\frac{\partial v_y}{\partial \tau} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = a_y \tag{2}$$

We divide the velocity v_x , v_y into stationary components U_s , V_s and non-stationary components u_d , v_d .

$$v_x = U_s + u_d \tag{3}$$ $$v_y = V_s + v_d \tag{4}$$

Furthermore we write,

$$U_s = U + u_s \tag{5}$$ $$V_s = v_s \tag{6}$$

where u_d , v_d , u_s , $v_s \ll U$. If we neglect the small quantities of higher order, the Euler's equations may be written as follows.

$$\left(\frac{\partial u_d}{\partial \tau} + U\frac{\partial u_d}{\partial x}\right) + \left[\frac{\partial}{\partial x}(u_d \cdot u_s) + v_s\frac{\partial u_d}{\partial y} + v_d\frac{\partial u_s}{\partial y}\right] + \left(U\frac{\partial u_s}{\partial x}\right) = a_x \tag{7}$$

$$\left(\frac{\partial v_d}{\partial t} + U\frac{\partial v_d}{\partial x}\right) + \left[\frac{\partial}{\partial y}(v_d \cdot v_s) + u_s\frac{\partial v_d}{\partial x} + u_d\frac{\partial v_s}{\partial x}\right] + \left(U\frac{\partial v_s}{\partial x}\right) = a_y \tag{8}$$

Then the acceleration a_x , a_y may be divided into three components.

$$\frac{\partial u_d}{\partial t} + U\frac{\partial u_d}{\partial x} = a_{xd} \tag{9}$$

$$\frac{\partial v_d}{\partial t} + U\frac{\partial v_d}{\partial x} = a_{yd} \tag{10}$$

$$\frac{\partial}{\partial x}(u_d \cdot u_s) + v_s\frac{\partial u_d}{\partial y} + v_d\frac{\partial u_s}{\partial y} = a_{xds} \tag{11}$$

$$\frac{\partial}{\partial y}(v_d \cdot v_s) + u_s\frac{\partial v_d}{\partial x} + u_d\frac{\partial v_s}{\partial x} = a_{yds} \tag{12}$$

$$U\frac{\partial u_s}{\partial x} = a_{xs} \tag{13}$$

$$U\frac{\partial v_s}{\partial x} = a_{ys} \tag{14}$$

where,

$$a_x = a_{xd} + a_{xds} + a_{xs} \quad , \quad a_y = a_{yd} + a_{yds} + a_{ys}$$

The acceleration components a_{xd} , a_{yd} are due to the non-stationary purterbating velocity u_d , v_d , components a_{xs} , a_{ys} are due to u_s , v_s , and a_{xds} , a_{yds} are due to both stationary components u_s , v_s and non-stationary ones u_d , v_d . Since we are interested in non-stationary components only, we take the acceleration components except the stationary components into account.

1.1.3 Acceleration potential

We consider the acceleration potential Φ_d defined as $\partial\Phi_d/\partial x = a_{xd}$, $\partial\Phi_d/\partial y = a_{yd}$ for the acceleration components a_{xd} , a_{yd} and the acceleration potential Φ_{ds} defined as $\partial\Phi_{ds}/\partial x = a_{xds}$, $\partial\Phi_{ds}/\partial y = a_{yds}$ for the acceleration components a_{xds} , a_{yds} . Then the equation of continuity gives $\nabla^2\Phi_d = 0$. Let us consider the function Ψ_d defined by the equations; $\partial\Psi_d/\partial y = a_{xd}$, $-\partial\Psi_d/\partial x = a_{yd}$, then the function satisfies the equation $\nabla^2\Psi_d = 0$. The complex potential may be constructed in terms of the above mentioned functions Φ_a , Ψ_a as follows

$$\overline{W}_a = \Phi_a + i\Psi_a \tag{15}$$

Then the function $\overline{W}_a$ is analytic so long as the Euler's equations and the continuity equation are satisfied. This fact is remarkable in contrast with the fact that the complex potential in a velocity field is analytic only for irrotational flow. For this reason, the acceleration potential method can be the most powerful meaning for the analysis of rotational flow such as the non-stationary flow around blades where the shed-off vortices exist. On the other hand, the acceleration potential Φ_{as} does not satisfy the Laplacian equation, so we must use Eqs.(11),(12) directly as the fundermental equations. Between the fluctuating pressure p and the acceleration potential Φ_a , Φ_{as} we have the following relation.

$$p = -\rho\,(\Phi_a + \Phi_{as}) \tag{16}$$

1.1.4 Sinusoidal gusts

Consider a cascade of chordlength C (= 2), blade spacing t , stagger γ in the Z -plane as shown in Fig.0 at the commencement. For the cascade,we assume the gusts, in which the velocities v_x , v_y at $x=-\infty$ are given as follows (for the n-th blade)

For sinusoidal gusts in x -direction

$$\begin{aligned} v_x &= U + u_0 \exp\left[j\omega\left(\tau - \frac{x - nt\sin\gamma}{U}\right)\right] \\ v_y &= 0 \end{aligned} \tag{17}$$

For sinusoidal gusts in y -direction

$$\begin{aligned} v_x &= U \\ v_y &= v_0 \exp\left[j\omega\left(\tau - \frac{x - nt\sin\gamma}{U}\right)\right] \end{aligned} \tag{18}$$

Where we assume $u_0 \ll U$, $v_0 \ll U$, and U means the velocity of the main flow with an angle of attack β . (Fig.1)

1.1.5 Conformal mapping

Both the singular point method and the conformal mapping method are applicable for the determination of the acceleration potential for an unsteady flow. We take the latter method, in which the blade surfaces are mapped to a unit circle in ζ -plane. The mapping function is;

$$Z = (1/2g)\left[e^{-i\gamma} \log \frac{e^{\varepsilon}+\zeta}{e^{\varepsilon}-\zeta} + e^{i\gamma} \log \frac{\zeta + e^{-\varepsilon}}{\zeta - e^{-\varepsilon}} \right] \tag{19}$$

where

$$\tan\alpha = \tan\gamma \cdot \tanh\varepsilon \tag{19-1}$$

$$g = \pi/t = \cos\gamma \tanh^{-1}\frac{\cos\alpha}{\cosh\varepsilon} + \sin\gamma \tan^{-1}\frac{\sin\alpha}{\sinh\varepsilon} \tag{19-2}$$

The mapping constants α and ε are determined by Eqs.(19-1),(19-2). By using Eq.(19), the cascade in the Z -plane shown in Fig.0 at the commencement is mapped to a unit circle whose center is at the origin of the ζ -plane. The locations $x=\pm\infty$ in the Z-plane are mapped to the points $\pm e^{\varepsilon}$ in the ζ -plane respectively, and the leading and the trailing deges A , B are mapped to the points $\mp e^{i\alpha}$ (A',B') in the ζ -plane (Fig.2). In the latter part of this section, we will represent the complex acceleration potential in the Z -plane as a function of ζ .

1.1.6 Acceleration potential Φ_d due to non-stationary acceleration components a_{xd} , a_{yd}.

The non-stationary acceleration potential Φ_d due to the components a_{xd} , a_{yd} may be determined independently of stationary velocity components.

1.1.6.1 Boundary condition on the blade surface

A) In case of sinusoidal gusts in -direction

From Eqs.(8),(11), we obtain

$$v_x = U + u_s + u_d \quad (20) \qquad\qquad v_y = v_s + v_d \quad (21)$$

Here we divide the unsteady velocity components u_d , v_d into two parts as follows. For the 0-th blade, for example,

$$\text{(I)} \qquad u_{d1} = u_0 \exp\left[j\omega\left(\tau - \frac{x}{U}\right)\right] , \quad v_{d1} = 0 \tag{22}$$

$$\text{(II)} \quad u_{d2} = u_d - u_0 \exp\left[j\omega\left(\tau - \frac{x}{U}\right)\right], \quad v_{d2} = v_d \tag{23}$$

As the acceleration for the flow (I) equals zero everywhere, it derives no acceleration potential. So in order to get the pressure field it is sufficient to take only the velocity field (II) into account. The boundary condition may be written;

$$v_y = -\beta v_x \tag{24}$$

Also the stationary components satisfy the boundary condition;

$$v_s = -\beta (U + u_s) \tag{25}$$

The boundary condition for non-stationary components may be derived from Eqs.(24),(25) as follows.

$$v_d = -\beta u_d \tag{26}$$

By using u_{d2} , v_{d2} , Eq.(26) may be written as

$$v_{d2} = -\beta \left[u_{d2} + u_0 \exp\left\{ j\omega\left(\tau - \frac{x}{U}\right)\right\} \right] \tag{27}$$

The accelerations a_{xd} , a_{yd} may be written with u_{d2} , v_{d2} as;

$$a_{xd} = \partial u_{d2} / \partial \tau + U \, \partial u_{d2} / \partial x \tag{28}$$

$$a_{yd} = \partial v_{d2} / \partial \tau + U \, \partial v_{d2} / \partial x \tag{29}$$

The boundary condition for a_{xd} , a_{yd} is;

$$a_{yd} = -\beta a_{xd} \tag{30}$$

The unsteady quantities oscillate harmoniously with the same angular velocity ω . They may be written ; $u_{d2} = \bar{u}_{d2} e^{j\omega\tau}$, $v_{d2} = \bar{v}_{d2} e^{j\omega\tau}$, $a_{xd} = \bar{a}_{xd} e^{j\omega\tau}$, $a_{yd} = \bar{a}_{yd} e^{j\omega\tau}$ and so on. The quantities $\bar{u}_{d2}$, $\bar{v}_{d2}$, $\bar{a}_{xd}$, $\bar{a}_{yd}$ are complex ones with respect to j , and independent on time τ .

Then Eqs.(28),(29) give

$$\bar{a}_{xd} = j\omega\ \bar{u}_{d2} + U\ \partial \bar{u}_{d2} / \partial x \tag{31}$$

$$\bar{a}_{yd} = j\omega\ \bar{v}_{d2} + U\ \partial \bar{v}_{d2} / \partial x \tag{32}$$

The flow infinitely upstream of the cascade is given by Eq.(17), which results in $\bar{u}_{d2} = \bar{v}_{d2} = 0$ at $x = -\infty$. Considering this condition, we may integrate Eqs.(31),(32) as follows.

$$\bar{u}_{d2}(x, y) = (1/U)\exp(-j\omega x/U)\int_{-\infty}^{x} \bar{a}_{xd}(\xi, y)\, e^{j\omega\frac{\xi}{U}} d\xi \tag{33}$$

$$\bar{v}_{d2}(x, y) = (1/U)\exp(-j\omega x/U)\int_{-\infty}^{x} \bar{a}_{yd}(\xi, y)\, e^{j\omega\frac{\xi}{U}} d\xi \tag{34}$$

The boundary condition (27) is then written with $\bar{a}_{xd}$, $\bar{a}_{yd}$;

$$(1/U)\, e^{-jS_t x}\int_{-\infty}^{x} \bar{a}_{yd}(\xi, 0)\, e^{jS_t\xi} d\xi + (\beta/U)\, e^{-jS_t x}\int_{-\infty}^{x} \bar{a}_{xd}(\xi, 0)\, e^{jS_t\xi} d\xi$$

$$+ \beta u_0 e^{-jS_t x} = 0 \qquad (-1 \leq x \leq 1) \tag{35}$$

For $-1 \leq x \leq 1$, Eq.(30) may be used for the caluculation of Eq.(35), which results in;

$$\int_{-\infty}^{-1} [\bar{a}_{yd}(\xi, 0) + \beta\, \bar{a}_{xd}(\xi, 0)]\, e^{jS_t\xi} d\xi + \beta u_0 U = 0 \tag{36}$$

Equation (36) is the boundary condition of the velocity described in acceleration, and sufficient for any location on the blade surface to suffice the boundary condition, under the condition (30). Though we confind our attention to the 0-th blade, it is easily understood that Eq.(36)is the only velocity-boundary-condition for arbitrary n-th blade if we regard the periodicity along the cascade axis.

B) In case of sinusoidal gusts in y -direction

We also divide the unsteady velocity components u_d , v_d into

$$\text{(I)} \quad u_{d1} = 0 , \quad v_{d1} = v_0 \exp[j\omega(\tau - \frac{x}{U})] \tag{37}$$

$$\text{(II)} \quad u_{d2} = u_d , \quad v_{d2} = v_d - v_0 \exp[j\omega(\tau - \frac{x}{U})] \tag{38}$$

Since the acceleration is zero for the flow (I), it is sufficient to take the velocity field (II) into account. The boundary conditions are;

For the velocity

$$v_{d2} + \beta u_{d2} + v_0 \exp [j\omega(\tau - \frac{x}{U})] = 0 , \quad -1 \leq x \leq 1 \tag{39}$$

For the acceleration

$$a_{yd} = -\beta a_{xd} , \qquad -1 \leq x \leq 1 \tag{40}$$

The acceleration may be written with u_{d2} , v_{d2} as;

$$a_{xd} = \partial u_{d2} / \partial \tau + U \partial u_{d2} / \partial x \tag{41}$$

$$a_{yd} = \partial v_{d2} / \partial \tau + U \partial v_{d2} / \partial x \tag{42}$$

The equations (41) and (42) are the same as Eqs.(31),(32), which lead to Eqs.(33),(34). Then the boundary condition (39) may be written;

$$(1/U)\, e^{-jS_t x} \int_{-\infty}^{x} \bar{a}_{yd}(\xi,0) e^{jS_t \xi} d\xi + (\beta/U)\, e^{-jS_t x} \int_{-\infty}^{x} \bar{a}_{xd}(\xi,0) e^{jS_t \xi} d\xi$$

$$+ v_0\, e^{-jS_t x} = 0 , \qquad -1 \leq x \leq 1 \tag{43}$$

Substitution of Eq.(40) into Eq.(43) yields

$$\int_{-\infty}^{-1} [\bar{a}_{yd}(\xi,0) + \beta \bar{a}_{xd}(\xi,0)] e^{jS_t \xi} d\xi + v_0 U = 0 \tag{44}$$

1.1.6.2 Singularity of acceleration potential at the leading edge

In the flow around the flat-plate-blades which satisfies Kutta-Joukowsky's condition at the trailing edge, the velocity at the leading edge becomes infinity. Since the velocity field is singular at the leading edge, the acceleration with respect to time may be neglected compared with the acceleration with respect to space. Hence near the leading edge,

$$\bar{a}_{xd} = \partial \bar{\Phi}_d / \partial x = U \partial \bar{u}_{d2} / \partial x \tag{45}$$

$$\bar{a}_{y\alpha} = -\partial\bar{\Psi}_\alpha/\partial x = U\,\partial\bar{v}_{d2}/\partial x \tag{46}$$

As the velocities $\bar{u}_{d2}$, $\bar{v}_{d2}$ become zero at $x=-\infty$, the flow represented by $\bar{u}_{d2}$, $\bar{v}_{d2}$ is thought to be irrotational. Considering this and the equation of continuity, we obtain

$$\partial\bar{\Phi}_\alpha/\partial y = U\,\partial\bar{u}_{d2}/\partial y \tag{47}$$

$$\partial\bar{\Psi}_\alpha/\partial y = -U\,\partial\bar{v}_{d2}/\partial y \tag{48}$$

Hence,

$$\bar{\Phi}_\alpha + i\bar{\Psi}_\alpha = U(\bar{u}_{d2} - i\,\bar{v}_{d2}) \tag{49}$$

Equation (49) shows that the complex acceleration potential is proportional to the conjugate complex velocity near a singular point. In general, the complex velocity of the flow around a sharp edge at $Z=-1$ may be written

$$\bar{u}_{d2} - i\,\bar{v}_{d2} = K_1(Z+1)^{-\frac{1}{2}} \tag{50}$$

where K_1 is a constant. Using the mapping function (19) we may transform the region in the vicinity of the leading edge ($Z=-1$), $Z=-1+re^{i\theta}$, ($r\ll 1$, $0\le\theta\le 2\pi$) into the region in the ζ -plane,

$$\zeta = -e^{i\alpha} + r'\exp[i(\theta'+\alpha+\pi/2)],\ (r'\ll 1,\ 0\le\theta'\le\pi)$$

So Eq.(19) may be written in the vicinity of the leading edge (Fig.3)

$$Z+1 = K_2\exp[-2(\alpha+\pi/2)](\zeta+e^{i\alpha})^2 \tag{51}$$

where K_2 is a constant. Then the complex acceleration potential in the vicinity of the leading edge may be written;

$$\bar{\Phi}_\alpha + i\bar{\Psi}_\alpha = U K_1 K_2^{-\frac{1}{2}}\exp[i(\alpha+\pi/2)](\zeta+e^{i\alpha})^{-1} \tag{52}$$

That is, the acceleration potential has singularity of 1-st order (order of ζ^{-1}) at the leading edge.

1.1.6.4 Acceleration complex potential

The boundary conditions on the blade surface are shown by the same equations (30) and (32), for sinusoidal gusts in x and y direction respectively. Therefore the boundary condition for complex acceleration potential is expressed by

$$\beta \Phi_d - \Psi_d = Const. \quad , \quad -1 \leq x \leq 1 \tag{53}$$

In order to get the complex acceleration potential function [10],[11], which has a singular point of 1-st order at the leading edge, and whose imaginary part with respect to i is constant on the blade surface in the ζ -plane.

$$\overline{W}(\zeta) = i(\zeta - e^{i\alpha})/(\zeta + e^{i\alpha}) \tag{54}$$

Then the complex acceleration potential may be written ;

$$\overline{W}_d(\zeta) = (1 + i\beta) A \overline{W}(\zeta) \tag{55}$$

where A is a constant, which is real with respect to i , and complex with respect to j . Writing the real and imaginary part of $\overline{W}_d$ with respect to i with Φ_d , Ψ_d respectively, we obtain ;

$$\beta \Phi_d - \Psi_d = -(1 + \beta^2) A \, Imag[\overline{W}(\zeta)] \tag{56}$$

which satisfies the boundary condition (53) on the blade surface. The complex acceleration potential $\overline{W}_d$ defined by Eq.(55) also satisfies all the other conditions. The condition at the leading edge (singularity of 1-st order) is evidently satisfied, as the function $\overline{W}(\zeta)$ is defined by Eq.(54). The complex acceleration potential $\overline{W}_d$ satisfies also the Kutta-Joukowski's condition, for the acceleration potential should have a singular point at the trailing edge as stated in section 1.1.6.3, if the flow turning the trailing edge exists. The condition which should be satisfied at $x=-\infty$, namely, $\overline{W}_d=0$ at $x=-\infty$, will be adjusted later, by subtracting the value of $\overline{W}_d$ at $x=-\infty$ ($\zeta = -e^{\varepsilon}$) from Eq.(55). The constant A cannot be evaluated from the boundary conditions above mentioned, but from those of velocity, i.e., from Eqs(36) and (44), the value of A can be decided. The upper limit of integration in Eqs.(36),

(44) i.e. the point $x=-1$ should be considered exceptionally, since it is a singular point of acceleration potential. Eqs.(33),(44) may be written for $-\infty < x < -1$

$$\bar{u}_{d2}(x) = (1/v)\, e^{-jS_t x}$$
$$\times\left[\{\bar{\Phi}_d(x) - \bar{\Phi}_d(-\infty)\} e^{jS_t x} - jS_t \int_{-\infty}^{x} \{\bar{\Phi}_d(\xi) - \bar{\Phi}_d(-\infty)\} e^{jS_t \xi} d\xi \right] \quad (57)$$

$$\bar{v}_{d2}(x) = (1/v)\, e^{-jS_t x}$$
$$\times\left[\{\bar{\Psi}_d(x) - \bar{\Psi}_d(-\infty)\} e^{jS_t x} - jS_t \int_{-\infty}^{x} \{\bar{\Psi}_d(\xi) - \bar{\Psi}_d(-\infty)\} e^{jS_t \xi} d\xi \right] \quad (58)$$

Equation (49) is valid in the vicinity of the leading edge. Hence, for $0 < \delta \ll 1$ we obtain

$$\bar{u}_{d2}(-1+\delta) = \bar{u}_{d2}(-1-\delta) - [\bar{\Phi}_d(-1-\delta) - \bar{\Phi}_d(-1+\delta)]/v \quad (59)$$

$$\bar{v}_{d2}(-1+\delta) = \bar{v}_{d2}(-1-\delta) - [\bar{\Psi}_d(-1-\delta) - \bar{\Psi}_d(-1+\delta)]/v \quad (60)$$

Substituting Eqs.(59),(60) in Eqs.(57),(58) we obtain the velocity on the leading edge

$$\bar{u}_{d2}(-1+0) = (1/v)\, e^{jS_t} \times$$
$$\left[\{\bar{\Phi}_d(-1+0) - \bar{\Phi}_d(-\infty)\} e^{-jS_t} - jS_t \int_{-\infty}^{-1-0} \{\bar{\Phi}_d(\xi) - \bar{\Phi}_d(-\infty)\} e^{jS_t \xi} d\xi \right] \quad (61)$$

$$\bar{v}_{d2}(-1+0) = -(1/v)\, e^{jS_t} \times$$
$$\left[\{\bar{\Psi}_d(-1+0) - \bar{\Psi}_d(-\infty)\} e^{-jS_t} - jS_t \int_{-\infty}^{-1-0} \{\bar{\Psi}_d(\xi) - \bar{\Psi}_d(-\infty)\} e^{jS_t \xi} d\xi \right] \quad (62)$$

Then the boundary conditions (36),(44) will become

$$[\{\bar{\Psi}_d(-\infty) - \beta\bar{\Phi}_d(-\infty)\} - \{\bar{\Psi}_d(-1+0) - \beta\bar{\Phi}_d(-1+0)\}]\, e^{-jS_t} + jS_t \times$$
$$\int_{-\infty}^{-1-0} [\{\bar{\Psi}_d(\xi) - \beta\bar{\Phi}_d(\xi)\} - \{\bar{\Psi}_d(-\infty) - \beta\bar{\Phi}_d(-\infty)\}]\, e^{jS_t \xi} d\xi + \beta u_0 v = 0 \quad (63)$$

$$[\{\bar{\Psi}_d(-\infty) - \beta\bar{\Phi}_d(-\infty)\} - \{\bar{\Psi}_d(-1+0) - \beta\bar{\Phi}_d(-1+0)\}]\, e^{-jS_t} + jS_t \times$$

$$\int_{-\infty}^{-1-0} [\{ \bar{\Psi}_a(\xi) - \beta \bar{\Phi}_d(\xi) \} - \{ \bar{\Psi}_a(-\infty) - \beta \bar{\Phi}_d(-\infty) \}] e^{jS_t \xi} d\xi + v_0 \bar{U} = 0 \qquad (64)$$

Equation (55) gives

$$\bar{\Psi}_a - \beta \bar{\Phi}_d = (1+\beta^2) \bar{A} \ Imag [\bar{W}(\xi)]$$

where the terms of higher order than β^2 may be neglected. Then from Eqs. (63),(64), the value of $\bar{A}$ becomes

For sinusoidal gusts in x -direction

$$\bar{A} = \beta u_0 \bar{U} \times$$

$$[\{ Imag \bar{W}(-1+0) - Imag \bar{W}(-\infty) \} e^{-jS_t} - jS_t \int_{-\infty}^{-1-0} \{ Imag \bar{W}(\xi) - Imag \bar{W}(-\infty) \} e^{jS_t \xi} d\xi]^{-1} \qquad (65)$$

For sinusoidal gusts in y -direction

$$\bar{A} = v_0 \bar{U} \times$$

$$[\{ Imag \bar{W}(-1+0) - Imag \bar{W}(-\infty) \} e^{-jS_t} - jS_t \int_{-\infty}^{-1-0} \{ Imag \bar{W}(\xi) - Imag \bar{W}(-\infty) \} e^{jS_t \xi} d\xi]^{-1} \qquad (66)$$

Let us assume the constants $\tilde{A}_x$, $\tilde{A}_y$ defined as follows;

for sinusoidal gusts in x -direction

$$\bar{A} = \beta u_0 \bar{U} \tilde{A}_x \qquad (67)$$

for sinusoidal gusts in y -direction

$$\bar{A} = v_0 \bar{U} \tilde{A}_y \qquad (68)$$

Then the acceleration potential Φ_d will be

for sinusoidal gusts in x -direction

$$\Phi_d = \beta u_0 \bar{U} \tilde{A}_x [Real \bar{W}(x) - \beta Imag \bar{W}(x)] e^{j\omega t} \qquad (69)$$

for sinusoidal gusts in y -direction

$$\Phi_d = v_0 \bar{U} \tilde{A}_y [Real \bar{W}(x) - \beta Imag \bar{W}(x)] e^{j\omega t} \qquad (70)$$

1.1.7 Acceleration potential Φ_{ds} due to crossed acceleration components a_{xds} , a_{yds} .

Unsteady velocity components u_d , v_d were determined in the previous section by Eqs.(22),(23),(33) and (34). To begin with, we will determine the stationary velocity components.

1.1.7.1 Stationary velocity components u_s , v_s

The boundary condition on the blade surface may be given by Eq.(25), where the term βu_s may be neglected, since $\beta u_s \ll \beta \overline{U}$. Then Eq.(25) may be written;

$$v_s = -\beta \overline{U} \tag{71}$$

The complex conjugate velocity Ω_s satisfying Eq.(71) may be written;

$$\Omega_s = u_s - i v_s = A_s [\, \overline{W}(x) - \overline{W}(-\infty) \,] \tag{72}$$

where $\overline{W}(x)$ is given by Eq.(54) on the ζ -plane. The constant A_s is given by

$$A_s = \beta \overline{U} / \, Imag \, [\, \overline{W}(o) - \overline{W}(-\infty) \,] \tag{73}$$

It is easily seen that Eq.(72) satisfies Eq.(71), since $Imag\, \overline{W}(x)$ remains constant on the blade surface. Equation (72) also satisfies the condition at $x = -\infty$, where u_s , v_s should vanish.

1.1.7.2 Acceleration potential ϕ_{ds}

The stationary components u_s , v_s may be considered to be irrotational as above. But the non-stationary components u_d , v_d are rotational. If we put

$$\partial u_s / \partial y = \partial v_s / \partial x \; , \quad \partial u_d / \partial y = \partial v_x / \partial x + \omega_d$$

into Eq.(11), we obtain

$$a_{xds} = \partial / \partial x \, (\, u_d \cdot u_s \,) + \partial / \partial x \, (\, v_d \cdot v_s \,) + v_s \, \omega_d \tag{74}$$

where ω_d is the vorticity of the non-stationary velocity components. Integrating Eq.(74) under the condition $\phi_{ds}=0$ at $x=-\infty$, we get

$$\phi_{ds} = u_d \cdot u_s + v_d \cdot v_s + \int_{-\infty}^{x} v_s \omega_d \, dx \tag{75}$$

The second and third term in the right hand side of Eq.(75) have no effect on the resultant forces on the blade. It is apparent that the term $v_d \cdot v_s$ have the same value on up and down surfaces of the blades, and the integrand $v_s \cdot \omega_d$ of the third term is continuous across the blade surface. So the acceleration potential components related to the term $v_d \cdot v_s$ and $v_s \cdot \omega_d$ have the same value on up and down surface of the blade, which implies that they have no effect on the resultant forces on the blades. For brevity we will consider only the component $u_d \cdot u_s$. The stationary velocity u_s is given by Eq.(72) and the non-stationary velocity u_d may be written

for sinusoidal gusts in x-direction

$$u_d = (1 + \beta \tilde{A}_x I)\, u_0 \exp[j\omega(\tau - \frac{x}{U})] \tag{76}$$

for sinusoidal gusts in y-direction

$$u_d = v_0 \tilde{A}_y \, I \exp[j\omega(\tau - \frac{x}{U})] \tag{77}$$

where

$$I = [\mathrm{Real}\,\overline{W}(x) - \mathrm{Real}\,\overline{W}(-\infty)]e^{jS_t x} - j\omega\int_{-\infty}^{x}[\mathrm{Real}\,\overline{W}(\xi) - \mathrm{Real}\,\overline{W}(-\infty)]e^{jS_t \xi}d\xi \tag{78}$$

Consequently, the acceleration potential component responsible to the resultant forces is

for sinusoidal gusts in x-direction

$$\phi_{ds} = \beta u_0 \overline{U} \frac{\mathrm{Real}[\overline{W}(x) - \overline{W}(-\infty)]}{\mathrm{Imag}[\overline{W}(0) - \overline{W}(-\infty)]} \exp[j\omega(\tau - \frac{x}{U})] \tag{79}$$

for sinusoidal gusts in y-direction

$$\phi_{ds} = \beta v_0 \overline{U} \frac{\mathrm{Real}[\overline{W}(x) - \overline{W}(-\infty)]}{\mathrm{Imag}[\overline{W}(0) - \overline{W}(-\infty)]} \tilde{A}_y I \exp[j\omega(\tau - \frac{x}{U})] \tag{80}$$

where we have neglected the smaller quantities of order β^2. If we compare the crossed acceleration potential Φ_{ds} given by Eq. (80) for sinusoidal gusts in y-direction with the unsteady acceleration potential Φ_d given by Eq. (70), it will be seen that the crossed acceleration potential Φ_{ds}, which is an order of $\beta v_0 \overline{U}$, may be neglected as it is small quantity of higher order in comparison with the unsteady acceleration potential Φ_d, which is an order of $v_0 \overline{U}$. For the case of sinusoidal gusts in x-direction we must retain both of those two components given by Eqs. (69), (70) since they are the same order of $\beta u_0 \overline{U}$. In the actual machines, following Horlock [8], the value of βu_0 will be comparable order as the value of v_0 if the angle between the blade and the direction of the wakes from the upstream blades is small. That is the reason why we retain the components of order βu_0 for the case of sinusoidal gusts in x-direction. In conclusion we will consider next acceleration potential components relating to unsteady forces on the blades.

for sinusoidal gusts in x-direction

$$\Phi = \Phi_d \tag{81}$$

for sinusoidal gusts in y-direction

$$\Phi = \Phi_d + \Phi_{ds} \tag{82}$$

1.1.8 Unsteady pressure distribution on the blade

The unsteady pressure distribution responsible to unsteady forces is,

for sinusoidal gusts in x-direction

$$p = -\rho \beta u_0 \overline{U} \left\{ \widetilde{A}_x + e^{-jS_t x} / Imag\, [\overline{W}(0) - \overline{W}(-\infty)] \right\} \cdot Real\, [\overline{W}(x) - \overline{W}(-\infty)]\, e^{j\omega\tau} \tag{83}$$

for sinusoidal gusts in y-direction

$$p = -\rho v_0 \overline{U}\, \widetilde{A}_y\, Real\, [\overline{W}(x) - \overline{W}(-\infty)]\, e^{j\omega\tau} \tag{84}$$

1.1.9 Lift fluctuation

Integrating the pressure distribution given by Eqs.(83),(84), we get the lift fluctuation L per unit length of span. Coefficient of fluctuating lift C_L will be defined as;

For sinusoidal gusts in x-direction

$$C_L = L / [\rho \beta u_0 U (c/2)\, e^{j\omega\tau}] \tag{85}$$

For sinusoidal gusts in y-direction

$$C_L = L / [\rho v_0 U (c/2)\, e^{j\omega\tau}] \tag{86}$$

1.1.10 Comparison with Sears' results

Most of the analyses treating unsteady flows around airfoils are based on the vortex theories in the velocity field. Those theories are also linearized by the assumption of shed-off vortices and give equivalent result as the present theory. In this section will be introduced the Sears function $S(S_t)$ in order to show the equivalence with the vortex theory. From Eqs.(61) and (62)

$$\tilde{A}_y = [\{ \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=-1} - \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=-\infty} \} e^{-jS_t}$$
$$- jS_t \int_{-\infty}^{-1} \{ \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=\xi} - \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=-\infty} \} e^{jS_t\xi}]^{-1} \tag{87}$$

Representing the integral term by

$$I \equiv \int_{-\infty}^{-1} \{ \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=\xi} - \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}} \Big|_{x=-\infty} \} e^{jS_t\xi} d\xi$$

we can get for $\gamma = 0$, $\zeta = r e^{i\pi}$

$$I = \int_{-\infty}^{-1} \left(\frac{r-1-e^{\varepsilon}}{r+1-e^{\varepsilon}} - \frac{e^{\varepsilon}+1}{e^{\varepsilon}-1} \right) e^{jS_t\xi} d\xi \tag{88}$$

Eqs.(19.1),(19.2) give

$$\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1} = e^{\delta} \quad , \quad \frac{r-1-e^{\varepsilon}}{r+1-e^{\varepsilon}} = \sqrt{\frac{\sinh\{\delta(x-1)\}}{\sinh\{\delta(x+1)\}}}$$

Then Eq.(88) will be

$$I = \int_1^\infty \left(\sqrt{\frac{\sinh\{\xi(x+1)\}}{\sinh\{\xi(x-1)\}}} - e^{\xi} \right) e^{-jS_t x} dx$$

In case $\xi \to 0$

$$I = \int_1^\infty \left(\sqrt{\frac{x+1}{x-1}} - 1 \right) e^{-jS_t x} dx$$

$$= \int_1^\infty \frac{e^{-jS_t x}}{\sqrt{x^2-1}} dx + \int_1^\infty \frac{x e^{-jS_t x}}{\sqrt{x^2-1}} - \int_1^\infty e^{-jS_t x} dx$$

$$= K_0(jS_t) + K_1(jS_t) - e^{-jS_t}/jS_t \qquad (89)$$

Since

$$\mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}}\bigg|_{x=-\infty} = 1 \,, \qquad \mathrm{Real}\, \frac{\zeta - e^{i\alpha}}{\zeta + e^{i\alpha}}\bigg|_{x=-1} = 0$$

Eq.(87) can be written

$$\tilde{A}_Y = [-jS_t \{ K_0(jS_t) + K_1(jS_t) \}]^{-1} \qquad (90)$$

From Eqs.(82) and (84)

$$p = \rho v_0 U e^{j\omega t} \sqrt{(1-x)/(1+x)} \; \tilde{A}_Y \qquad (91)$$

$$L = 2\pi \rho v_0 U e^{j\omega t} \{ j\omega (K_0(jS_t) + K_1(jS_t)) \}^{-1} \qquad (92)$$

which agrees with the following result by Sears.

$$L = 2\pi \rho U v_0 S(S_t)$$

$$S(S_t) = \{ jS_t (K_0(jS_t) + K_1(jS_t)) \}^{-1} \qquad (93)$$

1.1.11 Computational results

The coefficients of fluctuating lift C_L for sinusoidal gusts in x -direction are shown in Figs.4-6. The chained line in Fig.4 shows Horlock's result [8] for isolated airfoil. As the solidity increases, the fluctuating lift diminishes to 1/2 ~ 1/3 times as large as that for a single airfoil. The magnitude of fluctuating lift diminishes fairly quickly as S_t increases in the range of the small values of S_t (= 0.0 ~ 0.4), where the phase angle delays at first and then advances in the case of low solidity cascade. Those phenomena cannot be seen for cascades of ordinary solidity. The lift increases as the stagger increases if the blade pitch is kept constant. In any case, the parameters of blade arrangement do not have so much effect on the phase angle of fluctuating lift as on the magnitude. The coefficients of fluctuating lift for sinusoidal gusts in y-direction are shown in Fig.7~9. The chained line in Fig.7 shows the result by Sears [2]. The effects of parameters of blade arrangement on the fluctuating lifts show a similar tendency to that in x-direction. We have a good agreement with Schorr & Reddy's results [7] for lifts on a blade of a cascade produced by gusts in y-direction as shown in Fig.10. The induced velocity u_{d2} on the blade surface presented by Eq.(57) is shown in Figs.11-13. They are normalized by βu_o for sinusoidal gusts in x-direction, by v_o for gusts in y-direction. It is one of the advantages of the present method that we can get accurate velocity distribution in comparison with the singular point method.

1.1.12 Conclusion

It was shown that the analysis in the acceleration field in combination with the conformal mapping method is very brief even for the case of a cascade. The parameters of blade arrangement affect considerably the fluctuating lift, suggesting that these effects should be taken into account in estimating the fluctuating aerodynamic forces due to viscous or potential interaction of cascades.

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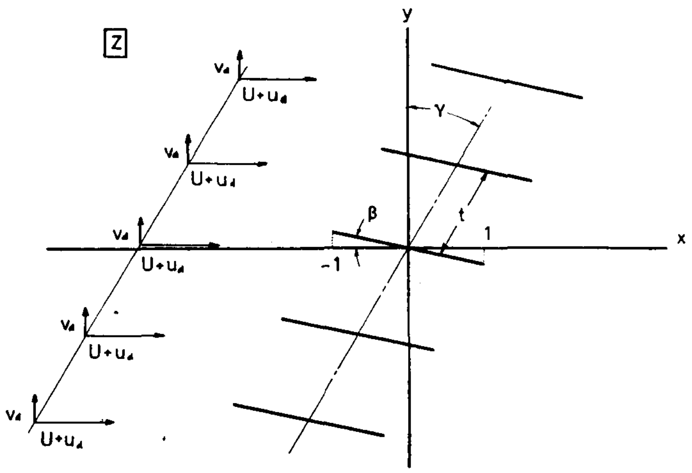

Fig.1 Cascade in gusts

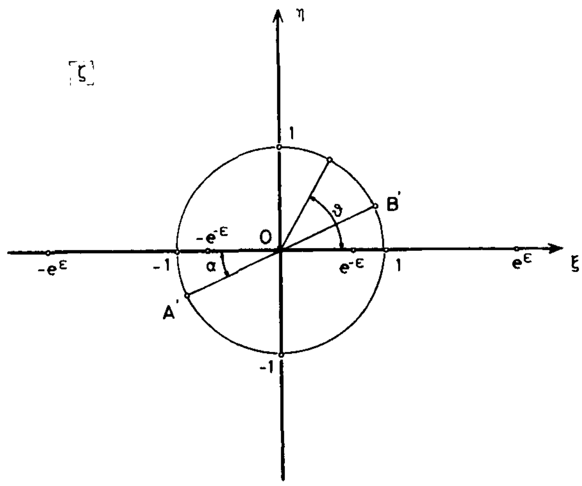

Fig.2 Mapping plane

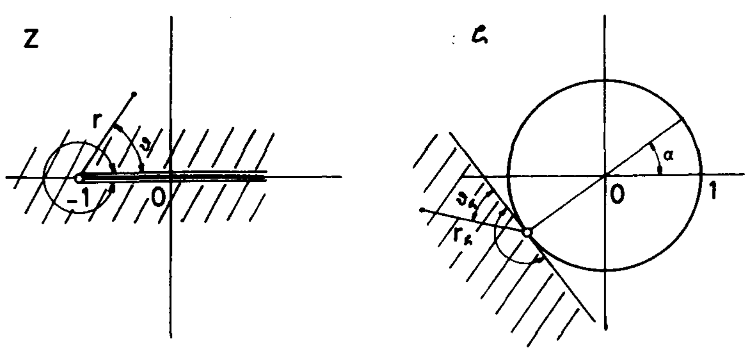

Fig.3 Mapping near the leading edge

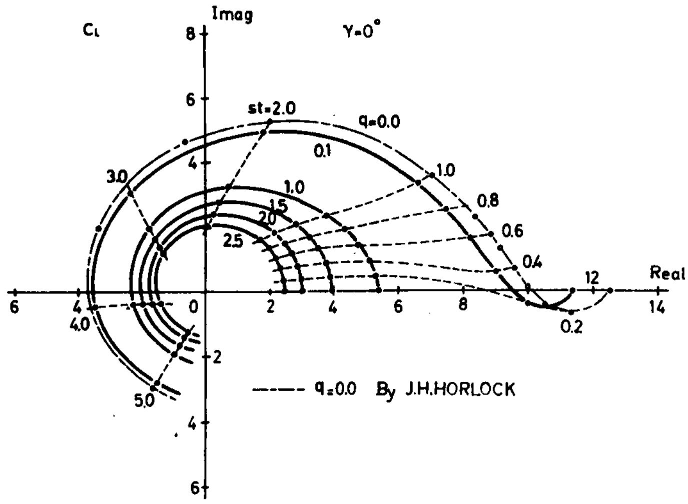

Fig.4 Fluctuating lift coefficient for sinusoidal gusts in χ-direction (γ = 0°)

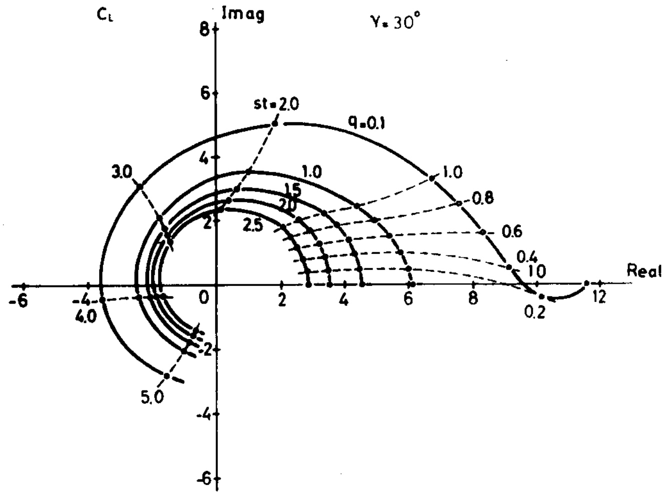

Fig.5 Fluctuating lift coefficient for sinusoidal gusts in χ-direction (γ = 30°)

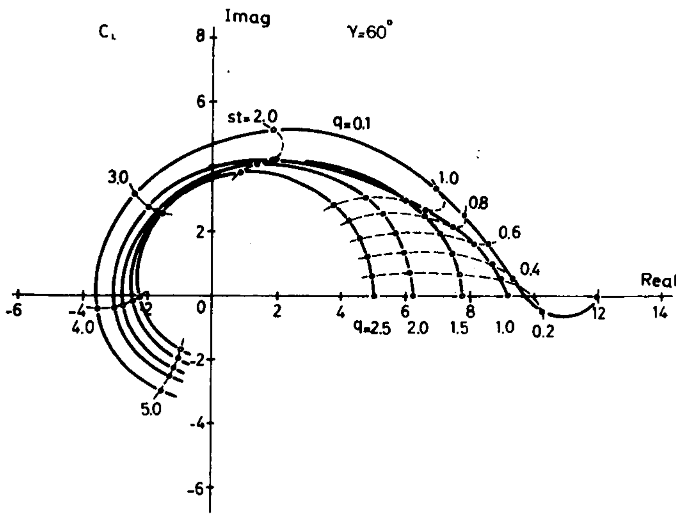

Fig.6 Fluctuating lift coefficient for sinusoidal gusts in χ-direction (γ = 60°)

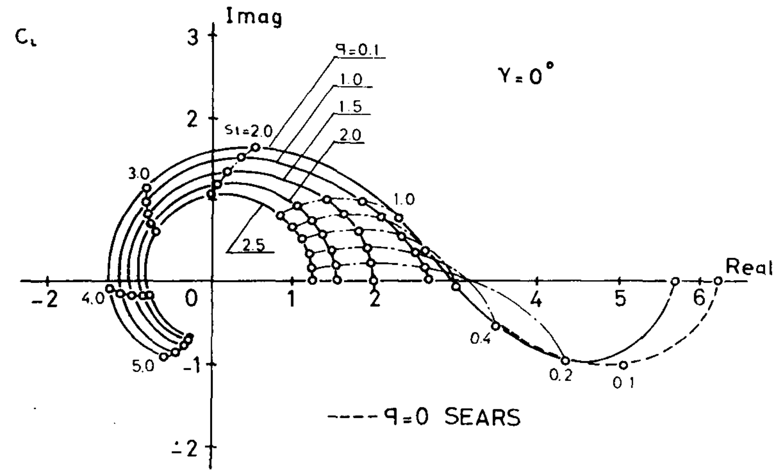

Fig. 7 Fluctuating lift coefficient for sinusoidal gusts in y-direction (γ = 0°)

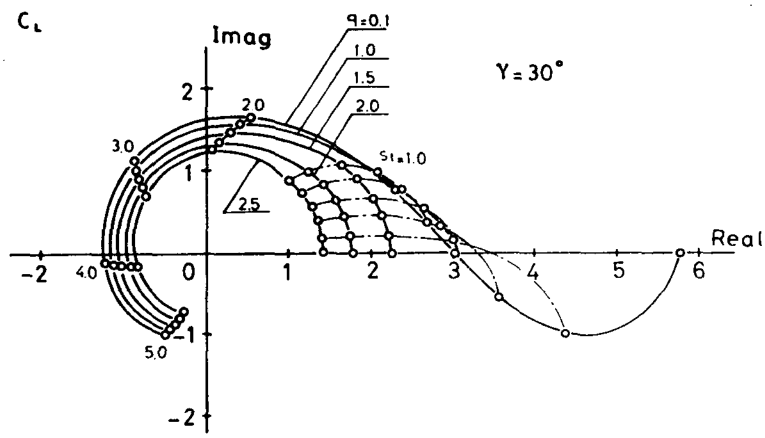

Fig. 8 Fluctuating lift coefficient for sinusoidal gusts in y-direction (γ = 30°)

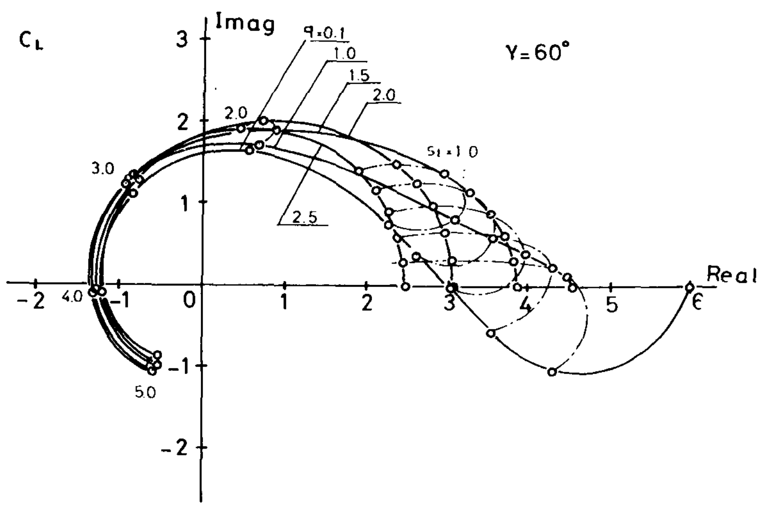

Fig. 9 Fluctuating lift coefficient for sinusoidal gusts in y-direction (γ = 60°)

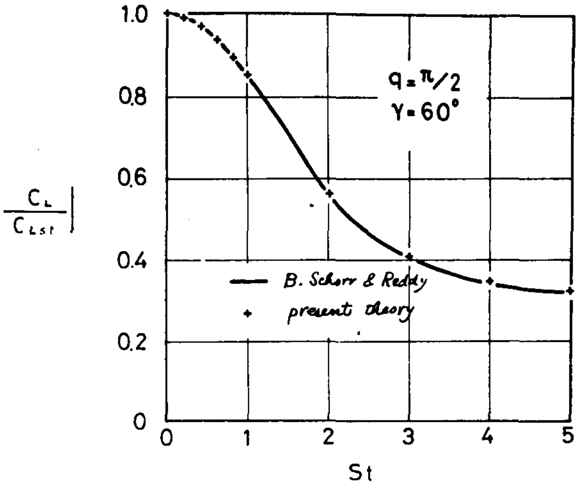

Fig.10 Comparison with Schorr & Reddy's results for sinusoidal gusts in y-direction (C_{Lst} ;lift coefficient for $S_t = 0$)

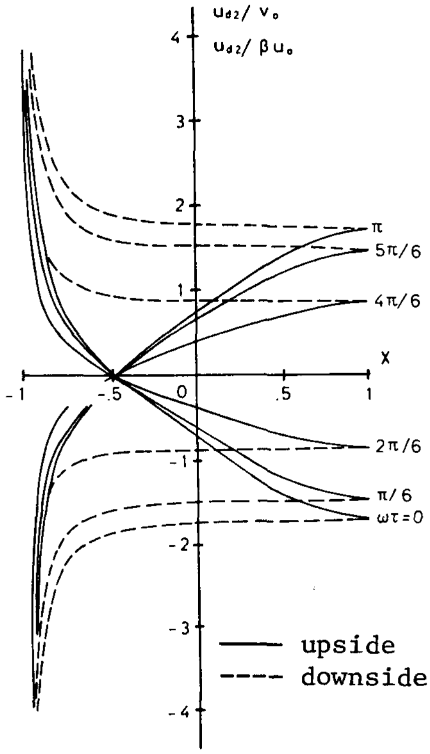

Fig.11 Instantaneous induced velocity for $S_t = 0$, $\gamma = 60°$, q = 2.0

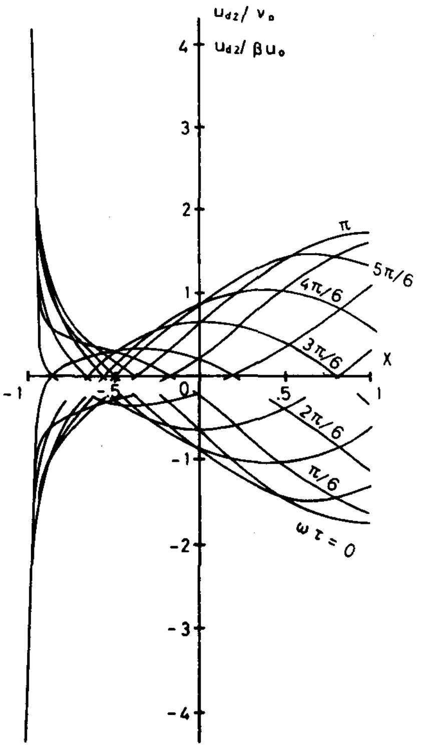

Fig.12 Instantanious induced velocity for $S_t = 1$, $\gamma = 60°$, q = 2.0 on upside surface

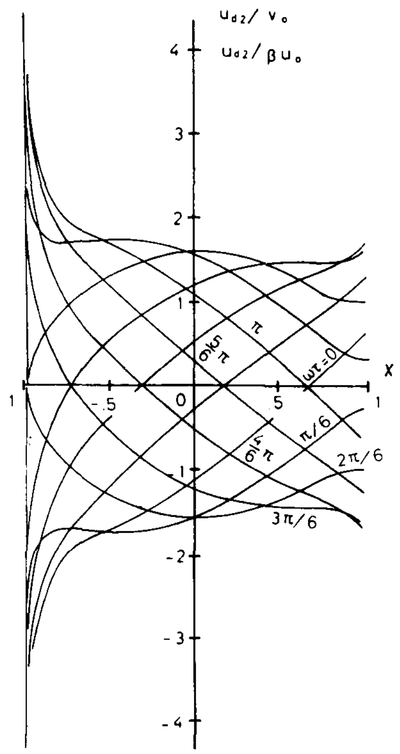

Fig.13 Instantanious induced velocity for $S_t = 1$, $\gamma = 60°$, q = 2.0 on downside surface

1.2 Sinusoidal gusts of different phase

1.2.1 Introduction

In the preceding section we have analized the acceleration field of cascades subjected to sinusoidal gusts in x and y -direction by means of acceleration potential method combined with conformal mapping method. It was assumed that the phase of the gusts are all the same for each blade of the cascade. In case of viscous wake interaction problems the purterbations are in the same phase only if the blade spacings of the cascades in front and after are the same. But the blade numbers of the stator and the rotor are ususlly different and the phase of the purterbations are different for each blade. This section presents the analysis of unsteady lifts on the cascade blades subjected to sinusoidal gusts by using the acceleration potential method in combination with conformal mapping method., in case the phase of the purterbation is different for each blades. It is assumed that the fluid is inviscid and incompressible and the purterbations are small compared to stationary amounts.

1.2.2 Sinusoidal gusts

Though the phase angle of the purterbations is usually different on each blades, it can be assumed that the phase of the purterbations are the same on every m blades, since in actual machines the phase is the same at least on every blade numbers of the cascade. For brevity we assume following sinusoidal gusts which have the phase difference of $2\pi/m$ between adjacent blades and flow through the cascade with an angle of attack β .

For sinusoidal gusts in x-direction

$$v_x = U + u_0 \exp[j\omega(\tau - \frac{nt\sin\delta}{U})]\, e^{\frac{2\pi n}{m}j}$$
$$v_y = 0 \tag{1}$$

For sinusoidal gusts in y -direction

$$v_x = U$$
$$v_y = v_0 \exp[j\omega(\tau - \frac{nt\sin\delta}{U})]\, e^{\frac{2\pi n}{m}j} \tag{2}$$

where m is an arbitrary integer, j the imaginary unit with respect to time and it is assumed that $u_0, v_0 \ll V$ and $\beta \ll 1$. (Fig.1)

1.2.3 Conformal mapping

In the preceding section is assumed that the purterbations have the same phase on each blades. Hence all the blades could be mapped to single unit circle on which the acceleration potential is given. In this place the blades are mapped to m unit circles and on each the acceleration potential is given so as to be considered the phase difference of the purterbations. Though the mapping function which maps the blades to concentric circles is available for $m=2$, we will introduce a mapping function which may be used for arbitrary integer m . The cascade on Z -plane which has the chordlength $c=2$, blade spacing t , staggering angle γ as shown in Fig.2 should be mapped to m unit circles arranged around the origin of ζ-plane. (Fig.3) Far upstream ($x=-\infty$) in Z-plane is mapped to the origin in ζ -plane and far downstream ($x=+\infty$) in Z -plane to infinity ($\zeta=\infty$) in ζ -plane. The displacement of m pitches along the cascade axis in Z -plane corresponds to one turning around the origin in ζ -plane. To get the mapping function the analytic complex function $W=Z$ representing the through flow is related to the flow around the unit circles with a source and a vortex at the origin of ζ -plane. The strengths Γ and Q of the vortex and the source respectively are decided so that the variation of the complex potential during the progression of m -pitches along the cascade axis equals to that during one turn around the origin in ζ -plane. Hence ,

$$(1/2\pi)(Q+i\Gamma) = \frac{m}{2g}\, e^{i\gamma}$$

where $g=\pi/t$. Vortices and sources are then placed at the images of the origin with respect to each unit circle in order to make the unit circles stream lines. The unit circles are numbered 0,1, $\cdots$,m-1 for convenience. If it were not for other unit circles, the arrangement of a logarithmic singularity of strength $(m/2g)e^{-i\gamma}$ at the image ζ_1 of the origin with respect to the unit circle 0 and that of strength $-(m/2g)e^{-i\gamma}$ at the center ζ_2 of the unit circle 0 makes the circumference of the unit circle 0 a stream line. In the same manner the singularities should be put at the image of the origin and at the center of other unit circles. The singularities in each unit circle thus arranged disturb the boundary conditions on other unit circles, for instance, the singularities in the unit circle 0 disturb the boundary conditions on the unit circles 1~ m-1. To compensate this disturbance the logarithmic singularities of strength $(m/2\xi)\, e^{i\delta}$ and $-(m/2\xi)\, e^{-i\delta}$ should be placed at the images $\zeta_{11}, \zeta_{12}, \cdots, \zeta_{1(m-1)}$ and $\zeta_{21}, \zeta_{22}, \cdots, \zeta_{2(m-1)}$ of the singularities in other unit circles 1~m-1 with respect to the unit circle 0. Those manipulations should be done for other unit circles 1 m-1. Those singularities disturb the boundary conditions again and the same adjustment should be repeated. The locations $\zeta_{a,b,c,\cdots}$ of the singularities are given as follows. The location $\zeta_{a,b,c,\cdots,k}$ is assumed to be known. Then the image $\zeta_{a,b,c\cdots k,\ell}$ of the location $\zeta_{a,b,\cdots,k}\, e^{\frac{2\pi\ell}{m}i}$ in the unit circle ℓ with respect to the unit circle 0 is found from

$$|\,\zeta_{abc\cdots k}\, e^{\frac{2\pi\ell}{m}i} - ai\,| \cdot |\,\zeta_{abc\cdots \ell} - ai\,| = 1$$

$$\frac{(\zeta_{abc\cdots k}\, e^{\frac{2\pi\ell}{m}i} - ai)}{|\,\zeta_{abc\cdots k}\, e^{\frac{2\pi\ell}{m}i} - ai\,|} = \frac{(\zeta_{abc\cdots k\ell} - ai)}{|\,\zeta_{abc\cdots k\ell} - ai\,|}$$

and hence,

$$\zeta_{abc\cdots k\ell} = ai + \frac{\zeta_{abc\cdots k}\, e^{\frac{2\pi\ell}{m}i} - ai}{|\,\zeta_{abc\cdots k}\, e^{\frac{2\pi\ell}{m}i} - ai\,|^2} \tag{3}$$

where

$$\zeta_1 = (a + 1/a)i \quad , \quad \zeta_2 = ai \tag{3.1}$$

The complex progression $\zeta_{abc\cdots}$ can be caluculated from the initial value (3-1) and the asymptotic equation (15). Those give the locations of the singularities in the unit circle 0 and the locations of the singularities in the unit circle n are given by $\zeta_{abc\cdots}\, e^{\frac{2\pi n}{m}i}$. Equating the complex potential thus given in ζ-plane and the complex potential $\overline{W} = Z$ of the through flow on Z -plane, we get the following mapping function.

$$\begin{aligned} Z = Z_0 &+ (me^{i\delta}/2\xi)\log\zeta + (m/2\xi)\sum_{n=1}^{m} \Big[\!\Big[e^{-i\delta}\log\frac{\zeta-\zeta_1 e^{\frac{2\pi n}{m}i}}{\zeta-\zeta_2 e^{\frac{2\pi n}{m}i}} \\ &+ \sum_{s=1}^{m-1}\Big[e^{i\delta}\log\frac{\zeta-\zeta_{1s} e^{\frac{2\pi n}{m}i}}{\zeta-\zeta_{2s} e^{\frac{2\pi n}{m}i}} + \sum_{t=1}^{m-1}\Big\{ e^{-i\delta}\log\frac{\zeta-\zeta_{1st} e^{\frac{2\pi n}{m}i}}{\zeta-\zeta_{2st} e^{\frac{2\pi n}{m}i}} \\ &+ \sum_{u=1}^{m-1}\Big(e^{i\delta}\log\frac{\zeta-\zeta_{1stu} e^{\frac{2\pi n}{m}i}}{\zeta-\zeta_{2stu} e^{\frac{2\pi n}{m}i}} + \cdots\cdots \quad \cdots\cdots\Big)\Big\}\Big]\Big]\!\Big] \end{aligned} \tag{4}$$

The distance a of the centers of the unit circles from the origin was decided by trial so that the unit circles are mapped to a cascade of chord length $C=2$ in Z-plane by the mapping function (16). The pararell displacement Z_0 is decided so as to get the correspondence of unit circle 0 with the blade on $-1 \leq x \leq 1$, $y=0$ on Z-plane. The series on the right hand side of Eq.(16) converges because $\zeta_{1st\cdots k} - \zeta_{2st\cdots k}$ converges to zero geometrically as the adjustment is repeated. The convergence is better for larger a , i.e., for larger blade spacing. For instance the equality $\zeta_{1ab\cdots} = \zeta_{2ab\cdots}$ is attained to five figures by eight repitition for $a=1.5$ and by five repitition for $a=2.5$ in case $m=2$.

1.2.4 Acceleration complex potential

The boundary condition for acceleration complex potential on the blade surface is reduced both for sinusoidal gusts in x and y-direction to,

$$\beta \Phi_a - \Psi_a = Const. \tag{5}$$

As shown in the preceding section the acceleration complex potential have a singularity of order ζ^{-1} at the leading edge in the mapping plane. In order to get the acceleration complex potential which has the singularity at the leading edge and which satisfies the boundary conditions , we will define a complex potential $\overline{W}(\zeta)$ which have the singularity of order ζ^{-1} at the leading edge and have constant imaginary part with respect to imaginary unit i on the blade surfaces, and moreover the values on the blade surfaces differs only by the factor $e^{\frac{2\pi}{m}i}$ between adjacent blade surfaces. In the first place doublets of momentum μ_n whose axis is directed peripherally at the leading edges A_n' in ζ-plane. Those doublets suffice the boundary condition on their own circles but disturbs that on other unit circles. Those disturbances are adjusted by putting doublets at the images with respect to each unit ciecle so as to make the imaginary part constant on each unit circle. The location of the images of the doublet at the leading edge ($ai + e^{i\beta'}$) of the unit circle 0 with respect to unit circles $1 \sim m-1$ are represented by ζ_1', ζ_2', ζ_3', $\cdots$ ζ_{m-1}' respectively. Moreover the locations of the images of the doublets at ζ_k' with respect to unit circle ℓ are written as $\zeta_{k,\ell}'$. In this manner the boundary conditions on the unit circles are adjusted by putting doublets at the images one after another. The locations of the doublets are

$$\zeta_0' = ai + e^{i\beta'} \tag{6}$$

$$\zeta_n' = ai\,e^{\frac{2\pi n}{m}i} + \frac{\zeta_0' - ai\,e^{\frac{2\pi n}{m}i}}{|\zeta_0' - ai\,e^{\frac{2\pi n}{m}i}|^2} \quad (n=1,2,\cdots,m-1) \tag{7}$$

$$\zeta'_{a,b,c,\cdots k,\ell} = ai\,e^{\frac{2\pi\ell}{m}i} + \frac{\zeta'_{abc\cdots k} - ai\,e^{\frac{2\pi\ell}{m}i}}{|\zeta'_{abc\cdots k} - ai\,e^{\frac{2\pi\ell}{m}i}|^2} \tag{8}$$

where $a \neq b$, $b \neq c$, $\cdots$ $k \neq \ell$

The strength A_{abc} of the doublet at ζ'_{abc} is,

$$A_0 = 1 \tag{9}$$

$$A_n = A_0 \,/\, |\zeta_0' - ai\,e^{\frac{2\pi n}{m}i}|^2 \qquad n = 1,2,\cdots m-1 \tag{10}$$

$$A_{ab\cdots k,\ell} = A_{ab\cdots k} \,/\, |\zeta'_{abc\cdots k} - ai\,e^{\frac{2\pi\ell}{m}i}|^2 \quad (a\neq b,\ b\neq c,\cdots,k\neq\ell) \tag{11}$$

The axial direction θ_{abc} of the doublet at $\zeta'_{a,b,c}$ is,

$$\theta_0 = \beta' + \pi/2 \tag{12}$$

$$\theta_n = 2\,\mathrm{Arg}(ai\,e^{\frac{2\pi n}{m}i} - \zeta_0') - (\pi + \theta_0) \qquad n = 1,2,\cdots,m-1 \tag{13}$$

$$\theta_{a,b,c,\cdots k,\ell} = 2\,\mathrm{Arg}(ai\,e^{\frac{2\pi\ell}{m}i} - \zeta'_{abc\cdots k}) - (\pi + \theta_{abc\cdots,k}) \qquad (a\neq b,\ b\neq c,\cdots,k\neq\ell) \tag{14}$$

Then the complex potential $\overline{W}(\zeta)$ can be expressed as follows.

$$\begin{aligned}\overline{W}(\zeta) = \sum_{n=0}^{m-1} \mu_n \Big[\!\Big[&\sum_{a=0}^{m-1} [A_a e^{i\theta_a} e^{\frac{2\pi n}{m}i} / (\zeta - \zeta_a' e^{\frac{2\pi n}{m}i})] \\ &+ \sum_{a=1}^{m-1}\sum_{b=0,\,b\neq a}^{m-1} [A_{a,b}\, e^{i\theta_{ab}} e^{\frac{2\pi n}{m}i} / (\zeta - \zeta'_{ab} e^{\frac{2\pi n}{m}i})] \\ &+ \sum_{a=1}^{m-1}\sum_{b=0,\,b\neq a}^{m-1}\sum_{c=0,\,c\neq b}^{m-1} [A_{abc}\, e^{i\theta_{abc}} e^{\frac{2\pi n}{m}i} / (\zeta - \zeta'_{abc} e^{\frac{2\pi n}{m}i})] \cdots \Big]\!\Big]\end{aligned} \tag{15}$$

where the notation $\sum_{b=0}^{m-1} b \neq a$ designates the summation except for $b=a$. From the relation

$$\overline{W}(x+nt\sin\gamma,\ y+nt\cos\gamma) = \overline{W}(x,y)\, e^{\frac{2\pi n}{m} j} \tag{16}$$

the constants μ_n will be,

$$\mu_n = e^{\frac{2\pi n}{m} j} \tag{17}$$

The complex potential $\overline{W}(\zeta)$ is doubly complex with respect to imaginary units i and j. The notations R_{e_i}, I_{m_i}, R_{e_j}, I_{m_j} are used to express the real and the imaginary part with respect to imaginary units i and j. The acceleration complex potential $\overline{W}_a$ may be expressed,

$$\overline{W}_a(\zeta) = (1+\beta i)\, A\, \overline{W}(\zeta) \tag{18}$$

Then the arguments in section 1.1.6.4 are valid unaltered and the acceleration potential can be represented by Eqs.(69) and (70) in section 1.1.6.4.

1.2.5 Crossed acceleration potential components

By simply replacing the complex conjugate velocity Ω_s given by Eq.(72) in section 1.1.7 with the following representation we can get the crossed acceleration potential components.

$$\Omega_s = A_s \left[\overline{W}'(\zeta) - \overline{W}'(-\infty)\right] \tag{19}$$

where $\overline{W}'(\zeta)$ is given by Eq.(15) with $\mu_n = 1$. The constant A_s is,

$$A_s = \beta \overline{U} \,/\, I_{m_i}\left[\overline{W}'(\zeta) - \overline{W}'(x=-\infty)\right] \tag{20}$$

Then the crossed accrleration potential is,

for sinusoidal gusts in x-direction

$$\phi_{as} = \beta u_0 \overline{U}\, \frac{R_{e_i}\left[\overline{W}'(x) - \overline{W}'(-\infty)\right]}{I_{m_i}\left[\overline{W}'(0) - \overline{W}'(-\infty)\right]}\, e^{j\omega\left(T - \frac{x - nt\sin\gamma}{\overline{U}}\right)}\, e^{\frac{2\pi n}{m} j} \tag{21}$$

for sinusoidal gusts in y -direction

$$\phi_{ds} = \beta U_0 U \tilde{A}_y I \frac{R_e i [\overline{W}'(x) - \overline{W}'(-\infty)]}{I_m i [\overline{W}'(0) - \overline{W}'(-\infty)]} e^{j\omega(\tau - \frac{x - m \lambda i \gamma}{U})} e^{\frac{2\pi n}{m} j} \tag{22}$$

1.2.6 Pressure fluctuation

The pressure fluctuation relating to the lift fluctuation is,

for sinusoidal gusts in x -direction

$$p = -\rho\beta u_0 U \, [\tilde{A}_x \, R_e i \, (\overline{W}(x) - \overline{W}(-\infty))$$

$$+\{R_e i \, (\overline{W}'(x) - \overline{W}'(-\infty)) / I_m i \, (\overline{W}'(x) - \overline{W}'(-\infty))\} e^{-j S_t x}] \, e^{j\omega\tau} \tag{23}$$

for sinusoidal gusts in y -direction

$$p = -\rho U_0 U \, \tilde{A}_y \, R_e i \, (\overline{W}(x) - \overline{W}(-\infty)) \, e^{j\omega\tau} \tag{24}$$

The lift fluctuation is given by the integraton of the pressure distribution. The fluctuating lift coefficient C_L is defined by Eqs. (85) and (86) in section 1.1.9.

1.2.7 Numerical examples

A) Sinusoidal gusts in x -direction

Figs.4 ~ 6 show the fluctuating lift coefficients C_L for $m=2$ (in case the phase of the gust differs by π between adjacent blades). The broken line in Fig.4 is given by Horlock[2] for isolated airfoil, which is near our results for large blade spacing. The lift fluctuation is smaller for finer blade spacing, which is the same tendency as for $m=1$. For finer blade spacing cascades the lift coefficient stretches upwards from the value for $\omega=0$ in case $m=1$, but goes once downwards in case $m=2$ as for the case of isolated airfoil. Fig.7 shows the effects of the phase difference for fixed cascade geometry ($\gamma=30°$, $\xi=2.0$). It can be seen that the phase difference enlarges the lift fluctuation and the effect is more remarkable for smaller frequency.

B) Sinusoidal gusts in y -direction

The fluctuating lift coefficient is shown in Figs.8～10. For $m=1$ the cascade effect works so as to enlarge the lift fluctuation, while it works contrary for $m=2$. This effect is more remarkable for smaller frequency. The broken line in Fig.8 shows the result for isolated airfoil [3]. In Fig. 11 can be seen the effects of the phase difference for fixed cascade geometry. The effect is the same as for sinusoidal gusts in x-direction. By the way, it has been checked that we can get the same result for $m=1$ as the preceding section by putting $\mu_n=1$ in Eq.(17).

1.2.8 Conclusion

It has been shown that we can analyze the lift fluctuation taking in account the phase difference of the gusts in entirely the same manner as for the case of the same phase, by only introducing a new mapping function and a complex potential.

References in 1.2

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[2] J.H.Horlock; Trans. ASME, Ser.D 90-4 (1968) PP.494

[3] W.R.Sears , J.Aern. Sci. 8-3 (1941) PP.104

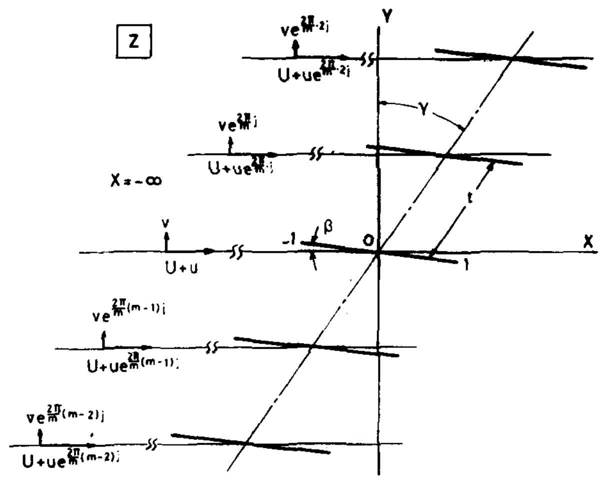

Fig.1 Cascade in gusts

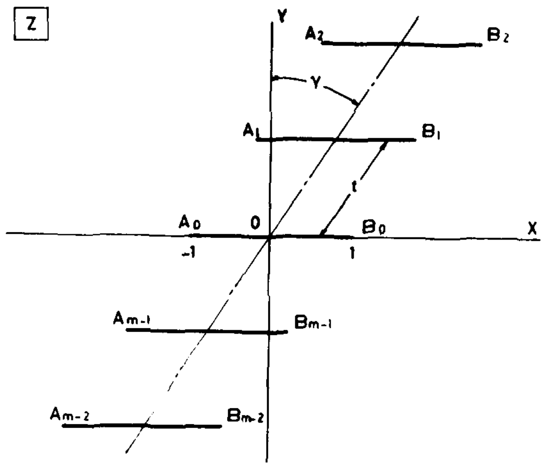

Fig.2 Physical plane

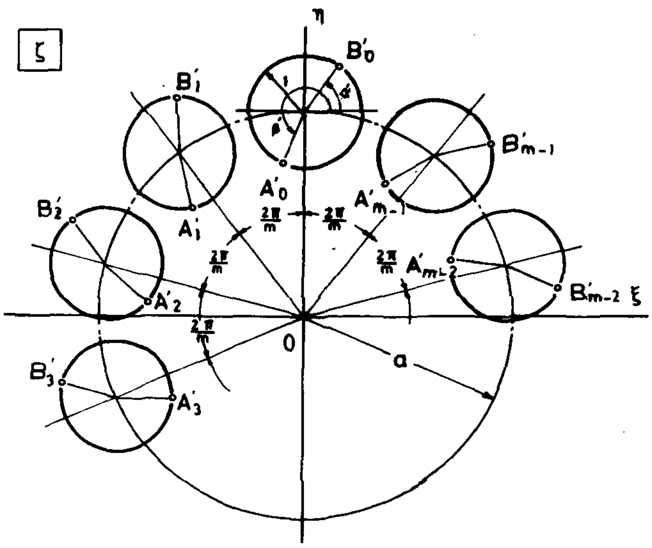

Fig.3 Mapping plane

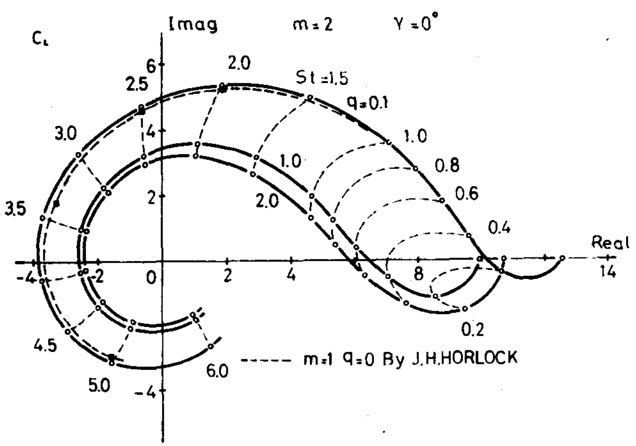

Fig.4 Fluctuating lift coefficient for sinusoidal gusts in x-direction (m = 2, γ = 0°)

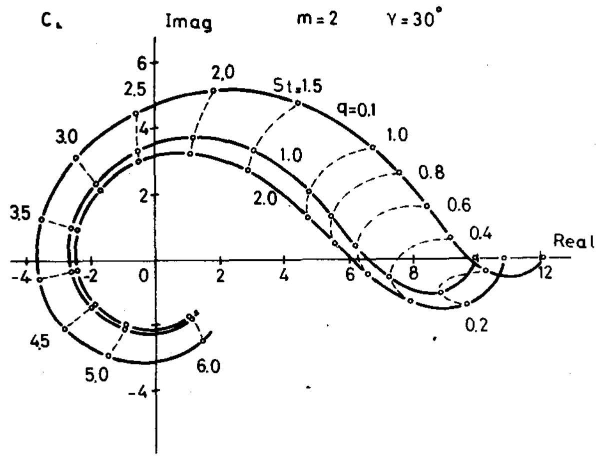

Fig.5 Fluctuating lift coefficient for sinusoidal gusts in x-direction (m = 2, γ = 30°)

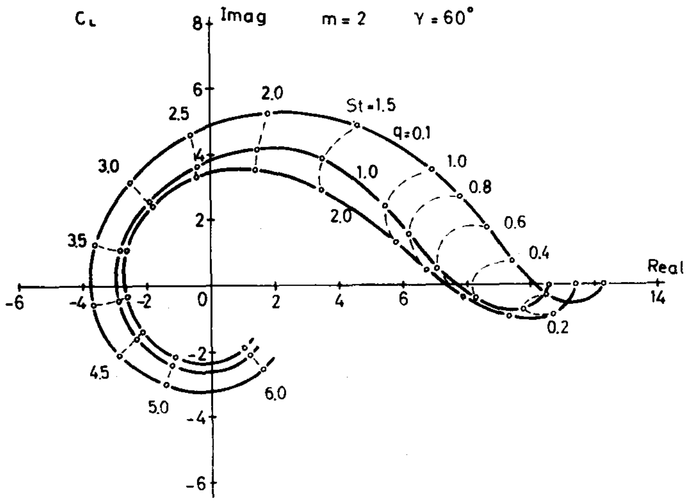

Fig.6 Fluctuating lift coefficient for sinusoidal gusts in x-direction (m = 2, γ = 60°)

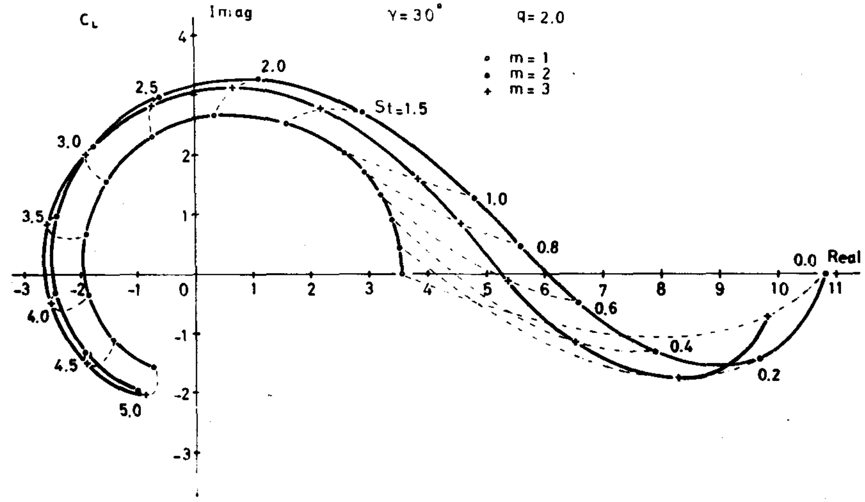

Fig.7 Effects of the phase difference for gusts in x-direction (γ = 30°, q = 2.0)

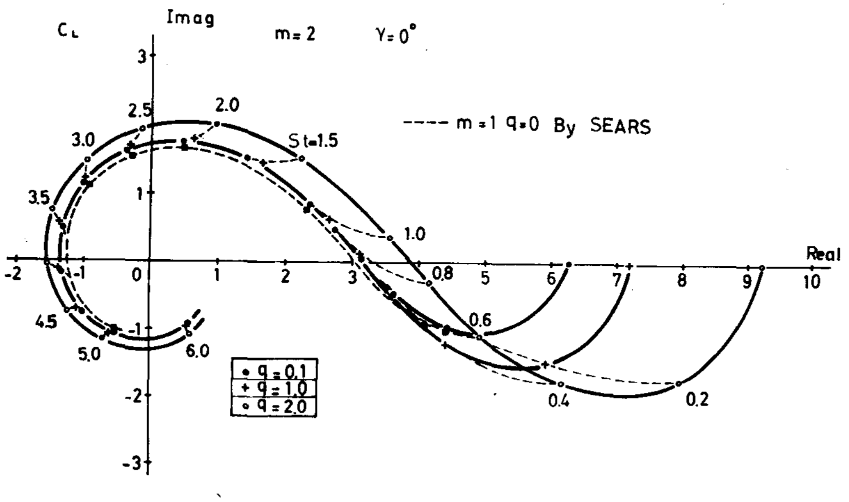

Fig.8 Fluctuating lift coefficient for sinusoidal gusts in y -direction (m = 2, γ = 0°)

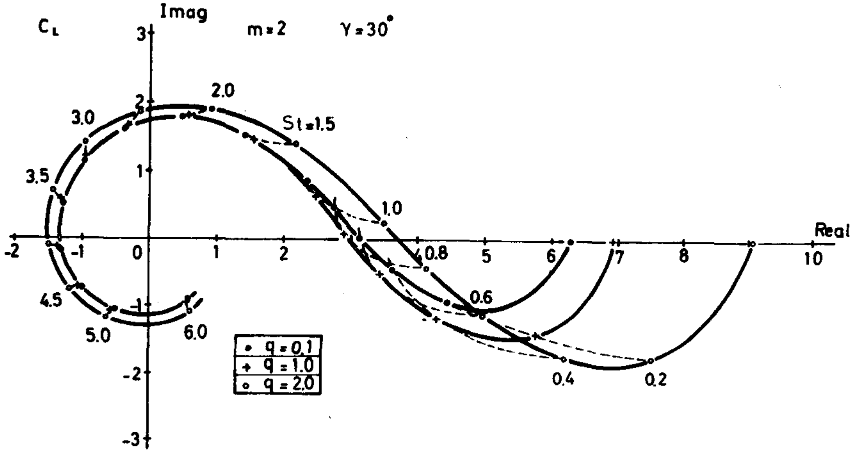

Fig.9 Fluctuating lift coefficient for sinusoidal gusts in y -direction (m = 2, γ = 30°)

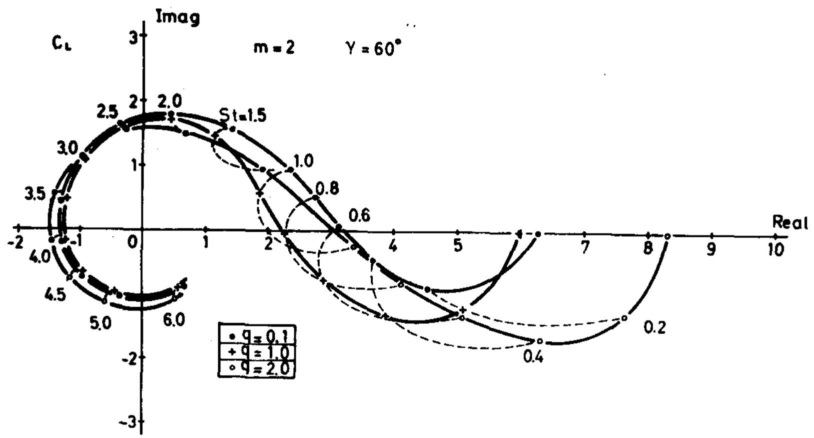

Fig.10 Fluctuating lift coefficient for sinusoidal gusts in y -direction (m = 2, γ = 60°)

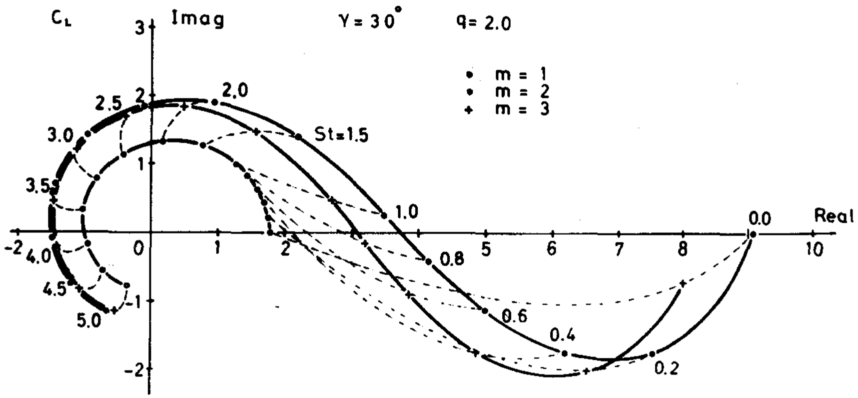

Fig.11 Effects of the phase difference for gusts in y -direction (γ = 30°, q = 2.0)

1.3 Applications to several flows

1.3.1 Introduction

In the preceding sections we restricted our attention to the sinusoidal gusts, which had relatively simple boundary condition of acceleration potential on the blade surfaces. This section treats the case of Kemp-type upwash, blade oscillation and transient flows in the same method. It is assumed that the fluid is inviscid and imcompressible, and that the disturbances are again in the same phase on each blade .

1.3.2 Kemp-type upwash

Though the boundary conditions for acceleration on the blade surfaces are utterly simple for sinusoidal gusts but they are not so simple in general. To show the way to treat general cases we will consider Kemp-type upwash in the first place. This flow was Firstly treated by Kemp & Sears [1] as the elementary velocity component induced by adjacent blade rows when they studied the potential interaction problem. The flow is characterized by the velocity disturbance on the blade surface;

$$v_d = v_0 \exp\{j(\omega\tau - \mu x/U)\} \tag{1}$$

where ω is real and μ is complex ($\mu = f + jk$). The case $\mu = 0$ corresponds to the case of translatory oscillation of the blades, and the case $\mu = \omega$ to sinusoidal gusts. It is assumed that the disturbance has no phase difference between adjacent blades and that the blades have no stationary lifts. The acceleration on the blade is,

$$\begin{aligned} a_{yd} &= \frac{\partial v}{\partial \tau} + U\frac{\partial v}{\partial x} \\ &= j(\omega - \mu)\, v_0\, e^{j(\omega\tau - \mu x/U)} \end{aligned} \tag{2}$$

where U designates the mainstream velocity. Hence, the boundary condition for the imaginary part ψ of the complex acceleration potential may be written for $\mu \neq 0$

$$\psi = -\int a_{yd}\, dx$$

$$= [\![\left(\frac{f\omega}{f^2+h^2} - 1 \right) - j \frac{h\omega}{f^2+h^2}]\!] \times \bar{v} v_0 e^{\frac{h}{\bar{v}}x} \left(\cos\frac{f}{\bar{v}}x - j \sin\frac{f}{\bar{v}}x \right) e^{j\omega\tau} + Const. \tag{3}$$

The acceleration potential ϕ is given as the real part of the analytical complex function $\bar{W}$ whose imaginary part ψ satisfies the boundary condition (3) and has finite value at $x=-\infty$. The cascade shown in Fig.0 at the commencement is mapped to a unit circle in ζ -plane shown in Fig.2 in section 1.1. by the mapping function represented by Eq. (19) in section 1.1. The acceleration complex potential $\bar{W}$ is given in ζ -plane. In the first place are given the complex functions $\bar{W}_1 = \phi_1 + i\psi_1$, $\bar{W}_2 = \phi_2 + i\psi_2$ whose imaginary parts ψ_1 and ψ_2 satisfy

$$\psi_1 = e^{\frac{h}{\bar{v}}x} \cos\frac{f}{\bar{v}}x \quad , \quad \psi_2 = e^{\frac{h}{\bar{v}}x} \sin\frac{f}{\bar{v}}x \tag{4}$$

Putting higher order singularities at the origin of ζ -plane, we have

$$\bar{W}_m = \sum_{\ell=1}^{\infty} \left(A_{m,\ell} + i B_{m,\ell} \right) / \zeta^{\ell} \qquad (m=1,2) \tag{5}$$

The constants $A_{m,\ell}$, $B_{m,\ell}$ can be given as the Fourier coefficients of the imaginary parts ψ_m on the blade surface. [2]

$$\left.\begin{matrix} A_{1,\ell} \\ A_{2,\ell} \end{matrix}\right\} = -(1/\pi) \int_0^{2\pi} \left\{ \begin{matrix} e^{\frac{h}{\bar{v}}x} \cos\frac{f}{\bar{v}}x \\ e^{\frac{h}{\bar{v}}x} \sin\frac{f}{\bar{v}}x \end{matrix} \right\} \sin\ell\theta \, d\theta$$

$$\left.\begin{matrix} B_{1,\ell} \\ B_{2,\ell} \end{matrix}\right\} = \frac{1}{\pi} \int_0^{2\pi} \left\{ \begin{matrix} e^{\frac{h}{\bar{v}}x} \cos\frac{f}{\bar{v}}x \\ e^{\frac{h}{\bar{v}}x} \sin\frac{f}{\bar{v}}x \end{matrix} \right\} \cos\ell\theta \, d\theta$$

where θ denotes the deviation angle on the unit circle and x denotes the corresponding location on the blade surface in Z -plane. If we define

$$\bar{W}_c = i(\zeta - e^{i\alpha}) / (\zeta + e^{i\alpha}) = \phi_c + i\psi_c \tag{6}$$

and

$$\bar{W} = \left\{ A_k \bar{W}_c + [\![\left(\frac{f\omega}{f^2+h^2} - 1 \right) - j \frac{f\omega}{f^2+h^2}]\!] \times v_0 \bar{v} \left(\bar{W}_1 - j \bar{W}_2 \right) \right\} e^{j\omega\tau} \tag{7}$$

the acceleration complex potential $\bar{W}$ satisfies the boundary condition on the blade surface. The constant A_k is real with respect to i and will be fixed from the velocity boundary conditions. The term $A_k \overline{W}_c$ represents the singularity at the leading edge and has constant imaginary part which represents the constant term in Eq.(3). Next let us determine the value of A_k from the velocity boundary condition on the blade surface. Integrating Eq.(2) we can express the velocity purterbation as

$$v_d = (1/U)\cdot e^{-jS_t x}\int_{-\infty}^{x} a_y(\xi,y)\, e^{jS_t\xi}d\xi \tag{8}$$

where $S_t = \frac{\omega C}{2U}$ ($C=2$; chordlength). The velocity purterbation on the blade surface is given by (1) and equating with (8) gives the boundary condition ;

$$\int_{-\infty}^{-1} \bar{a}_y(\xi,0)\, e^{jS_t\xi}d\xi = v_0 U\, e^{-j\frac{\omega-\mu}{U}} \tag{9}$$

where $a_y = \bar{a}_y e^{j\omega\tau}$. Integrating Eq.(9) by parts taking advantage of the relation $a_y = -\partial\Psi/\partial x$ and considering the singularity of Ψ_c at the leading edge , we find

$$A_k = v_0 U \Big[\!\Big[e^{-j\frac{\omega-\mu}{U}} - \Big[\Big(\frac{\omega f}{f^2+h^2}-1\Big) - j\frac{\omega h}{f^2+h^2}\Big]\times\Big[\Big\{(-\Psi_1(-1) + \Psi_1(-\infty))e^{-jS_t} + jS_t\int_{-\infty}^{-1-0}(\Psi_1(\xi)-\Psi_1(-\infty))e^{jS_t\xi}d\xi\Big\} - j\Big\{(-\Psi_2(-1)+\Psi_2(-\infty)) \times e^{-jS_t} + jS_t\int_{-\infty}^{-1-0}(\Psi_2(\xi)-\Psi_2(-\infty))e^{jS_t\xi}d\xi\Big\}\Big]\Big]\!\Big] \times \Big[\!\Big[(-\Psi_c(-1)+\Psi_c(-\infty)) \times e^{-jS_t} + jS_t\int_{-\infty}^{-1-0}(\Psi_c(\xi)-\Psi_c(-\infty))\, e^{jS_t\xi}d\xi \tag{10}$$

The fluctuating pressure p is given by

$$p = -\rho\phi = -\rho\, Re\Big[A_k\overline{W}_c + \Big\{\Big(\frac{\omega f}{f^2+h^2}-1\Big) - j\frac{\omega h}{f^2+h^2}\Big\} \times U v_0(\overline{W}_1 - j\overline{W}_2)\Big]e^{j\omega\tau} \tag{11}$$

The fluctuating lift is known through the integration of the pressure distribution. Normal velocity fluctuations have been treated above, which correspond to sinusoidal gusts in y -direction in the preceding sections. Considering the crossed components of the acceleration potential we can treat the Kemp-type upwash in x -direction. That is, the pararell velocity disturbance

$$v_x = \bar{U} + u_0 e^{j(\omega t - \mu x/\bar{U})}, \qquad v_y = 0 \tag{12}$$

inflows the cascade with stationary angle of attack β. The unsteady component can be treated in the same way as the above analysis and the crossed component can be treated in the similar way as shown for sinusoidal gusts in y-direction in the previous sections. The pressure fluctuation related to unsteady forces is,

$$p = -\rho \, [A_{kx} \; Re_i \, (\bar{W}_c(x) - \bar{W}_c(-\infty)) - \beta u_0 \bar{U} \; Re_i \, \{ (\bar{W}_1(x)$$
$$- \bar{W}_1(-\infty)) - j \, (\bar{W}_2(x) - \bar{W}_2(-\infty)) \} + \beta u_0 \bar{U} \, e^{-\frac{j\mu}{\bar{U}} x}$$
$$Re_i \, (\bar{W}_c(x) - \bar{W}_c(-\infty)) / Im_i \, (\bar{W}_c(0) - \bar{W}_c(-\infty)) \;] \; e^{j\omega t} \tag{13}$$

where $A_{kx} = -\frac{\beta u_0}{v_0} A_k$ and the constant A_k is given by Eq.(10). By putting $\mu = \omega$ we can see the results shown by Eq.(11),(13) agree with those by Eq.(81),(82) in section 1.1, the results for sinusoidal gusts in x and y-direction respectively. By taking $\mu \to 0$ we can get the results which agree with those for the translatory oscillation of the blades shown in the next section.

1.3.3 Translatory oscillation of the blades
(uniformly oscillating flow)

Though we can analyze the unsteady lifts on the blades under translatory oscillation or on the blades in uniformly oscillating flow as the limiting case $\mu \to 0$ of above mentioned Kemp-type upwash. We can show another way without Fourier series. The velocity disturbance on the blade surface is

$$v_\alpha = v_0 \, e^{j\omega t} \tag{14}$$

The acceleration due to the disturbance is,

$$a_{y\alpha} = j\omega \, v_0 \, e^{j\omega t} \tag{15}$$

Hence, the boundary condition for the imaginary part Ψ of the acceleration complex potential will be,

$$\Psi = j\omega\, v_0\, e^{j\omega\tau} \cdot x \quad + \quad Const. \tag{16}$$

We will introduce the following complex potential $\overline{W}_3$ whose imaginary part Ψ_3 satisfies $\Psi_3 = x$ on the blade surfaces.

$$\overline{W}_3 = (i/g)\log[(\zeta + e^{-\varepsilon})/(\zeta - e^{-\varepsilon})] = \Phi_3 + i\,\Psi_3 \tag{17}$$

Considering the singularity at the leading edge, we may write

$$\overline{W} = A'\,\overline{W}_c + j\omega\, v_0\, \overline{W}_3\, e^{j\omega\tau} \tag{18}$$

The acceleration complex potential $\overline{W}$ satisfies the singularity at the leading edge and the boundary condition (16) on the blade surface. The boundary condition on the blade surface for velocity is given by putting $\mu = 0$ in Eq.(9) and be written,

$$\int_{-\infty}^{-1} \overline{a}_y(\xi, 0)\, e^{jSt\xi}\, d\xi = v_0\, \overline{U}\, e^{-jSt} \tag{19}$$

The strength of the leading edge singularity A' is decided from Eqs. (18) and (19) as

$$A' = -v_0\overline{U}e^{-jSt}\{1 - jSt\,e^{jSt}\{(\Psi_3(-1) - \Psi_3(-\infty))e^{-jSt} - jSt\int_{-\infty}^{-1-0}(\Psi_3(\xi) - \Psi_3(-\infty))e^{+jSt\xi}d\xi\}] \times [(\Psi_c(-1) - \Psi_c(-\infty))e^{-jSt} - jSt\int_{-\infty}^{-1-0}(\Psi_c(\xi) - \Psi_c(-\infty))e^{jSt\xi}d\xi]^{-1} \tag{20}$$

The pressure fluctuation on the blade surface is

$$p = -\rho\, Re_i\, [A'(\overline{W}_c(x) - \overline{W}_c(-\infty)) + jSt\, v_0\overline{U}(\overline{W}_3(x) - \overline{W}_3(-\infty))]\, e^{j\omega\tau} \tag{21}$$

In case that the main stream velocity oscillates around mean value sinusoidally with time (uniformly oscillating flow in x -direction) , the boundary conditions on the blade surfaces are similar to that for translatory oscillation of the blades. We can analyze the lift fluctuation due to the flow by considering the crossed component of acceleration potential and by making use of the complex potential $\overline{W}_3$. The result is summerized as follows. When the unsteady flow of

$$v_x = \overline{U} + u_0 e^{j\omega\tau} \tag{22}$$

inflows to the cascade with an angle of attack β , the corresponding pressure distribution relating to the unsteady forces is

$$p = -\rho \left[Re_i \{ (\overline{W}_c(x) - \overline{W}_c(-\infty)) A_x' + j\beta u_0 \overline{U} St \right.$$

$$\times (\overline{W}_3(x) - \overline{W}_3(-\infty)) \} + \beta u_0 \overline{U}\, Re_i \{ \overline{W}_c(x) - \overline{W}_c(-\infty) \}$$

$$\left. / Im_i \{ \overline{W}_c(0) - \overline{W}_c(-\infty) \} \right] e^{j\omega\tau} \tag{23}$$

where $$A_x' = -(\beta u_0 / v_0)\, A' \tag{24}$$

Comparing Eqs.(12) with (22) we see that they agree by letting $\mu \to 0$ in Kemp-type upwash in x -direction.

1.3.4 Rotational oscillation

Consider the case that the blades oscillate rotationally with angle of attack β .

$$\beta = \beta_0 e^{j\omega\tau} \tag{25}$$

around zero lift angle of attack. The boundary condition on the blade surfaces is

$$v_n = -x\dot{\beta} = v_x \beta + v_y \tag{26}$$

and putting $v_x = \overline{U} + u_d$, $v_y = v_a$ gives,

$$v_d = -(U + j\omega x)\beta_0 e^{j\omega\tau} \tag{27}$$

Hence the boundary condition on the blade surface is,

$$a_{yd} = -j\omega(2U + j\omega x)\, U\beta_0\, e^{j\omega\tau} \tag{28}$$

Therefore the imaginary part of the acceleration complex potential should satisfy the following relation on the blade surface.

$$\psi = j S_t U \beta_0 \cdot 2Ux - \frac{1}{2} S_t^2 U^2 \beta_0 x^2 + Const. \tag{29}$$

The complex potential to satisfy the first term on the right hand side of Eq.(29) is given by Eq.(17). The method of putting higher order singularities at the origin in ζ-plane can be used to get the potential function to satisfy the second term of Eq.(29). That is,

$$\overline{W}_4 = \sum_{\ell=1}^{\infty} (A_{4,\ell} + i B_{4,\ell})/\zeta^{\ell} = \phi_4 + i\psi_4 \tag{30}$$

where

$$A_{4,\ell} = -(1/\pi)\int_0^{2\pi} x^2 \sin\ell\theta\, d\theta$$

$$B_{4,\ell} = (1/\pi)\int_0^{2\pi} x^2 \cos\ell\theta\, d\theta$$

Considering the singularity at the leading edge, we can write the acceleration complex potential as follows.

$$\overline{W} = A''\, \overline{W}_c + 2j S_t U^2 \beta_0 \overline{W}_3 - \frac{1}{2} S_t^2 U^2 \beta_0\, \overline{W}_4 \tag{31}$$

The velocity boundary condition is reduced from Eqs.(18) and (29) to,

$$\int_{-\infty}^{-1+0} \overline{a}_{yd}(\xi, 0)\, e^{jS_t\xi} d\xi = -(1 - jS_t)\, U^2 \beta_0\, e^{-jS_t} \tag{32}$$

which determine the constant A'' as follows.

$$A'' = U^2 \beta_0 [(1 - jS_t)e^{-jS_t} - 2jS_t\{(\psi_3(-1) - \psi_3(-\infty))\, e^{-jS_t}$$

$$- jSt \int_{-\infty}^{-1-0} (\Psi_3(\xi) - \Psi_3(-\infty)) e^{jSt\xi} d\xi \} + \frac{1}{2} St^2 \{ (\Psi_4(-1) - \Psi_4(-\infty))$$

$$\times e^{-jSt} - jSt \int_{-\infty}^{-1-0} (\Psi_4(\xi) - \Psi_4(-\infty)) e^{jSt\xi} d\xi \}] \times [(\Psi_c(-1) - \Psi_c(-\infty)) e^{-jSt} \tag{33}$$
$$- j\omega \int_{-\infty}^{-1-0} (\Psi_c(\xi) - \Psi_c(-\infty)) e^{jSt\xi} d\xi]^{-1}$$

Then the pressure fluctuation will be

$$p = -\rho R_e i \, [A'' \{ \overline{W}_c(x) - \overline{W}_c(-\infty) \} + 2jSt U^2 \beta_0 \{ \overline{W}_3(x)$$

$$- \overline{W}_3(-\infty) \} - \frac{1}{2} St^2 U^2 \beta_0 \{ \overline{W}_4(x) - \overline{W}_4(-\infty) \}] e^{j\omega t} \tag{34}$$

As can be seen from the examples above, the boundary condition for the imaginary part of the complex acceleration potential on the blade surface is determined from assumed velocity disturbance and the acceleration complex potential is given as the sum of the term $A \cdot \overline{W}_c$ which represents the singularity at the leading edge and the term $\overline{W}_m$ ($m = 1 \sim 4$) to satisfy the boundary condition for Ψ. The complex potential $\overline{W}_m$ may generally be given by putting higher order singularities at the origin of ζ -plane as shown by Eqs.(5) or (30). In case the normal velocity purterbation is given the crossed components of the acceleration potential can be neglected as a higher order small quantity. But for pararell velocity purterbations, the crossed acceleration components are of the same order as the unsteady components and have to be taken into account. The magnitude A of the singularity is determined by the velocity boundary condition afterwards.

1.3.5 Transient flows

The unsteady flows treated so far are all cyclic flows. Transient flows are treated in this section. It has been treated by Wagner [3], Küssner [4], and Karman & Sears [5] for the case of isolated airfoil. This section is intended to get the lift response on cascade blades by applying conformal mapping method to acceleration potential method as is used for oscillating flows.

A) Step gusts

Consider the case when the free vortex sheet inflows to cascade as shown in Fig.1. The velocity disturbance due to the vortex sheet can be separated into two components on the blade surface as follows.

$$v_x = \bar{U} \quad , \qquad v_y = v_0\, \sigma\,(\bar{U}\tau - x - 1) \tag{35}$$

$$v_x = \bar{U} + u_0\, \sigma\,(\bar{U}\tau - x - 1)\,, \qquad v_y = 0 \tag{36}$$

where σ designates the step function defined by

$$\sigma(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

In the first place is considered the step gust in y-direction given by Eq.(35). For brevity the angle of attack is considered to be zero. The velocity v_d induced by the blade cancells the velocity disturbance and hence,

$$v_d = -\,v_0\, \sigma\,(\bar{U}\tau - x - 1) \tag{37}$$

on the blade surface. Then the acceleration on the blade surface is,

$$a_{yd} = \partial v_d / \partial \tau + \bar{U}\, \partial v_d / \partial x$$

$$= -\bar{U} v_0\, \delta(\bar{U}\tau - x - 1) + \bar{U} v_0\, \delta(\bar{U}\tau - x - 1) = 0 \tag{38}$$

where $d\sigma(x)/dx = \delta(x)$ and $\delta(x)$ is the Dirac's δ function. Then the boundary condition for Ψ is,

$$\Psi = -\int a_y\, dx = Const. \tag{39}$$

and the acceleration complex potential which satisfies the singularity at the leading edge and the boundary condition (39) will be,

$$\bar{W} = A_{ys}(\tau)\, \bar{W}_c \tag{40}$$

where $A_{ys}(\tau)$ is a real function of time τ . The function cannot be determined from the boundary condition for acceleration but from velocity boundary condition. In case of oscillating flows the velocity components are related to acceleration through Eq. (8) and similar relations can be given through integration of Eq. (38) for general flows.

$$v_a(x,\tau) = g(x - \overline{v}\tau) + \left[\int a_y\,(\tau, c + \overline{v}\tau)\,d\tau\right]_{c = x - \overline{v}\tau} \tag{41}$$

where g is an arbitrary function of $x - \overline{v}\tau$ which designates the velocity disturbance flowing down with the mainstream velocity and may be written $g(x - \overline{v}\tau) = v_0\,\delta(\overline{v}\tau - x - 1)$ in this case. Since the gust has not reached the leading edge for $\tau < 0$, i.e., $v_a = 0$ for $\tau \leq 0$, Eq. (41) with Eq. (40) is reduced to

$$v_a(x,\tau) = -\left[\int_0^{\tau}\left(A_{ys}(\tau')\,\frac{\partial\psi_c}{\partial x}\Big|_{x = c + \overline{v}\tau'}\right)d\tau'\right]_{c = x - \overline{v}\tau} \tag{42}$$

The induced velocity (42) vanishes at $x = -\infty$ since $\frac{\partial\psi_c}{\partial x}\Big|_{x = -\infty} = 0$. From Eqs. (42) and (47) the velocity boundary condition on the blade surface may be written for $-1 \leq x \leq 1$, $\tau \geqq 0$

$$-v_0\,\delta(\overline{v}\tau - x - 1) = -\left[\int_0^{\tau}\left(A_{ys}(\tau')\,\frac{\partial\psi_c}{\partial x}\Big|_{x = c + \overline{v}\tau'}\right)d\tau'\right]_{c = x - \overline{v}\tau} \tag{43}$$

By the way, by putting $a_y = \overline{a}_y e^{j\omega\tau}$ in the second term in Eq. (41) and by assuming that infinite time has passed since the oscillation began, we can get Eq. (8). Eq. (43) is regarded as the integral equation for $A_{ys}(\tau)$ and by solving it numerically the unknown function $A_{ys}(\tau)$ can be known. The right hand side of Eq. (43) may be deformed as follows.

$$-\left[\int_0^{\tau}\left(A_{ys}(\tau')\,\frac{\partial\psi_c}{\partial x}\Big|_{x = c + \overline{v}\tau'}\right)d\tau'\right]_{c = x - \overline{v}\tau}$$

$$= -(1/\overline{v})\int_{x - \overline{v}\tau}^{x}\left(A_{ys}\left(\frac{x^* - x}{\overline{v}} + \tau\right)\frac{\partial\psi_c}{\partial x}\Big|_{x^*}\right)dx^*$$

Let us consider the case $-1 < x - \overline{v}\tau < x < 1$. Since $\psi_c = Const.$ on blade surface the right hand side of Eq. (43) equals to zero and since $\overline{v}\tau - x - 1 < 0$ the left hand side of Eq. (43) also equals to zero. Therefore Eq. (43) is satisfied regardless of the value of A_{ys}. Next is considered the case $x - \overline{v}\tau < -1 < x < 1$. Since $\frac{\partial\psi_c}{\partial x} = 0$ for $x > -1$, Eq. (43) may be written

$$v_0 = (1/\overline{v})\int_{x - \overline{v}\tau}^{-1+0}\left(A_{ys}\left(\frac{x^* - x}{\overline{v}} + \tau\right)\frac{\partial\psi_c}{\partial x}\Big|_{x^*}\right)dx^* \tag{44}$$

Eq.(44) may be integrated by parts taking into account the singularity of Ψ_c at the leading edge,

$$v_0 = (1/\bar{v}) \left\{ A_{ys}\left(\frac{-1+\bar{v}\tau-x}{\bar{v}}\right)\left(\Psi_c(-1+0) - \Psi_c(-\infty)\right) \right.$$

$$\left. - (1/\bar{v}) \int_{x-\bar{v}\tau}^{-1-0} A'_{ys}\left(\frac{x^*-x}{\bar{v}} - \tau\right)\left(\Psi_c(x^*) - \Psi_c(-\infty)\right) dx^* \right\} \tag{45}$$

and is reduced by putting $x - \bar{v}\tau = -1 - \delta$ and $\frac{x^* + 1 + \delta}{\bar{v}} = y$

$$v_0 = -(1/\bar{v}) \int_0^{\delta/\bar{v}} A'_{ys}(y)\left(\Psi_c(\bar{v}y - 1 - \delta) - \Psi_c(-1+0)\right) dy \tag{46}$$

Equation (46) should be satisfied for all $\delta > 0$. The value of Ψ_c can be known approximately for $0 < x^* \ll 1$ by considering the singularity of Ψ_c at the leading edge. For $0 < x^* \ll 1$

$$\Psi_c(-1 - x^*) = 1 + 2\sqrt{b}\, x^{*-\frac{1}{2}} \approx 2\sqrt{b}\, x^{*-\frac{1}{2}} \tag{47}$$

where

$$b = \frac{2\cosh^3\varepsilon \cos\gamma \cos\alpha (\tan^2\alpha + \tanh^3\varepsilon)}{\gamma(2\cos^2 2\alpha - 4\cosh 2\varepsilon \cos 2\alpha + \cosh 4\varepsilon + 1)}$$

Since $\Psi_c(-1+0) = 0$, Eq.(46) may be written for

$$-\frac{\bar{v} v_0}{2\sqrt{b}} = \int_0^{\delta/\bar{v}} A'_{ys}(y)\,(\delta - \bar{v}y)^{-\frac{1}{2}} dy \tag{48}$$

The right hand side of Eq.(48) should be a constant regardless of the value of $\delta/\bar{v} \ll 1$ and by putting $A'_{ys}(y) = a y^{-\frac{1}{2}}$ we see that

$$\int_0^{\delta/\bar{v}} A'_{ys}(y)(\delta - \bar{v}y)^{-\frac{1}{2}} dy = \frac{a}{\sqrt{\bar{v}}} \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx = \frac{a}{\sqrt{\bar{v}}} \frac{\Gamma^2(1/2)}{\Gamma(1)}$$

The constant a is given by

$$a = -\frac{\bar{v} v_0}{2\sqrt{b}} \frac{\Gamma(1)}{\Gamma^2(1/2)} \sqrt{\bar{v}}$$

from which

$$A'_{ys}(y) = -\frac{\bar{v} v_0}{2\sqrt{b}} \frac{\Gamma(1)}{\Gamma^2(1/2)} \sqrt{\frac{\bar{v}}{y}}$$

$$A_{ys}(y) = -\frac{\bar{U}v_0}{\sqrt{b}}\frac{\Gamma(1)}{\Gamma^2(1/2)}\sqrt{\bar{U}y} \tag{49}$$

The value of $A_{ys}(y)$ can be known numerically for general value of y. By putting $s/\bar{U} = m\Delta s$, $y = n\Delta s$ in Eq.(46) and applying trapesoidal formula, we have

$$\bar{U}v_0 = -\sum_{n=1}^{m} A'_{ys}(n\Delta s)\,\Psi_c(-1-(m-n)\bar{U}\Delta s)\,\Delta s$$

where $m = 1, 2, \cdots, \infty$ and $n = 1, 2, \cdots, m$. When the value of $A'_{ys}(y)$ for $y = n\Delta s$ are known ,

$$A'_{ys}(m\Delta s) = \frac{-\bar{U}v_0\left(1-\sum_{n=0}^{m-1} A'_{ys}(n\Delta s)\,\Psi_c(-1-(m-n)\Delta s)\Delta s\right)}{\Psi_c(-1)\,\Delta s} \tag{50}$$

where

$$\Delta s\,\Psi_c(-1) = \frac{1}{2}\int_0^{\Delta s}\Psi_c(-1-\bar{U}s)\,ds = 2\sqrt{b}\sqrt{\frac{\Delta s}{\bar{U}}} \tag{50-1}$$

and

$$A'_{ys}(0)\Delta s = \frac{1}{2}\int_0^{\Delta s}\left(-\frac{\bar{U}v_0}{2\sqrt{b}}\frac{\Gamma(1)}{\Gamma^2(\frac{1}{2})}\sqrt{\bar{U}}\,y^{-\frac{1}{2}}\right)dy = -\frac{\bar{U}v_0}{2\sqrt{b}}\frac{\Gamma(1)}{\Gamma^2(\frac{1}{2})}\sqrt{\bar{U}}\sqrt{\Delta s} \tag{50-2}$$

The value of A'_{ys} is known from Eq.(50) and by integration the value of A_{ys} is known. Fluctuating pressure then be written as follows.

$$p = -\rho\phi = -\rho A_{ys}\,\Phi_c \tag{51}$$

In case that the step gust in x-direction inflows to the cascade with an angle of attack β , the unsteady component of the acceleration potential can be known in the same way as for step gust in y-direction. Considering the crossed acceleration potential component we can get the pressure fluctuation on the blade surface as follows.

$$p = -\rho\left(A_{xs}(\tau)\Phi_c + \beta u_0\bar{U}\,\frac{\Phi_c(x) - \Phi_c(-\infty)}{\Psi_c(x) - \Psi_c(-\infty)}\,\delta(x-\bar{U}\tau)\right) \tag{52}$$

where

$$A_{xs}(\tau) = (\beta u_0/v_0)\,A_{ys}(\tau)$$

B) Uniform ramp flows

Consider the case that the normal velocity disturbance ramps uniformly in the entire flow region. (Fig.2) The boundary conditions on the blade surface are,

for $\tau < 0$

$$v_x = v_y = 0 \quad , \quad \overline{W} = 0$$

for $\tau > 0$

$$v_y = \tau/\tau_1 \;, \quad a_y = \frac{\partial v_y}{\partial \tau} + U\frac{\partial v_y}{\partial x} = 1/\tau_1 \;, \quad \psi = -\int a_y dx = -x/\tau_1 + Const.$$

Therefore the acceleration complex potential $\overline{W}$ is written as follows.

$$\overline{W} = A_{y\ell}\cdot\overline{W}_c - (1/\tau_1)\,\overline{W}_3 \tag{54}$$

where $A_{y\ell}$ is a real function of τ . In case $\tau > \tau_1$, $v_y = 1$, $a_y = 0$, $\psi = Const$ and hence

$$\overline{W} = A_{y\ell}\,\overline{W}_c \tag{56}$$

The function $A_{y\ell}$ is determined by the velocity boundary condition in the same manner as for step gust. The velocity boundary condition on the blade surface is

for $0 < \tau < \tau_1$

$$(\tau/\tau_1) = -\left[\int_0^{\tau}\left(A_{y\ell}(\tau')\frac{\partial\psi_c}{\partial x}\Big|_{x=c+U\tau'} - (1/\tau_1)\frac{\partial\psi_3}{\partial x}\Big|_{x=c+U\tau'}\right)d\tau'\right]_{c=x-U\tau} \tag{56}$$

for $\tau > \tau_1$

$$1 = -\left[\int_0^{\tau_1}\left(A_{y\ell}(\tau')\frac{\partial\psi_c}{\partial x}\Big|_{x=c+U\tau'} - (1/\tau_1)\frac{\partial\psi_3}{\partial x}\Big|_{x=c+U\tau'}\right)d\tau'\right]_{c=x-U\tau}$$

$$-\left[\int_{\tau_1}^{\tau} A_{y\ell}(\tau')\frac{\partial\psi_c}{\partial x}\Big|_{x=c+U\tau'}\,d\tau'\right]_{c=x-U\tau} \tag{57}$$

Putting $\tau' = \frac{x^*-x}{U} + \tau$ and considering the values of ψ_c and ψ_3 we can rewrite Eqs.(56) and (57) as follows.

for $x+1-U\tau > U\tau_1$

$$(x-U\tau+1)/\tau_1 = \int_{x-U\tau}^{-1}\left(A_{y\ell}\left(\frac{x^*-x}{U}+\tau\right)\frac{\partial\psi_c}{\partial x}\Big|_{x^*} - (1/\tau_1)\frac{\partial\psi_3}{\partial x}\Big|_{x^*}\right)dx^* \tag{58}$$

for $x+1-U\tau < U\tau_1$

$$1 = -(1/U)\left[\int_{x-U\tau}^{-1}\left(A_{y\ell}\left(\frac{x^*-x}{U}+\tau\right)\frac{\partial\psi_c}{\partial x}\Big|_{x^*}\right)dx^*\right.$$

$$-(1/\tau_1)\int_{x-U\tau}^{x-U(\tau-\tau_1)}\frac{\partial\psi_3}{\partial x}\Big|_{x^*}\,dx^* \tag{59}$$

By partially integratin Eqs.(58) and (59) considering the singularity at the leading edge, we get

for $\delta < \bar{U}\tau_1$

$$\frac{\delta - (\Psi_3(-1) - \Psi_3(-1-\delta))}{\tau_1} = \int_0^{\delta/\bar{U}} A'_{y\ell}(y)(\Psi_c(\bar{U}y-1-\delta) - \Psi_c(-1))\,dy \tag{60}$$

for $\delta > \bar{U}\tau_1$.

$$\frac{\bar{U}\tau_1 - (\Psi_3(\bar{U}\tau_1-1-\delta) - \Psi_3(-1-\delta))}{\tau_1} = \int_0^{\delta/\bar{U}} A'_{y\ell}(y)(\Psi_c(\bar{U}y-1-\delta) - \Psi_c(-1))\,dy \tag{61}$$

Equations (60) and (61) can be solved numerically in the similar way as for Eq.(46). After getting $A'_{y\ell}(y)$ we can get $A_{y\ell}(y)$ by numerical integration under $A_{y\ell}(0)=0$. Then the pressure is given by

for $\tau < \tau_1$

$$p = -\rho\,(A_{y\ell}(\tau)\Phi_c - (1/\tau_1)\,\Phi_3) \tag{62}$$

for $\tau > \tau_1$

$$p = -\rho\,A_{y\ell}(\tau)\,\Phi_c \tag{63}$$

Next let us consider the flow of uniform ramp in x -direction in which the chordwise velocity ramps uniformly and the mainstream has an angle of attack β . The unsteady component can be given in the same way as for uniform ramp in y -direction, and considering the crossed component we get the following results.

for $\tau < \tau_1$

$$p = -\rho\,[A_{x\ell}(\tau)\,\Phi_c(x) - (1/\tau_1)\,\Phi_3(x) + \beta\bar{U}\,(\Phi_c(x) - \Phi_c(-\infty))\,/\,(\Psi_c(x) - \Psi_c(-\infty))\cdot(\tau/\tau_1)] \tag{64}$$

for $\tau > \tau_1$

$$p = -\rho\,[A_{x\ell}(\tau)\,\Phi_c(x) + \beta\bar{U}\,\frac{\Phi_c(x) - \Phi_c(-\infty)}{\Psi_c(x) - \Psi_c(-\infty)}] \tag{65}$$

where $A_{x\ell} = -\beta\,A_{y\ell}(\tau)$

1.3.6 Numerical results

The lift coefficients for Kemp-type upwash are shown in Figs.3,4. The lift coefficients are defined by $C_L = L/\rho\beta u_0 \bar{U}(c/2)e^{j\omega\tau}$ (Kemp-type upwash in x -direction) and by $C_L = L/\rho U_0 \bar{U}(c/2)e^{j\omega\tau}$ (Kemp-type upwash in y -direction), where L is the unsteady lift. The fat lines in the figures show the results for uniformly oscillating flow ($\mu=0$) and sinusoidal gust ($\mu=\omega$). They agree with the results of section 1.1 or the results in section 1.3.3. (Figs.5,6) In Figs.5,6 are shown the coefficients for uniformly oscillating flow in x and y -direction. The lift coefficients are defined by the same equations as for Kemp-type upwash. In Fig.6 is shown the result by Karman & Sears for isolated airfoil. Fig.7 shows the lift coefficient for rotating oscillation of the blades. The lift coefficient is defined as $C_L = L/\rho\beta_0 \bar{U}^2(c/2)e^{j\omega\tau}$. The results by Karman & Sears for isolated airfoil are plotted in the figure. The lift responces to step gusts in x and y-direction are shown in Figs.8,9. The lift coefficients are defined by $C_L = L/\rho\beta u_0 \bar{U}(c/2)$ or $C_L = L/\rho U_0 \bar{U}(c/2)$ for gusts in x and y -direction respectively. It is seen that the lift grows to stationary value earlier for finer cascades. The broken line in Fig.9 shows the results by Karman & Sears for isolated airfoil. Lift responces to uniform ramp flows are shown in Figs.10 and 11. The definitions of lift coefficients are the same as for step gusts. In Fig.12 can be seen the effect of ramping time τ_1 . It can be seen the lift development in $\tau > \tau_1$ is affected little by τ_1 in case $\bar{U}\tau_1 < 0.4$. At the instant $\tau=0$ the lift grows to finite value suddenly and then grows uniformly till $\tau=\tau_1$. After growing to the peak value the lift falls down to finite value at $\tau=\tau_1$. After Wagner [3] , in case of isolated airfoil and in the limit of $\tau_1 \to 0$, the lift grows impulsively to infinite at the instant $\tau=0$ and then falls to half of the stationary value, and thenafter uniformly grows to stationary value. According to Fig.11 the lift force at $\tau<\tau_1$ is greater for smaller value of $\bar{U}\tau_1$ but falls deeper at $\tau=\tau_1$ and the lift growth after $\tau=\tau_1$ is not affected by $\bar{U}\tau_1$ so much. Lift response for finer cascades are shown for $\bar{U}\tau_1=0.2$. For finer cascade, the lift variation is concentrated to acceleration period ($0<\tau<\tau_1$) and the lift falls to nearly the stationary value at $\tau=\tau_1$. In Fig.10 is shown the lift response for uniform ramp flows in y -direction, which have similar tendency as for uniform ramp in x -direction. In any case it can be said that the lift responce is much quicker for finer cascades than for isolated airfoil.

1.3.7 Conclusion

It has been shown that we can easily analyze the unsteady flows through cascades by applying acceleration potential method in combination with conformal mapping method. By introducing the potential function of higher order singularities we can treat oscillatory flows of arbitrary velocity disturbance at the blade surfaces. Moreover the method is applied to several transient flows and the cascade effects on the lift responce were elucidated.

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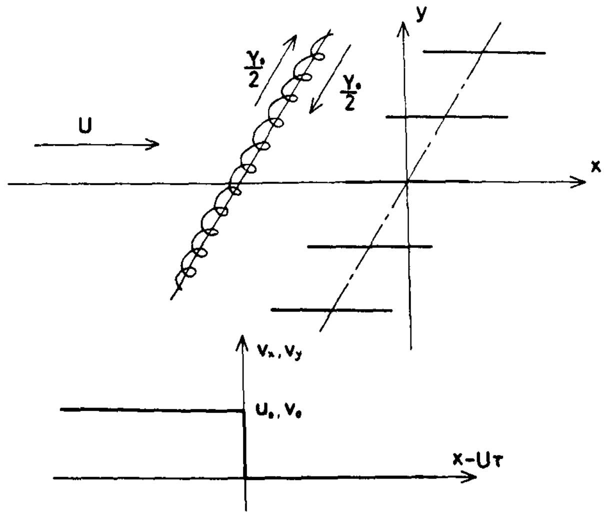

Fig.1 Step gusts

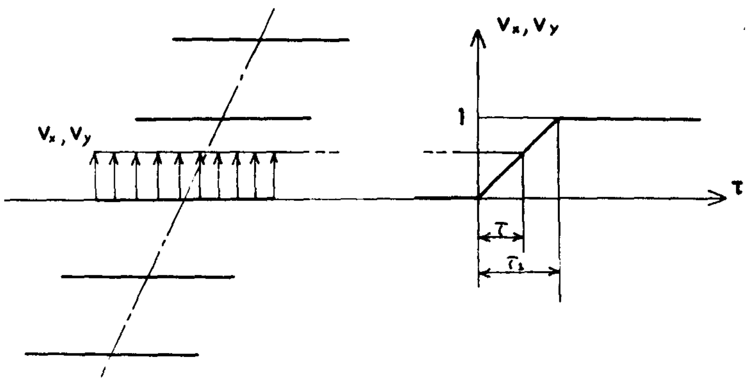

Fig.2 Uniform ramp flows

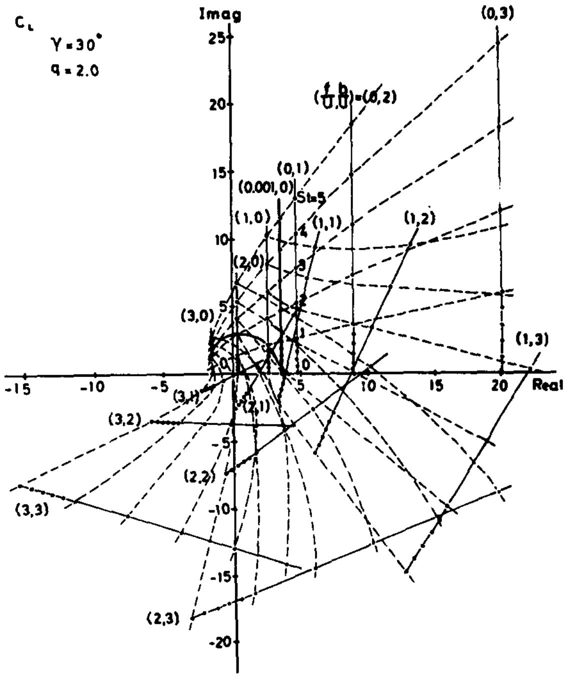

Fig.3 Kemp-type upwash in x -direction

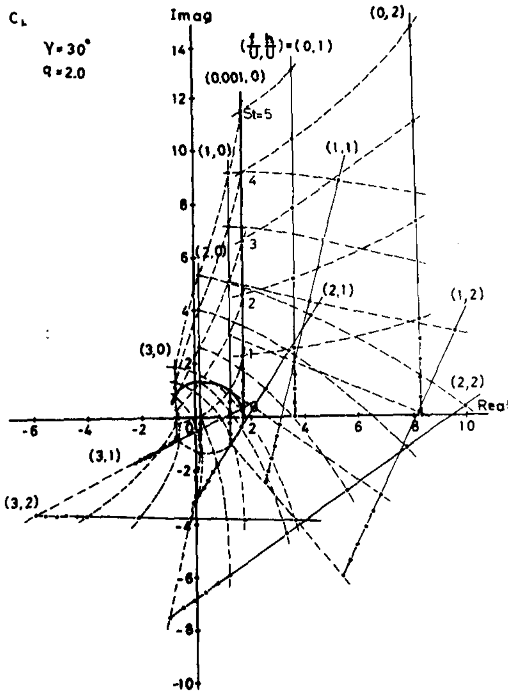

Fig.4 Kemp-type upwash in y -direction

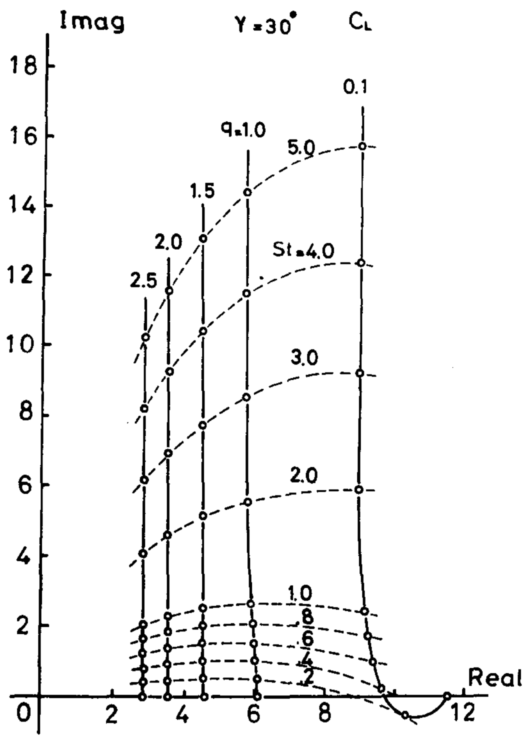

Fig.5 Uniformly oscillating flow in x-direction

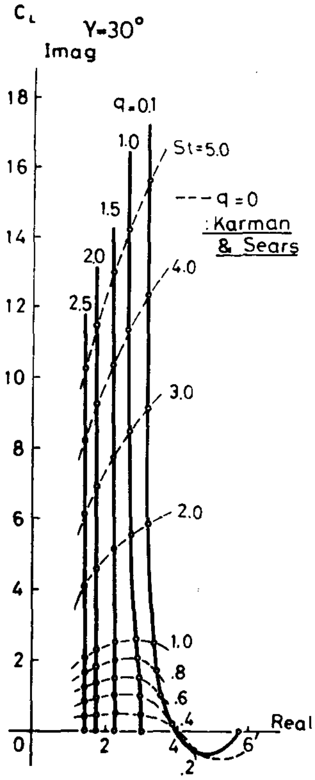

Fig.6 Uniformly oscillating flow in y-direction

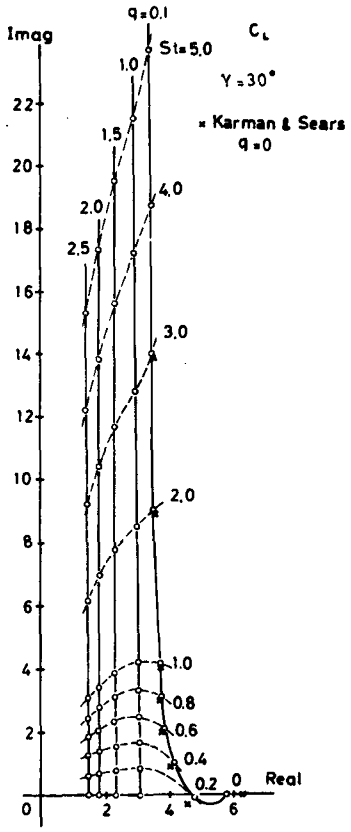

Fig.7 Rotational oscillation of the blades

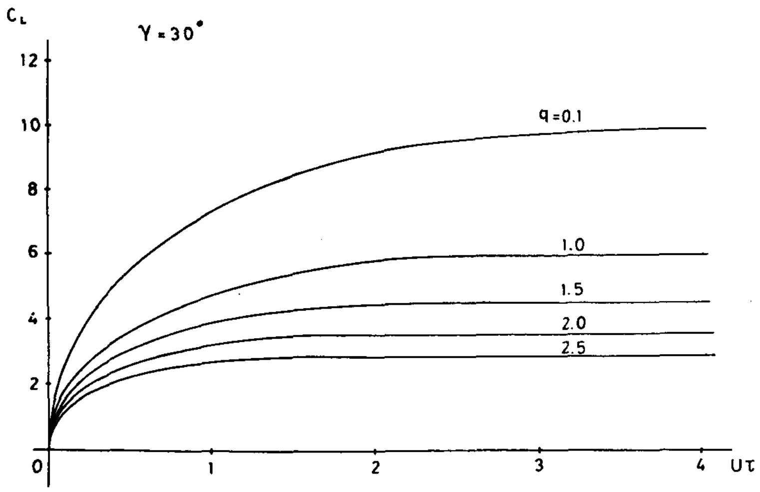

Fig.8 Step gusts in x-direction

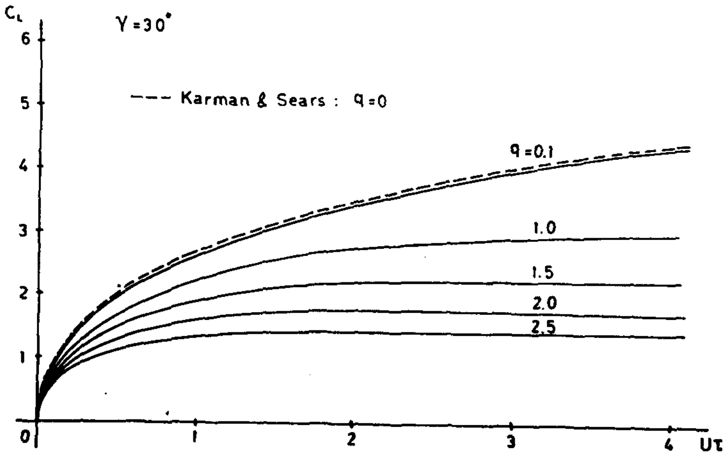

Fig.9 Step gusts in y-direction

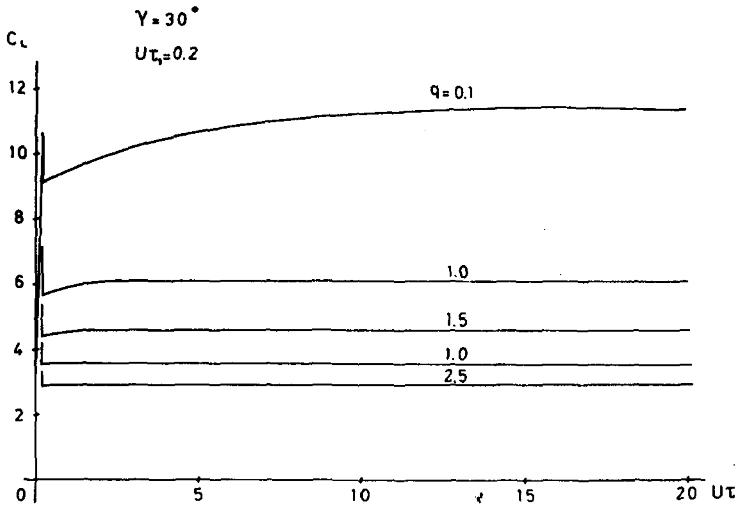

Fig.10 Uniform ramp flows in x-direction

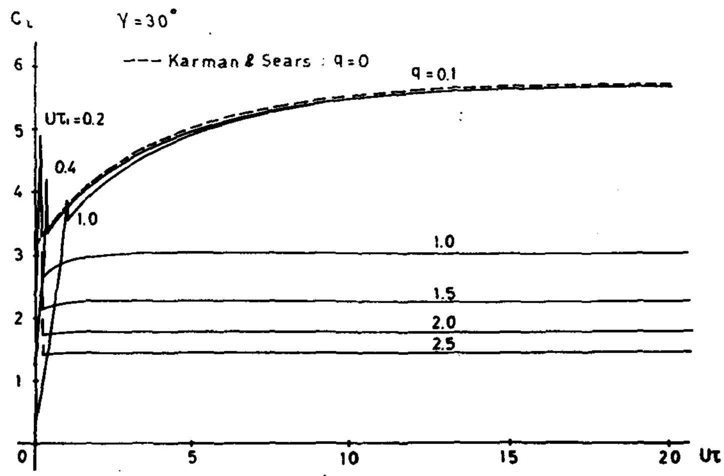

Fig.11 Uniform ramp flows in y-direction

Chapter 2 Incompressible Viscous Flow

2.1 Elementary solutions and applications to isolated airfoils

2.1.1 Introduction

There is a wealth of literature on the analysis of unsteady forces on the flat plate airfoils executing small oscillation in inviscid fluid. On the other hand, for the purpose of getting the insight into the viscous effects of stationary flow, Oseen's approximation has been adopted by several authors, for example, Bairstow (1923), Piercy & Winny (1933), Imai (1954), Tamada & Miyagi (1962), because of the simplicity that the total flow field can be treated by a single fundermental equation for the full Reynolds number range. In order to estimate the viscous flow effects on the flow around oscillating flat plates, Oseen's approximation has been adopted by Chu (1962) and Shen & Crimi (1965). However, both of them are not exact because the former doesn't take into account the viscous dissipation of the shed-off vortices entirely, and the latter neglects that of vortices flowing down on the flat plate. Recently, as an extension of the trailing edge problems, Brown & Daniels (1975) analyzed the flow around a oscillating flat plate by restricting themselves to the case of large Reynolds number, large Strouhal number and small oscillation amplitude. Their solution is based on the idea of dividing the Blasius boundary layer into several sublayers near the trailing edge and matching the solutions of the equations appropriate to each subdivision. The author believe that Oseen's approximation is still an useful tool for the study of the viscous effect on the unsteady flow around oscillating flat plates by virtue of its simplicity that a single linear equation is applied to the entire flow field. In this section the rigorous elementary solutions of the fundermental equations are introduced at first, and then applied to the problem of plunging motion of a flat plate airfoil and oscillation parallel to itself.

2.1.2 Fundermental equations

The equation of motion and the equation of continuity for the incompressible two dimensional flow are linearized on the assumption of small purterbation as follows.

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + X + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} + \bar{U}\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where $\bar{U}$, ν , p , X and Y are the main stream velocity, kinetic viscisity, static pressure, mass force in x and y direction respectively. Apparently $X = Y = 0$ everywhere except the blade surface. Eliminating u and v from Eqs.(1) and (3) one obtains

$$\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \tag{4}$$

The condition $X = Y = 0$ everywhere except the blade surface leads to

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{5}$$

We introduce the stream function ψ defined as

$$u = \partial\psi/\partial y \quad , \quad v = -(\partial\psi/\partial x) \tag{6}$$

Then, Eqs.(1) and (2) may be written

$$\frac{\partial}{\partial y}\left\{\frac{\partial\psi}{\partial t} + \bar{U}\frac{\partial\psi}{\partial x} - \nu\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right\} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$-\frac{\partial}{\partial x}\left\{\frac{\partial\psi}{\partial t} + \bar{U}\frac{\partial\psi}{\partial x} - \nu\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)\right\} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Putting

$$\frac{\partial\psi}{\partial t} + \bar{U}\frac{\partial\psi}{\partial x} - \nu\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = Q \quad , \quad p/\rho = P \tag{7}$$

the above equations are written as

$$\frac{\partial Q}{\partial y} = -\frac{\partial P}{\partial x} \quad , \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \tag{8}$$

Equation (8) expresses the Cauchy-Riemann relation and implies that $P+iQ$ is an analytic function of $z = x+iy$, where $i = \sqrt{-1}$. The flow of a source of strength $q\,\delta(x)\cdot\delta(y)$ at the origin of x-y plane satisfies the following equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = q\,\delta(x)\,\delta(y)$$

where δ designates the Dirac's δ function defined as $\delta(x)=0$ for $x \neq 0$ and $\int_{-\infty}^{+\infty}\delta(x)dx=1$. Introducing the velocity potential φ defined as

$$u = \partial\varphi/\partial x \quad , \quad v = \partial\varphi/\partial y$$

we can write the above equation as follows.

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = q\,\delta(x)\,\delta(y) \tag{9}$$

Now, supposing a sourse of strength $q\delta(x)\delta(y)/\Delta$ at the origin and a sink of strength $-q\delta(x)\delta(y)/\Delta - \frac{\partial}{\partial y}(q\delta(x)\delta(y)/\Delta)\cdot\Delta$ at (0 , $-\Delta$) and by letting $\Delta \to 0$, one obtains from Eq.(9) the following relation.

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = q\,\delta(x)\,\frac{d}{dy}\{\delta(y)\}$$

Above equation represents the flow induced by a doublet at the origin of x-y plane. Putting $\varphi \to p$ and $q \to Y_0$, the above equation is reduced to Eq.(4) with $X=0$, $Y=Y_0\delta(x)\delta(y)$. That is, as shown by Lamb, the mass force exerted in a flow field corresponds to a doublet in the pressure field. In other words, the exertion of the concentrated mass force Y_0 at the origin in y direction in a flow field produces a doublet of intensity Y_0 in y direction at the origin in the pressure field. This pressure field may be written as,

$$P + iQ = \frac{i\,Y_0}{2\pi\,(x+iy)} \tag{10}$$

2.1.3 Velocity field relating to the fluctuating lift

Assume a bound vortex of strength Γ at the origin of x-y plane at time $\tau=0$. According to the Helmholtz's theorem the free vortex of strength $-\Gamma$ appears at that instance and convected downstream with the free stream velocity U . In case of viscous fluid, the free vortex begins to dissipate as soon as it is produced. The velocity field induced by the bound vortex and the dissipating free vortex is given by

$$u = \left\{ \frac{\Gamma y}{2\pi r^2} - \frac{\Gamma y}{2\pi r'^2}\left(1 - e^{-\frac{r'^2}{4\nu\tau}}\right) \right\} \sigma(\tau)$$
$$v = \left\{ -\frac{\Gamma x}{2\pi r^2} + \frac{\Gamma(x-U\tau)}{2\pi r'^2}\left(1 - e^{-\frac{r'^2}{4\nu\tau}}\right) \right\} \sigma(\tau) \tag{11}$$

where $r^2 = x^2 + y^2$, $r'^2 = (x-U\tau)^2 + y^2$ and $\sigma(\tau)$ designates the step function defined as $\sigma(\tau)=0$ for $\tau<0$ and $\sigma(\tau)=1$ for $\tau>0$. The function is related to the Dirac's δ function as

$$d\sigma(\tau)\,/\,d\tau = \delta(\tau)$$

The flow field given by Eq.(11) satisfies the equation of continuity and the corresponding pressure field may be caluculated as follows. Considering the relations

$$\frac{\partial Q}{\partial y} = \frac{\partial u}{\partial \tau} + U\frac{\partial u}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -\frac{\Gamma U x y}{\pi r^4}\,\sigma(\tau) \tag{12}$$

$$-\frac{\partial Q}{\partial x} = \frac{\partial v}{\partial \tau} + U\frac{\partial v}{\partial x} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \left(-\frac{\Gamma U}{2\pi r^2} + \frac{\Gamma U x^2}{\pi r^4}\right)\sigma(\tau) \tag{13}$$

we can get the following equation from Eq.(7).

$$P + iQ = \frac{i\Gamma U}{2\pi}\,\frac{\sigma(\tau)}{x+iy} \tag{14}$$

Equation (14) represents a doublet in y direction at the origin in the pressure field. This implies that the velocity field given by Eq.(11) represents the flow field in which the concentrated force $Y=\rho\Gamma U$ is exerted at the moment $\tau=0$ and then kept unchanged thereafter. If we considered only the bound vortex at the origin of x-y plane, and not the shedding vortex, the corresponding pressure field would become, in the same way,

$$P + iQ = \frac{i\Gamma U}{2\pi}\left\{ \delta(\tau)\log(x+iy) + \frac{\sigma(\tau)}{x+iy} \right\} \tag{15}$$

The first term in the above equation, namely a vortex at the origin, does not appear in Eq.(10) and it violates the dynamic condition. The shedding vortex plays the role to cancel this unreasonable vortex in the pressure field, and the above argument gives another way of proof of Helmholtz's vortex theorem. Consider a harmoniously oscillating bound vortex $\Gamma e^{j\omega\tau}$ (j designates the imaginary unit with respect to time) at the origin. Then the pressure field will become

$$P + iQ = \frac{i\Gamma U}{2\pi} \left\{ j\omega \log(x+iy) + \frac{1}{x+iy} \right\} e^{j\omega\tau} \tag{16}$$

and hence one must consider the harmonious shedding of free vortices for this case also. Now we assume a bound vortex at $x = x_1$ whose strength oscillate harmoniously with time τ .

$$\Gamma = \Gamma_0(x_1)\, e^{j\omega\tau} dx_1 \tag{17}$$

The strength of the bound vortex varies $(d\Gamma/d\tau)\Delta\tau$ during the time interval $\Delta\tau$. Then the free vortex shed during the period $\Delta\tau$ spreads between x_1 and $x_1 + U\Delta\tau$. So the free vorticity distribution ε per unit length becomes

$$\frac{d\Gamma}{d\tau} = -\varepsilon U \qquad \text{or} \qquad \varepsilon = -\frac{j\omega}{U}\Gamma_0(x_1)\, e^{j\omega\tau}$$

The free vortex shed from x_1 at time $\tau - (\xi - x_1)/U$ reaches at $x = \xi$ at the instance $\tau = \tau$. Hence the distribution ε of this free vortex at $x = \xi$ is,

$$\varepsilon\, dx_1 = -\frac{j\omega}{U}\Gamma_0(x_1)\, e^{j\omega(\tau - \frac{\xi - x_1}{U})} dx_1 = -\frac{j\omega}{U}\Gamma_0(x_1)\, e^{-j\omega\frac{\xi - x_1}{U}} e^{j\omega\tau} dx_1 \tag{18}$$

Taking into account the dissipation of the vorticity during the period $(\xi - x_1)/U$ needed to flow down from $x = x_1$ to $x = \xi$, one gets the velocity at any point (x, y) in the flow field by the free vortices $\varepsilon\, d\xi$ at $x = \xi$ as follows.

$$\begin{aligned}
u &= \frac{\varepsilon y}{2\pi r^2} \left\{ 1 - \exp\left(-\frac{r^2 U}{4\nu(\xi - x_1)}\right) \right\} d\xi \\
v &= -\frac{\varepsilon(x-\xi)}{2\pi r^2} \left\{ 1 - \exp\left(-\frac{r^2 U}{4\nu(\xi - x_1)}\right) \right\} d\xi \\
r^2 &= (x-\xi)^2 + y^2
\end{aligned} \tag{19}$$

On the other hand, the velocity at (x , y) induced by the bound vortex of strength $\Gamma_0(x_1)\,dx_1$ is,

$$u = \frac{\Gamma_0(x_1)}{2\pi} e^{j\omega\tau} \frac{y}{r^2} dx_1 \quad , \quad v = -\frac{\Gamma_0(x_1)}{2\pi} e^{j\omega\tau} \frac{x - x_1}{r^2} dx_1 \tag{20}$$

The velocity induced by the above mentioned bound vortex has discontinuity across the blade surface or on ($x_1, 0$), which is not allowed for viscous fluid because of the coherence charactristics. To cancel the velocity gap is suitable the following velocity field.

$$u = -\frac{\Gamma_0(x_1)}{2\pi} e^{j\omega t} \beta e^{R_e(x-x_1)} K_1(\beta r) \frac{y}{r} dx_1$$

$$v = \frac{\Gamma_0(x_1)}{2\pi} e^{j\omega t} \cdot [\beta e^{R_e(x-x_1)} K_1(\beta r) \frac{x-x_1}{r} - R_e e^{R_e(x-x_1)} K_0(\beta r)] dx_1 \tag{21}$$

where $R_e = UC/4\nu$, $\beta^2 = R_e^2 + (j\omega/\nu) = R_e^2 + j(2R_e\omega/U)$ and K_0, K_1 designates the second kind modified Bessel functions of 0-th and 1-st order respectively. The velocity field (21) satisfies the continuity equation and the substitution into Eq.(7) gives $Q=0$ and correspondingly $P=0$. That is, the velocity field given by Eq.(21) represents one of the elementary solution satisfying Oseen's equations and makes no disturbance in the pressure field. Since $K_1(\beta r)$ is approximated by $1/(\beta r)$ for small βr , the velocity gap of the order $1/r$ in the velocity field (20) may be canceled by the superposition of the velocity field (21). The sum of the velocity fields (20) and (21) yields the velocity field of Oseenlet in y -direction extended for oscillating flow. Both velocity fields (19) of free vortex and (20),(21) of Oseenlet gives $u=0$ on $y=0$. Hence the induced velocity on the flat plate located on the x axis $-1 \le x \le 1$ is evidently normal to the blade surface and is given as follows.

$$v = \frac{e^{j\omega t}}{2\pi} \Big[\!\Big[\int_{-1}^{1} - \Gamma_0(x_1) [\{ \frac{1}{x-x_1} - \beta e^{R_e(x-x_1)} \mathrm{sgn}(x-x_1) K_1(\beta|x-x_1|) \} + R_e e^{R_e(x-x_1)} K_0(\beta|x-x_1|)] dx_1$$

$$+ \int_{-1}^{1} \int_{x_1}^{\xi} \frac{1}{x-\xi} \frac{j\omega}{U} e^{j\omega\frac{\xi-x_1}{U}} \Gamma_0(x_1) \{ 1 - e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}} \} d\xi dx_1 \Big]\!\Big] \tag{22}$$

The first term on the right hand side of the above equation is the velocity due to unsteady Oseenlet, and the second term to the free vortices. The bound vorticity distribution $\Gamma_0(x_1)$ is decided so that the induced velocity given by Eq.(22) becomes identical with the normal velocity of the oscillating airfoil. Then the total lift on the airfoils will be evaluated, for the sake of Eq.(14), as follows.

$$L = \rho U [\int_{-1}^{1} \Gamma_0(x_1) dx_1] e^{j\omega t} \tag{23}$$

In case of the steady flow around an airfoil set with a small angle of attack α in the uniform velocity $\bar{U}$, the boundary condition on the airfoil may be approximated by giving normal velocity $-\alpha\bar{U}$ on the plane $y=0$ and $-1 \leq x \leq 1$. The steady vortex distribution $\gamma(x_1)$ relating to the steady lift is given, by the following well known formula.

$$v = -\alpha\bar{U} = -\frac{1}{2\pi}\int_{-1}^{1} \gamma(x_1)\Big[\frac{1}{x-x_1} - R_e e^{R_e(x-x_1)} \operatorname{sgn}(x-x_1)\, K_1(R_e|x-x_1|)\Big\} \\ + R_e e^{R_e(x-x_1)} K_0(R_e|x-x_1|)\Big] dx_1 \tag{24}$$

Steady lift $\bar{L}$ will be,

$$\bar{L} = \rho\bar{U}\int_{-1}^{1} \gamma(x_1)\,dx_1 \tag{25}$$

By letting $\omega \to 0$ in Eq.(23) one obtains Eq.(24), whence the quasi-steady lift calculated from Eq.(23) coincides with the steady lift.

2.1.4 Velocity field relating to fluctuating drag

To begin with is considered the steady drag on the airfoil set at ($y=0$, $-1 \leq x \leq 1$) in the uniform velocity $\bar{U}$. The pressure field of the concentrated force $X = X_0\,\delta(x)\,\delta(y)$ in x-direction at the origin is verified to be expressed by a doublet of strength X_0 in x-direction in completely same manner as was previously shown for the force in y-direction.

$$P + iQ = \frac{X_0}{2\pi(x+iy)} \tag{26}$$

The velocity field of the Oseenlet in x-direction will be given;

$$u = \frac{q}{2\pi}\Big[\Big\{\frac{1}{r} - R_e e^{R_e x} K_1(\beta r)\Big\}\frac{x}{r} - R_e e^{R_e x} K_0(\beta r)\Big]$$
$$v = \frac{q}{2\pi}\Big[\Big\{\frac{1}{r} - R_e e^{R_e x} K_1(\beta r)\Big\}\frac{y}{r}\Big] \quad , \quad r^2 = x^2 + y^2 \tag{27}$$

It is easily shown that the velocity field given by Eq.(27) satisfies the equation of continuity, and the substitution of Eq.(27) in Eq.(7) gives

$$P + iQ = \frac{q\bar{U}}{2\pi(x+iy)} \tag{28}$$

The normal velocity v vanishes on $y=0$, hence the boundary condition for normal velocity on the airfoil is automatically satisfied. The allignment of the Oseenlet given by Eq.(27) on the airfoil surface induces the parallel velocity on the airfoil surface given as follows.

$$u = \frac{1}{2\pi}\int_{-1}^{1} \gamma(x_1)\left[\left\{\frac{1}{x-x_1} - R_e e^{R_e(x-x_1)} \operatorname{sgn}(x-x_1)\, K_1(R_e|x-x_1|)\right\} - R_e e^{R_e(x-x_1)} K_0(R_e|x-x_1|)\right] dx_1 \tag{29}$$

The induced parallel velocity u must cancel the uniform velocity U. For the sake of the vanishing velocity on the blade surface, by letting $u=-U$ in Eq.(29) one is ready to caluculate the Oseenlet distribution $\gamma(x_1)$, and the steady drag $\bar{D}$ results as follows.

$$\bar{D} = \rho U \int_{-1}^{1} \gamma(x_1)\, dx_1 \tag{30}$$

Next we proceed to the problem of the fluctuating drag. Consider a doublet in x-direction originated at the origin at $\tau=0$ and convected downstream with the uniform velocity U experiencing dissipation due to fluid viscosity. Such a flow field can be given by differentiating with respect to y the velocity field of the dissipating free vortex shown by the Eq.(11). The resulting velocities of dissipating free doublet of strength m become as follows.

$$\begin{aligned} u &= -\frac{m}{2\pi}\left[\frac{\{(x-U\tau)^2 - y^2\}}{r^4}\left(1 - e^{-\frac{r^2}{4\nu\tau}}\right) + \frac{1}{2\nu\tau}\frac{y^2}{r^2} e^{-\frac{r^2}{4\nu\tau}}\right]\delta(\tau) \\ v &= \frac{m}{2\pi}(x-U\tau)\left\{-\frac{2y}{r^4}\left(1 - e^{-\frac{r^2}{4\nu\tau}}\right) - \frac{1}{2\nu\tau}\frac{y^2}{r^2} e^{-\frac{r^2}{4\nu\tau}}\right]\delta(\tau) \end{aligned} \tag{31}$$

Substitution in Eq.(7) yields

$$\frac{\partial Q}{\partial y} = -\frac{m}{2\pi}\frac{x^2-y^2}{r^4}\delta(\tau) \quad , \quad \frac{\partial Q}{\partial x} = \frac{m x y}{\pi r^4}\delta(\tau)$$

and hence

$$P + iQ = \frac{m}{2\pi}\frac{1}{x+iy}\delta(\tau) \tag{32}$$

Equation (32) implies that the velocity field given by Eq.(31) represents the flow field to which the concentrated force $X=\rho m\delta(\tau)$ is applied.

Consider a fluctuating doublet of pressure field whose strength m varies sinusoidally, i.e.,

$$m = m_0(x_1)\, e^{j\omega\tau}\, dx_1$$

The duration of this doublet of strength m during $\Delta\tau$ in the pressure field results in the shedding out of doublets of total strength $m\Delta\tau$ which is spread over the interval $d\xi = U\Delta\tau$. The strength of the free doublet per unit length is, therefore,

$$\lambda = m / U \tag{33}$$

Since the free doublet on $x=\xi$ at time τ shed from $x=x_1$ should have left the point $x=x_1$ at time $\tau-(\xi-x_1)/U$, the strength λ of the free doublet should be,

$$\lambda\, d\xi = m_0(x_1)\, e^{j\omega(\tau-\frac{\xi-x_1}{U})} dx_1\, d\xi = m_0(x_1)\, e^{-j\omega\frac{\xi-x_1}{U}} e^{j\omega\tau} dx_1\, d\xi \tag{34}$$

Considering the viscous dissipation during the time interval $(\xi-x_1)/U$ to travel from x_1 to ξ with the velocity U , one obtains the velocity at (x, y , τ) due to the free doublet spread over the interval $d\xi$ at $x=\xi$ given above as follows.

$$u = -\frac{\lambda}{2\pi}\left[\frac{(x-\xi)^2-y^2}{r^4}\left\{1 - e^{-\frac{r^2 U}{4\nu(\xi-x_1)}}\right\} + \frac{U}{2\nu(\xi-x_1)}\frac{y^2}{r^2} e^{-\frac{r^2 U}{4\nu(\xi-x_1)}}\right]d\xi$$

$$v = -\frac{\lambda}{2\pi}(x-\xi)\left[\frac{2y}{r^4}\left\{1 - e^{-\frac{r^2 U}{4\nu(\xi-x_1)}}\right\} + \frac{U}{2\nu(\xi-x_1)}\frac{y^2}{r^2} e^{-\frac{r^2 U}{4\nu(\xi-x_1)}}\right]d\xi \tag{35}$$

$$r^2 = (x-\xi)^2 + y^2$$

On the airfoil surface $y=0$ and $-1 \leq x \leq 1$, the normal velocity vanishes and, by integrating the contribution of all the free doublet, which corresponds to the pressure doublet distributed over the airfoil surface $-1 \leq x \leq 1$, one gets the following parallel velocity.

$$u = \frac{e^{j\omega\tau}}{2\pi}\left[\int_{-1}^{1}\int_{x_1}^{\infty} - m_0(x_1)\frac{1}{(x-\xi)^2}\left\{1 - e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}}\right\} e^{-j\omega\frac{\xi-x_1}{U}} d\xi\, dx_1\right] \tag{36}$$

Equating the induced velocity u in Eq.(36) with the x-component of the airfoil oscillation velocity, the basis is given to evaluate the pressure doublet distribution $m_0(x_1)$. The fluctuating drag D on the blade is,

$$D = \rho U e^{j\omega t} \int_{-1}^{1} m_0(x_1) dx_1 \tag{37}$$

It could be easily seen by compareing Eq.(22) with Eq.(24) that the quasi-steady lift in the limit of $\omega \to 0$ coincides with the steady lift obtaind directly from the independent formulation. In case of fluctuating drag, though rather cumbersome, the coincidence of quasisteady drag with steady drag can be also verified as mentioned in appendix. The same line of manipulation leads to the following alternative form of Eq.(36),

$$\begin{aligned} u = \frac{e^{j\omega t}}{2\pi} \Big[\!\Big[\int_{-1}^{1} - m_0(x_1) \Big[\Big\{ \frac{1}{x-x_1} - \beta e^{R_e(x-x_1)} \mathrm{sgn}(x-x_1) \, K_1(\beta|x-x_1|) \Big\} \\ - R_e e^{R_e(x-x_1)} K_0(\beta|x-x_1|) \Big] dx_1 \\ - \int_{-1}^{1} \int_{x_1}^{\infty} \frac{1}{x-\xi} \frac{j\omega}{U} e^{-j\omega\frac{\xi-x_1}{U}} m_0(x_1) \Big\{ 1 - e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}} \Big\} d\xi \, dx_1 \Big]\!\Big] \end{aligned} \tag{38}$$

which is parallel to Eq.(22). The first term in Eq.(38) is due to Oseenlet in x -direction extended to unsteady flow and the second term corresponds to the shed-off vortex for lift fluctuation. They do not satisfy the continuity condition separately but do in ensemble.

2.1.5 Relation between free doublet and free vortex

Different way of deduction of fluctuating forces have been utilized in the preceding sections, namely, a combination of bound vortex and free vortex for lift, and only free travelling doublet for drag. Here will be described the identity of these two different way of explanation. In the first place, it will be shown that it is possible to discuss the lift fluctuation only with free doublet as is done for drag fluctuation. To make the discussion simple, we will show only for inviscid case here, though it is not so laborious work to take the viscous dissipation into account. The complex potential of a doublet shed from the origin with the velocity U at $t = 0$ will be given by

$$\overline{W} = \frac{im}{2\pi} \frac{\delta(t)}{(x - Ut) + iy} \tag{39}$$

The pressure field relating to the velocity field of Eq.(39) is obtained by substituting Eq.(39) into Eq.(7).

$$P + iQ = \frac{im}{2\pi}\,\delta(\tau)\,\frac{1}{x+iy} \tag{40}$$

which implies that the concentrated force of magnitude $\rho m\,\delta(\tau)$ was applied in y -direction at the origin. That is, the momentum of the applied force is proportional to the strength of the shed off doublet. Hence if the strength m of the pressure doublet at the origin oscillates sinusoidally with time, i.e.,

$$m = m_0\, e^{j\omega\tau} \tag{41}$$

the strength of the free doublet $\lambda(\xi,\tau)$ at $x-\xi$ at time τ will be caluculated by considering the time interval from $x=0$ to $x=\xi$ with velocity $\bar{U}$, i.e.,

$$\lambda = (m_0/U)\; e^{j\omega(\tau-\frac{\xi}{U})} \tag{42}$$

The velocity at (x_1 , 0) induced by the doublet on $d\xi$ at $x=\xi$ will be,

$$v = \frac{\lambda}{2\pi}\,\frac{d\xi}{(x_1-\xi)^2} = \frac{m_0}{2\pi U}\,\frac{1}{(x_1-\xi)^2}\, e^{j\omega(\tau-\frac{\xi}{U})} \tag{43}$$

The total induced velocity due to entire doublet distribution shed from the pressure doublet at the origin will be given by integrating Eq.(43);

$$v = \frac{m_0}{2\pi U}\, e^{j\omega\tau}\int_0^\infty \frac{e^{-j\omega\frac{\xi}{U}}}{(x_1-\xi)^2}\,d\xi$$

Integration by parts gives

$$v = \frac{m_0}{2\pi U}\, e^{j\omega\tau}\left[-\frac{1}{x} + \int_0^\infty \frac{j\omega}{U}\, e^{-j\omega\frac{\xi}{U}}\,\frac{1}{x_1-\xi}\,d\xi\,\right] \tag{44}$$

Next consider the model of the combination of bound vortex and shed-off vortex. The induced velocity at (x_1 , 0) and at time τ by the free vortex on $d\xi$ at $x=\xi$ is, by putting $\nu=0$, $x_1=0$, $x=x_1$ and $y=0$ in Eqs.(18) and (19).

Integrating above induced velocity over entire free vortex distribution and considering the induced velocity of the bound vortex $\Gamma_0 e^{j\omega\tau}$ at the origin, we obtain the velocity at (x_1 , 0) at time τ as,

$$v = \frac{\Gamma_0}{2\pi} e^{j\omega\tau} \left[-\frac{1}{x_1} + \frac{j\omega}{U} \int_0^\infty e^{-j\omega\frac{\xi}{U}} \frac{1}{x_1 - \xi} d\xi \right] \tag{45}$$

Two equations (44) and (45) are identical except for the factor $U\Gamma_0 / m_0$. This identity is attributed to the fact that we could integrate the free doublet taking advantage of the relation that the differentiation of the free vortex in x -direction results in free doublet in y -direction. The interchangeability of these two way of caluculation to evaluate the velocity field is easily verified to be extended to entire flow field, and not only on the blade surface, if the relation between doublet and free doublet and free vortex are skillfully taken into account. As for the case of drag fluctuation, doublet in x -direction is given by differentiation of vortex in y -direction and hence the relation cannot be used to the integration of free doublet in x -direction over the wake region. This is the reason why we may use only the free doublet model for the fluctuating drag problem.

2.1.6 Oscillation of airfoils parallel to the uniform velocity

Consider the oscillation of flat plate airfoils of chordlength 2, set parallel to the uniform velocity U . In case of Oseen's approximation, as shown by Shen & Crimi (1965), the boundary condition on the airfoils should be adjusted not on the instantaneous position of the airfoils but on $y=0$, $-1 \leq x \leq 1$, i.e., on the average position of the oscillating airfoil. Consider the oscillation of flat plate airfoil whose velocity is given by

(normal oscillation)

$$u = 0 \quad , \quad v = v_0 e^{j\omega\tau} \tag{46}$$

(parallel oscillation)

$$u = u_0 e^{j\omega\tau} \quad , \quad v = 0 \tag{47}$$

where u_0 , v_0 are constants.

2.1.6.1 Normal oscillation

Since $u = 0$ on the airfoil surface, we have only to consider Oseenlet in y -direction and shed off vortices. Distribution of the Oseenlet $\Gamma_0(x_1)$ will be decided from Eqs.(22),(46) to suffice

$$v_0 = \frac{1}{2\pi}\Big[\int_{-1}^{1} -\Gamma_0(x_1)\Big[\Big\{\frac{1}{x-x_1} - \beta e^{R_e(x-x_1)} sgn(x-x_1)\, K_1(\beta|x-x_1|)\Big\} + R_e e^{R_e(x-x_1)} K_0(\beta|x-x_1|)\Big]dx_1 + \int_{-1}^{1}\int_{x_1}^{\infty}\frac{1}{x-\xi}\frac{j\omega}{U} e^{-j\omega\frac{\xi-x_1}{U}}\Gamma_0(x_1)\Big\{1-e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}}\Big\}d\xi\, dx_1 \tag{48}$$

on $-1 \leq x \leq 1$, where $\Gamma_0(x_1)$ should be assumed to have the form

$$\Gamma_0(x_1) = A_0 \tan\frac{\theta}{2} + B_0 \cot\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\theta \tag{49}$$

where $x_1 = -\cos\theta$ and $0 \leq \theta \leq \pi$. The singularities of $\Gamma_0(x_1)$ at the leading and trailing edges (singularity of $s^{-\frac{1}{2}}$ where s designates the distance from the leading or trailing edge) will be confirmed in the same manner as adopted by Shen & Crimi. Differentiating both of Eq.(48) with respect to x we find

$$-\frac{1}{2\pi}\mathscr{P}\int_{-1}^{1}\frac{\Gamma_0(\xi)}{x-\xi}d\xi = f_1(x) + f_2(x) \tag{50}$$

where

$$f_1(x) = \frac{1}{2\pi R_e}\int_{-1}^{1}\Gamma_0(x_1)\Big[\frac{-1}{(x-\xi)^2} + \frac{R_e}{x-\xi} + e^{R_e(x-\xi)}\Big\{(R_e^2+\beta^2)K_0(\beta|x-\xi|) - 2\beta R_e\, sgn(x-\xi)\, K_1(\beta|x-\xi|) + \frac{1}{2}\beta^2 K_2(\beta|x-\xi|)\Big\}\Big]d\xi$$

and

$$f_2(x) = \frac{d}{dx}\Big[\int_{-1}^{1}\int_{x_1}^{\infty}\frac{1}{2\pi R_e(x-\xi)}\Big(-\frac{j\omega}{U}\Big) \times\Big[e^{-j\omega\frac{\xi-x_1}{U}}\Gamma_0(x_1)\Big\{1-e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}}\Big\}\Big]d\xi\, dx_1$$

The $\mathscr{P}$ preceding the integral signifies that th Cauchy's principal value is to be taken. Then $f_1(x)$, $f_2(x)$ are both bounded over the integral $-1 \leq x \leq 1$. According to Muskhelishvili (1946), the inverse of such Cauchy principal value integral equation should have the form

$$\Gamma_0(x) = \alpha_1\{(1-x)/(1+x)\}^{\frac{1}{2}} + \alpha_2\{(1+x)/(1-x)\}^{\frac{1}{2}} + g(x)$$

$$g(1) = g(-1) = 0$$

Letting $R_e \to \infty$ in Eq.(48), we have the same equation as Shen & Crimi and see that their model is correct in the limit $R_e \to \infty$. The behavior of the coefficients A_0 and B_0 as R_e goes to infinity have been fully examined by Shen & Crimi, according to whom $B_0 \to 0$ as $R_e \to \infty$. Hence we know that the singularity at the trailing edge vanishes and the Kutta-Joukowski condition is satisfied in the limit of $R_e \to \infty$. Numerical caluculations have been carried out by truncating the series of Eq.(49) at a finite number and adjusting the velocity by Eq.(48) at the same number of the points on the airfoils as the number of the unknowns in Eq. (49). Then the fluctuating lift $L e^{j\omega\tau}$ becomes

$$L = \rho U \int_{-1}^{1} \Gamma_0(x_1)\, dx_1 = \pi \rho U v_0 \left(A_0 + B_0 + \frac{A_1}{2}\right) \tag{51}$$

The fluctuating lift coefficient C_L is defined by

$$C_L = \frac{L}{\rho U v_0 (c/2)} = \pi \left(A_0 + B_0 + \frac{A_1}{2}\right) \tag{52}$$

where c is the chord length of the airfoil and supposed $c=2$ in the present section. As previously mentioned the quasisteady flow coincides with the steady flow of small angle of attack $\alpha = -v_0/U$. Then the quasisteady lift coefficient will be,

$$C_{L,\omega\to 0} = \frac{L_{\omega\to 0}}{\rho U^2 \alpha_0 (c/2)} = \pi \left(A_0 + B_0 + \frac{A_1}{2}\right) \tag{53}$$

The steady lift coefficient is defined as $L/(\frac{1}{2}\rho U^2 c)$ and equals to $2\pi\alpha$ for $R_e \to \infty$ while $C_{L,\omega\to 0}$ tends to 2π as $R_e \to \infty$, which implies that the quasisteady lift gives correct steady lift.

2.1.6.2 Parallel oscillation

Since $v=0$ on the airfoil surface we have only to distribute the oseenlet in x -direction over the airfoil surface. Distribution of the Oseenlet $m(x_1)$ will be decided from Eq.(38),(47) to suffice on $-1 \le x \le 1$,

$$u_0 = \frac{1}{2\pi}\Big[\!\Big[\int_{-1}^{1} -m(x_1)\Big[\Big\{ \frac{1}{x-x_1} - \beta e^{R_e(x-x_1)} \mathrm{sgn}(x-x_1)\, K_1(\beta|x-x_1|)\Big\} - R_e e^{R_e(x-x_1)} K_0(\beta|x-x_1|)\Big] dx_1 - \int_{-1}^{1}\int_{x_1}^{\infty} \frac{1}{x-\xi}\,\frac{j\omega}{U}\, e^{-j\omega\frac{\xi-x_1}{U}} m(x_1)\Big\{1 - e^{-\frac{U(x-\xi)^2}{4\nu(\xi-x_1)}}\Big\}\, d\xi\, dx_1 \Big]\!\Big] \tag{54}$$

In the same way as the plunging oscillation the distribution $m(x_1)$ is assumed to have the form

$$m(x_1) = A_0 \cot\frac{\theta}{2} + B_0 \tan\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin n\theta \tag{55}$$

Coefficients A_0 , B_0 , A_n are caluculated by means of Eq.(54). The singularities at the leading and trailing edges were assumed to be the same as for the normal oscillation from the same reason. The magnitude of the fluctuating drag $D e^{j\omega\tau}$ becomes

$$D = \rho U \int_{-1}^{1} m(x_1)\, dx_1 = \pi\left(A_0 + B_0 + \frac{A_1}{2}\right)\rho U u_0 \tag{56}$$

Fluctuating drag coefficient C_D is defined as

$$C_D = D / \rho u_0 U (c/2) e^{j\omega\tau} = \pi\left(A_0 + B_0 + \frac{A_1}{2}\right) \tag{57}$$

2.1.7 Discussion of numerical results

Numerical caluculations have been carried out for the two cases of oscillation mentioned in the previous section. Fig.1 shows the dependence of the quasisteady lift and drag (lift and drag for $\omega = 0$) on the Reynolds number. The broken line in the figure shows the asymptotic solution for steady drag $D_{\omega\to 0}$ given by Miyagi (1964). It demonstrates that the accuracy of the numerical caluculation is satisfactory. References for C_L is not shown in the figure for the following reason. The steady Oseen flow around an airfoil set with an angle of attack α tends to a separated flow in the limit $R_e \to \infty$ and hence the value of the lift will be a half of that caluculated from unseparated inviscid flow. This occurs when the boundary conditions are applied exactly on the airfoil surface, not on $y=0$, $-1 \le x \le 1$. Since we have applied the boundary conditions

$$u_{\omega\to 0} = -U \qquad v_{\omega\to 0} = -\alpha U \tag{58}$$

on $y=0$, $-1 \le x \le 1$ instead on the exact position of the airfoil, our model approaches the unseparated model of inviscid flow in the limit $R_e \to \infty$. Hence applying the boundary conditions (59) on $y=0$ and $-1 \le x \le 1$, we obtained the lift coefficient $C_{L,\omega\to 0}$ which correctly approaches inviscid unseparated value in the limit $R_e \to \infty$. Fig.(2) shows the fluctuating lift coefficient C_L on flat plate airfoils executing plunging motion. Reynolds number R_e and Strouhal number S_t are defined as follows.

$$R_e = UC/(4\nu) \quad , \qquad S_t = \omega C/(2U) \tag{59}$$

It is seen that the viscosity of the fluid have the effects to increase lift fluctuation. This tendency conflicts with the results of the multi-layered boundary layer theory of Brown & Daniels (1975). The theory is valid for sufficiently large Reynolds number and states that the Kutta-Joukowski condition is satisfied in the limit $R_e \to \infty$. According to this theory the fluid outside the small region of order $R_e^{-\frac{3}{4}}$ near the trailing edge doesn't follow the rapid oscillation of airfoils. In case that the Reynolds number is finite, the vicinity region works so as to permit the shift of the apparent stagnation point of the outer inviscid flow from the trailing edge in the direction to diminish the resultant lift. While, within Oseen's approximation, the singularity of the trailing edge is determined not from the physical condition but from the mathematical condition. The physical interpretation of this condition seems to claim that the infinite pressure difference across the blade surface in the immediate neighbourhood of the trailing edge is required to prohibit the flow turning the trailing edge. Hence the increase of the lift fluctuation due to viscosity in the present caluculation is likely to attributed to the increase of virtual mass attached to the vibrating flat plate. The present paper is intended to clarify the viscous effect using Oseen's approximation which has the fundamental advantage of simplicity that only one type of approximation is applied uniformly for the entire flow region. However, this simplicity resulted in the unreasonable flow model near the trailing edge. Nevertheless, since the theory of Brown & Daniels is restricted to the case of large Reynolds number and large Strouhal number, and since

Oseen's approximation is more appropriate than the boundary layer approximation for small Reynolds number, one may conclude that the viscosity works so as to increase the lift fluctuation below certain Reynolds number. Fig.(3) shows the fluctuating drag coefficient. Drag diminishes as the Reynolds number increases. The phase angle of the drag proceeds by $\pi/4$ than the phase angle of the oscillating velocity of the plates in the limit $St \to \infty$. This phenomena can be seen in the case of the oscillation of infinite plates parallel to itself in infinite fluid otherwise at rest. This means that, as St increase, the flow near the flat plate tends to that near the infinite plate except in the vicinity of the both edges. This fact is easily understood if we consider that the thickness of the surface effects diminishes as St increases.

2.1.8 Conclusion

Elementary solutions for unsteady lift and unsteady drag were introduced in velocity and pressure fields on the basis of Oseen's approximation. The elementary solution in the velocity field is expressed by a combination of an Oseenlet in y-direction and dissipating shed off vortices for fluctuating lift and dissipating shed off doublets for fluctuating drag. Secondly, by means of these elementary solutions the unsteady forces on the oscillating flat plate airfoil was caluculated.

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Appendix; Quasisteady drag and steady drag

Taking the limit $\omega \to 0$ in Eq.(36) we have

$$u = \frac{1}{2\pi}\int_{-1}^{1}\int_{x_1}^{\infty} - m_0(x_1)\ \frac{1}{(x-\xi)^2}\left\{1 - e^{-\frac{R_e(x-\xi)^2}{2(\xi-x_1)}}\right\} d\xi\, dx_1$$

$$= \int_{-1}^{1} I_1(x_1)\, dx_1 \tag{A}$$

Introducing the function $I_2(x_1)$ to Eq.(29)

$$u = \int_{-1}^{1} I_2(x_1)\, dx_1 \tag{B}$$

and putting $m_0(x_1) = g(x_1)$, let us show

$$I_1(x_1) = I_2(x_1)$$

The case $x > x_1$ will be described for the brevity, the entirely similar way of verification being valid also for $x < x_1$. Applying the formula for Bessel function;

$$K_n(z) = \frac{1}{2}\int_0^{\infty} e^{-\frac{z}{2}(\eta + \frac{1}{\eta})}\ \eta^{-n-1}\, d\eta$$

for $z = R_e(x - x_1)$, $n = 0$ or 1 we have

$$K_0(R_e|x-x_1|) = \frac{1}{2}\int_0^{\infty}\frac{1}{\eta}\ e^{-\frac{R_e}{2}(x-x_1)(\eta+\frac{1}{\eta})}\, d\eta$$

$$K_1(R_e|x-x_1|) = \frac{1}{2}\int_0^{\infty}\frac{1}{\eta^2}\ e^{-\frac{R_e}{2}(x-x_1)(\eta+\frac{1}{\eta})}\, d\eta$$

After transformation $\eta = (\xi - x_1)/(x - x_1)$ Eq.(B) becomes

$$I_2(x_1) = \frac{g(x_1)}{2\pi}\left[\!\left[\ \frac{1}{x-x_1} - \frac{R_e}{2}\, e^{R_e(x-x_1)}\int_{x_1}^{\infty}\left\{\frac{1}{\xi-x_1} - \frac{x_1-x}{(\xi-x_1)^2}\right\}\right.\right.$$

$$\left.\left.\times \exp\left[-\frac{R_e}{2}\left\{(\xi-x_1) + \frac{(x-x_1)^2}{\xi-x_1}\right\}\right] d\xi\ \right]\!\right] \tag{C}$$

For the while, Eq.(A) will be by partial integration

$$I_1(x_1) = \frac{m_0(x_1)}{2\pi}\left[\frac{1}{x-x_1} + \int_{x_1}^{\infty}\frac{1}{(x-\xi)^2}\ e^{-\frac{R_e(x-\xi)^2}{2(\xi-x_1)}}\right]$$

$$
= \frac{m_0(x_1)}{2\pi} \left[\frac{1}{x - x_1} + \left| -\frac{1}{\xi - x} e^{-\frac{R_e}{2}\frac{(x-\xi)^2}{\xi - x_1}} \right|_{x_1}^{\infty} - \int_{x_1}^{\infty} \frac{1}{\xi - x} \frac{R_e}{2} \left\{ \frac{2(\xi - x)}{\xi - x_1} - \frac{(\xi - x)^2}{(\xi - x_1)^2} \right\} \right.
$$

$$
\left. \times \, e^{-\frac{R_e}{2}\frac{(x-\xi)^2}{\xi - x_1}} d\xi \right] \tag{D}
$$

$$
= \frac{m_0(x_1)}{2\pi} \left[\!\left[\frac{1}{x - x_1} - \frac{R_e}{2} \cdot e^{R_e(x - x_1)} \int_{x_1}^{\infty} \left(\frac{1}{\xi - x_1} - \frac{x_1 - x}{(\xi - x_1)^2} \right) \exp\left[-\frac{R_e}{2} \left\{ (\xi - x_1) + \frac{(x - x_1)^2}{\xi - x_1} \right\} \right] d\xi \right.\right.
$$

Comparing Eq. (E) with (D) we see that $I_1(x_1) = I_2(x_1)$ if $g(x_1) = m_0(x_1)$, and hence it is veryfied that the quasisteady drag coincides with the steady drag.

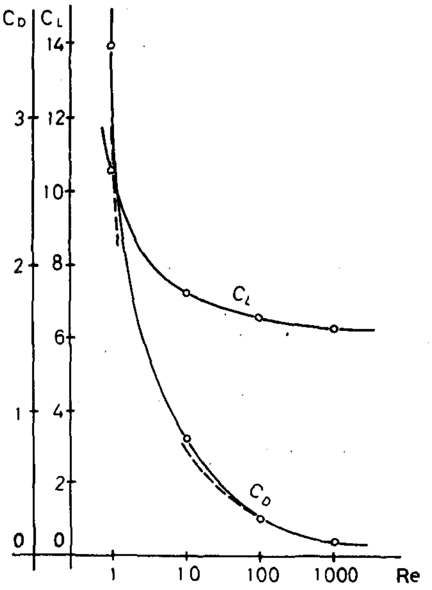

Fig.1 Quasi-steady lift and drag
-----, Miyagi

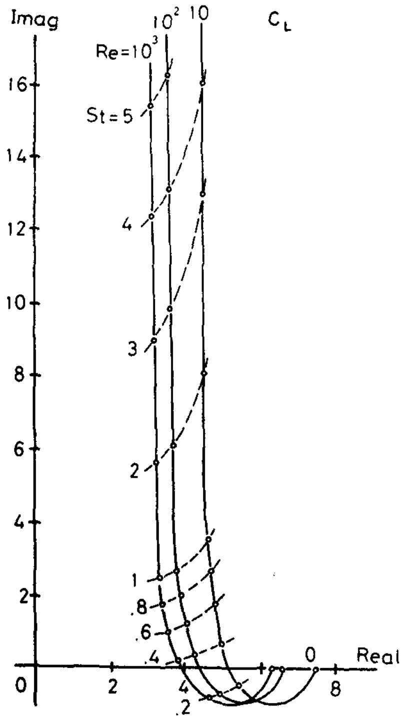

Fig.2 Fluctuating lift

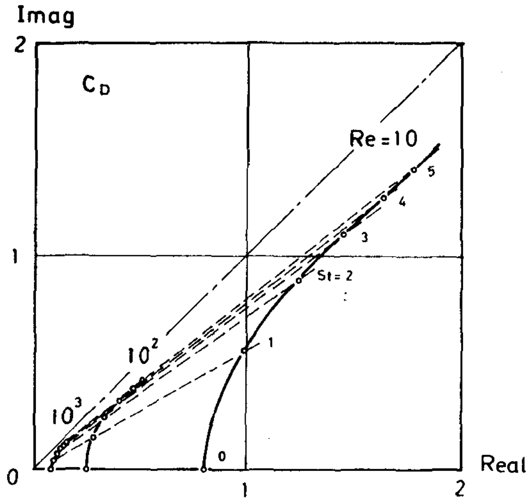

Fig.3 Fluctuating drag

2.2 Applications to cascade

2.2.1 Introduction

In the last section were analized the forces on a flat plate airfoil by introducing elementary solutions of linearlized Navier-Stokes equations in which the inertia terms are linearlized on the assumption of small amplitude of oscillation. This section is intended to apply the method to cascade problem. By the way the cascade blades of a turbomachine are subjected to unsteady forces due to interaction with viscous wakes from upstream cascades. Kemp & Sears[1] analyzed the unsteady forces by representing viscous wakes by inviscid sinusoidal gusts. The sinusoidal gusts that satisfy linearlized Navier-Stokes equation are given in this paper and the lift response to the gusts and the lift fluctuation due to translatory oscillation of the blades are numerically calculated. The fluid is assumed to be incompressible and flat-plate cascades with no steady lifts are considered.

2.2.2 Elementary solutions

Consider the case that fluctuating concentrated forces

$$g e^{j\omega\tau} \quad (1) \qquad \gamma e^{j\omega\tau} \quad (2)$$

in x and y direction respectively are applied at the origin. The velocity fields induced by the unsteady forces that satisfy the elementary equations are

due to $\gamma e^{j\omega\tau}$

$$u = -\frac{\gamma}{2\pi} e^{j\omega\tau} \left[\frac{y}{R^2} - R_e \frac{y}{R} e^{R_e x} K_1(\beta R) + \int_0^\infty \left(-\frac{j\omega}{U}\right) e^{-\frac{j\omega\xi}{U}} \frac{y}{R^2} \right.$$

$$\left. \times \left\{ 1 - e^{-\frac{R_e}{2}\frac{R'^2}{\xi}} \right\} d\xi \right] \equiv \frac{\gamma}{2\pi} e^{j\omega\tau} K'_{ur}(x, y)$$

$$v = \frac{\gamma}{2\pi} e^{j\omega\tau} \left[\frac{y}{R^2} + R_e e^{R_e x} \left\{ K_0(\beta R) - \frac{\beta}{R_e} \cdot \frac{x}{R} K_1(\beta R) \right\} - \int_0^\infty \left(-\frac{j\omega}{U}\right) \right. \tag{3}$$

$$\left. \times e^{-\frac{j\omega\xi}{U}} \frac{\xi - x}{R'^2} \left\{ 1 - e^{-\frac{R_e}{2}\frac{R'^2}{\xi}} \right\} d\xi \right] \equiv \frac{\gamma}{2\pi} e^{j\omega\tau} K'_{vr}(x, y)$$

due to $\gamma e^{j\omega\tau}$

$$u = -\frac{\gamma}{2\pi} e^{j\omega\tau}\left[-\frac{x}{R^2} + R_e e^{R_e x}\left\{\frac{\beta}{R_e}\frac{x}{R} K_1(\beta R) + K_0(\beta R)\right\} - \int_0^\infty \left(-\frac{j\omega}{U}\right)\right.$$

$$\left.\times e^{-\frac{j\omega\xi}{U}} \frac{x-\xi}{R'^2}\left\{1 - e^{-\frac{R_e}{2}\frac{R'^2}{\xi}}\right\} d\xi\right] \equiv \frac{\gamma}{2\pi} e^{j\omega\tau} K u'_{\gamma}(x, y)$$

$$v = -\frac{\gamma}{2\pi} e^{j\omega\tau}\left[\frac{y}{R^2} - \beta\frac{y}{R} e^{R_e x} K_1(\beta R) + \int_0^\infty \left(-\frac{j\omega}{U}\right) e^{-\frac{j\omega\xi}{U}} \frac{y}{R'^2}\right. \tag{4}$$

$$\left.\times\left\{1 - e^{-\frac{R_e}{2}\frac{R'^2}{\xi}}\right\} d\xi\right] \equiv \frac{\gamma}{2\pi} e^{j\omega\tau} K v'_{\gamma}(x, y)$$

where $R_e = U/2\nu$, $\beta = \{R_e^2 + (j\omega/U)^2\}^{\frac{1}{2}}$, $R^2 = x^2 + y^2$
$R'^2 = (x-\xi)^2 + y^2$ and K_0 and K_1 designates second kind modified Bessel functions of 0-th and 1-st order.

2.2.3 Superposition of elementary solutions

Equations (3) and (4) are velocity fields due to concentrated forces at $x = 0$, $y = 0$. The velocity disturbances at (x , y) induced by unsteady forces $\gamma_0(x_1) e^{jm\alpha} e^{j\omega\tau}$, $\gamma_0(x_1) e^{jm\alpha} e^{j\omega\tau}$ on the blade surfaces $x = x_1 - ma$, $y = -mb$ ($a = t\sin\gamma$, $b = t\cos\gamma$, $m = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, $-1 \le x_1 \le 1$) of the cascade of blade spacing t , stagger γ , chordlength $C = 2$ may be written as follows.

$$u(x, y) = \frac{e^{j\omega\tau}}{2\pi}\int_{-1}^{1}\left[\gamma_0(x_1)\left\{\sum_{m=-\infty}^{+\infty} K u'_{\gamma}(x - x_1 - ma, y - mb)\, e^{jm\alpha}\right\}\right.$$

$$\left. + \gamma_0(x_1)\left\{\sum_{m=-\infty}^{+\infty} K u'_{\gamma}(x - x_1 - ma, y - mb)\, e^{jm\alpha}\right\}\right] \tag{5}$$

$$v(x, y) = \frac{e^{j\omega\tau}}{2\pi}\int_{-1}^{1}\left[\gamma_0(x_1)\left\{\sum_{m=-\infty}^{+\infty} K v'_{\gamma}(x - x_1 - ma, y - mb)\, e^{jm\alpha}\right\}\right.$$

$$\left. + \gamma_0(x_1)\left\{\sum_{m=-\infty}^{+\infty} K v'_{\gamma}(x - x_1 - ma, y - mb)\, e^{jm\alpha}\right\}\right]$$

where the constant α designates the phase difference of the oscillation between adjacent blades. For convenience the kernels Ku'_{γ} , Ku'_{γ} , Kv'_{γ} , Kv'_{γ} are divided into potential parts $Ku'_{\gamma p}$, $Kv'_{\gamma p}$, $Ku'_{\gamma p}$, $Kv'_{\gamma p}$ and viscous parts $Ku'_{\gamma v}$, $Kv'_{\gamma v}$, $Ku'_{\gamma v}$, $Kv'_{\gamma v}$. That is,

$$Ku'_{\gamma p} = -y/R^2 + jSt\int_0^\infty e^{-jSt\xi}(y/R'^2)\, d\xi$$

$$K v'_{r p} = x / R^2 - j S_t \int_0^\infty e^{-j S_t \xi} (x - \xi) / R'^2 \, d\xi$$

$$K u'_{\xi p} = K u'_{r p} \quad , \quad K v'_{\xi p} = K u'_{r p}$$

$$K u'_{r v} = \beta (y/R) e^{R_e x} K_1(\beta R) - j S_t \int_0^\infty e^{-j S_t \xi} (y / R'^2) e^{-\frac{R_e}{2} \frac{R'^2}{\xi}} d\xi$$

$$K v'_{r v} = R_e e^{R_e x} K_0(\beta R) - \beta (x/R) K_1(\beta R) + j S_t \int_0^\infty e^{-j S_t \xi} (x - \xi) / R'^2 e^{-\frac{R_e}{2} \frac{R'^2}{\xi}} d\xi$$

$$K u'_{\xi v} = - R_e e^{R_e x} \left\{ \frac{\beta}{R_e} \frac{x}{R} K_1(\beta R) + K_0(\beta R) \right\} + j S_t \int_0^\infty e^{-j S_t \xi} \frac{x - \xi}{R'^2} e^{-\frac{R_e}{2} \frac{R'^2}{\xi}} d\xi$$

$$K v'_{\xi v} = - \beta \frac{y}{R} e^{R_e x} K_1(\beta R) + j S_t \int_0^\infty e^{-j S_t \xi} (y / R'^2) e^{-\frac{R_e}{2} \frac{R'^2}{\xi}} d\xi$$

and

$$K u'_r = K u'_{r p} + K u'_{r v} \quad , \quad K v'_r = K v'_{r p} + K v'_{r v}$$

$$K u'_\xi = K u'_{\xi p} + K u'_{\xi v} \quad , \quad K v'_\xi = K v'_{\xi p} + K v'_{\xi v}$$

where $S_t = \omega c / (2 U)$. In the limiting case $R_e \to \infty$, i.e., for inviscid flow the viscous parts vanish and the flow is represented by potential parts only. The potential parts can be superposed in the direction of cascade axis by the following relation. [2]

$$\sum_{m=-\infty}^{+\infty} \frac{e^{j m \alpha}}{\lambda + m} = \begin{cases} \pi e^{j(\pi - \alpha)} \operatorname{cosec} \pi \lambda & (\alpha \neq 0) \\ \pi \cot \pi \lambda & (\alpha = 0) \end{cases} \tag{6}$$

The infinite series;

$$H_x(x, y) \equiv \sum_{m=-\infty}^{+\infty} \frac{(x - m a) e^{j m \alpha}}{(x - m a)^2 + (y - m b)^2} \tag{7}$$

$$H_y(x, y) \equiv \sum_{m=-\infty}^{+\infty} \frac{(y - m b) e^{j m \alpha}}{(x - m a)^2 + (y - m b)^2} \tag{8}$$

can be expressed as follows.

for $\alpha \neq 0$

$$H_x(x, y) = - \frac{\pi}{2} \Big[\frac{1}{a + j b} \exp \Big\{ j (\pi - \alpha) \Big(- \frac{x + j y}{a + j b} \Big) \Big\} \times \operatorname{cosec} \Big(- \pi \frac{x + j y}{a + j b} \Big)$$

$$+ \frac{1}{a - j b} \exp \Big\{ j (\pi - \alpha) \Big(- \frac{x - j y}{a - j b} \Big) \Big\} \operatorname{cosec} \Big(- \pi \frac{x - j y}{a - j b} \Big) \Big]$$

$$H_y(x,y) = -\frac{\pi j}{2}\left[\frac{1}{a+jb}\exp\left\{j(\pi-\alpha)\left(-\frac{x+jy}{a+jb}\right)\right\}\csc\left(-\pi\frac{x+jy}{a+jb}\right)\right.$$

$$\left. - \frac{1}{a-jb}\exp\left\{j(\pi-\alpha)\left(-\frac{x-jy}{a-jb}\right)\right\}\csc\left(-\pi\frac{x-jy}{a-jb}\right)\right]$$

for $\alpha = 0$

$$H_x(x,y) = -\frac{\pi}{2}\left[\frac{1}{a+jb}\cot\left(-\pi\frac{x+jy}{a+jb}\right) + \frac{1}{a-jb}\cot\left(-\pi\frac{x-jy}{a-jb}\right)\right]$$

$$H_y(x,y) = -\frac{\pi j}{2}\left[\frac{1}{a+jb}\cot\left(-\pi\frac{x+jy}{a+jb}\right) - \frac{1}{a-jb}\cot\left(-\pi\frac{x-jy}{a-jb}\right)\right]$$

Then the potential parts are written

$$K_{u\gamma p}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{u\gamma p}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right]$$

$$= -H_y(x-x_1,\ y) + jSt\int_{x_1}^{\infty} e^{-jSt(\xi-x_1)} H_y(x-\xi,\ y)\,d\xi \tag{9}$$

$$K_{v\gamma p}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{v\gamma p}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right]$$

$$= H_x(x-x_1,\ y) - jSt\int_{x_1}^{\infty} e^{-jSt(\xi-x_1)} H_x(x-\xi,\ y)\,d\xi \tag{10}$$

$$K_{u\xi p}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{u\xi p}(x-x_1-ma, y-mb)\,e^{jm\alpha}\right] = K_{v\gamma p}(x-x_1,y) \tag{11}$$

$$K_{v\xi p}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{v\xi p}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right] = -K_{u\gamma p}(x-x_1,\ y) \tag{12}$$

Next let us consider about the superposition of the viscous parts. The viscous parts contain the Bessel function or exponential functions and all of them converge fairly quickly for usual blade spacing and Reynolds number. Then the following infinite series can be approximated by cutting off at finite terms accurately.

$$K_{u\gamma v}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{u\gamma v}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right] \tag{13}$$

$$K_{v\gamma v}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{v\gamma v}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right] \tag{14}$$

$$K_{u\xi v}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{u\xi v}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right] \tag{15}$$

$$K_{v\xi v}(x-x_1,\ y) \equiv \sum_{m=-\infty}^{+\infty}\left[K'_{v\xi v}(x-x_1-ma,\ y-mb)\,e^{jm\alpha}\right] \tag{16}$$

By the superposition of elementary solutions in the direction of cascade axis finite velocity is induced infinitely upstream of the cascade. That is,

$$\lim_{x\to-\infty} H_x(x,y) = \begin{cases} -\pi b/(a^2+b^2) & \alpha=0 \\ 0 & \alpha\neq 0 \end{cases}$$

$$\lim_{x\to-\infty} H_y(x,y) = \begin{cases} \pi a/(a^2+b^2) & \alpha=0 \\ 0 & \alpha\neq 0 \end{cases}$$

and the infinite integrals in Eqs.(9) (12) oscillate. This is because the shed off vortices induce finite velocity at $x=-\infty$, and we must subtract uniform velocity from entire flow field to cancel the induced velocity at infinitely upstream of the cascades. The induced velocities due to viscous part diminishes exponentially at $x=-\infty$ even for $\alpha=0$ and we have only to take the velocity components induced by potential part into account. Then the velocity disturbance will be written as follows.

$$u(x,y) = \frac{e^{j\omega t}}{2\pi}\int_{-1}^{1}[\gamma_0(x_1)\{K_{u\gamma p}(x-x_1,y) - K_{u\gamma p}(-\infty,y) + K_{u\gamma v}(x-x_1,y)\}$$
$$+\xi_0(x_1)\{K_{u\xi p}(x-x_1,y) - K_{u\xi p}(-\infty,y) + K_{u\xi v}(x-x_1,y)\}] \quad (17)$$

$$v(x,y) = \frac{e^{j\omega t}}{2\pi}\int_{-1}^{1}[\gamma_0(x_1)\{K_{v\gamma p}(x-x_1,y) - K_{v\gamma p}(-\infty,y) + K_{v\gamma v}(x-x_1,y)\}$$
$$+\xi_0(x_1)\{K_{v\xi p}(x-x_1,y) - K_{v\xi p}(-\infty,y) + K_{v\xi v}(x-x_1,y)\}] \quad (18)$$

2.2.4 Sinusoidal gusts

In this section will be given the sinusoidal dissipating gusts that satisfy the fundermental equations (Equations (1) and (2) in section 2.1) and approach to inviscid sinusoidal gusts by letting $R_e\to\infty$. Consider the flow given by

$$\begin{aligned} u &= u_0\, e^{j\varepsilon y}\, e^{\lambda(x-\alpha\cdot y)}\, e^{j\omega t} \\ v &= v_0\, e^{j\varepsilon y}\, e^{\lambda(x-\alpha\cdot y)}\, e^{j\omega t} \end{aligned} \quad (19)$$

where $\alpha = \tan\gamma$. Substitution of Eq.(19) to the continuity equation give

$$u_0 \lambda + v_0 (j\varepsilon - \lambda d) = 0 \tag{20}$$

Putting $p=0$, $\chi=0$ and $\Upsilon=0$ in Eq.(1) in section 2.1 and substituting Eq.(19) into the equation, we have the same equation shown below from either the first or the second of Eq.(19).

$$j\omega + \bar{U}\lambda = \nu \{ \lambda^2 + (j\varepsilon - d\cdot\lambda)^2 \}$$

from which

$$\lambda = Re/(1+d^2) + j\varepsilon d/(1+d^2) \pm \sqrt{\{ Re/(1+d^2) + j\varepsilon d/(1+d^2)\}^2 + (\varepsilon^2 + 2j\,Re\cdot St)/(1+d^2)} \tag{21}$$

The sinusoidal gust (19) satisfies the elementary Equations (1) and (2) in section 2.1 as far as the constants u_0 , v_0 and λ are selected to fulfill Eqs.(20) and (21). Next consider the case of $Re \to \infty$. Then Eq. (21) will be,

$$\lambda = \begin{cases} \text{(with + sign in front of the squareroot)} \\ 2Re/(1+d^2) + jSt \\ \text{(with - sign in front of the squareroot)} \\ -jSt \end{cases} \tag{22}$$

We should discard the plus sign in Eq.(22) because $\lambda \to +\infty$ as $Re \to \infty$ which is not the case of our interest. Then we will take only the minus sign in Eq.(21) thereafter. Equation (19) then will be for $Re \to \infty$

$$\begin{aligned} u &= u_0 \exp [\![j\omega \{ t - \frac{1}{\bar{U}} \{ x - (d + \frac{\varepsilon}{St}) y \} \}]\!] \\ v &= v_0 \exp [\![j\omega \{ t - \frac{1}{\bar{U}} \{ x - (d + \frac{\varepsilon}{St}) y \} \}]\!] \end{aligned} \tag{23}$$

which is the inviscid sinusoidal gust having constant phase on the line $x - (d + \varepsilon/St) y = Const.$. The angle δ made by the direction of the gust (the direction of equiphase line) and y axis is given by

$$\delta = \tan^{-1} (\tan\gamma + \varepsilon / St) \tag{24}$$

Equation (23) can be written on the line $x - y \cdot d = Const.$ as

$$u = u_0\, e^{\lambda c} \exp\{ j\omega (\tau + \varepsilon y/\omega)\}$$
$$v = v_0\, e^{\lambda c} \exp\{ j\omega (\tau + \varepsilon y/\omega)\} \tag{25}$$

from which it can be seen that the disturbances of constant amplitudes $u_0 e^{\lambda c}$ and $v_0 e^{\lambda c}$ are transmitted with the velocity ω/ε downwards along the line. (see Fig.1) We can get the following relation by considering Eq.(20)

$$u/v = u_0/v_0 = (\lambda d - j\varepsilon)/\lambda = \tan\gamma + \varepsilon/S_t = \tan\delta \tag{26}$$

which means that the direction of the velocity disturbance coincides with the direction of the eqiuphase line given by Eq.(24). In the last place let us verify that the direction of the equiphase line of the gusts coincides with the direction of the wakes from the upstream moving blades. Consider the velocity triangle for upstream moving blades shown in Fig.2, where $\bar{U}$ is the mainstream velocity, $\bar{V}$ the velocity of the upstream blades and $\bar{W}$ is the relative velocity with respect to the moving blades. The wakes are considered to extend in the direction of the relative velocity $\bar{W}$. From the velocity triangle we can get

$$V = U \cos\delta \,/\, \sin(\gamma - \delta)$$

The n-th order angular frequency ω_n can be written with the blade spacing t' of the upstream cascade,

$$\omega_n = 2\pi n \,/\, (t_r / V)$$

and from Eq.(25)

$$-\omega_n/\varepsilon = V \cos\gamma$$

Then

$$S_t/\varepsilon = \frac{\omega_n / U}{-\omega_n/(V\cos\gamma)} = -\frac{V}{U}\cos\gamma = \frac{\cos\delta \cdot \cos\gamma}{\sin(\delta - \gamma)}$$

and

$$\tan\gamma + \varepsilon / St = \tan\gamma + \frac{\sin(\gamma-\delta)}{\cos\delta\,\cos\gamma} = \tan\delta \tag{27}$$

which is the same as Eq.(24). That is, the direction δ of the sinusoidal gusts given by Eq.(19) coincides with the direction of the relative velocity to the moving blades, and with the direction of the velocity fluctuation. Hence it is expected that the sinusoidal dissipating gust (19) can represent the wakes from the upstream moving blades.

2.2.6 Numerical caluculations

Unsteady lifts on translatory oscillating blades of the cascades and on the cascade blades in the sinusoidal dissipating gusts are caluculated by singular point method. By equating the velocity disturbance given by Eqs.(17),(18) with;

for translatory oscillation of the blades

$$\begin{aligned} u(x,0) &= 0 \\ v(x,0) &= v_0 e^{j\omega t} \end{aligned} \qquad (-1 \leq x \leq 1) \tag{28}$$

for sinusoidal gusts

$$\begin{aligned} u(x,0) &= -u_0 e^{\lambda x} e^{j\omega t} \\ v(x,0) &= -v_0 e^{\lambda x} e^{j\omega t} \end{aligned} \qquad (-1 \leq x \leq 1) \tag{29}$$

so that the boundary condition on the blade surface is fulfilled, we get the simultaneous integral equations for the unknown functions $\gamma_0(x_1)$ and $\zeta_0(x_1)$. Though for inviscid flows the lifts vanish at the trailing edge due to Kutta's condition but for viscous flows the lift and drag distributions have singularity of order $s^{-\frac{1}{2}}$ (s ; distance from the trailing edge) at the trailing edge as well at the leading edge. And it has been pointed out in [3] that the coefficient of the term representing the singularity approaches to zero by letting $R_e \to \infty$ and this can be numerically ascertained. Then representing $\gamma_0(x_1)$ and $\zeta_0(x_1)$ by Glauert series with a term representing the singularity at the trailing edge and applying Eqs.(28),(29) on several points corresponding to the number of the terms of the series, we get a set of simultaneous linear equations for Glauert coefficients and by solving it we find the force distributions. The numerical integrations of Eqs.(17),(18) were made by the trigonometrical formula except near the logarithmic singularity caused by the term $k_0(\beta R)$ in Eqs.(3),(4), which was integrated analytically. As the value of the integrands vary largely near the singularities for large R_e, it was necessary to drop the pitch of the numerical integration near the singularities to save the computing time.

2.2.7 Numerical results

The coefficients of fluctuating lift C_L and drag C_D are defined as follows.

$$C_L = L / (\tfrac{1}{2} \rho U U_0 C) \tag{30}$$

$$C_D = D / (\tfrac{1}{2} \rho U U_0 C) \tag{31}$$

where L and D are the fluctuating lift and drag respectively. In the case of isolated airfoil the drag fluctuation is not produced by up and down oscillation only, but in case of cascades the velocity fluctuations parallel to the blade surface are induced by the lift fluctuations on the blades other than the reference blade and the drag fluctuates to cancel the parallel velocity disturbance on the blade. We have the next relation between the phase difference α of the sinusoidal gusts among adjacent blades and the constant ε relating to the period in the direction of the cascade axis.

$$\alpha = \varepsilon t \cos\gamma + 2\pi M$$

The arbitrary integer M is selected to be zero in the present calculations. The cascade geometry is fixed to $t = \pi$ and $\gamma = 30°$ of flat plate blades of chordlength $C = 2$ and the Glauert series is cut off at 5-th term.We could neglect the effects of the blades farther than the adjacent blades in the calculations of the viscous parts for the present cascade geometry and Reynolds number, and the series in Eqs.(13), (16) were cut off at $m = 1$. The fluctuating lift and drag coefficients due to the translatory oscillation of the blades are shown in Figs.3~10. Comparisons with the inviscid results by the acceleration potential method in chapter 1 are made for $\alpha = 0$ and $\alpha = \pi$. It is seen that the lift coefficient for $R_e = 10^3$ is very near to the inviscid value.

For any phase difference the fluid viscosity enlarges the amplitude of the lift fluctuation and has not so much effects on the phase of the lift fluctuation. Strictly speaking the lift and drag distributions have singularities at the trailing edge, but actually the effects of the singularities are small even for $Re=10$ as shown in Fig.5 and the term representing the trailing edge singularity is neglected throughout the present calculations. The lift coefficients have an unified tendency that $C_L \to j\infty$ as $St \to \infty$ for any phase difference but the trace pattern of the drag coefficient as the variation of St differs largely by the phase difference α. This is because the normal velocity fluctuation is mainly affected by the lifts of the reference blade itself and the effects of the virtual mass are apparent for lift fluctuation, but the parallel velocity fluctuation is mainly due to the lifts on the blades other than the reference blade and hence the drag fluctuation is largely affected by the phase difference of the oscillation. As a matter of course the drag fluctuation tends to zero in the inviscid limit. Fluctuating lift and drag coefficients for dissipating sinusoidal gusts are shown in Figs. 11~18. Inviscid results by the method in Chapter 1 are shown in Figs.11 and 13. The fluid viscosity enlarges the amplitude of lift fluctuation and affects the phase difference little as for the case of blade oscillation. The effects of the viscosity is remarkable for large St for every phase difference α. This is because the dissipation rate of the gusts is larger for larger St and the normal velocity is larger on the upstream half of the blade chord. The lift fluctuation for $Re=10$ is larger for larger value of α which is again considered to be because of larger damping rate of the gusts for larger α. In case $\alpha \neq 0$ the angle made by the isophase line and the blade surface tends to zero and the parallel velocity fluctuation will have a large value for small St. In order to cancel the large parallel velocity, large drag force works on the blade correspondingly to the value of Re, which affect the lift fluctuation. For this reason the lift fluctuation is largely affected by the fluid viscosity in the limit $St \to 0$. The drag coefficient C_D tends to infinity in the limit $St \to 0$ for the reason above mentioned. The phase of the drag coefficient leads with the increase of St, which is the same tendency for lift. The amplitude of the drag fluctuation is larger for larger value of α and tends to zero as $Re \to \infty$.

2.2.8 Conclusion

Viscous effects on the fluctuating lift and drag have been analyzed for the translatory oscillation of the blades and sinusoidal dissipating gusts on the basis of the linearized Navier-Stokes equations. It is found that the viscosity have an effect to enlarge the amplitude of the lift fluctuation but little effect on the phase of the lift. The drag fluctuations due to the up and down oscillation of the blades are also analyzed.

Refereces in section 2.2

[1] Kemp & Sears J. Aern. Sci. Vol.20, No.10, 585-597 (1953)

[2] D.S.Whitehead, A.R.C. R & M , No3386 (1962)

[3] Shen,S.F., Crimi,P. J.F.M. 23 585 (1965)

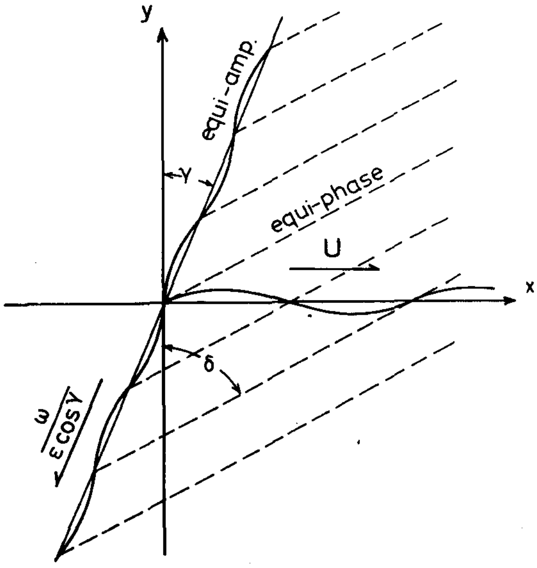

Fig.1 Sinusoidal gusts

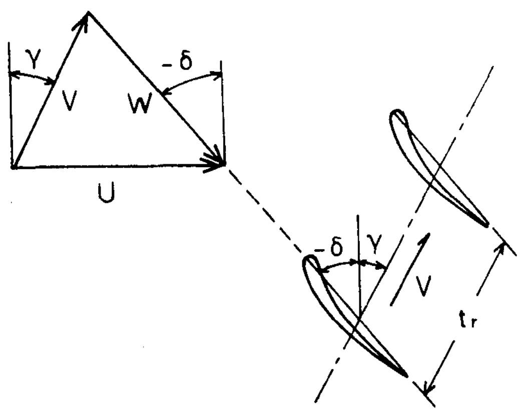

Fig.2 Velocity triangle

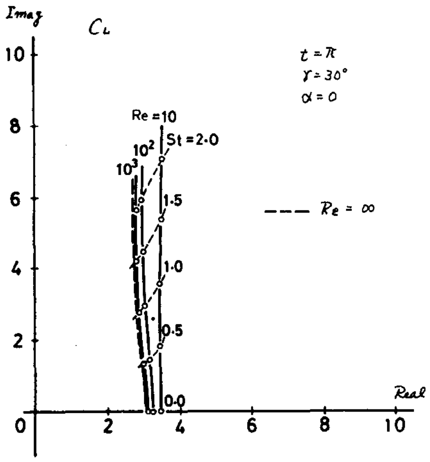

Fig.3 Lifts for translatory oscillation ($\alpha = 0$)

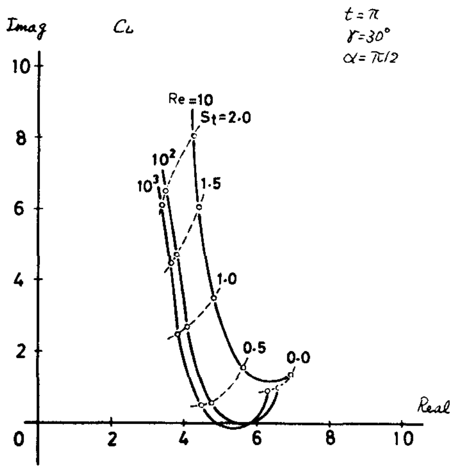

Fig.4 Lifts for translatory oscillation ($\alpha=\pi/2$)

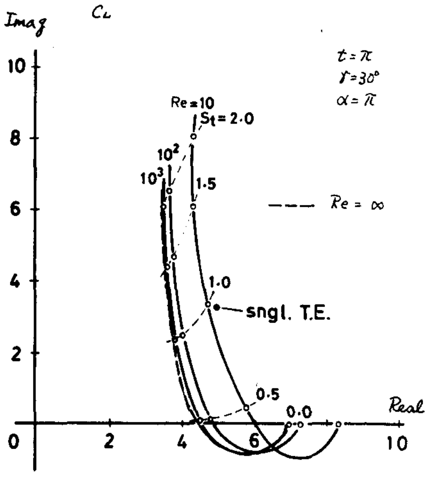

Fig.5 Lifts for translatory oscillation ($\alpha = \pi$)

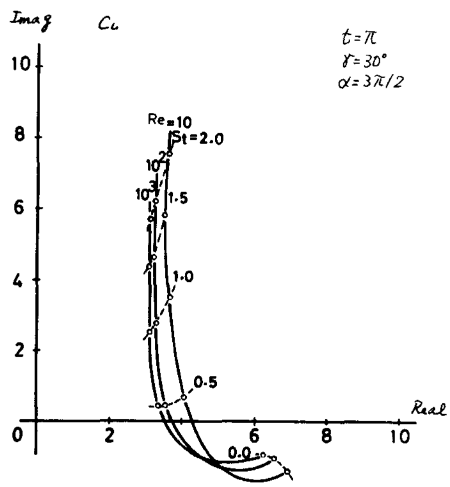

Fig.6 Lifts for translatory oscillation ($\alpha=3\pi/2$)

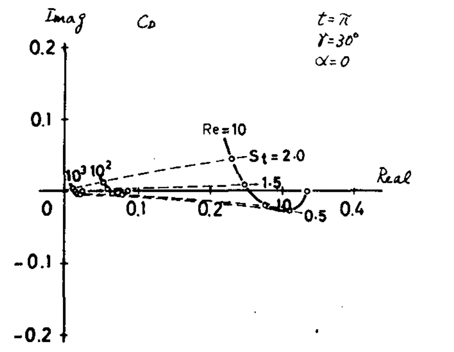

Fig.7 Drags for translatory oscillation ($\alpha = 0$)

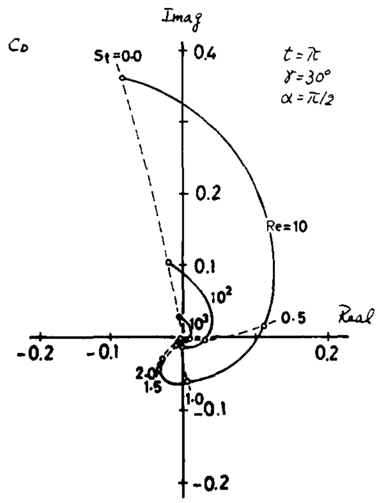

Fig.8 Drags for translatory oscillation ($\alpha = \pi/2$)

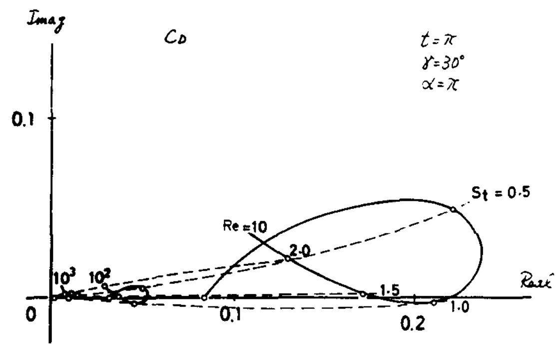

Fig.9 Drags for translatory oscillation ($\alpha = \pi$)

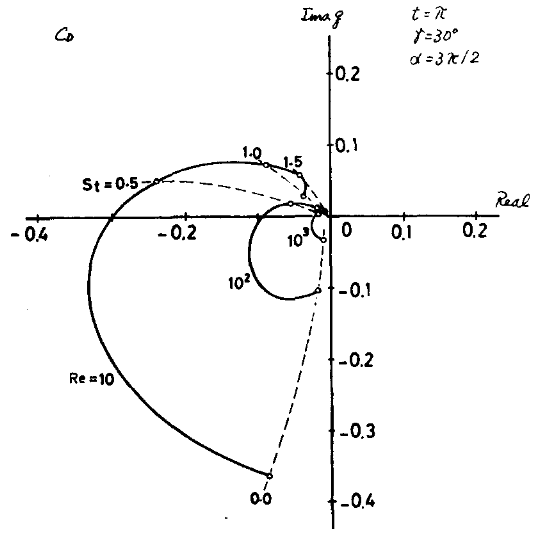

Fig.10 Drags for translatory oscillation ($\alpha = 3\pi/2$)

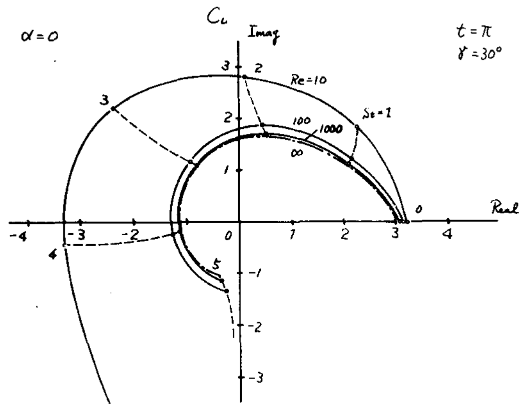

Fig.11 Lifts for sinusoidal gusts ($\alpha = 0$)

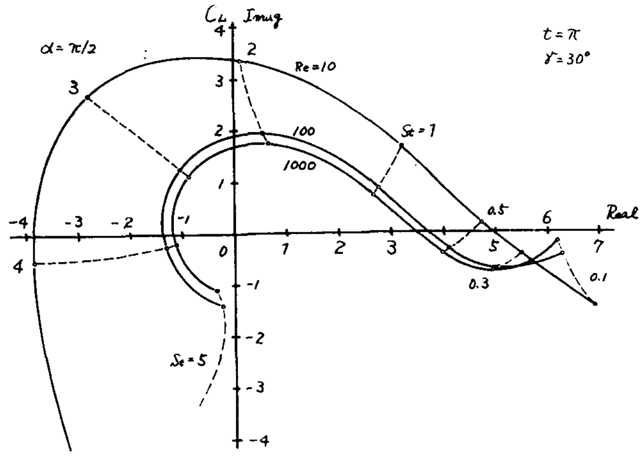

Fig.12 Lifts for sinusoidal gusts ($\alpha = \pi/2$)

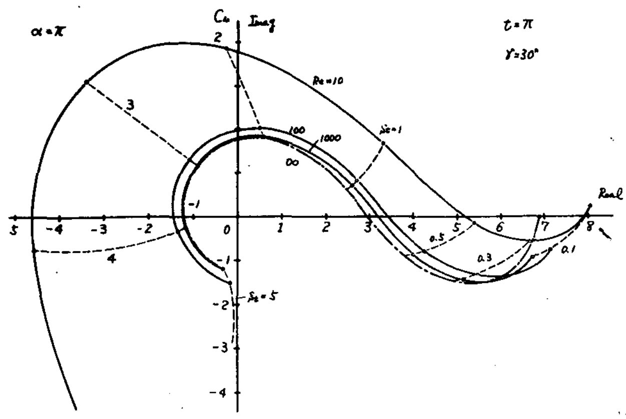

Fig.13 Lifts for sinusoidal gusts ($\alpha=\pi$)

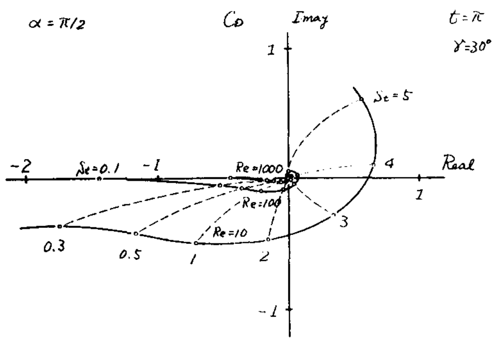

Fig.16 Drags for sinusoidal gusts ($\alpha=\pi/2$)

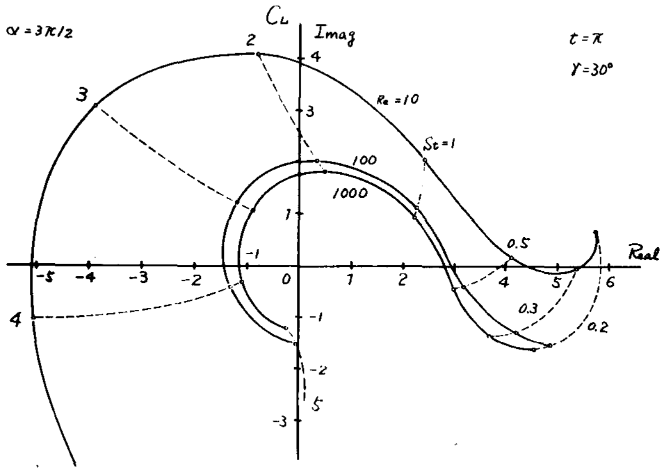

Fig.14 Lifts for sinusoidal gusts ($\alpha=3\pi/2$)

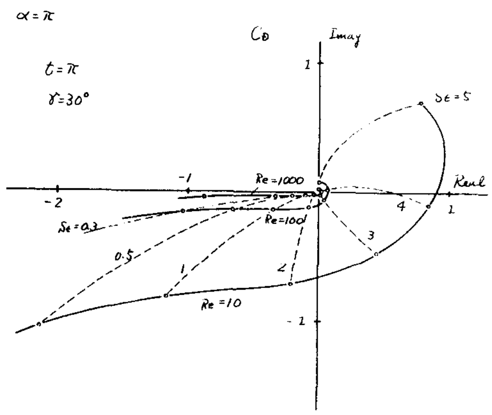

Fig.17 Drags for sinusoidal gusts ($\alpha=\pi$)

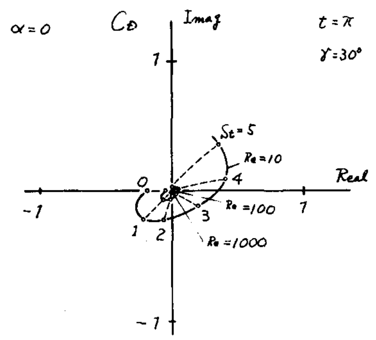

Fig.15 Drags for sinusoidal gusts ($\alpha=0$)

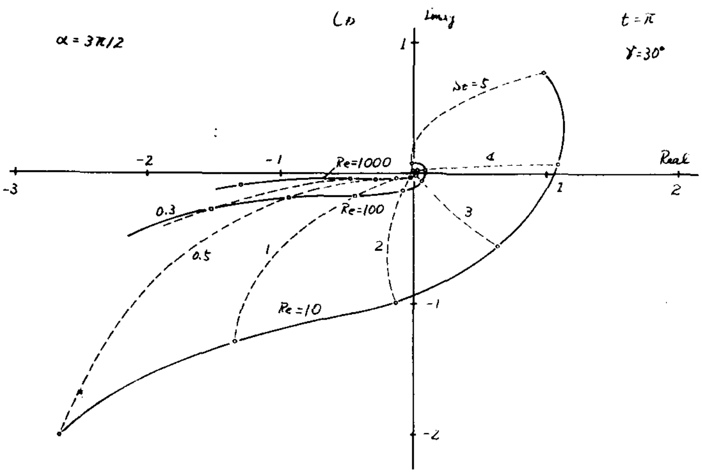

Fig.18 Drags for sinusoidal gusts ($\alpha=3\pi/2$)

Chapter 3 Compressible Viscous Flow

3.1 Actuator disk theory

3.1.1 Introduction

The unsteady cascades theories may be assorted into two groups; one is finite pitch theory and the other is actuator disk theory. In case of compressible flows the resonance phenomena, which is caused by the finiteness of the propagation velocity of the disturbance, is expected to be largely affected by the viscosity of the fluid. The finite pitch theory considering simultaniously compressibility and viscosity will be considerably complex one. This section gives an analysis of the viscous effects on the fluctuating lifts of three dimensional subsonic cascades by means of the actuator disk theory in which the pitch and the chordlength of the cascade are assumed to be infinitely small.

3.1.2 Basic equations

Let us assume that the pitch and the chordlength of the cascade are infinitely small and the cascade extends on y-z plane. We select x axis in the direction of cascade axis and z axis in the span direction as shown in Fig.1. The cascade is assumed to be non-staggered and to have no steady lift. The velocity is given by ($U+u$, v , w), the pressure by $p+p_0$ and the density by $\rho+\rho_0$, where U , p_0 and ρ_0 are the velocity, pressure and density of the uniform flow respectively. The purterbations are assumed to be small compared to the uniform quantities, i.e., u , v , $w \ll U$, $p \ll p_0$, $\rho \ll \rho_0$ and so on. Then the unsteady Navier-Stokes equation is linearized as;

$$\frac{\partial \mathbf{v}}{\partial t} + U\frac{\partial \mathbf{v}}{\partial x} = -\frac{1}{\rho_0}\,\mathrm{grad}\,p + \mathbf{F} + \nu\left(\frac{\partial^2 \mathbf{v}}{\partial x^2}+\frac{\partial^2 \mathbf{v}}{\partial y^2}+\frac{\partial^2 \mathbf{v}}{\partial z^2}\right)$$

$$+\,\nu(1+c')\,\mathrm{grad}\,(\mathrm{div}\,\mathbf{v}) \qquad (1)$$

where $\mathbf{F}(X, Y, Z)$ is the external force, $\mathbf{v}(u, v, w)$ the velocity purterbation, t the time, μ the viscosity, $\nu=\mu/\rho_0$ the kinetic viscosity, λ the bulk viscosity and $c'=\lambda/\mu$. The ratio c' is assumed to be $-2/3$ so that the pressure is given as the average of the principal stresses (Stokes' approximation). Though for the case of compressible viscous flow the viscosity varies as the temperature varies, we assume that the viscosity is constant and that the flow is isoentropic because we assume the small purterbating velocity and temperature. External force $\mathcal{F}$ is assumed to act only on the cascade plane ($x = 0$) and to be given as follows.

$$\mathcal{F} = (1/\rho_0)\,\overline{\mathcal{F}} \exp\left[2\pi j\,(n t + y/s + \varepsilon z/s)\right] \cdot \delta(x) \tag{2}$$

where $\overline{\mathcal{F}}\,(\overline{X}, \overline{Y}, \overline{Z})$ is the amplitude of the external force, j the imaginary unit ($= \sqrt{-1}$), n the number of the oscillations and S the wave length in y direction. If all the blades vibrates with the same phase , $S = \infty$. The quantity S/ε is the wave length in the span direction and $\varepsilon = 0$ for the case of two-dimensional flow. The function $\delta(x)$ is the impulse function defined as $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$. The continuity equation is linearized as follows.

$$\partial\rho/\partial t + \rho_0\,(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + \overline{U}\,(\partial\rho/\partial x) = 0 \tag{3}$$

Isoentropic condition is,

$$(1 + p/p_0) = (1 + \rho/\rho_0)^{\kappa}$$

so that

$$p/\rho = \kappa\,(p_0/\rho_0) = a_0^2 \tag{4}$$

where κ is the ratio of specific heats, a_0 is the sound velocity in the uniform flow. Equation (3) with Eq.(4) may be written as follows.

$$\rho_0\,(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) = -(1/a_0^2)(\partial p/\partial t + \overline{U}\,\partial p/\partial x) \tag{5}$$

3.1.3 Pressure fields in up and dounstream regions of the cascade

Differentiating the x, y, z components of Eq.(1) with $\mathcal{F} = 0$ in x , y , z direction respectively and then summing them and using Eq.(5), we have the following equation.

$$\Big[\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)+\frac{(2+C')\nu}{a_0^2}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)\Big(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\Big)$$

$$-\frac{1}{a_0^2}\Big(\frac{\partial}{\partial t}+\bar{U}\frac{\partial}{\partial x}\Big)^2\Big]\,p=0 \qquad (6)$$

Equation (6) is the basic equation for the pressure fields in up and downstream regions of the cascade. We assume the pressure as follows.

$$p=\sum_i A_i\exp\big[2\pi j(nt+y/s+\varepsilon z/s)\big]\exp(2\pi\alpha_i x/s) \qquad (7)$$

Putting Eq.(7) in Eq.(6) we have

$$\Big[2\pi(2+C')\frac{M_0^2}{R_e}\Big]\alpha_i^3+\Big[2\pi j(2+C')\frac{M_0^2}{R_e}k-M_0^2+1\Big]\alpha_i^2$$
$$-M_0^2\Big[2jk+2\pi(2+C')\frac{1+\varepsilon^2}{R_e}\Big]\alpha_i$$
$$+\Big[-2\pi j(2+C')\frac{1+\varepsilon^2}{R_e}k+k^2\Big]M_0^2-(1+\varepsilon^2)=0 \qquad (8)$$

where the dimensionless numbers are defined as $R_e=\bar{U}S/\nu$ (Reynolds number), $k=Sn/\bar{U}$ (reduced frequency), $M_0=\bar{U}/a_0$ (Mach number) with the characteristic length S (wave length in the direction of the cascade axis) and characteristic velocity $\bar{U}$. The three roots α_1 , α_2 , α_3 of Eq.(8) are complex ones, one of which have positive real part and the others negative real parts, and then are assumed $Real(\alpha_1)>0$, $Real(\alpha_2)<0$, $Real(\alpha_3)<0$ where $Real(\alpha)$ means the real part of α . Then the pressure should be summed with $i=1$ for $x<0$ and with $i=2,3$ for $x>0$ in Eq.(7) in order that the pressure fluctuation is finite at infinitely up and downstream of the cascade.

3.1.4 Velocity field in up and downstream regions of the cascade

Putting Eq.(5) in Eq.(1) with $\bar{F}=0$, we have

$$\frac{\partial v}{\partial t}+\bar{U}\frac{\partial v}{\partial x}-\nu\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\Big)=-\frac{1}{\rho_0}\,\mathrm{grad}\,(p)+$$

$$\frac{\nu}{a_0^2\rho_0}(1+C')\,\mathrm{grad}\Big(\frac{\partial p}{\partial t}+\bar{U}\frac{\partial p}{\partial x}\Big) \qquad (9)$$

The right hand side of Eq.(9) is known with Eq.(7). Then we assume the particular solutions of Eq.(9) as follows.

$$u = \sum_i B_i \exp[2\pi j(nt + y/s + \varepsilon z/s)] \exp(2\pi\alpha_i x/s)$$

$$v = \sum_i D_i \exp[2\pi j(nt + y/s + \varepsilon z/s)] \exp(2\pi\alpha_i x/s) \tag{10}$$

$$w = \sum_i F_i \exp[2\pi j(nt + y/s + \varepsilon z/s)] \exp(2\pi\alpha_i x/s)$$

where $\sum_i$ denotes the summation with $i=1$ for $x<0$ and $i=2,3$ for $x>0$. Putting the first of Eqs.(10) in the x component of Eq.(9), we have

$$B_i = -\frac{A_i}{\rho_0 U}\,\frac{\alpha_i[1 + 2\pi(1+C')(M_0^2/R_e)(\alpha_i + jk + j)]}{\alpha_i + jk - (2\pi/R_e)[\alpha_i^2 - (1+\varepsilon^2)]} \tag{11}$$

In the same manner the y and z components of Eq.(9) give

$$D_i = -\frac{A_i}{\rho_0 U}\,\frac{j + 2\pi(1+C')(M_0^2/R_e)(j\alpha_i - k)}{\alpha_i + jk - (2\pi/R_e)[\alpha_i^2 - (1+\varepsilon^2)]} \tag{12}$$

$$F_i = \varepsilon D_i \tag{13}$$

Next, let us assume the general solution of Eq.(9) (solution for $p=0$) as follows.

$$u = (C_i/\rho_0 U) \exp[2\pi j(n\tau + y/s + \varepsilon z/s)] \exp(2\pi\beta_i x/s)$$

$$v = (E_i/\rho_0 U) \exp[2\pi j(n\tau + y/s + \varepsilon z/s)] \exp(2\pi\beta_i x/s) \tag{14}$$

$$w = (G_i/\rho_0 U) \exp[2\pi j(n\tau + y/s + \varepsilon z/s)] \exp(2\pi\beta_i x/s)$$

Then all of the three components of Eq.(9) with $p=0$ will be reduced to,

$$(2\pi/R_e)\beta_i^2 + \beta_i - jk - (2\pi/R_e)(1+\varepsilon^2) = 0 \tag{15}$$

So the constants β_i are

$$\beta_i = (R_e/4\pi)\left[1 \pm \sqrt{1 + 4(2\pi/R_e)^2(1+\varepsilon^2) + 4j(2\pi/R_e)k}\right] \tag{16}$$

where β_1 is assumed to be given with plus sign in front of the root sign and β_2 with minus sign. Then $Real(\beta_1)>0$ and $Real(\beta_2)<0$, so $i=1$ for $x<0$ and $i=2$ for $x>0$ should be taken in Eqs. (14) so that the velocity purterbation is finite in far up and downstream of the cascade. The complex velocity purterbation is given as the sum of the particular solution (10) and the general solution (14). The particular solution given by Eq.(10) satisfies the continuity equation (5). The reason is Eqs.(10) $\sim$ (13) are given so as to satisfy the equation of motion (1) and the pressure equation (6) which is given by eliminating the velocities from Eq.(1) and the continuity equation (5). Therefore the general solution given by Eqs.(14) should satisfy the continuity equation for itself. Substitution of Eq.(14) into Eq.(5) with $p=0$ gives

$$\beta_i C_i + j E_i + j \varepsilon G_i = 0 \qquad (i=1,2) \tag{17}$$

3.1.5 Matching equations

Now the velocity and pressure fields have been decided separately in up and downstream regions of the cascade except for several constants. These constants will be decided by matching the flows across the cascade surface as follows. Firstly let us integrate the equation of motion (1) from $x=-0$ to $x=+0$. Considering the fact that the integration of the impulse function $\delta(x)$ gives the step function $\sigma(x)$ defined as $\sigma(x)=0$ for $x<0$ and $\sigma(x)=1$ for $x>0$, we have ,

$$\begin{aligned}
\bar{U}\{u\} &= -\frac{1}{\rho_0}\{p\} + \frac{1}{\rho_0}\bar{X}\exp[2\pi j(nt+y/s+\varepsilon z/s)] + \nu\left\{\frac{\partial u}{\partial x}\right\} \\
&\quad + \nu(1+c')\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right\} \\
\bar{U}\{v\} &= \frac{1}{\rho_0}\bar{Y}\exp[2\pi j(nt+y/s+\varepsilon z/s)] + \nu\left\{\frac{\partial v}{\partial x}\right\} + \nu(1+c')\left\{\frac{\partial u}{\partial y}\right\} \\
\bar{U}\{w\} &= \frac{1}{\rho_0}\bar{Z}\exp[2\pi j(nt+y/s+\varepsilon z/s)] + \nu\left\{\frac{\partial w}{\partial x}\right\} + \nu(1+c')\left\{\frac{\partial u}{\partial z}\right\}
\end{aligned} \tag{18}$$

where the notation $\{f(x)\} = f(+0) - f(-0)$ has been used. Equation (18) is considered to be the momentum equation in x, y and z direction. Secondly, the following relations are given through the same manipulation on the integrals of Eq.(1) with respect to x .

$$0 = \nu (2 r C') \{u\}, \quad 0 = \nu \{v\}, \quad 0 = \nu \{w\} \tag{19}$$

In the same manner, the continuity equation (5) gives

$$\rho_0 \{u\} = -(U / a_0^2) \{p\} \tag{20}$$

Since we consider only the lift force

$$\bar{X} = 0 \quad (21) \qquad \bar{Z} = 0 \quad (22)$$

The velocity component which is normal to the blade surface at the trailing edge should be given as

$$v = v_0 \exp [2\pi j (n\tau + y/S + \varepsilon z/S)] \tag{23}$$

which may be considered to be the vibrating velocity of the blades. The constants B_i , F_i and D_i can be expressed with A_i by Eqs. (11) ~ (13). The unknowns are A_i ($i = 1 \sim 3$), C_i , E_i , D_i ($i = 1, 2$), $\bar{X}$, $\bar{Y}$ and $\bar{Z}$. These twelve constants can be decided by making use of the two equations of Eqs.(17), three of Eqs.(18), three of Eqs.(19), and Eqs.(20),(21),(22) and (23). Then the fluctuating lift coefficient may be defined as follows.

$$C_L = \bar{Y} / (\rho_0 U v_0) \tag{24}$$

3.1.6 For the case of inviscid flow

As a particular case, we will consider the case of inviscid flow ($\nu = 0$). In the first place is considered the two dimensional flow ($\varepsilon = 0$). Then Eq.(8) reduces to the quadratic equation

$$(1 - M_0^2)\alpha_i^2 + (2j M_0^2 k)\alpha_i + (k^2 M_0^2 - 1) = 0$$

which results in

$$\alpha_i = \frac{j M_0^2 k \pm \sqrt{1 - M_0^2 + k^2 M_0^2}}{1 - M_0^2} \tag{25}$$

where $i = 1$ when the upper sign in front of the squareroot is taken and $i = 2$ when the lower one is taken. The constants A_3 , B_3 and D_3 vanish and Eqs.(11) and (12) become

$$B_i = -\frac{A_i}{\rho_0 \overline{U}} \frac{\alpha_i}{\alpha_i + jk} \tag{26}$$

$$D_i = -\frac{A_i}{\rho_0 \overline{U}} \cdot \frac{j}{\alpha_i + jk} \tag{27}$$

Moreover Eq.(15) is reduced to a linear equation and,

$$\beta_2 = -jk \tag{28}$$

The general solution of the velocity fluctuation vanish in $x<0$ and the constants C_1 and E_1 are zero. Equation (17) may be written

$$E_2 = j \beta_2 C_2 \tag{29}$$

Next the matching equations are considered. The constants B_i and D_i being expressed in A_i and E_2 in C_2 , the unknowns will be counted five (i.e., A_1 , A_2 , C_2 , $\overline{X}$, $\overline{Y}$). As Eq.(19) is inevitably satisfied, the matching equations are counted five, that is, two equations of Eqs.(18) and Eqs.(20),(21) and (23), and are sufficient to settle the five unknowns. Introducing the expression of the velocity and pressure, we get from Eq.(20)

$$C_2 = -\rho_0 \overline{U} (B_2 - B_1) - M_0^2 (A_2 - A_1) \tag{30}$$

Putting $\nu = 0$ in the expressions of velocity and pressure in Eq.(18) and using Eqs.(26) ~ (30) give

$$\overline{X} = (1 - M_0^2) (A_2 - A_1) \tag{31}$$

$$\overline{Y} = g(\alpha_2) A_2 - g(\alpha_1) A_1 \tag{32}$$

where

$$g(\alpha_i) = j(M_0^2-1)\alpha_i - 2M_0^2 k$$

(A) Resonance frequency

At the resonance frequency the aerodynamic damping force (i.e., fluctuating lift) will be zero for finite velocity amplitude at the outlet of the cascade. Then from Eqs.(21) and (24) at resonance frequency

$$\bar{X} = \bar{Y} = 0$$

which results

$$A_2 = A_1$$

and from Eq.(32)

$$\bar{Y} = j(M_0^2-1)(\alpha_2-\alpha_1)A_1 = 0$$

Hence, at resonance $\alpha_2 = \alpha_1$ and

$$k = \sqrt{1-M_0^2}\,/\,M_0 \tag{33}$$

which agrees with the resonance frequency so far known [1]. That is, the pressure purterbation at the origin at $\tau = 0$ spreads within the circle $(x-\bar{U}\tau)^2 + y^2 = a_0^2\tau^2$ at time τ . Therefore the propagating velocity a of the purterbation in the direction of the cascade axis is

$$a = y/t = a_0\sqrt{1-M_0^2}$$

Putting $n = a/S$, we have

$$k = Sn/\bar{U} = a/\bar{U} = \sqrt{1-M_0^2}\,/\,M_0$$

which agrees with Eq.(33). When the value in the squareroot of Eq.(25) is positive (subresonance), the constants α_1 , α_2 are complex and the pressure purterbation is cut-off far upstream or downstream of the cascade. When it is negative (superresonance), the constants α_1 , α_2 are imaginary and the pressure purterbation propagates to infinitely upstream and downstream of the cascade. It is seen from Eq.(28) that β_2 is imaginary and the velocity purterbation does not die away, which means the free vorticity from the cascade fills the region downstream of the cascade. On the other hand, for the case of viscous fluid, the constants α_i , β_i have non-zero real parts and hence the velocity purterbation dies away infinitely upstream or downstream of the cascade. For the three dimensional flow the resonance frequency will be,

$$k = \sqrt{(1+\varepsilon^2)(1-M_0^2)} \;/\; M_0$$

(B) Two dimensional incompressible flow ($M_0=0$, $\varepsilon=0$, $R_e=\infty$)

For imcompressible flow $\alpha_i=\pm 1$ and from Eqs.(31),(32)

$$\bar{X} = (A_2 - A_1) \qquad , \qquad \bar{Y} = j(A_2 + A_1)$$

Constants A_1 , A_2 are,

$$A_1 = A_2 = (\rho_0 \bar{U} v_0)[\,j - k/(1+jk)\,]^{-1} \tag{34}$$

Therefore,

$$C_L = \bar{Y} / (\rho_0 \bar{U} v_0) = 2j/[\,j - (j + 1/k)^{-1}] \tag{35}$$

which agrees with the results given by D.S.Whitehead [2].

(C) Three dimensional incompressible flow ($M_0=0$, $\varepsilon \neq 0$, $R_e=\infty$)

In this case the lift coefficient is found to be

$$C_L = \frac{2\sqrt{1+\varepsilon^2}\;(1+k^2+\varepsilon^2)}{(1+2k^2+2\varepsilon^2)\sqrt{1+\varepsilon^2} + jk} \tag{36}$$

which agrees with the result given through the integration of the elementary solution for unsteady lifts for three dimensional incompressible flow [3]. In this section we have seen that the procedure of the present analysis leads to the results so far given for the case of inviscid flow.

3.1.7 For the case of infinitely large Reynolds number

In the preceding section we have examined the case of perfectly inviscid flow. In this place we will study the case of the infinitely large Reynolds number for viscous fluid. For brevity two dimensional case ($\xi = 0$) is considered. The roots of Eq.(8) will be for

$$\begin{aligned} \alpha_{1,2} &= [\, j k M_0^2 \pm \sqrt{1 - M_0^2 - k^2 M_0^2}\,] / (1 - M_0^2) \\ \alpha_3 &= -(1 - M_0^2) R_e / [2\pi (2 + C') M_0^2] \end{aligned} \tag{37}$$

and $\alpha_3 \to \infty$ as $R_e \to \infty$. The roots of Eq.(15) are

$$\beta_1 = R_e / (2\pi) \quad , \quad \beta_2 = -jk$$

and $\beta_1 \to \infty$ as $R_e \to \infty$. Among the constants in the expression of velocity and pressure, A_3 , B_3 , P_3 and C_1 are order of $1/R_e$ and others are order of R_e^0 . Using Eq.(19) and putting Eq.(7),(10) and (14) in Eq.(20), we get

$$(M_0^2 / R_e)(-A_1 + A_2 + A_3) = 0$$

Hence, $A_1 = A_2$. From the first equation of Eqs.(19),

$$(1/R_e)(-B_1 + B_2 + B_3 - C_1 + C_2) = 0$$

and using Eqs.(11) and (12) we have

$$C_2 = B_1 - B_2 = -\frac{A_1}{\rho_0 \overline{U}} \left(\frac{\alpha_1}{\alpha_1 + jk} - \frac{\alpha_2}{\alpha_2 + jk} \right) \tag{39}$$

Equation (23) is reduced to

$$P_2 + P_3 + j\beta_2 C_2 = v_0$$

and by putting Eqs.(11) and (39) in the above equation we have;

$$v_0 = \frac{A_1}{\rho_0 \overline{U}} \left(\frac{k\alpha_2 - j}{\alpha_2 + jk} - \frac{\alpha_1 k}{\alpha_1 + jk} \right)$$

$$= \frac{A_1}{\rho_0 U} \frac{j(1+k^2)\sqrt{1-M_0^2-k^2M_0^2} - k}{1+k^2} \tag{40}$$

From the second of Eq.(19) we get

$$(1/R_e)(-D_1 + D_2 + D_3 - j\beta_1 C_1 + j\beta_2 C_2) = 0$$

which give

$$C_1 = \frac{2\pi}{R_e} \frac{A_1}{\rho_0 U} \frac{(1+k^2)(\alpha_2 - \alpha_1)}{\alpha_1\alpha_2 + jk(\alpha_2+\alpha_1) - k^2}$$

$$= \frac{2\pi U_0}{R_e} \frac{2(1+k^2)\sqrt{1-M_0^2-k^2M_0^2}}{j(1+2k^2)\sqrt{1-M_0^2-k^2M_0^2} - k^2} \tag{41}$$

The second of Eq.(18) is

$$(2\pi/R_e)[D_1\alpha_1 - D_2\alpha_2 - D_3\alpha_3 + j(1+c')(B_1 - B_2 - B_3)$$

$$+ jC_1(\beta_1^2 + 1 + c') - jC_2(\beta_2^2 + 1 + c')] = \bar{Y}/(\rho_0 \bar{U})$$

which will be for $R_e \to \infty$

$$\bar{Y}/(\rho_0 \bar{U}) = (2\pi/R_e)\, j\beta_1^2 C_1$$

In case of $R_e \to \infty$, the term $\nu\{\partial v/\partial x\}$ contributes most to the lift fluctuation among the terms of the second of Eq.(18). It is because $\{\partial v/\partial x\} \to \infty$ though $\nu \to 0$ as $R_e \to \infty$. From Eqs.(40) and (41) we have

$$C_L = \frac{2(1+k^2)\sqrt{1-M_0^2-k^2M_0^2}}{(1+2k^2)\sqrt{1-M_0^2-k^2M_0^2} + jk} \tag{42}$$

In reference [3] the fluctuating lift coefficient C_L is given by integrating along y axis the elementary solution for unsteady concentrated lifts in subsonic flow and the result agrees with Eq.(42). Equation (42) is also given by putting $\nu = 0$ at first and then using the procedure shown in the preceding section. Resonance frequency is given by putting $C_L = 0$ in Eq.(42).

$$k = \sqrt{1-M_0^2}\,/\,M_0$$

which agrees with Eq.(33). Putting $M_0 = 0$ in Eq.(42) give

$$C_L = 2(1+k^2)/(1+2k^2+jk) \qquad (43)$$

which agrees with Eqs.(35) and (36) with $\varepsilon = 0$. Both by Eq.(42) and by Eq.(43), $C_L \to 2$ for $k \to 0$, i.e., for infinitely small reduced frequency. On the other hand, $C_L \to 1$ for $k \to \infty$, i.e., for infinitely large frequency or for the case that all the blades vibrates in the same phase. That is, the value of the lift coefficient C_L is not affected by the compressibility of the fluid in the limit of $k \to 0$ or $k \to \infty$. Equation (42) is for $1 - M_0^2 - k^2 M_0^2 > 0$ i.e., for subresonance. For $1 - M_0^2 - k^2 M_0^2 < 0$ i.e., for superresonance

$$C_L = \frac{2(1+k^2)\sqrt{k^2 M_0^2 + M_0^2 - 1}}{(1+k^2)\sqrt{k^2 M_0^2 + M_0^2 - 1} + k} \qquad (44)$$

which implies that C_L is real for superresonance.

3.1.8 In case of viscous fluid

For inviscid flow we have seen the results of the present analyses agree with those so far given. In this section will be examined the effects of the viscosity. For brevity, the case $\varepsilon = 0$ and $M_0 \to 0$ is considered. Then $B_3 \to 0$, $D_3 \to 0$, $B_3 \alpha_3 \to -(R_e/2\pi)\rho_0 \bar{U} v_0/(2+C')$ and $D_3 \alpha_3 \to 0$. Considering these behaviours of the unknowns and from Eqs.(19) and (20) we see that $A_3 \to 0$ as $M_0 \to 0$. The second of Eqs.(18) is reduced to

$$\bar{Y} = (2\pi/R_e)(2+C')(D_1 + D_2)$$
$$+ (2\pi j/R_e)[(\beta_1^2 C_1 - \beta_2^2 C_2) + (1+C')(C_1 - C_2)] \qquad (45)$$

The constants C_1 and C_2 are found to be

$$C_1 = \rho_0 \bar{U} v_0 [2(1 - j\beta_2 C_2)/(1+jk) + j\beta_2 C_2]/j\beta_1$$

$$C_2 = \rho_0 \bar{U} v_0 [2/(1+jk) + 2jk\beta_1/(1+jk)]$$
$$/[2j\beta_2(1+jk) + j\beta_1 - 2k\beta_1\beta_2/(1+jk) - j\beta_1]$$

and D_1 , D_2 are

$$D_1 = (j\beta_2 C_2 - 1)(1 - jk)/(1 + jk)$$

$$D_2 = 1 - j\beta_2 C_2$$

Then the fluctuating lift coefficient C_L will be known from Eq.(45). By letting $k \to \infty$ we have

$$C_1 \to -(\rho_0 \overline{U} \cdot v_0)(j/\beta_1) \qquad C_2 \to -(\rho \overline{U} v_0)/(j/\beta_2)$$

$$D_1, \; D_2 \to 0$$

and the lift coefficient will be

$$C_L = (\rho_0 \overline{U} v_0)^{-1}(2\pi j/R_e)(\beta_1^2 C_1 - \beta_2^2 C_2) = \sqrt{8\pi jk/R_e + 1} \tag{46}$$

As in case of infinitely large Reynolds number, the term $\nu\{\partial v/\partial x\}$ is predominant in Eq.(18). Equation (46) shows that $C_L \to \sqrt{j} \times \infty$ as $k \to \infty$, which is thought to be the effect of the apparent mass added by the viscosity of the fluid. In case of $k = 0$, we get

$$C_1 = C_2 = -4\pi j/R_e \quad , \quad D_1 - D_2 = 0$$

and hence

$$C_L = 2\sqrt{16\pi^2/R_e^2 + 1} \tag{47}$$

3.1.10 Numerical results

Fluctuating lift coefficient C_L is shown in Figs.2~6. In Fig. 2~4 the effect of compressibility can be seen. The dash-dot lines in the figures show the results for inviscid flow (Eqs.(42),(43)), to which the viscous results converge as $R_e \to \infty$. The effect of viscosity is remarkable at large reduced frequency and near the resonance frequency. In case of viscous fluid, the lift coefficient have finite value at resonance frequency. Fig.5 shows the behaviour of C_L at large frequency. Lift coefficient C_L tends to 1 as $k \to \infty$ for inviscid flow but to $\sqrt{j} \times \infty$ for viscous flow. From Figs.2,6 and 7 can be seen the effect of the three-dimensionality. It deminishes C_L in small frequency range but increases near the resonance frequency. In Figs.8,9 the amplitudes of fluctuating velocity and pressure are given. For subresonance (Fig.8) the pressure fluctuation is cut off up and downstream of the cascade and the effects of the viscosity is not so remarkable. The velocity purterbation due to shed off vortices does not die away for inviscid flow and the effect of the viscous damping is remarkable in this region. In case of superresonance (Fig. 9) the pressure and the velocity fluctuations are transmitted to up and downstream for $Re \to \infty$ but die down for finite Reynolds number. The viscosity effects remarkably the pressure and velocity fluctuations both in up and downstream regions.

3.1.11 Conclusion

The effect of viscosity on unsteady lifts of three dimensional subsonic cascade was investigated by means of actuator disk method. The application limits due to the method are, 1. The pitch and the chordlength of the cascade are small. 2. The phase difference of the purterbations between adjacent blades should be small. 3. The reduced frequency based on the chordlength ($2\pi n c/U$, c ; chordlength) should be small. Among the effects of the viscosity, the viscous dissipation of the shed off vortices and the friction in the direction of cascade axis are considered, but not the friction in the direction of the mainstream (skin friction on the blade surface) because of the infinitely small chordlength. The results given tend to the inviscid solutions by letting $Re \to \infty$ and are rigorous solutions of linearlized unsteady Navier-Stokes equation. In spite of these limitations, authors believe that the results express the effects of the viscosity qualitatively so far as the unsteady lifts are concerned.

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[2] D.S.Whitehead; Vibration of cascade blades treated by actuator disk methods, Proc. Inst. Mech. Eng. 173-21 (1959) P.555

[3] Г.С.Самойлович: Нестационарное обтекание и аэроупругие колебания решеток турбомашин, Наука, (1969)

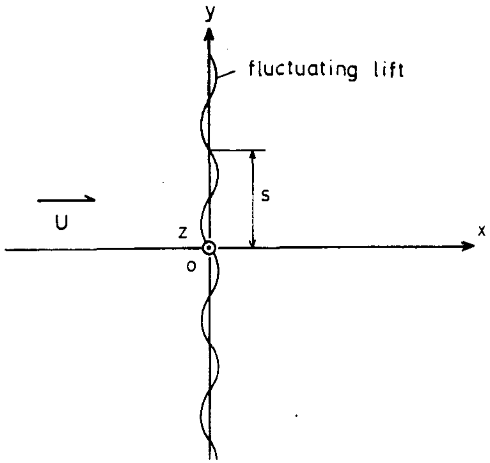

Fig.1 Actuator disk plane

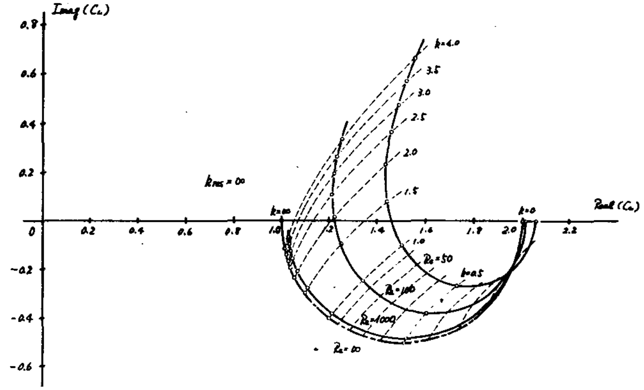

Fig.2 Fluctuating lift coefficient
for $\mathcal{E} = 0$, $M_0 \to 0$

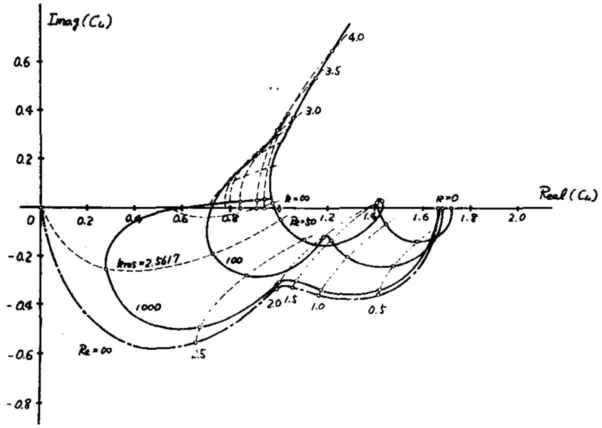

Fig.3 Fluctuating lift coefficient
for $\mathcal{E} = 0$, $M_0 = 0.4$
*k*res ; resonance frequency

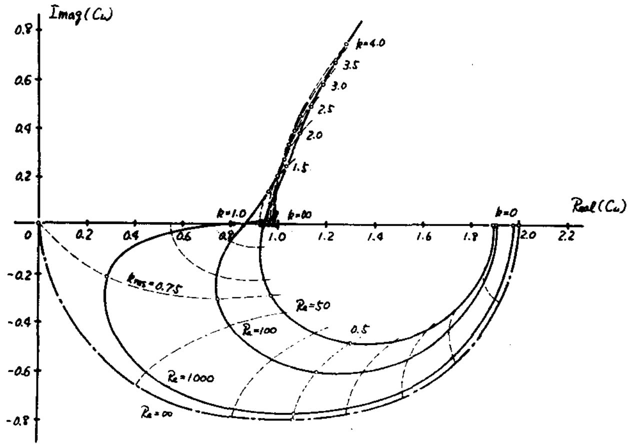

Fig.4 Fluctuating lift coefficient
for $\mathcal{E} = 0$, $M_0 = 0.8$

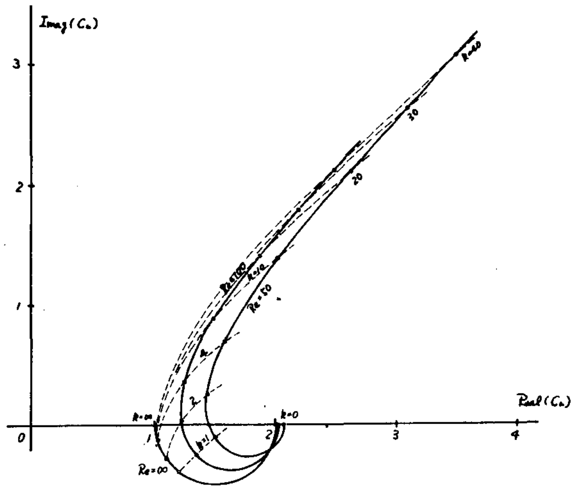

Fig.5 Fluctuating lift coefficient for $\mathcal{E} = 0$, $M_0 \to 0$ in case $k \to \infty$

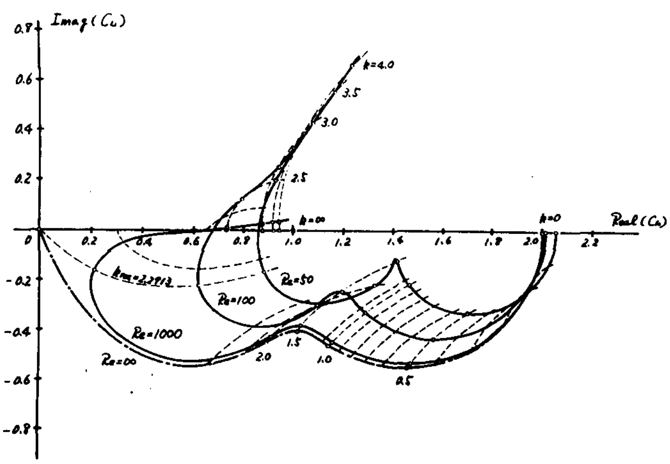

Fig.6 Fluctuating lift coefficient for $\mathcal{E} = 0.5$, $M_0 = 0.4$

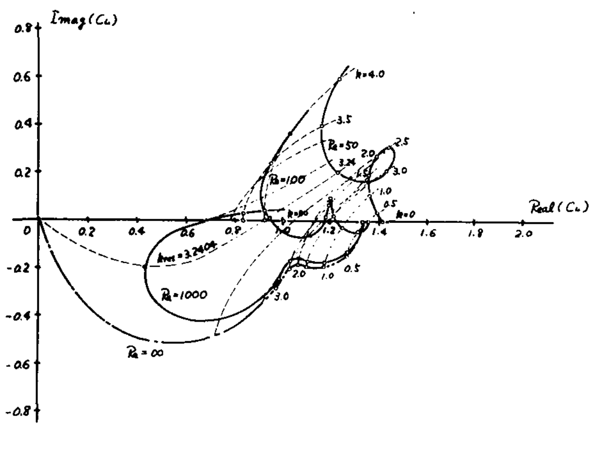

Fig.7 Fluctuating lift coefficient for $\mathcal{E} = 1.0$, $M_0 = 0.4$

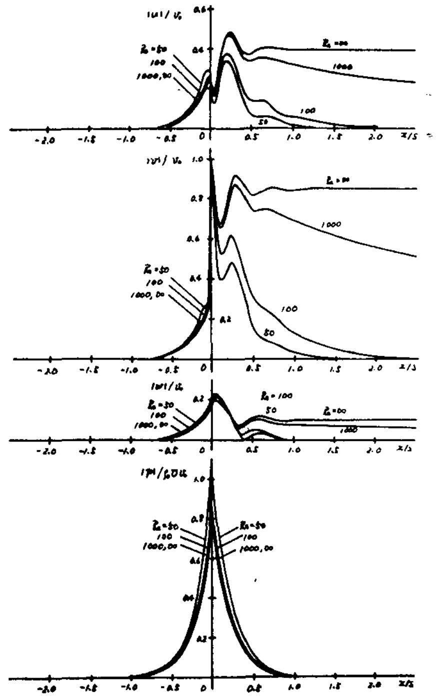

Fig.8 Amplitude of fluctuating pressure and velocity for $M_0 = 0.4, \mathcal{E} = 0.5, k = 2.0$ (subresonance)

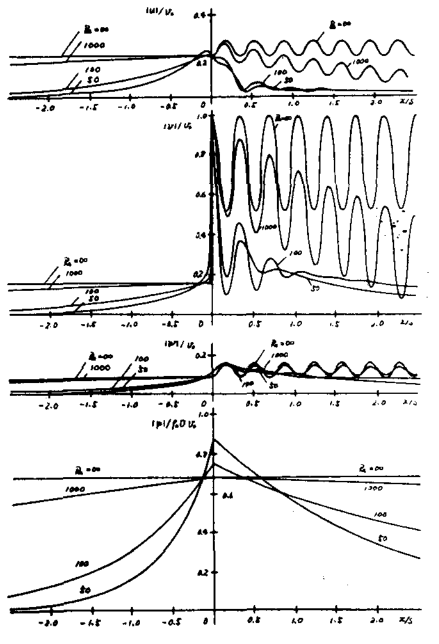

Fig.9 Amplitude of fluctuating pressure and velocity for $M_0 = 0.4$, $\mathcal{E} = 0.5$, $k = 3.0$ (superresonance)

3.2 Finite pitch cascade theory

3.2.1 Introduction

In the last section was developed a subsonic viscous unsteady actuator disk theory in order to analyze the effects of the viscisity on the lift fluctuations on unsteady subsonic cascade blades based on the linearlized Navier-Stokes equation, in which the blade spacing and the chordlength are assumed to be infinitely small. This section is intended to take the finiteness of the blade spacing and the chordlength into account. In order to take the finiteness of the blade spacing and the chordlength into account there can be another way such as an extension to a semi-actuator disk theory adopted by Tanida et. al. [1] but we employed the method of expanding the external forces represented by a series of δ functions into Fourier series and then applying the actuator disk theory to each term of the series by taking advantage of the mathematical strictness of the actuator disk theory that the coupling equations of the flow fields in front and after the actuator plane are directly lead from the fudamental equations. The singularities in the flowfield thus given agree with those so far given for inviscid or incompressible flow. The advantages of the present method are; 1. The elementary solutions are given in a form of a series and the infinite integration of the shed-off vortices necessary in the usual vortex method is avoided. 2. The numerical calculations of Fourier integral (infinite integration containing some singularities) necessary for the method applying Fourier transformation [2],[3] is avoided. 3. The effects of the viscosity is easily taken into account compared to the acceleration potential method. Therefore it seems that the present method can give a powerful calculation method even for compressible viscous flow compared to other method. It is assumed that the cascade is non-staggered and have no stationary lift.

3.2.2 Actuator disk solutions

Consider a flowfield around a row of concentrated external forces $L_0 e^{j\omega\tau}$ located on the y axis with pitch t as shown in Fig.1, where the main stream velocity U is directed in the direction of y axis, and j is the imaginary unit $\sqrt{-1}$ with respect to time, ω the angular velocity of oscillation, τ the time. Then the external force can be written as

$$F = \frac{2\pi}{t} L_0 e^{j\omega\tau} \delta(x) \left\{ \sum_{n=-\infty}^{+\infty} \delta\left(\frac{2\pi}{t}(y-nt)\right) \right\}$$

$$= \frac{L_0}{t} e^{j\omega\tau} \delta(x) \left(\sum_{n=-\infty}^{+\infty} e^{jn\frac{2\pi}{t}y} \right)$$

or

$$F = \sum_{n=-\infty}^{+\infty} F_n \tag{1}$$

$$F_n = (L_0/t) e^{jn\frac{2\pi}{t}y} \delta(x) e^{j\omega\tau} = \Upsilon_0 e^{jn\frac{2\pi}{t}y} \delta(x) e^{j\omega\tau} \tag{2}$$

The flowfield due to the distributed external force F_n can be given by putting

$$s = t/n \quad , \quad k = s\omega^*/U = t\,S_t/(2\pi n) , \quad R_e^* = US/\nu = 2R_e t/n$$

in the last section, where R_e^* and ω^* are expressed by R_e , ω respectively in the last section. The reduced frequency S_t and the Reynolds number R_e are defined by

$$S_t = \omega c/(2U) \qquad , \qquad R_e = Uc/(4\nu) \tag{3}$$

taking half chordlength $c/2 = 1$ as the reference length. Then the pressure field and the velocity field can be given as follows.

for $x < 0$

$$p = A_1 f_e(\tau, y) \exp(2\pi\alpha_1 x/s)$$

$$u = B_1 f_e(\tau, y) \exp(2\pi\alpha_1 x/s) + C_1/(\rho_0 U) f_e(\tau, y) \exp(2\pi\beta_1 x/s) \tag{4}$$

$$v = D_1 f_e(\tau, y) \exp(2\pi\alpha_1 x/s) + E_1/(\rho_0 U) f_e(\tau, y) \exp(2\pi\beta_1 x/s)$$

for $x > 0$

$$p = \sum_{i=2,3} A_i f_e(\tau, y) \exp(2\pi\alpha_i x/s)$$

$$u = \sum_{i=2,3} B_i f_e(\tau, y) \exp(2\pi\alpha_i x/s) + C_2/(\rho_0 U) f_e(\tau, y) \exp(2\pi\beta_1 x/s)$$

$$v = \sum_{i=2,3} D_i f_e(\tau, y) \exp(2\pi\alpha_i x/s) + E_2/(\rho_0 U) f_e(\tau, y) \exp(2\pi\beta_2 x/s) \tag{5}$$

$$f_e \equiv \exp(2\pi j y/s) e^{j\omega\tau}$$

where the constants α_i are the roots of the following cubic equation and it is assumed that $Real(\alpha_1)>0$, $Real(\alpha_2)<0$ and $Real(\alpha_3)<0$.

$$2\pi(2+C')(M_0^2/R_e^*)\,\alpha_i^3 + \{2\pi j(2+C')(M_0^2/R_e^*)k - M_0^2+1\}\,\alpha_i^2$$
$$- M_0^2\{2jk + 2\pi(2+C')/R_e^*\}\,\alpha_i + [\{k^2 - 2\pi j(2+C')\frac{k}{R_e^*}\}M_0^2 - 1] = 0 \qquad (6)$$

where C' is the ratio of the bulk viscosity λ to the viscosity μ and is assumed $C'=\lambda/\mu=-2/3$. The constants β_i are given by

$$\beta_i = (R_e^*/4\pi)\{\,1 \pm \sqrt{1 + 16\pi^2/R_e^{*2} + 8\pi jk/R_e^*}\,\} \qquad (7)$$

The thirteen constants in Eqs.(4),(5) are given by the following simultanious linear equations.

$$(2+C')[B_2\alpha_2 + B_3\alpha_3 - B_1\alpha_1] + (C_2\beta_2 - C_1\beta_1)/\rho_0\bar{U}$$
$$+ j(1+C')(D_2 + D_3 - D_1) = 0$$

$$-\frac{2\pi}{R_e^*}[D_2\alpha_2 + D_3\alpha_3 - D_1\alpha_1 + (E_2\beta_2 - E_1\beta_1)/\rho_0\bar{U}] = Y_0/(\rho_0\bar{U})$$

$$B_2 + B_3 - B_1 + (C_2 - C_1)/\rho_0\bar{U} = 0$$

$$D_2 + D_3 - D_1 + (E_2 - E_1)/\rho_0\bar{U} = 0 \qquad (8)$$

$$A_1 = A_2 + A_3$$

$$B_i = -\frac{A_i}{\rho_0\bar{U}}\,\frac{\alpha_i + 2\pi(1+C')(M_0^2/R_e^*)(\alpha_i^2 + j\alpha_i)}{\alpha_i + jk - (2\pi/R_e^*)(\alpha_i^2 - 1)}\,, \quad i=1,2,3$$

$$D_i = -\frac{A_i}{\rho_0\bar{U}}\,\frac{j + 2\pi(1+C')(M_0^2/R_e^*)(j\alpha_i - k)}{\alpha_i + jk - (2\pi/R_e^*)(\alpha_i^2 - 1)}\,, \quad i=1,2,3$$

$$E_i = j\beta_i C_i \qquad\qquad , \quad i=1,2$$

The velocity components represented by B_i and D_i are irrotational and affect the pressure field, and the components represented by C_i and E_i are rotational and do not affect the pressure field.

3.2.3 Inviscid and incompressible flow

The flowfield due to the external force F_n will be for $M_0 \to 0$ and $R_e \to \infty$;

for $x < 0$

$$
\begin{aligned}
p_n &= -\frac{j}{2} \Gamma_0 \exp(2\pi j \tfrac{n}{t} y) \exp(2\pi \tfrac{n}{t} x) e^{j\omega\tau} \\
u_n &= \frac{\Gamma_0}{\rho_0 U} \frac{j}{2(1+jk)} \exp(2\pi j \tfrac{n}{t} y) \exp(2\pi \tfrac{n}{t} x) e^{j\omega\tau} \qquad (9) \\
v_n &= -\frac{\Gamma_0}{\rho_0 U} \frac{1}{2(1+jk)} \exp(2\pi j \tfrac{n}{t} y) \exp(2\pi \tfrac{n}{t} x) e^{j\omega\tau}
\end{aligned}
$$

for $x > 0$

$$
\begin{aligned}
p_n &= -\frac{j}{2} \Gamma_0 \exp(2\pi j \tfrac{n}{t} y) \exp(-2\pi \tfrac{n}{t} x) e^{j\omega\tau} \\
u_n &= \frac{\Gamma_0}{\rho_0 U} \left\{ \frac{j}{2(1-jk)} \exp(-2\pi \tfrac{n}{t} x) + \frac{k}{k^2+1} \exp(-j S_t x) \right\} \exp(2\pi j \tfrac{n}{t} y) e^{j\omega\tau} \qquad (10) \\
v_n &= \frac{\Gamma_0}{\rho_0 U} \left\{ \frac{j}{2(1-jk)} \exp(-2\pi \tfrac{n}{t} x) + \frac{k^2}{k^2+1} \exp(-j S_t x) \right\} \exp(2\pi j \tfrac{n}{t} y) e^{j\omega\tau}
\end{aligned}
$$

where $k = \frac{S_t}{2\pi} \cdot \frac{t}{|n|}$. The velocity field is composed of only the irrotational component for $x<0$ and the first term of Eq.(10-1) and (10-2) are irrotational and the second rotational for $x>0$. The rotational component in $x>0$ vanish in the limit $S_t \to 0$ and is considered to represent the shed-off vortices from the bound vortices due to the fluctuating lift. Equations (9) and (10) are the flowfield due to the external force F_n and the flowfield due to $F = \sum_{n=-\infty}^{+\infty} F_n$ can be given by the superposition of Eqs.(9) and (10). Let us consider the pressure field due to the external force F . Assuming that the pressure fluctuation is zero at $x=-\infty$, we get the following pressure field.

$$
\begin{aligned}
p &= -\sum_{n=1}^{\infty} \left[\frac{j}{2} \Gamma_0 \exp\left\{ \frac{2\pi}{t}(jy + x) n \right\} - \sum_{n=1}^{\infty} \left[-\frac{j}{2} \Gamma_0 \exp\left\{ \frac{2\pi}{t}(-jy+x) n \right\} \right] \right] \times e^{j\omega\tau} \\
&= -\frac{j}{2} \Gamma_0 \frac{2j\, e^{\frac{2\pi}{t}x} \sin \frac{2\pi}{t} y}{1 - 2 e^{\frac{2\pi}{t}x} \cos \frac{2\pi}{t} y + e^{\frac{2\pi}{t}2x}} e^{j\omega\tau} \qquad (11)
\end{aligned}
$$

Equation (11) can be further deformed to

$$
p = \sum_{n=-\infty}^{+\infty} \frac{L_0}{2\pi} \frac{y - nt}{x^2 + (y-nt)^2} e^{j\omega\tau} \qquad (12)
$$

which means that the pressure field can be given as the velocity potential of a doublet series of strength $L_0 = t\Gamma_0$ distributed on y axis with the spacing t . This can be easily understood if we consider that the pressure field due to a single concentrated force is represented by the velocity potential of a doublet at the lift point whose strength equals to the strength of the external force. Next will be considered the velocity field. The irrotational component of the velocity field is,

for $x<0$

$$u=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\left[\frac{j\,\mathrm{sgn}(n)}{2(1+jk)}\exp\left(2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{j}{2}\left[\mathrm{sgn}(n)\frac{-jk}{1+jk}\exp\left(2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$+\frac{\Gamma_0}{\rho_0 U}\,\frac{j}{2}\,\frac{2j\,e^{\frac{2\pi}{t}x}\sin\frac{2\pi}{t}y}{1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}}\,e^{j\omega\tau} \tag{13}$$

$$v=-\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{1}{2(1+jk)}\exp\left(2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega\tau}$$

$$=-\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{j}{2}\left[\frac{-k}{1+jk}\exp\left(2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$-\frac{\Gamma_0}{\rho_0 U}\,\frac{1}{2}\,\frac{1-e^{\frac{2\pi}{t}2x}}{1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}}\,e^{j\omega\tau}$$

$$=-\frac{\Gamma_0}{\rho_0 U}\frac{j}{2}\left\{j+\frac{tS_t}{2\pi}\log\left(1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}\right)+\frac{jtS_t}{2\pi}\sum_{n=1}^{\infty}\frac{e^{\frac{2\pi n}{t}(x+jy)}+e^{\frac{2\pi n}{t}(x-jy)}}{n\left(\frac{2\pi}{tS_t}n+j\right)}\right]e^{j\omega\tau}$$

$$-\frac{\Gamma_0}{\rho_0 U}\cdot\frac{1}{2}\,\frac{1-e^{\frac{2\pi}{t}2x}}{1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}}\,e^{j\omega\tau}. \tag{14}$$

for $x>0$

$$u=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\left[\frac{j\,\mathrm{sgn}(n)}{2(1-jk)}\exp\left(-2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\left[\frac{j}{2}\mathrm{sgn}(n)\frac{jk}{1-jk}\exp\left(-2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$+\frac{\Gamma_0}{\rho_0 U}\,\frac{j}{2}\,\frac{2j\,e^{-\frac{2\pi}{t}x}\sin\frac{2\pi}{t}y}{1-2e^{-\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{-\frac{2\pi}{t}2x}}\,e^{j\omega\tau} \tag{15}$$

$$v=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\left[\frac{1}{2(1-jk)}\exp\left(-2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$=\frac{\Gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\left[\frac{1}{2}\frac{jk}{1-jk}\exp\left(-2\pi\frac{|n|}{t}x\right)\exp\left(2\pi j\frac{n}{t}y\right)\right]e^{j\omega\tau}$$

$$+\frac{\Gamma_0}{\rho_0 U}\,\frac{1}{2}\,\frac{1-e^{\frac{2\pi}{t}2x}}{1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}}\,e^{j\omega\tau}$$

$$=\frac{\Gamma_0}{\rho_0 U}\frac{j}{2}\left\{j-\frac{tS_t}{2\pi}\log\left(1-2e^{-\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{-\frac{2\pi}{t}2x}\right)+\frac{jtS_t}{2\pi}\sum_{n=1}^{\infty}\frac{e^{-\frac{2\pi n}{t}(x-jy)}+e^{-\frac{2\pi n}{t}(x+jy)}}{n\left(\frac{2\pi}{tS_t}n-j\right)}\right]e^{j\omega\tau}$$

$$+\frac{\Gamma_0}{\rho_0 U}\,\frac{1}{2}\,\frac{1-e^{\frac{2\pi}{t}2x}}{1-2e^{\frac{2\pi}{t}x}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}2x}}\,e^{j\omega\tau} \tag{16}$$

As shown by the second terms of Eqs.(13) (16), the irrotational components have the singularities of a row of bound vortices. The logarithmic singularlity shown by the first terms of Eqs.(14) and (16) is written for $y=0$ as;

$$\frac{\gamma_0}{\rho_0 U}\cdot\frac{j}{2}\frac{tS_t}{\pi}\log|x| = \frac{1}{2\pi}\frac{L_0}{\rho_0 U}\, jS_t \log|x| \tag{17}$$

which is caused by the finiteness of the strength of the free vortices at the shedding point. The rotational component reveals itself in the region $x>0$ and is written,

$$\begin{aligned} u &= \frac{\gamma_0}{\rho_0 U}\sum_{n=1}^{\infty}\frac{k}{k^2+1}\left\{\exp(2\pi j\frac{n}{t}y)-\exp(-2\pi j\frac{n}{t}y)\right\}e^{j\omega(\tau-\frac{x}{U})} \\ &= \frac{\gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{(\frac{t}{2\pi}S_t)\,n}{n^2+(\frac{t}{2\pi}S_t)^2}\exp(2\pi j\frac{n}{t}y)\,e^{j\omega(\tau-\frac{x}{U})} \\ &= -\frac{\gamma_0}{\rho_0 U}\frac{jtS_t}{2}\frac{e^{-tS_t(1-\frac{y}{t})}-e^{-S_t y}}{1-e^{-tS_t}}\,e^{j\omega(\tau-\frac{x}{U})} \end{aligned} \tag{18}$$

$$\begin{aligned} v &= \frac{\gamma_0}{\rho_0 U}\sum_{n=1}^{\infty}\left\{\frac{k^2}{k^2+1}\,2\cos(2\pi\frac{n}{t}y)+1\right\}e^{j\omega(\tau-\frac{x}{U})} \\ &= \frac{\gamma_0}{\rho_0 U}\frac{tS_t}{2}\frac{e^{-tS_t(1-\frac{y}{t})}+e^{-S_t y}}{1-e^{-tS_t}}\,e^{j\omega(\tau-\frac{x}{U})} \end{aligned} \tag{19}$$

which show that the rotational components of the disturbance flow down with the main flow velocity U . From Eq.(18) we get

$$u(y=0) - u(y=t) = \frac{\gamma_0}{\rho_0 U}\, jtS_t\, e^{j\omega(\tau-\frac{x}{U})} = \frac{L_0}{\rho_0 U}\, jS_t\, e^{j\omega(\tau-\frac{x}{U})} \tag{20}$$

which show that the shed-off vortices after cascades are represented by the rotational component.

It has been shown that the finite pitch solution can be correctly given by the superposition of the actuator disk solutions. The flowfield can be represented by convergent serieses of order $1/n^2$ after sorting out the singularities involved.

3.2.4 Inviscid compressible flow

The flow field due to the external force F_n will be for $R_e=\infty$ and $M_0\neq 0$, for $x<0$

$$p_n = -\frac{j\Upsilon_0}{2\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_1 x)e^{j\omega\tau}$$

$$u_n = \frac{\Upsilon_0}{\rho_0 U}\frac{j}{2}\frac{jkM_0^2+\sqrt{1-M_0^2(k^2+1)}}{(jk+\sqrt{1-M_0^2(k^2+1)})\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_1 x)e^{j\omega\tau} \quad (21)$$

$$v_n = \frac{\Upsilon_0}{\rho_0 U}\frac{j}{2}\frac{j}{jk+\sqrt{1-M_0^2(k^2+1)}}\frac{1-M_0^2}{\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_1 x)e^{j\omega\tau}$$

for $x>0$

$$p_n = -\frac{j\Upsilon_0}{2\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_2 x)e^{j\omega\tau}$$

$$u_n = \frac{\Upsilon_0}{\rho_0 U}\cdot\frac{j}{2}\frac{jkM_0^2-\sqrt{1-M_0^2(k^2+1)}}{jk-\sqrt{1-M_0^2(k^2+1)}}\frac{1}{\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_2 x)e^{j\omega\tau}$$
$$+\frac{\Upsilon_0}{\rho_0 U}\frac{k}{k^2+1}\exp(2\pi j\frac{n}{t}y)\exp(-jS_t x)e^{j\omega\tau} \quad (22)$$

$$v_n = \frac{\Upsilon_0}{\rho_0 U}\cdot\frac{j}{2}\frac{j}{jk-\sqrt{1-M_0^2(k^2+1)}}\frac{1-M_0^2}{\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi j\frac{n}{t}y)\exp(2\pi\frac{n}{t}\alpha_2 x)e^{j\omega\tau}$$
$$+\frac{\Upsilon_0}{\rho_0 U}\frac{k^2}{k^2+1}\exp(2\pi j\frac{n}{t}y)\exp(-jS_t x)e^{j\omega\tau}$$

where

$$\alpha_1 = \frac{jkM_0^2+\sqrt{(1-M_0^2)(1-k^2M_0^2)-M_0^4k^2}}{1-M_0^2}$$

$$\alpha_2 = \frac{jkM_0^2-\sqrt{(1-M_0^2)(1-k^2M_0^2)-M_0^4k^2}}{1-M_0^2}$$

Equations (21) and (22) are for $n>0$, and the flow for $n<0$ can be given by putting $n\to -n$, $y\to -y$, $\Upsilon_0\to -\Upsilon_0$ and $v_n = -v_n$ in Eqs.(21) and (22). The flowfield for F is known by the superposition of Eqs. (21) and (22) as follows.

for $x<0$

$$p = -\frac{j\Upsilon_0}{2}\sum_{n=-\infty}^{+\infty}\left[\frac{\operatorname{sgn}(n)}{\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi\frac{|n|}{t}\alpha_1 x)-\frac{\operatorname{sgn}(n)}{\sqrt{1-M_0^2}}\exp(2\pi\frac{|n|}{t}\frac{x}{\sqrt{1-M_0^2}})\right]\exp(2\pi j\frac{n}{t}y)e^{j\omega\tau}$$
$$-\frac{j\Upsilon_0}{2}\frac{1}{\sqrt{1-M_0^2}}\frac{2j\exp(\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}})\sin\frac{2\pi}{t}y}{1-2e^{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}}}e^{j\omega\tau}$$

$$u = \frac{\Upsilon_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{j\operatorname{sgn}(n)}{2}\left[\frac{jkM_0^2+\sqrt{1-M_0^2(k^2+1)}}{jk+\sqrt{1-M_0^2(k^2+1)}}\frac{1}{\sqrt{1-M_0^2(k^2+1)}}\exp(2\pi\frac{|n|}{t}\alpha_1 x)\right.$$
$$\left.-\frac{1}{\sqrt{1-M_0^2}}\exp(2\pi\frac{|n|}{t}\frac{x}{\sqrt{1-M_0^2}})\right]\exp(2\pi j\frac{n}{t}y)e^{j\omega\tau}$$

$$+\frac{\gamma_0}{\rho_0 U}\frac{j}{2\sqrt{1-M_0^2}}\frac{2j\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\sin\frac{2\pi}{t}y}{1-2\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\cos\frac{2\pi}{t}y+\exp\left\{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}}e^{j\omega\tau}$$

$$v=-\frac{\gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{1}{2}\Big[\frac{1}{jk+\sqrt{1-M_0^2(k^2+1)}}\frac{1-M_0^2}{\sqrt{1-M_0^2(k^2+1)}}\exp\left(2\pi\frac{|n|}{t}\alpha_1 x\right) \tag{23}$$

$$-\left(1-\frac{jk}{1+jk}\frac{1}{\sqrt{1-M_0^2}}\right)\exp\left(2\pi\frac{|n|}{t}\frac{x}{\sqrt{1-M_0^2}}\right)\Big]\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega\tau}$$

$$-\frac{\gamma_0}{\rho_0 U}\cdot\frac{1}{2}\frac{1-e^{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}}}{1-2\exp\left(\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right)\cos\frac{2\pi}{t}y+\exp\left(\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right)}e^{j\omega\tau}$$

$$-\frac{\gamma_0}{\rho_0 U}\frac{j}{2\sqrt{1-M_0^2}}\Big[j+\frac{tS_t}{2\pi}\log\left(1-2e^{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}}\cos\frac{2\pi}{t}y+e^{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}}\right)$$

$$+\frac{jtS_t}{2\pi}\sum_{n=1}^{\infty}\frac{\exp\left\{\frac{2\pi n}{t}\left(\frac{x}{\sqrt{1-M_0^2}}+jy\right)\right\}+\exp\left\{\frac{2\pi n}{t}\left(\frac{x}{\sqrt{1-M_0^2}}-jy\right)\right\}}{n\left(\frac{2\pi}{tS_t}n+j\right)}\Big]e^{j\omega\tau}$$

for $x>0$

$$p=-\frac{j\gamma_0}{2}\sum_{n=-\infty}^{+\infty}\Big[\frac{\mathrm{sgn}(n)}{\sqrt{1-M_0^2(k^2+1)}}\exp\left(2\pi\frac{|n|}{t}\alpha_2 x\right)-\frac{\mathrm{sgn}(n)}{\sqrt{1-M_0^2}}\exp\left(2\pi\frac{|n|}{t}\frac{-x}{\sqrt{1-M_0^2}}\right)\Big]$$

$$\times\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega\tau}-\frac{j\gamma_0}{2}\frac{1}{\sqrt{1-M_0^2}}\frac{2j\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\sin\frac{2\pi}{t}y}{1-2\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\cos\frac{2\pi}{t}y+\exp\left\{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}}e^{j\omega\tau}$$

$$u=\frac{\gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{j\,\mathrm{sgn}(n)}{2}\Big[\frac{jkM_0^2-\sqrt{1-M_0^2(k^2+1)}}{jk-\sqrt{1-M_0^2(k^2+1)}}\frac{1}{\sqrt{1-M_0^2(k^2+1)}}\exp\left(2\pi\frac{|n|}{t}\alpha_2 x\right)$$

$$-\frac{1}{\sqrt{1-M_0^2}}\exp\left(2\pi\frac{|n|}{t}\frac{-x}{\sqrt{1-M_0^2}}\right)\Big]\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega\tau}$$

$$+\frac{\gamma_0}{\rho_0 U}\frac{j}{2\sqrt{1-M_0^2}}\frac{2j\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\sin\frac{2\pi}{t}y}{1-2\exp\left\{\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\cos\frac{2\pi}{t}y+\exp\left\{\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}}e^{j\omega\tau}$$

$$-\frac{\gamma_0}{\rho_0 U}\frac{jtS_t}{2}\frac{e^{-tS_t\left(1-\frac{y}{t}\right)}-e^{-S_t y}}{1-e^{-tS_t}}e^{j\omega\left(\tau-\frac{x}{U}\right)}$$

$$v=-\frac{\gamma_0}{\rho_0 U}\sum_{n=-\infty}^{+\infty}\frac{1}{2}\Big[\frac{1}{jk-\sqrt{1-M_0^2(k^2+1)}}\frac{1-M_0^2}{\sqrt{1-M_0^2(k^2+1)}}\exp\left(2\pi\frac{|n|}{t}\alpha_2 x\right) \tag{24}$$

$$-\left(-1-\frac{jk}{1-jk}\frac{1}{\sqrt{1-M_0^2}}\right)\exp\left(2\pi\frac{|n|}{t}\frac{-x}{\sqrt{1-M_0^2}}\right)\Big]\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega\tau}$$

$$+\frac{\gamma_0}{\rho_0 U}\cdot\frac{1}{2}\frac{1-\exp\left\{-\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}}{1-2\exp\left\{-\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\cos\frac{2\pi}{t}y+\exp\left\{-\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}}e^{j\omega\tau}$$

$$+\frac{\gamma_0}{\rho_0 U}\frac{j}{2\sqrt{1-M_0^2}}\Big[j-\frac{tS_t}{2\pi}\log\left(1-2\exp\left\{-\frac{2\pi}{t}\frac{x}{\sqrt{1-M_0^2}}\right\}\cos\frac{2\pi}{t}y+\exp\left\{-\frac{2\pi}{t}\frac{2x}{\sqrt{1-M_0^2}}\right\}\right)$$

$$+\frac{jtS_t}{2\pi}\sum_{n=1}^{\infty}\frac{\exp\left\{-2\pi\frac{n}{t}\left(\frac{x}{\sqrt{1-M_0^2}}-jy\right)\right\}+\exp\left\{-2\pi\frac{n}{t}\left(\frac{x}{\sqrt{1-M_0^2}}+jy\right)\right\}}{n\left(\frac{2\pi}{tS_t}n-j\right)}\Big]e^{j\omega\tau}$$

$$+\frac{\gamma_0}{\rho_0 U_0}\frac{tS_t}{2}\frac{e^{-tS_t\left(1-\frac{y}{t}\right)}+e^{-S_t y}}{1-e^{-tS_t}}e^{j\omega\left(\tau-\frac{x}{U}\right)}$$

The series terms in Eqs.(23),(24) are convergent with $\mathcal{O}(1/n^2)$ at least and have no singularity. The pressure field has the singularity of the differential of the potential of vortices (i.e., doublets) scaled down by the factor $\sqrt{1-M_0^2}$ in x direction. The velocity field has singularities of the bound vortex row scaled down by the factor $\sqrt{1-M_0^2}$ in x direction and the logarithmic singularitirs due to shed-off vortices. It can be seen that the strength of the shed-off vortices composed of the rotational components is the same as for incompressible flow. These singularities agree with those given through the Fourier transformation method [4].

3.2.5 Incompressible viscous flow

As is shown in the preceding sections the singularities of the flow field can be known from the behaviour of the coefficients in Eqs.(4) and (5) in the limit of $n \to \infty$. That is, the singularities of order $1/x$ are produced from the constant terms at $n \to \infty$, the logarithmic singularities or velocity gaps from the terms having the coefficients of order $1/n$. Though it is not so cumbersome to get the full representations of the coefficients in case $M_0 \to 0$, we will show only the asymptotic form for $n \to \infty$, which is sufficient for the separation of the singularities.

$$
\begin{aligned}
A_1 &= A_2 = -\frac{j}{2}\gamma_0 + \mathcal{O}(1/n^2) \\
B_1 &= (\gamma_0/\rho_0 U)(j/2 + k/2) + \mathcal{O}(1/n^2) \\
B_2 &= (\gamma_0/\rho_0 U)(j/2 - k/2) + \mathcal{O}(1/n^2) \\
D_1 &= (\gamma_0/\rho_0 U)(-1/2 + jk/2) + \mathcal{O}(1/n^2) \\
D_2 &= (\gamma_0/\rho_0 U)(1/2 + jk/2) + \mathcal{O}(1/n^2) \\
C_1 &= \gamma_0(-j/2 - k/2) + \mathcal{O}(1/n^2) \\
C_2 &= \gamma_0(-j/2 + k/2) + \mathcal{O}(1/n^2) \\
E_1 &= \gamma_0(1/2 - jk/2 + R_e^*/8\pi) + \mathcal{O}(1/n^2) \\
E_2 &= \gamma_0(-1/2 - jk/2 + R_e^*/8\pi) + \mathcal{O}(1/n^2) \\
\alpha_1 &= 1, \quad \alpha_2 = -1, \quad \beta_1 = 1 + R_e^*/4\pi, \quad \beta_2 = -1 + R_e^*/4\pi
\end{aligned}
\tag{25}
$$

Representing the coefficients for n-th order harmonics as A_1^n for example and considering the behaviours of the coefficients for $n \to \infty$, we can separate the singularities as follows.

(A) Pressure field

$$p = \sum_{n=-\infty}^{+\infty} sgn(n) \left(A_{1,2}^n - \frac{\dot{J}}{2}\gamma_0 \right) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(-2\pi \frac{|n|}{t} |x|\right) e^{j\omega\tau}$$

$$+ \frac{\dot{J}}{2}\gamma_0 \frac{2j\, e^{\frac{2\pi}{t}x} \sin \frac{2\pi}{t} y}{1 - 2 e^{\frac{2\pi}{t}x} \cos \frac{2\pi}{t} y + e^{\frac{2\pi}{t} 2x}} e^{j\omega\tau} \tag{26}$$

where A_1^n should be taken for $x<0$, and A_2^n for $x>0$. The coefficients of the series in Eq.(26) are of order of $1/n^2$ and so the term has no singularity. The second term represents the singularity of a doublet row on y axis, which is identical with that for inviscid flow. The viscous effects on the pressure field are contained in the series term.

(B) Velocity field

$$u = \sum_{n=-\infty}^{+\infty} sgn(n) \left[\frac{C_{1,2}^n}{\rho_0 U} \exp\left(2\pi \frac{|n|}{t} \beta_{1,2}^n x\right) + B_{1,2}^n \exp\left(-2\pi \frac{|n|}{t} |x|\right) \right] e^{2\pi j \frac{n}{t} y} e^{j\omega\tau}$$

$$v = \sum_{n=-\infty}^{+\infty} \left[\frac{E_{1,2}^n}{\rho_0 U} \exp\left(2\pi \frac{|n|}{t} \beta_{1,2}^n x\right) + D_{1,2}^n \exp\left(-2\pi \frac{|n|}{t} |x|\right) \right] e^{2\pi j \frac{n}{t} y} e^{j\omega\tau} \tag{27}$$

The parallel velocity u for $x<0$ is considered first. Since $\beta_1^n = 1 + \frac{R_e^*}{4\pi}$ for $n \to \infty$,

$$\exp\left\{2\pi \frac{|n|}{t} \beta_1 x\right\} = \exp\left\{2\pi \frac{|n|}{t} x\right\} \cdot \exp\left\{R_e \cdot x\right\}$$

$$= \exp\left\{2\pi \frac{|n|}{t} x\right\} (1 + R_e \cdot x) = \exp\left\{2\pi \frac{|n|}{t} x\right\} + R_e \cdot x \exp\left\{2\pi \frac{|n|}{t} x\right\}$$

Therefore for $n \to \infty$ and $x \to 0$, we can write

$$\frac{C_1^n}{\rho_0 U} sgn(n) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} \beta_1^n x\right)$$

$$+ \quad B_1^{\,n} \, sgn(n) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

$$= R_e \cdot x \cdot \frac{C_1^{\,n}}{\rho_0 U} sgn(n) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

$$+ \left(\frac{C_1^{\,n}}{\rho_0 U} + B_1^{\,n}\right) sgn(n) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

The first term in the above equation tends to zero as $x \to 0$ and the coefficient of the second term is of order $1/n^2$. Hence it can be concluded that the first of Eqs.(27) has no singularity for $x \to 0$. The same arguments as above are valid for $x > 0$ and it can be shown that the parallel velocity u has no singularity for $x \to 0$. Next let us consider the normal velocity v. For $x < 0$ and $|x| \ll 1$, we can write,

$$\frac{E_1^{\,n}}{\rho_0 U} \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} \beta_1^{\,n} x\right)$$

$$+ \quad D_1^{\,n} \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

$$= R_e \cdot x \cdot \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

$$+ \left(\frac{E_1^{\,n}}{\rho_0 U} + D_1^{\,n}\right) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

The first term tends to zero as $x \to 0$ and this term constructs no singularity at $x \to 0$. Considering that

$$\frac{E_1^{\,n}}{\rho_0 U} + D_1^{\,n} = \frac{\gamma_0}{\rho_0 U} \frac{R_e}{4\pi} \frac{t}{n} + O(1/n^2)$$

we can write

$$\sum_{n=-\infty}^{+\infty} \left(\frac{E_1^{\,n}}{\rho_0 U} + D_1^{\,n}\right) \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right)$$

$$= \frac{\gamma_0 t}{\rho_0 U} \cdot \frac{R_e}{4\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{|n|+1} \exp\left(2\pi j \frac{n}{t} y\right) \exp\left(2\pi \frac{|n|}{t} x\right) + finite$$

$$= -\frac{L_0}{\rho_0 U} \cdot \frac{R_e}{4\pi} \log\left(1 - 2e^{\frac{2\pi}{t}x} \cos\frac{2\pi}{t} y + e^{\frac{2\pi}{t}2x}\right) + finite$$

Therefore the normal velocity v can be represented for $y = 0$ and $x \to 0$ as

$$v = -\frac{L_0}{\rho_0 U} \frac{R_e}{2\pi} \log|x| + finite$$

The logarithmic singularity thus separated coincides with that of unsteady Oseenlet for isolated unsteady lift. It can be shown that the same singularity exists for $x>0$. The second of Eqs.(27) can therefore be written as,

$$v = \sum_{n=-\infty}^{+\infty} \Big[\frac{E_{1,2}^{n}}{\rho_0 U} \exp\left(2\pi \frac{|n|}{t} \beta_{1,2}^{n} x\right) + \left(D_{1,2}^{n} - \frac{\gamma_0 t}{\rho_0 U} \frac{R_e}{4\pi} \frac{1}{|n|+1} \right)$$

$$\times \exp\left(-2\pi \frac{|n|}{t} |x|\right) \Big] \exp\left(2\pi j \frac{n}{t} y\right) e^{j\omega\tau}$$

$$+ \frac{\gamma_0 t}{\rho_0 U} \frac{R_e}{4\pi} \Big[1 - \log\left(1 - 2e^{-\frac{2\pi}{t}|x|} \cos \frac{2\pi}{t} y + e^{-\frac{2\pi}{t} 2|x|}\right) - \sum_{n=1}^{\infty} \frac{2e^{-2\pi \frac{n}{t}|x|} \cos \frac{2\pi}{t} n y}{n(n+1)} \Big] e^{j\omega\tau} \quad (28)$$

As examined above, there exists only the logarithmic singularity corresponding to that of Oseenlet, which is due to the terms of order $R_e^*/8\pi$ in the rotational component represented by E_1 and E_2 in Eqs.(4) and (5). The irrotational components have the singularity of the bound vortex row as for inviscid flow, which is canceled by the same singularity involved in the rotational component. The shed-off vortices dissipate at the instant of shedding and there cannot be seen the velocity gap or the logarithmic singularity due to the shed-off vortices for inviscid flow. In this way the irrotational components of the actuator disk solution correspond to the inviscid part of the elementary solution and the rotational components to the shed-off vortices and Oseenlet given in Chapter 2. The coefficients of the actuator disk solutions are expanded with respect to k and R_e^* for the purpose of the separation of the singularities. In order to get the convergence, the residual series after the separation of the singularities should be summed over to the term in which k and R_e^* are sufficiently small. That is, the number N of the cutting off should be $N \gg \frac{t S_t}{2\pi}$ and $N \gg 2 R_e S_t$ or N should be taken proportional to the values of t , S_t and R_e . Hence, in order to represent a large Reynolds number flows, the series should be taken to very high harmonics. Physically speaking, this is because high harmonics are needed in order to represent the large velocity gradient of high Reynolds number flow near the application point of the external force.

3.2.6 Compressible viscous flow

For inviscid or incompressible flows, the coefficient A_3 vanishes and the coefficients of the actuator disk solutions can be represented in the closed form. But for compressible and viscous flows, the representation of the coefficients would be complicated and we will show only the asymptotic representations for $n \to \infty$.

$$
\begin{aligned}
A_1 &= -\frac{j}{2} Y_0 \frac{R_e^*}{2\pi(2+C') M_0^2} + A_{12} R_e^{*2} + \theta(R_e^{*3}) \\
A_2 &= \frac{j}{2} Y_0 \frac{R_e^*}{2\pi(2+C') M_0^2} + A_{22} R_e^{*2} + \theta(R_e^{*3}) \\
A_3 &= -j Y_0 \frac{R_e^*}{2\pi(2+C') M_0^2} + A_{32} R_e^{*2} + \theta(R_e^{*3}) \\
B_1 &= \frac{Y_0}{\rho_0 U} (j/2 + B_{11} R_e^*) + \theta(R_e^{*2}) \\
B_2 &= \frac{Y_0}{\rho_0 U} (j/2 + B_{21} R_e^*) + \theta(R_e^{*2}) \\
B_3 &= \frac{Y_0}{\rho_0 U} B_{33} R_e^{*3} + \theta(R_e^{*4}) \\
D_1 &= \frac{Y_0}{\rho_0 U} (-1/2 + D_{11} R_e^*) + \theta(R_e^{*2}) \\
D_2 &= \frac{Y_0}{\rho_0 U} (1/2 + D_{21} R_e^*) + \theta(R_e^{*2}) \\
D_3 &= \frac{Y_0}{\rho_0 U} \frac{-1}{4\pi^2 (2+C')^2 M_0^2} R_e^{*2} + \theta(R_e^{*3}) \\
C_1 &= Y_0 (-j/2 + C_{11} R_e^*) + \theta(R_e^{*2}) \\
C_2 &= Y_0 (-j/2 + C_{21} R_e^*) + \theta(R_e^{*2}) \\
E_1 &= Y_0 (1/2 + E_{11} R_e^*) + \theta(R_e^{*2}) \\
E_2 &= Y_0 (-1/2 + E_{21} R_e^*) + \theta(R_e^{*2}) \\
\alpha_1 &= 1 + \frac{R_e^*}{4\pi(2+C')} \\
\alpha_2 &= -1 + \frac{R_e^*}{4\pi(2+C')} \\
\alpha_3 &= -jk - \frac{1}{2\pi(2+C') M_0^2} R_e^* \\
\beta_1 &= 1 + R_e^*/4\pi \quad , \quad \beta_2 = -1 + R_e^*/4\pi
\end{aligned}
\tag{29}
$$

where, $\qquad B_{11} + C_{11} = 0 \quad , \quad B_{21} + C_{21} = 0$

$$D_{11} + E_{11} = D_{21} + E_{21} \neq 0$$

The pressure field does not have the singularity of doublets which was found for inviscid or incompressible flows. It can be written as;

for $x<0$

$$p = \sum_{n=-\infty}^{+\infty} \left(A_1^n + \frac{j Y_0 R_e}{2\pi(2+C')M_0^2} \frac{n}{n^2+1} t \right) \exp\left(2\pi j \frac{n}{t} y\right) e^{j\omega\tau}$$

$$- \frac{t Y_0 R_e}{2(2+C')M_0^2} \frac{e^{-2\pi(1-\frac{y}{t})} - e^{-\frac{y}{t}}}{1-e^{-2\pi}} e^{j\omega\tau}$$

for $x>0$ (30)

$$p = \sum_{n=-\infty}^{+\infty} \left(A_2^n + A_3^n + \frac{j Y_0 R_e t}{2\pi(2+C')M_0^2} \frac{n}{n^2+1} \right) \exp\left(2\pi j \frac{n}{t} y\right) e^{j\omega\tau}$$

$$- \frac{t Y_0 R_e}{2(2+C')M_0^2} \frac{e^{-2\pi(1-\frac{y}{t})} - e^{-\frac{y}{t}}}{1-e^{-2\pi}} e^{j\omega\tau}$$

The coefficients of the serises in Eqs.(30) are of order of $1/n^2$ and the series are continuous across $y=0$, but the second term of Eqs.(30) makes the pressure gap of ;

$$p_{+0} - p_{-0} = \frac{t Y_0 R_e}{(2+C')M_0^2} e^{j\omega\tau} \qquad (31)$$

across $y=0$. The pressure difference approaches infinity in case $M_0 \to 0$ or $R_e \to \infty$. In case of $x \neq 0$, the pressure components relating to A_1 and A_2 are continuous across $y=0$ since $\alpha_1 \to 1$ and $\alpha_2 \to -1$ as $|n| \to \infty$ and the series converge geometrically.

Since $\alpha_3 \to -\frac{t}{n}\left(j\frac{S_t}{2\pi} + \frac{2R_e}{2\pi(2+C')M_0^2}\right)$, the pressure component relating to A_3 can be written;

$$p = \sum_{n=-\infty}^{+\infty} \left[A_3^n \exp\left(2\pi\alpha_3 \frac{n}{t} x\right) + \frac{j R_e Y_0 t}{\pi(2+C')M_0^2} \frac{n}{n^2+1} \exp\left\{-\left(jS_t + \frac{2R_e}{(2+C')M_0^2}\right)x\right\} \right]$$

$$\times \exp\left(2\pi j \frac{n}{t} y\right) e^{j\omega\tau}$$

$$- \frac{R_e Y_0 t}{(2+C')M_0^2} \frac{e^{-2\pi(1-\frac{y}{t})} - e^{-2\pi\frac{y}{t}}}{1-e^{-2\pi}} \exp\left\{-\left(jS_t + \frac{2R_e}{(2+C')M_0^2}\right)x\right\} e^{j\omega\tau} \qquad (32)$$

The first term of Eq.(32) is convergent and continuous across $y=0$ but the second term produces the pressure gap of

$$p_{+0} - p_{-0} = \frac{2R_e\gamma_0 t}{(2+c')M_0^2} \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} e^{j\omega\tau} \quad (33)$$

In case of $R_e \to \infty$ or $M_0 \to 0$ the coefficient of the exponential and the coefficient of x in the exponent tend to infinity and hence the pressure difference would be infinite at $x \to 0$ and then decrease exponentially in the downstream. It can be seen that the integral of the pressure difference equals to the lift in case $S_t = 0$. The reason of the pressure gap on the plane where no external force is applied will be explained below. Consider the velocity components v_3 corresponding to D_3 . The n-th order component v_3^n can be written for $n \to \infty$;

$$v_3^n = -\frac{\gamma_0}{\rho_0 U} \frac{R_e^2}{\pi^2(2+c')^2 M_0^2} \frac{t^2}{n^2+1} \exp\left(2\pi j \frac{n}{t} y\right) \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} \times e^{j\omega\tau}$$

and therefore the velocity component v_3 is continuous across $y = 0$. Differentiating the above equation with y , we have;

$$\frac{\partial v_3^n}{\partial y} = -\frac{\gamma_0}{\rho_0 U} \frac{2j R_e^2 t}{\pi(2+c')^2 M_0^2} \frac{n}{n^2+1} \exp\left(2\pi j \frac{n}{t} y\right) \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} \times e^{j\omega\tau}$$

and therefore

$$\frac{\partial v_3}{\partial y} = \sum_{n=-\infty}^{+\infty} \left\{\left(2\pi j \frac{n}{t} D_3^n + \frac{\gamma_0}{\rho_0 U} \frac{2j R_e^2 t}{\pi(2+c')^2 M_0^2} \frac{n}{n^2+1}\right) \exp\left(2\pi j \frac{n}{t} y\right)\right]$$

$$\times \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} e^{j\omega\tau}$$

$$-\frac{\gamma_0}{\rho_0 U} \frac{2R_e^2 t}{(2+c')^2 M_0^2} \frac{e^{-2\pi(1-\frac{y}{t})} - e^{-2\pi\frac{y}{t}}}{1-e^{-2\pi}} \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} e^{j\omega\tau}$$

Hence,

$$\left(\frac{\partial v_3}{\partial y}\right)_{+0} - \left(\frac{\partial v_3}{\partial y}\right)_{-0} = \frac{\gamma_0}{\rho_0 U} \frac{4R_e^2 t}{(2+c')^2 M_0^2} \exp\left\{-\left(jS_t + \frac{2R_e}{(2+c')M_0^2}\right)x\right\} e^{j\omega\tau} \quad (34)$$

Applying the isoentropic equation (Eq.(4) in section 3.1) on the linearlized Navier-Stokes equation (Eq.(1) in section 3.1), we have,

$$\frac{\partial v}{\partial \tau} + U\frac{\partial v}{\partial x} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\nu}{a_0^2 \rho_0}(1+c')\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial \tau} + U\frac{\partial p}{\partial x}\right) \quad (35)$$

which is parallel to Eq.(9) in section 3.1 and a_0 is the sound velocity in undisturbed flow. The quantities of $\partial v/\partial \tau$, $\partial v/\partial x$ and $\partial v^2/\partial^2 x$ are continuous across $y=0$ and the integrals of them from $y=-0$ to $y=+0$ are zero. Integrating each term of Eq.(35) from $y=-0$ to $y=+0$, we have;

$$-\nu\Delta\left(\frac{\partial U}{\partial y}\right) = -\frac{1}{\rho_0}\left[\Delta p - \frac{\nu}{a_0^2}(1+C')\left(\frac{\partial}{\partial t}\Delta p + U\frac{\partial}{\partial x}\Delta p\right)\right]$$

where the notation $\Delta f(y) = f(+0) - f(-0)$ is used. Applying Eq.(33) on the right hand side of above equation, we have;

$$\Delta\left(\frac{\partial U}{\partial y}\right) = \frac{Y_0}{\rho_0 U}\frac{4R_e^2 t}{(2+C')^2 M_0^2}\exp\left\{-\left(jS_t + \frac{2R_e}{(2+C')M_0^2}\right)x\right\}e^{j\omega t}$$

which agrees with Eq.(34). That is, the pressure discontinuity on $y=0$, $x>0$ is canceled by the discontinuity of velocity gradient and the elementary equations are satisfied. Next we can get the following pressure equation by eliminating the velocities from the linearlized Navier-Stokes equation, the equation of continuity and isentropic equation, which is parallel to Eq.(6) in section 3.1.

$$\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{(2+C')\nu}{a_0^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right) - \frac{1}{a_0^2}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\right]p = \rho_0\frac{\partial Y}{\partial y} \qquad (36)$$

where Y is the mass force in y direction and represented by $\rho_0 Y = L_0\,\delta(x)\,\delta(y)$ in this case. Then the right hand side of Eq.(36) is $L_0\,\delta(x)\,\delta'(y)$. In case $R_e \neq \infty$ and $M_0 \neq 0$, we have no pressure gap in $x<0$ and have that given by Eq.(33) in $x>0$. In order to examine the singularities of each term of Eq.(36) at the origin, we may represent the pressure difference in the vicinity of the origin as,

$$\Delta p = \frac{2R_e Y_0 t}{(2+C')M_0^2}\,\sigma(x)$$

from which

$$\frac{\partial p}{\partial y} = \frac{2R_e Y_0 t}{(2+C')M_0^2}\,\delta(y)\,\sigma(x)$$

and

$$\frac{(2+C')\nu}{a_0^2}\frac{\partial^2}{\partial y^2}\cdot U\frac{\partial}{\partial x}p = \gamma_0 t\,\delta(y)\,\delta(x) = L_0\,\delta(y)\,\delta(x)$$

That is, the singularity of the concentrated external force can be represented by the second term of Eq.(36) owing to the pressure difference at $x>0$. In case of $R_e=\infty$ or $M_0=0$ the external force is represented by the first term of Eq.(36) owing to the singularity of doublet in the pressure field. The first term of Eq.(36) is due to the pressure term of Navier-Stokes equation, the second to viscous terms and the third to inertia terms. Therefore it can be said that in case of $R_e=\infty$ or $M_0=0$ the external concentrated force is sustained by the pressure force and hence the singularity of doublet appears in the pressure field, and in case of $R_e\neq\infty$ and $M_0\neq 0$, by viscous force and hence no singularity appears in the pressure field. In case of $M_0=0$ even for $R_e\neq\infty$, the viscous terms do not affect the pressure field and hence the external force is sustained by the pressure force, which results in the doublet singularity in the pressure field.

Compareing Eq.(29) with (25), we can see that the same singularity as for incompressible flows appear in the velocity field. That is, the singularity due to the velocity components $B_{1,2}$ is canceled by that due to $C_{1,2}$ and hence the parallel velocity u can be represented by a convergent series if we sum up the series after adding those two components. The component relating to B_3 has no singularity since $B_3\sim\Theta(1/n^2)$. The logarithmic singularity can be separated from the velocity components due to $D_{1,2}$, $E_{1,2}$ and the normal velocity can be written as follows after separation of the singularity.

$$\begin{aligned}
v = \sum_{n=-\infty}^{+\infty} \Big[& \sum_i \frac{E_i^n}{\rho_0 U}\exp\left(2\pi\frac{|n|}{t}\beta_i^n x\right) + \sum_l D_l^n \exp\left(2\pi\frac{|n|}{t}\alpha_l^n x\right) \\
& - \frac{2\gamma_0 t}{\rho_0 U}(D_{11}+E_{11})\frac{R_e}{|n|+1}\exp\left(-2\pi\frac{|n|}{t}|x|\right)\Big]\exp\left(2\pi j\frac{n}{t}y\right)e^{j\omega t} \\
& + \frac{2\gamma_0 t}{\rho_0 U}(D_{11}+E_{11})R_e\Big[1-\log\left(1-2e^{-\frac{2\pi}{t}|x|}\cos\frac{2\pi}{t}y+e^{-\frac{2\pi}{t}2|x|}\right) \\
& - \sum_{n=1}^{\infty}\frac{2e^{-2\pi\frac{n}{t}|x|}}{n(n+1)}\cos\frac{2\pi}{t}ny\Big]e^{j\omega t}
\end{aligned} \tag{37}$$

where the summation should be taken with $i=1$ by $\sum_i$, with $l=1$ by $\sum_l$ for $x<0$, and with $i=2$ by $\sum_i$, with $l=2,3$ by $\sum_l$ for $x>0$.

In conclusion, if the fluid is simultaniously viscous and compressible, there appears no such singularity in the pressure field as can be seen for incompressible or inviscid flow, but the singularity of the velocity field is similar to that for incompressible flow.

3.2.7 Numerical examples

In the last sections we have separated the singularities out of the elementary solutions and expressed the residual by convergent series of order $1/m^2$. The series can be evaluated numerically by cutting off the series at finite terms corresponding to the accuracy expected. Present calculations have been made by cutting off the terms smaller than 0.05 percent of the sum of the series, summing 800 terms at most. The external force L_0 is assumed to distribute on the chord ($-1 \leq x \leq 1$), in order to take the finiteness of the chordlength into account. The distribution L_0 has the singularity of $[-(1+x)(1-x)]^{-\frac{1}{2}}$ at the leading and the trailing edges as well for incompressible flow. Then assuming the lift distribution by Glauert series reinforced by the term representing the trailing edge singularity and determining the Glauert coefficients from the boundary conditions on the blade surface, we can get the lift distribution on the blades. We have to take care to cancel the uniform induced velocity at infinitely upstream of the cascade by the calculation of the induced velocity. The numerical calculations have been made for up and down oscillation of the blades and for dissipating sinusoidal gusts. The dissipating gusts can be represented by the rotational component (relating to C_2 , E_2 in Eq.(5)) of the downstream actuator disk solutions, that are independent on the pressure field and hence on the Mach number. The elementary solutions given in the last sections are constructed for $\alpha = 0$. The analyses for $\alpha = 2\pi/M$ (M ; integer) can be made by arranging the elementary solutions for $t' = Mt'$ with the spacing t on y axis with the phase difference α . For general values of α , it seems to be necessary to represent the external force $\bar{F}$ with Fourier integral in stead of Fourier series, as made by Namba [5] for inviscid flow. The irrotational component of the elementary solution has finite parallel velocity on the blade surfaces except for $\alpha = 0, \pi$, and the drag fluctuations should be taken into account in order to cancel the parallel velocity. The effects of the drag fluctuation on the lift fluctuation is considered to be small in case the unsteady flow assumed has no parallel velocity disturbance on the blade surface or in case of high Reynolds number flow. For this reason the drag fluctuation has been entirely neglected in the present calculations. A cascade of flat plate blades of chordlength $C=2$, blade spacing t , stagger 0 and without steady lift is considered. Fig.2 shows the fluctuating lift coefficient for up and down oscillation of the blades in incompressible flow. The lift coefficient C_L is defined as $C_L = L/(\rho v_0 \bar{U}(c/2)e^{i\omega\tau})$, where v_0 is the amplitude of the oscillating velocity of the blades. Comparisons are made in the figure with the results by acceleration potential method for inviscid flow and with the results given by using the elementary solutions for incompressible viscous flow. It can be seen that the present analysis gives reasonable results for incompressible flow. Figs 3 and 4 show the results for inviscid compressible flow. Comparisons with the results by D.S.Whitehead [3] have been made in Fig.3 for blade oscillation and it can be seen that the results are satisfactory. The fluctuating lift coefficient for dissipating sinusoidal gusts is defined by the same equation as for blade oscillation, in which v_0 is assumed to be the normal velocity fluctuation at the midchord ($x=0$) of the blades. The trace pattern of C_L with the parameter S_t resembles to that by actuator disk theory, which suggests that the actuator disk theory can give a reasonable qualitative results. Figs.5 and 6 show the results for $M_0 \neq 0$ and $R_e \neq \infty$. It can be seen that the lift fluctuation has finite value at resonance frequency in case $R_e \neq \infty$, and that the inviscid solution can be given by letting $R_e \to \infty$ for viscous flow.

3.2.8 Conclusion

An analytical method of finite pitch subsonic viscous and unsteady lifts on cascade blades is given on the basis of actuator disk theory. The elementary solutions are represented by convergent series after separation of the singularities. It has been certified that the results agree with those so far given for inviscid or incompressible flow by numerical calculations.

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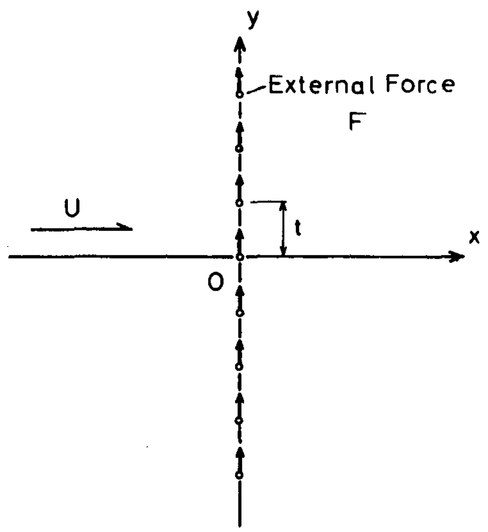

Fig.1 External force

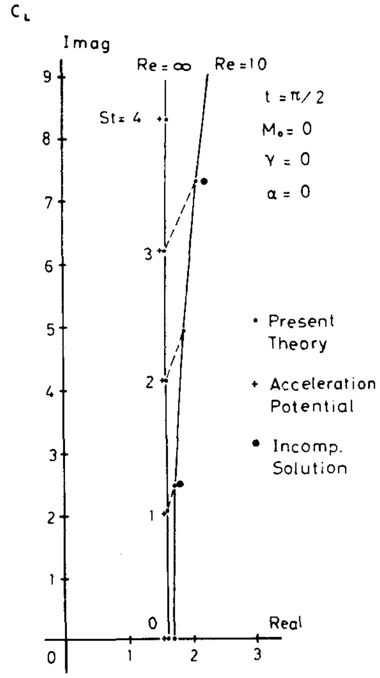

Fig.2 Incompressible limit for translatory oscillation

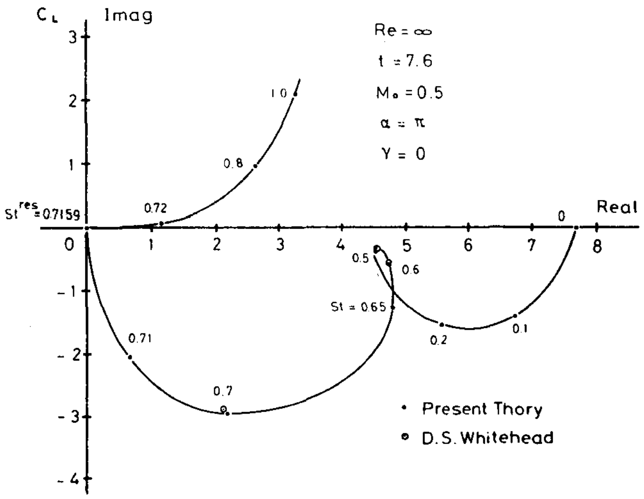

Fig.3 Inviscid compressible limit for translatory oscillation

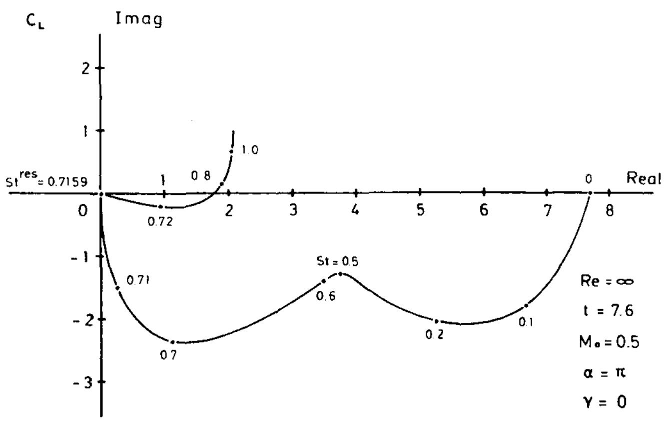

Fig.4 Inviscid compressible limit for sinusoidal gusts

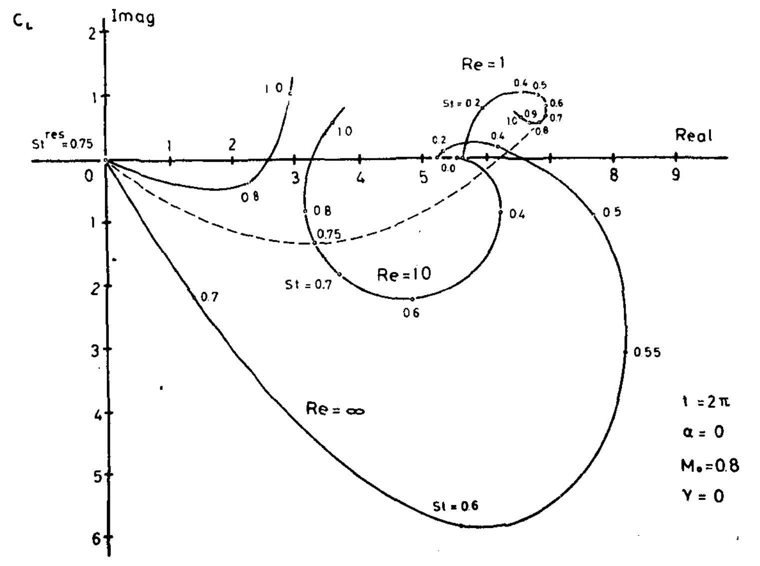

Fig.5 Viscous compressible case
for translatory oscillation

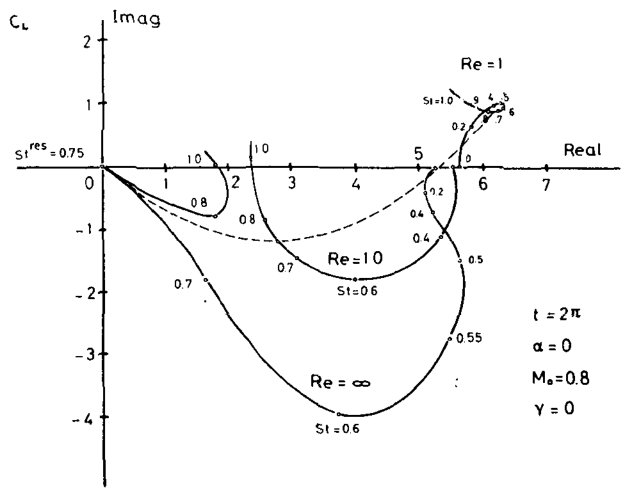

Fig.6 Viscous compressible case
for sinusoidal gusts

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