ON THE SINGULAR IMPULSIVE FUNCTIONAL INTEGRAL EQUATIONS INVOLVING NONLOCAL CONDITIONS

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Abstract

In this paper, of concern is the singular impulsive functional integral equations subject to nonlocal conditions in a Banach space. Sufficient conditions, ensuring the existence of solutions, are presented. An example is also given to illustrate the applications of the abstract results. Our results essentially extend some existing results in this area.

1. Introduction

In recent years, the theory of various functional integral equations in Banach spaces has been studied deeply due to their important values in sciences and technologies, and many significant results have been established (see, e.g., [13, 16, 23, 24, 26, 29, 30] and references therein). Let us point out that many systems involve memory effects can be modelled by functional integral equations. Moreover, functional integral equations describe systems with continuously distributed memory over the entire past of the system; consequently, they have features that are substantially different from those of memoryless systems (i.e. ordinary or partial differential equations and differential inclusions), and also very different from those of systems with concentrated memory effects (i.e. delay-differential equations, with either constant or variable delays). In particular, there has a significant development in the research area of impulsive integral equations; see for instance [4, 15]. Impulsive conditions arise in a variety of applications; as shown in, e.g., [2, 5, 21, 28], the dynamics of many evolutionary processes from some research fields are subject to abrupt changes of states at certain moments of time between intervals of continuous evolution, such changes can be well-approximated as being instantaneous changes as state, that is, in the form of "impulses".

Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. In the present paper we consider the singular impulsive functional integral equation involving a nonlocal condition in the form
(1.1)
$$\begin{cases} u(t) = T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds, & 0 \le t \le T,\ t \ne t_i, \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, \ldots, n, \end{cases}$$

in $\mathbb{X}$, where $\{T(t)\}_{t \ge 0}$ is a compact C_0-semigroup on $\mathbb{X}$, $0 < \alpha < 1$, $0 < t_1 < t_2 < \cdots < t_n < T$ are pre-fixed numbers, $\Delta u(t_i)$ represents the jump of the function u at t_i, which is defined by $u(t_i^+) - u(t_i^-)$, where $u(t_i^+) = \lim_{h \to 0^+} u(t_i + h)$ and $u(t_i^-) = \lim_{h \to 0^-} u(t_i + h)$ denote respectively the right and left limits of $u(t)$ at $t = t_i$. H, F, I_i $(i = 1, \ldots, n)$ are appropriate operators to be specified later. As can be seen, H constitutes a nonlocal condition and the integral equation in (1.1) is singular.

As usual, the solution $t \to u(t)$ with the points of discontinuity at the moments t_i $(i = 1, \ldots, n)$ follows that $u(t_i) = u(t_i^-)$, that is, at which it is continuous from the left.

We adopt the following concept of solution for (1.1).

DEFINITION 1.1. By a solution of (1.1) we mean a function $u \in PC([0, T]; \mathbb{X})$ satisfying the integral equation
(1.2)
$$u(t) = \begin{cases} T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds, & \text{if } t \in J_0, \\[2ex] T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} T(t-s) F(s, u(s))\, ds \\[1ex] \qquad\qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds \\[1ex] + \displaystyle\sum_{0 < t_i < t} T(t - t_i) I_i(u(t_i^-)), & \text{if } t \in J_i\ (i = 1, \ldots, n). \end{cases}$$

We refer to Section 2 for a complete definition of the set $PC([0, T]; \mathbb{X})$, and to, e.g., [7, Lemma 3.1] and [25, Lemma 3.3] for more details of Definition 1.1.

Interest in the problems incorporating nonlocal conditions stems mainly from the observation that nonlocal conditions have better effects in treating physical problems than the usual ones, see [8, 11, 12] and the references therein for more detailed information about the importance of nonlocal conditions in applications. Here, it is worth mentioning that much attention is attracted by questions of existence of solutions to the impulsive functional integral equations with nonlocal conditions in recent years, where the integral equations are regarded as mild solutions of the corresponding impulsive functional differential equations incorporating nonlocal initial conditions. For significant works along this line, see, e.g., [1, 6, 9, 10, 14, 22] and the references therein for more comments and citations. However, it is easy to see that much of these research

on the existence of solutions was done under the restriction that the impulsive item I_i $(i = 1, \ldots, n)$ are compact or Lipschitz continuous. This condition turns out to be quite restrictive and is not satisfied usually in practical applications. Thus, there naturally arises a question: "whether there exists a solution for the impulsive functional integral equations with nonlocal conditions when the impulsive item loss the compactness and Lipschitz continuity".

In this paper, among others, we will give an affirmative answer to this question. Some new ideas will be given to obtain the desired results. Sufficient conditions, ensuring the existence of solutions for the singular impulsive functional integral equation involving a nonlocal condition (1.1), are established. These conditions allows us to relax the compactness and Lipschitz continuity on the impulsive item I_i $(i = 1, \ldots, n)$. In fact, in the proof of one of main results we only need to suppose the continuity and the growth conditions on the impulsive item and nonlocal item and do not impose any other conditions. The main tools in our study are approximating technique in terms of the theory of compact C_0-semigroup and the fixed point theorems due to Schauder and Darbo-Sadovskii. Our results essentially extend some existing results in this area.

REMARK 1.1. As the reader will see, the hypotheses on the impulsive item and nonlocal item in our theorems are reasonably weak and different from those in many previous papers such as [1, 6, 9, 10, 14] (where the integral equations considered are defined as mild solutions of the corresponding impulsive functional differential equations incorporating nonlocal initial conditions and the results obtained are based upon stronger restrictions on the nonlocal item and the impulsive item), and the proofs provided are concise.

REMARK 1.2. Let us note that the approximating technique plays a key role in the proof of our main results, which enable us to get rid of the compactness or Lipschitz continuity of impulsive item and nonlocal item. Furthermore, this approach can be easily extended to other functional integral equations involving impulsive conditions and nonlocal conditions.

REMARK 1.3. We mention that in recent paper [25], the solutions of integral equation (1.2) are defined as mild solutions of the following impulsive fractional functional differential equation with nonlocal initial condition

$$(1.3) \quad \begin{cases} {}^c D_t^\alpha u(t) = Au(t) + F(t, u(t)), & t \in [0, T] \setminus \{t_1, t_2, \ldots, t_n\}, \\ u(0) = H(u), \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, \ldots, n, \end{cases}$$

where ${}^c D_t^\alpha$, $0 < \alpha < 1$, is the Caputo fractional derivative of order α and $A: D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of C_0-semigroup $\{T(t)\}_{t \geq 0}$. However, from the results in [19] it is easy to see that this concept of mild solutions of (1.3) is not appropriate

(though this fashion of definitions of mild solutions was used in some situations of previous research (see, e.g., [20])).

This work is organized as follows. In Section 2, we present some preliminaries and assumptions. Section 3 is devoted to our main results and their proof. Finally in Section 4, an example is given to illustrate the feasibility of our abstract results.

2. Preliminaries

Throughout this paper, we let $\mathcal{L}(\mathbb{X})$ be the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$ and M be a constant such that

$$M = \sup_{t \in [0, T+1]} \|T(t)\|_{\mathcal{L}(\mathbb{X})}.$$

Write

$$J_0 = [0, t_1], \quad J_i = (t_i, t_{i+1}], \quad i = 1, \ldots, n,$$

with $t_0 = 0$, $t_{n+1} = T$, and let u_i be the restriction of a function u to J_i ($i = 0, 1, \ldots, n$). Consider the set of functions

$$PC([0, T]; \mathbb{X})$$
$$= \{u \colon [0, T] \to \mathbb{X};\ u_i \in C(J_i; \mathbb{X}),\ i = 0, 1, \ldots, n, \text{ and } u(t_i^+)$$
$$\text{and } u(t_i^-) \text{ exist and satisfy } u(t_i) = u(t_i^-) \text{ for } i = 1, \ldots, n\}.$$

Endowed with the norm

$$\|u\|_{PC} = \max\left\{\sup_{t \in J_i}\|u_i(t)\|;\ i = 0, 1, \ldots, n\right\}.$$

It is easy to show that, with this norm, $PC([0, T]; \mathbb{X})$ is a Banach space (see [17]).

The following lemma will play an important role in this paper.

Lemma 2.1. *A set $B \subset PC([0, T]; \mathbb{X})$ is precompact in $PC([0, T]; \mathbb{X})$ if and only if, for each $i = 0, 1, \ldots, n$, the set $B|_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; \mathbb{X})$.*

For the sake of convenience, we put

$$\Omega_r = \{u \in PC([0, T]; \mathbb{X});\ \|u(t)\| \le r,\ \forall t \in [0, T]\},$$

where r is any positive constant, and list the assumptions to be used in this work as follows:

(H_F) $F\colon [0, T] \times \mathbb{X} \to \mathbb{X}$ is a Carathéodory function, and there exists a constant $\beta \in [0, \alpha)$ and a positive function $f_r(\cdot) \in L^{1/\beta}(0, T; \mathbb{R}^+)$ such that for a.e. $t \in [0, T]$ and all $x \in \mathbb{X}$ satisfying $\|x\| \le r$,

$$\|F(t, x)\| \le f_r(t), \quad \text{and} \quad \liminf_{r \to +\infty} \frac{\|f_r(t)\|_{L^{1/\beta}(0,T)}}{r} = \sigma < \infty.$$

(H_H) (i) $H\colon PC([0, T]; \mathbb{X}) \to \mathbb{X}$ is continuous, there exists a nondecreasing function $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|H(u)\| \le \Phi(r),$$

for all $u \in \Omega_r$, and

$$\liminf_{r \to +\infty} \frac{\Phi(r)}{r} = \mu < \infty.$$

(ii)   There is an $\varsigma \in (0, t_1)$ such that for any $u, w \in PC([0, T]; \mathbb{X})$ satisfying $u(t) = w(t)$ $(t \in [\varsigma, T])$, $H(u) = H(w)$.

(H'_H) There exists a positive constant L_H such that

$$\|H(u) - H(w)\| \le L_H \|u - w\|_{PC},$$

for all $u, w \in PC([0, T]; \mathbb{X})$.

(H_I) For every $i = 1, \ldots, n$, $I_i\colon \mathbb{X} \to \mathbb{X}$ is continuous, there exists a nondecreasing function $\Psi_i\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|I_i(y)\| \le \Psi_i(r),$$

for all $y \in \mathbb{X}$ satisfying $\|y\| \le r$, and

$$\liminf_{r \to +\infty} \frac{\Psi_i(r)}{r} = \gamma_i < \infty.$$

Define

$$\mathcal{C}_{\alpha,\beta}(t) := \frac{t^{\alpha - \beta}}{\Gamma(\alpha)} \left(\frac{1 - \beta}{\alpha - \beta}\right)^{1 - \beta}, \quad t \in [0, T].$$

It is not difficult to see that $\lim_{t \to 0+} \mathcal{C}_{\alpha,\beta}(t) = 0$.

REMARK 2.1.   Note that the assumption (H_H) (ii) is the case when the values of the solution $u(t)$ for t near zero do not affect $H(u)$. A case in point was presented in [12], where the operator H is given as follows:

$$H(u) = \sum_{i=1}^{p} C_i u(s_i),$$

where C_i $(i = 1, \ldots, p)$ are given constants and $0 < s_1 < \cdots < s_{p-1} < s_p < +\infty$ $(p \in \mathbb{N})$, which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

3. Main results

Let $m \geq 1$ be fixed. Define an operator Γ^α on $PC(0, T; \mathbb{X})$ by

$$
(\Gamma^\alpha u)(t) = \begin{cases}
T\left(\dfrac{1}{m}\right)T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}T(t-s)F(s, u(s))\,ds, & \text{if } t \in J_0, \\[2ex]
T\left(\dfrac{1}{m}\right)T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}T(t-s)F(s, u(s))\,ds \\[2ex]
\qquad\qquad\qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1}T(t-s)F(s, u(s))\,ds \\[2ex]
+ \displaystyle\sum_{0<t_i<t} T\left(\dfrac{1}{m}\right)T(t-t_i)I_i(u(t_i^-)), & \text{if } t \in J_i,
\end{cases}
$$

where $i = 1, \ldots, n$.

Firstly, we are in a position to show the following result.

Lemma 3.1. *Let the hypotheses* (H_F), (H_H) (i) *and* (H_I) *hold. Then for every* $m \geq 1$, *the operator* Γ^α *has at least a fixed point* $u_m \in PC(0, T; \mathbb{X})$ *provided that*

$$
(3.1) \qquad M\left(\mu + \sigma \mathcal{C}_{\alpha,\beta}(T)(n^{1-\alpha+\beta} + 1) + \sum_{i=1}^n \gamma_i\right) < 1.
$$

Proof. It is clear that $\Gamma^\alpha : PC(0, T; \mathbb{X}) \to PC(0, T; \mathbb{X})$. In the sequel, we prove that there is a positive number ρ such that Γ^α maps Ω_ρ into itself. In fact, if this is not the case, then for each $\rho \in \mathbb{N}$, there would exist $u_\rho \in \Omega_\rho$ and $t_\rho \in [0, T]$ such that $\|(\Gamma^\alpha u_\rho)(t_\rho)\| > \rho$. Thus, from Hölder inequality and the assumptions (H_F), (H_H) (i), (H_I) we deduce that

$$
\begin{aligned}
\rho &< \|(\Gamma^\alpha u_\rho)(t_\rho)\| \\
&\leq \left\|T\left(\frac{1}{m} + t_\rho\right)\right\|_{\mathcal{L}(\mathbb{X})} \|H(u_\rho)\| + \frac{1}{\Gamma(\alpha)} \int_0^{t_\rho} (t_\rho - s)^{\alpha-1} \|T(t_\rho - s)\|_{\mathcal{L}(\mathbb{X})} \|F(s, u_\rho(s))\|\,ds \\
&\leq M\Phi(\rho) + \frac{M}{\Gamma(\alpha)} \int_0^{t_\rho} (t_\rho - s)^{\alpha-1} f_\rho(s)\,ds \\
&\leq M\Phi(\rho) + M\mathcal{C}_{\alpha,\beta}(T)\|f_\rho\|_{L^{1/\beta}(0,T)},
\end{aligned}
$$

for the case when $t_\rho \in J_0$, and

$$
\begin{aligned}
\rho &< \|(\Gamma^\alpha u_\rho)(t_\rho)\| \\
&\leq \left\| T\left(\frac{1}{m} + t_\rho\right) \right\|_{\mathcal{L}(\mathbb{X})} \|H(u_\rho)\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{0 < t_i < t_\rho} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \|T(t_\rho - s)\|_{\mathcal{L}(\mathbb{X})} \|F(s, u_\rho(s))\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t_\rho} (t_\rho - s)^{\alpha-1} \|T(t_\rho - s)\|_{\mathcal{L}(\mathbb{X})} \|F(s, u_\rho(s))\| \, ds \\
&\quad + \sum_{0 < t_i < t_\rho} \left\| T\left(\frac{1}{m} + t_\rho - t_i\right) \right\|_{\mathcal{L}(\mathbb{X})} \|I_i(u_\rho(t_i))\| \\
&\leq M\Phi(\rho) + \frac{M}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f_\rho(s) \, ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_{t_k}^{t_\rho} (t_\rho - s)^{\alpha-1} f_\rho(s) \, ds + M \sum_{i=1}^{k} \Psi_i(\rho) \\
&\leq M\Phi(\rho) + \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} \|f_\rho\|_{L^{1/\beta}(0,T)} \sum_{i=1}^{k} (t_i - t_{i-1})^{\alpha-\beta} \\
&\quad + M\mathcal{C}_{\alpha,\beta}(T) \|f_\rho\|_{L^{1/\beta}(0,T)} + M \sum_{i=1}^{k} \Psi_i(\rho) \\
&\leq M\Phi(\rho) + M\mathcal{C}_{\alpha,\beta}(T) \|f_\rho\|_{L^{1/\beta}(0,T)} (n^{1-\alpha+\beta} + 1) + M \sum_{i=1}^{n} \Psi_i(\rho)
\end{aligned}
$$

for the case when $t_\rho \in J_k$ $(k = 1, \ldots, n)$. Dividing on both sides by ρ and taking the lower limit as $\rho \to \infty$ one has

$$
M\left(\mu + \sigma\mathcal{C}_{\alpha,\beta}(T)(n^{1-\alpha+\beta} + 1) + \sum_{i=1}^{n} \gamma_i\right) \geq 1,
$$

which contradicts (3.1). This means that for some positive integer $\rho > 0$, $\Gamma^\alpha(\Omega_\rho) \subset \Omega_\rho$.

Next, we shall prove that Γ^α is continuous on Ω_ρ. Let $\{u_q\}_{q=1}^\infty \subset \Omega_\rho$ be a sequence such that $u_q \to u$ as $q \to \infty$ on $PC(0, T; \mathbb{X})$. From the continuity of F with respect to second variable it is not difficult to see that

$$
F(s, u_q(s)) \to F(s, u(s)), \quad \text{a.e.} \quad s \in [0, T] \quad \text{as} \quad q \to \infty.
$$

Also, from the assumption (H_F) note that

$$\int_{t_i}^{t} (t-s)^{\alpha-1} f_\rho(s)\, ds \leq \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (T-t_i)^{\alpha-\beta} \| f_\rho \|_{L^{1/\beta}(0,T)} \quad (i = 0, 1, \ldots, n,\ t \geq t_i),$$

$$\int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} f_\rho(s)\, ds \leq \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (t_i-t_{i-1})^{\alpha-\beta} \| f_\rho \|_{L^{1/\beta}(0,T)} \quad (i = 1, \ldots, n).$$

Hence, by the continuity of operators H, I_i $(i = 1, \ldots, n)$, the Lebesgue dominated convergence theorem gives that for each $t \in [0, T]$,

$$\| (\Gamma^\alpha u_q)(t) - (\Gamma^\alpha u)(t) \| \to 0, \quad \text{as} \quad q \to \infty,$$

which implies that

$$\| (\Gamma^\alpha u_q)(t) - (\Gamma^\alpha u)(t) \|_{PC} \to 0, \quad \text{as} \quad q \to \infty.$$

That is to say that Γ^α is continuous on Ω_ρ.

Finally, to be able to apply Schauder's second fixed point theorem to obtain a fixed point of Γ^α, we need to prove that Γ^α is compact on Ω_ρ. Let us decompose the operator Γ^α as follows:

$$\Gamma^\alpha = \Gamma_H^\alpha + \Gamma_F^\alpha + \Gamma_I^\alpha,$$

where

$$(\Gamma_H^\alpha u)(t) = T\left(\frac{1}{m} \right) T(t) H(u), \quad t \in [0, T],$$

$$(\Gamma_F^\alpha u)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds, & \text{if} \quad t \in J_0, \\[2mm] \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds \\[2mm] \qquad\qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1} T(t-s) F(s, u(s))\, ds, \\[2mm] \qquad\qquad\qquad\qquad \text{if} \quad t \in J_i\ (i = 1, \ldots, n), \end{cases}$$

$$(\Gamma_I^\alpha u)(t) = \sum_{0 < t_i < t} T\left(\frac{1}{m} \right) T(t-t_i) I_i(u(t_i)), \quad t \in J_i\ (i = 1, \ldots, n).$$

From the assumption (H_H) (i) and the compactness of $T(t)$ for $t > 0$ we deduce that Γ_H^α which maps Ω_ρ into $PC(0, T; \mathbb{X})$ is compact. Also, by the assumption (H_F) and the compactness of $T(t)$ for $t > 0$ a standard argument yields that $\Gamma_F^\alpha|_{J_0}$ is compact on Ω_ρ.

Let $t \in J_1$ be fixed and $0 < \varepsilon_1 < \min\{t_1, t - t_1\}$. For $u \in \Omega_\rho$, we define the map $\Gamma_F^{\alpha, \varepsilon_1}$ by

$$
\begin{aligned}
(\Gamma_F^{m, \varepsilon_1} u)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1 - \varepsilon_1} (t_1 - s)^{\alpha - 1} T(t - s) F(s, u(s)) \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t - \varepsilon_1} (t - s)^{\alpha - 1} T(t - s) F(s, u(s)) \, ds \\
&= \frac{T(\varepsilon_1)}{\Gamma(\alpha)} \int_0^{t_1 - \varepsilon_1} (t_1 - s)^{\alpha - 1} T(t - \varepsilon_1 - s) F(s, u(s)) \, ds \\
&\quad + \frac{T(\varepsilon_1)}{\Gamma(\alpha)} \int_{t_1}^{t - \varepsilon_1} (t - s)^{\alpha - 1} T(t - \varepsilon_1 - s) F(s, u(s)) \, ds.
\end{aligned}
$$

Since $T(t)$ for $t > 0$ is compact, for each $t \in J_1$, the set

$$
\{(\Gamma_F^{\alpha, \varepsilon_1} u)(t); \, u \in \Omega_\rho, \, 0 < \varepsilon_1 < \min\{t_1, t - t_1\}\}
$$

is precompact in $\mathbb{X}$. Moreover, for $t \in J_1$, from Hölder inequality and the assumption (H_F) it follows that

$$
\begin{aligned}
\|(\Gamma_F^\alpha u)(t) - (\Gamma_F^{\alpha, \varepsilon_1} u)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1 - \varepsilon_1}^{t_1} (t_1 - s)^{\alpha - 1} T(t - s) F(s, u(s)) \, ds \right\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_{t - \varepsilon_1}^{t} (t - s)^{\alpha - 1} T(t - s) F(s, u(s)) \, ds \right\| \\
&\leq \frac{M}{\Gamma(\alpha)} \left(\int_{t_1 - \varepsilon_1}^{t_1} (t_1 - s)^{\alpha - 1} f_\rho(s) \, ds + \int_{t - \varepsilon_1}^{t} (t - s)^{\alpha - 1} f_\rho(s) \, ds \right) \\
&\leq 2M \mathcal{C}_{\alpha, \beta}(\varepsilon_1) \|f_\rho\|_{L^{1/\beta}(0,T)} \to 0 \quad \text{as} \quad \varepsilon_1 \to 0^+.
\end{aligned}
$$

Therefore, using the total boundedness we conclude that for each $t \in J_1$, $\{(\Gamma_F^\alpha u)(t); u \in \Omega_\rho\}$ is precompact in $\mathbb{X}$.

Note that

$$
(\Gamma_F^\alpha u)(t_1^+) = \int_0^{t_1} (t_1 - s)^{\alpha - 1} T(t_1 - s) F(s, u(s)) \, ds.
$$

The same idea can be used to prove that the set $\{(\Gamma_F^\alpha u)(t_1^+); u \in \Omega_\rho\}$ is precompact in $\mathbb{X}$. In fact, for each $0 < \varepsilon_2 < t_1$, the compactness of $T(t)$ for $t > 0$ implies that

$$
\begin{aligned}
&\left\{ \int_0^{t_1 - \varepsilon_2} (t_1 - s)^{\alpha - 1} T(t_1 - s) F(s, u(s)) \, ds; \, u \in \Omega_\rho \right\} \\
&= \left\{ T(\varepsilon_2) \int_0^{t_1 - \varepsilon_2} (t_1 - s)^{\alpha - 1} T(t_1 - \varepsilon_2 - s) F(s, u(s)) \, ds; \, u \in \Omega_\rho \right\}
\end{aligned}
$$

is precompact in $\mathbb{X}$. Thus, the result that the set $\{(\Gamma_F^\alpha u)(t_1^+); u \in \Omega_\rho\}$ is precompact in $\mathbb{X}$ follows from the estimate:

$$\left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} T(t_1 - s) F(s, u(s))\, ds - \int_0^{t_1-\varepsilon_2} (t_1 - s)^{\alpha-1} T(t_1 - s) F(s, u(s))\, ds \right\|$$

$$\leq \left\| \int_{t_1-\varepsilon_2}^{t_1} (t_1 - s)^{\alpha-1} T(t_1 - s) F(s, u(s))\, ds \right\|$$

$$\leq \int_{t_1-\varepsilon_2}^{t_1} (t_1 - s)^{\alpha-1} f_\rho(s)\, ds$$

$$\leq M \mathcal{C}_{\alpha,\beta}(\varepsilon_2) \| f_\rho \|_{L^{1/\beta}(0,T)}$$

$$\to 0 \quad \text{as} \quad \varepsilon_1 \to 0^+.$$

Next, we show the equicontinuity of $\{(\Gamma_F^\alpha u)(\cdot); \cdot \in \overline{J}_1, u \in \Omega_\rho\}$. Let $u \in \Omega_\rho$, $s_1, s_2 \in \overline{J}_1$, $s_1 < s_2$ and $\delta > 0$ be small enough. We put

$$h_1 = \frac{1}{\Gamma(\alpha)} \int_0^{t_1-\delta} (t_1 - s)^{\alpha-1} \| T(s_2 - s) - T(s_1 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

$$h_2 = \frac{1}{\Gamma(\alpha)} \int_{t_1-\delta}^{t_1} (t_1 - s)^{\alpha-1} \| T(s_2 - s) - T(s_1 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

and put

$$h_3 = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{s_1-\delta} (s_2 - s)^{\alpha-1} \| T(s_2 - s) - T(s_1 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

$$h_4 = \frac{1}{\Gamma(\alpha)} \int_{s_1-\delta}^{s_1} (s_2 - s)^{\alpha-1} \| T(s_2 - s) - T(s_1 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

$$h_5 = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{s_1} ((s_1 - s)^{\alpha-1} - (s_2 - s)^{\alpha-1}) \cdot \| T(s_1 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds$$

$$\quad + \frac{1}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2 - s)^{\alpha-1} \| T(s_2 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

for the case when $s_1 > t_1$, and

$$h_6 = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{s_2} (s_2 - s)^{\alpha-1} \| T(s_2 - s) \|_{\mathcal{L}(\mathbb{X})} \| F(s, u(s)) \|\, ds,$$

for the case when $s_1 = t_1$. Then, making use of the assumption (H_F) we obtain

$$h_1 \leq \frac{1}{\Gamma(\alpha)} \| T(s_2 - s_1 + \delta) - T(\delta) \|_{\mathcal{L}(\mathbb{X})} \int_0^{t_1-\delta} (t_1 - s)^{\alpha-1} \| T(s_1 - s - \delta) \|_{\mathcal{L}(\mathbb{X})} f_\rho(s)\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (t_1^{(\alpha-\beta)/(1-\beta)} - \delta^{(\alpha-\beta)/(1-\beta)})^{1-\beta}$$

$$\quad \times \| T(s_2 - s_1 + \delta) - T(\delta) \|_{\mathcal{L}(\mathbb{X})} \| f_\rho \|_{L^{1/\beta}(0,T)},$$

$$h_2 \leq \frac{2M}{\Gamma(\alpha)} \int_{t_1-\delta}^{t_1} (t_1-s)^{\alpha-1} f_\rho(s)\, ds \leq 2M\mathcal{C}_{\alpha,\beta}(\delta)\|f_\rho\|_{L^{1/\beta}(0,T)},$$

$$h_3 \leq \frac{1}{\Gamma(\alpha)} \|T(s_2-s_1+\delta)-T(\delta)\|_{\mathcal{L}(\mathbb{X})} \int_{t_1}^{s_1-\delta} (s_2-s)^{\alpha-1}\|T(s_1-s-\delta)\|_{\mathcal{L}(\mathbb{X})} f_\rho(s)\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} ((s_2-t_1)^{(\alpha-\beta)/(1-\beta)}-(s_2-s_1+\delta)^{(\alpha-\beta)/(1-\beta)})^{1-\beta}$$

$$\times \|T(s_2-s_1+\delta)-T(\delta)\|_{\mathcal{L}(\mathbb{X})}\|f_\rho\|_{L^{1/\beta}(0,T)},$$

$$h_4 \leq \frac{2M}{\Gamma(\alpha)} \int_{s_1-\delta}^{s_1} (s_2-s)^{\alpha-1} f_\rho(s)\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} ((s_2-s_1+\delta)^{(\alpha-\beta)/(1-\beta)}-(s_2-s_1)^{(\alpha-\beta)/(1-\beta)})^{1-\beta}\|f_\rho\|_{L^{1/\beta}(0,T)},$$

$$h_5 \leq \frac{M}{\Gamma(\alpha)} \int_{t_1}^{s_1} ((s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}) f_\rho(s)\, ds + \frac{M}{\Gamma(\alpha)} \int_{s_1}^{s_2} (s_2-s)^{\alpha-1} f_\rho(s)\, ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \left\{\int_{t_1}^{s_1} ((s_1-s)^{(\alpha-1)/(1-\beta)}-(s_2-s)^{(\alpha-1)/(1-\beta)})\, ds\right\}^{1-\beta}\|f_\rho\|_{L^{1/\beta}(0,T)}$$

$$+ \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} (s_2-s_1)^{\alpha-\beta}\|f_\rho\|_{L^{1/\beta}(0,T)}$$

$$\leq \frac{M}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\|f_\rho\|_{L^{1/\beta}(0,T)}$$

$$\times \{((s_1-t_1)^{(\alpha-\beta)/(1-\beta)}+(s_2-s_1)^{(\alpha-\beta)/(1-\beta)}-(s_2-t_1)^{(\alpha-\beta)/(1-\beta)})^{1-\beta}+(s_2-s_1)^{\alpha-\beta}\},$$

$$h_6 \leq \frac{M}{\Gamma(\alpha)} \int_{t_1}^{s_2} (s_2-s)^{\alpha-1} f_\rho(s)\, ds \leq M\mathcal{C}_{\alpha,\beta}(s_2-t_1)\|f_\rho\|_{L^{1/\beta}(0,T)}.$$

This together with the fact that the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology shows that h_i $(i = 1, \ldots, 6)$ tend to zero as $s_2 \to s_1$ and $\delta \to 0^+$.

Note that

$$\|(\Gamma_F^\alpha u)(s_2) - (\Gamma_F^\alpha u)(s_1)\| \leq \begin{cases} h_1 + h_2 + h_3 + h_4 + h_5, & \text{if} \quad t_1 < s_1, \\ h_1 + h_2 + h_6, & \text{if} \quad t_1 = s_1. \end{cases}$$

Therefore, we obtain that $\{(\Gamma_F^\alpha u)(\cdot);\, u \in \Omega_\rho\}$ is equicontinuous on $\overline{J}_1$. Hence, applying the Arzela–Ascoli theorem we can conclude that $\Gamma_F^\alpha|_{\overline{J}_1}$ is compact on Ω_ρ. The same idea can be used to prove that for each $i = 2, \ldots, n$, $\Gamma_F^\alpha|_{\overline{J}_i}$ is also compact on Ω_ρ.

Finally, to prove the compactness of the operator Γ_I^α, we note that for every $t \in \overline{J}_i$ $(i = 1, \ldots, n)$,

$$\left\{T\left(\frac{1}{m}\right)T(t-t_i)I_i(u(t_i)),\, u \in \Omega_\rho\right\}\bigg|_{\overline{J}_i}$$

is relatively compact in $\mathbb{X}$ due to the compactness of $T(t)$ for $t > 0$. And, a direct calculation yields

$$\left\| T\left(\frac{1}{m}\right)T(t-t_i)I_i(u(t_i)) - T\left(\frac{1}{m}\right)T(s-t_i)I_i(u(t_i)) \right\|$$

$$= \left\| T\left(\frac{1}{m}\right)T(s-t_i)(T(t-s)-T(0))I_i(u(t_i)) \right\|$$

$$\leq M\left\| (T(t-s)-T(0))T\left(\frac{1}{m}\right)I_i(u(t_i)) \right\|$$

for $t_i \leq s < t \leq t_{i+1}$, $i = 1,\ldots,n$. This together with the strong continuity of $T(t)$ yields that on $\overline{J}_i$, $i = 1, \ldots, n$, $\{T(1/m)T(t-t_i)I_i(u(t_i)), \; u \in \Omega_\rho\}$ are equicontinuous. Thus, the Arzela–Ascoli theorem indicates that the operator Γ_I^α is compact and hence Γ^α is compact on Ω_ρ. Consequently, we can make use of Schauder's fixed point theorem to deduce that for each $m \geq 1$, Γ^α has at least a fixed point $u_m \in \Omega_\rho$. The proof is then complete. $\square$

We now return to the problem (1.1). One of our main results in this paper is the following theorem.

Theorem 3.1. *Let the hypotheses in* Lemma 3.1 *and hypothesis* (H_H) (ii) *hold. Then problem* (1.1) *has at least one solution.*

Proof.   We proceed in two steps.

STEP 1.   We show that the set $\{u_m\}_{m=1}^\infty$ is precompact in $PC(0, T; \mathbb{X})$.

Let $\eta \in (0, \varsigma)$ be fixed, where ς is the constant in the assumption (H_H) (ii). Firstly, from the compactness of $T(t)$ for $t > 0$ and the assumption (H_H) (i) it is not difficult to see that for each $t \in (0, T]$, the set $\{T(t)T(1/m)H(u_m), m \geq 1\}$ is precompact in $\mathbb{X}$, which together with the strong continuity of $T(t)$ yields that for the case when $s_1, s_2 \in [\eta, t_1]$, $s_1 \leq s_2$,

$$\left\| T(s_2)T\left(\frac{1}{m}\right)H(u_m) - T(s_1)T\left(\frac{1}{m}\right)H(u_m) \right\|$$

$$= \left\| (T(s_2-\eta) - T(s_1-\eta))T(\eta)T\left(\frac{1}{m}\right)H(u_m) \right\|$$

$$\to 0, \quad \text{as} \quad s_2 \to s_1,$$

uniformly for $m \geq 1$, and for the case when $s_1, s_2 \in \overline{J}_i$, $s_1 \leq s_2$ $(i = 1, \ldots, n)$,

$$\left\| T(s_2)T\left(\frac{1}{m}\right)H(u_m) - T(s_1)T\left(\frac{1}{m}\right)H(u_m) \right\|$$

$$= \left\| (T(s_2-t_i) - T(s_1-t_i))T(t_i)T\left(\frac{1}{m}\right)H(u_m) \right\|$$

$$\to 0, \quad \text{as} \quad s_2 \to s_1,$$

uniformly for $m \geq 1$. Therefore, in view of Arzela–Ascoli theorem we deduce that the set $\{T(t)T(1/m)H(u_m), \ m \geq 1\}|_{[\eta,t_1]}$ is precompact in $C([\eta, t_1]; \mathbb{X})$ and for each $i = 1, \ldots, n$, the set $\{T(t)T(1/m)H(u_m), \ m \geq 1\}|_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; \mathbb{X})$. Put

$$
(Eu_m)(t) =
\begin{cases}
\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u_m(s)) \, ds, & \text{if} \quad t \in J_0, \\[2ex]
\dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} T(t-s) F(s, u_m(s)) \, ds \\[2ex]
\qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1} T(t-s) F(s, u_m(s)) \, ds, \\[2ex]
\qquad\qquad\qquad \text{if} \quad t \in J_i, \ i = 1, \ldots, n.
\end{cases}
$$

Then, the same idea with the proof of Lemma 3.1 can be used to prove that the set $\{Eu_m, \ m \geq 1\}|_{J_0}$ is precompact in $C(J_0; \mathbb{X})$ and for each $i = 1, \ldots, n$, $\{Eu_m, \ m \geq 1\}|_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; \mathbb{X})$.

In what follows, we consider the set

$$
\left\{ \sum_{0 < t_i < t} T\left(\frac{1}{m}\right) T(t - t_i) I_i(u_m(t_i)), \ m \geq 1 \right\}, \quad i = 1, \ldots, n.
$$

Notice that for $t \in J_0$,

$$
u_m(t) = T(t) T\left(\frac{1}{m}\right) H(u_m) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u_m(s)) \, ds
$$

$$
= T(t) T\left(\frac{1}{m}\right) H(u_m) + (Eu_m)(t)|_{J_0}.
$$

Then, from the argument above we see that for any $\eta \leq \tilde{\eta} < t_1$, $\{u_m; \ m \geq 1\}|_{[\tilde{\eta}, t_1]}$ is precompact in $C([\eta, t_1]; \mathbb{X})$. Without loss of generality, we let

(3.2) $$u_m \to u \quad \text{in} \quad C([\tilde{\eta}, t_1]; \mathbb{X})$$

as $m \to \infty$, which implies in particular that

$$u_m(t_1) \to u(t_1) \quad \text{in} \quad \mathbb{X},$$

as $m \to \infty$, that is, the set $\{u_m(t_1); \ m \geq 1\}$ is compact in $\mathbb{X}$. This together with the

assumption (H_I) and the strong continuity of $T(t)$ concludes that

$$\left\| I_1(u(t_1)) - T\left(\frac{1}{m}\right) I_1(u_m(t_1)) \right\|$$

$$\leq \left\| I_1(u(t_1)) - T\left(\frac{1}{m}\right) I_1(u(t_1)) \right\| + \left\| T\left(\frac{1}{m}\right) I_1(u(t_1)) - T\left(\frac{1}{m}\right) I_1(u_m(t_1)) \right\|$$

$$\leq \left\| I_1(u(t_1)) - T\left(\frac{1}{m}\right) I_1(u(t_1)) \right\| + M \| I_1(u(t_1)) - I_1(u_m(t_1)) \|$$

$$\to 0 \quad \text{as} \quad m \to \infty,$$

which implies that the set $\{T(1/m)I_1(u_m(t_1)); \ m \geq 1\}$ is relatively compact in $\mathbb{X}$. Since $T(t)$ is compact for $t > 0$, for each $t \in \overline{I}_1$ the set $\{T(1/m)T(t-t_1)I_1(u_m(t_1)); \ m \geq 1\}$ is also relatively compact in $\mathbb{X}$. On the other hand, for $s_1, s_2 \in \overline{J}_1$, $s_1 \leq s_2$, from the compactness of the set $\{u_m(t_1); \ m \geq 1\}$ and the strong continuity of $T(t)$ we get

$$\left\| T\left(\frac{1}{m}\right) T(s_2 - t_1) I_1(u_m(t_1)) - T\left(\frac{1}{m}\right) T(s_1 - t_1) I_1(u_m(t_1)) \right\|$$

$$\leq \left\| T\left(\frac{1}{m} + s_1 - t_1\right) \right\|_{\mathcal{L}(\mathbb{X})} \|(T(s_2 - s_1) - I) I_1(u_m(t_1))\|$$

$$\leq M \|(T(s_2 - s_1) - I) I_1(u_m(t_1))\|$$

$$\to 0, \quad \text{as} \quad s_2 \to s_1,$$

uniformly for $m \geq 1$. Therefore, in view of Arzela–Ascoli theorem we find that $\{T(1/m)T(t-t_1)I_1(u_m(t_1)); \ m \geq 1\}_{\overline{J}_1}$ is precompact in $C(\overline{J}_1; \mathbb{X})$. A similar argument enable us to conclude that for each $i = 2, \dots, n$,

$$\left\{ \sum_{0 < t_i < t} T\left(\frac{1}{m}\right) T(t - t_i) I_i(u_m(t_i)), \ m \geq 1 \right\}\bigg|_{J_i}$$

is precompact in $C(\overline{J}_i; \mathbb{X})$.

Finally, to prove that the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $PC(0, T; \mathbb{X})$, it will suffice to show that the set

$$\left\{ T(t) T\left(\frac{1}{m}\right) H(u_m), m \geq 1 \right\}\bigg|_{[0,\eta]}$$

is precompact in $C([0, \eta]; \mathbb{X})$. Let ς be the constant in the assumption (H_H) (ii). Write

$$\tilde{u}_m(t) = \begin{cases} u_m(t) & \text{if} \quad t \in [\varsigma, T], \\ u_m(\varsigma) & \text{if} \quad t \in [0, \varsigma]. \end{cases}$$

Then, from (3.2) we may assume, without loss of generality, that

$$\tilde{u}_m \to u \quad \text{in} \quad PC(0, T; \mathbb{X}),$$

as $m \to \infty$. Thus, from the continuity of operator H and the strong continuity of $T(t)$ we get

$$
\begin{aligned}
\left\| H(u) - T\left(\frac{1}{m}\right) H(u_m) \right\| &= \left\| H(u) - T\left(\frac{1}{m}\right) H(\tilde{u}_m) \right\| \\
&\leq \left\| H(u) - T\left(\frac{1}{m}\right) H(u) \right\| + \left\| T\left(\frac{1}{m}\right)(H(u) - H(\tilde{u}_m)) \right\| \\
&\leq \left\| H(u) - T\left(\frac{1}{m}\right) H(u) \right\| + M\|H(u) - H(\tilde{u}_m)\| \\
&\to 0 \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

which implies that the set $\{T(1/m)H(u_m), \, m \geq 1\}$ is relatively compact in $\mathbb{X}$. This together with the strong continuity of $T(t)$ concludes that for $s_1, s_2 \in [0, \eta]$, $s_1 \leq s_2$,

$$
\begin{aligned}
&\left\| T(s_2) T\left(\frac{1}{m}\right) H(u_m) - T(s_1) T\left(\frac{1}{m}\right) H(u_m) \right\| \\
&\leq \left\| (T(s_2) - T(s_1)) T\left(\frac{1}{m}\right) H(u_m) \right\| \\
&\to 0 \quad \text{as} \quad s_2 \to s_1,
\end{aligned}
$$

uniformly for $m \geq 1$. Consequently, we conclude that the set $\{T(t)T(1/m)H(u_m), \, m \geq 1\}|_{[0,\eta]}$ is precompact in $C([0, \eta]; \mathbb{X})$ and hence the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $PC(0, T; \mathbb{X})$.

STEP 2. Since the set $\{u_m\}_{m=1}^{\infty}$ is precompact in $PC(0, T; \mathbb{X})$, there is a subsequence $\{u_{m_j}\}$ and there is a $u \in PC(0, T; \mathbb{X})$ such that $u_{m_j} \to u$ in $PC(0, T; \mathbb{X})$ as $j \to \infty$. Note that $u_{m_j} \in PC(0, T; \mathbb{X})$ satisfies the integral equation

$$
u_{m_j}(t) = \begin{cases}
T\left(\dfrac{1}{m_j}\right) T(t) H(u_{m_j}) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s, u_{m_j}(s))\, ds, \\
\hfill \text{if} \quad t \in J_0, \\
T\left(\dfrac{1}{m_j}\right) T(t) H(u_{m_j}) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0 < t_i < t} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} T(t-s) F(s, u_{m_j}(s))\, ds \\
\hfill + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1} T(t-s) F(s, u_{m_j}(s))\, ds \\
+ \displaystyle\sum_{0 < t_i < t} T\left(\dfrac{1}{m_j}\right) T(t-t_i) I_i(u_{m_j}(t_i^-)), \hfill \text{if} \quad t \in J_i,
\end{cases}
$$

where $i = 1, \ldots, n$. Letting $j \to \infty$ one has that u is one solution of problem (1.1). This completes the proof. $\qquad\square$

Theorem 3.2.   *Let the hypotheses* (H_F), (H'_H) *and* (H_I) *hold. Then problem* (1.1) *has at least one solution provided that*

$$(3.3) \qquad M\left(L_H + \sigma \mathcal{C}_{\alpha,\beta}(T)(n^{1-\alpha+\beta} + 1) + \sum_{i=1}^{n} \gamma_i\right) < 1,$$

Proof.   We proceed in two steps.

STEP 1.   Let $m \geq 1$ be fixed. Consider the operator Γ^α: $PC(0,T;\mathbb{X}) \to PC(0,T;\mathbb{X})$ which is defined by

$$(\Gamma^\alpha u)(t) = \begin{cases} T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) F(s,u(s))\, ds, \qquad \text{if}\quad t \in J_0, \\[2mm] T(t)H(u) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} T(t-s) F(s,u(s))\, ds \\[2mm] \qquad\qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_i}^t (t-s)^{\alpha-1} T(t-s) F(s,u(s))\, ds \\[2mm] + \displaystyle\sum_{0<t_i<t} T\left(\dfrac{1}{m}\right) T(t-t_i) I_i(u(t_i^-)), \qquad \text{if}\quad t \in J_i\ (i=1,\ldots,n), \end{cases}$$

we will prove that Γ^α has a fixed point. Firstly, from Hölder inequality and the assumptions (H_F), (H'_H), (H_I) we infer that for $t \in J_k$ $(k=1,\ldots,n)$,

$$\|(\Gamma^\alpha u)(t)\|$$

$$\leq \|T(t)\|_{\mathcal{L}(\mathbb{X})} \|H(u)\| + \frac{1}{\Gamma(\alpha)} \sum_{0<t_i<t} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1} \|T(t-s)\|_{\mathcal{L}(\mathbb{X})} \|F(s,u(s))\|\, ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} \|T(t-s)\|_{\mathcal{L}(\mathbb{X})} \|F(s,u(s))\|\, ds$$

$$+ \sum_{0<t_i<t} \left\| T\left(\frac{1}{m} + t - t_i\right) \right\|_{\mathcal{L}(\mathbb{X})} \|I_i(u(t_i))\|$$

$$\leq M(L_H \rho + H(0)) + M\mathcal{C}_{\alpha,\beta}(T) \|f_\rho\|_{L^{1/\beta}(0,T)} (k^{1-\alpha+\beta} + 1) + M \sum_{i=1}^{k} \Psi_i(\rho).$$

Therefore, an application of the same idea with the proof of Lemma 3.1 together with condition (3.3) gives that there exists a positive number ρ such that Γ^α maps Ω_ρ into itself. Moreover, similarly to the proof of Lemma 3.1 we can also deduce, in view of the continuity of F with respect to second variable and H, I_i $(i=1,\ldots,n)$, that Γ^α is continuous on Ω_ρ.

Now, we assume that the operators Γ_F^α, Γ_I^α are defined by the same as in Lemma 3.1. From the proof of Lemma 3.1 note that the operators Γ_F^α, Γ_I^α are compact on Ω_ρ.

Write

$$(\Gamma'_H u)(t) = T(t)H(u), \quad t \in [0, T].$$

Then from the assumption (H'_H) it follows that

$$\|(\Gamma'_H u)(t) - (\Gamma'_H w)(t)\| \le \|T(t)\|_{\mathcal{L}(\mathbb{X})}\|H(u) - H(w)\|$$
$$\le ML_H\|u - v\|_{PC}, \quad \text{for} \quad u, w \in \Omega_\rho,$$

which yields that

(3.4) $\|(\Gamma'_H u)(t) - (\Gamma'_H w)(t)\|_{PC} \le ML_H\|u - v\|_{PC}, \quad \text{for} \quad u, w \in \Omega_\rho.$

That is, Γ' is Lipschitz continuous on Ω_ρ. Since the condition (3.3) implies that $ML_H < 1$, the operator $\Gamma^\alpha = \Gamma'_H + \Gamma^\alpha_F + \Gamma^\alpha_I$ is an α-contraction on Ω_ρ. Therefore, applying the Darbo-Sadovskii's fixed point theorem [3] we deduce that for each $m \ge 1$, the operator Γ^α has at least a fixed point $u_m \in \Omega_\rho$.

STEP 2.   Consider the set $\{u_m\}_{m=1}^\infty$. Since for each $m \ge 1$, u_m satisfies the integral equation

$$u_m(t) = T(t)H(u_m) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)F(s, u_m(s))\,ds, \quad \text{for} \quad t \in J_0,$$

from our hypotheses it is not difficult to see that for each $t \in (0, t_1]$, the set $\{u_m(t)\}_{m=1}^\infty$ is relatively compact (cf. the proof of Lemma 3.1). This yields in particular that there is a subsequence $\{u_{m_j}(t_1)\}_{j=1}^\infty$ and there is a u_0 such that

$$u_{m_j}(t_1) \to u_0 \quad \text{in} \quad X,$$

as $j \to \infty$, which together with the strong continuity of $T(t)$ and the continuity of I_1 implies that

$$\left\| I_1(u_0) - T\left(\frac{1}{m_j}\right)I_1(u_{m_j}(t_1)) \right\|$$
$$\le \left\| I_1(u_0) - T\left(\frac{1}{m_j}\right)I_1(u_0) \right\| + M\|I_1(u_0) - I_1(u_{m_j}(t_1))\|$$
$$\to 0, \quad \text{as} \quad j \to \infty.$$

That is, the set $\{T(1/m_j)I_1(u_{m_j}(t_1))\}_{j=1}^\infty$ is relatively compact in $\mathbb{X}$. Since $T(t)$ for $t > 0$ is compact, for each $t \in J_0$ the set $\{T(1/m_j)T(t-t_1)I_1(u_{m_j}(t_1))\}_{j=1}^\infty$ is also relatively compact in $\mathbb{X}$. Moreover, from the compactness of the set $\{T(1/m_j)I_1(u_{m_j}(t_1))\}_{j=1}^\infty$ and

the strong continuity of $T(t)$ note that for $s_1, s_2 \in \overline{J}_1$, $s_1 \leq s_2$,

$$\left\| T\left(\frac{1}{m_j}\right) T(s_2 - t_1) I_1(u_{m_j}(t_1)) - T\left(\frac{1}{m_j}\right) T(s_1 - t_1) I_1(u_{m_j}(t_1)) \right\|$$

$$\leq \|T(s_1 - t_1)\|_{\mathcal{L}(\mathbb{X})} \left\| (T(s_2 - s_1) - I) T\left(\frac{1}{m_j}\right) I_1(u_{m_j}(t_1)) \right\|$$

$$\leq M \left\| (T(s_2 - s_1) - I)\left[T\left(\frac{1}{m_j}\right) I_1(u_{m_j}(t_1)) \right] \right\|$$

$$\to 0, \quad \text{as} \quad s_2 \to s_1,$$

uniformly for $j \geq 1$, which implies that the set $\{T(1/m_j)T(\cdot - t_1) I_1(u_{m_j}(t_1)); \cdot \in \overline{J}_1, j \geq 1\}$ is equicontinuous on $\overline{J}_1$. Hence, using the Arzela–Ascoli theorem we have that the set $\{T(1/m_j)T(t - t_1) I_1(u_{m_j}(t_1)); j \geq 1\}|_{\overline{J}_1}$ is precompact in $C(\overline{J}_1; \mathbb{X})$. By a similar argument it follows that for each $i = 2, \ldots, n$, $\left\{ \sum_{0 < t_i < t} T(1/m)T(t - t_i) I_i(u_m(t_i)), m \geq 1 \right\}\big|_{J_i}$ is precompact in $C(\overline{J}_i; \mathbb{X})$.

Let $\alpha(\cdot)$ stand for the Hausdorff measure of noncompactness (see [3]). Then from the argument above and (3.4) we have

$$\alpha(\{u_{m_j}\}_{j=1}^{\infty}) \leq ML_H \alpha(\{u_{m_j}\}_{j=1}^{\infty}),$$

which together with the fact that the condition (3.3) implies that $ML_H < 1$ yields that

$$\alpha(\{u_{m_j}\}_{j=1}^{\infty}) = 0.$$

That is to say that the set $\{u_{m_j}\}_{j=1}^{\infty}$ is precompact in $PC(0, T; \mathbb{X})$. Thus, we may suppose without loss of generality that

$$u_{m_j} \to u \quad \text{in} \quad PC(0, T; \mathbb{X})$$

as $j \to \infty$, and the same reason with the last proof of Theorem 3.1 can conclude that the limit u is one solution of problem (1.1). This completes the proof. $\square$

4. An example

In this section, we present an example, which does not aim at generality but indicates how our theorems can be applied to concrete problem.

Let $\alpha = 1/2$, $\mathbb{X} = L^2[0, \pi]$, and the operators $A \colon D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be defined by

$$D(A) = \{u \in \mathbb{X}; u, u' \text{ are absolutely continuous, } u'' \in \mathbb{X}, \text{ and } u(0) = u(\pi) = 0\},$$

$$A = \frac{\partial^2}{\partial x^2}.$$

Clearly, A generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$ and the semigroup generated by A is compact, analytic and $\|T(t)\|_{\mathcal{L}(\mathbb{X})} \leq e^{-t}$ for all $t \geq 0$ (see [18, 27]).

Define

$$F(t, u(t, x)) = \frac{\sin u(t, x)}{t^{1/3}},$$

$$I_i(u(t_i^-, x)) = \frac{u(t_i^-, x)}{1 + u(t_i^-, x)},$$

$$H(u(\cdot, x)) = u_0(x) + \sum_{i=1}^{p} C_i u(s_i, x),$$

where C_i $(i = 1, \ldots, p)$ are given constants, and $0 < s_1 < \cdots < s_{p-1} < s_p < T$ and $0 < t_1 < t_2 < \cdots < t_n < T$ are pre-fixed numbers.

Then, the hypotheses (H_F), (H_H) and (H_I) hold with

$$\frac{1}{3} < \beta < \frac{1}{2}, \quad f_r(t) = t^{-1/3},$$

$$\Psi_i(r) = 1 \ (i = 1, \ldots, n), \quad \Phi(r) = \|u_0\|_{L^2[0,\pi]} + r \sum_{i=1}^{p} |C_i|,$$

$$\sigma = 0, \quad \gamma_i = 0 \ (i = 1, \ldots, n), \quad \mu = \sum_{i=1}^{p} |C_i|.$$

Hence, when $|C_i|$ $(i = 1, \ldots, p)$ are small enough such that the condition (3.1) is satisfied, the corresponding problem has at least one solution due to Theorem 3.1.

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