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BUBBLETONS ARE NOT EMBEDDED

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Abstract

We discuss constant mean curvature bubbletons in Euclidean 3-space via dressing with simple factors, and prove that single-bubbletons are not embedded.

Introduction

A key feature of an integrable system is the presence of an algebraic transformation method which generates new solutions from old ones. In particular even by starting with a trivial solution one obtains a hierarchy of interesting global solutions. For the KdV equation one thus obtains the solitons via a Bäcklund transform. Solitons are solitary traveling waves with localized energy that are stable when interacting with each other. Many of the modern techniques in integrable systems theory stem from classical surface theory, developed by Bäcklund, Bianchi and Darboux amongst others for the structure equations of special surface classes.

Away from umbilic points the structure equation of a constant mean curvature (CMC) surface is the sinh-Gordon equation, whose trivial solution gives rise to the round cylinder. The term 'bubbleton' is due to Sterling and Wente [19], and the bubbletons are the solitons of the sinh-Gordon equation. The single-bubbletons are obtained by transforming the round cylinder by a Bianchi–Bäcklund transform. The resulting transformed CMC cylinder globally looks like the round cylinder except for a localized part in which bubble-like pieces are glued into the underlying surface, see Fig. 1.1. A video of how bubbles interact when they move through each other can be seen at [15].

Recently the classical transformations have received a treatment from the modern point of view of dressing [2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 21]. Bubbletons can be realized by dressing the round cylinder by a class of very simple maps, called simple factors [21]. By repeatedly applying the Bianchi–Bäcklund transformation to the round cylinder one produces the 'multi-bubbletons' classified by Sterling and Wente [19]. While graphics of bubbletons clearly suggest that they are not embedded (see Fig. 1.1), there does not seem to be a direct proof of this fact in the literature. An indirect proof when the target is the 3-sphere is given in [10].

The purpose here is to prove that no member in the infinite family of single-bubbletons is embedded. This is done by showing that all single-bubbletons possess

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a planar curve which is not homologous to zero on the surface. It is shown that this curve has turning number at least three, which implies that the surface cannot be embedded. We reveal how the choice of the 'singularity' in the simple factor is reflected in the geometry of the resulting bubbleton. The monotone sequence of the singularities are indexed by an integer $K \in \mathbb{N}$ for $K \geq 2$, and K is the number of 'bubbles' of the bubbleton, as $2K-1$ turns out to be the turning number of the planar curve.

1. The round cylinder

If $\mathbb{i} = \sqrt{-1}$, then in the spinor representation of Euclidean 3-space $\mathbb{R}^3$ we identify $\mathbb{R}^3 \cong \mathfrak{su}_2$ via

$$(x_1, x_2, x_3) \cong \begin{pmatrix} \mathbb{i}x_3 & x_1 + \mathbb{i}x_2 \\ -x_1 + \mathbb{i}x_2 & -\mathbb{i}x_3 \end{pmatrix}.$$

The extended frame of a round cylinder (up to isometry and conformal change of coordinate) is

$$F_\lambda(z) = \begin{pmatrix} \cos \mu_\lambda & \mathbb{i}\lambda^{-1/2} \sin \mu_\lambda \\ \mathbb{i}\lambda^{1/2} \sin \mu_\lambda & \cos \mu_\lambda \end{pmatrix}, \tag{1.1}$$

where

$$\mu_\lambda = \mu_\lambda(z) = \frac{\pi}{2}(z\lambda^{-1/2} + \bar{z}\lambda^{1/2}). \tag{1.2}$$

The Sym–Bobenko formula [20, 1] for a CMC surface in Euclidean 3-space $\mathbb{R}^3$ is a formula of the immersion in terms of its extended frame F_λ. In our conventions [18], the associated family with constant mean curvature $H \in \mathbb{R}^\times$ is given by

$$f_\lambda(z) = -2\mathbb{i}\lambda H^{-1} F'_\lambda(z) F_\lambda^{-1}(z) \tag{1.3}$$

where F'_λ denotes the derivative with respect to λ. If we pick one member of the associated family f_λ with $\lambda \in \mathbb{S}^1$ and choose $\lambda = 1$, and insert the extended frame (1.1) of the round cylinder, we obtain

$$\begin{aligned} f_1(x, y) &= H^{-1} \begin{pmatrix} \mathbb{i} \sin^2(\pi x) & -\cos(\pi x) \sin(\pi x) - \pi \mathbb{i} y \\ \cos(\pi x) \sin(\pi x) - \pi \mathbb{i} y & -\mathbb{i} \sin^2(\pi x) \end{pmatrix} \\ &\cong \frac{1}{2H} \begin{pmatrix} \sin(2\pi x) \\ -2\pi y \\ 1 - \cos(2\pi x) \end{pmatrix}. \end{aligned}$$

This is clearly a round cylinder which is generated by a circle in the x_1x_3-plane of radius $1/(2|H|)$ centered at the point $(0, 1/(2H))$ parallel translated along the x_2-axis. Any

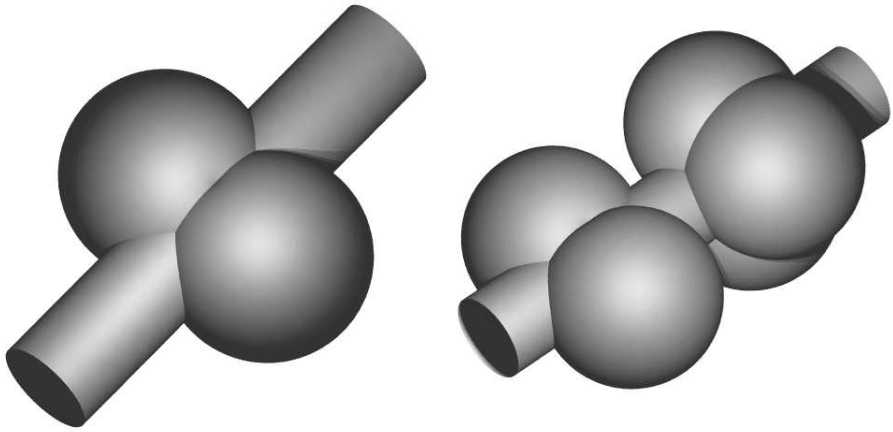

Fig. 1.1. Parts of a two-lobed single-bubbleton, and a multi-bubbleton with 2 and 3 lobes. Further graphics of bubbletons can be viewed at [16, 17].

curve $x_2 \equiv c$ for some constant $c \in \mathbb{R}$ is not contractible on the cylinder. By restricting the x-coordinate to any interval of length one we obtain an embedded round cylinder.

In general, or when the parametrization is less explicit, one can describe the period problem as follows. Suppose we have an extended frame F_λ and an associated family of CMC surfaces as in (1.3). Then periodicity $f_\lambda(z+\tau) = f_\lambda(z)$ for all $z \in \mathbb{C}$ can in general not hold for all $\lambda \in \mathbb{S}^1$. But if we fix $\lambda_0 \in \mathbb{S}^1$, then periodicity reads

$$F'_{\lambda_0}(z+\tau)F^{-1}_{\lambda_0}(z+\tau) = F'_{\lambda_0}(z)F^{-1}_{\lambda_0}(z). \tag{1.4}$$

If we define the monodromy matrix $M_\lambda(\tau)$ with respect to the translation $z \mapsto z+\tau$ of F_λ by

$$M_\lambda(\tau) = F_\lambda(z+\tau)F^{-1}_\lambda(z) \tag{1.5}$$

then the period problem (1.4) is equivalent to

$$M_{\lambda_0}(\tau) = \pm\mathbb{1} \quad \text{and} \quad M'_{\lambda_0}(\tau) = 0. \tag{1.6}$$

The monodromy matrix is not well defined, since it depends on the choice of a base point, and so is only defined up to conjugacy. However, the periodicity conditions (1.6) are invariant under conjugation.

Since we need some of the above in the special case of the round cylinder, let us specialize again to this case. If F_λ is the extended frame (1.1) of a round cylinder, we choose the base point $z_0 = 0$ and note that $F_\lambda(0) = \mathbb{1}$ for all $\lambda \in \mathbb{C}^\times$. As before pick

$\lambda_0 = 1$. The monodromy of $F_\lambda(z)$ with respect to the translation $\tau\colon z \mapsto z+1$ is then

$$M_\lambda(\tau) = F_\lambda(1), \tag{1.7}$$

and a quick computation confirms that $F_1(1) = -\mathbb{1}$ and $F_1'(1) = 0$.

2. Simple factors

There is a deformation technique in the theory of harmonic maps called *dressing* [4]. In particular, dressing by specific very simple maps corresponds to the classical Bianchi–Bäcklund transformation [14], and amounts to adding 'bubbletons' to the standard round cylinder—these simple maps are called *simple factors* [21]. Let us briefly review the theory of simple factors in the context of CMC surfaces in $\mathbb{R}^3$. Let $\pi_L\colon \mathbb{C}^2 \to L$ be the hermitian projection onto a line $L \in \mathbb{CP}^1$, and $\pi_L^\perp = \mathbb{1} - \pi_L$. For $\alpha \in \mathbb{C}^\times$, set

$$\psi_{L,\alpha}(\lambda) = \pi_L + \frac{\alpha - \lambda}{1 - \bar\alpha\lambda}\pi_L^\perp. \tag{2.1}$$

To normalize we make the determinant equal to 1 and do a Gram–Schmidt factorization at $\lambda = 0$ to obtain $(\det \psi_{L,\alpha}(0))^{-1/2}\psi_{L,\alpha}(0) = QR$ with $Q \in \mathrm{SU}_2$ and $R \in \mathrm{SL}_2$ upper triangular with positive real entries on the diagonal. A *simple factor* is a map of the form

$$h_{L,\alpha} = (\det \psi_{L,\alpha})^{-1/2} Q^{-1}\psi_{L,\alpha}. \tag{2.2}$$

By Proposition 4.2 in [21] dressing by simple factors is explicit, and adapted to the case at hand in Theorem 1.2 in [11]: Generally, suppose that F_λ is an extended frame, and $h_{L,\alpha}$ a simple factor with $\alpha \in \mathbb{C}^\times$, $|\alpha| < 1$, and $L \in \mathbb{P}^1$. Then the dressed extended frame is given by dressing on an r-circle with $r < |\alpha| < 1$, and is

$$h_{L,\alpha} \# F_\lambda = h_{L,\alpha} F_\lambda h_{\tilde L,\alpha}^{-1} \quad \text{with} \quad \tilde L = \overline{F_\alpha(z)}^t L. \tag{2.3}$$

We next show that to obtain the single-bubbletons we can choose diagonal simple factors with very specific singularities α.

Lemma 2.1. *Up to isometry and conformal coordinate change any single-bubbleton can be obtained by dressing the round cylinder by a simple factor $h_{L,\alpha}$ with line $L = [1:0]$, so of the form*

$$h_{L,\alpha} = \sqrt{\frac{1-\alpha\lambda}{\alpha-\lambda}}\begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\alpha-\lambda}{1-\alpha\lambda} \end{pmatrix}, \tag{2.4}$$

with real $\alpha \in (0, 1)$ *given by*

$$\alpha = 2K^2 - 1 - 2K\sqrt{K^2-1} \quad \textit{for some integer} \quad K \geq 2. \tag{2.5}$$

Proof. If we want to dress the extended frame of the round cylinder, then we will have to choose the line L and the singularity α in such a way that the resulting bubbleton remains periodic with the same period as the underlying round cylinder. To get the conditions on L and α we will need to look at the monodromy of the dressed extended frame (2.3), with respect to the same translation $\tau\colon z \mapsto z+1$. Since $F_\lambda(0) = \mathbb{1}$ for all $\lambda \in \mathbb{C}^\times$, for the dressed frame we also have $h_{L,\alpha} \# F_\lambda|_{z=0} = \mathbb{1}$ since $\tilde{L} = L$ for $z = 0$. Hence with the monodromy $M_\lambda(\tau)$ in (1.7) of the round cylinder we obtain for the bubbleton monodromy

$$h_{L,\alpha} \# F_\lambda|_{z=1} = h_{L,\alpha} M_\lambda(\tau) h_{L,\alpha}^{-1} \Longleftrightarrow \overline{M_\alpha(\tau)}^t L = L. \tag{2.6}$$

Thus the condition on the line L is that it has to be an eigenline of $\overline{M_\alpha(\tau)}^t$. Now SU_2 acts transitively on $\mathbb{CP}^1$ and $h_{UL,\alpha} = U h_{L,\alpha} U^{-1}$ for any $U \in \mathrm{SU}_2$. Since dressing by $h_{L,\alpha}$ and $U h_{L,\alpha} U^{-1}$ give the same surface up to isometry and conformal coordinate change, we may choose without loss of generality the line $L = [1 : 0]$.

The monodromy of an extended frame of a CMC surface should be holomorphic in $\lambda \in \mathbb{C}^\times$ and unitary for $\lambda \in \mathbb{S}^1$. Clearly away from $\lambda = \alpha, 1/\alpha$ we have

$$\overline{h_{L,\alpha}(1/\bar{\lambda})}^t = h_{L,\alpha}^{-1}(\lambda)$$

so $h_{L,\alpha} M_\lambda(\tau) h_{L,\alpha}^{-1}$ is unitary on the unit circle, if we demand that

$$|\alpha| \neq 1. \tag{2.7}$$

Further $h_{L,\alpha} M_\lambda(\tau) h_{L,\alpha}^{-1}$ is holomorphic for all $\lambda \in \mathbb{C}^\times$ away from $\lambda = \alpha, \alpha^{-1}$. To make these two singularities removable, we impose the condition that $M_\alpha(\tau) = F_\alpha(1) = \pm\mathbb{1}$, or equivalently that $\mu_\alpha(1) = \pm 1$ for the function μ_λ in (1.2). This is equivalent to there existing an integer $K \in \mathbb{Z}$ such that

$$\alpha^{-1/2} + \alpha^{1/2} = 2K. \tag{2.8}$$

Rewriting this as a quadratic equation we obtain for each $K \in \mathbb{Z}$ two real solutions

$$\alpha_\pm = 2K^2 - 1 \pm 2K\sqrt{K^2-1}. \tag{2.9}$$

First observe that $\alpha_- = \alpha_+^{-1}$. Now $\psi_{L,\alpha^{-1}} = \psi_{L,\alpha}^{-1}$ and

$$h_{L,\alpha^{-1}} = \sqrt{\frac{\alpha-\lambda}{1-\alpha\lambda}}\sqrt{\frac{1-\alpha\lambda}{\alpha-\lambda}} h_{L,\alpha}^{-1} = \sqrt{\frac{\alpha-\lambda}{1-\alpha\lambda}}\sqrt{\frac{1-\alpha\lambda}{\alpha-\lambda}} h_{UL,\alpha} \quad \text{for} \quad U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But since

$$\sqrt{\frac{\alpha-\lambda}{1-\alpha\lambda}}\sqrt{\frac{1-\alpha\lambda}{\alpha-\lambda}}\bigg|_{\lambda=1} = -1 \quad \text{and} \quad \frac{\partial}{\partial\lambda}\bigg|_{\lambda=1}\left(\sqrt{\frac{\alpha-\lambda}{1-\alpha\lambda}}\sqrt{\frac{1-\alpha\lambda}{\alpha-\lambda}}\right) = 0,$$

dressing by $h_{L,\alpha}$ and $h_{L,\alpha^{-1}}$ gives the same bubbleton up to isometry and conformal coordinate change. Thus α_+ and α_- give the same bubbleton, so we may omit the subscript, restrict to non-negative integers $K \geq 0$, and set $\alpha = \alpha_-$, so that α is as in (2.5).

Then $\alpha^{-1} - \alpha = (\alpha^{-1/2} + \alpha^{1/2})(\alpha^{-1/2} - \alpha^{1/2}) = 2K(\alpha^{-1/2} - \alpha^{1/2})$, and consequently

$$\alpha^{-1/2} - \alpha^{1/2} = 2\sqrt{K^2 - 1}.$$

Note that when $K = 0, \pm 1$ then $\alpha = -1, 1, 1$ respectively, and it is not hard to see that $\alpha \notin \mathbb{S}^1$ when $|K| \geq 2$. Hence the condition (2.7) that $|\alpha| \neq 1$ requires that we impose $K \neq 0, \pm 1$, and consequently we may restrict to the case $K \geq 2$. To see that the singularities at $\lambda = \alpha, 1/\alpha$ are now apparent, we use L'Hôpital's rule to obtain

$$\begin{aligned}
\lim_{\lambda\to\alpha} h_{L,\alpha}F_\lambda(1)h_{L,\alpha}^{-1} &= \lim_{\lambda\to\alpha}\begin{pmatrix} \cos\mu_\lambda(1) & \dfrac{1-\alpha\lambda}{\alpha-\lambda}\mathbb{i}\lambda^{-1/2}\sin\mu_\lambda(1) \\ \dfrac{\alpha-\lambda}{1-\alpha\lambda}\mathbb{i}\lambda^{1/2}\sin\mu_\lambda(1) & \cos\mu_\lambda(1)\end{pmatrix} \\
&= \begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix} + \mathbb{i}\alpha^{-1/2}(1-\alpha^2)\lim_{\lambda\to\alpha}\frac{\sin\mu_\lambda(1)}{\alpha-\lambda}\begin{pmatrix}0&1\\0&0\end{pmatrix} \\
&= \begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix} \pm \mathbb{i}\alpha^{-1/2}4K(K^2-1)\begin{pmatrix}0&1\\0&0\end{pmatrix},
\end{aligned}$$

with sign depending on the parity of K. A similar computation, or using $\overline{F_{1/\bar\lambda}}^t = F_\lambda^{-1}$ shows that $\lambda = 1/\alpha$ is also a removable singularity. □

To obtain any multi-bubbleton of finite type one can dress by a finite product $\prod h_{L,\alpha_j}$ of simple factors, but we may use each integer $K \geq 2$ only once, since otherwise the singularities are no longer removable in the dressed monodromy. Hence on a multi-bubbleton each lobe number K can appear only once.

3. The main result

With the preparations of the preceding two sections we can now prove our main result.

Theorem 3.1. *A single-bubbleton is not embedded.*

Proof. From (2.3) the extended frame of a bubbleton is $h_{L,\alpha}F_\lambda h_{\tilde L,\alpha}^{-1}$ with

$$\tilde L = \overline{F_\alpha(z)}^t\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\bar\mu_\alpha\\ -\mathbb{i}\alpha^{-1/2}\sin\bar\mu_\alpha\end{pmatrix}. \tag{3.1}$$

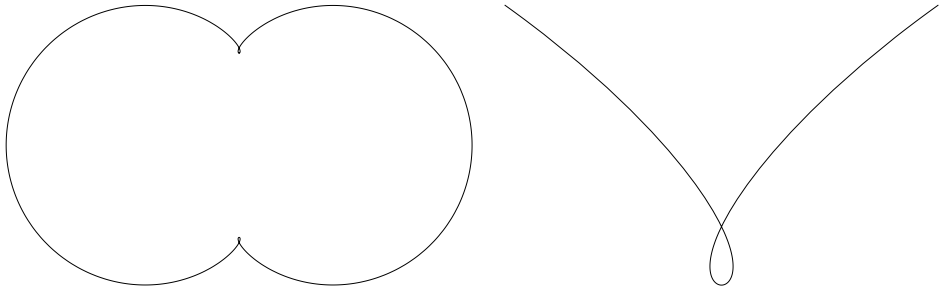

Fig. 3.1. Planar curve on the 2-lobed single-bubbleton. It has turning number 3. On the right, a magnified view of one of the two small loops of the curve.

We next explicitly compute the parameter curve $y = 0$ on a bubbleton. Since single-bubbletons are immersions [19], parameter curves are immersed. We may set the mean curvature to $H = -1/2$. Inserting the bubbleton frame into the Sym–Bobenko formula (1.3) gives three terms

$$\mathbb{i} h'_{L,\alpha} h_{L,\alpha}^{-1} + \mathbb{i} h_{L,\alpha} F'_\lambda F_\lambda^{-1} h_{L,\alpha}^{-1} + \mathbb{i} h_{L,\alpha} F_\lambda h_{\tilde{L},\alpha}^{-1\prime} h_{\tilde{L},\alpha} F_\lambda^{-1} h_{L,\alpha}^{-1}|_{\lambda=1,y=0}. \tag{3.2}$$

In the following we will compute these three terms. First, we have for $h_{L,\alpha}$ in (2.4) that

$$\mathbb{i} h'_{L,\alpha} h_{L,\alpha}^{-1}|_{\lambda=1} = \frac{K}{2\sqrt{K^2-1}} \begin{pmatrix} -\mathbb{i} & 0 \\ 0 & \mathbb{i} \end{pmatrix}. \tag{3.3}$$

The second term is

$$\mathbb{i} h_{L,\alpha} F'_\lambda F^{-1} h_{L,\alpha}^{-1}|_{\lambda=1,y=0} = \frac{1}{2} \begin{pmatrix} -\mathbb{i}\sin^2(\pi x) & -\frac{1}{2}\sin(2\pi x) \\ \frac{1}{2}\sin(2\pi x) & \mathbb{i}\sin^2(\pi x) \end{pmatrix}. \tag{3.4}$$

If $L = [a : b]$, then with respect to the standard basis of $\mathbb{C}^2$ the projection π_L is

$$\pi_L = \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}.$$

Hence for the line $\tilde{L}$ in (3.1) we obtain

$$\pi_{\tilde{L}} = \begin{pmatrix} \dfrac{|\cos\mu_\alpha|^2}{|\cos\mu_\alpha|^2 + \alpha^{-1}|\sin\mu_\alpha|^2} & \dfrac{\mathbb{i}\alpha^{-1/2}\sin\mu_\alpha\cos\bar{\mu}_\alpha}{|\cos\mu_\alpha|^2 + \alpha^{-1}|\sin\mu_\alpha|^2} \\ \dfrac{-\mathbb{i}\alpha^{-1/2}\sin\bar{\mu}_\alpha\cos\mu_\alpha}{|\cos\mu_\alpha|^2 + \alpha^{-1}|\sin\mu_\alpha|^2} & \dfrac{\alpha^{-1}|\sin\mu_\alpha|^2}{|\cos\mu_\alpha|^2 + \alpha^{-1}|\sin\mu_\alpha|^2} \end{pmatrix}.$$

The function μ_λ defined in (1.2) evaluated at $\lambda = \alpha$ along $y = 0$ reads $\mu_\alpha = \pi K x$. Then $\psi_{\tilde{L},\alpha}(\lambda)$ in (2.1) along $y = 0$ computes to

$$\psi_{\tilde{L},\alpha}(\lambda)|_{y=0} = \begin{pmatrix} \dfrac{\lambda\alpha^2-2\alpha+\lambda+(\alpha^2-1)\lambda\cos(2K\pi x)}{2(\alpha\lambda-1)(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} & \dfrac{\mathrm{i}(\alpha-1)\sqrt{\alpha}(\lambda+1)\sin(2K\pi x)}{2(\alpha\lambda-1)(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} \\ -\dfrac{\mathrm{i}(\alpha-1)\sqrt{\alpha}(\lambda+1)\sin(2K\pi x)}{2(\alpha\lambda-1)(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} & -\dfrac{\alpha^2-2\lambda\alpha+(\alpha^2-1)\cos(2K\pi x)+1}{2(\alpha\lambda-1)(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} \end{pmatrix}.$$

Evaluating at $\lambda = 0$ gives

$$\psi_{\tilde{L},\alpha}(0)|_{y=0} = \begin{pmatrix} \dfrac{\alpha}{\alpha\cos^2(K\pi x)+\sin^2(K\pi x)} & \dfrac{\mathrm{i}(1-\alpha)\sqrt{\alpha}\sin(2K\pi x)}{2(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} \\ \dfrac{\mathrm{i}(\alpha-1)\sqrt{\alpha}\sin(2K\pi x)}{2(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} & \dfrac{\alpha^2+(\alpha^2-1)\cos(2K\pi x)+1}{2(\alpha\cos^2(K\pi x)+\sin^2(K\pi x))} \end{pmatrix}.$$

Now $\det \psi_{\tilde{L},\alpha}(0)|_{y=0} = \alpha \neq 0$, and Gram–Schmidt on $\alpha^{-1/2}\psi_{\tilde{L},\alpha}(0)|_{y=0} = QR$ gives

$$Q = \begin{pmatrix} \dfrac{2\sqrt{\alpha}}{\sqrt{(\alpha+1)^2-(\alpha-1)^2\cos^2(2K\pi x)}} & \dfrac{\mathrm{i}(\alpha-1)\sin(2K\pi x)}{\sqrt{(\alpha-1)^2\sin^2(2K\pi x)+4\alpha}} \\ \dfrac{\mathrm{i}(\alpha-1)\sin(2\pi Kx)}{\sqrt{(\alpha+1)^2-(\alpha-1)^2\cos^2(2K\pi x)}} & \dfrac{2\sqrt{\alpha}}{\sqrt{(\alpha-1)^2\sin^2(2K\pi x)+4\alpha}} \end{pmatrix}.$$

Putting everything together gives that $h^{-1}_{\tilde{L},\alpha}|_{y=0} = \sqrt{\det \psi_{\tilde{L},\alpha}}\,\psi^{-1}_{\tilde{L},\alpha} Q$ is equal to

$$\begin{pmatrix} \dfrac{\sqrt{2}\sqrt{\alpha}((\alpha+1)(\lambda-1)-(\alpha-1)(\lambda+1)\cos(2K\pi x))}{\sqrt{\lambda-\alpha}\sqrt{\alpha\lambda-1}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} & -\dfrac{\mathrm{i}(\alpha^2-1)\sin(2K\pi x)}{\sqrt{\lambda-\alpha}\sqrt{\alpha\lambda-1}\sqrt{(\alpha-1)^2\sin^2(2K\pi x)+4\alpha}} \\ \dfrac{\mathrm{i}\sqrt{2}(\alpha^2-1)\lambda\sin(2K\pi x)}{\sqrt{\lambda-\alpha}\sqrt{\alpha\lambda-1}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} & \dfrac{\sqrt{2}\sqrt{\alpha}((\alpha+1)(\lambda-1)+(\alpha-1)(\lambda+1)\cos(2K\pi x))}{\sqrt{\lambda-\alpha}\sqrt{\alpha\lambda-1}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} \end{pmatrix}.$$

Differentiating with respect to λ gives that $h^{-1'}_{\tilde{L},\alpha}|_{y=0,\lambda=1}$ is equal to

$$\begin{pmatrix} \dfrac{\sqrt{2}\sqrt{\alpha}(\alpha+1)}{\sqrt{-(\alpha-1)^2}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} & \dfrac{\mathrm{i}\sqrt{\alpha-1}(\alpha+1)\sin(2K\pi x)}{2\sqrt{1-\alpha}\sqrt{(\alpha-1)^2\sin^2(2K\pi x)+4\alpha}} \\ \dfrac{\mathrm{i}\sqrt{\alpha-1}(\alpha+1)\sin(2K\pi x)}{\sqrt{2-2\alpha}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} & \dfrac{\sqrt{2}\sqrt{\alpha}(\alpha+1)}{\sqrt{-(\alpha-1)^2}\sqrt{-\cos(4K\pi x)(\alpha-1)^2+\alpha(\alpha+6)+1}} \end{pmatrix},$$

and consequently

$$h_{\tilde{L},\alpha}^{-1\prime} h_{\tilde{L},\alpha}|_{y=0,\lambda=1} = \begin{pmatrix} -\dfrac{\alpha^2+(\alpha+1)^2\cos(2K\pi x)-1}{2(\cos(2K\pi x)(\alpha-1)^2+\alpha^2-1)} & -\dfrac{\mathrm{i}\sqrt{\alpha}(\alpha+1)\sin(2K\pi x)}{\cos(2K\pi x)(\alpha-1)^2+\alpha^2-1} \\ \dfrac{\mathrm{i}\sqrt{\alpha}(\alpha+1)\sin(2K\pi x)}{\cos(2K\pi x)(\alpha-1)^2+\alpha^2-1} & \dfrac{\alpha^2+(\alpha+1)^2\cos(2K\pi x)-1}{2(\cos(2K\pi x)(\alpha-1)^2+\alpha^2-1)} \end{pmatrix}.$$

Thus the final contribution to the curve $y = 0$ comes from

$$\mathrm{i} h_{L,\alpha} F_\lambda h_{\tilde{L},\alpha}^{\prime -1} h_{\tilde{L},\alpha} F_\lambda^{-1} h_{L,\alpha}^{-1}|_{y=0,\lambda=1} = \begin{pmatrix} \mathrm{i}u & -v \\ v & -\mathrm{i}u \end{pmatrix} \tag{3.5}$$

with the real valued functions u and v given by

$$u = \frac{(\alpha+1)((\alpha-1)\cos(2\pi Kx)-\alpha-1)((\sqrt{\alpha}-1)^2\cos(2\pi(K+1)x)+(\sqrt{\alpha}+1)^2\cos(2\pi(K-1)x)+2(\alpha-1)\cos(2\pi x))}{2(\alpha-1)((\alpha-1)^2(-\cos(4\pi Kx))+\alpha^2+6\alpha+1)},$$

$$v = \frac{(\alpha+1)((\sqrt{\alpha}-1)^2\sin(2\pi(K+1)x)-(\sqrt{\alpha}+1)^2\sin(2\pi(K-1)x)+2(\alpha-1)\sin(2\pi x))((\alpha-1)\cos(2\pi Kx)-\alpha-1)}{2(\alpha-1)((\alpha-1)^2(-\cos(4\pi Kx))+\alpha^2+6\alpha+1)}.$$

Inspection of the three summands in (3.2) computed in (3.3), (3.4) and (3.5) show that the $y = 0$ curve on the bubbleton is a planar curve, since the off-diagonal terms do not have an imaginary part. Combining these terms then gives the planar curve $x \mapsto (X(x), Z(x))$ with

$$X(x) = \frac{\sin(2\pi x)}{4} - \frac{(\alpha+1)((\sqrt{\alpha}-1)^2\sin(2\pi(K+1)x)-(\sqrt{\alpha}+1)^2\sin(2\pi(K-1)x)+2(\alpha-1)\sin(2\pi x))((\alpha-1)\cos(2\pi Kx)-\alpha-1)}{2(\alpha-1)^3\cos(4\pi Kx)-(\alpha-1)(\alpha(\alpha+6)+1)},$$

$$Z(x) = \frac{((\alpha^2-1)\cos(2\pi Kx)-(\alpha+1)^2)((\sqrt{\alpha}-1)^2\cos(2\pi(K+1)x)+(\sqrt{\alpha}+1)^2\cos(2\pi(K-1)x)+2(\alpha-1)\cos(2\pi x))}{2(\alpha-1)((\alpha-1)^2(-\cos(4\pi Kx))+\alpha(\alpha+6)+1)} - \frac{\sin^2(\pi x)}{2} - \frac{K}{2\sqrt{K^2-1}}.$$

The turning number of this immersed planar curve $[0, 1] \to \mathbb{R}^2$, $x \mapsto (X(x), Z(x))$ computes to

$$\frac{1}{2\pi}\int_0^1 \frac{X'(x)Z''(x) - X''(x)Z'(x)}{X'^2(x) + Z'^2(x)}\, dx = 2K - 1.$$

Since $K \geq 2$, the planar curve has self intersections, and thus the bubbleton is not embedded. Plots of the curves for $K = 2, 3, 4, 5$ are shown in Figs. 3.1 and 3.2. □

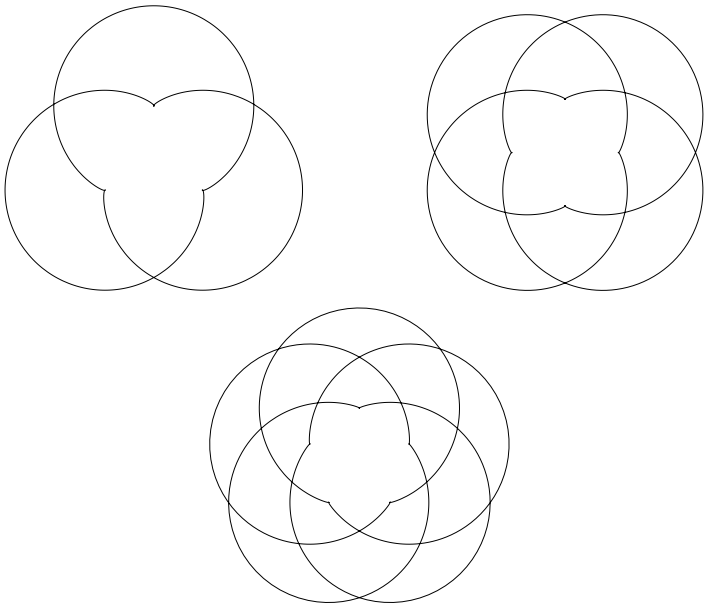

Fig. 3.2. Planar curves on the 3, 4 and 5-lobed single-bubbletons, with turning numbers 5,7,9 respectively. What appear to be cusps on the immersed curves are in fact small loops, as in the curve in Fig. 3.1.

One can also dress a multiply wrapped round cylinder by simple factors and obtain topologically different single-and multi-bubbletons, than if one just dresses the embedded cylinder. But if the undressed cylinder is embedded, then the dressed cylinder again has embedded ends [19, 9, 13]. Hence far away from the bubbles, one can take a planer cross section of a bubbleton and get a curve that is almost circular with turning number equal to one. Flowing this curve towards a bubble it deforms into a space curve which at one instance becomes planar and has turning number at least three. The fact that single-bubbletons have a smooth closed planar curvature line, or equivalently a reflective symmetry is what makes our proof work for single-bubbletons. It would be interesting to extend the result and find a more conceptual proof that also works for multi-bubbletons.

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