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ON VECTOR VALUED SIEGEL MODULAR FORMS OF DEGREE 2 WITH SMALL LEVELS

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Abstract

In this paper, we show that the space of vector valued Siegel modular forms of $\Gamma_0(N) \subset \mathrm{Sp}(2, \mathbb{Z})$ with respect to the symmetric tensor of degree 2 has a simple unified structure for $N = 2, 3, 4$. Each structure is similar to the structure of the full modular group.

1. Introduction

On the structure theorem of Siegel modular forms of degree 2, Igusa [10, 11] determined the structure of Siegel modular forms with respect to the full modular group $\mathrm{Sp}(2, \mathbb{Z})$. There are five generators of weight 4, 6, 10, 12 and 35. The first four generators are algebraically independent and the square of the last generator is in the subring generated by first four. Recently, Aoki and Ibukiyama [3] indicated that the rings of Siegel modular forms with small levels have similar structures. That is, on the ring of Siegel modular forms of degree 2 with respect to the congruent subgroup of level $N = 1, 2, 3, 4$ (for $N = 3, 4$, taking Neven-type case with character), there are five generators, among which four generators are algebraically independent and the square of the other generator is in the subring generated by first four.

On the structure of vector valued Siegel modular forms of degree 2 with respect to the symmetric tensor of degree 2, Satoh [12] and Ibukiyama [9] determined the structure with respect to the full modular group. There are ten generators with some relations. In this paper, we determine the structures of vector valued Siegel modular forms with small levels. Their structures are similar to the structure with respect to the full modular group.

2. Main theorem

In this section, we state two main theorems. The first one is on the structure of complex valued Siegel modular forms and the second one is on the structure of vector valued Siegel modular forms. Today we have already known several kinds of proofs of the first one. For example, In Aoki [1], we proved the first one by using the restriction maps to Jacobi forms. In this paper we give another new proof of the first one, that

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is available for the second one. The idea of our new proof is an application of the restriction map to the diagonal component, that is called Witt operator in the paper by Ibukiyama [7]. This idea was given by van der Geer [4] and by the author [2] independently but almost simultaneously.

2.1. Complex valued case. For a positive integer g, we denote the Siegel upper half plane of degree g by

$$\mathbb{H}_g := \{Z = {}^tZ \in \mathrm{M}_g(\mathbb{C}) \mid \mathrm{Im}\, Z > 0\}.$$

The symplectic group

$$\mathrm{Sp}(g,\mathbb{R}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_{2g}(\mathbb{R}) \;\middle|\; {}^tM J_g M = J_g := \begin{pmatrix} O_g & -E_g \\ E_g & O_g \end{pmatrix} \right\}$$

acts on $\mathbb{H}_g$ transitively by

$$\mathbb{H}_g \ni Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_g.$$

For $M \in \mathrm{Sp}(g,\mathbb{R})$, $k \in \mathbb{Z}$ and a holomorphic function $F\colon \mathbb{H}_g \to \mathbb{C}$, we write

$$(F|_k M)(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle).$$

Put

$$\mathrm{Sp}(g,\mathbb{Z}) := \mathrm{Sp}(g,\mathbb{R}) \cap \mathrm{M}_{2g}(\mathbb{Z}).$$

Let Γ be a finite index subgroup of $\mathrm{Sp}(g,\mathbb{Z})$ and let $\psi\colon \Gamma \to \mathbb{C}^\times$ be a character. We denote by 1 the constant character.

For a holomorphic function $F\colon \mathbb{H}_g \to \mathbb{C}$ and $k \in \mathbb{Z}$, we say F is a Siegel modular form of weight k with a character ψ if F satisfies the following two conditions:
(M1) $\psi(M)F(Z) = (F|_k M)(Z)$ for any $M \in \Gamma$.
(M2) F is bounded for each cusps.
We remark that, if $g \geq 2$, the condition (M2) is induced from the condition (M1) by Koecher principle. We denote by $A_k(\Gamma,\psi)$ the space of all Siegel modular forms of weight k with a character ψ. Put $A_k(\Gamma) := A_k(\Gamma,1)$ and $A_*(\Gamma) := \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma)$. The space $A_*(\Gamma)$ is a graded ring.

Put

$$\Gamma_0^{(g)}(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g,\mathbb{Z}) \;\middle|\; C \equiv O_g \pmod N \right\}$$

for any natural number $N \in \mathbb{N} := \{1,2,3,\dots\}$. We denote by $\psi_3^{(g)}$ the character defined by $\psi_3^{(g)}(M) = (-3/\det(D))$ and by $\psi_4^{(g)}$ the character defined by $\psi_4^{(g)}(M) = (-1/\det(D))$. We put

$$\Gamma_{0,\psi_N}^{(g)}(N) := \{M \in \Gamma_0^{(g)}(N) \mid \psi_N^{(g)}(M) = 1\}$$

for $N = 3, 4$.

In this paper, our interest is the case $g = 2$ and $N = 1, 2, 3, 4$. From now on, we denote the coordinate of $\mathbb{H}_2$ by

$$Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$$

and set $q := \exp(2\pi\sqrt{-1}\tau)$, $\zeta := \exp(2\pi\sqrt{-1}z)$ and $p := \exp(2\pi\sqrt{-1}\omega)$. In foregoing cases, the structure of $A_*(\Gamma)$ is already known.

Theorem 1. *For each $\Gamma = \mathrm{Sp}(2,\mathbb{Z})$, $\Gamma_0^{(2)}(2)$, $\Gamma_{0,\psi_3}^{(2)}(3)$ or $\Gamma_{0,\psi_4}^{(2)}(4)$, the graded ring $A_*(\Gamma)$ is generated by five modular forms. The first four generators are algebraically independent and the square of the last generator is in the subring generated by the first four.*

Γ	*The weights of the first four generators*	*The weights of the last generator*	*References*
$\mathrm{Sp}(2,\mathbb{Z})$	4, 6, 10, 12	35	*Igusa* [10, 11]
$\Gamma_0^{(2)}(2)$	2, 4, 4, 6	19	*Ibukiyama* [7]
$\Gamma_{0,\psi_3}^{(2)}(3)$	1, 3, 3, 4	14	*Ibukiyama* [7] *Aoki–Ibukiyama* [3]
$\Gamma_{0,\psi_4}^{(2)}(4)$	1, 2, 2, 3	11	*Hayashida–Ibukiyama* [6]

In this paper, we denote by $A_*^\natural(\Gamma)$ the subring of $A_*(\Gamma)$ generated by the first four generators.

Today we know several methods to construct these generators. One of the simplest construction of the first four generators is by the Maass lift of Jacobi forms of index 1. The author feels that the simplest construction of the last generator is by the Rankin–Cohen–Ibukiyama differential operator (cf. [3, 5, 8]). Namely, four algebraically independent modular forms $F_j \in A_{k_j}(\Gamma)$ $(j = 1, 2, 3, 4)$ induce a new modular form

$$[F_1, F_2, F_3, F_4] := \det \begin{pmatrix} k_1F_1 & k_2F_2 & k_3F_3 & k_4F_4 \\ \dfrac{\partial F_1}{\partial \tau} & \dfrac{\partial F_2}{\partial \tau} & \dfrac{\partial F_3}{\partial \tau} & \dfrac{\partial F_4}{\partial \tau} \\ \dfrac{\partial F_1}{\partial z} & \dfrac{\partial F_2}{\partial z} & \dfrac{\partial F_3}{\partial z} & \dfrac{\partial F_4}{\partial z} \\ \dfrac{\partial F_1}{\partial \omega} & \dfrac{\partial F_2}{\partial \omega} & \dfrac{\partial F_3}{\partial \omega} & \dfrac{\partial F_4}{\partial \omega} \end{pmatrix} \in A_{k_1+k_2+k_3+k_4+3}(\Gamma)$$

and if we choose the first four generators as F_j, we have the last generator.

In Section 4, we prove Theorem 1 and give the generating function of $\dim_{\mathbb{C}} A_k(\Gamma)$.

2.2. Vector valued case. Let s be a non-negative integer, V be a $(s+1)$-dimensional $\mathbb{C}$-vector space and $\rho\colon \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ be a rational representation. It is well-known that ρ is a rational irreducible representation if and only if $\rho = \rho_{k,s} := \mathrm{Sym}^s \otimes \det^k$. For the sake of simplicity, in this paper, we fix a coordinate of $\mathrm{Sym}^s \otimes \det^k$ as follows: put $V := \mathbb{C}^{s+1}$ and $\rho_{k,s}(A) := (\det A)^k \rho_{0,s}(A)$, where $\rho_{0,s}(A)$ is defined by

$$(u^s, u^{s-1}v, \ldots, v^s) = (x^s, x^{s-1}y, \ldots, y^s)\rho_{0,s}(A) \quad ((u, v) = (x, y)A).$$

For $M \in \mathrm{Sp}(2,\mathbb{R})$ and a holomorphic function $F\colon \mathbb{H}_2 \to \mathbb{C}^{s+1}$, we write

$$(F|_\rho M)(Z) := \rho(CZ + D)^{-1} F(M\langle Z\rangle).$$

We say F is a Siegel modular forms of weight ρ with a character ψ if F satisfies the condition $\psi(M)F(Z) = (F|_\rho M)(Z)$ for any $M \in \Gamma$. We remark that this F is bounded at each cusps by Koecher principle. We denote by $A_{k,s}(\Gamma, \psi)$ the space of all Siegel modular forms of weight $\rho_{k,s}$ with a character ψ. Put $A_{k,s}(\Gamma) := A_{k,s}(\Gamma, 1)$. We remark $A_{k,0}(\Gamma) = A_k(\Gamma)$. It is easy to show that if s is odd and if $-E_4 \in \Gamma$, then $A_{k,s}(\Gamma) = \{0\}$. Put $A_{*,s}(\Gamma) := \bigoplus_{k\in\mathbb{Z}} A_{k,s}(\Gamma)$. The space $A_{*,s}(\Gamma)$ is a graded module of $A^\natural_*(\Gamma)$.

The aim of this paper is to determine the structure of $A_{*,2}(\Gamma)$. The structure of $A_{*,2}(\mathrm{Sp}(2,\mathbb{Z}))$ was already determined by Satoh [12] and Ibukiyama [9]. There are ten generators, whose weights are

$$\begin{array}{llll}
10 = 4 + 6, & 16 = 6 + 10, & 21 = 4 + 6 + 10 + 1, & \\
14 = 4 + 10, & 18 = 6 + 12, & 23 = 4 + 6 + 12 + 1, & \\
16 = 4 + 12, & 22 = 10 + 12, & 27 = 4 + 10 + 12 + 1 & \text{and} \\
 & & 29 = 6 + 10 + 12 + 1. &
\end{array}$$

To show this, they used the dimension formula of modular forms. In this paper we will give this result by another way. By our way, we can determine the module structure of $A_{*,2}(\Gamma)$ for $\Gamma = \Gamma_0^{(2)}(2), \Gamma_{0,\psi_3}^{(2)}(3)$ or $\Gamma_{0,\psi_4}^{(2)}(4)$.

Theorem 2. *For each* $\Gamma = \mathrm{Sp}(2,\mathbb{Z}), \Gamma_0^{(2)}(2), \Gamma_{0,\psi_3}^{(2)}(3)$ *or* $\Gamma_{0,\psi_4}^{(2)}(4)$, *the graded* $A^\natural_*(\Gamma)$-*module* $A_{*,2}(\Gamma)$ *is generated by ten modular forms.*

Γ	*The weights of generators (type 1)*	*The weights of generators (type 2)*	*References*
$\mathrm{Sp}(2,\mathbb{Z})$	10, 14, 16, 16, 18, 22	21, 23, 27, 29	*Satoh* [12] *Ibukiyama* [9]
$\Gamma_0^{(2)}(2)$	6, 6, 8, 8, 10, 10	11, 13, 13, 15	*This paper*
$\Gamma_{0,\psi_3}^{(2)}(3)$	4, 4, 5, 6, 7, 7	8, 9, 9, 11	
$\Gamma_{0,\psi_4}^{(2)}(4)$	3, 3, 4, 4, 5, 5	6, 7, 7, 8	

All generators are constructed by differential operators. When $N = 1$, the generators of type 1 was constructed by Satoh [12] and the generators of type 2 was constructed by Ibukiyama [9]. For each $N = 2, 3, 4$, we can construct all generators according to the way by Satoh and Ibukiyama.

From $F_j \in A_{k_j}(\Gamma)$ $(j = 1, 2)$, we have a new modular form

$$[F_1, F_2] := \begin{pmatrix} k_1 F_1 \dfrac{\partial F_2}{\partial \tau} - k_2 F_2 \dfrac{\partial F_1}{\partial \tau} \\ k_1 F_1 \dfrac{\partial F_2}{\partial z} - k_2 F_2 \dfrac{\partial F_1}{\partial z} \\ k_1 F_1 \dfrac{\partial F_2}{\partial \omega} - k_2 F_2 \dfrac{\partial F_1}{\partial \omega} \end{pmatrix} \in A_{k_1+k_2,2}(\Gamma).$$

If we choose two distinct generators from the first four generators of $A_{*,0}(\Gamma)$, we get generators of type 1. We remark that these generators of type 1 are not independent. There is a relation so called Jacobi identity:

$$k_1F_1[F_2, F_3] + k_2F_2[F_3, F_1] + k_3F_3[F_1, F_2] = 0 \quad (F_j \in A_{k_j}(\Gamma)).$$

From $F_j \in A_{k_j}(\Gamma)$ $(j = 1, 2, 3)$, we have a new modular form

$$\begin{aligned}[F_1, F_2, F_3] := {}& k_1 F_1 \begin{pmatrix} \dfrac{\partial F_2}{\partial \tau}\dfrac{\partial F_3}{\partial z} - \dfrac{\partial F_2}{\partial z}\dfrac{\partial F_3}{\partial \tau} \\ 2\left(\dfrac{\partial F_2}{\partial \tau}\dfrac{\partial F_3}{\partial \omega} - \dfrac{\partial F_2}{\partial \omega}\dfrac{\partial F_3}{\partial \tau}\right) \\ \dfrac{\partial F_2}{\partial z}\dfrac{\partial F_3}{\partial \omega} - \dfrac{\partial F_2}{\partial \omega}\dfrac{\partial F_3}{\partial z} \end{pmatrix} - k_2 F_2 \begin{pmatrix} \dfrac{\partial F_1}{\partial \tau}\dfrac{\partial F_3}{\partial z} - \dfrac{\partial F_1}{\partial z}\dfrac{\partial F_3}{\partial \tau} \\ 2\left(\dfrac{\partial F_1}{\partial \tau}\dfrac{\partial F_3}{\partial \omega} - \dfrac{\partial F_1}{\partial \omega}\dfrac{\partial F_3}{\partial \tau}\right) \\ \dfrac{\partial F_1}{\partial z}\dfrac{\partial F_3}{\partial \omega} - \dfrac{\partial F_1}{\partial \omega}\dfrac{\partial F_3}{\partial z} \end{pmatrix} \\ &+ k_3 F_3 \begin{pmatrix} \dfrac{\partial F_1}{\partial \tau}\dfrac{\partial F_2}{\partial z} - \dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial \tau} \\ 2\left(\dfrac{\partial F_1}{\partial \tau}\dfrac{\partial F_2}{\partial \omega} - \dfrac{\partial F_1}{\partial \omega}\dfrac{\partial F_2}{\partial \tau}\right) \\ \dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial \omega} - \dfrac{\partial F_1}{\partial \omega}\dfrac{\partial F_2}{\partial z} \end{pmatrix} \in A_{k_1+k_2+k_3+1,2}(\Gamma).\end{aligned}$$

If we choose three distinct generators from the first four generators of $A_{*,0}(\Gamma)$, we have generators of type 2. There is a relation

$$k_1F_1[F_2, F_3, F_4] + k_2F_2[F_3, F_4, F_1] + k_3F_3[F_4, F_1, F_2] + k_4F_4[F_1, F_2, F_3] = 0$$
$$(F_j \in A_{k_j}(\Gamma)).$$

In Section 5, we prove Theorem 2 and give the generating function of $\dim_{\mathbb{C}} A_{k,2}(\Gamma)$.

3. Generalized Witt operators

3.1. Witt modular forms. From now on, we assume that Γ satisfies a condition

$$\Gamma = \gamma_0^{-1}\Gamma\gamma_0,$$

where

$$\gamma_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For a while, we assume $\gamma_0 \notin \Gamma$. In this case, let $\tilde{\Gamma}$ be a subgroup of $\mathrm{Sp}(2, \mathbb{Z})$ generated by Γ and γ_0 and let ψ be a character of $\tilde{\Gamma}$ defined by $\psi(M) = 1$ for any $M \in \Gamma$ and $\psi(\gamma_0) = -1$. Then we have a decomposition

$$A_k(\Gamma) = A_k(\tilde{\Gamma}) \oplus A_k(\tilde{\Gamma}, \psi)$$

by

$$A_k(\Gamma) \ni F = \left(\frac{F + F|_k\gamma_0}{2}\right) + \left(\frac{F - F|_k\gamma_0}{2}\right).$$

Because $\gamma_0 \in \tilde{\Gamma}$, if we admit modular forms with character, we may assume the translation formula with respect to γ_0 always holds. Namely, we investigate $A_k(\tilde{\Gamma})$ and $A_k(\tilde{\Gamma}, \psi)$ separately, instead of investigating $A_k(\Gamma)$ directly.

For $M' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(1, \mathbb{R})$, let

$$\gamma_1(M') := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_2(M') := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

For $\Gamma \subset \mathrm{Sp}(2, \mathbb{Z})$, we define a subgroup of $\mathrm{SL}(2, \mathbb{Z})$ by

$$\Gamma' := \{M' \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma_1(M') \in \Gamma\}.$$

For ψ, that is a character of Γ, we define a character of Γ' by

$$\psi'(M') := \psi(\gamma_1(M')).$$

Because $\gamma_0^{-1}\gamma_1(M')\gamma_0 = \gamma_2(M')$, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$, we have $\gamma_2(M') \in \Gamma$ and $\psi'(M') = \psi(\gamma_2(M'))$.

We consider $\mathbb{H}_1 \times \mathbb{H}_1$ to be a subset of $\mathbb{H}_2$ by

$$\iota\colon \mathbb{H}_1 \times \mathbb{H}_1 \ni (\tau, \omega) \mapsto \begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix} \in \mathbb{H}_2$$

and let

$$G := \{M \in \mathrm{Sp}(2, \mathbb{R}) \mid M\langle \mathbb{H}_1 \times \mathbb{H}_1 \rangle = \mathbb{H}_1 \times \mathbb{H}_1\}$$

be the isotropy group of $\mathbb{H}_1 \times \mathbb{H}_1$. By direct calculation, we can show that G is generated by γ_0 and $\gamma_1(M')$, where M' runs over $\mathrm{SL}(2, \mathbb{R})$. Therefore, if $F \in A_k(\Gamma)$, then $\iota^* F$ is invariant not only with respect to γ_0 but also with respect to $M' \in \Gamma'$ for each variable.

For a holomorphic function $f\colon \mathbb{H}_1 \times \mathbb{H}_1 \to \mathbb{C}$ and $k, l \in \mathbb{Z}$, we say f is a Witt modular form of weight (k, l) with respect to Γ' and ψ' if f satisfies the following two conditions:

(1) For any fixed $\omega_0 \in \mathbb{H}_1$, the function $f(\tau, \omega_0)$ on $\tau \in \mathbb{H}_1$ belongs to $A_k(\Gamma', \psi')$.

(2) For any fixed $\tau_0 \in \mathbb{H}_1$, the function $f(\tau_0, \omega)$ on $\omega \in \mathbb{H}_1$ belongs to $A_l(\Gamma', \psi')$.

We denote by $\mathrm{W}_{k,l}(\Gamma', \psi')$ the space of all Witt modular forms of weight (k, l) with respect to Γ' and ψ'. By Witt [13, Satz A], we have

$$\mathrm{W}_{k,l}(\Gamma', \psi') = A_k(\Gamma', \psi') \otimes_{\mathbb{C}} A_l(\Gamma', \psi').$$

We say $f \in \mathrm{W}_{k,k}(\Gamma', \psi')$ is symmetric or skew-symmetric if $f(\tau, \omega) = f(\omega, \tau)$ or $f(\tau, \omega) = -f(\omega, \tau)$, respectively. We denote the space of all symmetric or skew-symmetric forms by $\mathrm{W}_k^{\mathrm{sym}}(\Gamma', \psi')$ or $\mathrm{W}_k^{\mathrm{skew}}(\Gamma', \psi')$, respectively. It is easy to show that

$$\mathrm{W}_{k,k}(\Gamma', \psi') = \mathrm{W}_k^{\mathrm{sym}}(\Gamma', \psi') \oplus \mathrm{W}_k^{\mathrm{skew}}(\Gamma', \psi').$$

3.2. Differential operators. For a complex domain X, we denote by $\mathrm{Hol}(X, \mathbb{C}^s)$ the set of all holomorphic functions from X to $\mathbb{C}^s$. For $r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, define a differential operator $D_r\colon \mathrm{Hol}(\mathbb{H}_2, \mathbb{C}^s) \to \mathrm{Hol}(\mathbb{H}_1 \times \mathbb{H}_1, \mathbb{C}^s)$ by

$$(D_r(F))(\tau, \omega) := \left(\frac{\partial^r F}{\partial z^r}\right)\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix},$$

and put

$$A_{k,s}(\Gamma, \psi; r) := \{F \in A_{k,s}(\Gamma, \psi) \mid D_t(F) = 0 \text{ for any } t < r\}.$$

The generalized Witt operator D_r induces an exact sequence

$$0 \to A_{k,s}(\Gamma, \psi; r+1) \to A_{k,s}(\Gamma, \psi; r) \xrightarrow{D_r} \mathrm{Hol}(\mathbb{H}_1 \times \mathbb{H}_1, \mathbb{C}^s)$$

and we have a dimension formula

$$\dim_{\mathbb{C}} A_{k,s}(\Gamma, \psi) = \sum_{r=0}^{\infty} \dim_{\mathbb{C}} D_r(A_{k,s}(\Gamma, \psi; r)).$$

From the next section, we will calculate an upper bound of the dimension of $D_r(A_{k,s}(\Gamma, \psi; r))$ and hence we will have an upper bound of the dimension of $A_{k,s}(\Gamma, \psi)$ for each foregoing Γ. Therefore, if we can construct sufficiently many modular forms, we can show this upper bound is the true dimension of $A_{k,s}(\Gamma, \psi)$.

Before the separate calculation, here we show one proposition. The translation formulas of $\gamma_1(M')$ and $\gamma_2(M')$ induces that the image of above D_r is in the space of Witt modular forms. When $s = 0$ and $s = 2$, we have the following proposition.

Proposition 3. *There exist exact sequences as follows*:

(1) *When* $s = 0$,

(1a) *If* $(-1)^k \psi(\gamma_0) = 1$,

$$0 \to A_k(\Gamma, \psi; r+1) \to A_k(\Gamma, \psi; r) \xrightarrow{D_r} \mathrm{W}^{\mathrm{sym}}_{k+r}(\Gamma', \psi').$$

(1b) *If* $(-1)^k \psi(\gamma_0) = -1$,

$$0 \to A_k(\Gamma, \psi; r+1) \to A_k(\Gamma, \psi; r) \xrightarrow{D_r} \mathrm{W}^{\mathrm{skew}}_{k+r}(\Gamma', \psi').$$

(2) *When* $s = 2$,

(2a) *If* $(-1)^k \psi(\gamma_0) = 1$,

$$0 \to A_{k,2}(\Gamma, \psi; r+1) \to A_{k,2}(\Gamma, \psi; r) \xrightarrow{D_r} \mathrm{W}_{k+r+2,k+r}(\Gamma', \psi') \oplus \mathrm{W}^{\mathrm{sym}}_{k+r+1}(\Gamma', \psi').$$

(2b) *If* $(-1)^k \psi(\gamma_0) = -1$,

$$0 \to A_{k,2}(\Gamma, \psi; r+1) \to A_{k,2}(\Gamma, \psi; r) \xrightarrow{D_r} \mathrm{W}_{k+r+2,k+r}(\Gamma', \psi') \oplus \mathrm{W}^{\mathrm{skew}}_{k+r+1}(\Gamma', \psi').$$

We omit the proof. However, here we assume $\gamma_1(-E_2) \in \Gamma$ and remark some comments about the above proposition. When $s = 0$, if $(-1)^{k+r} \psi(\gamma_1(-E_2)) = -1$, then $\mathrm{W}^{\mathrm{sym}}_{k+r}(\Gamma', \psi') = \{0\}$. Hence, for example, if k is even, $\psi = 1$ and $\gamma_0 \in \Gamma$, we can sharpen the above exact sequence to

$$0 \to A_k(\Gamma; 2r+2) \to A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'),$$

where we denote $A_k(\Gamma; r) := A_k(\Gamma, 1; r)$. When $s = 2$, the image of D_r is a vector valued function, strictly. But, by γ_0, the first entry equals to the third entry up to the sign. Therefore, in the above proposition, we denote the image of D_r by the direct sum of two spaces. Moreover, for example, if k is even, r is odd, $\psi = 1$ and $\gamma_0 \in \Gamma$, then $\mathrm{W}^{\mathrm{sym}}_{k+r}(\Gamma') = \{0\}$, that means the second entry of the image of D_r is zero. Hence the image of D_r is determined only from the first entry and we can denote

$$0 \to A_{k,2}(\Gamma; 2r+1) \to A_{k,2}(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma').$$

4. Proof of Theorem 1

4.1. Case $N = 1$. First, we consider the simplest case, that is, we set $N = 1$, $\Gamma = \mathrm{Sp}(2, \mathbb{Z})$, $\Gamma' = \mathrm{SL}(2, \mathbb{Z})$. In this case, the structure theorem is well known as Igusa's theorem. When $N = 1$, because $\gamma_1(-E_2) \in \Gamma$, Proposition 3 induces the following proposition immediately.

Proposition 4. *There exist exact sequences as follows*:
(1) *If k is even, $A_k(\Gamma) = A_k(\Gamma; 0)$ (by definition) and*

$$0 \to A_k(\Gamma; 2r+2) \to A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma').$$

(2) *If k is odd, $A_k(\Gamma) = A_k(\Gamma; 1)$ and*

$$0 \to A_k(\Gamma; 2r+3) \to A_k(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma').$$

To study the image $D_r(A_k(\Gamma; r))$ more precisely, we will investigate Fourier coefficients of modular forms. For $f \in A_k(\Gamma')$, put the Fourier coefficients of f by

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n) q^n.$$

and let

$$A_k(\Gamma'; r) := \{ f \in A_k(\Gamma') \mid a_f(n) = 0 \text{ for } n < r \}.$$

Let

$$\mathrm{W}_{k,l}(\Gamma'; r) := A_k(\Gamma'; r) \otimes_{\mathbb{C}} A_l(\Gamma'; r)$$

be a subspace of $\mathrm{W}_k(\Gamma')$ and let

$$\mathrm{W}^{\mathrm{sym}}_k(\Gamma'; r) := \mathrm{W}_{k,k}(\Gamma'; r) \cap \mathrm{W}^{\mathrm{sym}}_k(\Gamma')$$

and

$$\mathrm{W}^{\mathrm{skew}}_k(\Gamma'; r) := \mathrm{W}_{k,k}(\Gamma'; r) \cap \mathrm{W}^{\mathrm{skew}}_k(\Gamma').$$

For $F \in A_k(\Gamma)$, put the Fourier coefficients of F by

$$F(Z) = \sum_{n,l,m \in \mathbb{Z}} a(n, l, m) q^n \zeta^l p^m.$$

Because

$$(D_r(F))(\tau,\omega) := \sum_{n,m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} (2\pi\sqrt{-1}l)^r a(n,l,m)\right) q^n p^m,$$

if $F \in A_k(\Gamma; r)$, for any $n \in \mathbb{Z}$, $m \in \mathbb{Z}$ and $t < r$,

$$\sum_{l\in\mathbb{Z}} l^t a(n,l,m) = 0.$$

Let

$$\gamma_3(x) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_4(x) := \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix}.$$

Lemma 5. *The Fourier coefficients of F satisfy the following properties*:

(1) *If $4nm - l^2 < 0$, $n < 0$ or $m < 0$, then $a(n,l,m) = 0$.*

(2) $a(n,-l,m) = (-1)^k a(n,l,m)$.

(3) $a(m,l,n) = (-1)^k a(n,l,m)$.

(4) *$a(n + xl + x^2 m, l + 2xm, m) = a(n,l,m)$ for any $x \in \mathbb{Z}$. Therefore, if $|l| > |m|$, then there exist $n', l' \in \mathbb{Z}$ such that $n' < n$ and $a(n', l', m) = a(n,l,m)$.*

(5) *$a(n, l + 2xn, m + xl + x^2 n) = a(n,l,m)$ for any $x \in \mathbb{Z}$. Therefore, if $|l| > |n|$, then there exist $m', l' \in \mathbb{Z}$ such that $m' < m$ and $a(n, l', m') = a(n,l,m)$.*

(6) *If k is odd, then $a(n,0,m) = 0$ and $a(n,l,n) = 0$.*

Proof. (1) is well-known as the Koecher principle. On the equation $F|_k M = F$, by setting $M = \gamma_1(-E_2)$, γ_0, $\gamma_3(x)$ and $\gamma_4(x)$, we have (2), (3), (4) and (5). From (2) and (3), we have (6). □

From this lemma, we have the next lemma, that is easy but the key of our proof.

Lemma 6. *The Fourier coefficients of F has the following properties*:

(1) *If k is even, $F \in A_k(\Gamma; 2r)$ and $\min\{n, m\} < r$, then $a(n,l,m) = 0$ for any l.*

(2) *If k is odd, $F \in A_k(\Gamma; 2r+1)$ and $\min\{n, m\} < r + 2$, then $a(n,l,m) = 0$ for any l.*

Proof. We will give a proof by induction on r. First, we show (1). When $r = 0$, the assertion is trivial. Therefore we assume (1) holds for r and prove it also holds for $r + 1$. Put

$$b(n,l,m) := \begin{cases} 2a(n,l,m) & (\text{if } l \neq 0), \\ a(n,0,m) & (\text{if } l = 0). \end{cases}$$

Because $a(m, l, n) = a(n, l, m)$, it is sufficient to show that $b(r, l, m) = 0$. Because $F \in A_k(\Gamma; 2(r + 1))$, for any $m \in \mathbb{Z}$ and $t \in \{0, 1, \ldots, r\}$, we have

$$\sum_{l=0}^{2\sqrt{rm}} l^{2t} b(r, l, m) = 0.$$

When $m = r$, from Lemma 5 (5) and the assumption of the induction, we have

$$\sum_{l=0}^{r} l^{2t} b(r, l, r) = 0$$

for $t \in \{0, 1, \ldots, r\}$. Hence, by the Vandermonde formula, we have $b(r, l, r) = 0$. For general m, we can show $b(r, l, m) = 0$ by induction on m.

Next, we consider (2). When $r = 0$, from Lemma 5 (5), we have $a(1, 0, m) = -a(1, 0, m)$ and $a(1, 1, m) = a(1, -1, m) = -a(1, 1, m)$, hence $a(1, 0, m) = a(1, 1, m) = 0$. Then the assertion holds because $a(1, l, m) = a(1, l-2, m-l+1)$. Therefore we assume (2) holds for r and prove it also holds for $r+1$. In this case, put $b(n, l, m) := la(n, l, m)$. When $F \in A_k(\Gamma; 2r + 1)$, for any $n, m \in \mathbb{Z}$ and $t \in \{0, 1, \ldots, r - 1\}$, we have

$$\sum_{l=1}^{2\sqrt{nm}} l^{2t} b(n, l, m) = 0.$$

From Lemma 5 (5) (6), we can show (2) by analogous procedure to (1). $\square$

By this lemma, we see the image of D_r is contained in a smaller space and immediately we have the following proposition and corollary.

Proposition 7. *There exist exact sequences as follows*:

(1) *If k is even*, $A_k(\Gamma) = A_k(\Gamma; 0)$ (*by definition*) *and*

$$0 \to A_k(\Gamma; 2r + 2) \to A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

(2) *If k is odd*, $A_k(\Gamma) = A_k(\Gamma; 1)$ *and*

$$0 \to A_k(\Gamma; 2r + 3) \to A_k(\Gamma; 2r + 1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r + 2).$$

Corollary 8. *We have an upper bound for the dimension of $A_k(\Gamma)$.*

(1) *If k is even*, $\dim_{\mathbb{C}} A_k(\Gamma) \le \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r)$.

(2) *If k is odd*, $\dim_{\mathbb{C}} A_k(\Gamma) \le \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r + 2)$.

To calculate the right-hand sides of this corollary, we discuss the structure of Witt modular forms. It is classically well known that $A_*(\Gamma')$ is generated by two algebraically independent modular forms e_4 and e_6, where e_4, e_6 are the Eisenstein series of weights 4 and 6. Ramanujan's delta function

$$\Delta(\tau) = \eta(\tau)^{24} = \frac{e_4(\tau)^3 - e_6(\tau)^2}{1728}$$

is a unique cusp form of weight 12. It is also well known that $A_k(\Gamma'; r) = \Delta^r A_{k-12r}(\Gamma')$. Therefore, the bigraded ring of Witt modular forms $\bigoplus_{k,l\in\mathbb{Z}} \mathrm{W}_{k,l}(\Gamma')$ is generated by four algebraically independent forms $e_4(\tau)$, $e_6(\tau)$, $e_4(\omega)$, $e_6(\omega)$. Especially, we have

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{sym}}(\Gamma') = \mathbb{C}[e_4(\tau)e_4(\omega), e_6(\tau)e_6(\omega), e_4(\tau)^3e_6(\omega)^2 + e_6(\tau)^2e_4(\omega)^3],$$

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{sym}}(\Gamma'; r) = (\Delta(\tau)\Delta(\omega))^r \left(\bigoplus_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{sym}}(\Gamma')\right)$$

and

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{skew}}(\Gamma'; r) = (e_4(\tau)^3e_6(\omega)^2 - e_6(\tau)^2e_4(\omega)^3)\left(\bigoplus_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{sym}}(\Gamma'; r)\right).$$

Therefore we have

$$\sum_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{sym}}(\Gamma'; r)x^k = \frac{x^{12r}}{(1-x^4)(1-x^6)(1-x^{12})}$$

and

$$\sum_{k\in\mathbb{Z}} \mathrm{W}_k^{\mathrm{skew}}(\Gamma'; r)x^k = \frac{x^{12(r+1)}}{(1-x^4)(1-x^6)(1-x^{12})}.$$

Hence, if k is even, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}_{k+2r}^{\mathrm{sym}}(\Gamma'; r))x^k &= \sum_{r=0}^{\infty} \frac{x^{12r-2r}}{(1-x^4)(1-x^6)(1-x^{12})} \\ &= \frac{1}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

If k is odd, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}_{k+2r+1}^{\mathrm{skew}}(\Gamma'; r+2))x^k &= \sum_{r=0}^{\infty} \frac{x^{12(r+3)-(2r+1)}}{(1-x^4)(1-x^6)(1-x^{12})} \\ &= \frac{x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

Consequently, we have an upper bound of the dimension of $A_k(\Gamma)$:

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} A_k(\Gamma))x^k \preceq \frac{1+x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})},$$

where $\preceq$ means that the dimension of $A_k(\Gamma)$ is not greater than the coefficient of x^k on the formal power series development of the right-hand side. Namely, if we construct algebraically independent modular forms of weight 4, 6, 10, 12, and if we construct a modular form of weight 35, we finish the proof of Theorem 1 for $N=1$. Indeed, Igusa [10, 11] constructed these modular forms from theta functions. We denote normalized generators by E_4, E_6, Δ_{10}, Δ_{12} and Δ_{35}. We remark

$$\begin{aligned} &E_4 \in A_4(\Gamma;0), && D_0(E_4) = e_4(\tau)e_4(\omega), && E_6 \in A_6(\Gamma;0), && D_0(E_6) = e_4(\tau)e_6(\omega),\\ &\Delta_{10} \in A_{10}(\Gamma;2), && D_2(\Delta_{10}) = \Delta(\tau)\Delta(\omega), && \Delta_{12} \in A_{12}(\Gamma;0), && D_0(\Delta_{12}) = \Delta(\tau)\Delta(\omega) \end{aligned}$$

and

$$\Delta_{35} \in A_{35}(\Gamma;1), \quad D_1(\Delta_{35}) = (e_4(\tau)^3 e_6(\omega)^2 - e_6(\tau)^2 e_4(\omega)^3)(\Delta(\tau)\Delta(\omega))^2.$$

This means D_{2r} and D_{2r+1} in Proposition 7 are surjective. Therefore, E_4, E_6, Δ_{10} and Δ_{12} are algebraically independent.

4.2. Case $N = 2$. Second, we consider the case $N=2$. Namely, we set $\Gamma := \Gamma_0^{(2)}(2)$ and $\Gamma' = \Gamma_0^{(1)}(2)$.

When $N=2$, the obstruction on our way is that there are more than one cusp. Therefore, we should observe the behavior of a modular form at each cusp at the same time.

Let

$$M_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} O_2 & -E_2 \\ 2E_2 & O_2 \end{pmatrix} \quad \text{and} \quad M_1' := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}.$$

For $F \in A_k(\Gamma;r)$ and $f \in A_k(\Gamma')$, it is easy to show that $F|_k M_1 \in A_k(\Gamma;r)$ and $f|_k M_1' \in A_k(\Gamma')$. For $f \in A_k(\Gamma')$, put the Fourier coefficients of f by

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n)q^n$$

and

$$(f|_k M_1')(\tau) = \sum_{n\in\mathbb{Z}} b_f(n)q^n.$$

We define

$$A_k(\Gamma';r) := \{ f \in A_k(\Gamma') \mid a_f(n) = 0,\ b_f(n) = 0 \text{ for } n < r\}$$

and apply the way in the previous subsection to F and $F|_k M_1$. Because Lemma 5 and Lemma 6 hold not only for F but also for $F|_k M_1$, we have the following proposition and corollary in a similar way in the previous section

Proposition 9. *There exist exact sequences as follows*:
(1) *If k is even*, $A_k(\Gamma) = A_k(\Gamma; 0)$ (*by definition*) *and*

$$0 \to A_k(\Gamma; 2r+2) \to A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

(2) *If k is odd*, $A_k(\Gamma) = A_k(\Gamma; 1)$ *and*

$$0 \to A_k(\Gamma; 2r+3) \to A_k(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

Corollary 10. *We have an upper bound for the dimension of $A_k(\Gamma)$.*
(1) *If k is even*, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r)$.
(2) *If k is odd*, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2)$.

It is well known that $A_k(\Gamma'; r) = (\eta(\tau)^8 \eta(2\tau)^8)^r A_{k-8r}(\Gamma')$ and that $A_*(\Gamma')$ is generated by two algebraically independent modular forms of weight 2 and 4. Hence, if k is even, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sum_{r=0}^{\infty} (\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r)) x^k &= \sum_{r=0}^{\infty} \frac{x^{8r-2r}}{(1-x^2)(1-x^4)(1-x^4)} \\ &= \frac{1}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.\end{aligned}$$

If k is odd, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sum_{r=0}^{\infty} (\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2)) x^k &= \sum_{r=0}^{\infty} \frac{x^{8(r+2)+4-(2r+1)}}{(1-x^2)(1-x^4)(1-x^4)} \\ &= \frac{x^{19}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.\end{aligned}$$

Consequently, we have an upper bound of the dimension of $A_k(\Gamma)$:

$$\sum_{k\in\mathbb{Z}} (\dim_{\mathbb{C}} A_k(\Gamma)) x^k \leq \frac{1+x^{19}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.$$

Namely, if we construct algebraically independent modular forms of weight 2,4,4,6 and if we construct a modular forms of weight 19, we finish the proof of Theorem 1 for $N = 2$. Indeed, Ibukiyama [7] constructed these modular forms from theta functions.

4.3. Case $N = 3$. Third, we consider the case $N = 3$. Namely, we set $\Gamma := \Gamma^{(2)}_{0,\psi_3}(3)$, $\Gamma' = \Gamma^{(1)}_{0,\psi_3}(3)$, $\tilde{\Gamma} := \Gamma^{(2)}_0(3)$, $\tilde{\Gamma}' = \Gamma^{(1)}_0(3)$, $\psi := \psi^{(2)}_3$ and $\psi' := \psi^{(1)}_3$. Let

$$M_1 := \frac{1}{\sqrt{3}}\begin{pmatrix} O_2 & -E_2 \\ 3E_2 & O_2 \end{pmatrix} \quad \text{and} \quad M_1' := \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}.$$

When $N = 3$, the obstruction on our way is $\gamma_0 \notin \Gamma$. Therefore, we decompose

$$A_k(\Gamma) = A_k(\tilde{\Gamma}) \oplus A_k(\tilde{\Gamma}, \psi)$$

and apply our process to $A_k(\tilde{\Gamma})$ and $A_k(\tilde{\Gamma}, \psi)$. If k is even, then $A_k(\tilde{\Gamma}') = A_k(\Gamma')$ and $\mathrm{W}_k(\tilde{\Gamma}') = \mathrm{W}_k(\Gamma')$. If k is odd, then $A_k(\tilde{\Gamma}') = A_k(\Gamma', \psi')$ and $\mathrm{W}_k(\tilde{\Gamma}') = \mathrm{W}_k(\Gamma', \psi')$. In a similar way in the previous section, we have the following proposition.

Proposition 11. *There exist exact sequences as follows*:

(1) *If k is even,*

(1a) $A_k(\tilde{\Gamma}) = A_k(\tilde{\Gamma}; 0)$ (*by definition*) *and*

$$0 \to A_k(\tilde{\Gamma}; 2r+2) \to A_k(\tilde{\Gamma}; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

(1b) $A_k(\tilde{\Gamma}, \psi) = A_k(\tilde{\Gamma}, \psi; 1)$ *and*

$$0 \to A_k(\tilde{\Gamma}, \psi; 2r+3) \to A_k(\tilde{\Gamma}, \psi; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

(2) *If k is odd,*

(2a) $A_k(\tilde{\Gamma}) = A_k(\tilde{\Gamma}; 1)$ *and*

$$0 \to A_k(\tilde{\Gamma}; 2r+3) \to A_k(\tilde{\Gamma}; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

(2b) $A_k(\tilde{\Gamma}, \psi) = A_k(\tilde{\Gamma}, \psi; 0)$ (*by definition*) *and*

$$0 \to A_k(\tilde{\Gamma}, \psi; 2r+2) \to A_k(\tilde{\Gamma}, \psi; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

Because $\dim_{\mathbb{C}} A_k(\Gamma) = \dim_{\mathbb{C}} A_k(\tilde{\Gamma}) + \dim_{\mathbb{C}} A_k(\tilde{\Gamma}, \psi)$, we have the following corollary immediately.

Corollary 12. *We have an upper bound*

$$\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r) + \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

It is well known that $A_k(\Gamma'; r) = (\eta(\tau)^6\eta(3\tau)^6)^r A_{k-6r}(\Gamma')$ and that $A_*(\Gamma')$ is generated by two algebraically independent modular forms of weight 1 and 3. Hence, we have

$$\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r))x^k = \sum_{r=0}^{\infty}\frac{x^{6r-2r}}{(1-x)(1-x^3)(1-x^3)}$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^3)(1-x^4)}$$

and

$$\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2))x^k = \sum_{r=0}^{\infty}\frac{x^{6(r+2)+3-(2r+1)}}{(1-x)(1-x^3)(1-x^3)}$$
$$= \frac{x^{14}}{(1-x)(1-x^3)(1-x^3)(1-x^4)}.$$

Consequently, we have an upper bound of the dimension of $A_k(\Gamma)$:

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} A_k(\Gamma))x^k \leq \frac{1+x^{14}}{(1-x)(1-x^3)(1-x^3)(1-x^4)}.$$

Namely, if we construct algebraically independent modular forms of weight 1,3,3,4 and if we construct a modular forms of weight 14, we finish the proof of Theorem 1 for $N = 3$. Indeed, Ibukiyama [7] constructed these modular forms from theta functions.

4.4. Case $N = 4$. Finally, we consider the case $N = 4$. Namely, we set $\Gamma := \Gamma^{(2)}_{0,\psi_4}(4)$, $\Gamma' = \Gamma^{(1)}_{0,\psi_4}(4)$, $\tilde{\Gamma} := \Gamma^{(2)}_0(4)$, $\tilde{\Gamma}' = \Gamma^{(1)}_0(4)$, $\psi := \psi^{(2)}_4$, $\psi' := \psi^{(1)}_4$ and decompose

$$A_k(\Gamma) = A_k(\tilde{\Gamma}) \oplus A_k(\tilde{\Gamma}, \psi).$$

Let

$$M_1 := \frac{1}{2}\begin{pmatrix} O_2 & -E_2 \\ 4E_2 & O_2 \end{pmatrix}, \quad M_1' := \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix},$$
$$M_2 := \begin{pmatrix} E_2 & O_2 \\ 2E_2 & E_2 \end{pmatrix} \quad \text{and} \quad M_2' := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

When $N = 4$, the obstruction on our way is $F|_k M_2 \notin A_k(\Gamma)$, even when $F \in A_k(\Gamma)$. However, $\gamma_0 \in M_2^{-1}\Gamma M_2$ and we can apply the way in the previous section not only to F, $F|_k M_1$ but also to $F|_k M_2$. Here, for $f \in A_k(\Gamma')$, we put the Fourier coefficients of f by

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n) q^n,$$
$$(f|_k M_1')(\tau) = \sum_{n\in\mathbb{Z}} b_f(n) q^n,$$
$$(f|_k M_2')(\tau) = \sum_{n\in\mathbb{Z}} c_f(n) q^{n/2}$$

and redefine

$$A_k(\Gamma'; r) := \{ f \in A_k(\Gamma') \mid a_f(n) = 0,\ b_f(n) = 0,\ c_f(n) = 0 \text{ for } n < r\}.$$

Then, in a similar way in the previous section, we have the following proposition and corollary.

Proposition 13. *There exist exact sequences as follows*:

(1) *If k is even,*

(1a) $A_k(\tilde{\Gamma}) = A_k(\tilde{\Gamma}; 0)$ (*by definition*) *and*

$$0 \to A_k(\tilde{\Gamma}; 2r+2) \to A_k(\tilde{\Gamma}; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

(1b) $A_k(\tilde{\Gamma}, \psi) = A_k(\tilde{\Gamma}, \psi; 1)$ *and*

$$0 \to A_k(\tilde{\Gamma}, \psi; 2r+3) \to A_k(\tilde{\Gamma}, \psi; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

(2) *If k is odd,*

(2a) $A_k(\tilde{\Gamma}) = A_k(\tilde{\Gamma}; 1)$ *and*

$$0 \to A_k(\tilde{\Gamma}; 2r+3) \to A_k(\tilde{\Gamma}; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

(2b) $A_k(\tilde{\Gamma}, \psi) = A_k(\tilde{\Gamma}, \psi; 0)$ (*by definition*) *and*

$$0 \to A_k(\tilde{\Gamma}, \psi; 2r+2) \to A_k(\tilde{\Gamma}, \psi; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r).$$

Corollary 14. *We have an upper bound*

$$\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r) + \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2).$$

It is well known that $A_k(\Gamma'; r) = (\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4)^r A_{k-5r}(\Gamma')$ and that $A_*(\Gamma')$ is generated by two algebraically independent modular forms of weight 1 and 2. Hence, we have

$$\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r}(\Gamma'; r))x^k = \sum_{r=0}^{\infty}\frac{x^{5r-2r}}{(1-x)(1-x^2)(1-x^2)}$$
$$= \frac{1}{(1-x)(1-x^2)(1-x^2)(1-x^3)}$$

and

$$\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}(\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+2))x^k = \sum_{r=0}^{\infty}\frac{x^{5(r+2)+2-(2r+1)}}{(1-x)(1-x^2)(1-x^2)}$$
$$= \frac{x^{11}}{(1-x)(1-x^2)(1-x^2)(1-x^3)}.$$

Consequently, we have an upper bound of the dimension of $A_k(\Gamma)$:

$$\sum_{k\in\mathbb{Z}}(\dim_{\mathbb{C}} A_k(\Gamma))x^k \leq \frac{1+x^{11}}{(1-x)(1-x^2)(1-x^2)(1-x^3)}.$$

We can show this upper bound coincides with the true dimension by constructing generators. Indeed, Hayashida and Ibukiyama [6] constructed these generators from theta functions.

5. Proof of Theorem 2

Our proof of Theorem 2 is almost similar to the proof of Theorem 1. But, because each Fourier coefficient is not a scalar but a vector, we need small modification.

5.1. Case $N = 1$. First, we give a lemma corresponding to Lemma 5. For $F \in A_{k,2}(\Gamma)$, put the Fourier coefficients of F by

$$F(Z) = \sum_{n,l,m\in\mathbb{Z}} a(n,l,m)q^n\zeta^l p^m$$

and denote

$$a(n,l,m) = \begin{pmatrix} a_1(n,l,m) \\ a_2(n,l,m) \\ a_3(n,l,m) \end{pmatrix}.$$

Lemma 15. *The Fourier coefficients of F satisfy the following equations*:
(1) *If $4nm - l^2 < 0$, $n < 0$ or $m < 0$, then $a(n,l,m) = \mathbf{0}$.*

(2) $a(n,-l,m) = (-1)^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a(n,l,m)$.

(3) $a(m,l,n) = (-1)^k \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} a(n,l,m)$.

(4) $a(n+xl+x^2m, l+2xm, m) = \begin{pmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 1 \end{pmatrix} a(n,l,m)$ *for any* $x \in \mathbb{Z}$.

(5) $a(n, l+2xn, m+xl+x^2n) = \begin{pmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ x^2 & x & 1 \end{pmatrix} a(n,l,m)$ *for any* $x \in \mathbb{Z}$.

(6a) *If* k *is even, then we have*

$$\begin{aligned} a_2(n,0,m) &= 0, \\ a_1(n,l,n) &= a_3(n,l,n), \\ a_1(n,n,m) &= a_2(n,n,m), \\ a_2(n,m,m) &= a_3(n,m,m). \end{aligned}$$

(6b) *If* k *is odd, then we have*

$$\begin{array}{ll} a_1(n,0,m) = 0, & a_3(n,0,m) = 0, \\ a_1(n,l,n) = -a_3(n,l,n), & a_2(n,l,n) = 0, \\ a_1(n,n,m) = 0, & a_2(n,n,m) = 2a_3(n,n,m), \\ a_3(n,m,m) = 0, & a_2(n,m,m) = 2a_1(n,m,m). \end{array}$$

Proof. This lemma is proved in the same manner as Lemma 5. For example, we can show $a_1(n,n,m) = a_2(n,n,m)$ on (6a) by substituting $l = n$ and $x = -1$ on (2) and (5). $\square$

Lemma 16. *The Fourier coefficients of* F *has the following properties*:

(1a) *Suppose* k *is even and* $F \in A_{k,2}(\Gamma; 2r)$.
If $\min\{n-1, m\} < r$, *then* $a_1(n,l,m) = 0$.
If $\min\{n, m\} < r+1$, *then* $a_2(n,l,m) = 0$.
If $\min\{n, m-1\} < r$, *then* $a_3(n,l,m) = 0$.

(1b) *Suppose* k *is even and* $F \in A_{k,2}(\Gamma; 2r+1)$.
If $\min\{n, m\} < r+1$, *then* $a(n,l,m) = \mathbf{0}$.

(2a) *Suppose* k *is odd and* $F \in A_{k,2}(\Gamma; 2r)$.
If $\min\{n-1, m\} < r+1$, *then* $a_1(n,l,m) = 0$.
If $\min\{n, m\} < r+1$, *then* $a_2(n,l,m) = 0$.
If $\min\{n, m-1\} < r+1$, *then* $a_3(n,l,m) = 0$.

(2b) *Suppose* k *is odd and* $F \in A_{k,2}(\Gamma; 2r+1)$.
If $\min\{n-1, m\} < r+1$, *then* $a_1(n,l,m) = 0$.

If $\min\{n, m\} < r + 1$, *then* $a_2(n, l, m) = 0$.
If $\min\{n, m - 1\} < r + 1$, *then* $a_3(n, l, m) = 0$.

Proof. This lemma is proved in the same manner as Lemma 6. □

Let

$$\mathrm{W}_{k,l}(\Gamma'; r, s) := A_k(\Gamma'; r) \otimes_{\mathbb{C}} A_l(\Gamma'; s)$$

be a subspace of $\mathrm{W}_{k,l}(\Gamma')$. We have the following proposition immediately.

Proposition 17. $A_{k,2}(\Gamma) = A_{k,2}(\Gamma; 0)$ (*by definition*) *and there exist exact sequences as follows*:
(1) *If k is even,*

$$0 \to A_{k,2}(\Gamma; 2r + 1) \to A_{k,2}(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma'; r + 1, r),$$

and

$$0 \to A_{k,2}(\Gamma; 2r + 2) \to A_{k,2}(\Gamma; 2r + 1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma'; r + 1).$$

(2) *If k is odd,*

$$0 \to A_{k,2}(\Gamma; 2r + 1) \to A_{k,2}(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r + 1),$$

and

$$0 \to A_{k,2}(\Gamma; 2r + 2) \to A_{k,2}(\Gamma; 2r + 1) \xrightarrow{D_{2r+1}} \mathrm{W}_{k+2r+3,k+2r+1}(\Gamma'; r + 2, r + 1).$$

When $s = 2$, we need one more lemma.

Lemma 18. *When $k = 10(r + 1)$, we have $D_{2r+1}(A_{k,2}(\Gamma; 2r + 1)) = 0$, although* $\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma'; r + 1) = 1$.

Proof. Set $k = 10(r + 1)$. From previous proposition, it is easy to show that

$$D_{2r+1}(A_{k,2}(\Gamma; 2r + 1)) \subset \mathbb{C}(\Delta(\tau)\Delta(\omega))^{r+1}, \quad \dim_{\mathbb{C}} A_{k,2}(\Gamma; 2r + 1) \leq 1,$$
$$D_{2r+1}(A_{k+4,2}(\Gamma; 2r + 1)) \subset \mathbb{C}e_4(\tau)e_4(\omega)(\Delta(\tau)\Delta(\omega))^{r+1}, \quad \dim_{\mathbb{C}} A_{k+4,2}(\Gamma; 2r + 1) \leq 1$$

and

$$D_{2r+1}(A_{k+6,2}(\Gamma; 2r + 1)) \subset \mathbb{C}e_6(\tau)e_6(\omega)(\Delta(\tau)\Delta(\omega))^{r+1}, \quad \dim_{\mathbb{C}} A_{k+6,2}(\Gamma; 2r + 1) \leq 1.$$

Because

$$D_0([E_4,\Delta_{10}])=0,\quad D_1([E_4,\Delta_{10}])=4e_4(\tau)e_4(\omega)\Delta(\tau)\Delta(\omega),$$
$$D_0([E_6,\Delta_{10}])=0\quad\text{and}\quad D_1([E_6,\Delta_{10}])=4e_6(\tau)e_6(\omega)\Delta(\tau)\Delta(\omega),$$

we have

$$A_{k+4,2}(\Gamma;2r+1)=\mathbb{C}\Delta_{10}^r[E_4,\Delta_{10}],$$
$$D_{2r+1}(\Delta_{10}^r[E_4,\Delta_{10}])=4(2r+1)!!\,e_4(\tau)e_4(\omega)(\Delta(\tau)\Delta(\omega))^{r+1}$$

and

$$A_{k+6,2}(\Gamma;2r+1)=\mathbb{C}\Delta_{10}^r[E_6,\Delta_{10}],$$
$$D_{2r+1}(\Delta_{10}^r[E_6,\Delta_{10}])=6(2r+1)!!\,e_6(\tau)e_6(\omega)(\Delta(\tau)\Delta(\omega))^{r+1}.$$

Assume the existence of $F\in A_{k,2}(\Gamma;2r+1)$ such that $D_{2r+1}(F)=(\Delta(\tau)\Delta(\omega))^{r+1}$. Then we have

$$4(2r+1)!!\,E_4F=\Delta_{10}^r[E_4,\Delta_{10}],$$
$$6(2r+1)!!\,E_6F=\Delta_{10}^r[E_6,\Delta_{10}]$$

and

$$6E_6\Delta_{10}^r[E_4,\Delta_{10}]-4E_4\Delta_{10}^r[E_6,\Delta_{10}]=0.$$

By Jacobi identity, it means $[E_4,E_6]=0$. It is a contradiction. Thus we have $D_{2r+1}(A_{k,2}(\Gamma;2r+1))=0$. □

Corollary 19. *We have an upper bound as follows*:
(1a) *If k is even and* $k\not\equiv 0 \pmod{10}$,

$$\dim_{\mathbb{C}} A_{k,2}(\Gamma)\le\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma';r+1,r)+\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma';r+1).$$

(1b) *If* $k\equiv 0 \pmod{10}$,

$$\dim_{\mathbb{C}} A_{k,2}(\Gamma)\le\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma';r+1,r)+\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma';r+1)-1.$$

(2) *If k is odd,*

$$\dim_{\mathbb{C}} A_{k,2}(\Gamma)\le\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}_{k+2r+3,k+2r+1}(\Gamma';r+2,r+1)+\sum_{r=0}^{\infty}\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma';r+1).$$

Proof. It is a easy consequence of Proposition 17 and Lemma 18. $\square$

Because $\dim_{\mathbb{C}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma'; r+1, r) = \dim_{\mathbb{C}} \mathrm{W}_{k-10r-10,k-10r}(\Gamma'; 0, 0)$, we need to calculate the dimension of $\mathrm{W}_{k-10,k}(\Gamma'; 0, 0)$. Namely, we need to calculate the coefficients $c_{k-10,k}$ in

$$\frac{1}{(1-x^4)(1-x^6)(1-y^4)(1-y^6)} = \sum_{k,l\in\mathbb{N}_0} c_{k,l}\, x^k y^l.$$

Because

$$\begin{aligned}
&\frac{1}{(1-x^4)(1-x^6)(1-y^4)(1-y^6)} \\
&= \left(\frac{1}{(1-x^4)(1-y^4)}\right) \times \left(\frac{1}{(1-x^6)(1-y^6)}\right) \\
&= \left(\frac{\cdots + x^{20} + x^{16} + x^{12} + x^8 + x^4 + 1 + y^4 + y^8 + y^{12} + y^{16} + y^{20} + \cdots}{1 - x^4 y^4}\right) \\
&\quad \times \left(\frac{\cdots + x^{30} + x^{24} + x^{18} + x^{12} + x^6 + 1 + y^6 + y^{12} + y^{18} + y^{24} + y^{30} + \cdots}{1 - x^6 y^6}\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0} c_{k-10,k}\, x^{k-10} y^k \\
&= \frac{\cdots + x^{20}y^{30} + x^8y^{18} + y^{10} + x^6y^{16} + x^{18}y^{28} + \cdots}{(1-x^4y^4)(1-x^6y^6)} \\
&= \frac{\cdots + x^{20}y^{30} + x^8y^{18}}{(1-x^4y^4)(1-x^6y^6)} + \frac{y^{10}}{(1-x^4y^4)(1-x^6y^6)} + \frac{x^6y^{16} + x^{18}y^{28} + \cdots}{(1-x^4y^4)(1-x^6y^6)} \\
&= \frac{x^8y^{18}}{(1-x^4y^4)(1-x^6y^6)(1-x^{12}y^{12})} + \frac{y^{10}}{(1-x^4y^4)(1-x^6y^6)} \\
&\quad + \frac{x^6y^{16}}{(1-x^4y^4)(1-x^6y^6)(1-x^{12}y^{12})}.
\end{aligned}$$

Hence, if k is even, we have an upper bound

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}} (\dim_{\mathbb{C}} A_{k,2}(\Gamma)) x^k \\
&\leq \frac{x^{16} + x^{18}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})} + \frac{x^{10}}{(1-x^4)(1-x^6)(1-x^{10})} \\
&\quad + \frac{x^{10}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})} - \frac{x^{10}}{(1-x^{10})} \\
&= \frac{(x^{10} + x^{14} + 2x^{16} + x^{18} + x^{22}) - (x^{20} + x^{22} + x^{26} + x^{28}) + x^{32}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.
\end{aligned}$$

We can show this upper bound coincides with the true dimension by constructing generators by differential operators (p. 4). Because D_{2r} and D_{2r+1} in Proposition 17 are surjective except when $k = 10(r+1)$, Jacobi identity is the only relation between these generators.

If k is odd, we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}+1} (\dim_{\mathbb{C}} A_{k,2}(\Gamma))x^k \\
&\le \frac{x^{27}+x^{29}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})} + \frac{x^{21}}{(1-x^4)(1-x^6)(1-x^{10})} \\
&\quad + \frac{x^{23}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})} \\
&= \frac{(x^{21}+x^{23}+x^{27}+x^{29})-x^{33}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.
\end{aligned}$$

We can show this upper bound coincides with the true dimension by constructing generators by differential operators (p. 5).

5.2. Case $N = 2, 3, 4$. Now we have already studied all the technique to prove our theorem. By similar calculation, we have an upper bound of the dimension of modular forms and we can show that it coincides with the true dimension by constructing generators by differential operators.

When $N = 2$, we have the following proposition.

Proposition 20. *The space $A_{k,2}(\Gamma) = A_{k,2}(\Gamma; 0)$ has the following properties*:

(1) *If k is even, there are exact sequences*

$$0 \to A_{k,2}(\Gamma; 2r+1) \to A_{k,2}(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma'; r+1, r)$$

and

$$0 \to A_{k,2}(\Gamma; 2r+2) \to A_{k,2}(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma'; r+1).$$

(2) *If k is odd, there are exact sequences*

$$0 \to A_{k,2}(\Gamma; 2r+1) \to A_{k,2}(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma'; r+1)$$

and

$$0 \to A_{k,2}(\Gamma; 2r+2) \to A_{k,2}(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}_{k+2r+3,k+2r+1}(\Gamma'; r+2, r+1).$$

(3) *When $k = 6(r+1)$, we have $D_{2r+1}(A_{k,2}(\Gamma; 2r+1)) = 0$, although*

$$\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma'; r+1) = 1.$$

Hence, if k is even, we have an upper bound

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}}(\dim_{\mathbb{C}} A_{k,2}(\Gamma))x^k \\
&\le \frac{x^6+x^8}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)} + \frac{x^6}{(1-x^2)(1-x^4)(1-x^6)} \\
&\quad + \frac{x^6}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)} - \frac{x^6}{(1-x^6)} \\
&= \frac{(2x^6+2x^8+2x^{10})-(x^{10}+2x^{12}+x^{14})+x^{16}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.
\end{aligned}$$

If k is odd, we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}+1}(\dim_{\mathbb{C}} A_{k,2}(\Gamma))x^k \\
&\le \frac{x^{13}+x^{15}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)} + \frac{x^{13}}{(1-x^2)(1-x^4)(1-x^6)} \\
&\quad + \frac{x^{11}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)} \\
&= \frac{(x^{11}+2x^{13}+x^{15})-x^{17}}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.
\end{aligned}$$

When $N=3$ or $N=4$, we have the following proposition.

Proposition 21. *We decompose*

$$A_{k,2}(\Gamma) = A_{k,2}(\tilde{\Gamma};0) \oplus A_k(\tilde{\Gamma},\psi;0).$$

Each decomposed space has the following properties:
(1a) *If k is even, there are exact sequences*

$$0 \to A_{k,2}(\tilde{\Gamma};2r+1) \to A_{k,2}(\tilde{\Gamma};2r) \xrightarrow{D_{2r}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma',\psi';r+1,r)$$

and

$$0 \to A_{k,2}(\tilde{\Gamma};2r+2) \to A_{k,2}(\tilde{\Gamma};2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma',\psi';r+1).$$

(1b) *If k is even, there are exact sequences*

$$0 \to A_{k,2}(\tilde{\Gamma},\psi;2r+1) \to A_{k,2}(\tilde{\Gamma},\psi;2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma',\psi';r+1)$$

and

$$0 \to A_{k,2}(\tilde{\Gamma},\psi;2r+2) \to A_{k,2}(\tilde{\Gamma},\psi;2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}_{k+2r+3,k+2r+1}(\Gamma',\psi';r+2,r+1).$$

(2a) *If k is odd, there are exact sequences*

$$0 \to A_{k,2}(\tilde{\Gamma}; 2r+1) \to A_{k,2}(\tilde{\Gamma}; 2r) \xrightarrow{D_{2r}} \mathrm{W}^{\mathrm{skew}}_{k+2r+1}(\Gamma', \psi'; r+1)$$

and

$$0 \to A_{k,2}(\tilde{\Gamma}; 2r+2) \to A_{k,2}(\tilde{\Gamma}; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}_{k+2r+3,k+2r+1}(\Gamma', \psi'; r+2, r+1).$$

(2b) *If k is odd, there are exact sequences*

$$0 \to A_{k,2}(\tilde{\Gamma}, \psi; 2r+1) \to A_{k,2}(\tilde{\Gamma}, \psi; 2r) \xrightarrow{D_{2r}} \mathrm{W}_{k+2r+2,k+2r}(\Gamma', \psi'; r+1, r)$$

and

$$0 \to A_{k,2}(\tilde{\Gamma}, \psi; 2r+2) \to A_{k,2}(\tilde{\Gamma}, \psi; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma', \psi'; r+1).$$

(3) *When $k = 4(r+1)$ $(N = 3)$ or when $k = 3(r+1)$ $(N = 4)$, we have* $D_{2r+1}(A_{k,2}(\tilde{\Gamma}; 2r+1)) = 0$, *although* $\dim_{\mathbb{C}} \mathrm{W}^{\mathrm{sym}}_{k+2r+2}(\Gamma'; r+1) = 1$.

Hence, when $N = 3$, from (1a) (2b) and (3), we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}))x^k + \sum_{k\in 2\mathbb{Z}+1}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}, \psi))x^k \\
&\le \frac{x^4+x^6}{(1-x)(1-x^3)(1-x^3)(1-x^4)} + \frac{x^4}{(1-x)(1-x^3)(1-x^4)} \\
&\quad + \frac{x^4}{(1-x)(1-x^3)(1-x^3)(1-x^4)} - \frac{x^4}{(1-x^4)} \\
&= \frac{(2x^4+x^5+x^6+2x^7)-(x^7+2x^8+x^{10})+x^{11}}{(1-x)(1-x^3)(1-x^3)(1-x^4)}
\end{aligned}$$

and from (1b) and (2a), we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}+1}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}))x^k + \sum_{k\in 2\mathbb{Z}}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}, \psi))x^k \\
&\le \frac{x^9+x^{11}}{(1-x)(1-x^3)(1-x^3)(1-x^4)} + \frac{x^9}{(1-x)(1-x^3)(1-x^4)} \\
&\quad + \frac{x^8}{(1-x)(1-x^3)(1-x^3)(1-x^4)} \\
&= \frac{(x^8+2x^9+x^{11})-x^{12}}{(1-x)(1-x^3)(1-x^3)(1-x^4)}.
\end{aligned}$$

The first case corresponds to the generators of type 1 and the last case corresponds to the generators of type 2.

When $N = 4$, from (1a) (2b) and (3), we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}))x^k + \sum_{k\in 2\mathbb{Z}+1}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}, \psi))x^k \\
&\le \frac{x^3+x^4}{(1-x)(1-x^2)(1-x^2)(1-x^3)} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)} \\
&\quad + \frac{x^3}{(1-x)(1-x^2)(1-x^2)(1-x^3)} - \frac{x^3}{(1-x^3)} \\
&= \frac{(2x^3+2x^4+2x^5)-(x^5+2x^6+x^7)+x^8}{(1-x)(1-x^2)(1-x^2)(1-x^3)}
\end{aligned}$$

and from (1b) and (2a), we have

$$\begin{aligned}
&\sum_{k\in 2\mathbb{Z}+1}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}))x^k + \sum_{k\in 2\mathbb{Z}}(\dim_{\mathbb{C}} A_{k,2}(\tilde{\Gamma}, \psi))x^k \\
&\le \frac{x^7+x^8}{(1-x)(1-x^2)(1-x^2)(1-x^3)} + \frac{x^7}{(1-x)(1-x^2)(1-x^3)} \\
&\quad + \frac{x^6}{(1-x)(1-x^2)(1-x^2)(1-x^3)} \\
&= \frac{(x^6+2x^7+x^8)-x^9}{(1-x)(1-x^2)(1-x^2)(1-x^3)}.
\end{aligned}$$

The first case corresponds to the generators of type 1 and the last case corresponds to the generators of type 2.

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