ON POSITIVE QUATERNIONIC KÄHLER MANIFOLDS
WITH $b_4 = 1$

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Abstract

Let M be a positive quaternionic Kähler manifold of dimension $4m$. In earlier papers, Fang and the first author showed that if the symmetry rank is greater than or equal to $[m/2] + 3$, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. The goal of this paper is to give a more refined classification result for positive quaternionic Kähler manifolds (in particular, of relatively low dimension or with even m) whose fourth Betti number equals one. To be precise, we show in this paper that if the symmetry rank of M with $b_4(M) = 1$ is no less than $[m/2] + 2$ for $m \geq 5$, then M is isometric to $\mathbf{HP}^m$.

1. Introduction and main results

A compact quaternionic Kähler manifold M is a Riemannian manifold of real dimension $4m$ whose holonomy group is contained in the Lie group $Sp(m)Sp(1)$ in $SO(4m)$ for $m \geq 2$. Such a manifold is called *positive* if it has the positive scalar curvature. It is known that every quaternionic Kähler manifold is Einstein. So it is common to define a 4-dimensional quaternionic Kähler manifold to be both Einstein with non-zero scalar curvature and self-dual. While many complete, non-compact, non-symmetric quaternionic Kähler manifolds with negative scalar curvature are known to exist, so far the only known examples of compact positive quaternionic Kähler manifolds are symmetric (see some similarities in [10] and [12] for positively curved Riemannian manifolds). Moreover, a theorem of Alekseevsky asserts that there are no other compact homogeneous positive quaternionic Kähler manifolds (e.g., see [1]).

According to a result of LeBrun and Salamon in [13], every positive quaternionic Kähler manifold M is simply connected and the second homotopy group π_2 is a finite group with 2-torsion, trivial or $\mathbf{Z}$. More precisely, M is isometric to $\mathbf{HP}^m$ (resp. $Gr_2(\mathbf{C}^{m+2})$) if $\pi_2(M) = 0$ (resp. $\pi_2(M) = \mathbf{Z}$). So its second Betti number is always less than or equal to 1. Furthermore, for such quaternionic Kähler manifolds of dimension $4m$ all odd Betti numbers vanish, so that the Euler characteristic of the manifold is always positive. Recall

also that there exists a nice relationship between Betti numbers of different degrees such as

$$\sum_{i=0}^{m-1} 2(6i(m-i-1) - (m-1)(m-3))b_{2i} = m(m-1)b_{2m}$$

(e.g., see [13] for more details).

On the other hand, it is also one of the interesting problems to classify positive quaternionic Kähler manifolds in terms of the rank of its isometry group. Strictly speaking, the symmetry rank sym-rank(M, g) (or simply sym-rank(M)) of a Riemannian manifold with a Riemannian metric g is defined as the rank of the isometry group Isom(M, g). Equivalently, it can be defined as the largest number r such that a r-dimensional torus acts effectively and isometrically on M. This concept was first introduced by Grove and Searle in [6] in order to measure the amount of symmetry of M.

It is not so hard to see that the symmetry rank sym-rank(M) of a positive quaternionic Kähler manifold of dimension $4m$ is less than or equal to $m+1$. In [2] Bielawski classified positive quaternionic Kähler manifolds of dimension $4m$ with isometry rank equal to $m + 1$. Moreover, in earlier papers [4] and [9], Fang and the first author gave a classification result of positive quaternionic Kähler manifolds with certain symmetry. That is, we showed that if the symmetry rank is greater than or equal to $[m/2] + 3$, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. In fact, there have been some concrete classification results of positive quaternionic Kähler manifolds of low dimension. For examples, Hitchin proved in [8] that every positive quaternionic Kähler 4-manifold must be isometric to $\mathbf{CP}^2$ and S^4. In case of dimension 8, Poon and Salamon showed in [14] that every positive quaternionic Kähler manifold should be isometric to $\mathbf{HP}^2$, $Gr_2(\mathbf{C}^4)$ or $G_2/SO(4)$, i.e., the Wolf spaces. Moreover, in [7] H. Herrera and R. Herrera gave the classification of positive quaternionic Kähler 12-dimensional manifolds under an isometric S^1-action. As a consequence of their classification, such a manifold is isometric to $\mathbf{HP}^3$, $Gr_2(\mathbf{C}^5)$ or $\widetilde{Gr_4(\mathbf{R}^7)}$. Here $\widetilde{Gr_4(\mathbf{R}^7)}$ means the oriented real Grassmannian manifold of dimension 12.

The goal of this paper is to give a more refined classification result for positive quaternionic Kähler manifolds whose fourth Betti number equals one. To be precise, in this paper we show the following theorem:

Theorem 1.1. *Let M be a positive quaternionic Kähler manifold of dimension* $4m$ *with* $b_4(M) = 1$. *If the symmetry rank of M satisfies*

$$\text{sym-rank}(M) \geq \left[\frac{m}{2}\right] + 2, \quad m \geq 5,$$

then M is isometric to $\mathbf{HP}^m$.

Finally, a remark is in order. After having submitted this paper for publication, in the subsequent paper [11] we were able to improve the lower bound of the symmetry

rank in Theorem 1.1 by one for positive quaternionic Kähler manifolds of real dimension $4m \geq 40$ (here, m is an even integer greater than or equal to 10) with $b_4 = 1$. On the other hand, note that Theorem 1.1 applies to any positive quaternionic Kähler manifold of real dimension ≥ 20 with $b_4(M) = 1$, under the stated condition of symmetry rank. The method of the paper [11] is to use a more delicate argument of Frankel type to positive quaternionic Kähler manifolds with certain symmetry rank which works well only for higher dimensional positive quaternionic Kähler manifolds. So we remark that the methods of two papers are essentially of different nature.

We organize this paper as follows. In Section 2, we set up basic terminology and prove several important results for the proof of our main theorem. In Section 3, we give a proof of Theorem 1.1 through the stratification of the connected components of the fixed point sets. Finally we remark that this paper is very much influenced by the ideas in the papers [3] and [4].

2. Preparatory results

In this section we set up basic terminology and prove several important results for the proof of our main Theorem 1.1. Throughout this paper, all Lie group actions on a Riemannian manifold are assumed to be effective and isometric.

Now we begin with the connectedness theorem of Fang and the first author in [3] and [9]. To do so, first recall that a map $f: N \to M$ between two manifolds is called *h-connected* if the induced map $f_*: \pi_i(N) \to \pi_i(M)$ is an isomorphism for all $i < h$ and an epimorphism for $i = h$. If f is an imbedding this is equivalent to saying that up to homotopy M can be obtained from $f(N)$ by attaching cells of dimension $\geq h+1$.

Theorem 2.1. *Let M be a positive quaternionic Kähler manifold of dimension $4m$. If N is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \hookrightarrow M$ is $(2n - m + 1)$-connected. Furthermore, if there is a Lie group G acting isometrically on M and fixing N pointwise, then the inclusion map is $(2n - m + 1 + \delta(G))$-connected, where $\delta(G)$ is the dimension of the principal orbit of G.*

The first statement of Theorem 2.1 is due to Fang ([3], [4]), while its second statement is the extension to the case with group action which is due to the first author ([9]). The latter can be also considered to be an extension of the connectedness theorem of Wilking in [15] and independently Fang, Mendonça, and Rong in [5] for positively curved manifolds to positive quaternionic Kähler manifolds.

We also need the following lemma.

Lemma 2.2. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with an isometric T^{m-1}-action. Then there always exists an isolated fixed point of the T^{m-1}-action.*

Proof.   We will show this lemma by contradiction. To do so, suppose that there exists a fixed point component N of dimension 4 in the fixed point set under the T^{m-1}-action. If we use a stratification of the fixed point sets under the T^k-subaction of T^{m-1}-action we can consider a sequence of positive connected quaternionic Kähler submanifolds as follows.

$$(2.1) \qquad x \in N = N_1^4 \subset N_2^8 \subset \cdots \subset N_{m-1}^{4(m-1)} \subset M,$$

where each quaternionic submanifold N_i^{4i} admits an isometric T^{i-1}-subaction of T^{m-1}-action on M. So, in particular, $N = N_1^4$ is contained in positive quaternionic Kähler strata $N_2^8 \subset N_3^{12}$ such that N_3^{12} admits an isometric T^2-action. If N_2^8 is isometric $G_2/SO(4)$ then N should be isometric to $\mathbf{CP}^2$, since it is known that the only 4-dimensional positive quaternionic Kähler manifold in $G_2/SO(4)$ is $\mathbf{CP}^2$. But then we claim that the T^2-action on N_3^{12} should have a fixed point outside N. Indeed, otherwise the Euler characteristic χ of N coincides with that of N_3^{12}, i.e., $\chi(N_3^{12}) = 3$. This implies that $b_6(N_3^{12}) = 1$. On the other hand, since we have the relation $b_6(N_3^{12}) = 2b_2(N_3^{12})$ (e.g., see [13]), $b_6(N_3^{12})$ should be even. This is a contradiction.

Next if N_2^8 is isometric to $\mathbf{HP}^2$, then it follows from Theorem 2.1 that the inclusion of N_2^8 into N_3^{12} is at least 3-connected. Hence N_3^{12} is also isometric to $\mathbf{HP}^3$. But then, since the Euler characteristic of N_2^8 (resp. N_3^{12}) is 3 (resp. 4), there should be an isolated fixed point of the T^2-action on N_3^{12} outside N_2^8. But that fixed point is also an isolated fixed point of the T^{m-1}-action on M. Hence we are done in this case. If N_2^8 is isometric to $Gr_2(\mathbf{C}^4)$, then, as in the above case, N_3^{12} is isometric to $Gr_2(\mathbf{C}^4)$. Since the Euler characteristic of N_2^8 (resp. N_3^{12}) is 6 (resp. 10), it follows from Theorem 0.1 of Frankel type in [3] there should be an isolated fixed point of the T^2-action on N_3^{12} outside N_2^8 which is also an isolated fixed point of the T^{m-1}-action on M.

Therefore we may assume that there is another fixed point component N' of dimension ≥ 4 outside N in the fixed point set under the T^2-action on N_3^{12}. But then N' and N_2^8 would intersect to each other by Theorem 0.1 of Frankel type in [3]. Hence N' is contained in N_2^8. If we apply the Theorem 0.1 of Frankel type in [3] to N and N' in N_2^8 once again, we can easily derive a contradiction. This completes the proof of Lemma 2.2.   □

Next we need the following Proposition 2.3 which will be useful to prove some important results at several places of this paper. Its statement can be found in Proposition 2.3 in [3] whose proof can be referred to the paper [7] of Herrera and Herrera. However, their paper contains the proof for more general results. For the sake of reader's convenience, we briefly sketch a different and interesting proof which seems to be known to experts.

To do so, first let us recall some definitions. Let G be a connected Lie subgroup of the isometry group Isom(M). For any $x \in M$, the isotropy group G_x is a subgroup of the holonomy group $(Sp(m)Sp(1))_x$ at x. So the isotropy representation is determined by

two homomorphisms

$$\rho_x \colon G_x \to Sp(m)_x, \quad \bar{\rho}_x \colon G_x \to Sp(1)_x.$$

We then say that an isometric G-action on M is *of quaternionic type* if $\bar{\rho}_x \colon G_x \to Sp(1)_x$ is trivial for any G-fixed point $x \in M$ (see [3], Section 2 for more details about the torus action of quaternionic type on a quaternionic Kähler manifold).

Proposition 2.3. *Let M be a positive quaternionic Kähler manifold of dimension 12 with an isometric T^r-action. If $r = 4$ or $r = 3$ and T^3-action is of quaternionic type then M is isometric to* $\mathbf{HP}^3$ *or* $Gr_2(\mathbf{C}^5)$.

Proof.   To show it, we consider the two cases, depending on $b_2(M)$. If $b_2(M) \geq 1$ then it follows from a result of LeBrun and Salamon that M is isometric to $Gr_2(\mathbf{C}^{m+2})$. So we are done.

Now we assume that $b_2(M) = 0$. Note that the T^r-action on M has only isolated fixed points, since r is greater than or equal to 3. At each isolated fixed point x, by considering the isotropy representation on the tangent space of M at x we see that there are exactly three positive quaternionic Kähler submanifolds of dimension 8 passing through x and also three positive quaternionic Kähler submanifolds of dimension 4 passing through x. Since every 8-dimensional positive quaternionic Kähler manifold is known to be isometric to either $\mathbf{HP}^2$ or $G_2/SO(4)$ or $Gr_2(\mathbf{C}^4)$. If any one of the 8-dimensional positive quaternionic Kähler submanifold is isometric to $\mathbf{HP}^2$ then M should be isometric to $\mathbf{HP}^3$, since the inclusion of the 8-dimensional submanifold into M is at least 3-connected by the connected Theorem 2.1. As in the proof of Lemma 2.2, we can conclude that the 8-dimensional positive quaternionic Kähler submanifold cannot be isometric to $G_2/SO(4)$.

Next we assume that all of the 8-dimensional positive quaternionic Kähler submanifolds are isometric to $Gr_2(\mathbf{C}^4)$. Then we first claim that the Euler characteristic of M is equal to 10. Since all the fixed points are isolated, it suffices to show that the action has 10 isolated fixed points. To see it, recall that at each isolated fixed point x, there are exactly three positive quaternionic Kähler submanifolds of dimension 8 passing through x and that each 8-dimensional positive quaternionic Kähler submanifold contains two 4-dimensional positive quaternionic Kähler submanifolds passing through x. In what follows, we call such a 4-dimensional positive quaternionic Kähler submanifold a *triangle*, and its terminology seems reasonable in view of the moment map image of $\mathbf{CP}^2$ in symplectic geometry.

Now fix such an 8-dimensional positive quaternionic Kähler submanifold N isometric $Gr_2(\mathbf{C}^4)$. Then N contains six isolated fixed point, called *vertices* and denoted $v_1, v_2, \ldots, v_6$, of the T^{r-1}-subaction of T^r-action, since the Euler characteristic of $Gr_2(\mathbf{C}^4)$ equals 6. We next show that there are exactly four more isolated fixed points outside N. To do so, let k be the number of vertices outside N. Since there are exactly

three triangles passing through each vertex and every triangle shares at least one vertex in N, a simple combinatorial argument says that there is no triangle lying outside N and that there are at least $3k/2$ triangles which does not lie in N. Thus we should have $3k/2 \leq 6$, so that $k \leq 4$. Finally since the Euler characteristic of M is even, it is easy to see that actually $k = 4$ and so we can finish the proof of the claim.

Next we assume that the Euler characteristic of M equals 10. Then we show that M is isometric to $Gr_2(\mathbf{C}^5)$. We show it by proving $b_2(M) \neq 0$. If $b_2(M) = 0$ then it follows from $b_6(M) = 2b_2(M)$ that $b_6(M) = 0$. Recall next that the Euler characteristic of M is given by the sum of the sign ± 1 associated to 10 isolated fixed points. So in our case the signature is zero. Since the signature of $Gr_2(\mathbf{C}^4)$ equals 2, there are exactly four $+1$ and exactly two -1 vertices among the six vertices $v_1, v_2, \ldots, v_6$. Moreover, since the signature of M is zero, there are exactly three vertices v_7, v_8, v_9 whose sign are all -1. Again a simple combinatorial argument gives rise to a contradiction to the signature of M. This completes the proof of Proposition 2.3. $\qquad\square$

We also need the following Proposition 2.4 which is analogous to Theorem B in [3]. Indeed, Fang proved the same result with a weaker assumption $m - 2$ on the lower bound of the symmetry rank, but a stronger assumption $m \geq 10$ on the dimension. A key ingredient with which we are able to weaken the dimension condition in Theorem B of [3] is the refined connectedness Theorem 2.1 above.

Proposition 2.4. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ with an isometric T^{m-1}-action of quaternionic type. If m is greater than or equal to 5, then M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

Proof. Since the Euler characteristic of M is not zero, there exists a fixed point $x \in M$ of the T^{m-1}-action. As in the proof of Lemma 2.2, if we use a stratification of fixed point sets under the T^k-subaction of T^{m-1}-action we can find a quaternionic Kähler manifold N containing x which lies in the fixed point set under the S^1-subaction of T^{m-1}.

Now we need to consider two cases, depending on the codimension of the submanifold N. If the codimension of N in M is 4, then it follows from Theorem 1.2 of Fang that M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. Hence we are done in this case.

On the other hand, if the codimension of N in M is at least 8 then we can consider a sequence of connected quaternionic Kähler submanifolds as follows.

$$(2.2) \qquad \{x\} = N_1^0 \subset N_2^4 \subset N_3^8 \subset \cdots \subset N_{m-1}^{4(m-2)} = N \subset M,$$

where for each $1 \leq i \leq m - 1$ the quaternionic submanifold $N_i^{4(i-1)}$ admits an isometric T^i-subaction of T^{m-1}-action on M. Observe that the codimension of N in M is actually equal to 8 by considering the chain (2.2) more closely. Next if we apply the

connectedness Theorem 2.1 to the above chain (2.2), we obtain

$$(2.3) \qquad \pi_2(N_2^8) \cong \pi_2(N_3^{12}) \cong \cdots \cong \pi_2(N_{m-2}^{4(m-2)}) \cong \pi_2(M),$$

provided that m is greater than or equal to 5. Since the isotropy group of the sub-manifold $N_{m-2}^{4(m-2)}$ has rank 1 by construction, N_3^{12} can be assumed to admit an isometric T^3-action which is of quaternionic type. Thus it follows from Proposition 2.3 that N is isometric to $\mathbf{HP}^3$ or $Gr_2(\mathbf{C}^5)$. Thus by (2.3), $\pi_2(M)$ is either 0 or $\mathbf{Z}$. Hence M is also isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$ by the rigidity result of LeBrun and Salamon in [13]. This completes the proof. □

We close this section with the following lemma necessary for the proof of Theorem 1.1 which is an immediate consequence of Theorem 1.2 in [4].

Lemma 2.5. *Let M be a positive quaternionic Kähler manifold of dimension $4m$ ($m \geq 3$) with $b_4(M) = 1$ which admits an isometric S^1-action. If N is a positive quaternionic Kähler submanifold of codimension 4 of M in the fixed point set of the S^1-action, then M is isometric to $\mathbf{HP}^m$.*

Proof. It follows from Theorem 1.2 of [4] that M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+4})$. Since $b_4(Gr_2(\mathbf{C}^{m+2}))$ is not equal to 1, M should be isometric to $\mathbf{HP}^m$. □

3. Proof of Theorem 1.1

The goal of this section is to give a proof of our main Theorem 1.1. To do so, we assume first that m ($m \geq 6$) is even and let $k = [m/2] + 2$. We may assume without loss of generality that there is no stratum of codimension 4 by Theorem 1.2 of Fang in [4] or Lemma 2.5.

In what follows, we denote by $\mathrm{Fix}(T^k, M)$ the fixed point set under the action of T^k on M. Let x be a fixed point of T^k-action on M, and let N be a positive quaternionic Kähler submanifold passing through x of M of the lowest codimension ≥ 8. Then N should admit an isometric T^{k-1}-subaction of the T^k-action on M. For $1 \leq i \leq k$, let $N_i = \mathrm{Fix}(T^i, M)_0$ be a connected component of the fixed point set $\mathrm{Fix}(T^i, M)$. Then there is a chain of positive connected quaternionic Kähler submanifolds of M as follows.

$$(3.1) \qquad x \in N_k \subset N_{k-1} \subset \cdots \subset N_1 = N \subset M = N_0.$$

Clearly each positive quaternionic Kähler manifold N_i admits an isometric T^{k-i}-action for each $0 \leq i \leq k$. Then we can show the following lemma.

Lemma 3.1. *Either the T^k-action on M always has an isolated fixed point or M is isometric to $\mathbf{HP}^m$.*

Proof. We assume that M is not isometric to $\mathbf{HP}^m$. We shall show the lemma by contradiction. So, suppose that there is no isolated fixed point under the T^k-action. Then we should have $\dim N_k \geq 4$ and so we have $\dim N_1 = \dim N \geq 4k$. By assumption, we also have $\dim N_1 \leq 4(m-2)$. This implies that the inclusion of N_1 into M is at least 6-connected by Theorem 2.1, since we have

$$2 \cdot \frac{1}{4} \dim N_1 - \frac{1}{4} \dim M + 2 = \frac{1}{2} \dim N_1 - m + 2$$
$$\geq 2k - m + 1 = 2\left(\frac{m}{2} + 2\right) - m + 2 = 6.$$

Hence by a theorem of Whitehead we have $b_4(N_1) = b_4(M) = 1$.

Next we show that we may assume that the difference $\dim N_1 - \dim N_2$ is greater than or equal to 8. To see it, if we assume that there is a connected positive quaternionic Kähler submanifold of codimension 4 in N_1 whose isotropy group is a circle of the T^{k-1}-action on N_1. Then since $b_4(N_1) = 1$, by Lemma 2.5 N_1 is isometric to $\mathbf{HP}^l$ for some $l \leq m - 2$. This in turn implies that $\pi_2(M) = 0$, so that M is isometric to $\mathbf{HP}^m$. Thus we have a contradiction. Hence we can conclude

$$\dim N_1 - \dim N_2 \geq 8.$$

This implies $\dim N_1 \leq 4m - 8$. Since $\dim N_2 \geq 4(k-1)$, we can also show that the inclusion of N_2 into N_1 is also at least 6-connected, since we have

$$2 \cdot \frac{1}{4} \dim N_2 - \frac{1}{4} \dim N_1 + 2 \geq 2(k-1) - (m-2) + 2$$
$$= 2\left(\frac{m}{2} + 2 - 1\right) - m + 4 = 6.$$

Hence again we have $b_4(N_2) = b_4(N_1) = 1$. A similar argument as above shows that

$$\dim N_2 - \dim N_3 \geq 8, \quad \dim N_3 \leq 4(m-6), \quad \text{and} \quad \dim N_3 \geq 4(k-2).$$

Repeating this arguments, for $i \geq 0$ we have the following relations:

(3.2) $\dim N_i - \dim N_{i+1} \geq 8, \quad \dim N_i \leq 4(m-2i), \quad \text{and} \quad \dim N_i \geq 4(k-i+1).$

Moreover, from (3.2) we have

$$2 + \frac{1}{2}(k - i + 1) \leq 2 + \frac{1}{8} \dim N_i \leq k - i.$$

Thus we have

$$k - i = \frac{m}{2} + 2 - i \geq 5, \quad \text{i.e.,} \quad 0 \leq i \leq \frac{m-6}{2}.$$

In case of $m = 6$, by the assumption on the symmetry rank, M admits an isometric T^5-action. It is also true by the above discussion that there exists a positive quaternionic submanifold N of codimension at least 8 (i.e., of dimension at most 16) with the symmetry rank at least 4. But as in the previous case the codimension of N in M must be exactly 8. Thus $\dim N_5 = 0$. But it is impossible in view of the assumption we started with.

Next we need to deal with the case of even $m \geq 8$. If m is greater than or equal to 8, again there exists a chain of positive connected quaternionic Kähler submanifolds of the form

$$(3.3) \qquad \begin{aligned} N_k &\subset N_{k-1} \subset \cdots \subset N_{(m-2)/2} \subset N_{(m-4)/2} \\ &\subset N_{(m-6)/2} \subset N_{(m-8)/2} \subset \cdots \subset N_1 = N \subset M = N_0, \end{aligned}$$

where $\dim N_i - \dim N_{i+1} \geq 8$ for $0 \leq i \leq (m-8)/2$ and N_j admits an isometric T^{k-j}-action for each $0 \leq j \leq k$. By the connectedness Theorem 2.1 as above, we have $b_4(N_i) = b_4(M) = 1$ for $0 \leq i \leq (m-6)/2$. Furthermore, since the dimension N_k is assumed to be at least 4, it follows from the chain (3.3) that the difference $\dim N_i - \dim N_{i+1}$ is, in fact, exactly same as 8 for $0 \leq i \leq (m-8)/2$. Hence $N_{(m-6)/2}$ has dimension 24, admits an isometric at least T^5-action, and satisfies $b_4(N_{(m-6)/2}) = 1$. But then the T^5-action on $N_{(m-6)/2}$ has an isolated fixed point by Lemma 2.2, and this isolated fixed point is also an isolated fixed point of the T^k-action on M. This is a contradiction to our assumption. Alternatively, by considering the chain (3.3) directly, we can show that N_k would be actually an isolated fixed point. This is again a contradiction to the assumption that there is no isolated fixed point. This completes the proof of Lemma 3.1. $\qquad\square$

Now let x be such an isolated fixed point of T^k-action on M as in Lemma 3.1. Then we may assume without loss of generality that there is a positive quaternionic Kähler submanifold N of M passing through x whose dimension is no more than $4(m-2)$. By assumption, in this case we have $\dim N_k = 0$ and thus $\dim N_1 \geq 4(k-1)$.

We first assume that m is greater than or equal to 8. Thus there is a chain of positive connected quaternionic Kähler submanifolds as in (3.3) such that $\dim N_i - \dim N_{i+1} \geq 8$ for $0 \leq i \leq (m-6)/2$. As in the proof of Lemma 3.1, we see that the difference $\dim N_i - \dim N_{i+1}$ equals 8 for $0 \leq i \leq (m-6)/2$. Moreover, by construction, for $0 \leq i \leq (m-6)/2$ the inclusion of N_i into N_{i-1} is at least 6-connected (N_{-1} is assumed to be an empty set), so that we have $b_4(N_i) = b_4(M) = 1$ and $\pi_2(N_i) \cong \pi_2(M)$. Hence for $m \geq 8$ we have a positive quaternionic Kähler submanifold N of dimension 24 whose fourth Betti number equals 1 and the symmetry rank is at least 5. On the other hand, since the isotropy group of N_1 has rank one, the T^{k-1}-action on N_1 can be assumed to be of quaternionic type without loss of generality. Therefore the T^5-subaction on the 24-dimensional quaternionic Kähler manifold is also of quaternionic type. Hence the proof of Theorem 1.1 for the case of $m \geq 8$ now follows from Prop-

osition 2.4 and the assumption $b_4(M) = 1$.

Now it remains to consider the case of $m = 6$. For this case, it suffices to prove the following lemma.

Lemma 3.2. *Let M be a positive quaternionic Kähler manifold of dimension* 24 *with $b_4(M) = 1$ and the symmetry rank ≥ 5. Let N be a positive quaternionic Kähler manifold of codimension at least* 8 *with an isometric T^4-action as above. Then M is isometric to* $\mathbf{HP}^6$.

Proof.   First note that by considering the chain (3.1) the codimension of N in M is exactly 8. Then we need to consider the following two cases, depending on the second Betti number $b_2(N)$. If $b_2(N) \geq 1$ then it follows from the theorem of LeBrun and Salamon that N should be isometric to $Gr_2(\mathbf{C}^6)$. Since by construction $\pi_2(N) \cong \pi_2(M)$ and $\pi_2(Gr_2(\mathbf{C}^6)) \cong \mathbf{Z}$, we have $\pi_2(M) \cong \mathbf{Z}$. But this implies that M is isometric to $Gr_2(\mathbf{C}^{m+2})$ by the theorem of LeBrun and Salamon again. Since $b_4(Gr_4(\mathbf{C}^{m+2}))$ is strictly greater than 1, this case does not occur.

Next assume that $b_2(N) = 0$. Then the isometric T^4-action has an isolated fixed points as before. Thus for each fixed point $x \in M$, there are exactly four positive quaternionic Kähler submanifolds of dimension 12 equipped with an isometric T^3-action passing through x, and exactly six positive quaternionic Kähler manifolds of dimension 8 equipped with an isometric S^1-action passing through x. According to the classification of positive quaternionic Kähler manifolds by Herrera and Herrera in [7], every 12-dimensional positive quaternionic Kähler manifold N' with an isometric S^1-action is isometric to either $\mathbf{HP}^3$ or $\widetilde{Gr_4(\mathbf{R}^7)}$ or $Gr_2(\mathbf{C}^5)$. If N' is isometric to $\mathbf{HP}^3$, then $\pi_2(M)$ is trivial, so that M should be isometric to $\mathbf{HP}^m$. Hence we are done. If N' is isometric to $\widetilde{Gr_4(\mathbf{R}^7)}$, then it follows from Proposition 2.3 that N' would be isometric to either $\mathbf{HP}^3$ or $Gr_2(\mathbf{C}^5)$, which does not make any sense at all. Thus it remains to consider the case that N' is isometric to $Gr_2(\mathbf{C}^5)$. But in this case $\pi_2(N') = \pi_2(M)$ is isomorphic to $\mathbf{Z}$. Hence M is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. But since $b_4(Gr_2(\mathbf{C}^{m+2}))$ is not equal to 1, M should be isometric to $\mathbf{HP}^m$. Note that instead of using the result of Herrera and Herrera in [7] as above, one may directly use Proposition 2.3 to show that M is isometric to $\mathbf{HP}^m$. This completes the proof of Lemma 3.2.   □

The proof for the case of odd $m \geq 5$ is completely parallel to that of even $m \geq 6$. So let us highlight only the points different from the case of even m. First of all, since m is odd, we need to let $k = (m+3)/2 = [m/2]+2$. Then the inclusion from N_{i+1} into N_i in the proof of Lemma 3.1 is now at least 5-connected for $0 \leq i \leq (m-5)/2$. Then we can use a 20-dimensional positive quaternionic Kähler manifold with $b_4 = 1$ and the symmetry rank at least 4 in order to finish the proof for $m \geq 7$. Finally we also need to consider the case $m = 5$ which is analogous to Lemma 3.2. But in this case there is a 12-dimensional positive quaternionic Kähler submanifold N' with the T^3-subaction of quaternionic type. Hence we can apply Proposition 2.3 together with the assumption

$b_4(M) = 1$ to conclude that M is indeed isometric to $\mathbf{HP}^5$. This completes the proof of Theorem 1.1.

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