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A TRANSCENDENTAL APPROACH TO KOLLÁR'S INJECTIVITY THEOREM

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Abstract

We treat Kollár's injectivity theorem from the analytic (or differential geometric) viewpoint. More precisely, we give a curvature condition which implies Kollár type cohomology injectivity theorems. Our main theorem is formulated for a compact Kähler manifold, but the proof uses the space of harmonic forms on a Zariski open set with a suitable complete Kähler metric. We need neither covering tricks, desingularizations, nor Leray's spectral sequence.

Contents

1. Introduction

In [28], János Kollár proved the following theorem. We call it *Kollár's original injectivity theorem* in this paper.

Theorem 1.1 (cf. [28, Theorem 2.2]). *Let X be a smooth projective variety defined over an algebraically closed field of characteristic zero and let L be a semi-ample line bundle on X. Let s be a nonzero holomorphic section of $L^{\otimes k}$ for some $k > 0$. Then*

$$\times s\colon H^q(X, K_X \otimes L^{\otimes m}) \to H^q(X, K_X \otimes L^{\otimes m+k})$$

is injective for every $q \geq 0$ and every $m \geq 1$, where K_X is the canonical line bundle of X. Note that $\times s$ is the homomorphism induced by the tensor product with s.

The following theorem is the main result of this paper. It is an analytic formulation of Kollár type cohomology injectivity theorem.

Theorem 1.2 (Main theorem). *Let X be an n-dimensional compact Kähler manifold. Let (E, h_E) (resp. (L, h_L)) be a holomorphic vector (resp. line) bundle on X with*

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a smooth hermitian metric h_E (resp. h_L). Let F be a holomorphic line bundle on X with a singular hermitian metric h_F. Assume the following conditions.
(i) *There exists a subvariety Z of X such that h_F is smooth on $X \setminus Z$.*
(ii) $\sqrt{-1}\,\Theta(F) \geq -\gamma$ *in the sense of currents, where γ is a smooth* $(1,1)$*-form on X.*
(iii) $\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F)) \geq_{\mathrm{Nak}} 0$ *on $X \setminus Z$.*
(iv) $\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F) - \varepsilon\,\mathrm{Id}_E \otimes \Theta(L)) \geq_{\mathrm{Nak}} 0$ *on $X \setminus Z$ for some positive constant ε.*
Here, $\geq_{\mathrm{Nak}} 0$ *means the Nakano semi-positivity. Let s be a nonzero holomorphic section of L. Then the multiplication homomorphism*

$$\times s\colon H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F) \otimes L)$$

is injective for every $q \geq 0$, where $\mathcal{J}(h_F)$ is the multiplier ideal sheaf associated to the singular hermitian metric h_F of F.

The formulation of Theorem 1.2 was inspired by Ohsawa's injectivity theorem (see [35]). Although the assumptions in Theorem 1.2 may look artificial for algebraic geometers, our main theorem is useful and have potentiality for various generalizations. As a direct consequence of Theorem 1.2, we have the following corollary.

Corollary 1.3. *Let X be an n-dimensional compact Kähler manifold. Let (E, h_E) (resp. (L, h_L)) be a holomorphic vector (resp. line) bundle on X with a smooth hermitian metric h_E (resp. h_L). Let F be a holomorphic line bundle on X. Assume the following conditions.*
(a) *There exists an effective Cartier divisor D on X such that $\mathcal{O}_X(D) \simeq F^{\otimes k}$ for some positive integer k.*
(b) $\sqrt{-1}\,\Theta(E) \geq_{\mathrm{Nak}} 0$.
(c) $\sqrt{-1}(\Theta(E) - \varepsilon\,\mathrm{Id}_E \otimes \Theta(L)) \geq_{\mathrm{Nak}} 0$ *for some positive constant ε.*
Let s be a nonzero holomorphic section of L. Then the multiplication homomorphism

$$\times s\colon H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}) \to H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J} \otimes L)$$

is injective for every $q \geq 0$, where $\mathcal{J} = \mathcal{J}((1/k)D)$ is the multiplier ideal sheaf associated to $(1/k)D$ (cf. Definition 2.8).

One of the advantages of our formulation is that we are released from sophisticated algebraic geometric methods such as desingularizations, covering tricks, Leray's spectral sequence, and so on both in the proof and in various applications (see, for example, the proof of Proposition 4.1). The main ingredient of our proof of Theorem 1.2 is Nakano's identity (see Proposition 2.16).

We note that there are many contributors (Kollár, Esnault–Viehweg, Kawamata, Ambro, ...) to this kind of cohomology injectivity theorem. We just mention that the first result was obtained by Tankeev [38, Proposition 1]. It inspired Kollár to obtain his famous injectivity theorem (see [28] or Theorem 1.1). After [28], many generalizations of Theorem 1.1 were obtained (see the books [7] and [29]). Kollár did not refer to [6] in [29]. However, we think that [6] is the first paper where Kollár's injectivity theorem is proved (and generalized) by differential geometric arguments.

Let us recall Enoki's theorem [6, Theorem 0.2], which is a very special case of Theorem 1.2, for the reader's convenience. To recover Corollary 1.4 from Theorem 1.2, it is sufficient to put $E = \mathcal{O}_X$, $F = L^{\otimes m}$, and $L = L^{\otimes k}$. The reader who reads Japanese can find [9] useful. It is a survey on Enoki's injectivity theorem.

Corollary 1.4 (Enoki). *Let X be an n-dimensional compact Kähler manifold and let L be a semi-positive holomorphic line bundle on X. Suppose $L^{\otimes k}$, $k > 0$, admits a nonzero global holomorphic section s. Then*

$$\times s\colon H^q(X, K_X \otimes L^{\otimes m}) \to H^q(X, K_X \otimes L^{\otimes m+k})$$

is injective for every $m > 0$ and every $q \geq 0$.

We recall Enoki's idea of the proof in [6] because we will use the same idea to prove Theorem 1.2.

1.5. Enoki's proof. From now on, we assume that $k = m = 1$ for simplicity. It is well known that the cohomology group $H^q(X, K_X \otimes L^{\otimes l})$ is represented by the space of harmonic forms $\mathcal{H}^{n,q}(L^{\otimes l}) = \{u\colon$ smooth $L^{\otimes l}$-valued (n, q)-form on X such that $\bar\partial u = 0$, $D''^{*}_{L^{\otimes l}} u = 0\}$, where $D''^{*}_{L^{\otimes l}}$ is the formal adjoint of $\bar\partial$. We take $u \in \mathcal{H}^{n,q}(L)$. Then, $\bar\partial(su) = 0$ because s is holomorphic. We can check that $D''^{*}_{L^{\otimes 2}}(su) = 0$ by using Nakano's identity and the semi-positivity of L. Thus, s induces $\times s$: $\mathcal{H}^{n,q}(L) \to \mathcal{H}^{n,q}(L^{\otimes 2})$. Therefore, the required injectivity is obvious.

Enoki's theorem contains Kollár's original injectivity theorem (cf. Theorem 1.1) by the following well-known lemma.

Lemma 1.6. *Let L be a semi-ample line bundle on a smooth projective manifold X. Then L is semi-positive.*

Proof. There exists a morphism $f = \Phi_{|L^{\otimes m}|}\colon X \to \mathbb{P}^N$ induced by the complete linear system $|L^{\otimes m}|$ for some $m > 0$ because L is semi-ample. Let h be a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ with positive definite curvature. Then $(f^*h)^{1/m}$ is a smooth hermitian metric on L whose curvature is semi-positive. □

REMARK 1.7. Let X be a complex analytic space and let $\mathcal{E}$ be a coherent sheaf on X. In order to prove $H^p(X, \mathcal{E}) = 0$, it is sufficient to construct a homomorphism φ: $\mathcal{E} \to \mathcal{F}$ of coherent sheaves on X such that the induced map $H^p(X, \mathcal{E}) \to H^p(X, \mathcal{F})$

is *injective* and that $H^p(X, \mathcal{F}) = 0$. This simple observation plays crucial roles for various vanishing theorems on toric varieties (see, for example, [10] and [11]). Anyway, injectivity theorems sometimes are very useful in proving various vanishing theorems. See the proof of Corollary 4.7 below.

We quickly review Kollár's proof of his injectivity theorem in [29], which is much simpler than Kollár's original proof in [28], for the reader's convenience.

1.8. Kollár's proof. Let X be a smooth projective n-fold and let L be a (not necessarily semi-ample) line bundle on X. Let s be a non-zero holomorphic section of $L^{\otimes 2}$. Assume that $D = (s = 0)$ is a *smooth* divisor on X for simplicity. We can take a double cover $\pi : Z \to X$ ramifying along D. By the Hodge decomposition, we obtain a surjection $H^q(Z, \mathbb{C}_Z) \to H^q(X, \mathcal{O}_Z)$ for every q. By taking the anti-invariant part of the covering involution, we obtain that $H^q(X, G) \to H^q(X, L^{-1})$ is surjective for every q, where $\pi_*\mathbb{C}_Z = \mathbb{C}_X \oplus G$ is the eigen-sheaf decomposition. It is not difficult to see that there exists a factorization $H^q(X, G) \to H^q(X, L^{-1} \otimes \mathcal{O}_X(-D)) \to H^q(X, L^{-1})$ for every q. Therefore, $\times s : H^q(X, K_X \otimes L) \to H^q(X, K_X \otimes L \otimes \mathcal{O}_X(D))$ is injective by the Serre duality. In general, D is not necessarily *smooth*. So, we have to use sophisticated algebraic geometric methods such as desingularizations, relative vanishing theorems, Leray's spectral sequences, and so on, even when X is smooth and L is free.

REMARK 1.9. As we saw in Subsection 1.8, thanks to the Serre duality, the injectivity of $H^q(X, K_X \otimes L) \to H^q(X, K_X \otimes L \otimes \mathcal{O}_X(D))$ is equivalent to the surjectivity of $H^{n-q}(X, L^{-1} \otimes \mathcal{O}_X(-D)) \to H^{n-q}(X, L^{-1})$. However, injectivity seems to be much better and more natural for some applications and generalizations. See Section 4.

Roughly speaking, Kollár's geometric proof in [29] (and Esnault–Viehweg's proof in [7]) depends on the Hodge decomposition, or the degeneration of the Hodge to de Rham type spectral sequence. So, it works only when E is a unitary flat vector bundle (see [29, 9.17 Remark]). On the other hand, our analytic proof (and the proofs in [6], [35], and [37]) relies on the harmonic representation of the cohomology groups. We do not know the true relationship between the geometric proof and the analytic one.

1.10. More advanced topics. In [8], we prove a relative version of Theorem 1.2. In that case, X is not necessarily compact. When X is not compact, a *locally* square integrable differential form u on X is not necessarily *globally* square integrable. So, we use the Ohsawa–Takegoshi twisted version of Nakano's identity to control the asymptotic behavior of the L^2-norm of u around the boundary of X. Thus, we need much more analytic methods for the relative setting.

In [19, Chapter 2], [12], and [18, Sections 5 and 6], we develop the geometric approach (see Subsection 1.8) to obtain a very important generalization of Kollár's injectivity theorem. In those papers, we consider mixed Hodge structures on compact support cohomology groups. Roughly speaking, the decomposition

$$H_c^n(X \setminus \Sigma, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X, \Omega_X^p(\log \Sigma) \otimes \mathcal{O}_X(-\Sigma))$$

where X is a smooth projective variety and Σ is a simple normal crossing divisor on X produces a generalization of Kollár type cohomology injectivity theorem. The reader can find a thorough treatment of our geometric approach in [19, Chapter 2]. We have already obtained many applications for the log minimal model program in [17], [19], [13], [14], [15], [22], [16], [18], and [20].

By our experience, we know that Kollár type injectivity theorems play crucial roles for the study of base point free theorems and the abundance conjecture for log canonical pairs (cf. [23], [21], and so on).

We summarize the contents of this paper. In Section 2, we fix notation and collect basic results. Section 3 is the proof of the main theorem: Theorem 1.2. We will represent the cohomology groups by the spaces of harmonic forms on a Zariski open set with a suitable complete Kähler metric. We will use L^2-estimates for $\bar{\partial}$-equations on complete Kähler manifolds (see Lemma 3.2). It is a key point of our proof. In Section 4, we treat Kollár type injectivity theorem, Esnault–Viehweg type injectivity theorem, and Kawamata–Viehweg–Nadel type vanishing theorem as applications of Theorem 1.2. We recommend the reader to compare them with usual algebraic geometric ones. We note that we discuss them in a more general relative setting in [8].

2. Preliminaries

In this section, we collect basic definitions and results in algebraic and analytic geometries. For details, see, for example, [4].

2.1. Singular hermitian metric. Let L be a holomorphic line bundle on a complex manifold X.

DEFINITION 2.2 (Singular hermitian metric). A *singular hermitian metric* on L is a metric which is given in every trivialization $\theta\colon L|_{\Omega} \simeq \Omega \times \mathbb{C}$ by

$$\|\xi\| = |\theta(\xi)|e^{-\varphi(x)}, \quad x \in \Omega, \ \xi \in L_x,$$

where $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ. Here, $L^1_{\mathrm{loc}}(\Omega)$ is the space of the locally integrable functions on Ω.

The following singular hermitian metrics play important roles in the study of higher dimensional algebraic varieties.

EXAMPLE 2.3. Let $D = \sum \alpha_j D_j$ be a divisor with coefficients $\alpha_j \in \mathbb{N}$. Then $\mathcal{O}_X(D)$ is equipped with a natural singular hermitian metric as follows. Let f be a local section of $\mathcal{O}_X(D)$, viewed as a meromorphic function such that $\operatorname{div}(f) + D \geq 0$. We define $\|f\|^2 = |f|^2 \in [0, \infty]$. If g_j is a generator of the ideal of D_j on an open set $\Omega \subset X$, then the weight corresponding to this metric is $\varphi = \sum_j \alpha_j \log|g_j|$. It is obvious that this metric is a smooth hermitian metric on $X \setminus D$ and its curvature is zero on $X \setminus D$. Let L be a holomorphic line bundle on X. Assume that $L^{\otimes k} \simeq M \otimes \mathcal{O}_X(D)$ for some holomorphic line bundle M and an effective divisor D on X. As above, $\mathcal{O}_X(D)$ is equipped with a natural singular hermitian metric h_D. Let h_M be any smooth hermitian metric on M. Then L has a singular hermitian metric $h_L := h_M^{1/k} h_D^{1/k}$. Note that h_L is smooth outside D and $\Theta_{h_L}(L) = (1/k)\Theta_{h_M}(M)$ on $X \setminus D$.

2.4. Multiplier ideal sheaf. The notion of multiplier ideal sheaves introduced by Nadel [32] is very important in recent developments of complex and algebraic geometries (cf. [31, Part three]).

DEFINITION 2.5 ((Quasi-)plurisubharmonic function and multiplier ideal sheaf). A function $u\colon \Omega \to [-\infty, \infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is called *plurisubharmonic* (psh, for short) if
1. u is upper semi-continuous, and
2. for every complex line $L \subset \mathbb{C}^n$, $u|_{\Omega \cap L}$ is subharmonic on $\Omega \cap L$, that is, for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, \Omega^c)$, the function u satisfies the mean inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi)\, d\theta.$$

Let X be an n-dimensional complex manifold. A function $\varphi\colon X \to [-\infty, \infty)$ is said to be *plurisubharmonic* (psh, for short) if there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $\varphi|_{U_i}$ is plurisubharmonic on U_i ($\subset \mathbb{C}^n$) for every i. A *smooth strictly plurisubharmonic function* ψ on X is a smooth function on X such that $\sqrt{-1}\,\partial\bar{\partial}\psi$ is a positive definite smooth (1, 1)-form. A *quasi-plurisubharmonic* (quasi-psh, for short) function is a function φ which is locally equal to the sum of a psh function and of a smooth function. If φ is a quasi-psh function on a complex manifold X, the *multiplier ideal sheaf* $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{J}(\varphi)) = \{ f \in \mathcal{O}_X(U);\ |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf of $\mathcal{O}_X$. See, for example, [4, (5.7) Proposition].

REMARK 2.6. By the assumption (ii) in Theorem 1.2, we may assume that the weight of the singular hermitian metric h_F is a quasi-psh function on every trivialization.

So, we can define multiplier ideal sheaves locally and check that they are independent of trivializations. Thus, we can define the multiplier ideal sheaf globally and denote it by $\mathcal{J}(h_F)$, which is an abuse of notation. It is a coherent ideal sheaf on X.

EXAMPLE 2.7. Let $X = \{z \in \mathbb{C} \mid |z| < r\}$ for some $0 < r < 1$ and let L be a trivial line bundle on X. We consider a singular hermitian metric $h_L = \exp(\sqrt{-\log|z|^2})$ of L. Then h_L is smooth outside the origin $0 \in X$. The weight of h_L is $\varphi = -(1/2)\sqrt{-\log|z|^2}$ and φ is a psh function on X. The Lelong number of φ at 0 is

$$\liminf_{z \to 0} \frac{\varphi(z)}{\log|z|} = 0.$$

Thus, we have $\mathcal{J}(h_L) \simeq \mathcal{O}_X$ by Skoda. Note that φ is smooth outside 0, which is an analytic subvariety of X. However, φ does not have analytic singularities around 0.

DEFINITION 2.8. Let X be a complex manifold and let $D = \sum \alpha_j D_j$ be an effective $\mathbb{Q}$-divisor on X. Let g_j be a generator of the ideal of D_j on an open set $\Omega \subset X$. We put $\mathcal{J}(D) := \mathcal{J}(\varphi)$, where $\varphi = \sum_j \alpha_j \log|g_j|$. Since $\mathcal{J}(\varphi)$ is independent of the choice of the generators g_j's, $\mathcal{J}(D)$ is a well-defined coherent ideal sheaf on X. We call $\mathcal{J}(D)$ the multiplier ideal sheaf associated to the effective $\mathbb{Q}$-divisor D. We say that the divisor D is *integrable* at a point $x_0 \in X$ if the function $\prod |g_j|^{-2\alpha_j}$ is integrable on a neighborhood of x_0, equivalently, $\mathcal{J}(D)_{x_0} = \mathcal{O}_{X,x_0}$. Let D' be another effective $\mathbb{Q}$-divisor on X. Then, $\mathcal{J}(D) = \mathcal{J}(D + \varepsilon D')$ for $0 < \varepsilon \ll 1$, $\varepsilon \in \mathbb{Q}$.

REMARK 2.9. In Definition 2.8, D is integrable at x_0 if and only if the pair (X, D) is *Kawamata log terminal* (*klt*, for short) in a neighborhood of x_0 (cf. [30, Definition 2.34]).

EXAMPLE 2.10. Let h_L be the singular hermitian metric defined in Example 2.3. Then the weight of the singular hermitian metric h_L is a quasi-psh function on every trivialization. Therefore, the multiplier ideal sheaf $\mathcal{J}(h_L)$ is well-defined and $\mathcal{J}(h_L) = \mathcal{J}((1/k)D)$.

2.11. Hermitian and Kähler geometries. We collect the basic notion and results of hermitian and Kähler geometries (see also [4]).

DEFINITION 2.12 (Chern connection and its curvature form). Let X be a complex hermitian manifold and let (E, h) be a holomorphic hermitian vector bundle on X. Then there exists the *Chern connection* $D = D_{(E,h)}$, which can be split in a unique way as a sum of a (1, 0) and of a (0, 1)-connection, $D = D'_{(E,h)} + D''_{(E,h)}$. By the definition of the Chern connection, $D'' = D''_{(E,h)} = \bar{\partial}$. We obtain the *curvature form* $\Theta(E) = \Theta_{(E,h)} = \Theta_h := D^2_{(E,h)}$. The subscripts might be suppressed if there is no danger of confusion.

Let U be a small open set of X and let (e_λ) be a local holomorphic frame of $E|_U$. Then the hermitian metric h is given by the hermitian matrix $H=(h_{\lambda\mu})$, $h_{\lambda\mu}=h(e_\lambda,e_\mu)$, on U. We have $h(u,v)={}^t uH\bar{v}$ on U for smooth sections u, v of $E|_U$. This implies that $h(u,v)=\sum_{\lambda,\mu}u_\lambda h_{\lambda\mu}\bar{v}_\mu$ for $u=\sum e_i u_i$ and $v=\sum e_j v_j$. Then we obtain that $\sqrt{-1}\,\Theta_h(E)=\sqrt{-1}\,\bar\partial(\overline{H}^{-1}\partial\overline{H})$ and ${}^t\overline{(\sqrt{-1}\,{}^t\Theta_h(E)H)}=\sqrt{-1}\,{}^t\Theta_h(E)H$ on U.

DEFINITION 2.13 (Inner product). Let X be an n-dimensional complex manifold with the hermitian metric g. We denote by ω the *fundamental form* of g. Let (E,h) be a hermitian vector bundle on X, and u, v are E-valued (p,q)-forms with measurable coefficients, we set

$$\|u\|^2=\int_X |u|^2\,dV_\omega,\quad \langle\!\langle u,v\rangle\!\rangle=\int_X\langle u,v\rangle\,dV_\omega,$$

where $|u|$ is the pointwise norm induced by g and h on $\bigwedge^{p,q}T^*_X\otimes E$, and $dV_\omega=(1/n!)\omega^n$. More explicitly, $\langle u,v\rangle\,dV_\omega={}^tu\wedge H\,\overline{*v}$, where $*$ is the *Hodge star operator* relative to ω and H is the (local) matrix representation of h. When we need to emphasize the metrics, we write $|u|_{g,h}$, and so on.

Let $L^{p,q}_{(2)}(X,E)$ $(=L^{p,q}_{(2)}(X,(E,h)))$ be the space of square integrable E-valued (p,q)-forms on X. The inner product was defined in Definition 2.13. When we emphasize the metrics, we write $L^{p,q}_{(2)}(X,E)_{g,h}$, where g (resp. h) is the hermitian metric of X (resp. E). As usual one can view D' and D'' as closed and densely defined operators on the Hilbert space $L^{p,q}_{(2)}(X,E)$. The formal adjoints D'^*, D''^* also have closed extensions in the sense of distributions, which do not necessarily coincide with the Hilbert space adjoints in the sense of Von Neumann, since the latter ones may have strictly smaller domains. It is well known, however, that the domains coincide if the hermitian metric of X is complete. See Lemma 2.17 below.

DEFINITION 2.14 (Nakano positivity and semi-positivity). Let (E,h) be a holomorphic vector bundle on a complex manifold X with a smooth hermitian metric h. Let Ξ be a $\mathrm{Hom}(E,E)$-valued $(1,1)$-form such that ${}^t\overline{({}^t\Xi h)}={}^t\Xi h$. Then Ξ is said to be *Nakano positive* (resp. *Nakano semi-positive*) if the hermitian form on $T_X\otimes E$ associated to ${}^t\Xi h$ is positive definite (resp. semi-definite). We write $\Xi>_{\mathrm{Nak}}0$ (resp. $\geq_{\mathrm{Nak}}0$). We note that $\Xi_1>_{\mathrm{Nak}}\Xi_2$ (resp. $\Xi_1\geq_{\mathrm{Nak}}\Xi_2$) means that $\Xi_1-\Xi_2>_{\mathrm{Nak}}0$ (resp. $\geq_{\mathrm{Nak}}0$). A holomorphic vector bundle (E,h) is said to be *Nakano positive* (resp. *Nakano semi-positive*) if $\sqrt{-1}\,\Theta(E)>_{\mathrm{Nak}}0$ (resp. $\geq_{\mathrm{Nak}}0$). We usually omit "Nakano" when E is a line bundle.

DEFINITION 2.15 (Graded Lie bracket). Let $C^\infty\big(X,\bigwedge^{p,q}T^*_X\otimes E\big)$ be the space of the smooth E-valued (p,q)-forms on X. If A, B are the endomorphisms of pure degree of the graded module $M^\bullet = C^\infty\big(X, \bigwedge^{\bullet,\bullet} T_X^* \otimes E\big)$, their *graded Lie bracket* is defined by

$$[A, B] = AB - (-1)^{\deg A \deg B} BA.$$

Let us recall Nakano's identity, which is one of the main ingredients of the proof of our main theorem: Theorem 1.2.

Proposition 2.16 (Nakano's identity). *We further assume that g is Kähler. Let*

$$\Delta' = D'D'^* + D'^*D'$$

and

$$\Delta'' = D''D''^* + D''^*D''$$

be the complex Laplace operators acting on E-valued forms. Then

$$\Delta'' = \Delta' + [\sqrt{-1}\,\Theta(E), \Lambda],$$

where Λ is the adjoint of $\omega \wedge \cdot$.

The following lemma is now classical. See, for example, [1, Lemme 4.3].

Lemma 2.17 (Density lemma). *If g is complete, then $C_0^{p,q}(X, E)$ is dense in* $\operatorname{Dom} D''^* \cap \operatorname{Dom} \bar\partial$ *with respect to the graph norm*

$$u \mapsto \|u\| + \|\bar\partial u\| + \|D''^* u\|,$$

where $C_0^{p,q}(X, E)$ is the space of the E-valued smooth (p,q)-forms on X with compact supports and $\operatorname{Dom} D''^*$ (*resp.* $\operatorname{Dom}\bar\partial$) *is the domain of D''^** (*resp.* $\bar\partial$).

Combining Proposition 2.16 with Lemma 2.17, we obtain the following formula.

Proposition 2.18. *Let u be a square integrable E-valued (n, q)-form on X with* $\dim X = n$ *and let g be a complete Kähler metric on X. Let ω be the fundamental form of g. Assume that $\sqrt{-1}\,\Theta(E) \geq_{\mathrm{Nak}} -c\,\mathrm{Id}_E \otimes \omega$ for some constant c. Then we obtain that*

$$\|D''^* u\|^2 + \|\bar\partial u\|^2 = \|D'^* u\|^2 + \langle\!\langle \sqrt{-1}\,\Theta(E) \Lambda u, u \rangle\!\rangle$$

for every $u \in \operatorname{Dom} D''^* \cap \operatorname{Dom}\bar\partial$.

The final remark in this section will play crucial roles in the proof of the main theorem: Theorem 1.2. The proof is an easy calculation (cf. [1, Lemme 3.3]).

REMARK 2.19. Let g' be another hermitian metric on X such that $g' \geq g$ and ω' be the fundamental form of g'. Let u be an E-valued (n, q)-form with measurable coefficients. Then, we have $|u|^2_{g',h}\, dV_{\omega'} \leq |u|^2_{g,h}\, dV_{\omega}$, where $|u|_{g',h}$ (resp. $|u|_{g,h}$) is the pointwise norm induced by g' and h (resp. g and h). If u is an E-valued $(n, 0)$-form, then $|u|^2_{g',h}\, dV_{\omega'} = |u|^2_{g,h}\, dV_{\omega}$. In particular, $\|u\|^2$ is independent of g when u is an $(n, 0)$-form.

3. Proof of the main theorem

In this section, we prove the main theorem: Theorem 1.2. The idea is very simple. We represent the cohomology groups by the space of harmonic forms on $X \setminus Z$ (not on X!). The manifold $X \setminus Z$ is not compact. However, it is a complete Kähler manifold and all hermitian metrics are smooth on $X \setminus Z$. So, there are no difficulties on $X \setminus Z$. Note that we do not need the difficult regularization technique for quasi-psh functions on Kähler manifolds (cf. [1, Théorème 9.1]).

Proof of Theorem 1.2. Since X is compact, there exists a complete Kähler metric g' on $Y := X \setminus Z$ such that $g' > g$ on Y. We sketch the construction of g' because we need some special properties of g' in the following proof. The next lemma is well known. See, for example, [2, Lemma 5].

Lemma 3.1. *There exists a quasi-psh function ψ on X such that $\psi = -\infty$ on Z with logarithmic poles along Z and ψ is smooth outside Z.*

Without loss of generality, we can assume that $\psi < -e$ on X. We put $\varphi = 1/\log(-\psi)$. Then φ is a quasi-psh function on X and $\varphi < 1$. Thus, we can take a positive constant α such that $\sqrt{-1}\,\partial\bar\partial\varphi + \alpha\omega > 0$ on Y. Let g' be the Kähler metric on Y whose fundamental form is $\omega' = \omega + (\sqrt{-1}\,\partial\bar\partial\varphi + \alpha\omega)$. We will show that

$$\omega' \geq \partial(\log(\log(-\psi))) \wedge \bar\partial(\log(\log(-\psi)))$$

if we choose $\alpha \gg 0$. We have

$$\bar\partial\varphi = -\frac{-\bar\partial\psi/(-\psi)}{(\log(-\psi))^2},$$

and

$$\begin{aligned}
\partial\bar\partial\varphi &= 2\frac{-\partial\psi/(-\psi) \wedge -\bar\partial\psi/(-\psi)}{(\log(-\psi))^3} - \frac{\partial(-\bar\partial\psi/(-\psi))}{(\log(-\psi))^2} \\
&= 2\frac{-\partial\psi/(-\psi) \wedge -\bar\partial\psi/(-\psi)}{(\log(-\psi))^3} - \frac{-\partial\bar\partial\psi/(-\psi)}{(\log(-\psi))^2} + \frac{-\partial\psi \wedge (-\bar\partial\psi)/(-\psi)^2}{(\log(-\psi))^2} \\
&= 2\frac{-\partial\psi/(-\psi) \wedge -\bar\partial\psi/(-\psi)}{(\log(-\psi))^3} + \frac{\partial\bar\partial\psi/(-\psi)}{(\log(-\psi))^2} + \frac{-\partial\psi \wedge (-\bar\partial\psi)/(-\psi)^2}{(\log(-\psi))^2}.
\end{aligned}$$

On the other hand,

$$\partial(\log(\log(-\psi))) = \frac{-\partial\psi/(-\psi)}{\log(-\psi)}.$$

Therefore,

$$\partial(\log(\log(-\psi))) \wedge \bar{\partial}(\log(\log(-\psi))) = \frac{-\partial\psi \wedge (-\bar{\partial}\psi)/(-\psi)^2}{(\log(-\psi))^2}.$$

This implies

$$\omega' \geq \partial(\log(\log(-\psi))) \wedge \bar{\partial}(\log(\log(-\psi)))$$

if $\alpha \gg 0$. Therefore, g' is a *complete* Kähler metric on Y by Hopf–Rinow because $\log(\log(-\psi))$ tends to $+\infty$ on Z. More precisely,

$$\eta := \frac{1}{\sqrt{2}} \log(\log(-\psi))$$

is a smooth exhaustive function on Y such that $|d\eta|_{g'} \leq 1$. We fix these Kähler metrics throughout this proof. In general,

$$L^{n,q}_{(2)}(Y, E \otimes F) = L^{n,q}_{(2)}(Y, E \otimes F)_{g', h_E h_F} = \overline{\operatorname{Im} \bar{\partial}} \oplus \mathcal{H}^{n,q}(E \otimes F) \oplus \overline{\operatorname{Im} D''^{*}_{E\otimes F}},$$

where

$$\mathcal{H}^{n,q}(E \otimes F) := \left\{u \in L^{n,q}_{(2)}(Y, E \otimes F) \;\middle|\; \bar{\partial}u = D''^{*}_{E\otimes F}u = 0\right\}$$

is the space of the $E \otimes F$-valued harmonic (n, q)-forms. We note that $u \in \mathcal{H}^{n,q}(E \otimes F)$ is smooth by the regularization theorem for the elliptic operator $\Delta''_{E\otimes F} = D''^{*}_{E\otimes F}\bar{\partial} + \bar{\partial}D''^{*}_{E\otimes F}$. The claim below is more or less known to experts (cf. [36, Section 2], [37, Proposition 4.6] and [34, Theorem 4.13]). We write it for the reader's convenience.

Claim 1. *We have the following equalities and an isomorphism of cohomology groups for every* $q \geq 0$.

$$\overline{\operatorname{Im} \bar{\partial}} = \operatorname{Im} \bar{\partial}, \quad \overline{\operatorname{Im} D''^{*}_{E\otimes F}} = \operatorname{Im} D''^{*}_{E\otimes F},$$

and

$$H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \simeq \frac{L^{n,q}_{(2)}(Y, E \otimes F) \cap \operatorname{Ker} \bar{\partial}}{\operatorname{Im} \bar{\partial}}.$$

If the claim is true, then $H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F)) \simeq \mathcal{H}^{n,q}(E \otimes F)$ because $L^{n,q}_{(2)}(Y, E \otimes F) \cap \operatorname{Ker} \bar{\partial} = \operatorname{Im} \bar{\partial} \oplus \mathcal{H}^{n,q}(E \otimes F)$.

Proof of Claim 1. First, let $X=\bigcup_{i\in I}U_i$ be a finite Stein cover of X such that each U_i is small. We can assume that there is a small Stein open set V_i of X such that $U_i \Subset V_i$ for every i (see the proof of Lemma 3.2). We denote this cover by $\mathcal{U}=\{U_i\}_{i\in I}$. By Cartan and Leray, we obtain

$$H^q(X, K_X\otimes E\otimes F\otimes \mathcal{J}(h_F))\simeq \check{H}^q(\mathcal{U}, K_X\otimes E\otimes F\otimes \mathcal{J}(h_F)),$$

where the right hand side is the Čech cohomology group calculated by $\mathcal{U}$. Let $\{\rho_i\}_{i\in I}$ be a partition of unity associated to $\mathcal{U}$. We put $U_{i_0i_1\cdots i_q}=U_{i_0}\cap\cdots\cap U_{i_q}$. Then $U_{i_0i_1\cdots i_q}$ is Stein. Let $u=\{u_{i_0i_1\cdots i_q}\}$ such that $u_{i_0i_1\cdots i_q}\in\Gamma(U_{i_0i_1\cdots i_q}, K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))$ and $\delta u=0$, where δ is the coboundary operator of Čech complexes. We put $u^1=\{u^1_{i_0\cdots i_{q-1}}\}$ with $u^1_{i_0\cdots i_{q-1}}=\sum_i\rho_i u_{ii_0\cdots i_{q-1}}$. Then $\delta u^1=u$ and $\delta(\bar\partial u^1)=0$. Thus, we can construct u^2 such that $\delta u^2=\bar\partial u^1$ as above by using $\{\rho_i\}$. By repeating this process, we obtain $\bar\partial u^q\in L^{n,q}_{(2)}(Y, E\otimes F)\cap\operatorname{Ker}\bar\partial$ by Remark 2.19 because X is compact. By the standard diagram chasing, we have a homomorphism

$$\bar\alpha\colon \check{H}^q(\mathcal{U}, K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))\to\frac{L^{n,q}_{(2)}(Y, E\otimes F)\cap\operatorname{Ker}\bar\partial}{\operatorname{Im}\bar\partial}.$$

On the other hand, we take $w\in L^{n,q}_{(2)}(Y, E\otimes F)\cap\operatorname{Ker}\bar\partial$. We put $w^0=\{w_{i_0}\}$, where $w_{i_0}=w|_{U_{i_0}\setminus Z}$. We will use C_i to represent some positive constants independent of w. By Lemma 3.2 below, we have $w^1=\{w^1_{i_0}\}$ such that $\bar\partial w^1=w$ on each $U_{i_0}\setminus Z$ with

$$\|w^1\|^2:=\sum_i\int_{U_i\setminus Z}|w^1_i|^2_{g',h_Eh_F}\le C_1\int_{X\setminus Z}|w|^2_{g',h_Eh_F}=C_1\|w\|^2.$$

Since $\bar\partial(\delta w^1)=0$, we can obtain w^2 such that $\bar\partial w^2=\delta w^1$ on each $U_{i_0i_1}\setminus Z$ with

$$\|w^2\|^2:=\sum_{\{i,j\}\subset I}\int_{U_{ij}\setminus Z}|w^2_{ij}|^2_{g',h_Eh_F}\le C_2\|w^1\|^2.$$

By repeating this procedure, we obtain w^q such that $\bar\partial w^q=\delta w^{q-1}$ with $\|w^q\|^2\le C_q\|w^{q-1}\|^2$. In particular, $\|\delta w^q\|^2\le C_0\|w\|^2$. We put $\beta(w):=\delta w^q=:\{v_{i_0\cdots i_q}\}$. Then $\bar\partial v_{i_0\cdots i_q}=0$ and $\|v_{i_0\cdots i_q}\|^2<\infty$. Thus, $v_{i_0\cdots i_q}\in\Gamma(U_{i_0\cdots i_q}, K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))$ and $\delta(\beta(w))=0$. Note that an $E\otimes F$-valued holomorphic $(n,0)$-form on $U\setminus Z$, where U is an open subset of X, with a finite L^2 norm can be extended to an $E\otimes F$-valued holomorphic $(n,0)$-form on U (see also Remark 2.19). Therefore, we have a homomorphism

$$\bar\beta\colon\frac{L^{n,q}_{(2)}(Y, E\otimes F)\cap\operatorname{Ker}\bar\partial}{\operatorname{Im}\bar\partial}\to\check{H}^q(\mathcal{U}, K_X\otimes E\otimes F\otimes\mathcal{J}(h_F))$$

by the standard diagram chasing. It is not difficult to see that $\bar\alpha$ and $\bar\beta$ induce the desired isomorphism by the above arguments.

Next, we note that $\overline{\operatorname{Im}\bar\partial} = \operatorname{Im}\bar\partial$ if and only if $\overline{\operatorname{Im} D''^{*}_{E\otimes F}} = \operatorname{Im} D''^{*}_{E\otimes F}$ (cf. [25, Theorem 1.1.1]). Thus, it is sufficient to prove that $\overline{\operatorname{Im}\bar\partial} = \operatorname{Im}\bar\partial$. Let $w \in \overline{\operatorname{Im}\bar\partial}$. Then there exists a sequence $\{v_k\} \subset \operatorname{Im}\bar\partial$ such that $\|w - \bar\partial v_k\|^2 \to 0$ if $k \to \infty$. By the above construction, $\|\beta(w - \bar\partial v_k)\|^2 \to 0$ when $k \to \infty$. This implies that $\beta(w - \bar\partial v_k) \to 0$ uniformly on every compact subset of X. Therefore, the image of w in $\check{H}^q(\mathcal{U}, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is zero because $\check{H}^q(\mathcal{U}, K_X \otimes E \otimes F \otimes \mathcal{J}(h_F))$ is a finite dimensional, separated, Fréchet space (cf. [24, Chapter VIII, Section A, 19. Theorem]). Thus, $w \in \operatorname{Im}\bar\partial$ by the above isomorphism. For the details of the topology on $\mathcal{F}$ and $H^q(X, \mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf on a complex manifold X, see [26, §55 Coherent Analytic Sheaves as Fréchet Sheaves]. $\square$

There are various formulations for L^2-estimates for $\bar\partial$-equations, which originated from Hörmander's paper [25]. The following one is suitable for our purpose. We used it in the proof of Claim 1.

Lemma 3.2 (L^2-estimates for $\bar\partial$-equations on complete Kähler manifolds). *Let $U \Subset V$ be small Stein open sets of X. If $u \in L^{n,q}_{(2)}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ with $\bar\partial u = 0$, then there exists $v \in L^{n,q-1}_{(2)}(U \setminus Z, E \otimes F)_{g', h_E h_F}$ such that $\bar\partial v = u$. Moreover, there exists a positive constant C independent of u such that*

$$\int_{U\setminus Z} |v|^2_{g', h_E h_F} \le C \int_{U\setminus Z} |u|^2_{g', h_E h_F}.$$

Proof. We can assume that $\omega' = \sqrt{-1}\partial\bar\partial\Psi$ on V because V is a small Stein open set. Then $(E \otimes F, h_E h_F e^{-\Psi})$ is Nakano positive and

$$C_1 h_E h_F \le h_E h_F e^{-\Psi} \le C_2 h_E h_F$$

for some positive constants C_1 and C_2 on $U \setminus Z$. Note that Ψ is a bounded function on X by the construction of g'. It is obvious that $\sqrt{-1}\,\Theta_{(E\otimes F, h_E h_F e^{-\Psi})} \ge_{\mathrm{Nak}} \mathrm{Id}_E \otimes \omega'$ on $U \setminus Z$ by the assumption (iii) in Theorem 1.2. Let w be an $E \otimes F$-valued (n, q)-form on $U \setminus Z$ with measurable coefficients. We write

$$\|w\|^2 = \int_{U\setminus Z} |w|^2_{g', h_E h_F}\, dV_{\omega'} \quad \text{and} \quad \|w\|_0^2 = \int_{U\setminus Z} |w|^2_{g', h_E h_F e^{-\Psi}}\, dV_{\omega'}.$$

Then $\|w\|$ is finite if and only if $\|w\|_0$ is finite. By the well-known L^2 estimates for $\bar\partial$-equations (cf. [1, Théorème 4.1, Remarque 4.2] or [4, (5.1) Theorem]), we obtain an $E \otimes F$-valued $(n, q-1)$-form v on $U \setminus Z$ such that $\bar\partial v = u$ and $\|v\|_0^2 \le C_0 \|u\|_0^2$, where C_0 is a positive constant independent of u. We note that g' is not a complete Kähler metric on $U \setminus Z$ but $U \setminus Z$ is a complete Kähler manifold (cf. [1, Thèoréme 0.2]).

Therefore, we obtain

$$C_1\|v\|^2 \leq \|v\|_0^2 \leq C_0\|u\|_0^2 \leq C_0C_2\|u\|^2.$$

So, it is sufficient to put $C = C_0C_2/C_1$. □

Therefore, we obtain

$$L^{n,q}_{(2)}(Y, E\otimes F) = \operatorname{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(E\otimes F) \oplus \operatorname{Im} D''^{*}_{E\otimes F}.$$

Thus, $H^q(X, K_X\otimes E\otimes F\otimes \mathcal{J}(h_F)) \simeq \mathcal{H}^{n,q}(E\otimes F)$.

Let $U \Subset V$ be small Stein open sets of X. Then there exists a smooth strictly psh function Φ on V such that $(L, h_Le^{-\Phi})$ is semi-positive on V. In this situation, $C'_1 \leq e^{-\Phi} \leq C'_2$ on U for some positive constants C'_1 and C'_2. By applying the same argument as in Lemma 3.2 to $(E\otimes F\otimes L, h_Eh_Fh_Le^{-\Psi-\Phi})$, we obtain

$$L^{n,q}_{(2)}(Y, E\otimes F\otimes L) = \operatorname{Im}\bar{\partial} \oplus \mathcal{H}^{n,q}(E\otimes F\otimes L) \oplus \operatorname{Im} D''^{*}_{E\otimes F\otimes L}$$

and

$$H^q(X, K_X\otimes E\otimes F\otimes \mathcal{J}(h_F)\otimes L) \simeq \mathcal{H}^{n,q}(E\otimes F\otimes L)$$

similarly.

Claim 2. *The multiplication homomorphism*

$$\times s\colon \mathcal{H}^{n,q}(E\otimes F) \to \mathcal{H}^{n,q}(E\otimes F\otimes L)$$

is well-defined for every $q \geq 0$.

Proof. By Proposition 2.18, we obtain

$$\|D''^{*}_{E\otimes F}u\|^2 + \|\bar{\partial}u\|^2 = \|D'^{*}u\|^2 + \langle\!\langle \sqrt{-1}\,\Theta(E\otimes F)\Lambda u, u\rangle\!\rangle$$

for $u \in \operatorname{Dom} D''^{*}_{E\otimes F} \cap \operatorname{Dom}\bar{\partial}$, where Λ is the adjoint of $\omega'\wedge\cdot$. We note that the Kähler metric g' on Y is complete. If $u \in \mathcal{H}^{n,q}(E\otimes F)$, then $D'^{*}u = 0$ and $\langle\sqrt{-1}(\Theta(E)+\mathrm{Id}_E\otimes\Theta(F))\Lambda u, u\rangle = 0$ by the assumption (iii). By (iv), we have $\langle\sqrt{-1}(\mathrm{Id}_E\otimes\Theta(L))\Lambda u, u\rangle \leq 0$. When $u \in \mathcal{H}^{n,q}(E\otimes F)$, $\bar{\partial}(su) = 0$ by the Leibniz rule and $D'^{*}(su) = sD'^{*}u = 0$ because s is an L-valued holomorphic $(0, 0)$-form. Since $|s|^2_{h_L}$ is a smooth function on X, there exists a positive number C such that $|s|^2_{h_L} < C$ everywhere on X. Therefore,

$$\int_Y |su|^2_{g',h_Lh_Eh_F}\,dV_{\omega'} < C\int_Y |u|^2_{g',h_Eh_F}\,dV_{\omega'} < \infty.$$

So, su is square integrable. We can also see that $su \in \operatorname{Dom} D''^{*}_{E\otimes F\otimes L}$ by using Lemma 2.17 and Proposition 2.18 since $|s|^2_{h_L} < C$ everywhere on X. Thus, we obtain

$$\| D''^{*}_{E\otimes F\otimes L}(su)\|^2 = \langle\!\langle \sqrt{-1}\,\Theta(E\otimes F\otimes L)\,\Lambda(su), su\rangle\!\rangle$$

by Proposition 2.18. We note that

$$\begin{aligned}
&\sqrt{-1}(\Theta(E) + \mathrm{Id}_E \otimes \Theta(F) + \mathrm{Id}_E \otimes \Theta(L))\\
&\geq_{\mathrm{Nak}} (1+\varepsilon)\,\mathrm{Id}_E \otimes \Theta(L)\\
&\geq_{\mathrm{Nak}} -c'\,\mathrm{Id}_E \otimes \omega'
\end{aligned}$$

on $Y = X \setminus Z$ for some constant c'. On the other hand,

$$\langle \sqrt{-1}\,\Theta(E\otimes F\otimes L)\,\Lambda(su), su\rangle = |s|^2 \langle \sqrt{-1}(\mathrm{Id}_E \otimes \Theta(L))\,\Lambda\, u, u\rangle \leq 0,$$

where $|s|$ is the pointwise norm of s with respect to h_L. Therefore, $D''^{*}_{E\otimes F\otimes L}(su) = 0$. This implies that $su \in \mathcal{H}^{n,q}(E\otimes F\otimes L)$. We finish the proof of the claim. □

By the above claims, the theorem is obvious because

$$\times s \colon \mathcal{H}^{n,q}(E\otimes F) \to \mathcal{H}^{n,q}(E\otimes F\otimes L)$$

is injective for every q. □

We close this section with the proof of Corollary 1.3.

Proof of Corollary 1.3. We put $h_F := h_D^{1/k}$ as in Example 2.3, where h_D is the natural singular hermitian metric on $\mathcal{O}_X(D)$. Then h_F is smooth on $X \setminus D$, $\sqrt{-1}\,\Theta(F) \geq 0$ in the sense of currents, and $\mathcal{J}(h_F) = \mathcal{J}((1/k)D)$. Therefore, we can apply Theorem 1.2. □

4. Applications: injectivity and vanishing theorems

In this section, we treat only a few applications of Theorem 1.2. We recommend the reader to see the results in [37] and the arguments in [33, Chapter V, §3] for other formulations and generalizations. See also [8] for various applications and generalizations in a more general relative setting. For applications in the log minimal model program, which can not be covered by the results in this paper, see [16], [18], [19], and so on.

The following formulation is due to Kollár (cf. [29, 10.13 Theorem]). He stated this result for the case where E is a trivial line bundle and (X, Δ) is klt, that is, $\mathcal{J}(\Delta) \simeq \mathcal{O}_X$.

Proposition 4.1 (Kollár type injectivity theorem). *Let $f\colon X\to Y$ be a proper surjective morphism from a compact Kähler manifold X to a normal projective variety Y. Let L be a holomorphic line bundle on X and let D be an effective divisor on X such that $f(D)\neq Y$. Assume that $L\equiv f^*M+\Delta$, where M is a nef and big $\mathbb{Q}$-divisor on Y and Δ is an effective $\mathbb{Q}$-divisor on X. Let (E,h_E) be a Nakano semi-positive holomorphic vector bundle on X. Then*

$$H^q(X, K_X\otimes E\otimes L\otimes \mathcal{J}(\Delta))\to H^q(X, K_X\otimes E\otimes L\otimes \mathcal{O}_X(D)\otimes \mathcal{J}(\Delta))$$

is injective for every $q\geq 0$, where $\mathcal{J}(\Delta)$ is the multiplier ideal sheaf associated to the effective $\mathbb{Q}$-divisor Δ.

Proof. By taking $P\in \mathrm{Pic}^0(X)$ suitably, we have $L\otimes P\sim_{\mathbb{Q}} f^*M+\Delta$. We can assume that $L\sim_{\mathbb{Q}} f^*M+\Delta$ by replacing L (resp. E) with $L\otimes P$ (resp. $E\otimes P^{-1}$). By Kodaira's lemma (see [30, Proposition 2.61]), we can further assume that M is ample (cf. Definition 2.8). Let $h:=\Phi_{|mM|}\colon Y\to \mathbb{P}^N$ be the embedding induced by the complete linear system $|mM|$ for a large integer m. Then $\mathcal{O}_Y(mM)\simeq h^*\mathcal{O}_{\mathbb{P}^N}(1)$. We can take an effective divisor A on $\mathbb{P}^N$ such that $\mathcal{O}_{\mathbb{P}^N}(A)\simeq \mathcal{O}_{\mathbb{P}^N}(l)$ for some positive integer l and $D'=f^*h^*A-D$ is an effective divisor on X. We add D' to D and can assume that $D=f^*h^*A$. Under these extra assumptions, we can easily construct hermitian metrics satisfying the assumptions in Theorem 1.2 (see Example 2.3). We finish the proof of the proposition. □

REMARK 4.2 (Numerical equivalence). In the above proposition, we note that $L\equiv f^*M+\Delta$ means $c_1(L)=c_1(f^*M+\Delta)$ in $H^2(X,\mathbb{R})$, where c_1 is the first Chern class of $\mathbb{Q}$-divisors or line bundles.

REMARK 4.3. Proposition 4.1 is a generalization of [29, 10.13 Theorem], which is stated for a compact Kähler manifold. However, the proof of [29, 10.13 Theorem] given in [29] works only for *projective manifolds*. In [29, 10.17.3 Claim], we need an ample divisor on X to prove local vanishing theorems.

The following proposition is a reformulation of [7, 5.12. Corollary b)] from the analytic viewpoint. It is essentially the same as Proposition 4.1. In [7], E is trivial and $\mathcal{J}\simeq \mathcal{O}_X$.

Proposition 4.4 (Esnault–Viehweg type injectivity theorem). *Let X be a smooth projective variety and let D be an effective divisor on X. Let (E,h_E) be a Nakano semi-positive holomorphic vector bundle and let L be a holomorphic line bundle on X. Assume that $L^{\otimes k}(-D)$ is nef and abundant, that is, $\kappa(L^{\otimes k}(-D))=\nu(L^{\otimes k}(-D))$, for some positive integer k. Let B be an effective divisor on X such that*

$$H^0(X,(L^{\otimes k}(-D))^{\otimes l}\otimes \mathcal{O}_X(-B))\neq 0$$

for some $l > 0$. Then

$$H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}) \to H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J} \otimes \mathcal{O}_X(B))$$

is injective for every q, where $\mathcal{J} := \mathcal{J}((1/k)D)$ is the multiplier ideal sheaf associated to the effective $\mathbb{Q}$-divisor $(1/k)D$.

Proof. Let $\pi : Z \to X$ be a projective birational morphism from a smooth projective variety Z with the following properties: (i) There exists a proper surjective morphism between smooth projective varieties $f : Z \to Y$ with connected fibers, and (ii) there is a nef and big $\mathbb{Q}$-divisor M on Y such that $\pi^*(L^{\otimes k}(-D)) \sim_{\mathbb{Q}} f^*M$. For the proof, see [27, Proposition 2.1]. On the other hand, $R^i\pi_*(K_{Z/X} \otimes \mathcal{J}((1/k)\pi^*D)) = 0$ for $i > 0$ and $\pi_*(K_Z \otimes \mathcal{J}((1/k)\pi^*D)) \simeq K_X \otimes \mathcal{J}((1/k)D)$ by [31, Theorem 9.2.33, and Example 9.6.4]. We note that (π^*E, π^*h_E) is Nakano semi-positive on Z. So, we can assume that $X = Z$ without loss of generality. It is not difficult to see that $f(B) \neq Y$ by the assumption that $H^0(X, (L^{\otimes k}(-D))^{\otimes l} \otimes \mathcal{O}_X(-B)) \neq 0$ for some $l > 0$. Thus, this proposition follows from Proposition 4.1. □

The referee pointed out that Proposition 4.4 is sharper than [5, Theorem 3.1].

REMARK 4.5. In this remark, we use the notation in [5, Theorem 3.1]. Let k be a positive integer such that $k > \lambda$. We take general members $D_1, \dots, D_k$ of $H^0(X, A \otimes \mathfrak{a})$ and put

$$D = \frac{\lambda}{k}(D_1 + \cdots + D_k).$$

Then $L - D$ is nef and abundant and $L - D - \epsilon B$ is $\mathbb{Q}$-effective for some $0 < \epsilon < 1$. By the construction, we have $\mathcal{J}(D) = \mathcal{J}(X, \mathfrak{a}^\lambda)$ (cf. [31, Proposition 9.2.28]). Therefore, by Proposition 4.4 for $E = \mathcal{O}_X$, we obtain that

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, \mathfrak{a}^\lambda)) \to H^i(X, \mathcal{O}_X(K_X + L + B) \otimes \mathcal{J}(X, \mathfrak{a}^\lambda))$$

is injective for every i.

REMARK 4.6 (Vanishing theorem and torsion-freeness). Proposition 4.1 gives some generalizations of Kollár's vanishing and torsion-free theorems. We do not pursue them here because we discuss them in a more general relative setting in [8]. We just mention that [29, 10.15 Corollary] holds for $K_X \otimes E \otimes \mathcal{J}(\Delta)$, where we use the same notation as in Proposition 4.1. We note [31, Example 9.5.9] when we restrict the multiplier ideal sheaf $\mathcal{J}(\Delta)$ to a *general* hypersurface. Related topics are in [7, 6.12 Corollary, and 6.17 Corollary] and [5, Section 3].

By combining Proposition 4.1 with Serre's vanishing theorem, we obtain the next corollary. It may be better to be called Nadel type vanishing theorem.

Corollary 4.7 (Kawamata–Viehweg type vanishing theorem). *Let X be a smooth projective variety and let L be a holomorphic line bundle on X. Assume that $L \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-divisor on X and Δ is an effective $\mathbb{Q}$-divisor on X. Let (E, h_E) be a Nakano semi-positive holomorphic vector bundle on X. Then $H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}(\Delta)) = 0$ for $q \geq 1$. Moreover, if Δ is integrable outside finitely many points, then $H^q(X, K_X \otimes E \otimes L) = 0$ for $q \geq 1$.*

Proof. We use Proposition 4.1 under the assumption that $Y = X$ and $f = \mathrm{id}_X$. We take an effective ample divisor D on X and apply Proposition 4.1. Then we obtain that

$$H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}(\Delta)) \to H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}(\Delta) \otimes \mathcal{O}_X(mD))$$

is injective for $m > 0$ and $q \geq 0$. By Serre's vanishing theorem, we have $H^q(X, K_X \otimes E \otimes L \otimes \mathcal{J}(\Delta)) = 0$ for $q \geq 1$. When Δ is integrable outside finitely many points, $\mathcal{O}_X/\mathcal{J}(\Delta)$ is a skyscraper sheaf. Therefore, $H^q(X, K_X \otimes E \otimes L \otimes \mathcal{O}_X/\mathcal{J}(\Delta)) = 0$ for $q \geq 1$. By combining it with the above mentioned vanishing result, we obtain the desired result. □

The final result is a slight generalization of Demailly's formulation of Kawamata–Viehweg type vanishing theorem.

Corollary 4.8 (cf. [3, Main Theorem]). *Let L be a holomorphic line bundle on an n-dimensional projective manifold X. Assume that some positive power $L^{\otimes k}$ can be written $L^{\otimes k} \simeq M \otimes \mathcal{O}_X(D)$, where M is a nef line bundle and D is an effective divisor such that $(1/k)D$ is integrable on $X \setminus B$. Let $\nu = \nu(M)$ be the numerical dimension of the nef line bundle M. Let (E, h_E) be a Nakano semi-positive holomorphic vector bundle on X. Then $H^q(X, K_X \otimes E \otimes L) = 0$ for $q > n - \min\{\max\{\nu, \kappa(L)\}, \operatorname{codim} B\}$.*

Sketch of the proof. By the standard slicing arguments, we can reduce it to the case where $\min\{\max\{\nu, \kappa(L)\}, \operatorname{codim} B\} = \dim X$. We note that $\operatorname{codim} B = \infty$ if and only if $B = \emptyset$. By Kodaira's lemma (cf. [30, Lemma 2.60, Proposition 2.61]), we can further reduce it to the case when M is ample. We note that if A is a general smooth very ample Cartier divisor on X then

$$0 \to K_X \otimes E \otimes L \to K_X \otimes E \otimes L \otimes \mathcal{O}_X(A) \to K_A \otimes E|_A \otimes L|_A \to 0$$

is exact and $\mathcal{J}((1/k)D)|_A = \mathcal{J}((1/k)D|_A)$. In particular, $(1/k)D|_A$ is integrable on $A \setminus B|_A$. For the details of these reduction arguments, see the first and second steps in the proof of the main theorem in [3]. Therefore, this corollary follows from the previous corollary: Corollary 4.7. □

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