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## REAL CHARACTERS IN BLOCKS

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### Abstract

We consider real versions of Brauer's  $k(B)$  conjecture, Olsson's conjecture and Eaton's conjecture. We prove the real version of Eaton's conjecture for 2-blocks of groups with cyclic defect group and for the principal 2-blocks of groups with trivial real core. We also characterize  $G$ -classes, real and rational  $G$ -classes of the defect group of  $B$ .

### 1. Introduction

Several authors have been investigating real classes, characters and blocks of finite groups, see e.g. [1, 2, 5, 8, 9, 11, 13, 14, 15, 17, 19, 20, 21, 24, 25]. The aim of this note is to formulate real versions of Brauer's  $k(B)$  conjecture, see [3], Olsson's conjecture, see [28], and Eaton's conjecture, see [10], for 2-blocks. We give special cases when we can prove the real versions of them. The last part of the paper deals with fusion of elements of defect groups.

### 2. Notations and terminology

In this note  $G$  will always denote a finite group,  $p$  a prime integer, which is 2 except for the last section of the paper. Let  $(R, k, F)$  be a  $p$ -modular system, where  $R$  is a complete discrete valuation ring with quotient field  $k$  of characteristic zero and residue class field  $F$  of characteristic  $p$ . We assume that  $k$  and  $F$  are splitting fields of all the subgroups of  $G$ . We may also assume that  $k$  is a subfield of the complex numbers. Complex conjugation acts on  $\text{Irr}(G)$ . A character is *real* if it is conjugate to itself, in other words if it is real valued. An element of  $G$  is *real* if it is conjugate to its inverse. An element  $x$  of a subgroup  $H$  of  $G$  we call *H-real*, if it can be conjugated to its inverse by an element from the subgroup  $H$ .

We say that the *conjugacy class*  $C$  is *real* if it is equal to the class of the inverse elements of the class. We use the notation  $\text{Cl}_r(G)$  for the set of these classes. A  $p$ -block  $B$  is called *real* if it contains the complex conjugate of an irreducible ordinary character (and hence of all irreducible characters) in the block. It is known, see e.g. [23, Theorem 3.33], that a real 2-block always contains real valued irreducible ordinary and Brauer characters, as well. We use the notation  $\text{Irr}_{rv}(G)$  and  $\text{Irr}_{rv}(B)$  for

the set of real valued irreducible ordinary characters in  $G$  and in  $B$ , respectively. Let  $k_{rv}(G)$  and  $k_{rv}(B)$  stand for the sizes of these sets. We use the notation  $k_{i,rv}(B)$  for the number of real valued irreducible characters of height  $i$  in the  $p$ -block  $B$ . By Brauer's permutation lemma the number of real conjugacy classes of the group  $G$  is equal to  $k_{rv}(G)$ . We use the notation  $\text{Bl}(G \mid D)$  for the set of  $p$ -blocks of  $G$  with defect group  $D$ ,  $D^{(n)}$  stands for the  $n$ -th derived subgroup of  $D$ . For constructing examples we used the GAP system, see [7], and we also describe these groups with their GAP notation.

### 3. The real conjectures

Unless otherwise stated, let  $p = 2$ . Let  $G$  be a finite group,  $B$  a real 2-block of  $G$  with defect group  $D$ .

**Conjecture 1** (Weak real version of Brauer's conjecture). *We conjecture that  $k_{rv}(B)$  is bounded from above by the number of  $G$ -real elements of  $D$ .*

**Conjecture 2** (Strong real version of Brauer's conjecture). *We conjecture that  $k_{rv}(B)$  is bounded from above by the number of  $N_G(D)$ -real elements of  $D$ .*

**Conjecture 3** (Real version of Olsson's conjecture). *We conjecture that  $k_{0,rv}(B)$  is bounded from above by the number of  $N_G(D)/D'$ -real elements of  $D/D'$ .*

**Conjecture 4** (Real version of Eaton's conjecture). *We conjecture that  $\sum_{i=0}^n k_{i,rv}(B)$  is bounded from above by the number of  $N_G(D)/D^{(n+1)}$ -real elements of  $D/D^{(n+1)}$ .*

REMARK 5. One could not replace in Conjecture 2 the  $N_G(D)$  by  $D$ . The smallest example is a group of order 24 which is the pullback of maps  $S_3 \rightarrow C_2$  and  $Q_8 \rightarrow C_2$ , (with GAP notations it is `SmallGroup(24, 4)`). In this group there are two 2-blocks. The nonprincipal block  $B$  has a normal defect group  $D \simeq C_4$ , where there are just two  $D$ -real elements, however  $k_{rv}(B) = 4$ . (In fact in this group all characters in  $\text{Irr}(G)$  are real). However, we do not know any such example for the principal block, or for blocks of maximal defect.

REMARK 6. If every irreducible character is real in the group  $G$  then we get stronger versions of the non-real conjectures, see Remark 7, namely  $k(B)$  ( $k_0(B)$ ,  $\sum_{i=0}^n k_i(B)$ ) are bounded from above by the number of elements of the defect group  $D$  of  $B$ , ( $D/D'$  or  $D/D^{(n+1)}$ ) that are real inside  $N_G(D)$  ( $N_G(D)/D'$  or  $N_G(D)/D^{(n+1)}$ ) respectively. Of course Conjecture 4 implies Conjectures 1, 2 and 3.

REMARK 7. If every irreducible character of a group  $G$  is real, it does not follow that the normalizer of its Sylow 2-subgroup also has this property. Let  $G = \text{SmallGroup}(96, 185)$ . This group has selfnormalizing Sylow 2-subgroups. In the

principal block of  $G$  all the 14 irreducible characters are real valued, its Sylow 2-subgroup has also 14 irreducible characters, but only 12 of them are real. This example also shows that an element can be real in one of the Sylow 2-subgroups, but not real in an other Sylow 2-subgroup, since one can find order 4 elements in this with that property. In this group all 32 elements of the Sylow 2-subgroup are real in  $G$ , but only 28 of them are real in  $N_G(S) = S$ .

In the next remark we show that the  $p$ -analogue of Conjecture 1 is not true for  $p > 2$ , and since the defect group is abelian the other Conjectures 2, 3 and 4 also cannot hold:

REMARK 8. Let  $G = M_{11}$ ,  $p = 11$  and let  $B$  be the principal block. Then  $|D| = 11$ ,  $k_{rv}(B) = 3$ , but in  $D$  there is only one  $G$ -real element. This group is also an example for the fact that the number of real valued irreducible characters can be different in the Brauer correspondent blocks with cyclic defect group if  $p > 2$ . Let  $b \in \text{Bl}(N_G(D) \mid D)$  be the Brauer correspondent of  $B$ . Then  $k_{rv}(b) = 1$ . If  $p = 2$  and the defect group is noncyclic then Brauer correspondent blocks might have different number of real valued irreducible characters: let us take the same group  $G$ , then the principal 2-block has 6, however its Brauer correspondent block has 5 real valued irreducible characters.

**4. Nilpotent groups, symmetric groups and blocks with central defect groups**

**Proposition 9** (The nilpotent groups). *A stronger form of Conjecture 4 (hence 3, 2 and 1) holds for nilpotent groups. If  $G$  is either a 2-group or abelian, then in Conjecture 3 there is equality.*

Proof. If  $G$  is nilpotent then every 2-block is of maximal defect, and by [8] the only real 2-block of maximal defect is the principal block  $B_0$ . Then  $\text{Irr}(B_0) = \text{Irr}(G_2)$ , where  $G_2 \in \text{Syl}_2(G)$ . Characters of height  $n$  of  $G_2$  are those of degree  $2^n$ . This is at most the  $n$ -th character degree of  $G_2$ . By [12, Lemma 5.12], all irreducible characters of height at most  $n$  contain  $G_2^{(n+1)}$  in their kernels, hence  $\sum_{i=0}^n k_{i,rv}(B_0) \leq |\text{Irr}_{rv}(G_2/G_2^{(n+1)})|$ , which is at most the number of  $G_2/G_2^{(n+1)}$ -real elements in  $G_2/G_2^{(n+1)}$ . □

**Proposition 10** (The symmetric groups). *Conjectures 2 and 3 hold for the symmetric groups.*

Proof. (a) Since every irreducible character of the symmetric group is real valued and since its Sylow 2-subgroup also has this property by [16, Theorem 4.4.8], Conjecture 2 for the principal 2-blocks reduces in this case to the non-real  $k(B)$  conjecture, which holds by [28]. In [26] it is proved that the defect group  $D$  of each block  $B$  of

weight  $w$  of  $S_n$  is isomorphic to the Sylow  $p$ -subgroup of  $S_{pw}$  and there is a canonical height preserving bijection between the irreducible characters of  $B$  and that of the principal block of  $S_{pw}$ . Thus if  $p = 2$  then in  $D$  each element is also real, and again by [28], Conjecture 2 holds also for nonprincipal 2-blocks of  $S_n$ .

(b) Olsson’s conjecture also holds for  $S_n$  by [28]. Thus by similar arguments as above, Conjecture 3, also holds. □

REMARK 11. A positive answer to Eaton’s conjecture for  $S_n$ , would imply a positive answer to Conjecture 4.

**Proposition 12** (Blocks with central defect groups). *Conjecture 4 (and hence, Conjectures 1, 2 and 3) holds for central defect groups. In fact we prove a slightly stronger statement: the strong forms of the conjectures holds for 2-blocks with defect group  $D$ , where  $G = DC_G(D)$ :*

Proof. Let  $B \in \text{Bl}(G \mid D)$  be a 2-block of  $G$ , where  $G = DC_G(D)$ . By [23, Theorem 9.12]  $|\text{IBr}(B)| = 1$  and there is a bijection between  $\text{Irr}(D)$  and  $\text{Irr}(B)$  mapping  $\zeta$  to  $\theta_\zeta$ , where  $\theta_\zeta(g) = \zeta(g_2)\theta(g_2)$ , if  $g_2 \in D$ , otherwise it is zero. Here  $\theta$  is the unique character in  $\text{Irr}(B)$  containing  $D$  in its kernel, and  $\text{IBr}(B) = \{\theta^0\}$ . Moreover  $ht(\theta_\zeta) = n$  iff  $\zeta(1) = 2^n$ . If  $B$  is a real 2-block then  $\theta$  is a real valued character and  $\theta_\zeta$  is real valued if and only if  $\zeta$  is real valued. Thus  $k_{r_v}(B) = k_{r_v}(D)$  and  $\sum_{i=0}^n k_{i,r_v}(B) = \sum_{i=0}^n k_{i,r_v}(D) \leq |\text{Irr}_{r_v}(D/D^{(n+1)})|$  by Proposition 9, this is at most the number of  $D/D^{(n+1)}$ -real elements of  $D/D^{(n+1)}$ . □

REMARK 13. It is easy to see that if the above conjectures are true for the direct factors of a group then they are also true for the direct product: a tensor product of characters is real iff each component is real, if we have defect classes  $C_1 \in \text{Cl}(B_1, D_1)$  and  $C_2 \in \text{Cl}(B_2, D_2)$ , then the pair  $(c_1, c_2) \in C_1 \times C_2$  is a defect class of  $B_1 \otimes B_2$ . The defect of the character  $\chi_1 \otimes \chi_2$  is the sum of the defects of  $\chi_1$  and  $\chi_2$ . The height of the product of characters is the sum of the heights. The number of real elements in  $D_1 \times D_2$  is just the product of the numbers of real elements in the direct components.

**5. Blocks with cyclic defect groups**

For a block  $B \in \text{Bl}(G)$  we consider the pairs  $(x, \theta)$  with  $x \in G$  a  $p$ -element  $\theta \in \text{IBr}(b)$ , where  $b \in \text{Bl}(C_G(x))$  such that  $b^G = B$ . As in [21], we call the  $G$ -conjugacy classes of these pairs, denoted by  $(x, \theta)^G$ , the *columns of  $B$* . A column  $(x, \theta)^G$  is called *real* if  $(x, \theta)^G = (x^{-1}, \bar{\theta})^G$ . In [21, Lemma 1.1] it is proved that  $k_{r_v}(B)$  is equal to the number of real columns of  $B$ .

We will use Dade’s description [6, Theorem 68.1] of  $p$ -blocks with cyclic defect groups only for the special case  $p = 2$ :

Let  $B$  be a 2-block with cyclic defect group  $D = \langle x \rangle$  of order  $2^a$ ,  $D_i = \langle x^{2^i} \rangle$ ,  $C_i = C_G(D_i)$ ,  $N_i = N_G(D_i)$ , for  $i = 0, \dots, a - 1$ . Let  $B_0 \in \text{Bl}(N_G(D) \mid D)$  be the

Brauer correspondent block of  $B$ . Let  $b_0 \in \text{Bl}(C_G(D) \mid D)$  with  $b_0^{N_0} = B_0$ . Such blocks are conjugate in  $N_0$ . Similarly let  $b_i = b_0^{C_i}$ , then every block of  $C_i$  that induces  $B$  is conjugate to  $b_i$  in  $N_i$ . Let  $\theta^i$  be the unique irreducible Brauer character of  $b_i$  for  $i = 0, \dots, a - 1$ . The inertia subgroup of  $b_i$  in  $N_i$  is  $C_i$  for  $i = 0, \dots, a - 1$ , and also  $|\text{IBr}(B)| = 1$ . Let  $\text{IBr}(B) = \{\phi\}$ . First we prove:

**Lemma 14.** *With the notation above, we have that  $(x, \theta^0)$ ,  $(x^k, \theta^0)$  for  $k$  odd,  $(x^2, \theta^1)$ ,  $(x^{2k}, \theta^1)$  for  $k$  odd,  $\dots$ ,  $(x^{2^{a-1}}, \theta^{a-1})$ ,  $(x^{2^{(a-1)k}}, \theta^{a-1})$  for  $k$  odd, and  $(1, \phi)$  are representatives of the columns of  $B$ . If  $x^{2^i}$  is the smallest power of  $x$  which is  $G$ -real then representatives of the real columns of  $B$  are among those columns whose first component is a power of  $x^{2^i}$ .*

*Proof.* If the first components of two pairs generate different subgroups, then they cannot be conjugate. Let us take the pair  $(y, \psi)$ , where  $y$  generates  $D_j$  and  $j < a$ . Then the block of  $\psi$  is conjugate to  $b_j$  in  $N_j$ , so  $\psi$  is conjugate to  $\theta^j$  in  $N_j$ . The conjugation takes  $y$  to another generator of  $D_j$ , i.e. to  $x^{2^j k}$ , where  $k$  is odd. If the first component is 1, then the second component must be  $\phi$ . □

**Corollary 15.** *Let  $G$  be a finite group, let  $B$  be a real 2-block with cyclic defect group  $D$ . Then Conjecture 1 holds for  $G$ .*

*Proof.* We use [21, Lemma 1.1], Lemma 14 and the notations above. Then the number of  $G$ -real elements in  $D$  is exactly  $2^{a-i}$ .

We have that the representatives of real columns of  $B$  are  $(1, \phi)$  and some of those columns whose first component is an element of  $D_i$  and if it generates  $D_j$  then the second component is  $\theta^j$ . Their number is at most the number of elements of  $D_i$ , which is  $2^{a-i}$ . □

**Corollary 16.** *Let  $D$  be a cyclic normal 2-subgroup of  $G$ . Then Conjecture 2 and hence Conjecture 4 also holds for blocks  $B \in \text{Bl}(G \mid D)$ .*

**REMARK 17.** Using similar arguments for the  $p > 2$  case, one gets for block with cyclic defect groups that  $k_{r_v}(B) \leq l(B) \cdot |\{G\text{-real elements in } D\}|$ . This could be considered as some kind of real analogue of the so called “Trace inequality” in [27, Proposition 2, p. 272].

To prove Conjecture 2 for 2-blocks with cyclic defect group we will need the following lemma (the  $p$ -analogue of it for  $p > 2$  is not true, and if  $p = 2$ , but the defect group is noncyclic then the analogous result is not true either, see Example 8):

**Lemma 18.** *Let  $G$  be a finite group, let  $B \in \text{Bl}(G \mid D)$  be a real 2-block with cyclic defect group  $D$  and let  $B_0 \in \text{Bl}(N_G(D) \mid D)$  be its Brauer correspondent block. Then  $k_{rv}(B) = k_{rv}(B_0)$ .*

*Proof.* We use the same notation as in the introduction to this section. By [21, Lemma 1.1] and Lemma 14 it is enough to prove that if  $(x^{2^j k}, \theta^j)$  represents a real column of the block  $B$ , (recall that  $\theta^j \in \text{IBr}(b_j)$  and  $b_j \in \text{Bl}(C_G(D_j) \mid D)$ ), and  $\tilde{b}_j \in \text{Bl}(N_{C_G(D_j)}(D) \mid D) = \text{Bl}(C_{N_G(D)}(D_j) \mid D)$  is the Brauer correspondent of  $b_j$  containing the single irreducible Brauer character  $\tilde{\theta}^j$ , then  $(x^{2^j k}, \tilde{\theta}^j)$  belongs to a real column of  $B_0$  and this correspondence defines a bijection of real columns of  $B_0$  and  $B$ .

Let  $z \in G$  such that  $((x^{2^j k})^z, \theta^{jz}) = ((x^{2^j k})^{-1}, \overline{\theta^j})$ . Then  $\overline{\theta^j} \in \text{IBr}(\overline{b_j})$ . This block's Brauer correspondent in  $N_{C_G(D_j)}(D)$  is  $\overline{\tilde{b}_j}$ , that contains the unique irreducible Brauer character  $\overline{\tilde{\theta}^j}$ . Since blocks of  $C_{N_G(D)}(D_j)$  that induce  $B_0$  are conjugate in  $N_{N_G(D)}(D_j) = N_G(D)$ , there exists an element  $z_1 \in N_G(D)$  with  $\tilde{b}_j^{z_1} = \overline{\tilde{b}_j}$ . Then  $b_j^{z_1} = \overline{b_j}$  and  $\tilde{\theta}^{jz_1} = \overline{\tilde{\theta}^j}$ . But then  $b_j^{zz_1} = b_j$ . But the inertia group of  $b_j$  in  $N_j$  is  $C_j$ , thus  $zz_1 \in C_j$  and so  $(x^{2^j k})^{zz_1} = x^{2^j k}$ , and  $(x^{2^j k})^{-1} = (x^{2^j k})^z = (x^{2^j k})^{z_1}$ , and hence  $((x^{2^j k})^{z_1}, \tilde{\theta}^{jz_1}) = ((x^{2^j k})^{-1}, \overline{\tilde{\theta}^j})$ . Thus it represents a real column of  $B_0$ . The remaining column of  $B$  containing  $(1, \phi)$  is real and the corresponding column containing  $(1, \tilde{\phi})$  in  $B_0$  is also real. So we are done.  $\square$

Now we have:

**Theorem 19.** *Let  $G$  be a finite group, let  $B \in \text{Bl}(G \mid D)$  be a 2-block with cyclic defect group  $D$ . Then Conjecture 2 and hence Conjecture 4 also holds for  $B$ .*

*Proof.* Using Lemma 18 and Corollary 16, we have that  $|\text{Irr}_{rv}(B)| = |\text{Irr}_{rv}(B_0)|$  is bounded from above by the number of the  $N_G(D)$ -real elements of  $D$ , thus we are done.  $\square$

## 6. Groups with odd real core

In [11] we defined the *real core*  $R(G)$  of  $G$  as the subgroup generated by the real elements of odd order.

Our main result is the following:

**Theorem 20.** *If  $|R(G)|$  is odd then Conjecture 4 holds (hence also Conjectures 3, 2 and 1) for the principal 2-block of  $G$ . In particular if any of the following cases occur Conjecture 4 holds for the principal 2-block of  $G$ .*

- (a) *The commutator subgroup  $G'$  is 2-nilpotent.*
- (b)  *$G = O_{2,2,2}(G)$ . (In fact this is equivalent to  $|R(G)|$  being odd.)*

(c)  $G$  is solvable and its Sylow 2-subgroup is abelian.  
 Moreover, we may replace in Conjecture 4  $N_G(D)$  by  $D$ .

Proof. If  $|R(G)|$  is odd then by [11]  $G = O_{2',2,2}(G)$ , in particular  $G$  is solvable. Let  $B_0$  be the principal 2-block of  $G$ . Then  $\text{Irr}_{rv}(B_0) = \text{Irr}_{rv}(G/O_2(G))$ . Let  $\bar{G} = G/O_2(G)$ . Then  $\bar{S} \in \text{Syl}_2(\bar{G})$  is normal.

STEP 1: For every real element  $x$  there exists a 2-element  $g$  such that  $x^g = x^{-1}$ :

Let  $g = g_2 g_{2'}$  if  $x^{g_2 g_{2'}} = x^{-1}$ , then an appropriate 2-power of  $g$  is already a 2'-element, which centralizes  $x$ . Thus  $g_{2'}$  acts on  $x$  trivially, and we are done.

STEP 2:  $R(\bar{G}) = 1$ :

If  $x \in \bar{G}$  is a real element of odd order, then by Step 1 there is a 2-element  $g$  inverting  $x$ . Since  $\bar{S} \triangleleft \bar{G}$ ,  $[x, g] \in \bar{S} \cap \langle x \rangle = 1$ . Thus  $x^{-1} = x$ , and so  $x = 1$ .

STEP 3: Every real element in  $\bar{G}$  is a 2-element, hence it lies in  $\bar{S} \in \text{Syl}_2(\bar{G})$ :

Let  $x = x_2 x_{2'}$  be a real element in  $\bar{G}$ . Then  $x_2^{-1} x_{2'}^{-1} = x^{-1} = x^g = x_2^g x_{2'}^g$ , thus  $x_2$  and  $x_{2'}$  are both real. By Step 2,  $x_{2'} = 1$ .

Thus  $|\text{Irr}_{rv}(B_0)| = |\text{Irr}_{rv}(\bar{G})| = |\text{Cl}_r(\bar{G})| \leq |\{x \in \bar{G} \mid x \text{ real}\}| = |\{x \in \bar{S} \in \text{Syl}_2(\bar{G}) \mid x \text{ real in } \bar{S}\}|$ .

We prove Conjecture 4 by induction on  $r$ . Let  $r = 0$ . An irreducible character  $\chi \in \text{Irr}(B_0)$  is of height zero iff its degree is odd. We have that  $\chi \in \text{Irr}(\bar{G})$ , and  $\chi_{\bar{S}}$  has only linear constituents, hence  $\bar{S}' \leq \ker(\chi)$ . Thus  $k_{0,rv}(B_0) \leq |\text{Irr}_{rv}(\bar{G}/\bar{S}')| = |\text{Cl}_r(\bar{G}/\bar{S}')| \leq |\{x \in \bar{G}/\bar{S}' \mid x \text{ real in } \bar{G}/\bar{S}'\}|$ . If there would be a real 2'-element in  $\bar{G}/\bar{S}'$  then by Proposition 5.3 in [11] there would be a real 2'-element in  $\bar{G} \setminus \bar{S}'$ , which is not the case. Thus there are also no real elements in  $\bar{G}/\bar{S}'$  whose 2'-part is not 1. Thus each real element belongs to  $\bar{S}/\bar{S}'$ , and by Step 1 this element is  $\bar{S}/\bar{S}'$ -real. Thus we are done for  $r = 0$ .

Let us suppose that Conjecture 4 is true for  $r < n$ . If  $\chi \in \text{Irr}(B_0)$  is of height  $n$ , then its degree has 2-part  $2^n$ . Then all constituents of  $\chi_{\bar{S}}$  have degree  $2^n$ . By [12, Theorem 5.12], they contain in their kernels  $\bar{S}^{(n+1)}$ , thus  $\chi$  also contains it in its kernel. Similarly all irreducible characters of  $\bar{G}$  of smaller height also contain it in their kernels. Hence  $\sum_{i=0}^n k_{i,rv}(B_0) \leq |\text{Irr}_{rv}(\bar{G}/\bar{S}^{(n+1)})| = |\text{Cl}_r(\bar{G}/\bar{S}^{(n+1)})| \leq |\{x \in \bar{G}/\bar{S}^{(n+1)} \mid x \text{ real}\}| = |\{x \in \bar{S}/\bar{S}^{(n+1)} \mid x \text{ real in } \bar{S}/\bar{S}^{(n+1)}\}|$ . Hence Conjecture 4 holds.

(a) Since  $R(G) \leq G'$  by [11], if  $G'$  is 2-nilpotent, then  $|R(G)|$  is odd.

(b) This is equivalent to  $|R(G)|$  odd by [11].

(c) By the Hall–Higman lemma,  $\bar{S}$  is normal, thus we have case (b). □

**Corollary 21.** *If  $S \in \text{Syl}_2(G)$  is normal then Conjecture 4 holds for  $G$ , since then each block is of maximal defect, and the only real 2-block of maximal defect is the principal block, hence we can apply Theorem 20 (b).*



## 7. Computer results

We have checked Conjecture 4 for the principal 2-block with GAP [7] for the small groups library. We also checked Conjecture 2 for the principal 2-block for the 26 sporadic simple groups. For these blocks the respective conjectures were true.

We also checked Conjecture 2 and Conjecture 3 with the help of GAP for all 2-blocks of groups up to order 1536 except for groups of orders 856, 1048, 1112, 1192, 1304, 1352, 1384, 1432, 1448, where our computational methods did not work (Conway polynomials are not yet known). We did not find any counterexamples for these conjectures among the investigated groups.

## 8. $B$ -classes and $G$ -classes of $D$

Let now  $p$  be an arbitrary prime number.

First we prove the following

**Lemma 22.** *Let  $B$  be a  $p$ -block of  $G$  with defect group  $D$ . Then for every  $x \in D$  there exists a  $\chi \in \text{Irr}(B)$  with  $\chi(x) \neq 0$ .*

*Proof.* Let us suppose by contradiction that there exists an element  $x \in D$  with  $\chi(x) = 0$ , for every  $\chi \in \text{Irr}(B)$ . If we can prove that there exists a trivial source  $FG$ -module  $M$  in this block with vertex  $D$ , then by [18, p. 175, Lemma 2.16] this is liftable to a trivial source  $RG$ -module  $\tilde{M}$  and its character is nonzero on the elements of the vertex of  $M$ , contradicting our assumption.

If  $D$  is normal in  $G$  then by [18, p. 247, Lemma 10.3] all simple modules in  $B$  are trivial source modules in  $B$  with vertex  $D$ . If  $D$  is not normal then the Brauer correspondent  $b \in \text{Bl}(N_G(D) \mid D)$  of  $B$  has the property that every simple module  $S$  in it is a trivial source module with vertex  $D$ . Let us lift a simple  $FN_G(D)$ -module  $S$  in  $b$  to an  $RN_G(D)$ -module  $\tilde{S}$ . Then its Green correspondent,  $f(\tilde{S})$  is a trivial source module with vertex  $D$  and by [4, p. 466, Theorem 59.9],  $f(\tilde{S})$  belongs to the block  $B$ . So we are done.  $\square$

**DEFINITION 23.** Let  $B$  be a  $p$ -block of the finite group  $G$  with defect group  $D$ . We say that two elements  $x, y \in D$  are in the same  $B$ -class, iff for every irreducible character  $\chi \in \text{Irr}(B)$ ,  $\chi(x) = \chi(y)$ .

We have the following result:

**Theorem 24.** *Let  $G$  be a finite group with  $p$ -block  $B \in \text{Bl}(G \mid D)$ . Then the  $B$ -classes of the defect group  $D$  are exactly the  $G$ -classes  $\text{Cl}_G(D)$  of  $D$  under conjugation.*

*Proof.* If two elements  $x, y \in D$  are conjugate in  $G$ , then of course they are also in the same  $B$ -class. Let us suppose now that  $x, y \in D$  are in the same  $B$ -class,

but they are not conjugate in  $G$ . Then by the strong block orthogonality relation, see [23, p.106, Corollary 5.11]  $\sum_{\chi \in \text{Irr}(B)} \chi(x)\overline{\chi(y)} = 0$ . Using that  $x, y$  are in the same  $B$ -class this gives us  $\sum_{\chi(x) \in \text{Irr}(B)} |\chi(x)|^2 = 0$ . Hence  $\chi(x) = 0$  for every  $\chi \in \text{Irr}(B)$ . By Lemma 22, this is not possible.  $\square$

**DEFINITION 25.** Let  $B$  be a  $p$ -block of a finite group  $G$  with defect group  $D$ . We say that the element  $x \in D$  is  $B$ -real if  $\chi(x)$  is real for every  $\chi \in \text{Irr}(B)$ . An element  $x \in D$  is  $B$ -rational, if  $\chi(x)$  is rational for every  $\chi \in \text{Irr}(B)$ .

**Corollary 26.** *Let  $B \in \text{Bl}(G \mid D)$  and let  $x \in D$ . Then  $x$  is  $B$ -real iff it is real in  $G$ .*

**Corollary 27.** *Let  $B \in \text{Bl}(G \mid D)$  and let  $x \in D$ . Then  $x$  is  $B$ -rational iff it is rational in  $G$ .*

**Corollary 28.** *Let  $F$  be a field containing  $\mathbb{Q}$ . Let  $B \in \text{Bl}(G \mid D)$  and let  $x \in D$ . Then  $\chi(x) \in F$  for every  $\chi \in \text{Irr}(G)$  if and only if  $\chi(x) \in F$  for every  $\chi \in \text{Irr}(B)$ .*

We have also the following

**Theorem 29.** *Let  $B \in \text{Bl}(G \mid D)$ . The restrictions  $\chi_D$  of  $\chi \in \text{Irr}(B)$  to the defect group  $D$  generate the vector space of the restrictions of all complex  $G$ -class functions to  $D$ .*

*Proof.* Let us choose representatives  $x_i \in C_i \cap D, i = 1, \dots, t$  of  $G$ -conjugacy classes  $C_i$  intersecting  $D$ . We want to prove that if we restrict the character table of  $G$  to these columns and to those rows which belong to the block  $B$ , then these columns are independent. It implies that this submatrix has rank  $t$ , hence, it has also  $t$  independent rows. But then any complex vector of length  $t$  can be expressed by these rows and we are done. Let us suppose that the above mentioned  $t$  columns are dependent. Then there are coefficients  $\alpha_1, \dots, \alpha_t$ , not all zero with the property that  $\sum_{i=1}^t \alpha_i \chi(x_i) = 0$ , for all  $\chi \in \text{Irr}(B)$ . By [22, Lemma 4.6, Chapter 5] the subsum, where  $x_i$ -s belong to any  $p$ -section is also zero. But the  $x_i$ -s all belong to different  $p$ -sections, thus  $\alpha_i \chi(x_i) = 0$  for every  $i = 1, \dots, t$  and every  $\chi \in \text{Irr}(B)$ . By Lemma 22 we see that there is no  $x_i$  where every  $\chi \in \text{Irr}(B)$  vanishes. Hence  $\alpha_i = 0$  for all  $i = 1, \dots, t$ . Thus the columns of the above restricted matrix are independent and we are done.  $\square$

In this way we get another proof of the following:

**Corollary 30.** *For a block  $B \in \text{Bl}(G \mid D)$ , the number of  $G$ -conjugacy classes  $|\text{Cl}_G(D)|$  of its defect group, is a lower bound for the number  $k(B)$ .*

EXAMPLE 31. It is not true, however that  $B$ -classes (hence  $G$ -classes) of the defect group  $D$  are the same as  $b$ -classes (hence  $N_G(D)$ -classes) for the Brauer correspondent block  $b \in \text{Bl}(N_G(D) \mid D)$  even for 2-blocks  $B \in \text{Bl}(G \mid D)$  with cyclic defect group. Let  $G = \text{SmallGroup}(288, 375)$ . Then the third 2-block has cyclic defect group of order 8, it contains four  $G$ -real (hence  $B$ -real) elements and only two  $N_G(D)$ -real (hence  $b$ -real) elements.

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### References

- [1] R. Brauer: *Some applications of the theory of blocks of characters of finite groups*, IV, J. Algebra **17** (1971), 489–521.
- [2] R. Brauer: *Representations of finite groups*; in *Lectures on Modern Mathematics* **1**, Wiley, New York, 133–175, 1963.
- [3] R. Brauer: *Number theoretical investigations on groups of finite order*; in *Proceedings of the International Symposium on Algebraic Number Theory*, Tokyo and Nikko, 1955, Science Council of Japan, Tokyo, 55–62, 1956.
- [4] C.W. Curtis and I. Reiner: *Methods of Representation Theory*, II, Wiley, New York, 1987.
- [5] S. Dolfi, G. Navarro and P.H. Tiep: *Primes dividing the degrees of the real characters*, Math. Z. **259** (2008), 755–774.
- [6] L. Dornhoff: *Group Representation Theory*, Part B, Dekker, New York, 1972.
- [7] The GAP Group: *GAP—Groups, Algorithms and Programming*, Version 4.4.12, <http://www.gap-system.org>, 2008.
- [8] R. Gow: *Real 2-blocks of characters of finite groups*, Osaka J. Math. **25** (1988), 135–147.
- [9] R. Gow and J. Murray: *Real 2-regular classes and 2-blocks*, J. Algebra **230** (2000), 455–473.
- [10] C.W. Eaton: *Generalisations of conjectures of Brauer and Olsson*, Arch. Math. (Basel) **81** (2003), 621–626.
- [11] L. Héthelyi and E. Horváth: *Galois actions on blocks and classes of finite groups*, J. Algebra **320** (2008), 660–679.
- [12] I.M. Isaacs: *Character Theory of Finite Groups*, Academic Press, New York, 1976.
- [13] I.M. Isaacs, G. Malle and G. Navarro: *Real characters of  $p'$ -degree*, J. Algebra **278** (2004), 611–620.
- [14] I.M. Isaacs and G. Navarro: *Group elements and fields of character values*, J. Group Theory **12** (2009), 635–650.
- [15] S. Iwasaki: *On finite groups with exactly two real conjugate classes*, Arch. Math. (Basel) **33** (1979), 512–517.
- [16] G. James and A. Kerber: *The Representation Theory of the Symmetric Group*, Addison-Wesley Publishing Co., Reading, MA, 1981.
- [17] S.G. Kolesnikov and Ja. N. Nuzhin: *On strong reality of finite simple groups*, Acta Appl. Math. **85** (2005), 195–203.
- [18] P. Landrock: *Finite Group Algebras and Their Modules*, London Mathematical Society Lecture Note Series **84**, Cambridge Univ. Press, Cambridge, 1983.

- [19] A. Moretó and G. Navarro: *Groups with three real valued irreducible characters*, Israel J. Math. **163** (2008), 85–92.
- [20] J. Murray: *Involutions, extended defect groups of real 2-blocks and vertex theory*, preprint.
- [21] J. Murray: *Real subpairs and Frobenius–Schur indicators of characters in 2-blocks*, J. Algebra **322** (2009), 489–513.
- [22] H. Nagao and Y. Tsushima: *Representations of Finite Groups*, Academic Press, Boston, MA, 1989.
- [23] G. Navarro: *Characters and Blocks of Finite Groups*, London Mathematical Society Lecture Note Series **250**, Cambridge Univ. Press, Cambridge, 1998.
- [24] G. Navarro, L. Sanus and P.H. Tiep: *Groups with two real Brauer characters*, J. Algebra **307** (2007), 891–898.
- [25] G. Navarro, L. Sanus and P.H. Tiep: *Real characters and degrees*, Israel J. Math., to appear.
- [26] J.B. Olsson: *McKay numbers and heights of characters*, Math. Scand. **38** (1976), 25–42.
- [27] J.B. Olsson: *Inequalities for block-theoretic invariants*; in Representations of Algebras (Puebla, 1980), Lecture Notes in Math. **903**, Springer, Berlin, 270–284, 1981.
- [28] J.B. Olsson: *On the number of characters in blocks of finite general linear, unitary and symmetric groups*, Math. Z. **186** (1984), 41–47.

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