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ON THE DISTRIBUTION OF αp MODULO ONE FOR PRIMES p OF A SPECIAL FORM

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Abstract

In this paper it is proved that for any irrational α and some $0 < \theta \leq 1.5/100$, there are infinitely many primes p such that $p+2$ has at most two prime factors and $\|\alpha p+\beta\| < p^{-\theta}$ which improves K. Matomäki's result $\theta < 1/1000$.

1. Introduction

Let α be a irrational real number and $\|x\|$ denote the distance from x to nearest integers. Earlier work about the distribution of the fractional parts of the sequence αp was considered by I.M. Vinogradov [16] who showed that for any real number β, there are infinitely many primes p such that if $\theta = 1/5 - \varepsilon$, then

$$\|\alpha p+\beta\| < p^{-\theta}, \tag{1}$$

where and below $\varepsilon > 0$ is arbitrarily small. Later the exponent θ was improved by several authors (Harman [3, 4], Jia [8, 9], Heath-Brown and Jia [5]). So far the best result is given by Matomäki with $\theta = 1/3 - \varepsilon$.

Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes p such that $p+2$ is a prime too. Up to now this conjecture is still open, but many approximation to it established. One of the most interesting of them is due to J.R. Chen [2], who showed in 1973 that there exist infinitely many primes p such that $p+2 = P_2$.

In [14] Todorova and Tolev considered the distribution of αp modulo one with primes of the form specified above, and showed that for $\theta = 1/100$, there are infinitely many solutions in primes p to (1) such that $p+2 = P_4$. Later Matomäki [11] has shown that this actually holds with $p+2 = P_2$ and $\theta = 1/1000$.

In this paper, our purpose is to improve the range θ and we shall prove the following result.

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Theorem 1.1. *Let* $\alpha \in \mathcal{R} \setminus \mathcal{Q}$, $\beta \in \mathcal{R}$ *and* $0 < \theta \leq 1.5/100$. *Then there are infinitely many primes* p *satisfying* $p + 2 = P_2$ *and such that*

$$\|\alpha p + \beta\| < p^{-\theta}. \tag{2}$$

NOTATION. Let α be a real number with a rational approximation a/q satisfying

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad \text{where} \quad (a, q) = 1, \quad \text{and} \quad q \geq 1.$$

Here $K \geq 1$, $k \sim H$ means $H < k \leq 2H$ and $0 < \theta \leq 1.5/100$. As usual let $\Lambda(n)$ and $\phi(n)$ respectively denote Von Mangoldt's function and Euler's function. For simplicity instead of $m \equiv n \pmod{k}$, $e^{2\pi i x}$ we write $m \equiv n(k)$, $e(x)$ respectively. Letter C is a positive constant, which is not necessarily the same at each occurrence.

2. Some lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([11, Theorem 1]). *For any well-factorable function* λ *of level* D, *we have*

$$\begin{aligned} &\sum_{\substack{d\sim D \\ (d,c)=1}} \lambda_d \sum_{k\sim H} c_k \sum_{\substack{n\sim x \\ n\equiv c(d)}} \Lambda(n) e(\alpha n k) \\ &\ll H(\log x)^C x^{3/4+\varepsilon} \left(\frac{x}{q} + \frac{q}{H} + D^2 + x^{7/9+4\varepsilon} + \min\left\{ D^{4+20\varepsilon}, \frac{x}{D} \right\} \right)^{1/4-\varepsilon}. \end{aligned} \tag{3}$$

Lemma 2.2 ([10, 13]). *Let* $x > 1$, $z = x^{1/u}$. *Then for* $u \geq 1$, *we have*

$$\sum_{\substack{n\leq x \\ (n,P(z))=1}} 1 = w(u)\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right),$$

where $w(u)$ *is determined by the following differential-difference equation*

$$\begin{cases} w(u) = \dfrac{1}{u}, & \text{if} \quad 1 < u \leq 2, \\ (uw(u))' = w(u-1), & \text{if} \quad u \geq 2. \end{cases}$$

Lemma 2.3 ([13]). *For any given constant* $A > 10$, *there exists a constant* $B = B(A) > 0$ *such that*

$$\sum_{d\leq D} \max_{(l,d)=1} \max_{y\leq x} \left| \sum_{\substack{k\leq E(x) \\ (k,d)=1}} g(x,k) H(y;k,d,l) \right| \ll \frac{x}{\log^A x},$$

where

$$H(y;k,d,l)=\sum_{\substack{kp\le y\\ kp\equiv l(d)}}1-\frac{1}{\phi(d)}\sum_{kp\le y}1,$$

$$\frac{1}{2}\le E(x)\ll x^{1-\vartheta},\quad 0<\vartheta\le 1,$$

$$g(x,k)\ll d_r(k),\quad D=x^{1/2}\log^{-B}x.$$

Lemma 2.4 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_1(y)$ *be a positive function depending on* x *and satisfying* $r_1(y)\ll x^{\vartheta}$ *for* $y\le x$. *Then we have*

$$\sum_{d\le D}\max_{(l,d)=1}\max_{y\le x}\left|\sum_{\substack{k\le E(x)\\ (k,d)=1}}g(x,k)H(kr_1(y);k,d,l)\right|\ll\frac{x}{\log^A x}.$$

Lemma 2.5 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_2(y)$ *be a positive function depending on* x, y *and satisfying* $kr_2(y)\ll x$ *for* $k\le E(x)$, $y\le x$. *Then we have*

$$\sum_{d\le D}\max_{(l,d)=1}\max_{y\le x}\left|\sum_{\substack{k\le E(x)\\ (k,d)=1}}g(x,k)H(kr_2(y);k,d,l)\right|\ll\frac{x}{\log^A x}.$$

3. Proof of Theorem 1.1

As in [14] we begin with a periodic function $\chi(t)$ with period 1 such that

$$\chi(t)\begin{cases}\in(0,1) & \text{if}\quad -\Delta<t<\Delta,\\ =0 & \text{if}\quad \Delta\le t\le 1-\Delta,\end{cases}$$

and which has a Fourier series

$$\chi(t)=\Delta+\sum_{|k|>0}g(k)e(kt) \tag{4}$$

with coefficients satisfying

$$\begin{aligned}&g(0)=\Delta,\\ &g(k)\ll\Delta,\quad\text{for all}\quad k,\\ &\sum_{|k|>H}|g(k)|\ll N^{-1},\end{aligned}\tag{5}$$

where

$$\Delta=\Delta(N)=N^{-\theta}\quad\text{and}\quad H=\Delta^{-1}\log^2 N. \tag{6}$$

Next we will use sieve methods. As usual, for any sequence $\mathcal{E}$ of integers weighted by the numbers $f_n, n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{\substack{n\in\mathcal{E}\\(n,P(z))=1}} f_n,$$

and denote by $\mathcal{E}_d$ be the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod d$. We write

$$P(z) = \prod_{p<z} p$$

and

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Let further

$$C_0 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right),$$

and we will use the following form of the linear sieve due to Iwaniec [6].

Lemma 3.1. *Let* $2 \le z \le D^{1/2}$ *and let* $s = \log D/\log z$. *If*
(A_1) $|\mathcal{E}_d| = (\omega(d)/d)X + r(\mathcal{E}, d),\ \mu(d) \neq 0$;
(A_2) $\sum_{z_1\le p<z_2} \omega(p)/p = \log(\log z_2/\log z_1) + O(1/\log z_1),\ z_2 > z_1 \ge 2$,
where $\omega(d)$ *is a multiplicative function,* $0 \le \omega(p) < p$, $X > 1$ *is independent of* d. *Then*

$$S(\mathcal{E}, z) \le XV(z)(F(s) + o(1)) + \sum_{l<L}\sum_{d|P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$
$$S(\mathcal{E}, z) \ge XV(z)(f(s) - o(1)) - \sum_{l<L}\sum_{d|P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$

where $L = O(1)$, $\lambda^{\pm}$ *are well-factorable bounded functions of level* D, $f(s)$, $F(s)$ *are determined by the following differential-difference equation*

$$\begin{cases} F(s) = \dfrac{2e^{\gamma}}{s}, \quad f(s) = 0, & \text{if} \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & \text{if} \quad s \ge 2, \end{cases}$$

where γ *denote the Euler's constant.*

So, if we define $\mathcal{A}$ to be the sequence of integers $n \leq N$ weighted by

$$a_n = \begin{cases} \chi(\alpha(n-2)+\beta) & \text{if} \quad n-2 \in \mathbb{P}, \\ 0 & \text{else.} \end{cases}$$

Then to prove Theorem 1.1, it suffice to show that

$$S(\mathcal{A}, N^{1/3}) = \sum_{\substack{p+2\leq N \\ (p+2, P(N^{1/3}))=1}} \chi(\alpha p + \beta) > 0. \tag{7}$$

However, we cannot quite do that, but need to use a more sophisticated weighted sieve method. Indeed following Cai (see [1], Lemma 5), let $k = 1/12$, $l = 1/3.1$, and we consider

$$S \geq \sum_{\substack{n\in\mathcal{A} \\ (n,N^{1/12})=1}} a_n \left(1 - \frac{1}{2} \sum_{\substack{N^{1/12}\leq p<N^{1/3.1} \\ p|n}} 1 - \frac{1}{2} \sum_{\substack{n=p_1p_2p_3 \\ N^{1/12}\leq p_1<N^{1/3.1} \\ N^{1/3.1}\leq p_2<(N/p_1)^{1/2}}} 1 - \sum_{\substack{n=p_1p_2p_3 \\ N^{1/3.1}\leq p_1<p_2<(N/p_1)^{1/2}}} 1 \right)$$
$$+ O(N^{11/12}).$$

Here we notice that the weight of n is a_n if and only if n has no prime factors $< N^{1/3.1}$ in which case clearly $n = P_2$. If the weight of n is $a_n/2$, then a_n has one prime factor in the interval $[N^{1/12}, N^{1/3.1})$ and the third, fourth sum is 0. But this again implies that $n = P_2$. Thus the weight of n is positive only if

$$n = P_2, \quad n-2 \in \mathbb{P} \quad \text{and} \quad \|\alpha(n-2)+\beta\| < N^{-\theta},$$

and so it is enough to show that $S > 0$.

Using the sieve notation, we can write

$$\begin{aligned} S &\geq S(\mathcal{A}, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12}\leq p<N^{1/3.1}} S(\mathcal{A}_p, N^{1/12}) - \frac{1}{2} \sum_{\substack{N^{1/12}\leq p_1<N^{1/3.1} \\ N^{1/3.1}\leq p_2<(N/p_1)^{1/2}}} S(\mathcal{A}_{p_1p_2}, p_2) \\ &\quad - \sum_{N^{1/3.1}\leq p_1<p_2<(N/p_1)^{1/2}} S(\mathcal{A}_{p_1p_2}, p_2) + O(N^{11/12}) \\ &=: S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 - S_4 + O(N^{11/12}). \end{aligned} \tag{8}$$

Consider a square-free number d. If $2 \mid d$, then we write $|\mathcal{A}_d| = |r(\mathcal{A}, d)| \le 1$. Otherwise we have by the Fourier expansion of $\chi(n)$

$$\begin{aligned}
|\mathcal{A}_d| &= \sum_{\substack{md \le N \\ md-2 \in \mathbb{P}}} \chi(\alpha(md-2)+\beta) \\
&= \sum_{\substack{p \le N-2 \\ p \equiv -2(d)}} \chi(\alpha p + \beta) \\
&= \sum_{\substack{p \le N \\ p \equiv -2(d)}} \left(\Delta + \Delta \sum_{0<|k|<H} c_k e(\alpha k p) + O(N^{-1}) \right) \\
&= \Delta \left(\frac{\operatorname{Li} N}{\phi(d)} + R_1(d) + R_2(d) + O\left(\frac{N}{d(\log N)^C} \right) \right),
\end{aligned}$$

where $c_k \ll 1$, and

$$\begin{aligned}
R_1(d) &= \sum_{\substack{p \le N \\ p \equiv -2(d)}} 1 - \frac{\operatorname{Li} N}{\phi(d)}, \\
R_2(d) &= \sum_{\substack{p \le N \\ p \equiv -2(d)}} \sum_{0<|k|<H} c_k e(\alpha k p).
\end{aligned}$$

Applying Bombieri–Vinogradov theorem (see [7], Theorem 17.1) implies that

$$\sum_{d \le N^{1/2}/\log^C N} |R_1(d)| \ll \frac{N}{\log^A N}.$$

On the other hand, Lemma 2.1 implies that for a well-factorable function λ of level $D < N^{1/2}/(H^2 \log^C N)$, we get

$$\sum_{d \le D} \lambda_d R_2(d) \ll \frac{N}{\log^A N},$$

when $N = q^2$, where a/q is a convergent to α with a large enough denominator.

Therefore we apply Lemma 3.1 with

$$\omega(d) = \begin{cases} 0 & \text{if} \quad 2 \mid d, \\ \dfrac{d}{\phi(d)} & \text{otherwise}, \end{cases} \quad X = \Delta \operatorname{Li} N, \quad \text{and} \quad D < \frac{N^{1/2}}{H^2 \log^C N},$$

to S_1 and obtain

$$\begin{aligned}S_1 &\geq \Delta \operatorname{Li} N V(N^{1/12}) f(6-24\theta)(1+o(1))\\ &= \frac{8}{1-4\theta}\left(\log(5-24\theta) + \int_3^{5-24\theta} \frac{1}{t}\,dt \int_2^{t-1} \frac{\log(s-1)}{s}\,ds\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (9)\\ &\geq 13.471\frac{C_0 \Delta N}{\log^2 N}.\end{aligned}$$

Since $(\mathcal{A}_p)_d = \mathcal{A}_{pd}$, we can use Lemma 3.1 also to S_2 by using the same method. In this case one faces the sum

$$\sum_{N^{1/12} \leq p < N^{1/3.1}} \sum_{d \leq D} \lambda_d R_2(pd),$$

by Remark 10 in [11], the above sum is at most

$$\ll \frac{N}{\log^A N},$$

then

$$\omega(d) = \begin{cases} 0 & \text{if} \quad 2 \mid d,\\ \dfrac{d}{\phi(d)} & \text{otherwise,}\end{cases} \qquad X = \frac{\Delta \operatorname{Li} N}{\phi(p)}, \quad \text{and} \quad D < \frac{N^{1/2}}{pH^2 \log^C N}.$$

And applying partial summation, prime number theory, we have

$$\begin{aligned}S_2 &\leq \sum_{N^{1/12} \leq p < N^{1/3.1}} \frac{\Delta \operatorname{Li} N}{\phi(p)} V(N^{1/12}) F\left(6 - 24\theta - 12\frac{\log p}{\log N}\right)(1+o(1))\\ &= 6\int_{1/12}^{1/3.1} \frac{F(6-24\theta-12t)}{t}\,dt\,\frac{C_0 \Delta N}{\log^2 N}(1+o(1))\\ &= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)}\left(1 + \int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right)\right.\\ &\qquad \left. + \int_{(1-8\theta)/4}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (10)\\ &= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)} \int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right.\\ &\qquad \left. + \int_{1/12}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1))\\ &\leq 21.3643\frac{C_0 \Delta N}{\log^2 N}.\end{aligned}$$

For the sum S_3, we write

$$\begin{aligned} S_3 &= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{np_1p_2\le N\\ np_1p_2-2\in\mathbb{P},\,(n,P(p_2))=1}} \chi(\alpha(np_1p_2-2)+\beta)\\ &= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{p=np_1p_2-2\\ 1\le n\le N/(p_1p_2),\,(n,P(p_2))=1}} \chi(\alpha p+\beta)\\ &\le \sum_{N^{1/3.1}\le p_2<N^{11/24}} \sum_{\substack{1\le n\le N^{11/12}/p_2\\ (n,P(p_2))=1}} \sum_{\substack{p=np_1p_2-2\\ N^{1/12}\le p_1<\min(N^{1/3.1},N/(np_2))}} 1. \end{aligned}$$

Let's consider the set

$$\mathcal{E}=\left\{e \;\middle|\; e=np_2,\ N^{1/3.1}\le p_2<N^{11/24},\ 1\le n\le \frac{N^{11/12}}{p_2},\ (n,\,P(p_2))=1\right\}.$$

By the definition of the set $\mathcal{E}$, it is easy to see that for every $e\in\mathcal{E}$, p_2 is determined by e uniquely. Let $p_2=r(e)$, then we have

$$N^{1/3.1}\le r(e)<N^{11/24} \quad\text{and}\quad er(e)<N.$$

Let

$$\mathcal{L}=\left\{l \;\middle|\; l=ep_1-2,\ e\in\mathcal{E},\ N^{1/12}\le p_1<\min\left(N^{1/3.1},\frac{N}{np_2}\right)\right\}.$$

Then

$$N^{1/3.1}<e<N^{11/12} \quad\text{for}\quad e\in\mathcal{E}$$

and

$$|\mathcal{E}|\le N^{11/12}, \qquad \sum_{l\in\mathcal{L},\,l\le N^{1/3.1}} 1 \ll N^{11/12},$$

and also we have

$$S_3\le S(\mathcal{L},z)+O(N^{11/12}) \quad\text{for}\quad z\le N^{1/3}. \tag{11}$$

We write

$$z^2=D=N^{1/2}\log^{-B}N,$$

then

$$S(\mathcal{L},z)\le 8\frac{C_0|\mathcal{L}|}{\log N}+R_3+R_4, \tag{12}$$

where

$$R_3=\sum_{\substack{d\le D\\(d,N)=1}}\left|\sum_{\substack{e\in\mathcal{E}\\(e,d)=1}}\left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)\\ ep_1\equiv 2(d)}}1-\frac{1}{\phi(d)}\sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)}1\right)\right|,$$

$$R_4=\sum_{d\le D,\,(d,N)=1}\frac{1}{\phi(d)}\sum_{\substack{e\in\mathcal{E}\\(e,d)>1}}\sum_{N^{1/12}\le p_1<\min(N^{1/3.1},\,N/e)}1.$$

Let

$$Q(k)=\sum_{e=k,\,e\in\mathcal{E}}1,$$

then

$$R_3=\sum_{d\le D}\left|\sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)=1}}Q(k)\left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)\\ kp_1\equiv 2(d)}}1-\frac{1}{\phi(d)}\sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)}1\right)\right|,$$

$$R_4=\sum_{d\le D}\frac{1}{\phi(d)}\sum_{\substack{N^{1/3}<k<N^{11/12}\\(k,d)>N^{1/3.1}}}\sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)}1.$$

It is easy to show

$$Q(k)\le 1.$$

Then we have

$$\begin{aligned}R_4&\ll\sum_{d\le D}\frac{1}{\phi(d)}\sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)>N^{1/3.1}}}\frac{N}{k}\\&\ll N\sum_{d\le D}\frac{1}{\phi(d)}\sum_{h|d,\,h\ge N^{1/3.1}}\sum_{\substack{k<N^{11/12}\\(k,d)=h}}\frac{1}{k}\\&\ll N\log N\sum_{d\le D}\frac{1}{\phi(d)}\sum_{h|d,\,h\ge N^{1/3.1}}\frac{1}{h}\\&\ll N\log N\sum_{N^{1/3.1}\le h\le D}\frac{1}{h\phi(h)}\sum_{d\le D/h}\frac{1}{\phi(d)}\\&\ll N^{2.1/3.1}\log^2 N,\end{aligned}\tag{13}$$

and

$$R_3\le R_5+R_6+R_7,\tag{14}$$

where

$$R_5 = \sum_{d \le D,\, (d,N)=1} \left| \sum_{\substack{N^{1/3.1} < k < N^{2.1/3.1} \\ (k,d)=1}} Q(k) \left(\sum_{\substack{p_1 < N^{1/3.1} \\ kp_1 \equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{p_1 < N^{1/3.1}} 1 \right) \right|,$$

$$R_6 = \sum_{d \le D,\, (d,N)=1} \left| \sum_{\substack{N^{2.1/3.1} < k < N^{11/12} \\ (k,d)=1}} Q(k) \left(\sum_{\substack{kp_1 < N \\ kp_1 \equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{kp_1 < N} 1 \right) \right|,$$

$$R_7 = \sum_{d \le D,\, (d,N)=1} \left| \sum_{\substack{N^{1/3.1} < k < N^{11/12} \\ (k,d)=1}} Q(k) \left(\sum_{\substack{p_1 < N^{1/12} \\ kp_1 \equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{p_1 < N^{1/12}} 1 \right) \right|.$$

Due to Lemma 2.3–2.5,

$$R_j \ll \frac{N}{\log^4 N}, \quad j = 5, 6, 7. \tag{15}$$

By Lemma 2.2 and prime number theorem, we have

$$\begin{aligned}
|\mathcal{L}| &= \sum_{e \in \mathcal{E}} \sum_{N^{1/12} \le p_1 < N^{1/3.1}} 1 \\
&= \sum_{N^{1/12} \le p_1 < N^{1/3.1} \le p_2 < (N/p_1)^{1/2}} \sum_{\substack{1 \le n \le N/(p_1 p_2) \\ (n, P(p_2))=1}} 1 + O(N^{11/12}) \\
&< (1 + o(1)) \sum_{N^{1/12} \le p_1 < N^{1/3.1} \le p_2 < (N/p_1)^{1/2}} w\left(\frac{\log(N/(p_1 p_2))}{\log p_2} \right) \frac{N}{p_1 p_2 \log p_2} \\
&\quad + O(N^{11/12}) \\
&\le \left(\int_{1/12}^{1/3.1} \frac{dt}{t} \int_{1/3.1}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{N}{\log N}.
\end{aligned} \tag{16}$$

By (11)–(16), we obtain

$$\begin{aligned}
S_3 &\le 8 \left(\int_{1/12}^{1/3.1} \frac{dt}{t} \int_{1/3.1}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{C_0 \Delta N}{\log^2 N} (1 + o(1)) \\
&\le 5.52946 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned}$$

We also use the same idea to S_4,

$$
\begin{aligned}
S_4 &\le 8\left(\int_{1/3.1}^{1/3} \frac{dt}{t} \int_t^{(1-t)/2} \frac{ds}{s(1-t-s)}\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\
&\le 0.018745\frac{C_0 \Delta N}{\log^2 N}.
\end{aligned} \tag{18}
$$

Combining (7)–(10), (17) and (18), then we obtain

$$
S > S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 - S_4 \gg \frac{\Delta N}{\log^2 N},
$$

which concludes the proof of Theorem 1.1.

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