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<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 49(4) P.993-P.1004</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2012-12</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/23416">https://doi.org/10.18910/23416</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/23416</td>
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ON THE DISTRIBUTION OF $\alpha p$ MODULO ONE FOR PRIMES $p$ OF A SPECIAL FORM

SAN-YING SHI

(Received February 25, 2011)

Abstract
In this paper it is proved that for any irrational $\alpha$ and some $0 < \theta \leq 1.5/100$, there are infinitely many primes $p$ such that $p + 2$ has at most two prime factors and $\|\alpha p + \beta\| < p^{-\theta}$ which improves K. Matomäki’s result $\theta < 1/1000$.

1. Introduction

Let $\alpha$ be a irrational real number and $\|x\|$ denote the distance from $x$ to nearest integers. Earlier work about the distribution of the fractional parts of the sequence $\alpha p$ was considered by I.M. Vinogradov [16] who showed that for any real number $\beta$, there are infinitely many primes $p$ such that if $\theta = 1/5 - \varepsilon$, then

\[(1) \quad \|\alpha p + \beta\| < p^{-\theta},\]

where and below $\varepsilon > 0$ is arbitrarily small. Later the exponent $\theta$ was improved by several authors (Harman [3, 4], Jia [8, 9], Heath-Brown and Jia [5]). So far the best result is given by Matomäki with $\theta = 1/3 - \varepsilon$.

Let $P_r$ denote an almost prime with at most $r$ prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes $p$ such that $p + 2$ is a prime too. Up to now this conjecture is still open, but many approximation to it established. One of the most interesting of them is due to J.R. Chen [2], who showed in 1973 that there exist infinitely many primes $p$ such that $p + 2 = P_2$.

In [14] Todorova and Tolev considered the distribution of $\alpha p$ modulo one with primes of the form specified above, and showed that for $\theta = 1/100$, there are infinitely many solutions in primes $p$ to (1) such that $p + 2 = P_4$. Later Matomäki [11] has shown that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$. In this paper, our purpose is to improve the range $\theta$ and we shall prove the following result.

2010 Mathematics Subject Classification. 11K60, 11N36.
Theorem 1.1. Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), \( \beta \in \mathbb{R} \) and \( 0 < \theta \leq 1.5/100 \). Then there are infinitely many primes \( p \) satisfying \( p + 2 = P_2 \) and such that
\[
\|\alpha p + \beta\| < p^{-\theta}.
\]

Notation. Let \( \alpha \) be a real number with a rational approximation \( a/q \) satisfying
\[
\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}, \quad \text{where} \quad (a, q) = 1, \quad \text{and} \quad q \geq 1.
\]
Here \( K \geq 1, k \sim H \) means \( H < k \leq 2H \) and \( 0 < \theta \leq 1.5/100 \). As usual let \( \Lambda(n) \) and \( \phi(n) \) respectively denote Von Mangoldt’s function and Euler’s function. For simplicity instead of \( m \equiv n \pmod{k} \), \( e^{2\pi it} \) we write \( m \equiv n(k) \), \( e(x) \) respectively. Letter \( C \) is a positive constant, which is not necessarily the same at each occurrence.

2. Some lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([11, Theorem 1]). For any well-factorable function \( \lambda \) of level \( D \), we have
\[
\sum_{d \sim D} \lambda_d \sum_{k \sim H} \sum_{n=1}^{\infty} \frac{\Lambda(n)\phi(ank)}{n^{1+\epsilon}} \ll H(\log x)^{3/4+\epsilon} \left( \frac{x}{q} + \frac{q}{H} + D^2 + x^{7/9+4\epsilon} + \min \left\{ D^{4+20\epsilon}, \frac{x}{D} \right\} \right)^{1/4-\epsilon}.
\]

Lemma 2.2 ([10, 13]). Let \( x > 1, z = x^{1/u} \). Then for \( u \geq 1 \), we have
\[
\sum_{n \leq x \atop (n, P(z))=1} 1 = w(u) \frac{x}{\log z} + O \left( \frac{x}{\log^2 z} \right),
\]
where \( w(u) \) is determined by the following differential-difference equation
\[
\begin{align*}
w(u) &= \frac{1}{u}, & \text{if} \quad 1 < u \leq 2, \\
(wu(u))' &= w(u - 1), & \text{if} \quad u \geq 2.
\end{align*}
\]

Lemma 2.3 ([13]). For any given constant \( A > 10 \), there exists a constant \( B = B(A) > 0 \) such that
\[
\sum_{d \leq D} \max_{(l,d)=1} \max_{y \leq t} \left| \sum_{k \leq E(x) \atop (k,d)=1} g(x, k)H(y; k, d, l) \right| \ll \frac{x}{\log^A x}.
\]
where

\[ H(y; k, d, l) = \sum_{k_p \leq y} 1 - \frac{1}{\phi(d)} \sum_{k_p \leq y} 1, \]

\[ \frac{1}{2} \leq E(x) \ll x^{1-\vartheta}, \quad 0 < \vartheta \leq 1, \]

\[ g(x, k) \ll d_r(k), \quad D = x^{1/2} \log^{-\theta} x. \]

**Lemma 2.4 ([13]).** Let the condition of Lemma 2.3 be given and \( r_1(y) \) be a positive function depending on \( x \) and satisfying \( r_1(y) \ll x^\vartheta \) for \( y \leq x \). Then we have

\[ \sum_{d \leq D} \max_{(l, d) = 1} \max_{y \leq x} \left| \sum_{k \leq E(x)} g(x, k) H(kr_1(y); k, d, l) \right| \ll \frac{x}{\log^A x}. \]

**Lemma 2.5 ([13]).** Let the condition of Lemma 2.3 be given and \( r_2(y) \) be a positive function depending on \( x, y \) and satisfying \( kr_2(y) \ll x \) for \( k \leq E(x), y \leq x \). Then we have

\[ \sum_{d \leq D} \max_{(l, d) = 1} \max_{y \leq x} \left| \sum_{k \leq E(x)} g(x, k) H(kr_2(y); k, d, l) \right| \ll \frac{x}{\log^A x}. \]

3. **Proof of Theorem 1.1**

As in [14] we begin with a periodic function \( \chi(t) \) with period 1 such that

\[ \chi(t) \begin{cases} \in (0, 1) & \text{if } -\Delta < t < \Delta, \\ = 0 & \text{if } \Delta \leq t \leq 1 - \Delta. \end{cases} \]

and which has a Fourier series

\[ \chi(t) = \Delta + \sum_{|k| > 0} g(k)e(kt) \]

with coefficients satisfying

\[ g(0) = \Delta, \]

\[ g(k) \ll \Delta, \quad \text{for all } k, \]

\[ \sum_{|k| > H} |g(k)| \ll N^{-1}, \]

where

\[ \Delta = \Delta(N) = N^{-\vartheta} \quad \text{and} \quad H = \Delta^{-1} \log^2 N. \]
Next we will use sieve methods. As usual, for any sequence $\mathcal{E}$ of integers weighted by the numbers $f_n$, $n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{n \in \mathcal{E}} f_n,$$

and denote by $\mathcal{E}_d$ be the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod{d}$. We write

$$P(z) = \prod_{p < z} p$$

and

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Let further

$$C_0 = \prod_{p \geq 2} \left(1 - \frac{1}{(p-1)^2}\right),$$

and we will use the following form of the linear sieve due to Iwaniec [6].

**Lemma 3.1.** Let $2 \leq z \leq D^{1/2}$ and let $s = \log D / \log z$. If

(A1) $|\mathcal{E}_d| = (\omega(d)/d)X + r(\mathcal{E}, d)$, $\mu(d) \neq 0$;

(A2) $\sum_{z_1 < p < z_2} \omega(p)/p = \log(\log z_2/\log z_1) + O(1/\log z_1)$, $z_2 > z_1 \geq 2$, where $\omega(d)$ is a multiplicative function, $0 \leq \omega(p) < p$, $X > 1$ is independent of $d$. Then

$$S(\mathcal{E}, z) \leq XV(z)(F(s) + o(1)) + \sum_{l \leq L} \sum_{d|P(z)} \lambda^+_l(d)r(\mathcal{E}, d),$$

$$S(\mathcal{E}, z) \geq XV(z)(f(s) - o(1)) - \sum_{l \leq L} \sum_{d|P(z)} \lambda^-_l(d)r(\mathcal{E}, d),$$

where $L = O(1)$, $\lambda^\pm$ are well-factorable bounded functions of level $D$, $f(s)$, $F(s)$ are determined by the following differential-difference equation

$$\begin{cases}
F(s) = \frac{2e^\gamma}{s}, & f(s) = 0, \\
(sF(s))' = f(s-1), & (sf(s))' = F(s-1)
\end{cases} \quad \text{if } 0 < s \leq 2,$$

where $\gamma$ denote the Euler’s constant.
So, if we define $\mathcal{A}$ to be the sequence of integers $n \leq N$ weighted by

$$a_n = \begin{cases} \chi(n - 2) & \text{if} \quad n - 2 \in \mathbb{P}, \\ 0 & \text{else}. \end{cases}$$

Then to prove Theorem 1.1, it suffice to show that

$$S(\mathcal{A}, N^{1/3}) = \sum_{p + 2 \leq N \atop (p + 2, p(N^{1/3})) = 1} \chi(\alpha p + \beta) > 0. \quad (7)$$

However, we cannot quite do that, but need to use a more sophisticated weighted sieve method. Indeed following Cai (see [1], Lemma 5), let $k = 1/12$, $l = 1/3.1$, and we consider

$$S \geq \sum_{n \in \mathcal{A} \atop (n, N^{1/12}) = 1} a_n \left( 1 - \frac{1}{2} \sum_{N^{1/12} \leq p < N^{1/3.1} \atop p \mid n} 1 - \frac{1}{2} \sum_{n = p_1 p_2 p_3 \atop N^{1/12} \leq p_1 < N^{1/3.1} \atop N^{1/3.1} \leq p_2 < (N/p_1)^{1/2}} 1 - \sum_{n = p_1 p_2 p_3 \atop N^{1/3.1} \leq p_1 < p_2 < (N/p_1)^{1/2}} 1 \right)$$

$$+ O(N^{11/12}).$$

Here we notice that the weight of $n$ is $a_n$ if and only if $n$ has no prime factors $< N^{1/3.1}$ in which case clearly $n = p_2$. If the weight of $n$ is $a_n/2$, then $a_n$ has one prime factor in the interval $[N^{1/12}, N^{1/3.1})$ and the third, fourth sum is 0. But this again implies that $n = p_2$. Thus the weight of $n$ is positive only if

$$n = p_2, \quad n - 2 \in \mathbb{P} \quad \text{and} \quad \|\alpha(n - 2) + \beta\| < N^{-\theta},$$

and so it is enough to show that $S > 0$.

Using the sieve notation, we can write

$$S \geq S(\mathcal{A}, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \leq p < N^{1/3.1}} S(\mathcal{A}_p, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \leq p_2 < (N/p_1)^{1/2}} S(\mathcal{A}_{p_1 p_2}, p_2)$$

$$- \sum_{N^{1/3.1} \leq p_1 < p_2 < (N/p_1)^{1/2}} S(\mathcal{A}_{p_1 p_2}, p_2) + O(N^{11/12}) \quad (8)$$

$$=: S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 - S_4 + O(N^{11/12}).$$
Consider a square-free number \( d \). If \( 2 \mid d \), then we write \( |\mathcal{A}_d| = |r(\mathcal{A}, d)| \leq 1 \). Otherwise we have by the Fourier expansion of \( \chi(n) \)

\[
|\mathcal{A}_d| = \sum_{\substack{md \leq N \\text{md-2}\in\mathbb{P}}} \chi(\alpha(md - 2) + \beta) \\
= \sum_{\substack{p \leq N - 2 \\text{p-2}\in\mathbb{P}}} \chi(\alpha p + \beta) \\
= \sum_{\substack{p \leq N \\text{p}\in\mathbb{P}}} \left( \Delta + \Delta \sum_{0 < |k| < H} c_k e(\alpha kp) + O(N^{-1}) \right) \\
= \Delta \left( \frac{\text{Li} \ N \phi(d)}{\phi(d)} + R_1(d) + R_2(d) + O\left(\frac{N}{d(\log N)^C}\right) \right),
\]

where \( c_k \ll 1 \), and

\[
R_1(d) = \sum_{\substack{p \leq N \\text{p}\in\mathbb{P}}} 1 - \frac{\text{Li} \ N \phi(d)}{\phi(d)}, \\
R_2(d) = \sum_{\substack{p \leq N \\text{p}\in\mathbb{P}}} \sum_{0 < |k| < H} c_k e(\alpha kp).
\]

Applying Bombieri–Vinogradov theorem (see [7], Theorem 17.1) implies that

\[
\sum_{d \leq N^{1/2}/\log^C N} |R_1(d)| \ll \frac{N}{\log^4 N}.
\]

On the other hand, Lemma 2.1 implies that for a well-factorable function \( \lambda \) of level \( D < N^{1/2}/(H^2 \log^C N) \), we get

\[
\sum_{d \leq D} \lambda_d R_2(d) \ll \frac{N}{\log^4 N},
\]

when \( N = q^2 \), where \( a/q \) is a convergent to \( \alpha \) with a large enough denominator.

Therefore we apply Lemma 3.1 with

\[
\omega(d) = \begin{cases} 
0 & \text{if } 2 \mid d, \\
\frac{d}{\phi(d)} & \text{otherwise},
\end{cases} \quad X = \Delta \text{Li} \ N, \quad \text{and } D < \frac{N^{1/2}}{H^2 \log^C N},
\]
to $S_1$ and obtain

$$S_1 \geq \Delta \text{Li} N V(N^{1/12}) f(6 - 24\theta)(1 + o(1))$$

\begin{equation}
\begin{aligned}
&= \frac{8}{1 - 4\theta} \left( \log(5 - 24\theta) + \int_{1/3}^{5-24\theta} \frac{1}{t} \frac{\log(s - 1)}{s} ds \right) C_0 \Delta N \log^2 N \left(1 + o(1)\right) \\
&\geq 13.471 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned}
\end{equation}

Since $(A_p)_d = A_{pd}$, we can use Lemma 3.1 also to $S_2$ by using the same method. In this case one faces the sum

$$\sum_{N^{1/12} \leq p < N^{1/3}} \sum_{d \leq D} \lambda_d R_2(pd),$$

by Remark 10 in [11], the above sum is at most

$$\ll \frac{N}{\log^A N},$$

then

$$\omega(d) = \begin{cases} 0 & \text{if } 2 \not| d, \\ \frac{d}{\phi(d)} & \text{otherwise, } \quad X = \frac{\Delta \text{Li} N}{\phi(p)}, \quad \text{and } D < \frac{N^{1/2}}{pH^2 \log^C N}.
\end{cases}$$

And applying partial summation, prime number theory, we have

$$S_2 \leq \sum_{N^{1/12} \leq p < N^{1/3}} \frac{\Delta \text{Li} N}{\phi(p)} V(N^{1/12}) F \left( 6 - 24\theta - 12 \frac{\log p}{\log N} \right)(1 + o(1))$$

$$= 6 \int_{1/12}^{1/3,1} \frac{F(6 - 24\theta - 12t)}{t} dt \frac{C_0 \Delta N}{\log^2 N} \left(1 + o(1)\right)$$

\begin{equation}
\begin{aligned}
&= 8 \left( \int_{1/12}^{1/3,1} \frac{dt}{t(1 - 4\theta - 2t)} \left(1 + \int_2^{5 - 24\theta - 12t} \frac{\log(s - 1)}{s} ds \right) \\
&\quad + \int_{(1 - 8\theta)/4}^{1/12} \frac{1}{t(1 - 4\theta - 2t)} dt \right) C_0 \Delta N \log^2 N \left(1 + o(1)\right) \\
&\leq 21.3643 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned}
\end{equation}
For the sum $S_3$, we write

$$S_3 = \sum_{N^{1/12} \leq p_1 < N^{1/3}, 1 \leq p_2 < (N/p_1)^{1/2}} \sum_{np_1 p_2 \leq N} \chi(\alpha(np_1 p_2 - 2) + \beta)$$

$$= \sum_{N^{1/12} \leq p_1 < N^{1/3}, 1 \leq p_2 < (N/p_1)^{1/2}} \sum_{1 \leq n \leq N/(p_1 p_2), (n, P(p_2)) = 1} \chi(\alpha p + \beta)$$

$$\leq \sum_{N^{1/12} \leq p_2 < N^{11/24}} \sum_{1 \leq n \leq N^{11/12} / p_2} \sum_{1 \leq p = np_1 p_2 \leq 2} 1.$$ 

Let’s consider the set

$$E = \left\{ e \mid e = np_2, N^{1/3, 1} \leq p_2 < N^{11/24}, 1 \leq n \leq \frac{N^{11/12}}{p_2}, (n, P(p_2)) = 1 \right\}.$$

By the definition of the set $E$, it is easy to see that for every $e \in E$, $p_2$ is determined by $e$ uniquely. Let $p_2 = r(e)$, then we have

$$N^{1/3, 1} \leq r(e) < N^{11/24} \quad \text{and} \quad e r(e) < N.$$ 

Let

$$L = \left\{ l \mid l = ep_1 - 2, e \in E, N^{11/12} \leq p_1 < \min\left( N^{1/3, 1}, \frac{N}{np_2} \right) \right\}.$$ 

Then

$$N^{1/3, 1} < e < N^{11/12} \quad \text{for} \quad e \in E$$

and

$$|E| \leq N^{11/12}, \quad \sum_{l \in L, l \leq N^{1/3, 1}} 1 \ll N^{11/12},$$

and also we have

(11) \quad $S_3 \leq S(L, z) + O(N^{11/12})$ \quad for \quad $z \leq N^{1/3}$.

We write

$$z^2 = D = N^{1/2} \log^B N,$$

then

(12) \quad $S(L, z) \leq 8 \frac{C_0 |L|}{\log N} + R_3 + R_4,$
It is easy to show

\[ R_3 = \sum_{d \leq D} \sum_{\substack{p_1 < N^{1/12} \leq p \leq N^{1/12} \atop (d,N) = 1}} \left( \sum_{\substack{e \in \mathcal{E} \atop (e,d) = 1}} \sum_{p_1 \leq 2(d)} 1 - \frac{1}{\phi(d)} \right) \sum_{N^{1/12} < p_1 < \min(N^{1/3}, N/e)} 1 \right), \]

\[ R_4 = \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{\frac{N^{1/12} < k < N^{1/12}}{(k,d) = 1}} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1. \]

Let

\[ Q(k) = \sum_{e = k, e \in \mathcal{E}} 1, \]

then

\[ R_3 = \sum_{d \leq D} \sum_{\frac{N^{1/12} < k < N^{1/12}}{(k,d) = 1}} Q(k) \left( \sum_{\frac{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)}{kp_1 = 2(d)}} 1 - \frac{1}{\phi(d)} \right) \sum_{N^{1/12} < p_1 < \min(N^{1/3}, N/k)} 1 \right), \]

\[ R_4 = \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{\frac{N^{1/12} < k < N^{1/12}}{(k,d) = 1}} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1. \]

It is easy to show

\[ Q(k) \leq 1. \]

Then we have

\[ R_4 \ll \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{\frac{N^{1/3} < k < N^{1/12} \atop (k,d) = 1} \sum_{N^{1/3} \leq h \leq N^{1/3} \atop (k,d) = h}} \frac{N}{k} \]

\[ \ll N \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{h|d, h \geq N^{1/3} \atop (k,d) = h} \sum_{N^{1/12} \leq k \leq N^{1/3}} \frac{1}{k} \]

\[ \ll N \log N \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{h|d, h \geq N^{1/3}} \frac{1}{h} \]

\[ \ll N \log N \sum_{h \leq D \atop h|d, h \geq N^{1/3}} \frac{1}{h} \phi(h) \sum_{d \leq D/h} \frac{1}{\phi(d)} \]

\[ \ll N^{2/3} \log^2 N, \]

and

\[ R_3 \leq R_5 + R_6 + R_7, \]
where

\[
R_5 = \sum_{d \leq D, (d,N)=1} \left| \sum_{\substack{N^{1/3} < k < N^{2/3} / N \leq 2(d) \to 1}} Q(k) \left( \sum_{p_1 < N^{1/3}} \frac{1}{\phi(d) \sum_{p_1 < N^{1/3}}} \right) \right|
\]

\[
R_6 = \sum_{d \leq D, (d,N)=1} \left| \sum_{\substack{N^{1/3} < k < N^{11/12} / N \leq 2(d) \to 1}} Q(k) \left( \sum_{p_1 < N} \frac{1}{\phi(d) \sum_{p_1 < N}} \right) \right|
\]

\[
R_7 = \sum_{d \leq D, (d,N)=1} \left| \sum_{\substack{N^{1/3} < k < N^{11/12} / N \leq 2(d) \to 1}} Q(k) \left( \sum_{p_1 < N^{11/12}} \frac{1}{\phi(d) \sum_{p_1 < N^{11/12}}} \right) \right|
\]

Due to Lemma 2.3–2.5,

\[
(15) \quad R_j \ll \frac{N}{\log^4 N}, \quad j = 5, 6, 7.
\]

By Lemma 2.2 and prime number theorem, we have

\[
|\mathcal{L}| = \sum_{e \in \mathcal{E}} \sum_{N^{1/12} \leq p_1 < N^{1/3}} 1
\]

\[
= \sum_{N^{1/12} \leq p_1 < N^{1/3}} \sum_{1 \leq n \leq N/(p_1 p_2) / (n, P(p_2)) = 1} 1 + O(N^{11/12})
\]

\[
< (1 + o(1)) \sum_{N^{1/12} \leq p_1 < N^{1/3}} \sum_{p_1 < N^{1/3}} \frac{1}{\phi(p_1)} \left( \frac{\log(N/(p_1 p_2))}{\log p_2} \right) \frac{N}{p_1 p_2 \log p_2}
\]

\[
+ O(N^{11/12})
\]

\[
\leq \left( \int_{1/12}^{1/3} \frac{dt}{t} \int_{1/3}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{N}{\log N}.
\]

By (11)–(16), we obtain

\[
S_3 \leq 8 \left( \int_{1/12}^{1/3} \frac{dt}{t} \int_{1/3}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{C_0 \Delta N}{\log^2 N} (1 + o(1))
\]

\[
\leq 5.52946 \frac{C_0 \Delta N}{\log^2 N}.
\]
We also use the same idea to $S_4$,

\begin{align}
S_4 &\leq 8 \left( \int_{1/3}^{1} \frac{dt}{t} \int_{t}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{C_0 \Delta N}{\log^2 N} (1 + o(1)) \\
&\leq 0.018745 \frac{C_0 \Delta N}{\log^2 N}.
\end{align}

Combining (7)–(10), (17) and (18), then we obtain

\[ S > S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 \gg \frac{\Delta N}{\log^2 N}, \]

which concludes the proof of Theorem 1.1.

ACKNOWLEDGMENT. The author thanks the referee for his/her helpful comments. This work is supported by The National Science Foundation of China (grant no. 11071186) and by Natural Science Foundation of Anhui province (Grant No. 1208085QA01).

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