ON THE DISTRIBUTION OF $\alpha p$ MODULO ONE FOR PRIMES $p$ OF A SPECIAL FORM

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(Received February 25, 2011)

Abstract

In this paper it is proved that for any irrational $\alpha$ and some $0 < \theta \leq 1.5/100$, there are infinitely many primes $p$ such that $p + 2$ has at most two prime factors and $\|\alpha p + \beta\| < p^{-\theta}$ which improves K. Matomäki’s result $\theta < 1/1000$.

1. Introduction

Let $\alpha$ be a irrational real number and $\|x\|$ denote the distance from $x$ to nearest integers. Earlier work about the distribution of the fractional parts of the sequence $\alpha p$ was considered by I.M. Vinogradov [16] who showed that for any real number $\beta$, there are infinitely many primes $p$ such that if $\theta = 1/5 - \varepsilon$, then

$$\|\alpha p + \beta\| < p^{-\theta},$$

where and below $\varepsilon > 0$ is arbitrarily small. Later the exponent $\theta$ was improved by several authors (Harman [3, 4], Jia [8, 9], Heath-Brown and Jia [5]). So far the best result is given by Matomäki with $\theta = 1/3 - \varepsilon$.

Let $P_r$ denote an almost prime with at most $r$ prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes $p$ such that $p + 2$ is a prime too. Up to now this conjecture is still open, but many approximation to it established. One of the most interesting of them is due to J.R. Chen [2], who showed in 1973 that there exist infinitely many primes $p$ such that $p + 2 = P_2$.

In [14] Todorova and Tolev considered the distribution of $\alpha p$ modulo one with primes of the form specified above, and showed that for $\theta = 1/100$, there are infinitely many solutions in primes $p$ to (1) such that $p + 2 = P_4$. Later Matomäki [11] has shown that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$.

In this paper, our purpose is to improve the range $\theta$ and we shall prove the following result.

2010 Mathematics Subject Classification. 11K60, 11N36.
**Theorem 1.1.** Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$ and $0 < \theta \leq 1.5/100$. Then there are infinitely many primes $p$ satisfying $p + 2 = P_2$ and such that

$$\|\alpha p + \beta\| < p^{-\theta}.$$  

**NOTATION.** Let $\alpha$ be a real number with a rational approximation $a/q$ satisfying

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2},$$

where $(a, q) = 1$, and $q \geq 1$.

Here $K \geq 1$, $k \sim H$ means $H < k \leq 2H$ and $0 < \theta \leq 1.5/100$. As usual let $\Lambda(n)$ and $\phi(n)$ respectively denote Von Mangoldt’s function and Euler’s function. For simplicity instead of $m \equiv n \pmod{k}$, $e^{2\pi i x}$ we write $m \equiv n(k)$, $e(x)$ respectively. Letter $C$ is a positive constant, which is not necessarily the same at each occurrence.

**2. Some lemmas**

In order to prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1 ([11, Theorem 1]).** For any well-factorable function $\lambda$ of level $D$, we have

$$\sum_{\substack{d \sim D \pmod{c} \geq 1}} \lambda(d) \sum_{k \sim H} \sum_{\substack{n \sim x \atop n \equiv c(d)}} \Lambda(n) e(ank)$$

(3)

$$\ll H (\log x)^C \sum_{\substack{n \sim x \atop (n, P(z)) = 1}} 1 = w(u) \frac{x}{\log z} + O \left( \frac{x}{\log^2 z} \right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases}
    w(u) = \frac{1}{u}, & \text{if } 1 < u \leq 2, \\
    (uw(u))' = w(u - 1), & \text{if } u \geq 2.
\end{cases}$$

**Lemma 2.2 ([10, 13]).** Let $x > 1$, $z = x^{1/a}$. Then for $u \geq 1$, we have

$$\sum_{\substack{n \leq x \atop (n, P(z)) = 1}} 1 = w(u) \frac{x}{\log z} + O \left( \frac{x}{\log^2 z} \right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases}
    w(u) = \frac{1}{u}, & \text{if } 1 < u \leq 2, \\
    (uw(u))' = w(u - 1), & \text{if } u \geq 2.
\end{cases}$$

**Lemma 2.3 ([13]).** For any given constant $A > 0$, there exists a constant $B = B(A) > 0$ such that

$$\sum_{d \leq D} \max_{(l,d) = 1} \max_{y \leq t} \left| \sum_{\substack{k \leq E(y) \atop (k,d) = 1}} g(x, k) H(y; k, d, l) \right| \ll \frac{x}{\log^A x}.$$
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where

$$H(y; k, d, l) = \sum_{k_p \leq y} 1 - \frac{1}{\phi(d)} \sum_{k_p \leq l(d)} 1,$$

$$\frac{1}{2} \leq E(x) \ll x^{1-\vartheta}, \quad 0 < \vartheta \leq 1,$$

g(x, k) \ll d_r(k), \quad D = x^{1/2} \log^{-6} x.

**Lemma 2.4 ([13]).** Let the condition of Lemma 2.3 be given and $r_1(y)$ be a positive function depending on $x$ and satisfying $r_1(y) \ll x^\vartheta$ for $y \leq x$. Then we have

$$\sum_{d \leq D} \max_{l(d) = 1} \max_{y \leq x} \sum_{k \leq E(x)} g(x, k)H(kr_1(y); k, d, l) \ll \frac{x}{\log^4 x}.$$

**Lemma 2.5 ([13]).** Let the condition of Lemma 2.3 be given and $r_2(y)$ be a positive function depending on $x$, $y$ and satisfying $kr_2(y) \ll x$ for $k \leq E(x)$, $y \leq x$. Then we have

$$\sum_{d \leq D} \max_{l(d) = 1} \max_{y \leq x} \sum_{k \leq E(x)} g(x, k)H(kr_2(y); k, d, l) \ll \frac{x}{\log^4 x}.$$

### 3. Proof of Theorem 1.1

As in [14] we begin with a periodic function $\chi(t)$ with period 1 such that

$$\chi(t) =\begin{cases} 
  (0, 1) & \text{if} -\Delta < t < \Delta, \\
  0 & \text{if} \Delta \leq t \leq 1 - \Delta.
\end{cases}$$

and which has a Fourier series

(4) \quad \chi(t) = \Delta + \sum_{|k|>0} g(k)e(kt)

with coefficients satisfying

$$g(0) = \Delta,$$

g(k) \ll \Delta, \quad \text{for all } k,

(5) \quad \sum_{|k|>H} |g(k)| \ll N^{-1},

where

(6) \quad \Delta = \Delta(N) = N^{-\vartheta} \quad \text{and} \quad H = \Delta^{-1} \log^2 N.
Next we will use sieve methods. As usual, for any sequence $\mathcal{E}$ of integers weighted by the numbers $f_n, n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{n \in \mathcal{E}, (n, P(z)) = 1} f_n,$$

and denote by $\mathcal{E}_d$ be the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod{d}$. We write

$$P(z) = \prod_{p < z} p,$$

and

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Let further

$$C_0 = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right),$$

and we will use the following form of the linear sieve due to Iwaniec [6].

**Lemma 3.1.** Let $2 \leq z \leq D^{1/2}$ and let $s = \log D/\log z$. If

(A1) $|\mathcal{E}_d| = (\omega(d)/d)X + r(\mathcal{E}, d), \mu(d) \neq 0$;

(A2) $\sum_{2 \leq p < s, \omega(p)/p = \log(\log z_2/\log z_1) + O(1/\log z_1), z_2 > z_1 \geq 2,$

where $\omega(d)$ is a multiplicative function, $0 \leq \omega(p) < p$, $X > 1$ is independent of $d$. Then

$$S(\mathcal{E}, z) \leq XV(z)(F(s) + o(1)) + \sum_{l < L} \sum_{d|P(z)} \lambda_1^+(d)r(\mathcal{E}, d),$$

$$S(\mathcal{E}, z) \geq XV(z)(f(s) - o(1)) - \sum_{l < L} \sum_{d|P(z)} \lambda_1^+(d)r(\mathcal{E}, d),$$

where $L = O(1), \lambda_1^\pm$ are well-factorable bounded functions of level $D$, $f(s), F(s)$ are determined by the following differential-difference equation

$$\begin{cases}
F(s) = \frac{2e^s}{s}, & f(s) = 0, \quad \text{if } 0 < s \leq 2, \\
(sF(s))' = f(s - 1), & (sf(s))' = F(s - 1), \quad \text{if } s \geq 2,
\end{cases}$$

where $\gamma$ denote the Euler’s constant.
So, if we define \( A \) to be the sequence of integers \( n \leq N \) weighted by

\[
a_n = \begin{cases} 
\chi(\alpha(n - 2) + \beta) & \text{if } n - 2 \in \mathbb{P}, \\
0 & \text{else.}
\end{cases}
\]

Then to prove Theorem 1.1, it suffice to show that

\[
S(A, N^{1/3}) = \sum_{p + 2 \leq N \atop (p + 2, p(\alpha + p^2)) = 1} \chi(\alpha p + \beta) > 0.
\]

However, we cannot quite do that, but need to use a more sophisticated weighted sieve method. Indeed following Cai (see [1], Lemma 5), let \( k = 1/12, l = 1/3.1 \), and we consider

\[
S \geq \sum_{n \in A \atop (n, N^{1/12}) = 1} a_n \left( 1 - \frac{1}{2} \sum_{N^{1/12} \leq p < N^{1/3.1}} 1 \right)
\]

Here we notice that the weight of \( n \) is \( a_n \) if and only if \( n \) has no prime factors \( < N^{1/3.1} \) in which case clearly \( n = P_2 \). If the weight of \( n \) is \( a_n/2 \), then \( a_n \) has one prime factor in the interval \([N^{1/12}, N^{1/3.1}]\) and the third, fourth sum is 0. But this again implies that \( n = P_2 \). Thus the weight of \( n \) is positive only if

\[ n = P_2, \; n - 2 \in \mathbb{P} \quad \text{and} \quad \|\alpha(n - 2) + \beta\| < N^{-\theta}, \]

and so it is enough to show that \( S > 0 \).

Using the sieve notation, we can write

\[
S \geq S(A, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \leq p < N^{1/3.1}} S(A_p, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \leq p_1, p_2 < N^{1/3.1}} S(A_{p_1, p_2}, N^{1/12}) - \sum_{N^{1/3.1} \leq p_1 < p_2 < (N/p_1)^{1/2}} S(A_{p_1, p_2}, p_2) + O(N^{11/12})
\]

\[
= S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 + O(N^{11/12}).
\]
Consider a square-free number $d$. If $2 \mid d$, then we write $|\mathcal{A}_d| = |r(\mathcal{A}, d)| \leq 1$. Otherwise we have by the Fourier expansion of $\chi(n)$

$$
\begin{align*}
|\mathcal{A}_d| &= \sum_{md \leq N \atop md - 2 \in \mathcal{P}} \chi(\alpha(md - 2) + \beta) \\
&= \sum_{p \leq N - 2 \atop p = 2(d)} \chi(\alpha p + \beta) \\
&= \sum_{p \leq N \atop p = 2(d)} \left( \Delta + \Delta \sum_{0 < |k| < H} c_k e(\alpha kp) + O(N^{-1}) \right) \\
&= \Delta \left( \frac{\text{Li} N}{\phi(d)} + R_1(d) + R_2(d) + O\left( \frac{N}{d(\log N)^C} \right) \right),
\end{align*}
$$

where $c_k \ll 1$, and

$$
\begin{align*}
R_1(d) &= \sum_{p \leq N \atop p = 2(d)} 1 - \frac{\text{Li} N}{\phi(d)}, \\
R_2(d) &= \sum_{p \leq N \atop p = 2(d)} \sum_{0 < |k| < H} c_k e(\alpha kp).
\end{align*}
$$

Applying Bombieri–Vinogradov theorem (see [7], Theorem 17.1) implies that

$$
\sum_{d \leq N^{1/2}/\log^C N} |R_1(d)| \ll \frac{N}{\log^A N}.
$$

On the other hand, Lemma 2.1 implies that for a well-factorable function $\lambda$ of level $D < N^{1/2}/(H^2 \log^C N)$, we get

$$
\sum_{d \leq D} \lambda_d R_2(d) \ll \frac{N}{\log^A N},
$$

when $N = q^2$, where $a/q$ is a convergent to $\alpha$ with a large enough denominator.

Therefore we apply Lemma 3.1 with

$$
\omega(d) = \begin{cases} 
0 & \text{if } 2 \mid d, \\
\frac{d}{\phi(d)} & \text{otherwise},
\end{cases} \quad X = \Delta \text{Li} N, \quad \text{and} \quad D < \frac{N^{1/2}}{H^2 \log^C N},
$$
to $S_1$ and obtain

$S_1 \geq \Delta \text{Li} N V(N^{1/12}) f(6 - 24\theta)(1 + o(1))$

(9) $= \frac{8}{1 - 4\theta} \left( \log(5 - 24\theta) + \int_3^{5-24\theta} \frac{1}{t} \, dt \int_2^{t-1} \frac{\log(s - 1)}{s} \, ds \right) \frac{C_0\Delta N}{\log^2 N} (1 + o(1))$

$\geq 13.471 \frac{C_0\Delta N}{\log^2 N}$.

Since $(A_\rho)_d = A_{pd}$, we can use Lemma 3.1 also to $S_2$ by using the same method. In this case one faces the sum

$$\sum_{N^{1/12} \leq p < N^{1/3}} \sum_{d \leq D} \lambda_d R_2(pd),$$

by Remark 10 in [11], the above sum is at most

$$\ll \frac{N}{\log^4 N},$$

then

$$\omega(d) = \begin{cases} 
0 & \text{if } 2 \mid d, \\
\frac{d}{\phi(d)} & \text{otherwise,}
\end{cases} \quad X = \frac{\Delta \text{Li} N}{\phi(p)}, \quad \text{and} \quad D < \frac{N^{1/2}}{p H^2 \log^C N}.$$

And applying partial summation, prime number theory, we have

$$S_2 \leq \sum_{N^{1/12} \leq p < N^{1/3}} \frac{\Delta \text{Li} N}{\phi(p)} V(N^{1/12}) F \left( 6 - 24\theta - 12 \frac{\log p}{\log N} \right) (1 + o(1))$$

$$= 6 \int_{1/12}^{1/3.1} \frac{F(6 - 24\theta - 12t)}{t} \, dt \frac{C_0\Delta N}{\log^2 N} (1 + o(1))$$

$$= 8 \left( \int_{1/12}^{1/3.1} \frac{dt}{t(1 - 4\theta - 2t)} \left( 1 + \int_2^{5-24\theta-12t} \frac{\log(s - 1)}{s} \, ds \right) \right)$$

(10) $$+ \int_{1/12}^{(1-8\theta)/4} \int_2^{5-24\theta-12t} \frac{dt}{t(1 - 4\theta - 2t)} \left( \frac{C_0\Delta N}{\log^2 N} (1 + o(1)) \right)$$

$$= 8 \left( \int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1 - 4\theta - 2t)} \int_2^{5-24\theta-12t} \frac{\log(s - 1)}{s} \, ds \right)$$

$$+ \int_{1/12}^{1/3.1} \frac{dt}{t(1 - 4\theta - 2t)} \left( \frac{C_0\Delta N}{\log^2 N} (1 + o(1)) \right)$$

$$\leq 21.3643 \frac{C_0\Delta N}{\log^2 N}.$$
For the sum $S_3$, we write

$$S_3 = \sum_{N^{1/2} \leq p_1 < N^{1/3}, 1 \leq p_2 < N/N^{1/2}} \sum_{np_1 p_2 \leq N} \chi(\alpha(np_1 p_2 - 2) + \beta)$$

$$= \sum_{N^{1/2} \leq p_1 < N^{1/3}, 1 \leq p_2 < N/N^{1/2}} \sum_{1 \leq n \leq N/(p_1 p_2), (p_1 p_2) = 1} \chi(\alpha p + \beta)$$

$$\leq \sum_{N^{1/3} \leq p_2 < N^{11/24}} \sum_{1 \leq n \leq N^{11/24}/p_2} \sum_{p = np_1 p_2 - 2} 1.$$ 

Let’s consider the set

$$\mathcal{E} = \left\{ e \mid e = np_2, N^{1/3, 1} \leq p_2 < N^{11/24}, 1 \leq n \leq \frac{N^{11/12}}{p_2}, (n, P(p_2)) = 1 \right\}.$$

By the definition of the set $\mathcal{E}$, it is easy to see that for every $e \in \mathcal{E}$, $p_2$ is determined by $e$ uniquely. Let $p_2 = r(e)$, then we have

$$N^{1/3, 1} \leq r(e) < \frac{N^{11/24}}{p_2} \quad \text{and} \quad er(e) < N.$$ 

Let

$$\mathcal{L} = \left\{ l \mid l = ep_1 - 2, e \in \mathcal{E}, N^{1/12} \leq p_1 < \min\left(N^{1/3, 1}, \frac{N}{np_2}\right) \right\}.$$ 

Then

$$N^{1/3, 1} < e < N^{11/12} \quad \text{for} \quad e \in \mathcal{E}$$

and

$$|\mathcal{E}| \leq N^{11/12}, \quad \sum_{l \in \mathcal{L}, l \leq N^{1/3, 1}} 1 \ll N^{11/12},$$

and also we have

(11) \quad $S_3 \leq S(\mathcal{L}, z) + O(N^{11/12})$ \quad for \quad $z \leq N^{1/3}$.

We write

$$z^2 = D = N^{1/2} \log^B N,$$

then

(12) \quad $S(\mathcal{L}, z) \leq \frac{C_0 |L|}{\log N} + R_3 + R_4,$
where

\[ R_3 = \sum_{d \leq D} \sum_{(d,N) = 1} \left( \sum_{e \in \mathcal{E}} \frac{1}{\phi(d)} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/e)} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/e)} 1 \right), \]

\[ R_4 = \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1. \]

Let

\[ Q(k) = \sum_{e=k, e \in \mathcal{E}} 1, \]

then

\[ R_3 = \sum_{d \leq D} \sum_{N^{1/12} \leq k < N^{1/12}} Q(k) \left( \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1 \right), \]

\[ R_4 = \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{N^{1/12} \leq p_1 < \min(N^{1/3}, N/k)} 1. \]

It is easy to show

\[ Q(k) \leq 1. \]

Then we have

\[ R_4 \ll \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{N^{1/3}, k < N^{1/12}} \frac{N}{k} \]

\[ \ll N \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{h|d, h \geq N^{1/3}, k < N^{1/12}} \frac{1}{h} \]

\[ \ll N \log N \sum_{d \leq D} \frac{1}{\phi(d)} \sum_{h|d, h \geq N^{1/3}} \frac{1}{h} \]

\[ \ll N \log N \sum_{N^{1/3} \leq h \leq D} \frac{1}{h \phi(h)} \sum_{d \leq D/h} \frac{1}{\phi(d)} \]

\[ \ll N^{2.1/3.1} \log^2 N, \]

and

\[ R_3 \leq R_5 + R_6 + R_7, \]
where

\[ R_5 = \sum_{d \leq D, (d, N) = 1} \left| \sum_{N^{1/3} < k < N^{2/3}} Q(k) \left( \sum_{p_1 < N^{1/3}} \frac{1}{\phi(d)} \sum_{p_1 < N^{1/3}} 1 \right) \right|, \]

\[ R_6 = \sum_{d \leq D, (d, N) = 1} \left| \sum_{N^{1/3} < k < N^{1/12}} Q(k) \left( \sum_{k_p < N} \frac{1}{\phi(d)} \sum_{k_p = 2(d)} 1 \right) \right|, \]

\[ R_7 = \sum_{d \leq D, (d, N) = 1} \left| \sum_{N^{1/3} < k < N^{1/12}} Q(k) \left( \sum_{p_1 < N^{1/12}} \frac{1}{\phi(d)} \sum_{p_1 = 2(d)} 1 \right) \right|. \]

Due to Lemma 2.3–2.5,

\[ (15) \quad R_j \ll \frac{N}{\log^4 N}, \quad j = 5, 6, 7. \]

By Lemma 2.2 and prime number theorem, we have

\[ |\mathcal{L}| = \sum_{e \in \mathcal{E}} \sum_{N^{1/12} \leq p_1 < N^{1/3}} 1 = \sum_{N^{1/12} \leq p_1 < N^{1/3}} \sum_{\substack{p_2 < (N/p_1)^{3/2} \leq \pi(N/p_1) - 1 \atop (n, p_1 p_2) = 1}} 1 + O(N^{11/12}), \]

\[ < (1 + o(1)) \sum_{N^{1/12} \leq p_1 < N^{1/3}} \sum_{p_2 < (N/p_1)^{1/2}} w(N/(p_1 p_2)) \left( \frac{\log(N/(p_1 p_2))}{\log p_2} \right) \frac{N}{p_1 p_2 \log p_2} + O(N^{11/12}), \]

\[ \leq \left( \int_{1/12}^{1/3} \frac{dt}{t} \int_{1/3}^{(1-t)/2} \frac{ds}{s(1 - t - s)} \right) N \frac{\log^2 N}{\log N}. \]

By (11)–(16), we obtain

\[ S_3 \leq 8 \left( \int_{1/12}^{1/3} \frac{dt}{t} \int_{1/3}^{(1-t)/2} \frac{ds}{s(1 - t - s)} \right) \frac{C_0 \Delta N}{\log^2 N} (1 + o(1)) \]

\[ \leq 5.52946 \frac{C_0 \Delta N}{\log^2 N}. \]
We also use the same idea to $S_4$,

\[
S_4 \leq 8 \left( \int_{1/3.1}^{1/3} \frac{dt}{t} \int_{s}^{(1-t)/2} \frac{ds}{s(1-t-s)} \right) \frac{C_0 N \log N}{N} (1 + o(1))
\]

(18)

\[
\leq 0.018745 \frac{C_0 \Delta N}{\log^2 N}.
\]

Combining (7)–(10), (17) and (18), then we obtain

\[
S > S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 \gg \frac{\Delta N}{\log^2 N},
\]

which concludes the proof of Theorem 1.1.

ACKNOWLEDGMENT. The author thanks the referee for his/her helpful comments. This work is supported by The National Science Foundation of China (grant no. 11071186) and by Natural Science Foundation of Anhui province (Grant No. 1208085QA01).

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