Shi, S.-Y.
Osaka J. Math.
49 (2012), 993–1004

ON THE DISTRIBUTION OF αp MODULO ONE FOR PRIMES p OF A SPECIAL FORM

SAN-YING SHI

(Received February 25, 2011)

Abstract

In this paper it is proved that for any irrational α and some $0 < \theta \leq 1.5/100$, there are infinitely many primes p such that $p + 2$ has at most two prime factors and $\|\alpha p + \beta\| < p^{-\theta}$ which improves K. Matomäki's result $\theta < 1/1000$.

1. Introduction

Let α be a irrational real number and $\|x\|$ denote the distance from x to nearest integers. Earlier work about the distribution of the fractional parts of the sequence αp was considered by I.M. Vinogradov [16] who showed that for any real number β, there are infinitely many primes p such that if $\theta = 1/5 - \varepsilon$, then

$$\|\alpha p + \beta\| < p^{-\theta}, \tag{1}$$

where and below $\varepsilon > 0$ is arbitrarily small. Later the exponent θ was improved by several authors (Harman [3, 4], Jia [8, 9], Heath-Brown and Jia [5]). So far the best result is given by Matomäki with $\theta = 1/3 - \varepsilon$.

Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. Up to now this conjecture is still open, but many approximation to it established. One of the most interesting of them is due to J.R. Chen [2], who showed in 1973 that there exist infinitely many primes p such that $p + 2 = P_2$.

In [14] Todorova and Tolev considered the distribution of αp modulo one with primes of the form specified above, and showed that for $\theta = 1/100$, there are infinitely many solutions in primes p to (1) such that $p + 2 = P_4$. Later Matomäki [11] has shown that this actually holds with $p + 2 = P_2$ and $\theta = 1/1000$.

In this paper, our purpose is to improve the range θ and we shall prove the following result.

2010 Mathematics Subject Classification. 11K60, 11N36.

Theorem 1.1. *Let $\alpha \in \mathcal{R} \setminus \mathcal{Q}$, $\beta \in \mathcal{R}$ and $0 < \theta \le 1.5/100$. Then there are infinitely many primes p satisfying $p + 2 = P_2$ and such that*

$$\|\alpha p + \beta\| < p^{-\theta}. \tag{2}$$

NOTATION. Let α be a real number with a rational approximation a/q satisfying

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad \text{where} \quad (a, q) = 1, \quad \text{and} \quad q \ge 1.$$

Here $K \ge 1$, $k \sim H$ means $H < k \le 2H$ and $0 < \theta \le 1.5/100$. As usual let $\Lambda(n)$ and $\phi(n)$ respectively denote Von Mangoldt's function and Euler's function. For simplicity instead of $m \equiv n \pmod{k}$, $e^{2\pi i x}$ we write $m \equiv n(k)$, $e(x)$ respectively. Letter C is a positive constant, which is not necessarily the same at each occurrence.

2. Some lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([11, Theorem 1]). *For any well-factorable function λ of level D, we have*

$$\begin{aligned} &\sum_{\substack{d \sim D \\ (d,c)=1}} \lambda_d \sum_{k \sim H} c_k \sum_{\substack{n \sim x \\ n \equiv c(d)}} \Lambda(n) e(\alpha n k) \\ &\ll H(\log x)^C x^{3/4+\varepsilon} \left(\frac{x}{q} + \frac{q}{H} + D^2 + x^{7/9+4\varepsilon} + \min\left\{ D^{4+20\varepsilon}, \frac{x}{D} \right\} \right)^{1/4-\varepsilon}. \end{aligned} \tag{3}$$

Lemma 2.2 ([10, 13]). *Let $x > 1$, $z = x^{1/u}$. Then for $u \ge 1$, we have*

$$\sum_{\substack{n \le x \\ (n, P(z))=1}} 1 = w(u) \frac{x}{\log z} + O\left(\frac{x}{\log^2 z} \right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases} w(u) = \dfrac{1}{u}, & \text{if} \quad 1 < u \le 2, \\ (u w(u))' = w(u-1), & \text{if} \quad u \ge 2. \end{cases}$$

Lemma 2.3 ([13]). *For any given constant $A > 10$, there exists a constant $B = B(A) > 0$ such that*

$$\sum_{d \le D} \max_{(l,d)=1} \max_{y \le x} \left| \sum_{\substack{k \le E(x) \\ (k,d)=1}} g(x, k) H(y; k, d, l) \right| \ll \frac{x}{\log^A x},$$

where

$$H(y;k,d,l) = \sum_{\substack{kp\le y\\ kp\equiv l(d)}} 1 - \frac{1}{\phi(d)}\sum_{kp\le y} 1,$$

$$\frac{1}{2} \le E(x) \ll x^{1-\vartheta}, \quad 0 < \vartheta \le 1,$$

$$g(x,k) \ll d_r(k), \quad D = x^{1/2}\log^{-B} x.$$

Lemma 2.4 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_1(y)$ *be a positive function depending on* x *and satisfying* $r_1(y) \ll x^{\vartheta}$ *for* $y \le x$. *Then we have*

$$\sum_{d\le D}\max_{(l,d)=1}\max_{y\le x}\left|\sum_{\substack{k\le E(x)\\(k,d)=1}} g(x,k)H(kr_1(y);k,d,l)\right| \ll \frac{x}{\log^A x}.$$

Lemma 2.5 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_2(y)$ *be a positive function depending on* x, y *and satisfying* $kr_2(y) \ll x$ *for* $k \le E(x)$, $y \le x$. *Then we have*

$$\sum_{d\le D}\max_{(l,d)=1}\max_{y\le x}\left|\sum_{\substack{k\le E(x)\\(k,d)=1}} g(x,k)H(kr_2(y);k,d,l)\right| \ll \frac{x}{\log^A x}.$$

3. Proof of Theorem 1.1

As in [14] we begin with a periodic function $\chi(t)$ with period 1 such that

$$\chi(t)\begin{cases}\in (0,1) & \text{if} \quad -\Delta < t < \Delta,\\ = 0 & \text{if} \quad \Delta \le t \le 1-\Delta,\end{cases}$$

and which has a Fourier series

$$\chi(t) = \Delta + \sum_{|k|>0} g(k)e(kt) \tag{4}$$

with coefficients satisfying

$$\begin{aligned} &g(0) = \Delta,\\ &g(k) \ll \Delta, \quad \text{for all} \quad k,\\ &\sum_{|k|>H} |g(k)| \ll N^{-1}, \end{aligned} \tag{5}$$

where

$$\Delta = \Delta(N) = N^{-\theta} \quad \text{and} \quad H = \Delta^{-1}\log^2 N. \tag{6}$$

Next we will use sieve methods. As usual, for any sequence $\mathcal{E}$ of integers weighted by the numbers f_n, $n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{\substack{n \in \mathcal{E} \\ (n, P(z))=1}} f_n,$$

and denote by $\mathcal{E}_d$ be the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod{d}$. We write

$$P(z) = \prod_{p<z} p$$

and

$$V(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Let further

$$C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

and we will use the following form of the linear sieve due to Iwaniec [6].

Lemma 3.1. *Let $2 \le z \le D^{1/2}$ and let $s = \log D/\log z$. If*
(A_1) $|\mathcal{E}_d| = (\omega(d)/d)X + r(\mathcal{E}, d)$, $\mu(d) \neq 0$;
(A_2) $\sum_{z_1 \le p < z_2} \omega(p)/p = \log(\log z_2/\log z_1) + O(1/\log z_1)$, $z_2 > z_1 \ge 2$,
where $\omega(d)$ is a multiplicative function, $0 \le \omega(p) < p$, $X > 1$ is independent of d. Then

$$S(\mathcal{E}, z) \le XV(z)(F(s) + o(1)) + \sum_{l<L} \sum_{d|P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$
$$S(\mathcal{E}, z) \ge XV(z)(f(s) - o(1)) - \sum_{l<L} \sum_{d|P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$

where $L = O(1)$, $\lambda^{\pm}$ are well-factorable bounded functions of level D, $f(s)$, $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \dfrac{2e^{\gamma}}{s}, \quad f(s) = 0, & \text{if} \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & \text{if} \quad s \ge 2, \end{cases}$$

where γ denote the Euler's constant.

So, if we define $\mathcal{A}$ to be the sequence of integers $n \le N$ weighted by

$$a_n = \begin{cases} \chi(\alpha(n-2)+\beta) & \text{if} \quad n-2 \in \mathbb{P}, \\ 0 & \text{else.} \end{cases}$$

Then to prove Theorem 1.1, it suffice to show that

$$S(\mathcal{A}, N^{1/3}) = \sum_{\substack{p+2 \le N \\ (p+2, P(N^{1/3}))=1}} \chi(\alpha p + \beta) > 0. \tag{7}$$

However, we cannot quite do that, but need to use a more sophisticated weighted sieve method. Indeed following Cai (see [1], Lemma 5), let $k = 1/12$, $l = 1/3.1$, and we consider

$$S \ge \sum_{\substack{n \in \mathcal{A} \\ (n, N^{1/12})=1}} a_n \left(1 - \frac{1}{2} \sum_{\substack{N^{1/12} \le p < N^{1/3.1} \\ p \mid n}} 1 - \frac{1}{2} \sum_{\substack{n = p_1 p_2 p_3 \\ N^{1/12} \le p_1 < N^{1/3.1} \\ N^{1/3.1 \le p_2 < (N/p_1)^{1/2}}}} 1 - \sum_{\substack{n = p_1 p_2 p_3 \\ N^{1/3.1} \le p_1 < p_2 < (N/p_1)^{1/2}}} 1 \right) + O(N^{11/12}).$$

Here we notice that the weight of n is a_n if and only if n has no prime factors $< N^{1/3.1}$ in which case clearly $n = P_2$. If the weight of n is $a_n/2$, then a_n has one prime factor in the interval $[N^{1/12}, N^{1/3.1})$ and the third, fourth sum is 0. But this again implies that $n = P_2$. Thus the weight of n is positive only if

$$n = P_2, \quad n-2 \in \mathbb{P} \quad \text{and} \quad \|\alpha(n-2)+\beta\| < N^{-\theta},$$

and so it is enough to show that $S > 0$.

Using the sieve notation, we can write

$$\begin{aligned} S \ge S(\mathcal{A}, N^{1/12}) &- \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/3.1}} S(\mathcal{A}_p, N^{1/12}) - \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < N^{1/3.1} \\ N^{1/3.1} \le p_2 < (N/p_1)^{1/2}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &- \sum_{N^{1/3.1} \le p_1 < p_2 < (N/p_1)^{1/2}} S(\mathcal{A}_{p_1 p_2}, p_2) + O(N^{11/12}) \\ &=: S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 + O(N^{11/12}). \end{aligned} \tag{8}$$

Consider a square-free number d. If $2 \mid d$, then we write $|\mathcal{A}_d| = |r(\mathcal{A}, d)| \leq 1$. Otherwise we have by the Fourier expansion of $\chi(n)$

$$\begin{aligned}
|\mathcal{A}_d| &= \sum_{\substack{md \leq N \\ md-2 \in \mathbb{P}}} \chi(\alpha(md-2)+\beta) \\
&= \sum_{\substack{p \leq N-2 \\ p \equiv -2(d)}} \chi(\alpha p + \beta) \\
&= \sum_{\substack{p \leq N \\ p \equiv -2(d)}} \left(\Delta + \Delta \sum_{0<|k|<H} c_k e(\alpha k p) + O(N^{-1}) \right) \\
&= \Delta \left(\frac{\operatorname{Li} N}{\phi(d)} + R_1(d) + R_2(d) + O\left(\frac{N}{d(\log N)^C} \right) \right),
\end{aligned}$$

where $c_k \ll 1$, and

$$\begin{aligned}
R_1(d) &= \sum_{\substack{p \leq N \\ p \equiv -2(d)}} 1 - \frac{\operatorname{Li} N}{\phi(d)}, \\
R_2(d) &= \sum_{\substack{p \leq N \\ p \equiv -2(d)}} \sum_{0<|k|<H} c_k e(\alpha k p).
\end{aligned}$$

Applying Bombieri–Vinogradov theorem (see [7], Theorem 17.1) implies that

$$\sum_{d \leq N^{1/2}/\log^C N} |R_1(d)| \ll \frac{N}{\log^A N}.$$

On the other hand, Lemma 2.1 implies that for a well-factorable function λ of level $D < N^{1/2}/(H^2 \log^C N)$, we get

$$\sum_{d \leq D} \lambda_d R_2(d) \ll \frac{N}{\log^A N},$$

when $N = q^2$, where a/q is a convergent to α with a large enough denominator.

Therefore we apply Lemma 3.1 with

$$\omega(d) = \begin{cases} 0 & \text{if} \quad 2 \mid d, \\ \dfrac{d}{\phi(d)} & \text{otherwise,} \end{cases} \qquad X = \Delta \operatorname{Li} N, \quad \text{and} \quad D < \frac{N^{1/2}}{H^2 \log^C N},$$

to S_1 and obtain

$$
\begin{aligned}
S_1 &\geq \Delta \operatorname{Li} N V(N^{1/12}) f(6-24\theta)(1+o(1)) \\
&= \frac{8}{1-4\theta}\left(\log(5-24\theta) + \int_3^{5-24\theta} \frac{1}{t}\,dt \int_2^{t-1} \frac{\log(s-1)}{s}\,ds\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (9)\\
&\geq 13.471 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned}
$$

Since $(\mathcal{A}_p)_d = \mathcal{A}_{pd}$, we can use Lemma 3.1 also to S_2 by using the same method. In this case one faces the sum

$$\sum_{N^{1/12} \leq p < N^{1/3.1}} \sum_{d \leq D} \lambda_d R_2(pd),$$

by Remark 10 in [11], the above sum is at most

$$\ll \frac{N}{\log^A N},$$

then

$$\omega(d) = \begin{cases} 0 & \text{if} \quad 2 \mid d, \\ \dfrac{d}{\phi(d)} & \text{otherwise,} \end{cases} \qquad X = \frac{\Delta \operatorname{Li} N}{\phi(p)}, \quad \text{and} \quad D < \frac{N^{1/2}}{pH^2 \log^C N}.$$

And applying partial summation, prime number theory, we have

$$
\begin{aligned}
S_2 &\leq \sum_{N^{1/12} \leq p < N^{1/3.1}} \frac{\Delta \operatorname{Li} N}{\phi(p)} V(N^{1/12}) F\left(6-24\theta-12\frac{\log p}{\log N}\right)(1+o(1)) \\
&= 6\int_{1/12}^{1/3.1} \frac{F(6-24\theta-12t)}{t}\,dt\, \frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\
&= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)}\left(1+\int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right)\right. \\
&\qquad \left. + \int_{(1-8\theta)/4}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (10)\\
&= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)} \int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right. \\
&\qquad \left. + \int_{1/12}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\
&\leq 21.3643 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned}
$$

For the sum S_3, we write

$$\begin{aligned} S_3 &= \sum_{N^{1/12} \le p_1 < N^{1/3.1} \le p_2 < (N/p_1)^{1/2}} \sum_{\substack{np_1p_2 \le N \\ np_1p_2 - 2 \in \mathbb{P},\ (n, P(p_2))=1}} \chi(\alpha(np_1p_2 - 2) + \beta) \\ &= \sum_{N^{1/12} \le p_1 < N^{1/3.1} \le p_2 < (N/p_1)^{1/2}} \sum_{\substack{p = np_1p_2 - 2 \\ 1 \le n \le N/(p_1p_2),\ (n, P(p_2))=1}} \chi(\alpha p + \beta) \\ &\le \sum_{N^{1/3.1} \le p_2 < N^{11/24}} \sum_{\substack{1 \le n \le N^{11/12}/p_2 \\ (n, P(p_2))=1}} \sum_{\substack{p = np_1p_2 - 2 \\ N^{1/12} \le p_1 < \min(N^{1/3.1}, N/(np_2))}} 1. \end{aligned}$$

Let's consider the set

$$\mathcal{E} = \left\{ e \;\middle|\; e = np_2,\ N^{1/3.1} \le p_2 < N^{11/24},\ 1 \le n \le \frac{N^{11/12}}{p_2},\ (n, P(p_2)) = 1 \right\}.$$

By the definition of the set $\mathcal{E}$, it is easy to see that for every $e \in \mathcal{E}$, p_2 is determined by e uniquely. Let $p_2 = r(e)$, then we have

$$N^{1/3.1} \le r(e) < N^{11/24} \quad \text{and} \quad er(e) < N.$$

Let

$$\mathcal{L} = \left\{ l \;\middle|\; l = ep_1 - 2,\ e \in \mathcal{E},\ N^{1/12} \le p_1 < \min\left(N^{1/3.1}, \frac{N}{np_2}\right) \right\}.$$

Then

$$N^{1/3.1} < e < N^{11/12} \quad \text{for} \quad e \in \mathcal{E}$$

and

$$|\mathcal{E}| \le N^{11/12}, \qquad \sum_{l \in \mathcal{L},\ l \le N^{1/3.1}} 1 \ll N^{11/12},$$

and also we have

$$S_3 \le S(\mathcal{L}, z) + O(N^{11/12}) \quad \text{for} \quad z \le N^{1/3}. \tag{11}$$

We write

$$z^2 = D = N^{1/2} \log^{-B} N,$$

then

$$S(\mathcal{L}, z) \le 8\frac{C_0|\mathcal{L}|}{\log N} + R_3 + R_4, \tag{12}$$

where

$$R_3 = \sum_{\substack{d\le D\\(d,N)=1}} \left| \sum_{\substack{e\in\mathcal{E}\\(e,d)=1}} \left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)\\ ep_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)} 1 \right) \right|,$$

$$R_4 = \sum_{d\le D,\,(d,N)=1} \frac{1}{\phi(d)} \sum_{\substack{e\in\mathcal{E}\\(e,d)>1}} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},\,N/e)} 1.$$

Let

$$Q(k) = \sum_{e=k,\,e\in\mathcal{E}} 1,$$

then

$$R_3 = \sum_{d\le D} \left| \sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k) \left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)\\ kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)} 1 \right) \right|,$$

$$R_4 = \sum_{d\le D} \frac{1}{\phi(d)} \sum_{\substack{N^{1/3}<k<N^{11/12}\\(k,d)>N^{1/3.1}}} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)} 1.$$

It is easy to show

$$Q(k) \le 1.$$

Then we have

$$\begin{aligned} R_4 &\ll \sum_{d\le D} \frac{1}{\phi(d)} \sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)>N^{1/3.1}}} \frac{N}{k} \\ &\ll N \sum_{d\le D} \frac{1}{\phi(d)} \sum_{h|d,\,h\ge N^{1/3.1}} \sum_{\substack{k<N^{11/12}\\(k,d)=h}} \frac{1}{k} \\ &\ll N\log N \sum_{d\le D} \frac{1}{\phi(d)} \sum_{h|d,\,h\ge N^{1/3.1}} \frac{1}{h} \\ &\ll N\log N \sum_{N^{1/3.1}\le h\le D} \frac{1}{h\phi(h)} \sum_{d\le D/h} \frac{1}{\phi(d)} \\ &\ll N^{2.1/3.1}\log^2 N, \end{aligned} \tag{13}$$

and

$$R_3 \le R_5 + R_6 + R_7, \tag{14}$$

where

$$R_5 = \sum_{d\le D,\,(d,N)=1} \left| \sum_{\substack{N^{1/3.1}<k<N^{2.1/3.1}\\(k,d)=1}} Q(k)\left(\sum_{\substack{p_1<N^{1/3.1}\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{p_1<N^{1/3.1}} 1 \right)\right|,$$

$$R_6 = \sum_{d\le D,\,(d,N)=1} \left| \sum_{\substack{N^{2.1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k)\left(\sum_{\substack{kp_1<N\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{kp_1<N} 1 \right)\right|,$$

$$R_7 = \sum_{d\le D,\,(d,N)=1} \left| \sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k)\left(\sum_{\substack{p_1<N^{1/12}\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{p_1<N^{1/12}} 1 \right)\right|.$$

Due to Lemma 2.3–2.5,

$$R_j \ll \frac{N}{\log^4 N}, \quad j = 5, 6, 7. \tag{15}$$

By Lemma 2.2 and prime number theorem, we have

$$\begin{aligned}
|\mathcal{L}| &= \sum_{e\in\mathcal{E}} \sum_{N^{1/12}\le p_1<N^{1/3.1}} 1 \\
&= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{1\le n\le N/(p_1p_2)\\(n,P(p_2))=1}} 1 + O(N^{11/12}) \\
&< (1+o(1)) \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} w\left(\frac{\log(N/(p_1p_2))}{\log p_2}\right)\frac{N}{p_1p_2\log p_2} \\
&\quad + O(N^{11/12}) \\
&\le \left(\int_{1/12}^{1/3.1}\frac{dt}{t}\int_{1/3.1}^{(1-t)/2}\frac{ds}{s(1-t-s)}\right)\frac{N}{\log N}.
\end{aligned} \tag{16}$$

By (11)–(16), we obtain

$$\begin{aligned}
S_3 &\le 8\left(\int_{1/12}^{1/3.1}\frac{dt}{t}\int_{1/3.1}^{(1-t)/2}\frac{ds}{s(1-t-s)}\right)\frac{C_0\Delta N}{\log^2 N}(1+o(1)) \\
&\le 5.52946\frac{C_0\Delta N}{\log^2 N}.
\end{aligned}$$

We also use the same idea to S_4,

$$
\begin{aligned}
S_4 &\le 8\left(\int_{1/3.1}^{1/3} \frac{dt}{t} \int_t^{(1-t)/2} \frac{ds}{s(1-t-s)}\right) \frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\
&\le 0.018745 \frac{C_0 \Delta N}{\log^2 N}.
\end{aligned} \tag{18}
$$

Combining (7)–(10), (17) and (18), then we obtain

$$
S > S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 - S_4 \gg \frac{\Delta N}{\log^2 N},
$$

which concludes the proof of Theorem 1.1.

ACKNOWLEDGMENT. The author thanks the referee for his/her helpful comments. This work is supported by The National Science Foundation of China (grant no. 11071186) and by Natural Science Foundation of Anhui province (Grant No. 1208085QA01).

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Nanjing University
Nanjing 210093
P.R. China
and
Hefei University of Technology
Hefei 230009
P.R. China
e-mail: vera123_99@hotmail.com