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ON CHARTS WITH TWO CROSSINGS II

Dedicated to Professor Akio Kawauchi for his 60th birthday

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Abstract

Let Γ be a chart with at most two crossings. In this paper, we show that if Γ is a 2-minimal generalized n -chart with $n \geq 5$, then Γ contains at least $4n - 10$ black vertices. And we show that if the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart. Hence the closure is a ribbon surface.

1. Introduction

S. Kamada introduced *charts* which correspond to surface braids [4], [5]. Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices. Kamada also introduced *C-moves* which are local modifications of charts in a disk. A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

A surface in \mathbb{R}^4 is called a *ribbon surface* if it is the boundary of an immersed handlebody with singularities which are mutually disjoint disks such that the preimage of each disk is a union of a proper disk of the domain and a disk in the interior of the domain, a handlebody. In the words of charts, a ribbon surface is the closure of a surface braid which corresponds to a *ribbon chart* where a ribbon chart is a chart which is C-move equivalent to a chart without white vertices [4].

Kamada showed that any 3-chart is a ribbon chart [4]. Nagase and Hirota extended Kamada's result: Any 4-chart with at most one crossing is a ribbon chart [7]. We showed that any n -chart with at most one crossing is a ribbon chart [11].

For a set X in a space, let $Cl(X)$ be the closure of the set X .

Let Γ be a chart. Let e_1 and e_2 be edges of Γ which connect two white vertices w_1 and w_2 where possibly $w_1 = w_2$. Suppose that the union $e_1 \cup e_2$ bounds an open disk E . Then $Cl(E)$ is called a *bigon* provided that any edge containing w_1 or w_2 does

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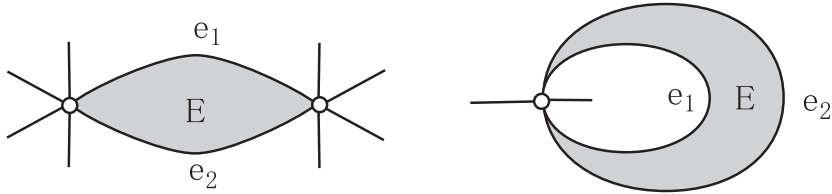


Fig. 1. The edges e_1 and e_2 do not contain crossings.

not intersect the open disk E (see Fig. 1). Since e_1 and e_2 are edges of Γ , they do not contain any crossings.

Let Γ be a chart. Let $w(\Gamma)$, $f(\Gamma)$ and $b(\Gamma)$ be the number of white vertices, the number of free edges and the number of bigons in Γ respectively. Let $C(\Gamma) = (w(\Gamma), -f(\Gamma), -b(\Gamma))$. The triplet $C(\Gamma)$ is called an *extended complexity* of the chart Γ (see [4] for complexities of charts).

For each non-negative integer k , let $c(\Gamma)$ be the number of crossings in a chart Γ and $C_k = \{\Gamma \mid c(\Gamma) \leq k\}$. A chart Γ in C_k is said to be *k-minimal* if its extended complexity is minimal among the charts in C_k which are C-move equivalent to the chart Γ with respect to the lexicographical order of the triad of the integers [11].

We showed that if a 2-minimal 4-chart contains exactly two crossings, then it contains at least eight black vertices [9]. It is well known that if the closure of the surface braid represented by a 4-chart is one sphere, then the chart contains exactly six black vertices. Hence we showed that any 4-chart with at most two crossings is a ribbon chart if the chart corresponds to a surface braid whose closure is one sphere [9]. We give another proof of this theorem [13].

Let Γ be a chart. For each label m , we denote by Γ_m the subgraph of Γ consisting of edges of label m and their vertices. In this paper,

crossings are vertices of Γ but we do not consider crossings as vertices of the subgraph Γ_m .

A chart Γ with a white vertex is called a *generalized n-chart* if there exist two non-negative integers $p < q$ with $n = q - p$ such that

- (i) Γ_i does not have a white vertex except for $p < i < q$, and
- (ii) the both Γ_{p+1} and Γ_{q-1} have white vertices.

In this paper the following are main results:

Theorem 1.1. *Let Γ be a 2-minimal generalized n -chart. If $n \geq 5$, then Γ contains at least $4n - 10$ black vertices.*

Theorem 1.2. *Let Γ be a chart with at most two crossings. If the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart. Hence the closure is a ribbon surface.*

The 2-twist spun trefoil is represented by a chart with six white vertices and three crossings. It is well known that the 2-knot is not a ribbon surface. By Theorem 1.2, the chart representing the 2-knot must possess at least three crossings.

On the other hand, Hasegawa showed that if a chart representing a 2-knot is minimal, then the chart must possess at least six white vertices [2], where a minimal chart means its complexity $(w(\Gamma), -f(\Gamma))$ is minimal among the charts C-move equivalent to the chart with respect to the lexicographic order of pairs of integers. We know that there does not exist a minimal chart with one, two nor three white vertices. We show that there does not exist a minimal chart with five white vertices [8]. We show that the minimal chart with four white vertices is a ribbon chart, or a disjoint union of free edges, hoops and a chart representing a “turned T^2 -link of Hopf link” [3] and [14].

Using the result in this paper, we get the following [15]: If Γ is a chart with at most three crossings and if the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart, or a disjoint union of free edges, hoops and a chart representing a 2-twist spun trefoil. The chart with six white vertices and three crossings representing a 2-twist spun trefoil is “primitive” k -minimal chart in some sense for $k \geq 3$. We study the properties of k -minimal charts and such primitive charts.

2. Preliminaries

Let n be a positive integer. An n -chart is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

- (i) Every vertex has degree 1, 4, or 6.
- (ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.
- (iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i , where the orientation and label of each arc are inherited from the edge containing the arc.
- (iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

A vertex of degree 1, 4, and 6 is called a *black vertex*, a *crossing*, and a *white vertex* respectively (see Fig. 2). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward or outward is called a *middle arc* at the white vertex (see Fig. 2 (c)). There are two middle arcs in a small neighborhood of each white vertex.

C-moves are local modifications of charts in a disk (see [1], [6] for the precise definition). Kamada originally defined CI-moves as follows (A C-I-M2 move and a C-I-R2 move as shown in Fig. 3 are special cases of CI-moves): A chart Γ is obtained from a chart Γ' by a *CI-move*, if there exists a disk D such that

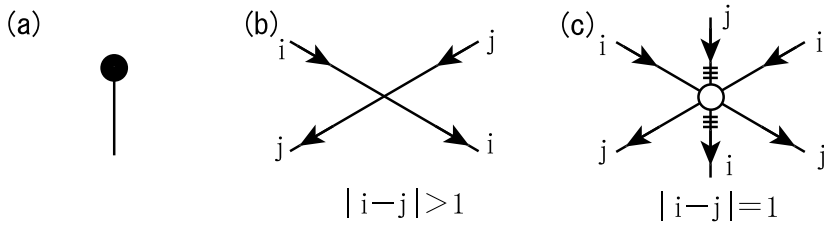


Fig. 2. (a) a black vertex, (b) a crossing, (c) a white vertex. Each arc with three transversal short arcs is a middle arc.

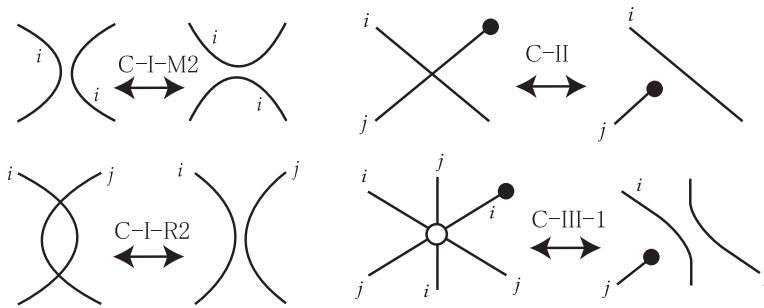


Fig. 3. For the C-III-1 move, the edge containing the black vertex does not contain a middle arc in the left figure.

- (i) the two charts Γ and Γ' intersect the boundary of D transversely or do not intersect the boundary of D ,
 - (ii) $\Gamma \cap D^c = \Gamma' \cap D^c$, and
 - (iii) neither of $\Gamma \cap D$ nor $\Gamma' \cap D$ contains a black vertex,
- where $(\dots)^c$ is the complement of (\dots) .

Let Γ be a chart. An *edge* of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ . On the other hand, an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m . A closed edge of Γ_m is called a *ring* if it contains a crossing but does not contain a white vertex nor a black vertex. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). An edge of Γ or Γ_m is called a *free edge* if it has two black vertices. An edge of Γ or Γ_m is called a *terminal edge* if it has a white vertex and a black vertex. Note that free edges and terminal edges may contain crossings of Γ .

To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk. In this paper,

all charts are contained in the 2-sphere S^2 .

We have the special point in the 2-sphere S^2 , called *the point at infinity*, denoted by ∞ . In this paper, all charts are contained in a disk which does not contain the point at infinity ∞ .

A hoop is said to be *simple* if one of the complementary domain of the hoop does not contain any white vertices.

We can assume that any k -minimal charts Γ satisfy the following five assumptions (cf. [10] and [11]):

ASSUMPTION 1. Any terminal edge of Γ_m does not contain a crossing. Hence any terminal edge of Γ_m is a terminal edge of Γ and any terminal edge of Γ_m contains a middle arc.

ASSUMPTION 2. Any free edge of Γ_m does not contain a crossing. Hence any free edge of Γ_m is a free edge of Γ .

ASSUMPTION 3. All free edges and simple hoops in Γ are moved into a small neighborhood U_∞ of the point at infinity ∞ .

ASSUMPTION 4. Each complementary domain of any ring must contain at least one white vertex.

ASSUMPTION 5. Hence we can assume that the subgraph obtained from Γ by omitting free edges and simple hoops does not meet the set U_∞ . And also we can assume that Γ does not contain free edges nor simple hoops, otherwise mentioned. Therefore we can assume that if an edge of Γ_m contains a black vertex, then it is a terminal edge and that each complementary domain of any hoops and rings of Γ contains a white vertex, otherwise mentioned.

Furthermore as shown in [10], we can also assume the following assumption:

ASSUMPTION 6. The point at infinity ∞ is moved in any complementary domain of Γ .

For a set X in a space, let $Int(X)$, $\partial(X)$ be the interior, the boundary of the set X respectively.

3. Tangles

For each graph G in S^2 , let (see Fig. 4)

$M(G)$ = the maximal subgraph of G without vertices of degree 1,

$Out(G)$ = the complementary domain of $M(G)$ containing the point at infinity ∞ ,

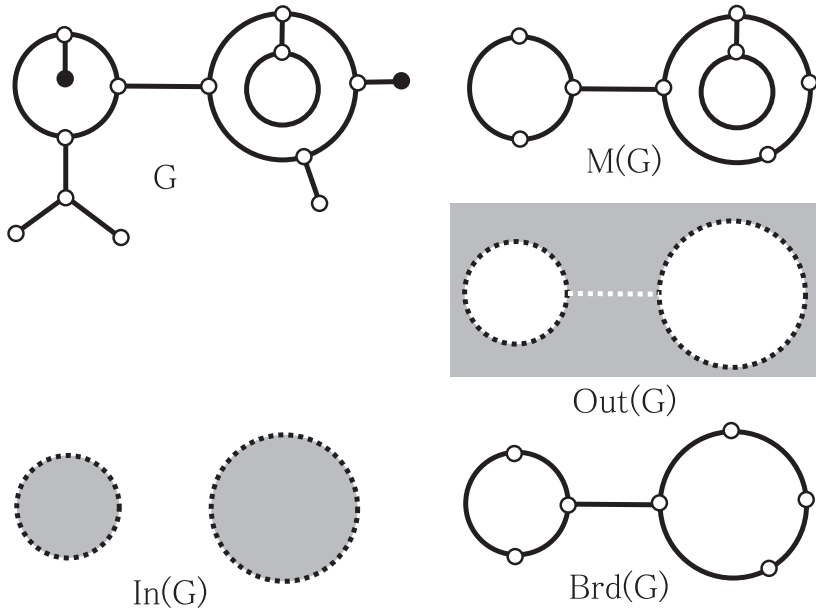


Fig. 4. $Out(G)$ and $In(G)$ are shaded areas.

$$In(G) = (Cl(Out(G)))^c, \text{ and}$$

$$Brd(G) = M(G) \cap Cl(Out(G)).$$

Lemma 3.1 ([11, Lemma 5.1]). *Let G be a connected graph in S^2 . Let D be a disk containing G . Then the following hold:*

- (1) $Out(G)$ is an open disk.
- (2) Each connected component of $In(G)$ is an open disk whose closure is a disk.
- (3) A regular neighbourhood of $In(G) \cup G$ in S^2 is a disk, and so is a regular neighbourhood of $In(G) \cup G$ in D .

Let Γ be a chart. For a subset X in Γ , let

$$w(X) = \text{the number of white vertices in } X.$$

Let Γ be a chart and D a disk. The pair $(D \cap \Gamma, D)$ is called a *tangle* if it satisfies the following two conditions:

- (1) ∂D does not contain any white vertices, black vertices nor crossings of the chart Γ , and
- (2) ∂D transversely intersects edges of Γ .

Let Γ be a chart, $(D \cap \Gamma, D)$ a tangle and $G_i = D \cap \Gamma_i$ ($i = 1, 2, \dots$). The tangle $(D \cap \Gamma, D)$ is called a *T-tangle* of label n (tangle with at most three labels) if

it satisfies the following two conditions:

- (i) $G_i = \emptyset$ except for $n - 1 \leq i \leq n + 1$.
- (ii) $w(D \cap \Gamma) \geq 1$ but D does not contain any crossing.

If $In(G_n) = \emptyset$ then we say that the T -tangle is *linear*. If $Cl(In(G_n))$ is a disk then we say that the T -tangle is *cellular*.

Let $(D \cap \Gamma, D)$ be a T -tangle of label n . If an edge e of Γ_n intersects ∂D , then $e \cap D$ is called an *exceptional arc* of the T -tangle.

Lemma 3.2 ([12, Lemma 4.2]). *Any linear T -tangle in a k -minimal chart possesses at least two exceptional arcs.*

Lemma 3.3. *Let $(D \cap \Gamma, D)$ be a linear T -tangle of label n with exactly two exceptional arcs in a k -minimal chart Γ . Then we have*

- (1) *each white vertex in D is contained in a terminal edge of label n , and*
- (2) *there exists a unique arc in $D \cap \Gamma_n$ connecting the two points $\partial D \cap \Gamma_n$ such that all the white vertices in the arc are contained in terminal edges.*

Proof. For (1). Let G be a connected component of $D \cap \Gamma_n$. Since the T -tangle is linear, G is a tree. Then $\partial D \cap G$ consists of two points by Lemma 3.2. Now consider the two points $\partial D \cap \Gamma_n$ as vertices of G . Let B be the number of terminal edges in G which is equal to the number of black vertices in G , W the number of white vertices in G , and E the number of edges in G . Since each white vertex in G is of degree 3, we have $3W + (B + 2) = 2E$. Since G is a tree, we have the Euler characteristic $(W + B + 2) - E = 1$. Thus $3W + B + 2 = 2(W + B + 1)$. Namely $W = B$. Since the chart is k -minimal, each white vertex in G is contained in at most one terminal edges of label n by Assumption 1. Hence the equality $W = B$ implies that each white vertex in G is contained in a terminal edge of label n .

For (2). By taking all terminal edges off from G , we get a unique simple arc. □

4. Tiny cellular T -tangles

Lemma 4.1. *Let $(D \cap \Gamma, D)$ be a T -tangle of label n in a k -minimal chart Γ . Let G be the closure of a connected component of $(D \cap \Gamma_n) - Cl(In(D \cap \Gamma_n))$. If G is not a terminal edge, then it is a tree containing at least two points in $Brd(D \cap \Gamma_n) \cup \partial D$.*

Proof. If G is an arc, then G is either a terminal edge or an arc containing two points in $Brd(D \cap \Gamma_n) \cup \partial D$. Hence we can assume that G is a tree containing a white vertex.

Suppose that G contains at most one point in $Brd(D \cap \Gamma_n) \cup \partial D$. Let D' be a regular neighborhood of $Cl(In(D \cap \Gamma_n))$ in D , $G' = G \cap Cl(D - D')$, and N a regular neighborhood of G' in $Cl(D - D')$. Then $N \cap \Gamma_n = G'$ and $\partial N \cap \Gamma_n$ contains at most one point. Since G contains a white vertex, $w(N \cap \Gamma) \geq 1$. Since G' is a tree, N is a disk.

Since $(D \cap \Gamma, D)$ is a T -tangle of label n , $(N \cap \Gamma, N)$ is a T -tangle of label n with at most one exceptional arc. Since G is a tree, $(N \cap \Gamma, N)$ is linear. This contradicts Lemma 3.2. Hence G contains at least two points in $\text{Brd}(D \cap \Gamma_n) \cup \partial D$. \square

A tangle $(D_1 \cap \Gamma, D_1)$ contains a tangle $(D_2 \cap \Gamma, D_2)$ provided that $D_1 \supset D_2$.

Let Γ be a chart, and $(D \cap \Gamma, D)$ a cellular T -tangle of label n . The tangle $(D \cap \Gamma, D)$ is tiny provided that the closure of each component of $(D - \text{Cl}(\text{In}(D \cap \Gamma_n))) \cap \Gamma$ is

- (i) an arc connecting a point in ∂D and a point in $\text{Brd}(D \cap \Gamma_n)$, or
- (ii) a terminal edge of label n .

NOTE. For any cellular T -tangle of label n , let X be the union of all the terminal edges of label n in D each of which intersects $\text{Cl}(\text{In}(D \cap \Gamma_n))$, and N a regular neighborhood of $\text{Cl}(\text{In}(D \cap \Gamma_n)) \cup X$ in D . Then $(N \cap \Gamma, N)$ is a tiny cellular T -tangle of label n .

Lemma 4.2. *Let $(D \cap \Gamma, D)$ be a non-linear T -tangle of label n with p exceptional arcs in a k -minimal chart Γ . Then $(D \cap \Gamma, D)$ contains a tiny cellular T -tangle with at most p exceptional arcs.*

Proof. Since $(D \cap \Gamma, D)$ is not linear, $\text{In}(D \cap \Gamma_n) \neq \emptyset$. Let Z be a connected component of $D \cap \Gamma_n$ such that $\text{In}(Z)$ contains a connected component of $\text{In}(D \cap \Gamma_n)$. Then $Z \cap \partial D$ consists of at most p points.

Let $D^* = \text{Cl}(\text{In}(Z))$ and Y the union of the closures of connected components of $Z - \text{Cl}(\text{In}(D \cap \Gamma_n))$ each of which is not a terminal edge (see Fig. 5). By Lemma 3.1 (2), $D^* = \text{Cl}(\text{In}(Z))$ consists of disjoint disks. And Y consists of disjoint trees.

Suppose $Y = \emptyset$. Then the closure of a connected component of $Z - \text{Cl}(\text{In}(D \cap \Gamma_n))$ is a terminal edge, and $\text{Cl}(\text{In}(Z))$ is a disk. Let N be a regular neighborhood of $\text{In}(Z) \cup Z$ in D . Then $(N \cap \Gamma, N)$ is a tiny cellular T -tangle without exceptional arcs. Hence we have a desired result. We can assume $Y \neq \emptyset$.

Let q be the number of points in $D^* \cap Y$. For $i = 1, 2, 3, \dots$, let

$d_i =$ the number of connected components of D^* containing i points in $D^* \cap Y$,

$t_i =$ the number of trees in Y containing i points in $D^* \cap Y$.

Then we have

$$(1) \quad \sum_{i=1}^{\infty} i \times d_i = \sum_{i=1}^{\infty} i \times t_i = q.$$

Since $Y \neq \emptyset$, we have $q \geq 1$. Since $D^* \cup Y$ is contractible, by Euler formula we have

$$(2) \quad \sum_{i=1}^{\infty} d_i + \sum_{i=1}^{\infty} t_i - q = 1.$$

By using the equation (1) and the equation obtained by doubling each side of the equation (2), we have

$$\begin{aligned} 2 &= 2 \sum_{i=1}^{\infty} d_i + 2 \sum_{i=1}^{\infty} t_i - 2q \\ &= 2 \sum_{i=1}^{\infty} d_i + 2 \sum_{i=1}^{\infty} t_i - \left(\sum_{i=1}^{\infty} i \times d_i + \sum_{i=1}^{\infty} i \times t_i \right) \\ &= \sum_{i=1}^{\infty} (2-i)d_i + \sum_{i=1}^{\infty} (2-i)t_i = d_1 - \sum_{i=3}^{\infty} (i-2)d_i + t_1 - \sum_{i=3}^{\infty} (i-2)t_i. \end{aligned}$$

Thus we have

$$(3) \quad \sum_{i=3}^{\infty} (i-2)d_i + \sum_{i=3}^{\infty} (i-2)t_i = d_1 + t_1 - 2.$$

By Lemma 4.1, if the closure of a connected component of $(D \cap \Gamma_n) - Cl(In(D \cap \Gamma_n))$ is not a terminal edge, then it contains at least two points in $Brd(D \cap \Gamma_n) \cup \partial D$. This implies that for a connected component G of Y , if $D^* \cap G$ consists of one point, then G contains a point in ∂D . Thus each tree in Y contributing to t_1 must contain a point in ∂D . Since there are at most p connected components of Y intersecting ∂D , we have $t_1 \leq p$.

We shall show that there exists an integer $1 \leq j \leq p$ with $d_j \neq 0$.

If $p = 1$, then $t_1 \leq 1$. Since the left side of the equation (3) is non negative, we have $d_1 + t_1 - 2 \geq 0$. Hence $d_1 \geq 2 - t_1 \geq 2 - 1 = 1$. Therefore $d_1 \neq 0$. We can assume $p \geq 2$.

Suppose that $d_i = 0$ for $i = 1, 2, \dots, p$. By the equation (1), we have $\sum_{i=1}^{\infty} i \times d_i = q \geq 1$. Thus there exists an integer $j > p \geq 2$ with $d_j \neq 0$. Hence for the left side of the equation (3) we have

$$(4) \quad \sum_{i=3}^{\infty} (i-2)d_i + \sum_{i=3}^{\infty} (i-2)t_i \geq \sum_{i=j}^{\infty} (i-2)d_i \geq j-2 > p-2.$$

On the other hand, for the right side of the equation (3) we have

$$d_1 + t_1 - 2 = t_1 - 2.$$

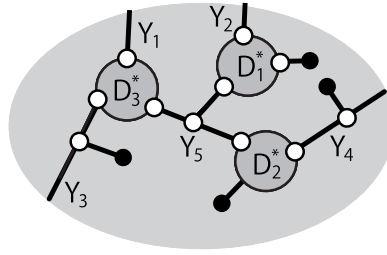


Fig. 5. $p = 4, q = 7, D_1^*$ and D_2^* are disks in D^* containing two points in $D^* \cap Y, D_3^*$ is a disk in D^* containing three points in $D^* \cap Y, Y_1, Y_2, Y_3$ and Y_4 are trees in Y containing one point in $D^* \cap Y, Y_5$ is a tree in Y containing three points in $D^* \cap Y, d_1 = 0, d_2 = 2, d_3 = 1, t_1 = 4, t_2 = 0, t_3 = 1.$

Since $t_1 \leq p,$ we have

$$(5) \quad d_1 + t_1 - 2 \leq p - 2.$$

We have a contradiction comparing (4) and (5). Therefore there exists an integer $1 \leq j \leq p$ with $d_j \neq 0.$

Since $d_j \neq 0$ for some integer $1 \leq j \leq p,$ there exists a connected component N of $Cl(In(D \cap \Gamma_n))$ such that N intersects at most p connected components in $Y.$ By Lemma 3.1 (2), N is a disk. Let X be the union of terminal edges in $D \cap \Gamma_n$ intersecting $N.$ Let N^* be a regular neighborhood of $N \cup X.$ Then $(N^* \cap \Gamma, N^*)$ is a tiny cellular T -tangle with at most p exceptional arcs. □

5. T_2 -tangles

Let Γ be a chart. A tangle $(D \cap \Gamma, D)$ is called an *NS-tangle of label m* (new significant tangle) if it satisfies the following two conditions:

- (i) If $i \neq m,$ then $\partial D \cap \Gamma_i$ is at most one point, and
- (ii) $w(D \cap \Gamma) \geq 1$ and D contains at most one crossing.

Lemma 5.1 ([12, Theorem 3.5]). *There does not exist any NS-tangle in a k -minimal chart $\Gamma.$*

Let $(D \cap \Gamma, D)$ be a T -tangle of a chart $\Gamma.$ If s is the number of labels in $\{i \mid \partial D \cap \Gamma_i \neq \emptyset\},$ then the T -tangle is called a T_s -tangle. Thus a T -tangle means a T_0 -tangle, a T_1 -tangle, a T_2 -tangle or a T_3 -tangle.

NOTE. Since T_0 -tangles and T_1 -tangles are NS-tangles, there do not exist any T_0 -tangles nor T_1 -tangles in a k -minimal chart by Lemma 5.1.

Lemma 5.2 ([12, Theorem 5.4]). *Let $(D \cap \Gamma, D)$ be a tiny cellular T_2 -tangle of label n in a k -minimal chart Γ which possesses exceptional arcs.*

- (1) *The tangle possesses at least two exceptional arcs.*
- (2) *If the tangle possesses exactly two exceptional arcs, then D contains at least two terminal edges of label n .*

Let Γ be a chart, $X \subset \Gamma$. Let

$$\alpha(X) = \min\{i \mid \Gamma_i \cap X \neq \emptyset\},$$

$$\beta(X) = \max\{i \mid \Gamma_i \cap X \neq \emptyset\}.$$

Lemma 5.3 (Boundary condition lemma ([12, Lemma 4.1])). *Let $(D \cap \Gamma, D)$ be a tangle in a k -minimal chart Γ such that D does not contain any crossing. Let $a = \alpha(\partial D \cap \Gamma)$ and $b = \beta(\partial D \cap \Gamma)$. Then $D \cap \Gamma_i = \emptyset$ except for $a \leq i \leq b$.*

Lemma 5.4. *Let $(D \cap \Gamma, D)$ be a non-linear T_2 -tangle of label n in a k -minimal chart Γ . If the T_2 -tangle possesses exactly two exceptional arcs, then the tangle possesses at least two terminal edges of label n .*

Proof. Since the T_2 -tangle possesses an exceptional arc, there exists an integer $\varepsilon \in \{+1, -1\}$ with $\partial D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\varepsilon}$. Thus we have $D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\varepsilon}$ by the Boundary condition lemma (Lemma 5.3). Hence the tangle $(D \cap \Gamma, D)$ contains a tiny cellular T_2 -tangle $(D' \cap \Gamma, D')$ with at most two exceptional arcs by Lemma 4.2.

By Lemma 5.2 (1), the tangle $(D' \cap \Gamma, D')$ possesses exactly two exceptional arcs. By Lemma 5.2 (2), there exist at least two terminal edges of label n in D . □

By Lemmas 3.3 and 5.4, we have the following corollary:

Corollary 5.5. *Let $(D \cap \Gamma, D)$ be a T_2 -tangle of label n with exactly two exceptional arcs in a k -minimal chart Γ . Then the following hold:*

- (1) *The disk D contains at least one terminal edge of label n .*
- (2) *If D contains exactly one terminal edge of label n , then $(D \cap \Gamma, D)$ is linear.*

6. Charts with at most three crossings

Let Γ be a chart, D a disk. Let m be a label with $D \cap \Gamma_m \neq \emptyset$. A connected component G of $D \cap \Gamma_m$ is a *two-color component* of label m in D provided that

- (i) $G \cap \partial D$ consists of at most one point,
- (ii) there exists an integer $\delta \in \{+1, -1\}$ such that all the white vertices in G are contained in $\Gamma_{m+\delta}$, and
- (iii) G is not an arc contained in a terminal edge.

Note that a two-color component may contain a crossing.

Lemma 6.1 ([12, Lemma 3.6]). *Let Γ be a k -minimal chart and D a disk. Then for any two-color component G in D , $G \cup \text{In}(G)$ contains at least two crossings.*

Let G be a graph. Then an edge e in G is called a *cut edge* of G provided that $G - e$ is disconnected.

Lemma 6.2. *Let Γ be a k -minimal chart and G a two-color component of label m in a disk D such that*

- (1) $G \cap \partial D = \emptyset$, and
- (2) G contains a cut edge.

Then Γ contains at least four crossings.

Proof. Let e be a cut edge of G . Since by Assumption 6 we can move the point at infinity ∞ to any complementary domain of Γ , we can assume $e \subset \text{Cl}(\text{Out}(G))$. Since e is a cut edge of G , $\text{Cl}(G - e)$ consists of two connected components, say G_1 and G_2 . For $i = 1, 2$ let N_i be a regular neighbourhood of $G_i \cup \text{In}(G_i)$ and $G'_i = N_i \cap G$. Then N_i is a disk by Lemma 3.1 (3). Thus G'_i is a two-color component in N_i . Hence by Lemma 6.1, each of $G'_1 \cup \text{In}(G'_1)$ and $G'_2 \cup \text{In}(G'_2)$ contains at least two crossings. Now $e \subset \text{Cl}(\text{Out}(G))$ implies $N_1 \cap N_2 = \emptyset$. Therefore Γ contains at least four crossings. \square

Lemma 6.3. *Let Γ be a k -minimal chart with at most three crossings. Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$. Then*

- (1) *each of Γ_α and Γ_β is connected,*
- (2) *each of $\text{Brd}(\Gamma_\alpha)$ and $\text{Brd}(\Gamma_\beta)$ is a simple closed curve, and*
- (3) *$\text{Brd}(\Gamma_\alpha) \cap \text{Brd}(\Gamma_\beta)$ consists of two crossings.*

Proof. Let G_α be a connected component of Γ_α . Let N be a regular neighbourhood of $G_\alpha \cup \text{In}(G_\alpha)$. Since G_α is connected, N is a disk by Lemma 3.1 (3). Let $D^* = \text{Cl}(S^2 - N)$ where S^2 is the 2-sphere. Then D^* is a disk, too.

Now $\alpha = \alpha(\Gamma)$ implies that any white vertices in G_α are contained in $\Gamma_\alpha \cap \Gamma_{\alpha+1}$. Thus G_α is a two-color component of label α in the disk N .

Since there are at most three crossings, G_α does not contain a cut edge by Lemma 6.2. By Assumption 5, G_α is not a free edge. Thus G_α is not a tree. Now $\text{Cl}(\text{In}(G_\alpha))$ consists of disks by Lemma 3.1 (2). Since G_α does not contain a cut edge, $\text{Cl}(\text{In}(G_\alpha))$ consists of only one disk. Hence we have

- (i) $\text{Brd}(G_\alpha)$ is a simple closed curve.

Suppose that $\text{Brd}(G_\alpha)$ contains at most one crossing. Since G_α does not contain a cut edge, $\partial D^* \cap (\Gamma - \Gamma_{\alpha+1})$ is at most one point (see Fig. 6 (a)).

By Lemma 6.1, $G_\alpha \cup \text{In}(G_\alpha)$ contains at least two crossings. Since there are at most three crossings, D^* contains at most one crossing.

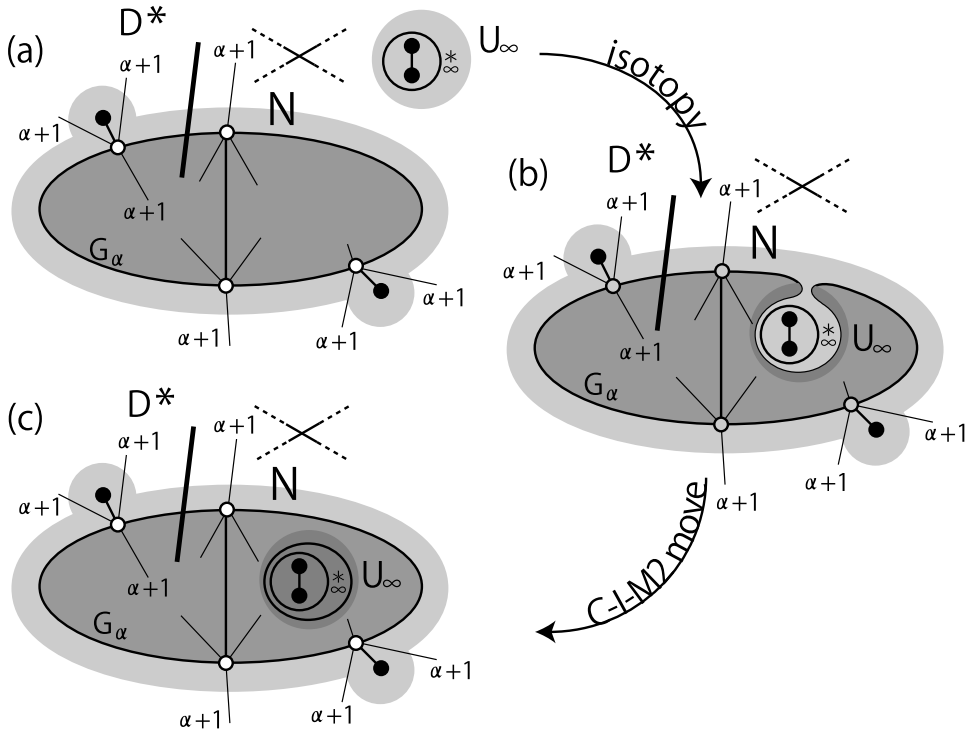


Fig. 6.

As mentioned in Assumption 6, by applying C-I-M2 moves we can push the neighbourhood U_∞ out from D^* without increasing the complexity of the chart (see Fig. 6 (c)). Then $(D^* \cap \Gamma, D^*)$ is an NS-tangle. This contradicts Lemma 5.1. Therefore

- (ii) $Brd(G_\alpha)$ contains at least two crossings.

Since each connected component of Γ_α contains at least two crossings and since there are at most three crossings, there exists only one connected component in Γ_α . Thus $G_\alpha = \Gamma_\alpha$. Thus Γ_α is connected.

Since Γ_α is connected and since Γ_α satisfies (i), Γ_α satisfies (2).

Similarly we can show that Γ_β is connected, satisfies (2) and $Brd(\Gamma_\beta)$ contains at least two crossings.

Since there are at most three crossings, $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ must intersect. Since $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ are simple closed curves, $Brd(\Gamma_\alpha) \cap Brd(\Gamma_\beta)$ consists of an even number of points. Since there are at most three crossings, $Brd(\Gamma_\alpha) \cap Brd(\Gamma_\beta)$ consists of two points. □

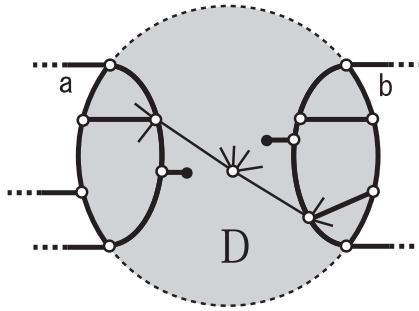


Fig. 7. The shaded area is the disk \$D\$.

7. 2-minimal charts

Lemma 7.1 ([11, Theorem 2]). *Any \$n\$-chart with at most one crossing is a ribbon chart.*

Let \$\Gamma\$ be a chart and \$D\$ a disk. Let \$a = \alpha(D \cap \Gamma)\$ and \$b = \beta(D \cap \Gamma)\$. The disk \$D\$ is called an *\$N\$-rectangle* if it satisfies the following four conditions (see Fig. 7):

- (i) \$D\$ does not contain any crossing,
- (ii) both of \$\partial D \cap \Gamma_a\$ and \$\partial D \cap \Gamma_b\$ are connected,
- (iii) \$\partial D \cap \Gamma \subset \Gamma_a \cup \Gamma_b\$, and
- (iv) there exists an arc in \$D \cap \Gamma\$ connecting a point in \$D \cap \Gamma_a\$ and a point in \$D \cap \Gamma_b\$.

From now on throughout this section, we assume that

- (i) \$\Gamma\$ is a 2-minimal chart with exactly two crossings,
- (ii) \$\Gamma\$ is not a ribbon chart, and
- (iii) \$\alpha = \alpha(\Gamma)\$ and \$\beta = \beta(\Gamma)\$.

By Lemma 6.3, each of \$Brd(\Gamma_\alpha)\$ and \$Brd(\Gamma_\beta)\$ is a simple closed curve containing the two crossings. Let

\$\Delta_\alpha\$ = the closure of the complementary domain of the simple closed curve \$Brd(\Gamma_\alpha)\$ such that \$\Delta_\alpha\$ does not contain the point at infinity \$\infty\$,

\$\Delta_\beta\$ = the closure of the complementary domain of the simple closed curve \$Brd(\Gamma_\beta)\$ such that \$\Delta_\beta\$ does not contain the point at infinity \$\infty\$.

Let \$v_1\$ and \$v_2\$ be the crossings in \$\Gamma\$. Let \$N_1 = N(v_1)\$ and \$N_2 = N(v_2)\$ be regular neighborhoods of \$v_1\$ and \$v_2\$ respectively, and \$N = N_1 \cup N_2\$ (see Fig. 8). Let

$$\begin{aligned}
 P_1 &= (\Gamma_\alpha - Int(N)) \cap \Delta_\beta, & P_3 &= (\Gamma_\alpha - Int(N)) \cap Cl(\Delta_\beta^c), \\
 P_2 &= (\Gamma_\beta - Int(N)) \cap \Delta_\alpha, & P_4 &= (\Gamma_\beta - Int(N)) \cap Cl(\Delta_\alpha^c), \\
 Q_1 &= (\Delta_\alpha \cap \Delta_\beta) - Int(N), & Q_3 &= (Cl(\Delta_\alpha^c) \cap Cl(\Delta_\beta^c)) - Int(N), \\
 Q_2 &= (\Delta_\alpha \cap Cl(\Delta_\beta^c)) - Int(N), & Q_4 &= (Cl(\Delta_\alpha^c) \cap \Delta_\beta) - Int(N).
 \end{aligned}$$

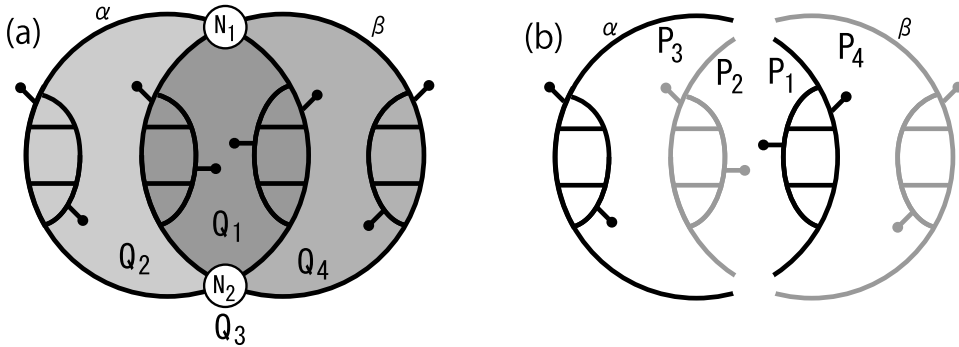


Fig. 8.

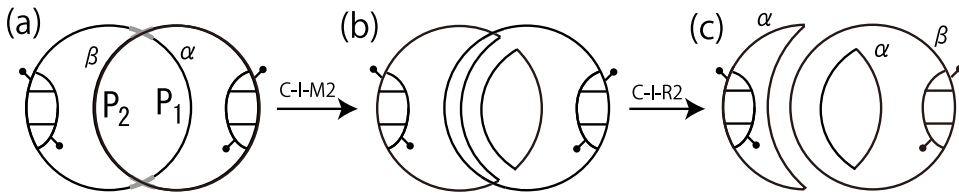


Fig. 9.

Lemma 7.2. *There are two N-rectangles among Q_1, Q_2, Q_3 and Q_4 . Moreover among P_1, P_2, P_3 and P_4 , three of them contain white vertices.*

Proof. We show our lemma by three steps.

STEP 1. We claim that P_1 or P_2 contains a white vertex. For, suppose that neither P_1 nor P_2 contains a white vertex. Apply a C-I-M2 move between two points in P_1 along the arc P_2 further apply a C-I-R2 move so that we can eliminate the crossings v_1 and v_2 (see Fig. 9 (c)). Hence Γ can be modified to a chart without crossings by C-moves. By Lemma 7.1, Γ is a ribbon chart. This contradicts the assumption (ii) of this section: Γ is not a ribbon chart. Hence one of P_1 and P_2 contains a white vertex. Without loss of generality we can assume that P_1 contains a white vertex.

STEP 2. We claim that Q_1 or Q_4 is an N-rectangle. For, suppose that neither Q_1 nor Q_4 is an N-rectangle. Then for $i = 1, 4$, there exists a simple arc l_i in Q_i connecting a point in ∂N_1 and a point in ∂N_2 with $l_i \cap \Gamma = \emptyset$ (see Fig. 10). Let D be the closure of the connected component of $\Delta_\beta - (l_1 \cup l_4 \cup N)$ containing P_1 . Then $\partial D \cap \Gamma \subset \Gamma_\alpha$. Since $P_1 \subset D$ and since P_1 contains a white vertex, we have $w(D \cap \Gamma) \geq 1$. Since D does not contain any crossing, $(D \cap \Gamma, D)$ is an NS-tangle of label α . This contradicts Lemma 5.1. Hence one of Q_1 and Q_4 is an N-rectangle. Without loss of generality we can assume that Q_1 is an N-rectangle.

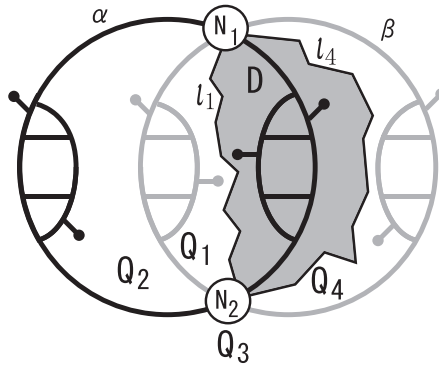


Fig. 10.

STEP 3. Hence both of P_1 and P_2 contain white vertices. We can show that one of P_3 and P_4 contains a white vertex by the same way as the one in Step 1. Hence among P_1, P_2, P_3 and P_4 , three of them contain white vertices.

If P_3 contains a white vertex, then we can show that one of Q_2 and Q_3 is an N-rectangle in the same way as the one in Step 2. If P_4 contains a white vertex, then we can show that one of Q_3 and Q_4 is an N-rectangle in the same way as the one in Step 2. Therefore two of Q_1, Q_2, Q_3 and Q_4 are N-rectangles. \square

Lemma 7.3. *Both of Δ_α and Δ_α^c contain white vertices of Γ_i for any label i ($\alpha + 2 \leq i \leq \beta - 2$), or both of Δ_β and Δ_β^c contain white vertices of Γ_i for any label i ($\alpha + 2 \leq i \leq \beta - 2$).*

Proof. By Lemma 7.2, two of Q_1, Q_2, Q_3 and Q_4 are N-rectangles. Without loss of generality we can assume that Q_1 is an N-rectangle. There exists an integer j in $\{2, 3, 4\}$ such that Q_j is an N-rectangle.

For the case $j = 2$, we have $Q_1 \subset \Delta_\beta$ and $Q_2 \subset Cl(\Delta_\beta^c)$. Since Q_1 is an N-rectangle, by the condition (iii) of N-rectangles, $\partial Q_1 \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_\beta$. By the condition (iv) of N-rectangles, there exists an arc γ in $Q_1 \cap \Gamma$ connecting a point in $\partial Q_1 \cap \Gamma_\alpha$ and a point in $\partial Q_1 \cap \Gamma_\beta$. Hence for each label i ($\alpha + 2 \leq i \leq \beta - 2$) there exists a white vertex in $\Gamma_i \cap \gamma$. Since $\partial Q_1 \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_\beta$, the white vertex of Γ_i is contained in $Int(\Delta_\beta)$. Since Q_2 is an N-rectangle, in a similar way as the one above we can show that there exists a white vertex of Γ_i ($\alpha + 2 \leq i \leq \beta - 2$) in Δ_β^c .

For the case $j = 3$ or 4 , we have $Q_1 \subset \Delta_\alpha$ and $Q_j \subset Cl(\Delta_\alpha^c)$. Similarly we can show that there exist white vertices of Γ_i for any label i ($\alpha + 2 \leq i \leq \beta - 2$) in Δ_α and Δ_α^c respectively. \square

A connected component G' of a graph G is called a *small component* of G if it satisfies $(In(G') - G') \cap G = \emptyset$. In Fig. 11, X is a small component of $X \cup Y$, but Y is not a small component of $X \cup Y$.

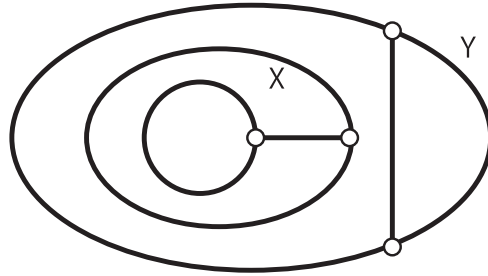


Fig. 11.

Lemma 7.4 ([12, Theorem 4.8]). *Let Γ be a k -minimal chart. Let G be a small component of Γ_n such that $G \cup \text{In}(G)$ does not contain any crossing. Then G contains at least two terminal edges of label n .*

Proposition 7.5. (1) *For any label i ($\alpha + 2 \leq i \leq \beta - 2$) the subgraph Γ_i contains at least four black vertices.*
 (2) *If $\alpha + 2 < \beta$, then $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices.*

Proof. (1) By Lemma 7.3 we can assume that both of Δ_α and Δ_α^c contain white vertices of Γ_i for any label i ($\alpha + 2 \leq i \leq \beta - 2$).

Let i be a label with $\alpha + 2 \leq i \leq \beta - 2$. Now $\partial\Delta_\alpha \subset \Gamma_\alpha$, $\alpha \neq i$ and $\alpha + 1 \neq i$ imply $\partial\Delta_\alpha \cap \Gamma_i = \emptyset$. Let G_i be a small component of $\Delta_\alpha \cap \Gamma_i$. Then G_i is a small component of Γ_i in $\text{Int}(\Delta_\alpha)$. Since $\text{Int}(\Delta_\alpha)$ does not contain any crossing, neither does $G_i \cup \text{In}(G_i)$. By Lemma 7.4, G_i contains at least two terminal edges of label i . Hence $\text{Int}(\Delta_\alpha)$ contains at least two terminal edges of label i .

Similarly we can show that Δ_α^c contains at least two terminal edges of label i . Hence Γ_i contains at least four black vertices.

(2) Since $\alpha + 2 < \beta$, we have $\alpha + 1 \neq \beta - 1$. By Lemma 7.2, three of P_1, P_2, P_3 and P_4 contain white vertices. Without loss of generality we can assume that all of P_1, P_2 and P_3 contain white vertices.

Since P_2 contains a white vertex and $P_2 \subset \Delta_\alpha \cap \Gamma_\beta$, the disk Δ_α contains a white vertex of $\Gamma_{\beta-1}$. Since $\partial\Delta_\alpha \subset \Gamma_\alpha$, $\alpha \neq \beta - 1$ and $\alpha + 1 \neq \beta - 1$, we have $\partial\Delta_\alpha \cap \Gamma_{\beta-1} = \emptyset$. In a similar way to (1) we can show that $\text{Int}(\Delta_\alpha)$ contains at least two terminal edges of label $\beta - 1$.

Since P_1 contains a white vertex and $P_1 \subset \Delta_\beta \cap \Gamma_\alpha$, the disk Δ_β contains a white vertex of $\Gamma_{\alpha+1}$. Similarly we can show that $\text{Int}(\Delta_\beta)$ contains at least two terminal edges of label $\alpha + 1$.

Since P_3 contains a white vertex and $P_3 \subset \text{Cl}(\Delta_\beta^c) \cap \Gamma_\alpha$, the open disk Δ_β^c contains a white vertex of $\Gamma_{\alpha+1}$. Similarly we can show that Δ_β^c contains at least two terminal edges of label $\alpha + 1$.

Therefore $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices. □

Proposition 7.6. *Both of Γ_α and Γ_β contain at least two black vertices.*

Proof. Let D_i be a regular neighborhood of $P_i \cup \text{In}(P_i)$ in $S^2 - \text{Int}(N_1 \cup N_2)$ ($i = 1, 2, 3, 4$). By Lemma 3.1 (3), D_i is a disk. By Lemma 7.2, three of P_1, P_2, P_3 and P_4 contain white vertices. Without loss of generality we can assume that all of P_1, P_2 and P_3 contain white vertices.

For $i = 1, 3$, we have $\partial D_i \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$. By the Boundary condition lemma (Lemma 5.3), $D_i \cap \Gamma_j = \emptyset$ except for $j \in \{\alpha, \alpha + 1\}$. Similarly for $i = 2, 4$, we have $D_i \cap \Gamma_j = \emptyset$ except for $j \in \{\beta, \beta - 1\}$.

Since P_i ($i = 1, 3$) contains a white vertex, $(D_i \cap \Gamma, D_i)$ is a T_2 -tangle of label α with two exceptional arcs. By Corollary 5.5 (1), the disk D_i ($i = 1, 3$) contains at least one terminal edge of label α . Hence Γ_α contains at least two black vertices.

Since P_2 contains a white vertex, $(D_2 \cap \Gamma, D_2)$ is a T_2 -tangle of label β with two exceptional arcs. By Corollary 5.5 (1), the disk D_2 contains at least one terminal edge of label β .

Suppose that the disk D_2 contains exactly one terminal edge of label β . By Corollary 5.5 (2), $(D_2 \cap \Gamma, D_2)$ is linear. Let e_1 and e_2 be the two exceptional arcs of $(D_2 \cap \Gamma, D_2)$. By Lemma 3.3, $D_2 \cap \Gamma_\beta$ consists of the two arcs e_1, e_2 and the terminal edge. Let w be the white vertex in the terminal edge. Since the terminal edge contains a middle arc at w by Assumption 1, both of e_1 and e_2 contain inward arcs at w or outward arcs at w . Hence P_4 contains a white vertex. Hence $(D_4 \cap \Gamma, D_4)$ is a T_2 -tangle of label β with two exceptional arcs. By Corollary 5.5 (1), the disk D_4 contains at least one terminal edge of label β . Hence Γ_β contains at least two black vertices. \square

8. Proof of Theorems 1.1 and 1.2

Lemma 8.1. *Let C be a hoop or a ring in a k -minimal chart Γ . Suppose that C contains exactly s crossings with $s \leq 3$. Then Γ contains at least $s + 4$ crossings.*

Proof. Let U_1 and U_2 be the connected components of $S^2 - C$. Then each of U_1 and U_2 contains a white vertex by Assumptions 3 and 4.

Suppose that U_i ($i = 1, 2$) contains at most one crossing. There are at most three edges transversely intersecting C . Let N_i be a disk in U_i such that $U_i - N_i$ is a very thin open annulus. Then we can assume that N_i contains a white vertex and that $\partial N_i \cap \Gamma$ consists of at most three points. Then for the edges intersecting ∂N_i there are two cases:

- (1) the labels of the edges are different each other, and
- (2) at least two labels of the edges are same.

In each case, $(N_i \cap \Gamma, N_i)$ is an NS-tangle. This contradicts Lemma 5.1. Hence U_i contains at least two crossings. Hence Γ contains at least $s + 4$ crossings. \square

The following corollary is a direct result of the above lemma.

Corollary 8.2. *Let Γ be a k -minimal chart with at most three crossings. Then Γ contains neither hoop nor ring.*

Proof of Theorem 1.1. Since Γ is a generalized n -chart, $w(\Gamma) \geq 1$. Since Γ is a 2-minimal chart, Γ contains at most two crossings and Γ is not a ribbon chart. By Lemma 7.1, Γ contains exactly two crossings. By Assumption 5, Γ does not contain any free edge. By Corollary 8.2, Γ contains neither hoop nor ring. Let $\alpha = \alpha(\Gamma)$, $\beta = \beta(\Gamma)$. Then $w(\Gamma_\alpha) \geq 1$, $w(\Gamma_\beta) \geq 1$. Since Γ is a generalized n -chart, we have $\beta - \alpha = n - 2$.

By Proposition 7.6, $\Gamma_\alpha \cup \Gamma_\beta$ contains at least four black vertices. Since $\beta - (\alpha + 2) = n - 4 \geq 5 - 4 = 1 > 0$, we have $\alpha + 2 < \beta$. By Proposition 7.5 (2), $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices. By Proposition 7.5 (1), for any label i ($\alpha + 2 \leq i \leq \beta - 2$) Γ_i contains at least four black vertices. We have that $\Gamma_{\alpha+2} \cup \Gamma_{\alpha+3} \cup \dots \cup \Gamma_{\beta-2}$ contains at least $4((\beta - 2) - (\alpha + 2) + 1)$ black vertices. Since $4((\beta - 2) - (\alpha + 2) + 1) = 4(\beta - \alpha - 3) = 4(n - 5)$, we have the number of black vertices of $\Gamma \geq 4(n - 5) + 4 + 6 = 4n - 10$. □

By [4, Remarks 8 (2)] we have the statement (1) in the following lemma.

Lemma 8.3. *Let Γ be an n -chart, and \hat{S}_Γ the closure of the surface braid obtained from Γ .*

- (1) *Let b be the number of black vertices of Γ . Then $\chi(\hat{S}_\Gamma) = 2n - b$.*
- (2) *Let $\iota_p^q(\Gamma)$ be the $(n + p + q)$ -chart obtained from Γ by shifting all labels i to $i + p$. Then the closure of the surface braid obtained from $\iota_p^q(\Gamma)$ contains at least $p + q + 1$ components.*
- (3) *Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$. Then \hat{S}_Γ contains at least $n - \beta + \alpha - 1$ components.*

Proof. We shall show the statement (2). Let \hat{S} be the closure of a surface braid obtained from $\iota_p^q(\Gamma)$. Then the surface \hat{S} is \hat{S}_Γ with p parallel spheres in front of \hat{S}_Γ and q parallel spheres behind \hat{S}_Γ (cf. [6, p. 183]). Therefore \hat{S} contains of at least $p + q + 1$ components.

We shall show the statement (3). Let Γ' be the $(\beta - \alpha + 2)$ -chart obtained from Γ by shifting all labels i to $i - \alpha + 1$. Then edges of Γ_α and edges of Γ_β change edges of Γ'_1 and edges of $\Gamma'_{\beta-\alpha+1}$ respectively. Hence $\alpha(\Gamma') = 1$ and $\beta(\Gamma') = \beta - \alpha + 1$.

Let $\Gamma'' = \iota_{\alpha-1}^{n-\beta-1}(\Gamma')$. Since $(\beta - \alpha + 2) + (\alpha - 1) + (n - \beta - 1) = n$, the chart Γ'' is an n -chart. Since edges of Γ'_1 and edges of $\Gamma'_{\beta-\alpha+1}$ change edges of Γ''_α and edges of Γ''_β respectively, the chart Γ'' is the same as the chart Γ . Since $(\alpha - 1) + (n - \beta - 1) + 1 = n - \beta + \alpha - 1$, \hat{S}_Γ contains of at least $n - \beta + \alpha - 1$ components by the statement (2) in this lemma. □

Lemma 8.4. *Let Γ' be an n -chart and Γ'' the n -chart obtained from Γ' by omitting all the free edges. Let $\hat{S}_{\Gamma'}$ and $\hat{S}_{\Gamma''}$ be the closures of the surface braids obtained from Γ' and Γ'' respectively. If $\hat{S}_{\Gamma'}$ is a disjoint union of spheres, then so is $\hat{S}_{\Gamma''}$.*

Proof. Since the chart Γ' is obtained by adding free edges to the chart Γ'' , the surface $\hat{S}_{\Gamma'}$ is obtained by attaching 1-handles from the surface $\hat{S}_{\Gamma''}$. Since $\hat{S}_{\Gamma'}$ is a disjoint union of spheres, so is $\hat{S}_{\Gamma''}$. □

Kamada showed that for $n = 1, 2, 3$ any n -chart is a ribbon chart [4]. We showed that if a 2-minimal 4-chart contains exactly two crossings, then it contains eight black vertices [9]. By the similar argument as above, we have the following remark:

REMARK 8.5. Let Γ be a k -minimal chart. Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$.

- (1) If $\beta - \alpha \leq 1$, then Γ is a ribbon chart.
- (2) If $\beta - \alpha = 2$ and if Γ is a 2-minimal chart with exactly two crossings, then it contains eight black vertices.

Proof of Theorem 1.2. Let n be the integer such that Γ is an n -chart. Let Γ' be a 2-minimal generalized n' -chart C-move equivalent to Γ . If Γ' contains at most one crossing, then by Lemma 7.1 Γ' is a ribbon chart, so is Γ .

Suppose that Γ' contains exactly two crossings. By Corollary 8.2, Γ' contains neither hoop nor ring. Let Γ'' be the n -chart obtained from Γ' by omitting all the free edges. Since Γ' is a 2-minimal generalized n' -chart, Γ'' is a 2-minimal generalized n' -chart.

Let $\alpha = \alpha(\Gamma'')$ and $\beta = \beta(\Gamma'')$. Since Γ'' contains neither hoops, rings nor free edges, both of Γ''_{α} and Γ''_{β} contain white vertices. Since Γ'' is a generalized n' -chart, we have $n' = \beta - \alpha + 2$.

Let \hat{S}_{Γ} , $\hat{S}_{\Gamma'}$ and $\hat{S}_{\Gamma''}$ be the closures of surface braids obtained from Γ , Γ' and Γ'' respectively. Since Γ is C-move equivalent to Γ' , \hat{S}_{Γ} is ambient isotopic to $\hat{S}_{\Gamma'}$ (cf. [6, Theorem 18.20]). The closure \hat{S}_{Γ} is a disjoint union of spheres, and so is $\hat{S}_{\Gamma'}$. By Lemma 8.4, $\hat{S}_{\Gamma''}$ is a disjoint union of spheres.

Since Γ'' is an n -chart, by Lemma 8.3 (3) $\hat{S}_{\Gamma''}$ contains at least $n - \beta + \alpha - 1$ spheres. Since $n' = \beta - \alpha + 2$, $\hat{S}_{\Gamma''}$ contains at least $n - n' + 1$ spheres. By Lemma 8.3 (1) $2(n - n' + 1) \leq \chi(\hat{S}_{\Gamma''}) = 2n -$ (the number of black vertices of Γ''). Hence Γ'' contains at most $2n' - 2$ black vertices.

Suppose $n' = 4$. Since $n' = \beta - \alpha + 2$, we have $\beta - \alpha = 2$. By Remark 8.5 (2), the chart contains at least eight black vertices. Hence Γ'' contains at least eight black vertices. On the other hand since $2n' - 2 = 2 \times 4 - 2 = 6$, the chart Γ'' contains at most six black vertices. This is a contradiction.

Suppose $n' \geq 5$. By Theorem 1.1, the chart Γ'' contains at least $4n' - 10$ black vertices. On the other hand the chart Γ'' contains at most $2n' - 2$ black vertices. Hence $4n' - 10 \leq 2n' - 2$. Hence $n' \leq 4$. This is a contradiction.

Therefore $n' \leq 3$. Since $n' = \beta - \alpha + 2$, we have $\beta - \alpha \leq 1$. By Remark 8.5 (1), the chart is a ribbon chart. Hence Γ'' is a ribbon chart, so is Γ . \square

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