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SIMPLE PROOFS OF SOME THEOREMS IN BLOCK THEORY OF FINITE GROUPS

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Abstract

We give simple proofs of Laradji's theorem on blocks with central defect groups, Watanabe's theorem on the Glauberman–Watanabe correspondences of blocks and Robinson's theorem on defect groups of p -blocks of p -solvable groups attaining Brauer's upper bound for the number of irreducible characters.

Introduction

In this paper all groups are finite groups. A block means a p -block for a fixed prime p . For a positive integer n , let $p^{v(n)}$ be the highest power of p dividing n . Laradji [7] has proved:

Theorem A. *Let Z be a central p -subgroup of a group G . Let χ be an irreducible character of G such that $v(\chi(1)) = v(|G : Z|)$. Then Z is a defect group of the block of G containing χ .*

Known proofs of this theorem ([7], [11], [13]) are rather complicated. Here we give a simple proof, which is analogous to the proof of Theorem 3.12 of [6].

Let S and G be groups such that S acts on G as automorphisms and that $(|S|, |G|) = 1$. Let B be an S -invariant block of G such that a defect group of B is centralized by S . In this situation Watanabe [15] has proved:

Theorem B. *Any irreducible character in B is S -invariant.*

Watanabe [15] has proved Theorem B by using a theorem of Dade [1]. Here we give a direct proof of Theorem B. Another direct proof, which uses the Glauberman correspondence, is found in Navarro [12].

Let B be a block of a group with a defect group D . A well-known conjecture of R.Brauer asserts that $k(B) \leq |D|$. For p -solvable groups, this conjecture has been proved by [4]:

Theorem C. *Let B be a p -block of a p -solvable group with a defect group D . Then $k(B) \leq |D|$. In particular, for a p -solvable group G with $O_p(G) = 1$, we have $k(G) \leq |D|$, where D is a Sylow p -subgroup of G .*

As to the equality in Theorem C, Robinson [14] has proved:

Theorem D. *Let B be a p -block of a p -solvable group with a defect group D . If $k(B) = |D|$, then D is abelian.*

We simplify Robinson's proof by using a theorem of Gallagher [3].

1. Proof of Theorem A

Proof of Theorem A. As in [7], we may assume that χ_Z is faithful. Let Z act by multiplication on the set of all conjugacy classes of G . Let $\{K_i\}$ be a complete set of representatives of Z -orbits. As in the proof of Theorem 3.12 of [6], we obtain

$$\frac{|G : Z|}{\chi(1)} = \sum_i \omega_\chi(\hat{K}_i) \chi(x_i^{-1}),$$

where $x_i \in K_i$ for each i . This shows that $|G : Z|/\chi(1)$ is a rational integer, which is coprime to p by our assumption. Let K be a sufficiently large algebraic number field and let P be a prime ideal of K lying over p . Then there exists i such that $\omega_\chi(\hat{K}_i) \chi(x_i^{-1}) \notin P$. This implies that χ has height 0 (cf. [2, IV 4.4]). So, if D is a defect group of the block of G containing χ , then $|D| = |Z|$. Thus $D = Z$. This completes the proof. \square

2. Proof of Theorem B

Watanabe [15, Proposition 1] proves essentially the following, which is stronger than Theorem B.

Theorem B'. *Suppose that a group S acts on a group G as automorphisms. Let B be an S -invariant block of G such that a defect group D of B is centralized by S . Assume that $(|S|, |N_G(D)/DC_G(D)|) = 1$. Then any irreducible character in B is S -invariant.*

In this section we give a direct proof of Theorem B'. Let $\Gamma = SG$ be the semi-direct product.

For a block b of a normal subgroup Y of a group X , let $\text{BL}(X|b)$ be the set of blocks of X covering b .

Lemma 1. *Let the notation be as in Theorem B'. Assume in addition that S is cyclic.*

- (i) If S is a p' -group, then $|\mathrm{BL}(\Gamma|B)| = |S|$.
(ii) If S is a p -group, let \hat{B} be a unique block of Γ covering B . Then \hat{B} has a defect group R such that $R = DC_R(D)$.

Proof. (i) Let \tilde{B} be the Brauer correspondent of B with respect to D in $N_G(D)$. By the Harris–Knörr theorem [5], it suffices to show $|\mathrm{BL}(N_\Gamma(D)|\tilde{B})| = |S|$. We note that $N_\Gamma(D) = S \ltimes N_G(D)$, \tilde{B} is S -invariant and D is a defect group of \tilde{B} .

A slight modification of the proof of Proposition 1 of [15] shows that there exists an S -invariant block, b say, of $C_G(D)$ covered by \tilde{B} . It is clear that a block of $N_\Gamma(D)$ covers \tilde{B} if and only if it covers one of the blocks in $\mathrm{BL}(C_\Gamma(D)|b)$. For each block $\beta \in \mathrm{BL}(C_\Gamma(D)|b)$, there exists a unique block in $\mathrm{BL}(N_\Gamma(D)|\tilde{B})$ which covers β . Thus it suffices to show the following:

- (1) $|\mathrm{BL}(C_\Gamma(D)|b)| = |S|$;
(2) No two distinct blocks in $\mathrm{BL}(C_\Gamma(D)|b)$ are $N_\Gamma(D)$ -conjugate.

(1) Let ζ be the canonical character of b . Since b is S -invariant, so is ζ . Since $C_\Gamma(D) = S \ltimes C_G(D)$ and S is cyclic, there exists an extension of ζ to $C_\Gamma(D)$. Let \mathcal{E} be the set of extensions of ζ to $C_\Gamma(D)$. For any $\eta \in \mathcal{E}$, let $B(\eta)$ be the block of $C_\Gamma(D)$ containing η . Then $B(\eta)$ covers b . Since $C_\Gamma(D)/C_G(D)$ is a p' -group, $B(\eta)$ has defect group $Z(D)$. Therefore η is the canonical character of $B(\eta)$. In particular, $B(\eta) \neq B(\eta')$ if $\eta, \eta' \in \mathcal{E}$ and $\eta \neq \eta'$. Clearly any block of $C_\Gamma(D)$ covering b is of the form $B(\eta)$ for some $\eta \in \mathcal{E}$. Since $|\mathcal{E}| = |S|$, (1) follows.

(2) We claim first that any $\eta \in \mathcal{E}$ is $N_G(D)_\zeta$ -invariant, where $N_G(D)_\zeta$ is the inertial group of ζ in $N_G(D)$. Indeed, for any $x \in N_G(D)_\zeta$, we have $\eta^x \in \mathcal{E}$. Thus $\eta^x = \eta \otimes \lambda_x$ for a unique $\lambda_x \in \mathrm{Irr}(C_\Gamma(D)/C_G(D)) = \mathrm{Irr}(S)$. Since $[C_\Gamma(D), N_G(D)_\zeta] \leq C_G(D)$, λ_x is $N_G(D)_\zeta$ -invariant. Therefore the map $x \mapsto \lambda_x$ is a group homomorphism from $N_G(D)_\zeta$ to $\mathrm{Irr}(S)$. Since this map is trivial on $C_G(D)$, it factors through $N_G(D)_\zeta/C_G(D)$. Since $(|S|, |N_G(D)_\zeta/C_G(D)|) = 1$, this map is a trivial homomorphism. Thus the claim is proved.

Now assume $B(\eta)^x = B(\eta')$ for $x \in N_\Gamma(D)$, $\eta, \eta' \in \mathcal{E}$. We may assume $x \in N_G(D)$. We have $\eta^x = \eta'$, so that $\zeta^x = \zeta$. Thus $x \in N_G(D)_\zeta$. Then $\eta = \eta'$ by the above, and (2) is proved. The proof of (i) is complete.

(ii) If $S = 1$, there is nothing to prove. So we assume $S > 1$. Let B' be the Harris–Knörr correspondent of \hat{B} over B in $N_\Gamma(D)$. Then B' and \hat{B} have a defect group in common. We have $N_\Gamma(D) = SN_G(D) = DC_\Gamma(D)N_G(D)$. So $N_\Gamma(D)/DC_\Gamma(D) \simeq N_G(D)/DC_G(D)$, which is a p' -group by assumption. Thus if β is a block of $DC_\Gamma(D)$ covered by B' , then a defect group R of β is a defect group of B' . Hence R is a defect group of \hat{B} . Now $D \leq R \leq DC_\Gamma(D)$, so that $R = DC_R(D)$. The proof is complete. \square

REMARK 1. As in the proof of Proposition 2 of [15], Lemma 1 (i) follows from [1].

REMARK 2. In Lemma 1, the conclusions of (i) and (ii) are in fact equivalent to the equality $S = S[B]$, where $S[B]$ is defined as in Proposition 1 of [15]. A proof will be given in a separate paper.

Proof of Theorem B'. We may assume that either S is a cyclic p' -group or a cyclic p -group.

Assume that S is a cyclic p' -group. Let ζ be any irreducible character in B . Let T be the inertial group of ζ in Γ . Since any block of Γ covering B contains an irreducible character lying over ζ , any block in $\text{BL}(\Gamma|B)$ is induced from a block in $\text{BL}(T|B)$ (cf. [10, Lemma 5.3.1 (ii)]). So $|\text{BL}(\Gamma|B)| \leq |\text{BL}(T|B)|$. Also, $|\text{BL}(T|B)| \leq k(T|\zeta) = |T/G| \leq |S|$, where $k(T|\zeta)$ denotes the number of irreducible characters of T lying over ζ . Thus Lemma 1 (i) yields $|T/G| = |S|$. Hence $T = \Gamma$ and ζ is S -invariant.

Assume that S is a cyclic p -group. Let \hat{B} and R be as in Lemma 1 (ii). Then by Lemma 4.14 (ii) of [8], any irreducible character in B is R -invariant. On the other hand, since \hat{B} is weakly regular with respect to G and B is Γ -invariant, RG/G is a Sylow p -subgroup of Γ/G by Fong's theorem. Thus $\Gamma = RG$. So any irreducible character in B is S -invariant. This completes the proof. \square

3. Proof of Theorem D

Theorem D is equivalent to the following theorem, cf. Remarks of [14].

Theorem D'. *Let G be a p -solvable group with $O_{p'}(G) = 1$ and $k(G) = |D|$, where D is a Sylow p -subgroup of G . Then D is abelian.*

Lemma 2 (Gallagher [3]). *Let N be a normal subgroup of a group G . Then $k(G) \leq k(G/N)k(N)$ and equality holds if and only if $C_G(x \bmod N) = C_G(x)N$ for all $x \in G$. Furthermore if equality holds, then every irreducible character of N is G -invariant.*

Proof. The first statement is (3) of [3]. If equality holds, then, as shown in the proof of (3) of [3, p.176], every conjugacy class of N is G -invariant. As is well-known, this implies that every irreducible character of N is G -invariant. \square

Proof of Theorem D'.. Let $N = O_{p,p'}(G)$. Then, since $O_{p'}(N) = O_{p'}(G/N) = 1$, by Theorem C and Lemma 2,

$$|D| = k(G) \leq k(G/N)k(N) \leq |G/N|_p |N|_p = |D|.$$

Thus equality holds throughout. So every irreducible character of N is G -invariant by Lemma 2. Then as in the proof of Lemma 3 of [14], we see $G = N$. Thus D is normal in G .

Let Φ be the Frattini subgroup of D . Then, as in Nagao [9], $O_{p'}(G/\Phi) = 1$. Thus by Theorem C and Lemma 2,

$$|D| = k(G) \leq k(G/\Phi)k(\Phi) \leq |G/\Phi|_p |\Phi| = |D|.$$

Thus equality holds throughout. Let $x \in D$. By Lemma 2, we have $C_G(x \bmod \Phi) = C_G(x)\Phi$. Since $D \leq C_G(x \bmod \Phi)$, we obtain $D \leq C_G(x)\Phi$. Thus $D = C_D(x)\Phi$ and $D = C_D(x)$. Since $x \in D$ is arbitrary, D is abelian. This completes the proof. \square

References

- [1] E.C. Dade: *Block extensions*, Illinois J. Math. **17** (1973), 198–272.
- [2] W. Feit: *The Representation Theory of Finite Groups*, North-Holland, Amsterdam, 1982.
- [3] P.X. Gallagher: *The number of conjugacy classes in a finite group*, Math. Z. **118** (1970), 175–179.
- [4] D. Gluck, K. Magaard, U. Riese and P. Schmid: *The solution of the $k(GV)$ -problem*, J. Algebra **279** (2004), 694–719.
- [5] M.E. Harris and R. Knörr: *Brauer correspondence for covering blocks of finite groups*, Comm. Algebra **13** (1985), 1213–1218.
- [6] I.M. Isaacs: *Character Theory of Finite Groups*, Academic Press, New York, 1976.
- [7] A. Laradji: *On characters with minimal defects*, J. Reine Angew. Math. **448** (1994), 27–29.
- [8] M. Murai: *Block induction, normal subgroups and characters of height zero*, Osaka J. Math. **31** (1994), 9–25.
- [9] H. Nagao: *On a conjecture of Brauer for p -solvable groups*, J. Math. Osaka City Univ. **13** (1962), 35–38.
- [10] H. Nagao and Y. Tsushima: *Representations of Finite Groups*, Academic Press, Boston, MA, 1989.
- [11] G. Navarro: *Characters and Blocks of Finite Groups*, Cambridge Univ. Press, Cambridge, 1998.
- [12] G. Navarro: *Actions and characters in blocks*, J. Algebra **275** (2004), 471–480.
- [13] G.R. Robinson: *Local structure, vertices and Alperin's conjecture*, Proc. London Math. Soc. (3) **72** (1996), 312–330.
- [14] G.R. Robinson: *On Brauer's $k(B)$ -problem for blocks of p -solvable groups with non-Abelian defect groups*, J. Algebra **280** (2004), 738–742.
- [15] A. Watanabe: *The Glauberman character correspondence and perfect isometries for blocks of finite groups*, J. Algebra **216** (1999), 548–565.

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