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SIMPLE PROOFS OF SOME THEOREMS IN BLOCK THEORY OF FINITE GROUPS

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Abstract

We give simple proofs of Laradji's theorem on blocks with central defect groups, Watanabe's theorem on the Glauberman–Watanabe correspondences of blocks and Robinson's theorem on defect groups of p-blocks of p-solvable groups attaining Brauer's upper bound for the number of irreducible characters.

Introduction

In this paper all groups are finite groups. A block means a p-block for a fixed prime p. For a positive integer n, let $p^{\nu(n)}$ be the highest power of p dividing n. Laradji [7] has proved:

Theorem A. *Let Z be a central p-subgroup of a group G. Let χ be an irreducible character of G such that $\nu(\chi(1)) = \nu(|G : Z|)$. Then Z is a defect group of the block of G containing χ.*

Known proofs of this theorem ([7], [11], [13]) are rather complicated. Here we give a simple proof, which is analogous to the proof of Theorem 3.12 of [6].

Let S and G be groups such that S acts on G as automorphisms and that $(|S|, |G|) = 1$. Let B be an S-invariant block of G such that a defect group of B is centralized by S. In this situation Watanabe [15] has proved:

Theorem B. *Any irreducible character in B is S-invariant.*

Watanabe [15] has proved Theorem B by using a theorem of Dade [1]. Here we give a direct proof of Theorem B. Another direct proof, which uses the Glauberman correspondence, is found in Navarro [12].

Let B be a block of a group with a defect group D. A well-known conjecture of R.Brauer asserts that $k(B) \leq |D|$. For p-solvable groups, this conjecture has been proved by [4]:

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Theorem C. *Let B be a p-block of a p-solvable group with a defect group D. Then $k(B) \leq |D|$. In particular, for a p-solvable group G with $O_{p'}(G) = 1$, we have $k(G) \leq |D|$, where D is a Sylow p-subgroup of G.*

As to the equality in Theorem C, Robinson [14] has proved:

Theorem D. *Let B be a p-block of a p-solvable group with a defect group D. If $k(B) = |D|$, then D is abelian.*

We simplify Robinson's proof by using a theorem of Gallagher [3].

1. Proof of Theorem A

Proof of Theorem A. As in [7], we may assume that χ_Z is faithful. Let Z act by multiplication on the set of all conjugacy classes of G. Let $\{K_i\}$ be a complete set of representatives of Z-orbits. As in the proof of Theorem 3.12 of [6], we obtain

$$\frac{|G : Z|}{\chi(1)} = \sum_i \omega_\chi(\hat{K}_i)\chi(x_i^{-1}),$$

where $x_i \in K_i$ for each i. This shows that $|G : Z|/\chi(1)$ is a rational integer, which is coprime to p by our assumption. Let K be a sufficiently large algebraic number field and let P be a prime ideal of K lying over p. Then there exists i such that $\omega_\chi(\hat{K}_i)\chi(x_i^{-1}) \notin P$. This implies that χ has height 0 (cf. [2, IV 4.4]). So, if D is a defect group of the block of G containing χ, then $|D| = |Z|$. Thus $D = Z$. This completes the proof. □

2. Proof of Theorem B

Watanabe [15, Proposition 1] proves essentially the following, which is stronger than Theorem B.

Theorem B′. *Suppose that a group S acts on a group G as automorphisms. Let B be an S-invariant block of G such that a defect group D of B is centralized by S. Assume that $(|S|, |N_G(D)/DC_G(D)|) = 1$. Then any irreducible character in B is S-invariant.*

In this section we give a direct proof of Theorem B′. Let $\Gamma = SG$ be the semidirect product.

For a block b of a normal subgroup Y of a group X, let $\mathrm{BL}(X|b)$ be the set of blocks of X covering b.

Lemma 1. *Let the notation be as in* Theorem B′. *Assume in addition that S is cyclic.*

(i) *If S is a p'-group, then* $|\mathrm{BL}(\Gamma|B)| = |S|$.
(ii) *If S is a p-group, let $\hat{B}$ be a unique block of Γ covering B. Then $\hat{B}$ has a defect group R such that $R = DC_R(D)$.*

Proof. (i) Let $\tilde{B}$ be the Brauer correspondent of B with respect to D in $N_G(D)$. By the Harris–Knörr theorem [5], it suffices to show $|\mathrm{BL}(N_\Gamma(D) \mid \tilde{B})| = |S|$. We note that $N_\Gamma(D) = S \ltimes N_G(D)$, $\tilde{B}$ is S-invariant and D is a defect group of $\tilde{B}$.

A slight modification of the proof of Proposition 1 of [15] shows that there exists an S-invariant block, b say, of $C_G(D)$ covered by $\tilde{B}$. It is clear that a block of $N_\Gamma(D)$ covers $\tilde{B}$ if and only if it covers one of the blocks in $\mathrm{BL}(C_\Gamma(D)|b)$. For each block $\beta \in \mathrm{BL}(C_\Gamma(D)|b)$, there exists a unique block in $\mathrm{BL}(N_\Gamma(D)|\tilde{B})$ which covers β. Thus it suffices to show the following:
(1) $|\mathrm{BL}(C_\Gamma(D) \mid b)| = |S|$;
(2) No two distinct blocks in $\mathrm{BL}(C_\Gamma(D) \mid b)$ are $N_\Gamma(D)$-conjugate.

(1) Let ζ be the canonical character of b. Since b is S-invariant, so is ζ. Since $C_\Gamma(D) = S \ltimes C_G(D)$ and S is cyclic, there exists an extension of ζ to $C_\Gamma(D)$. Let $\mathcal{E}$ be the set of extensions of ζ to $C_\Gamma(D)$. For any $\eta \in \mathcal{E}$, let $B(\eta)$ be the block of $C_\Gamma(D)$ containing η. Then $B(\eta)$ covers b. Since $C_\Gamma(D)/C_G(D)$ is a p'-group, $B(\eta)$ has defect group $Z(D)$. Therefore η is the canonical character of $B(\eta)$. In particular, $B(\eta) \neq B(\eta')$ if $\eta, \eta' \in \mathcal{E}$ and $\eta \neq \eta'$. Clearly any block of $C_\Gamma(D)$ covering b is of the form $B(\eta)$ for some $\eta \in \mathcal{E}$. Since $|\mathcal{E}| = |S|$, (1) follows.

(2) We claim first that any $\eta \in \mathcal{E}$ is $N_G(D)_\zeta$-invariant, where $N_G(D)_\zeta$ is the inertial group of ζ in $N_G(D)$. Indeed, for any $x \in N_G(D)_\zeta$, we have $\eta^x \in \mathcal{E}$. Thus $\eta^x = \eta \otimes \lambda_x$ for a unique $\lambda_x \in \mathrm{Irr}(C_\Gamma(D)/C_G(D)) = \mathrm{Irr}(S)$. Since $[C_\Gamma(D), N_G(D)_\zeta] \leq C_G(D)$, λ_x is $N_G(D)_\zeta$-invariant. Therefore the map $x \mapsto \lambda_x$ is a group homomorphism from $N_G(D)_\zeta$ to $\mathrm{Irr}(S)$. Since this map is trivial on $C_G(D)$, it factors through $N_G(D)_\zeta/C_G(D)$. Since $(|S|, |N_G(D)_\zeta/C_G(D)|) = 1$, this map is a trivial homomorphism. Thus the claim is proved.

Now assume $B(\eta)^x = B(\eta')$ for $x \in N_\Gamma(D)$, $\eta, \eta' \in \mathcal{E}$. We may assume $x \in N_G(D)$. We have $\eta^x = \eta'$, so that $\zeta^x = \zeta$. Thus $x \in N_G(D)_\zeta$. Then $\eta = \eta'$ by the above, and (2) is proved. The proof of (i) is complete.

(ii) If $S = 1$, there is nothing to prove. So we assume $S > 1$. Let B' be the Harris–Knörr correspondent of $\hat{B}$ over B in $N_\Gamma(D)$. Then B' and $\hat{B}$ have a defect group in common. We have $N_\Gamma(D) = SN_G(D) = DC_\Gamma(D)N_G(D)$. So $N_\Gamma(D)/DC_\Gamma(D) \simeq N_G(D)/DC_G(D)$, which is a p'-group by assumption. Thus if β is a block of $DC_\Gamma(D)$ covered by B', then a defect group R of β is a defect group of B'. Hence R is a defect group of $\hat{B}$. Now $D \leq R \leq DC_\Gamma(D)$, so that $R = DC_R(D)$. The proof is complete. $\square$

REMARK 1. As in the proof of Proposition 2 of [15], Lemma 1 (i) follows from [1].

REMARK 2. In Lemma 1, the conclusions of (i) and (ii) are in fact equivalent to the equality $S = S[B]$, where $S[B]$ is defined as in Proposition 1 of [15]. A proof will be given in a separate paper.

Proof of Theorem B$'$. We may assume that either S is a cyclic p'-group or a cyclic p-group.

Assume that S is a cyclic p'-group. Let ζ be any irreducible character in B. Let T be the inertial group of ζ in Γ. Since any block of Γ covering B contains an irreducible character lying over ζ, any block in $\mathrm{BL}(\Gamma|B)$ is induced from a block in $\mathrm{BL}(T|B)$ (cf. [10, Lemma 5.3.1 (ii)]). So $|\mathrm{BL}(\Gamma|B)| \leq |\mathrm{BL}(T|B)|$. Also, $|\mathrm{BL}(T|B)| \leq k(T|\zeta) = |T/G| \leq |S|$, where $k(T|\zeta)$ denotes the number of irreduciblle characters of T lying over ζ. Thus Lemma 1 (i) yields $|T/G| = |S|$. Hence $T = \Gamma$ and ζ is S-invariant.

Assume that S is a cyclic p-group. Let $\hat{B}$ and R be as in Lemma 1 (ii). Then by Lemma 4.14 (ii) of [8], any irreducible character in B is R-invariant. On the other hand, since $\hat{B}$ is weakly regular with respect to G and B is Γ-invariant, RG/G is a Sylow p-subgroup of Γ/G by Fong's theorem. Thus $\Gamma = RG$. So any irreducible character in B is S-invariant. This completes the proof. □

3. Proof of Theorem D

Theorem D is equivalent to the following theorem, cf. Remarks of [14].

Theorem D$'$. *Let G be a p-solvable group with $O_{p'}(G) = 1$ and $k(G) = |D|$, where D is a Sylow p-subgroup of G. Then D is abelian.*

Lemma 2 (Gallagher [3]). *Let N be a normal subgroup of a group G. Then $k(G) \leq k(G/N)k(N)$ and equality holds if and only if $C_G(x \bmod N) = C_G(x)N$ for all $x \in G$. Furthermore if equality holds, then every irreducible character of N is G-invariant.*

Proof. The first statement is (3) of [3]. If equality holds, then, as shown in the proof of (3) of [3, p. 176], every conjugacy class of N is G-invariant. As is well-known, this implies that every irreducible character of N is G-invariant. □

Proof of Theorem D$'$.. Let $N = O_{p,p'}(G)$. Then, since $O_{p'}(N) = O_{p'}(G/N) = 1$, by Theorem C and Lemma 2,

$$|D| = k(G) \leq k(G/N)k(N) \leq |G/N|_p |N|_p = |D|.$$

Thus equality holds throughout. So every irreducible character of N is G-invariant by Lemma 2. Then as in the proof of Lemma 3 of [14], we see $G = N$. Thus D is normal in G.

Let Φ be the Frattini subgroup of D. Then, as in Nagao [9], $O_{p'}(G/\Phi) = 1$. Thus by Theorem C and Lemma 2,

$$|D| = k(G) \le k(G/\Phi)k(\Phi) \le |G/\Phi|_p|\Phi| = |D|.$$

Thus equality holds throughout. Let $x \in D$. By Lemma 2, we have $C_G(x \mod \Phi) = C_G(x)\Phi$. Since $D \le C_G(x \mod \Phi)$, we obtain $D \le C_G(x)\Phi$. Thus $D = C_D(x)\Phi$ and $D = C_D(x)$. Since $x \in D$ is arbitrary, D is abelian. This completes the proof. $\square$

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