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# UNKNOTTING THE SPUN $T^2$ -KNOT OF A CLASSICAL TORUS KNOT

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## Abstract

We show that for the closure of a classical braid which satisfies certain conditions, the spun  $T^2$ -knot of the classical knot has the unknotting number one. This gives an alternative proof of the fact that the spun  $T^2$ -knot of a classical torus knot has the unknotting number one.

## 0. Introduction

A *surface knot* is the image of a smooth embedding of a closed connected surface into the Euclidean 4-space  $\mathbb{R}^4$ . Kanenobu and Marumoto [10] showed that the spun 2-knot of a classical torus knot has the unknotting number one. Hence it follows that the spun  $T^2$ -knot of a classical torus knot has the unknotting number one. Here, the *spun  $T^2$ -knot* of a classical knot  $K$  is the product of  $K$  in a 3-ball  $B^3$  with a circle  $S^1$ , embedded into  $\mathbb{R}^4$  via the natural embedding of  $B^3 \times S^1$  into  $\mathbb{R}^4$  ([15, 2]). In this paper, we show that for the closure of a classical braid which satisfies certain conditions, the spun  $T^2$ -knot of the classical knot has the unknotting number one (Theorem 3.1). Theorem 3.1 gives an alternative proof of the above-mentioned fact that the spun  $T^2$ -knot of a classical torus knot has the unknotting number one (Corollary 3.2). The proof of Theorem 3.1 is shown by a diagrammatic method, by using a surface link chart presenting the spun  $T^2$ -knot.

A surface link chart is a sort of finite graph in a 2-disk with some additional data ([5, 8, 9]). Any oriented surface knot is presented by a surface link chart ([7, 8, 9]). An unknotted surface knot is presented by an unknotted chart ([5, 9]). It is known [4] that any oriented surface knot  $S$  can be deformed to an unknotted surface knot by applying 1-handle surgeries along a finite number of mutually disjoint oriented 1-handles. The *unknotting number* of  $S$  is the minimum number of such 1-handles necessary to deform  $S$  to be unknotted. A *free edge* is an edge in a chart such that the end points are vertices of degree one. Applying a 1-handle surgery to an oriented surface knot  $S$  along a nice 1-handle is presented by adding a free edge to a surface link chart presenting  $S$  ([6]).

Theorem 3.1 is shown as follows. First, we obtain a surface link chart presenting the spun  $T^2$ -knot. Then we add a free edge to the chart, and deform it to an unknotted chart (a configuration consisting of free edges) by equivalence relations. We obtain a surface link chart presenting the spun  $T^2$ -knot, as follows. We showed in [14] (see Theorem 2.2) how to obtain a surface link chart which presents a surface knot in the form of a branched covering over the standard torus. We call such a surface knot a *torus-covering knot* ([13], see Definition 2.1). Since a spun  $T^2$ -knot is a torus-covering knot, we can obtain a surface link chart presenting the spun  $T^2$ -knot by [14].

This paper is organized as follows. In Section 1, we review a braided surface and its chart description, and prepare several notations. In Section 2, we review the definition of a torus-covering knot and Theorem 2.2. In Section 3, we show Theorem 3.1 and Corollary 3.2.

### 1. A braided surface and its chart description

A braided surface was defined in [16, 7, 9]. A surface braid is a braided surface with some boundary condition, and a notion of a chart was introduced [5, 9] to present a simple surface braid. Equivalent simple surface braids have distinct chart presentations. The notion of C-move equivalence between two charts of the same degree was introduced [5, 8, 9] to give the equivalence class of the chart which represents the equivalence class of a simple surface braid. The notion of a chart can be easily extended to a chart presenting a simple braided surface. In this section, we review a braided surface, and extend the notion of a chart description to a simple braided surface. We review the fact that any oriented surface knot is presented by the closure of a simple surface braid ([7, 9]); thus it is presented by a chart. In order to present a certain chart called an “oval nest”, we introduce a notation, and we prepare several equivalence relations between oval nests.

**DEFINITION 1.1.** A compact and oriented 2-manifold  $S$  embedded in a bidisk  $D_1 \times D_2$  properly and locally flatly is called a *braided surface* of degree  $m$  if  $S$  satisfies the following conditions:

- (i)  $p_2|_S: S \rightarrow D_2$  is a branched covering map of degree  $m$ ,
- (ii)  $\partial S$  is a closed  $m$ -braid in  $D_1 \times \partial D_2$ , where  $D_1, D_2$  are 2-disks, and  $p_2: D_1 \times D_2 \rightarrow D_2$  is the projection to the second factor.

Two braided surfaces are *equivalent* if there is a fiber-preserving ambient isotopy of  $D_1 \times D_2$  rel  $D_1 \times \partial D_2$  which carries one to the other. A braided surface  $S$  is called *simple* if  $\#(S \cap p_2^{-1}(x)) = m - 1$  or  $m$  for each  $x \in D_2$ . A braided surface  $S$  is called a *surface braid* if  $\partial S$  is the trivial closed braid. A surface braid  $Q_m \times D_2$  is called *trivial*, where  $Q_m$  is a set of  $m$  interior points of  $D_1$ .

When a simple braided surface  $S$  is given, we obtain a graph on  $D_2$ , as follows. Identify  $D_1$  with  $I \times I$ , where  $I = [0, 1]$ . Consider the singular set  $\text{Sing}(p_1(S))$  of the

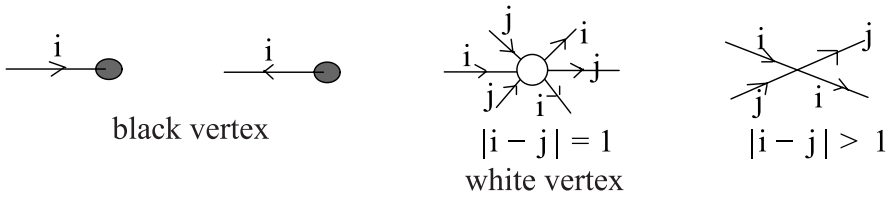


Fig. 1.1. Vertices in a chart.

image of  $S$  by the projection  $p_1$  to  $I \times D_2$ . Perturbing  $S$  if necessary, we can assume that  $\text{Sing}(p_1(S))$  consists of double point curves, triple points, and branch points. Moreover we can assume that the singular set of the image of  $\text{Sing}(p_1(S))$  by the projection to  $D_2$  consists of a finite number of double points such that the preimages belong to double point curves of  $\text{Sing}(p_1(S))$ . Thus the image of  $\text{Sing}(p_1(S))$  by the projection to  $D_2$  forms a finite graph  $\Gamma$  on  $D_2$  such that the degree of its vertex is either 1, 4 or 6. An edge of  $\Gamma$  corresponds to a double point curve, and a vertex of degree 1 (resp. 6) corresponds to a branch point (resp. triple point).

For such a graph  $\Gamma$  obtained from a simple braided surface  $S$ , we give orientations and labels to the edges of  $\Gamma$ , as follows. Let us consider a path  $\rho$  in  $D_2$  such that  $\rho \cap \Gamma$  is a point  $P$  of an edge  $e$  of  $\Gamma$ . Then  $S \cap p_2^{-1}(\rho)$  is a classical  $m$ -braid with one crossing in  $p_2^{-1}(\rho)$  such that  $P$  corresponds to the crossing of the  $m$ -braid. Let  $\sigma_1, \sigma_2, \dots, \sigma_{m-1}$  be the standard generators of the  $m$ -braid group  $B_m$ . Let  $\sigma_i^\epsilon$  ( $i \in \{1, 2, \dots, m-1\}$ ,  $\epsilon \in \{+1, -1\}$ ) be the presentation of  $S \cap p_2^{-1}(\rho)$ . Then label the edge  $e$  by  $i$ , and moreover give  $e$  an orientation such that the normal vector of  $\rho$  corresponds (resp. does not correspond) to the orientation of  $e$  if  $\epsilon = +1$  (resp.  $-1$ ). We call such an oriented and labeled graph a *chart of  $S$* .

In general, we define a chart on  $D_2$  as follows.

DEFINITION 1.2. Let  $m$  be a positive integer. A finite graph  $\Gamma$  on a 2-disk  $D_2$  is called a *chart of degree  $m$*  if it satisfies the following conditions:

- (i)  $\Gamma \cap \partial D_2$  consists of a finite number of vertices of degree 1.
- (ii) Every edge is oriented and labeled by an element of  $\{1, 2, \dots, m-1\}$ .
- (iii) Every vertex has degree 1, 4, or 6.
- (iv) The adjacent edges around each vertex in  $\text{Int}(D_2)$  are oriented and labeled as shown in Fig. 1.1, where we depict a vertex of degree 1 by a black vertex, and a vertex of degree 6 by a white vertex.

In a chart, an edge without end points is called a *loop*. An edge whose end points are black vertices is called a *free edge*. A configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an *oval nest*.

A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple braided surface presented by the chart. A chart presents a simple braided surface. In particular, a chart  $\Gamma$  such that  $\Gamma \cap \partial D_2 = \emptyset$  presents a simple surface braid.

When a chart  $\Gamma$  on  $D_2$  is given, we can reconstruct a simple braided surface  $S$  over  $D_2$  as follows. Let  $m$  be the degree of  $\Gamma$ , and let  $N(\Gamma)$  be a neighborhood of  $\Gamma$  in  $D_2$ . Let us consider a trivial braided surface  $S = Q_m \times (D_2 - N(\Gamma))$  over  $D_2 - N(\Gamma)$ , where  $Q_m$  is a set of  $m$  interior points of  $D_1$ . We extend  $S$  over a neighborhood of each edge as follows. Identify a neighborhood of an edge  $e$  with  $I \times I$  such that  $e$  is identified with  $\{1/2\} \times I$ . Let  $i$  be the label attached to  $e$ , and let  $\epsilon = +1$  (resp.  $-1$ ) if the orientation of  $e$  corresponds (resp. does not correspond) to the orientation of  $\{0\} \times I$ . Then let the braided surface  $S$  over the neighborhood of  $e$  be the braided surface which has a presentation  $\sigma_i^\epsilon \times I$  and the image of the double point curve of  $p_1(S)$  by the projection to  $D_2$  is  $e$ . Since  $\Gamma$  is as in Fig. 1.1 around each vertex,  $S$  can be extended naturally over a neighborhood of each vertex. See [3, 6, 9] for more details. Thus we can construct a simple braided surface  $S$  over  $D_2$  such that the original chart is a chart of  $S$ .

The boundary of a simple surface braid  $S$  consists of trivial closed  $m$ -braid. Consider a natural embedding of  $D_1 \times D_2$  in  $\mathbb{R}^4$ , and paste  $m$  disks to  $S$  to obtain an embedding of a closed surface in  $\mathbb{R}^4$ . The resulting surface is called the *closure* of  $S$ . It is known [7, 9] that any oriented surface knot is presented by the closure of a simple surface braid; thus it is presented by a chart  $\Gamma$  on  $D_2$  such that  $\Gamma \cap D_2 = \emptyset$ . We call such a chart presenting a surface link a *surface link chart*.

In [5, 9], a surface link chart is called simply a chart. However, in this paper we distinguish a “surface link chart” from a “chart”.

Two charts on  $D_2$  of the same degree are *C-move equivalent* if they are related by a finite sequence of ambient isotopies of  $D_2$  and C-moves (CI, CII, CIII-moves) as follows.

Let  $\Gamma$  and  $\Gamma'$  be two charts on  $D_2$  of the same degree. Then  $\Gamma'$  is said to be obtained from  $\Gamma$  (or  $\Gamma$  is said to be obtained from  $\Gamma'$ ) by a *CI-move*, *CII-move* or *CIII-move* if there exists a 2-disk  $E$  in  $D_2$  such that the loop  $\partial E$  is in general position with respect to  $\Gamma$  and  $\Gamma'$  and  $\Gamma \cap (D_2 - E) = \Gamma' \cap (D_2 - E)$  and the following condition holds: (CI) There are no black vertices in  $\Gamma \cap E$  nor  $\Gamma' \cap E$ .

A CI-move as in Fig. 1.2 is called a CI-move of type (1), (2) or (3) respectively; see [9] for the complete set of CI-moves.

(CII)  $\Gamma \cap E$  and  $\Gamma' \cap E$  are as in Fig. 1.3, where  $|i - j| > 1$ .

(CIII)  $\Gamma \cap E$  and  $\Gamma' \cap E$  are as in Fig. 1.4, where  $|i - j| = 1$ .

It is shown as a minor modification of [5, 8, 9] that two simple braided surfaces of the same degree are equivalent if and only if their charts are C-move equivalent. Two surface knots are *equivalent* if there is an ambient isotopy of  $\mathbb{R}^4$  which carries one to the other. Thus it follows that for two surface link charts of the same degree, their presenting surface knots are equivalent if the charts are C-move equivalent.

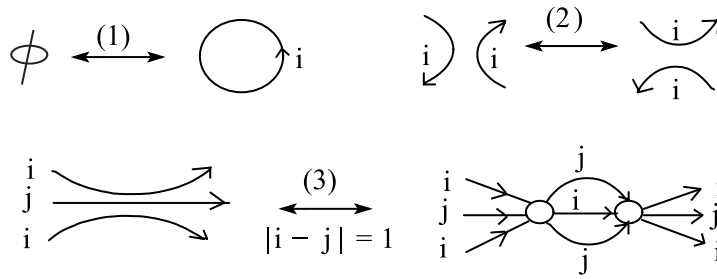


Fig. 1.2. CI-moves of types (1), (2) and (3).

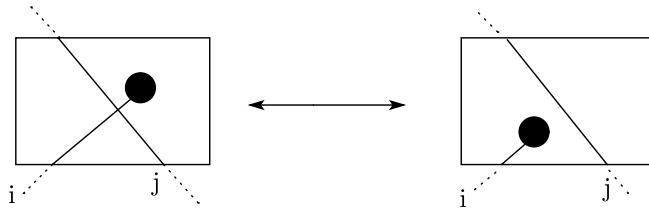


Fig. 1.3. CII-moves, where  $|i - j| > 1$ .

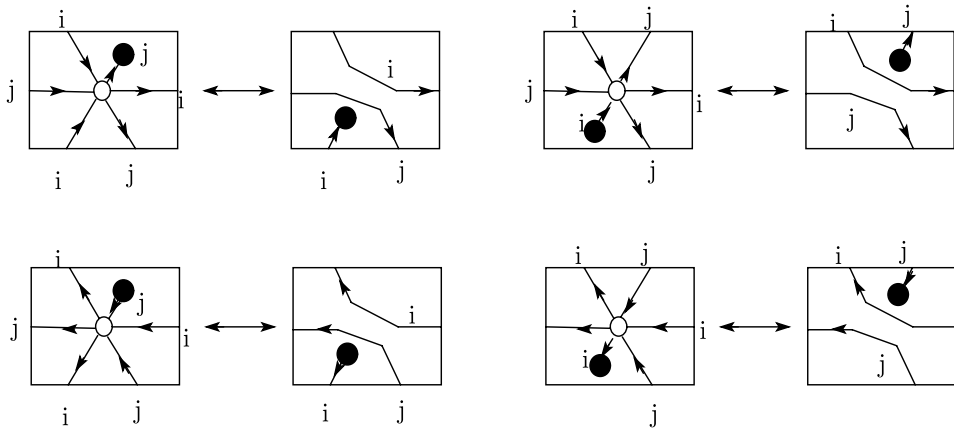


Fig. 1.4. CIII-moves, where  $|i - j| = 1$ .

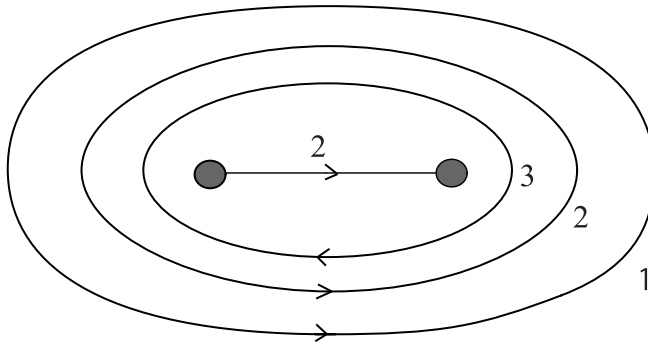


Fig. 1.5. An oval nest  $O(2; \bar{3}21)$ .

Throughout this paper, let us denote the oval nest with a free edge with the label  $i$  and its surrounding loops with the labels  $i_1, i_2, \dots, i_n$  and the orientation  $\epsilon_1, \epsilon_2, \dots, \epsilon_n$  from the free edge outward by  $O(i; i_1^* i_2^* \dots i_n^*)$ , where  $\epsilon_j = \pm 1$  and  $i_j^* = i_j$  (resp.  $\bar{i}_j$ ) if  $\epsilon_j = +1$  (resp.  $-1$ ) (see Fig. 1.5). In particular, let us denote the free edge  $O(i; \emptyset)$  by  $F_i$ . For  $0 < i < j$ , let us denote  $i(i+1) \dots j$  (resp.  $\bar{i}(\bar{i}+1) \dots \bar{j}$ ) by  $i \nearrow j$  (resp.  $\bar{i} \nearrow \bar{j}$ ), and for  $0 < j < i$ , let us denote  $i(i-1) \dots j$  (resp.  $\bar{i}(\bar{i}-1) \dots \bar{j}$ ) by  $i \searrow j$  (resp.  $\bar{i} \searrow \bar{j}$ ).

Let  $\Gamma_1$  and  $\Gamma_2$  be charts of the same degree in 2-disks  $D_1$  and  $D_2$  respectively, where  $D_i = [0, 1] \times [0, 1]$  for  $i = 1, 2$ . Identifying  $D_1$  with  $[0, 1] \times [0, 1/2]$  and  $D_2$  with  $[0, 1] \times [1/2, 1]$ , we have a new chart  $\Gamma_1 \cup \Gamma_2$  in  $D_1 \cup D_2 = [0, 1] \times [0, 1]$ . We will call it a *split union* of  $\Gamma_1$  and  $\Gamma_2$ , and use the notation  $\Gamma_1 \cup \Gamma_2$ .

Let us define the braid group relations between two sequences of integers as follows:

1.  $\emptyset \sim i \cdot \bar{i} \sim \bar{i} \cdot i$ , for a positive integer  $i$ ,
2.  $i \cdot j \sim j \cdot i$ , for positive integers  $i, j$  with  $|i - j| > 1$ ,
3.  $i \cdot j \cdot i \sim j \cdot i \cdot j$ , for positive integers  $i, j$  with  $|i - j| = 1$ .

In this paper, we will identify a braid  $\sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \dots \sigma_{i_n}^{\epsilon_n}$  with a sequence of integers  $i_1^* i_2^* \dots i_n^*$  with the braid group relations, where  $i_j^* = i_j$  (resp.  $\bar{i}_j$ ) if  $\epsilon_j = +1$  (resp.  $-1$ ). Then we have the following lemma.

**Lemma 1.3.** *For positive integers  $i, j$  and braids  $b, c$  such that  $\bar{b}ib = \bar{c}jc$ , the following oval nests are equivalent:*

$$(1.1) \quad O(i; b) \sim O(j; c).$$

Before the proof, we review the notions of a braid system of a chart and slide equivalence.

Let  $\Gamma$  be a chart of degree  $m$  on a 2-disk  $D_2$ . Let  $q_0$  be a fixed point on the boundary of  $D_2$ , and  $\Sigma(\Gamma)$  the set of black vertices in  $\Gamma$ . Let  $\mathfrak{A} = (a_1, a_2, \dots, a_n)$  be a *Hurwitz arc system* with the starting point set  $\Sigma(\Gamma)$  and the terminal point  $q_0$ , which is, for any  $i$  and  $j$ ,  $a_i \cap a_j = \{q_0\}$  and the normal vector of  $a_i$  points to  $a_{i+1}$ . For each

$i = 1, 2, \dots, n$ , consider a loop  $c_i$  in  $D_2 \setminus \Sigma(\Gamma)$  with the base point  $q_0$  such that it starts from  $q_0$  and goes along  $a_i$ , turns around the starting point of  $a_i$  (the black vertex in  $\Gamma$  which is at the other end of  $a_i$ ) anti-clockwise and comes back along  $a_i$  to  $q_0$ . Let  $\eta_i$  be the element of  $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$  represented by this loop  $c_i$ . The fundamental group is a free group of rank  $n$  generated by  $\eta_1, \eta_2, \dots, \eta_n$ . We call  $\eta_1, \eta_2, \dots, \eta_n$  the *Hurwitz generators* of  $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$  associated with  $\mathfrak{A}$ . A *braid system*  $\vec{b} = (b_1, b_2, \dots, b_n)$  of the chart  $\Gamma$  is an ordered  $n$ -tuple of elements of  $B_m$  such that each  $b_i$  is the  $m$ -braid represented by  $\eta_i$ , i.e.  $\eta_i$  in  $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$  represents the  $m$ -braid  $b_i$  in the simple surface braid of degree  $m$  which is represented by  $\Gamma$  on  $D_2$ .

Two braid systems are *slide equivalent* if we can transform one to the other by applying a finite sequence of the following equivalence relations:

$$(b_1, \dots, b_i, b_{i+1}, \dots, b_n) \sim (b_1, \dots, b_{i-1}, b_{i+1}, b_i^{-1}b_i b_{i+1}, b_{i+2}, \dots, b_n).$$

Two charts of the same degree are equivalent if and only if their braid systems are slide equivalent (see [7, Chapter 17 and Section 18.10]).

*Proof of Lemma 1.3.* We can take a braid system of  $\vec{b}$  of  $O(i; b)$  to be  $\vec{b} = (b^{-1}\sigma_i b, b^{-1}\sigma_i^{-1}b)$ . Since  $\bar{b}ib = \bar{c}jc$ , we have  $\vec{b} = (c^{-1}\sigma_j c, c^{-1}\sigma_j^{-1}c)$ , which is a braid system of  $O(j; c)$ . □

By Lemma 1.3, in particular the following equivalent deformations hold. We will prove several of them using C-moves. Let  $i, j$  be positive integers and  $b, b', c, c'$  be braids. For a positive integer  $k$ , Let  $k^* \in \{k, \bar{k}\}$ . If  $b = b'$ , then

(1.2)  $O(i; b) \sim O(i; b')$ .

(1.3)  $O(i; i^*) \sim O(i; \emptyset) = F_i$  (see Fig. 1.6),

(1.4)  $O(i; j^*) \sim O(i; \emptyset) = F_i$ , where  $|i - j| > 1$  (see Fig. 1.7),

(1.5)  $O(i; j) \sim O(j; \bar{i})$ , where  $|i - j| = 1$  (see Fig. 1.8).

If  $O(i; c) \sim O(j; c')$ , then

(1.6)  $O(i; cb) \sim O(j; c'b)$ .

Moreover, applying a CI-move of type (2) between the outermost loop labeled  $j$  of the oval nest  $O(i; b \cdot j^*)$  and the free edge  $F_j$ , we can see that

(1.7)  $O(i; b \cdot j^*) \cup F_j \sim O(i; b) \cup F_j$ ,

where  $b$  is a braid.



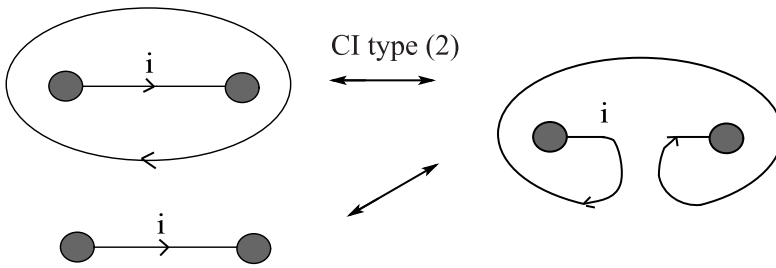


Fig. 1.6.  $O(i; \bar{i}) \sim O(i; \emptyset) = F_i$ .

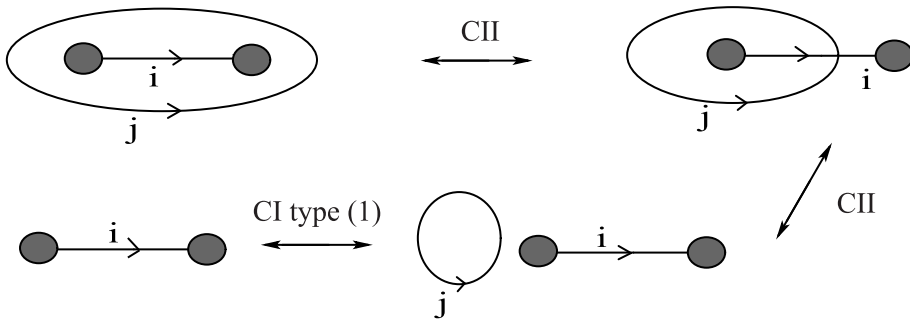


Fig. 1.7.  $O(i; j) \sim O(i; \emptyset) = F_i$ , where  $|i - j| > 1$ .

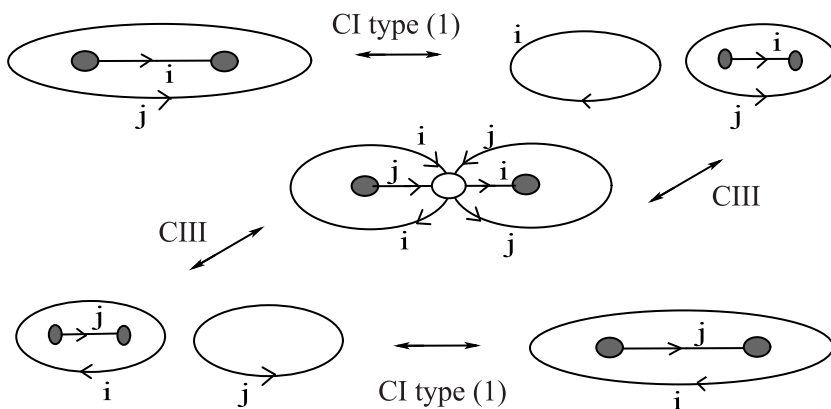


Fig. 1.8.  $O(i; j) \sim O(j; \bar{i})$ , where  $|i - j| = 1$ .

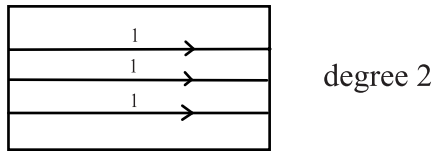


Fig. 2.1. A chart on  $T$  presenting the spun  $T^2$ -knot of a trefoil.

**2. A torus-covering knot and its chart description**

It is known [7, 9] that any oriented surface knot can be presented by a branched covering over the standard 2-sphere. A torus-covering knot was introduced in [13] as a new construction of a surface knot, by considering the standard torus instead of the standard 2-sphere. In this section, we give the definition of a torus-covering knot (see also [13]). The spun  $T^2$ -knot of a classical knot is a torus-covering knot. A torus-covering knot is presented by a chart on the standard torus  $T$ . We can obtain a surface link chart presenting a torus-covering knot from its chart on  $T$  ([14], see Theorem 2.2). Part of the obtained surface link chart is called a 1-handle chart. We obtain the 1-handle chart for the spun  $T^2$ -knot (Lemma 2.4).

Let  $T$  be a standard torus in  $\mathbb{R}^4$ , that is, the boundary of an unknotted solid torus in a 3-space in  $\mathbb{R}^4$ . Let us consider a tubular neighborhood  $N(T)$  of  $T$ , and identify  $N(T)$  with  $D^2 \times S^1 \times S^1$ , where  $D^2$  is a 2-disk, and  $S^1$  is a circle. The first  $S^1$  corresponds to the meridian, and the second  $S^1$  corresponds to the longitude of  $T$ . Let us identify  $S^1$  with  $I/\sim$ , where  $I = [0, 1]$  and  $0 \sim 1$ . For a manifold  $S$  in  $N(T)$ , let us denote by  $S \cap (D^2 \times I \times I)$  the manifold in  $D^2 \times I \times I$  obtained from  $S$  by cutting it at  $D^2 \times S^1 \times \{0\}$  and  $D^2 \times \{0\} \times S^1$ .

**DEFINITION 2.1.** A *torus-covering knot* is a surface knot  $S$  in  $\mathbb{R}^4$  such that  $S \subset N(T)$  and moreover  $S \cap (D^2 \times I \times I)$  is a simple braided surface.

By definition, a torus-covering knot  $S$  is presented by a chart on  $T$ . As we mentioned, for two charts of the same degree, their presenting braided surfaces are equivalent if the charts are C-move equivalent. Hence it follows that for two charts on  $T$  of the same degree, their presenting torus-covering knots are equivalent if the charts are C-move equivalent.

The *spun  $T^2$ -knot* of a classical knot  $K$  is the product of  $K$  in a 3-ball  $B^3$  with  $S^1$ , embedded into  $\mathbb{R}^4$  via the natural embedding of  $B^3 \times S^1$  into  $\mathbb{R}^4$  ([15, 2]). Identify  $S^1$  with the longitude of  $T$ . Since any classical knot is equivalent to a closed braid by Alexander's Theorem, the spun  $T^2$ -knot of any  $K$  is a torus-covering knot (see [13, Propositions 2.11]); see Fig. 2.1 for example.

Now we review a theorem, which shows how to obtain a surface link chart from a chart on  $T$  ([14]). A chart is presented by a simple braided surface. A simple braided surface is presented by a motion picture consisting of isotopic transformations

and hyperbolic transformations. A *motion picture* of a braided surface  $S \subset B^3 \times I$  is a one-parameter family  $\{\pi(S \cap (B^3 \times \{t\}))\}_{t \in I}$ , where  $\pi: B^3 \times I \rightarrow B^3$  is the projection (see [9]).

Let  $\{h_t\}_{t \in [0,1]}$  be an ambient isotopy of  $\mathbb{R}^3$ . For a classical link  $L$ , we have an isotopy (a one-parameter family)  $\{h_t(L)\}$  of classical links. We say that  $h_1(L)$  is obtained from  $L$  by an *isotopic transformation*, and we use the notation that  $L \rightarrow h_1(L)$  is an isotopic transformation (see [9, Section 9.1]).

Let  $L$  be a classical link in  $\mathbb{R}^3$ . A 2-disk  $B$  in  $\mathbb{R}^3$  is called a *band* attaching to  $L$  if  $L \cap B$  is a pair of disjoint arcs in  $\partial B$ . A *band set* attaching to  $L$  is a union  $\mathcal{B} = B_1 \cup B_2 \cup \dots \cup B_m$  of mutually disjoint bands  $B_1, B_2, \dots, B_m$  attaching to  $L$ . For a subset  $X$  of a space, let us denote by  $\text{Cl}(X)$  the closure of  $X$ . Define a link  $h(L; \mathcal{B})$  by

$$h(L; \mathcal{B}) = \text{Cl}((L \cup \partial \mathcal{B}) - (L \cap \mathcal{B})).$$

We say that the link  $h(L; \mathcal{B})$  is obtained from  $L$  by a *hyperbolic transformation* along  $\mathcal{B}$ , and we use the notation that  $L \rightarrow h(L; \mathcal{B})$  is a hyperbolic transformation (see [9, Section 9.1]).

For a classical  $m$ -braid  $c$ , let  $\iota_k^l(c)$  be the  $(m + k + l)$ -braid obtained from  $c$  by adding  $k$  (resp.  $l$ ) trivial strings before (resp. after)  $c$ , and put

$$\begin{aligned} \Pi_i^m &= \sigma_{m+1} \sigma_{m+2} \cdots \sigma_{m+i}, & \Pi_i^{\prime m} &= \sigma_{m-1} \sigma_{m-2} \cdots \sigma_{m-i}, \\ \Delta_m &= \Pi_{m-1}^m \Pi_{m-2}^m \cdots \Pi_1^m, & \Delta'_m &= \Pi_{m-1}^{\prime m} \Pi_{m-2}^{\prime m} \cdots \Pi_1^{\prime m}, \\ \Theta_m &= \sigma_m \cdot \Pi_{m-1}^{\prime m} \cdot \Pi_{m-1}^m \cdot \sigma_m \cdot \Pi_{m-2}^{\prime m} \cdot \Pi_{m-2}^m \cdots \sigma_m \cdot \Pi_1^{\prime m} \cdot \Pi_1^m \cdot \sigma_m. \end{aligned}$$

**Theorem 2.2** ([14]). *Let  $\Gamma_T$  be a chart of degree  $m$  on  $I \times I$ , obtained from a chart on  $T$  (of degree  $m$ ) by cutting  $T$  by the meridian and the longitude. Let  $a$  (resp.  $b$ ) be a classical  $m$ -braid presented by  $\Gamma_T \cap (I \times \{0\})$  (resp.  $\Gamma_T \cap (\{0\} \times I)$ ). Then the torus-covering knot presented by  $\Gamma_T$  is presented by a surface link chart  $\Gamma_S$  of degree  $2m$  as in Fig. 2.2. Here  $H_b$  is a chart of degree  $2m$  presenting the simple braided surface whose motion picture is as follows:*

$$\begin{aligned} \iota_0^m(b) &\rightarrow \iota_0^m(b) \cdot (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \xrightarrow{\quad} \iota_0^m(b) \cdot (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \\ &\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \iota_0^m(\bar{b}^*) \cdot \Theta_m \rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \cdot \iota_0^0(\bar{b}^*) \\ &\xrightarrow{\quad} (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \cdot \iota_0^0(\bar{b}^*) \rightarrow \iota_0^0(\bar{b}^*), \end{aligned}$$

where  $\rightarrow$  is an isotopic transformation and  $\xrightarrow{\quad}$  is a hyperbolic transformation along bands corresponding to the  $m$   $\sigma_m$ 's, and  $-(H_b)^*$  is the orientation-reversed mirror image of  $H_b$ , and  $\bar{b}^*$  is the  $m$ -braid obtained from the classical  $m$ -braid  $b$  by taking its mirror image and reversing all the crossings.

**DEFINITION 2.3.** We call  $H_b$  the 1-handle chart of  $\Gamma_T$ .

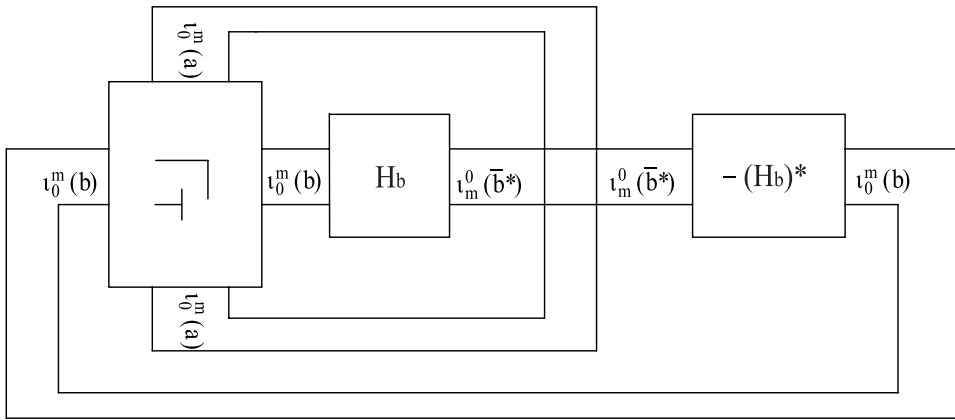


Fig. 2.2. The surface link chart  $\Gamma_S$  of degree  $2m$ .

Let us consider the spun  $T^2$ -knot of  $\hat{b}$ , where  $\hat{b}$  denotes the closure of a classical braid  $b$ . Let us determine  $\Gamma_T$  on  $I \times I$  to be a chart presenting the braided surface  $b \times I$ ; then the braids presented by  $\Gamma_T \cap (I \times \{0\})$  and  $\Gamma_T \cap (\{0\} \times I)$  are  $b$  and  $e$  respectively, where  $e$  is the trivial braid. The 1-handle chart of  $\Gamma_T$  is  $H_e$ . We obtain  $H_e$ , as follows.

**Lemma 2.4.** *Let  $e$  be the trivial  $m$ -braid. Then the 1-handle chart  $H_e$  is equivalent to the chart as follows:*

$$H_e \sim \bigcup_{k=0}^{m-1} O_k,$$

where  $O_k$  is the oval nest

$$O_k = O\left(m; \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \prod_{j=0}^{k-1} (m+1 \nearrow m+k-j)\right)$$

for  $k = 0, 1, 2, \dots, m-1$ . Note that for  $k = 0$ ,  $O_0 = O(m; \emptyset) = F_m$ .

**Proof.** By Theorem 2.2,  $H_e$  is a chart presenting the simple braided surface as follows:

$$\begin{aligned}
 (2.1) \quad e &\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \\
 &\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \\
 &\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \rightarrow e,
 \end{aligned}$$

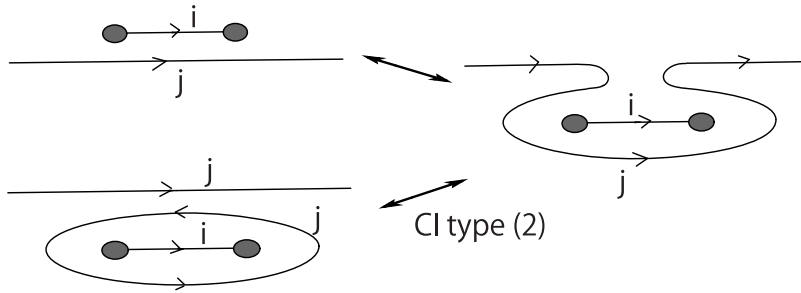


Fig. 2.3. Moving a free edge across an edge.

where  $\rightarrow$  means an isotopy transformation and  $\dot{\rightarrow}$  means a hyperbolic transformation along bands corresponding to the  $m$   $\sigma_m$ 's. Here  $e$  is the trivial  $2m$ -braid. Note that since (2.1) presents a simple surface braid,  $H_e$  does not have a boundary. The 1-handle chart  $H_e$  has  $m$  free edges, whose labels are all  $m$ . All the other edges have labels other than  $m$  and neither of them is connected with a black vertex. Draw  $H_e$  on  $[0, 1/2] \times [0, 1]$  such that we can read the braids  $e$ ,  $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m$ ,  $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$ ,  $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m$  and  $e$  of (2.1) at  $[0, 1/2] \times \{t_1\}, \dots, [0, 1/2] \times \{t_5\}$  respectively, where  $0 < t_1 < \dots < t_5 < 1$ . Let  $q_k$  ( $k = 0, 1, \dots, m - 1$ ) be the black vertex corresponding to the  $(m - k)$ -th  $\sigma_m$  of the braid  $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$ .

Let us denote by  $F_k$  the free edge connected with the black vertex  $q_k$ . Let us move the free edges into  $[1/2, 1] \times [0, 1]$  using CI-moves of type (2) as in Fig. 2.3 to be a split sum of oval nests and  $H'_e$ , where  $H'_e$  is the chart  $H_e - (\bigcup_{k=0}^{m-1} F_k)$ . Since  $H'_e$  has no black vertices, we can eliminate it by a CI-move. Thus we have a split sum of oval nests.

Let  $\tilde{O}_k$  ( $k = 0, 1, \dots, m - 1$ ) be the oval nest  $F_k$  becomes. We will see that each  $\tilde{O}_k$  is equivalent to the oval nest  $O_k$ . First we will obtain  $\tilde{O}_k$ . It suffices to see what edges  $F_k$  crosses as it moves into  $[1/2, 1] \times [0, 1]$ . We have

$$\Theta_m = \sigma_m \cdot \Pi_{m-1}^m \cdot \Pi_{m-1}^m \cdot \sigma_m \cdot \Pi_{m-2}^m \cdot \Pi_{m-2}^m \cdot \dots \cdot \sigma_m \cdot \Pi_1^m \cdot \Pi_1^m \cdot \sigma_m.$$

The first free edge  $F_0$  does not cross any edge. Hence  $\tilde{O}_0 = F_0$ . Then the second free edge  $F_1$  crosses edges representing  $\Pi_1^m \cdot \Pi_1^m = \sigma_{m-1} \sigma_{m+1}$ , so it becomes the oval nest  $\tilde{O}_1 = O(m; m - 1m + 1)$ . The third free edge  $F_2$  crosses edges representing  $\Pi_2^m \cdot \Pi_1^m \cdot \Pi_1^m = (\sigma_{m-1} \sigma_{m-2})(\sigma_{m+1} \sigma_{m+2}) \sigma_{m-1} \sigma_{m+1}$ . Hence it becomes the oval nest  $\tilde{O}_2 = O(m; (m - 1)(m - 2) \cdot (m + 1)(m + 2) \cdot (m - 1) \cdot (m + 1))$ . Repeating this step, we see that in general  $F_k$  crosses edges representing  $\Pi_k^m \cdot \Pi_k^m \cdot \Pi_{k-1}^m \cdot \Pi_{k-1}^m \cdot \dots \cdot \Pi_1^m \cdot \Pi_1^m$ , so it becomes an oval nest  $\tilde{O}_k = O(m; \prod_{j=0}^{k-1} ((m - 1 \searrow m - k + j) \cdot (m + 1 \nearrow m + k - j)))$  for  $k = 0, 1, \dots, m - 1$ .

We can show that if  $i + 1 < k$  then  $(i \searrow j)(k \nearrow l)$  can be transformed to  $(k \searrow l)(i \nearrow j)$  by the braid group relation 2, i.e.

$$(2.2) \quad (i \searrow j)(k \nearrow l) \sim (k \searrow l)(i \nearrow j).$$

Using (2.2), we see that  $\tilde{O}_k \sim O(m; \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \prod_{j=0}^{k-1} (m+1 \nearrow m+k-j))$ , which is  $O_k$ . □

### 3. Main theorem

An oriented surface knot is *unknotted* if it is equivalent to the connected sum of several standard tori. It is known [4] that any oriented surface knot  $S$  can be deformed to an unknotted surface knot by applying a finite number of oriented 1-handle surgeries. The *unknotting number* of  $S$  is the minimum number of oriented 1-handle surgeries necessary to deform it to be unknotted. In this section, we show Theorem 3.1 and Corollary 3.2.

**Theorem 3.1.** *Let  $S$  be the spun  $T^2$ -knot of a classical knot  $\hat{b}$ , where  $b$  is a classical  $m$ -braid ( $m > 1$ ) such that there exists a permutation  $\tau$  of degree  $m - 1$  which satisfies the following conditions:*

- (a1) *There is an integer  $r \in \{1, 2, \dots, m - 1\}$  such that for each  $k \in \{1, 2, \dots, m - 1\} - \{r\}$ ,  $\sigma_k \cdot b = b \cdot \sigma_{\tau(k)}$ , and*
- (a2) *For each  $i, j \in \{1, 2, \dots, m - 1\}$ , if  $i \neq j$ , then  $\tau^i(1) \neq \tau^j(1)$ . Note that then  $\tau^{m-1}(1) = 1$ .*

*Moreover assume that  $S$  is not unknotted. Then the unknotting number of  $S$  is one.*

By Theorem 3.1 we have an alternative proof of the fact [10] that the spun  $T^2$ -knot of a torus  $(p, q)$ -knot has the unknotting number one.

**Corollary 3.2.** *The spun  $T^2$ -knot of a classical torus  $(p, q)$ -knot has the unknotting number one.*

*Proof.* First we show that the spun  $T^2$ -knot is not unknotted. The knot group of the spun  $T^2$ -knot of a classical torus  $(p, q)$ -knot is isomorphic to the knot group of the classical torus  $(p, q)$ -knot ([15]). Hence we can see that the spun  $T^2$ -knot is not unknotted.

We determine the braid  $b$  and the permutation  $\tau$ , as follows. A classical torus  $(p, q)$ -knot is presented by the closure of the  $p$ -braid  $b = (\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q$ , where  $p$  and  $q$  are coprime integers and moreover  $p > 1$ . Let  $r$  be defined by  $q \bmod p$  such that  $r \in \{0, 1, 2, \dots, p - 1\}$ . Since  $p$  and  $q$  are coprime,  $r \neq 0$  and it follows that  $r \in \{1, 2, \dots, p - 1\}$ . Let us define a permutation  $\tau$  of degree  $p - 1$  by

$$\left( \begin{array}{cccccccc} 1 & 2 & \cdots & r-1 & r & r+1 & r+2 & \cdots & p-1 \\ p-r+1 & p-r+2 & \cdots & p-1 & p-r & 1 & 2 & \cdots & p-r-1 \end{array} \right).$$

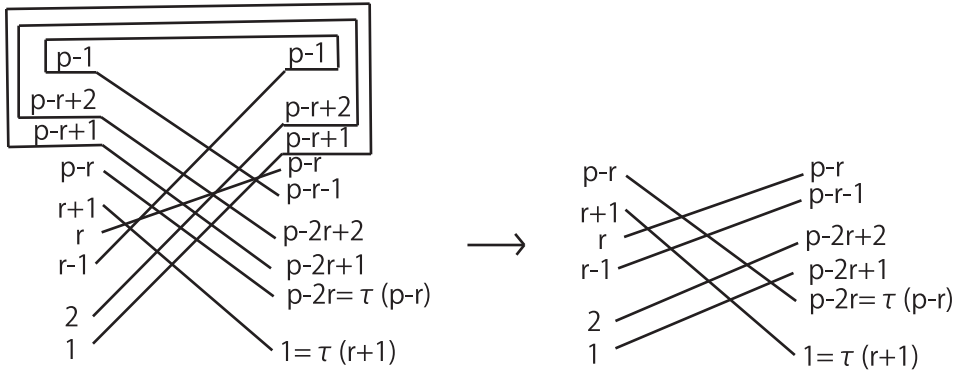


Fig. 3.1. The braid associated with  $\tau$  if  $r - 1 < p - r - 1$ .

We show that Condition (a1) of Theorem 3.1 holds, as follows. If  $k \neq 1$ , then we can show that  $\sigma_k(\sigma_1\sigma_2 \cdots \sigma_{p-1}) = (\sigma_1\sigma_2 \cdots \sigma_{p-1})\sigma_{k-1}$ . Similarly we have  $\sigma_1(\sigma_1\sigma_2 \cdots \sigma_{p-1})^2 = (\sigma_1\sigma_2 \cdots \sigma_{p-1})^2\sigma_{p-1}$ . From these two equations, we have  $\sigma_k \cdot b = b \cdot \sigma_{\tau(k)}$  for each  $k \in \{1, 2, \dots, p-1\} - \{r\}$ . Thus Condition (a1) holds.

Next we will show that  $\tau$  satisfies Condition (a2) of Theorem 3.1. Let us define Condition (a2)' as follows.

(a2)' The permutation  $\tau$  is associated with a classical braid  $c$  such that  $\hat{c}$  is a knot, i.e.  $\hat{c}$  is connected.

We can see that if the permutation  $\tau$  satisfies (a2)', then (a2) holds, as follows. If Condition (a2) does not hold, then  $\tau^i(1) = \tau^j(1)$  for some  $i, j \in \{1, 2, \dots, p-1\}$  with  $i \neq j$ . We can assume that  $j > i$ . Then we have  $\tau^{j-i}(1) = 1$ , where  $0 < j-i < p-1$ . On the other hand, if  $\tau$  is associated with a classical braid  $c$  such that  $\hat{c}$  is a knot, then  $\tau^k(1) \neq 1$  for any  $k$  with  $0 < k < p-1$ . This is a contradiction.

From now on we will show that  $\tau$  satisfies (a2)'. Since  $r \in \{1, 2, \dots, p-1\}$  with  $r = q \pmod p$ , and  $p$  and  $q$  are coprime integers, we see that  $r = 1$  or  $p$  and  $r$  are coprime. If  $r - 1 = p - r - 1$ , then  $p = 2r$ . Since  $r = 1$  or  $p$  and  $r$  are coprime, we have  $r = 1$  and  $p = 2$ . Then  $\tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , which is associated with the trivial braid of degree one, whose associated closed braid is a trivial knot. Hence we can assume that  $r - 1 \neq p - r - 1$ . If  $r - 1 < p - r - 1$ , then the permutation  $\tau$  is associated with a braid whose diagram is as in Fig. 3.1, where we omit the crossing information. Here we have  $\tau(r - j) = p - j$  for  $j = 1, 2, \dots, r - 1$  and  $\tau(p - j) = p - r - j$  for  $j = 1, 2, \dots, p - r - 1$ . Hence we have  $\tau^2(r - j) = \tau(p - j) = p - r - j$  for  $j = 1, 2, \dots, r - 1$ , which means that the  $(r - j)$ -th string of the closed braid is connected with the  $(p - j)$ -th string, which is connected with the  $(p - r - j)$ -th string. Hence we can assume that there is no  $(p - j)$ -th string, and the  $(r - j)$ -th string of the closed braid is connected with the  $(p - r - j)$ -th string, where  $j = 1, 2, \dots, r - 1$  (see Fig. 3.1).

Thus it suffices to show that the following permutation satisfies (a2)′:

$$(3.1) \quad \left( \begin{array}{cccccccc} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & p-r \\ p-2r+1 & p-2r+2 & \cdots & p-r & 1 & 2 & \cdots & p-2r \end{array} \right).$$

Similarly, if  $r-1 > p-r-1$ , then we have  $\tau^2(r-j) = \tau(p-j) = p-r-j$  for  $j = 1, 2, \dots, p-r-1$ . Hence it suffices to show that the following permutation satisfies (a2)′:

$$(3.2) \quad \left( \begin{array}{cccccccc} 1 & 2 & \cdots & 2r-p & 2r-p+1 & 2r-p+1 & \cdots & r \\ p-r+1 & p-r+2 & \cdots & r & 1 & 2 & \cdots & p-r \end{array} \right).$$

If  $r-1 < p-r-1$  (resp.  $r-1 > p-r-1$ ), then  $p-r > r$  (resp.  $r > p-r$ ). Hence together with  $1 \leq r \leq p-1$ , we can see that  $p-r > 1$  (resp.  $r > 1$ ). Thus the permutation (3.1) (resp. (3.2)) is associated with the  $m$ -braid  $c = (\sigma_1\sigma_2 \cdots \sigma_{m-1})^n$ , where  $m = p-r$  (resp.  $r$ ) is the degree of (3.1) (resp. (3.2)) with  $m > 1$ , and  $n = m - \tau(1) + 1$ . Since for (3.1) (resp. (3.2)) we have  $m = p-r$  (resp.  $r$ ) and  $\tau(1) = p-2r+1$  (resp.  $p-r+1$ ), it follows that  $(m, n) = (p-r, r)$  (resp.  $(m, n) = (r, 2r-p)$ ). Note that in both cases  $n > 0$ . Since  $r = 1$  or  $p$  and  $r$  are coprime, together with  $m > 1$  and  $n > 0$ , it follows that in both cases  $n = 1$  or  $m$  and  $n$  are coprime. If  $n = 1$ , then  $\hat{c}$  ( $c = \sigma_1\sigma_2 \cdots \sigma_{m-1}$ ) is a trivial knot, and if  $m$  and  $n$  are coprime, then  $\hat{c}$  ( $c = (\sigma_1\sigma_2 \cdots \sigma_{m-1})^n$ ) is a torus  $(m, n)$ -knot. Thus  $\tau$  satisfies (a2)′, and it follows that  $\tau$  satisfies (a2). Therefore the spun  $T^2$ -knot has the unknotting number one by Theorem 3.1. □

Proof of Theorem 3.1. We show that the unknotting number of  $S$  is one. Let  $\Gamma_S$  be a surface link chart presenting  $S$ . An *unknotted chart* is a chart presented by a configuration consisting of free edges ([5]). An unknotted oriented surface knot is presented by an unknotted chart ([5]). For an oriented surface knot, adding a free edge to the surface link chart corresponds to a nice 1-handle surgery, which is an oriented 1-handle surgery ([6]). Thus it suffices to see that the surface link chart obtained from  $\Gamma_S$  by adding a free edge is equivalent to an unknotted chart.

We will determine  $\Gamma_S$  by [14] (see Theorem 2.2). The chart  $\Gamma_T$  on  $I \times I$  presents the braided surface  $b \times I$ ; thus the braids presented by  $\Gamma_T \cap (I \times \{0\})$  and  $\Gamma_T \cap (\{0\} \times I)$  are  $b$  and  $e$  respectively (see Section 2). By Lemma 2.4 and (3.26) of Lemma 3.3, we can assume that the 1-handle chart  $H_e$  is as follows:

$$H_e = \bigcup_{k=0}^{m-1} O_k,$$

where

$$(3.3) \quad O_k = O(m-k; \overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k).$$



Let us define and  $O'_k$  as follows:

$$(3.4) \quad O'_k = O(m - k; \overline{m - k + 1} \nearrow \overline{m})(m + 1 \nearrow m + k) \cdot b.$$

The oval nest  $O'_k$  is obtained from  $O_k$  by adding loops describing  $b$  around it. By [14] (see Theorem 2.2), the surface link chart  $\Gamma_S$  obtained from  $\Gamma_T$  is as follows:

$$(3.5) \quad \Gamma_S = \bigcup_{i=0}^{m-1} O_i \cup \bigcup_{i=0}^{m-1} O'_i.$$

Remark that  $\Gamma_S$  is a ribbon chart of degree  $2m$  (see [5, 9]).

We will show that the surface link chart  $\Gamma_S$  can be deformed to an unknotted chart by adding a free edge.

STEP 1. We show that

$$O_{m-k-1} \cup O_{m-k} \cup F_{2m-k} \sim O_{m-k-1} \cup F_k \cup F_{2m-k},$$

for  $k \in \{1, 2, \dots, m - 1\}$ .

By (3.3) and (1.7), we have

$$\begin{aligned} &O_{m-k} \cup F_{2m-k} \\ &= O(k; \overline{k + 1} \nearrow \overline{m})(m + 1 \nearrow 2m - k) \cup F_{2m-k} \\ &\sim O(k; \overline{k + 1} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1) \cup F_{2m-k}. \end{aligned}$$

Let us denote  $O(k; \overline{k + 1} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1)$  by  $\tilde{O}_{m-k}$ . By (1.7) we have

$$O(k + 1; \emptyset) \cup O(k; \overline{k + 1}) \sim O(k + 1; \emptyset) \cup O(k; \emptyset).$$

Hence we have

$$(3.6) \quad O(k + 1; c) \cup O(k; \overline{k + 1} \cdot c) \sim O(k + 1; c) \cup O(k; c)$$

for a braid  $c$  by (1.6). By (3.3) and (3.6) we have

$$\begin{aligned} &O_{m-k-1} \cup \tilde{O}_{m-k} \\ &= O(k + 1; \overline{k + 2} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1) \\ &\quad \cup O(k; \overline{k + 1}) \cdot (\overline{k + 2} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1) \\ &\sim O(k + 1; \overline{k + 2} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1) \\ &\quad \cup O(k; \overline{k + 2} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1) \\ &= O_{m-k-1} \cup O(k; \overline{k + 2} \nearrow \overline{m})(m + 1 \nearrow 2m - k - 1). \end{aligned}$$

By (1.4) we see that

$$O(k; \overline{(k+2 \nearrow \overline{m})}(m+1 \nearrow 2m-k-1)) \sim O(k; \emptyset) = F_k.$$

Thus we have

$$O_{m-k-1} \cup O_{m-k} \cup F_{2m-k} \sim O_{m-k-1} \cup F_k \cup F_{2m-k}.$$

STEP 2. Similarly, we show that

$$O'_{m-k-1} \cup O'_{m-k} \cup F_{\tau(k)} \sim O'_{m-k-1} \cup F_{2m-k} \cup F_{\tau(k)},$$

for  $k \in \{1, 2, \dots, m-1\} - \{r\}$ .

By (a1),  $b^{-1}\sigma_k b = \sigma_{\tau(k)}$  for  $k \in \{1, 2, \dots, m-1\} - \{r\}$ . Hence we have

$$(3.7) \quad O(k; b) \sim F_{\tau(k)}$$

for  $k \in \{1, 2, \dots, m-1\} - \{r\}$  by Lemma 1.3.

Similarly to Step 1, using (3.27) of Lemma 3.3, we have

$$(3.8) \quad O_{m-k-1} \cup O_{m-k} \cup F_k \sim O_{m-k-1} \cup F_k \cup F_{2m-k}$$

for  $k \in \{1, 2, \dots, m-1\}$ . By (3.8) and (1.6), we have

$$(3.9) \quad O'_{m-k-1} \cup O'_{m-k} \cup O(k; b) \sim O'_{m-k-1} \cup O(k; b) \cup O(2m-k; b).$$

By (3.7),  $O(k; b) \sim F_{\tau(k)}$  for  $k \in \{1, 2, \dots, m-1\} - \{r\}$ . On the other hand, by (1.4) and  $2m-k > (m-1) + 1$ , we have  $O(2m-k; b) \sim O(2m-k; \emptyset) = F_{2m-k}$ . Hence together with (3.9), we see that

$$O'_{m-k-1} \cup O'_{m-k} \cup F_{\tau(k)} \sim O'_{m-k-1} \cup F_{2m-k} \cup F_{\tau(k)}$$

for  $k \in \{1, 2, \dots, m-1\} - \{r\}$ .

STEP 3. Let us denote Step 1 as follows:

$$\phi_l: O_{l-1} \cup O_l \cup F_{m+l} \rightarrow O_{l-1} \cup F_{m-l} \cup F_{m+l}$$

for  $l \in \{1, 2, \dots, m-1\}$ , and Step 2 as

$$\psi_l: O'_{l-1} \cup O'_l \cup F_{\tau(m-l)} \rightarrow O'_{l-1} \cup F_{m+l} \cup F_{\tau(m-l)}$$

for  $l \in \{1, 2, \dots, m-1\} - \{m-r\}$ .

We introduce several notations to make things easy to see. Let us define  $F^l$ ,  $F'^l$  and  $F''^l$  as follows:

$$(3.10) \quad F^l := F_{m-l},$$

$$(3.11) \quad F'^l := F_{m+l},$$

$$(3.12) \quad F''^l := F_{\tau(m-l)}$$

for  $l \in \{1, 2, \dots, m - 1\}$ . Moreover, for an integer  $s$ , let us define  $\tau_s$  to be

$$(3.13) \quad \tau_s := m - \tau^{-s}(r).$$

Step 1 is written as follows:

$$(3.14) \quad \phi_l: O_{l-1} \cup O_l \cup F^l \rightarrow O_{l-1} \cup F^l \cup F'^l,$$

for  $l \in \{1, 2, \dots, m - 1\}$ , and Step 2 is

$$(3.15) \quad \psi_l: O'_{l-1} \cup O'_l \cup F''^l \rightarrow O'_{l-1} \cup F'^l \cup F''^l,$$

for  $l \in \{1, 2, \dots, m - 1\} - \{m - r\}$ . Since by definition (3.13)  $m - r = \tau_0$ , Step 2 holds true for  $l \in \{1, 2, \dots, m - 1\} - \{\tau_0\}$ .

From now on we show that  $\Gamma_S$  can be deformed to an unknotted chart by adding a free edge  $F_r$ . Let us define charts  $I_0, I_1, \dots, I_{2m-4}$  of degree  $2m$ . First, define  $I_0$  as follows:

$$I_0 := \Gamma_S \cup F_r,$$

which is by (3.5) as follows:

$$(3.16) \quad \begin{aligned} I_0 = & O_0 \cup O_{\tau_1} \cup O_{\tau_2} \cup \dots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \\ & \cup O'_0 \cup O'_{\tau_1} \cup O'_{\tau_2} \cup \dots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \\ & \cup F_r. \end{aligned}$$

Note that by (a2),  $\{\tau_0, \tau_1, \dots, \tau_{m-2}\} = \{1, 2, \dots, m - 1\}$ . For  $n = 1, 2, \dots, m - 2$ , let us define  $I_{2n}$  as follows:

$$(3.17) \quad \begin{aligned} I_{2n} := & O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \dots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \cup F^{\tau_1} \cup F^{\tau_2} \cup \dots \cup F^{\tau_n} \\ & \cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \dots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \cup F'^{\tau_1} \cup F'^{\tau_2} \cup \dots \cup F'^{\tau_n} \\ & \cup F_r. \end{aligned}$$

And for  $n = 0, 1, 2, \dots, m - 3$ , let us define  $I_{2n+1}$  as follows:

$$(3.18) \quad I_{2n+1} := (I_{2n} - O'_{\tau_{n+1}}) \cup F'^{\tau_{n+1}}.$$

We will show that  $I_{2n+1}$  (resp.  $I_{2n+2}$ ) is obtained from  $I_{2n}$  (resp.  $I_{2n+1}$ ) by applying Steps 2 (resp. Steps 1) for  $n = 0, 1, \dots, m - 3$ .

When we have  $I_{2n}$  ( $n = 0, 1, \dots, m - 3$ ), there is an integer  $l_0 < \tau_{n+1}$  such that for any  $l$  with  $l_0 < l < \tau_{n+1}$ ,  $O'_l \not\subset I_{2n}$  and  $O'_{l_0} \subset I_{2n}$ . Note that such an  $l_0$  exists, for  $0 < \tau_{n+1}$  and  $O'_0 \subset I_{2n}$  for every  $n \in \{0, 1, \dots, m - 3\}$ . Since  $r = \tau^0(r) = \tau(m - (m - \tau^{-1}(r))) = \tau(m - \tau_1)$ , by the definition of  $F'^l$  (3.12) we have

$$(3.19) \quad F_r = F'^{\tau_1}.$$

For  $n = 0$ , by (3.16) we have  $l_0 = \tau_1 - 1$ , and by (3.19) we see that

$$(3.20) \quad I_0 \supset F'^{\tau_1} \cup O'_{\tau_1-1} \cup O'_{\tau_1}.$$

By the definitions (3.10) and (3.13), we have

$$F_{\tau^{-s}(r)} = F_{m-(m-\tau^{-s}(r))} = F^{m-\tau^{-s}(r)} = F^{\tau_s},$$

and by the definitions (3.12) and (3.13), we have

$$F_{\tau^{-s}(r)} = F_{\tau(\tau^{-(s+1)}(r))} = F'^{m-\tau^{-(s+1)}(r)} = F'^{\tau_{s+1}}.$$

Hence we have

$$(3.21) \quad F^{\tau_s} = F'^{\tau_{s+1}},$$

for each  $s$ . By the definition of  $I_{2n}$  (3.17) and (3.21) we have

$$(3.22) \quad \begin{aligned} I_{2n} = & O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \dots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \cup F'^{\tau_2} \cup F'^{\tau_3} \cup \dots \cup F'^{\tau_{n+1}} \\ & \cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \dots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \cup F'^{\tau_1} \cup F'^{\tau_2} \cup \dots \cup F'^{\tau_n} \\ & \cup F_r \end{aligned}$$

for  $n = 1, 2, \dots, m - 3$ . By (3.22) and (3.19), we can see that if  $F'^l \subset I_{2n}$ , then  $F'^l \subset I_{2n}$ . So together with (3.20), we have

$$(3.23) \quad \begin{aligned} I_{2n} \supset & F'^{l_0+1} \cup F'^{l_0+2} \cup \dots \cup F'^{\tau_{n+1}-1} \cup F'^{\tau_{n+1}} \\ & \cup O'_{l_0} \cup F'^{l_0+1} \cup F'^{l_0+2} \cup \dots \cup F'^{\tau_{n+1}-1} \cup O'_{\tau_{n+1}} \end{aligned}$$

for  $n = 0, 1, \dots, m - 3$ . By (3.22), we can see that if  $F'^l \subset I_{2n}$ , then  $l \in \{\tau_1, \tau_2, \dots, \tau_n\}$ . Hence  $l_0 + 1, l_0 + 2, \dots, \tau_{n+1} - 1 \in \{\tau_1, \tau_2, \dots, \tau_n\}$ . By (a2) and  $n \leq m - 3$ , none of  $l_0 + 1, l_0 + 2, \dots, \tau_{n+1} - 1, \tau_{n+1}$  is  $\tau_0$ . So we can apply Steps 2 (3.15) and its inverses to  $I_{2n}$  to deform  $O'_{\tau_{n+1}}$  to  $F'^{\tau_{n+1}}$ . The result is  $I_{2n+1}$  by the definition of  $I_{2n+1}$  (3.18):

$$(3.24) \quad \psi_{l_0+1} \circ \dots \circ \psi_{\tau_{n+1}-1} \circ \psi_{\tau_{n+1}}^{-1} \circ \psi_{\tau_{n+1}-1}^{-1} \circ \dots \circ \psi_{l_0+2}^{-1} \circ \psi_{l_0+1}^{-1}(I_{2n}) = I_{2n+1}$$

for  $n = 0, 1, \dots, m - 3$ .

By (3.16) and (3.17), we see that if  $O'_l \subset I_{2n}$ , then  $O_l \subset I_{2n}$ , and if  $F'^l \subset I_{2n}$ , then  $F^l \subset I_{2n}$ . Hence, by the definition of  $I_{2n+1}$  (3.18),

$$I_{2n+1} \supset O_{l_0} \cup F^{l_0+1} \cup F^{l_0+2} \cup \dots \cup F^{\tau_{n+1}-1} \cup O_{\tau_{n+1}} \\ \cup F'^{l_0+1} \cup F'^{l_0+2} \cup \dots \cup F'^{\tau_{n+1}}$$

for  $n = 0, 1, \dots, m - 3$ , where  $l_0$  is the same integer used in deforming  $I_{2n}$  to  $I_{2n+1}$ . And by the definitions (3.16), (3.17) and (3.18) we have

$$I_{2n+2} = (I_{2n} - O'_{\tau_{n+1}} - O_{\tau_{n+1}}) \cup F'^{\tau_{n+1}} \cup F^{\tau_{n+1}} \\ = (I_{2n+1} - O_{\tau_{n+1}}) \cup F^{\tau_{n+1}}$$

for  $n = 0, 1, \dots, m - 3$ . Similarly to (3.24), we can deform  $I_{2n+1}$  to  $I_{2n+2}$  by applying Steps 1 (3.14) and its inverses and deforming  $O_{\tau_{n+1}}$  to  $F^{\tau_{n+1}}$ :

$$(3.25) \quad \phi_{l_0+1} \circ \dots \circ \phi_{\tau_{n+1}-1} \circ \phi_{\tau_{n+1}} \circ \phi_{\tau_{n+1}-1}^{-1} \circ \dots \circ \phi_{l_0+2}^{-1} \circ \phi_{l_0+1}^{-1} (I_{2n+1}) = I_{2n+2}$$

for  $n = 0, 1, \dots, m - 3$ .

Thus, repeating Steps 2 (3.24) and Steps 1 (3.25) alternately  $m - 2$  times each, we have

$$I_{2(m-2)} = O_0 \cup O_{\tau_0} \cup \bigcup_{n=1}^{m-2} F^{\tau_n} \\ \cup O'_0 \cup O'_{\tau_0} \cup \bigcup_{n=1}^{m-2} F'^{\tau_n} \\ \cup F_r.$$

By (a2), we have  $\{\tau_1, \tau_2, \dots, \tau_{m-2}\} = \{1, 2, \dots, m - 1\} - \{\tau_0\} = \{1, 2, \dots, m - 1\} - \{m - r\}$ . Hence together with (3.10) and (3.11) we have

$$I_{2(m-2)} = O_0 \cup O_{m-r} \cup O'_0 \cup O'_{m-r} \cup \bigcup_{k \neq m, 2m-r} F_k,$$

where

$$O_{m-r} \sim O(2m - r; \overline{(2m - r - 1 \searrow \overline{m})(m - 1 \searrow r)})$$

by (3.27) of Lemma 3.3. On the other hand, by definition  $O_0 = F_m$ . Hence, we have free edges of all labels except  $2m - r$ , using which and (1.7) we can deform the oval nest  $O_{m-r}$  to the free edge  $F_{2m-r}$ .

Therefore  $\Gamma_S \cup F_r$  can be deformed to a chart containing  $\bigcup_{k=1}^{2m-1} F_k$ , using which and (1.7) we can deform  $\Gamma_S \cup F_r$  to have only free edges, which is an unknotted chart. □

**Lemma 3.3.** *The oval nest of Lemma 2.4*

$$O_k = O\left(m; \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \prod_{j=0}^{k-1} (m+1 \nearrow m+k-j)\right)$$

for  $k = 1, 2, \dots, m-1$ , is equivalent to the following:

$$(3.26) \quad O_k \sim O(m-k; \overline{(m-k+1 \nearrow m)})(m+1 \nearrow m+k)$$

$$(3.27) \quad \sim O(m+k; \overline{(m+k-1 \searrow m)})(m-1 \searrow m-k).$$

Proof. First, we will show that the braid  $\prod_{j=0}^{k-1} (m-1 \searrow m-k+j)$  is equivalent to  $\prod_{j=0}^{k-1} (m-k+j \searrow m-k)$ , i.e.

$$(3.28) \quad \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \sim \prod_{j=0}^{k-1} (m-k+j \searrow m-k).$$

For positive integers  $l, i_1, i_2$  with  $l \geq i_2 > i_1$ , we have  $(l \searrow i_1)i_2 \sim (i_2-1)(l \searrow i_1)$ . Hence we can see that

$$(3.29) \quad (l \searrow i_1)(l \searrow i_2) \sim (l-1 \searrow i_2-1)(l \searrow i_1).$$

By (3.29), we see that

$$\begin{aligned} & \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \\ &= (m-1 \searrow m-k) \cdot \prod_{j=1}^{k-1} (m-1 \searrow m-k+j) \\ &\sim \prod_{j=1}^{k-1} (m-2 \searrow m-k+j-1) \cdot (m-1 \searrow m-k) \\ &\sim \dots \\ &\sim \prod_{j=s-1}^{k-1} (m-s \searrow m-k+j-(s-1)) \prod_{j=k-(s-1)}^{k-1} (m-k+j \searrow m-k) \\ &= (m-s \searrow m-k) \prod_{j=s}^{k-1} (m-s \searrow m-k+j-(s-1)) \prod_{j=k-(s-1)}^{k-1} (m-k+j \searrow m-k) \end{aligned}$$

$$\begin{aligned}
&\sim \prod_{j=s}^{k-1} (m-s-1 \searrow m-k+j-s) \cdot (m-k+(k-s) \searrow m-k) \\
&\quad \cdot \prod_{j=k-(s-1)}^{k-1} (m-k+j \searrow m-k) \\
&= \prod_{j=s}^{k-1} (m-(s+1) \searrow m-k+j-s) \cdot \prod_{j=k-s}^{k-1} (m-k+j \searrow m-k) \\
&\sim \dots \\
&\sim (m-k) \prod_{j=1}^{k-1} (m-k+j \searrow m-k) \\
&= \prod_{j=0}^{k-1} (m-k+j \searrow m-k),
\end{aligned}$$

which is (3.28). Similarly, we have another equivalence relation:

$$(3.30) \quad \prod_{j=0}^{k-1} (m+1 \nearrow m+k-j) \sim \prod_{j=0}^{k-1} (m+k-j \nearrow m+k).$$

Note that for positive integers  $l, i_1, i_2$  with  $l \leq i_2 < i_1$ , we can easily show that  $(l \nearrow i_1)(l \nearrow i_2) \sim (l+1 \nearrow i_2+1)(l \nearrow i_1)$ .

Using (1.4), we can show that if  $m-1 > i$ , then

$$(3.31) \quad O(m; (i \searrow j) \cdot c) \sim O(m; c)$$

for a braid  $c$ . Similarly we can show that if  $m+1 < i$ , then

$$(3.32) \quad O(m; (i \nearrow j) \cdot c) \sim O(m; c).$$

By (3.28) and (3.30), we have

$$\begin{aligned}
O_k &= O \left( m; \prod_{j=0}^{k-1} (m-1 \searrow m-k+j) \prod_{j=0}^{k-1} (m+1 \nearrow m+k-j) \right) \\
&\sim O \left( m; \prod_{j=0}^{k-2} (m-k+j \searrow m-k) \cdot (m-1 \searrow m-k) \right. \\
&\quad \left. \cdot \prod_{j=0}^{k-2} (m+k-j \nearrow m+k) \cdot (m+1 \nearrow m+k) \right).
\end{aligned}$$

For  $j = 0, 1, \dots, k - 2$ , we have  $(m + k - j) - ((m - 1) + 1) = k - j > 0$ . Hence  $m + k - j > (m - 1) + 1$ . By (2.2), we have

$$O_k \sim O\left(m; \prod_{j=0}^{k-2} (m - k + j \searrow m - k) \cdot \prod_{j=0}^{k-2} (m + k - j \nearrow m + k) \cdot (m - 1 \searrow m - k)(m + 1 \nearrow m + k)\right).$$

By (3.31) and (3.32), we have

$$(3.33) \quad O_k \sim O(m; (m - 1 \searrow m - k)(m + 1 \nearrow m + k)).$$

Now we will show that

$$(3.34) \quad O(m; (m - 1 \searrow m - k) \cdot c) \sim O(m - k; \overline{(m - k + 1 \nearrow \bar{m})} \cdot c),$$

where  $c$  is a braid. For positive integers  $i_1, i_2$  with  $i_1 > i_2$ , by (1.5) and (1.6) we have  $O(i_1; (i_1 - 1 \searrow i_2)) \sim O(i_1 - 1; \overline{i_1} \cdot (i_1 - 2 \searrow i_2))$ , which is equivalent to  $O(i_1 - 1; (i_1 - 2 \searrow i_2) \cdot \overline{i_1})$  by (2.2). Thus we have

$$(3.35) \quad O(i_1; (i_1 - 1 \searrow i_2)) \sim O(i_1 - 1; (i_1 - 2 \searrow i_2) \cdot \overline{i_1}).$$

Using (3.35) and (1.6), we can see that

$$\begin{aligned} &O(m; (m - 1 \searrow m - k)) \\ &\sim O(m - 1; (m - 2 \searrow m - k) \cdot \bar{m}) \\ &\sim \dots \\ &\sim O(m - s; (m - s - 1 \searrow m - k) \cdot \overline{(m - s + 1 \nearrow \bar{m})}) \\ &\sim O(m - s - 1; (m - s - 2 \searrow m - k) \cdot \overline{(m - s)} \cdot \overline{(m - s + 1 \nearrow \bar{m})}) \\ &= O(m - s - 1; (m - s - 2 \searrow m - k) \cdot \overline{(m - s \nearrow \bar{m})}) \\ &\sim \dots \\ &\sim O(m - k; \overline{(m - k + 1 \nearrow \bar{m})}). \end{aligned}$$

Hence by (1.6), we have (3.34).

By (3.33) and (3.34), we have

$$\begin{aligned} O_k &\sim O(m; (m - 1 \searrow m - k)(m + 1 \nearrow m + k)) \\ &\sim O(m - k; \overline{(m - k + 1 \nearrow \bar{m})}(m + 1 \nearrow m + k)), \end{aligned}$$

which is (3.26).



By (3.33) and (2.2), we can see that

$$(3.36) \quad \begin{aligned} & O(m; (m-1 \searrow m-k) \cdot (m+1 \nearrow m+k)) \\ & \sim O(m; (m+1 \nearrow m+k) \cdot (m-1 \searrow m-k)). \end{aligned}$$

And similarly to (3.34), we can see that

$$(3.37) \quad O(m; (m+1 \nearrow m+k) \cdot c) \sim O(m+k; \overline{(m+k-1 \searrow \bar{m})} \cdot c),$$

for a braid  $c$ . Hence by (3.36) and (3.37) we have the other equivalence relation (3.27):

$$O_k \sim O(m+k; \overline{(m+k-1 \searrow \bar{m})} (m-1 \searrow m-k)). \quad \square$$

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