Nakamura, I.
Osaka J. Math.
49 (2012), 875–899

UNKNOTTING THE SPUN T^2-KNOT OF A CLASSICAL TORUS KNOT

INASA NAKAMURA

(Received May 11, 2009, revised February 15, 2011)

Abstract

We show that for the closure of a classical braid which satisfies certain conditions, the spun T^2-knot of the classical knot has the unknotting number one. This gives an alternative proof of the fact that the spun T^2-knot of a classical torus knot has the unknotting number one.

0. Introduction

A *surface knot* is the image of a smooth embedding of a closed connected surface into the Euclidean 4-space $\mathbb{R}^4$. Kanenobu and Marumoto [10] showed that the spun 2-knot of a classical torus knot has the unknotting number one. Hence it follows that the spun T^2-knot of a classical torus knot has the unknotting number one. Here, the *spun T^2-knot* of a classical knot K is the product of K in a 3-ball B^3 with a circle S^1, embedded into $\mathbb{R}^4$ via the natural embedding of $B^3 \times S^1$ into $\mathbb{R}^4$ ([15, 2]). In this paper, we show that for the closure of a classical braid which satisfies certain conditions, the spun T^2-knot of the classical knot has the unknotting number one (Theorem 3.1). Theorem 3.1 gives an alternative proof of the above-mentioned fact that the spun T^2-knot of a classical torus knot has the unknotting number one (Corollary 3.2). The proof of Theorem 3.1 is shown by a diagrammatic method, by using a surface link chart presenting the spun T^2-knot.

A surface link chart is a sort of finite graph in a 2-disk with some additional data ([5, 8, 9]). Any oriented surface knot is presented by a surface link chart ([7, 8, 9]). An unknotted surface knot is presented by an unknotted chart ([5, 9]). It is known [4] that any oriented surface knot S can be deformed to an unknotted surface knot by applying 1-handle surgeries along a finite number of mutually disjoint oriented 1-handles. The *unknotting number* of S is the minimum number of such 1-handles necessary to deform S to be unknotted. A *free edge* is an edge in a chart such that the end points are vertices of degree one. Applying a 1-handle surgery to an oriented surface knot S along a nice 1-handle is presented by adding a free edge to a surface link chart presenting S ([6]).

2010 Mathematics Subject Classification. Primary 57Q45; Secondary 57Q35.

Theorem 3.1 is shown as follows. First, we obtain a surface link chart presenting the spun T^2-knot. Then we add a free edge to the chart, and deform it to an unknotted chart (a configuration consisting of free edges) by equivalence relations. We obtain a surface link chart presenting the spun T^2-knot, as follows. We showed in [14] (see Theorem 2.2) how to obtain a surface link chart which presents a surface knot in the form of a branched covering over the standard torus. We call such a surface knot a *torus-covering knot* ([13], see Definition 2.1). Since a spun T^2-knot is a torus-covering knot, we can obtain a surface link chart presenting the spun T^2-knot by [14].

This paper is organized as follows. In Section 1, we review a braided surface and its chart description, and prepare several notations. In Section 2, we review the definition of a torus-covering knot and Theorem 2.2. In Section 3, we show Theorem 3.1 and Corollary 3.2.

1. A braided surface and its chart description

A braided surface was defined in [16, 7, 9]. A surface braid is a braided surface with some boundary condition, and a notion of a chart was introduced [5, 9] to present a simple surface braid. Equivalent simple surface braids have distinct chart presentations. The notion of C-move equivalence between two charts of the same degree was introduced [5, 8, 9] to give the equivalence class of the chart which represents the equivalence class of a simple surface braid. The notion of a chart can be easily extended to a chart presenting a simple braided surface. In this section, we review a braided surface, and extend the notion of a chart description to a simple braided surface. We review the fact that any oriented surface knot is presented by the closure of a simple surface braid ([7, 9]); thus it is presented by a chart. In order to present a certain chart called an "oval nest", we introduce a notation, and we prepare several equivalence relations between oval nests.

DEFINITION 1.1. A compact and oriented 2-manifold S embedded in a bidisk $D_1 \times D_2$ properly and locally flatly is called a *braided surface* of degree m if S satisfies the following conditions:
(i) $p_2|_S \colon S \to D_2$ is a branched covering map of degree m,
(ii) ∂S is a closed m-braid in $D_1 \times \partial D_2$, where D_1, D_2 are 2-disks, and p_2: $D_1 \times D_2 \to D_2$ is the projection to the second factor.
Two braided surfaces are *equivalent* if there is a fiber-preserving ambient isotopy of $D_1 \times D_2$ rel $D_1 \times \partial D_2$ which carries one to the other. A braided surface S is called *simple* if $\#(S \cap p_2^{-1}(x)) = m - 1$ or m for each $x \in D_2$. A braided surface S is called a *surface braid* if ∂S is the trivial closed braid. A surface braid $Q_m \times D_2$ is called *trivial*, where Q_m is a set of m interior points of D_1.

When a simple braided surface S is given, we obtain a graph on D_2, as follows. Identify D_1 with $I \times I$, where $I = [0, 1]$. Consider the singular set $\mathrm{Sing}(p_1(S))$ of the

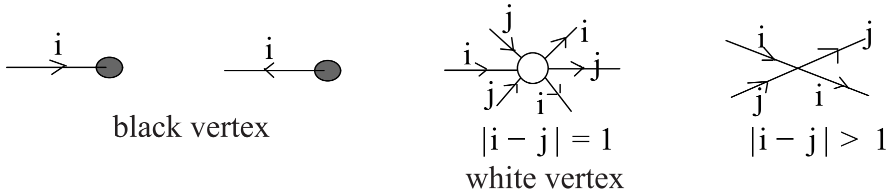

Fig. 1.1. Vertices in a chart.

image of S by the projection p_1 to $I \times D_2$. Perturbing S if necessary, we can assume that $\mathrm{Sing}(p_1(S))$ consists of double point curves, triple points, and branch points. Moreover we can assume that the singular set of the image of $\mathrm{Sing}(p_1(S))$ by the projection to D_2 consists of a finite number of double points such that the preimages belong to double point curves of $\mathrm{Sing}(p_1(S))$. Thus the image of $\mathrm{Sing}(p_1(S))$ by the projection to D_2 forms a finite graph Γ on D_2 such that the degree of its vertex is either 1, 4 or 6. An edge of Γ corresponds to a double point curve, and a vertex of degree 1 (resp. 6) corresponds to a branch point (resp. triple point).

For such a graph Γ obtained from a simple braided surface S, we give orientations and labels to the edges of Γ, as follows. Let us consider a path ρ in D_2 such that $\rho \cap \Gamma$ is a point P of an edge e of Γ. Then $S \cap p_2^{-1}(\rho)$ is a classical m-braid with one crossing in $p_2^{-1}(\rho)$ such that P corresponds to the crossing of the m-braid. Let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the m-braid group B_m. Let σ_i^{ϵ} ($i \in \{1, 2, \ldots, m-1\}$, $\epsilon \in \{+1, -1\}$) be the presentation of $S \cap p_2^{-1}(\rho)$. Then label the edge e by i, and moreover give e an orientation such that the normal vector of ρ corresponds (resp. does not correspond) to the orientation of e if $\epsilon = +1$ (resp. -1). We call such an oriented and labeled graph a *chart of S*.

In general, we define a chart on D_2 as follows.

DEFINITION 1.2. Let m be a positive integer. A finite graph Γ on a 2-disk D_2 is called a *chart* of degree m if it satisfies the following conditions:
(i) $\Gamma \cap \partial D_2$ consists of a finite number of vertices of degree 1.
(ii) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.
(iii) Every vertex has degree 1, 4, or 6.
(iv) The adjacent edges around each vertex in $\mathrm{Int}(D_2)$ are oriented and labeled as shown in Fig. 1.1, where we depict a vertex of degree 1 by a black vertex, and a vertex of degree 6 by a white vertex.

In a chart, an edge without end points is called a *loop*. An edge whose end points are black vertices is called a *free edge*. A configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an *oval nest*.

A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple braided surface presented by the chart. A chart presents a simple braided surface. In particular, a chart Γ such that $\Gamma \cap \partial D_2 = \emptyset$ presents a simple surface braid.

When a chart Γ on D_2 is given, we can reconstruct a simple braided surface S over D_2 as follows. Let m be the degree of Γ, and let $N(\Gamma)$ be a neighborhood of Γ in D_2. Let us consider a trivial braided surface $S = Q_m \times (D_2 - N(\Gamma))$ over $D_2 - N(\Gamma)$, where Q_m is a set of m interior points of D_1. We extend S over a neighborhood of each edge as follows. Identify a neighborhood of an edge e with $I \times I$ such that e is identified with $\{1/2\} \times I$. Let i be the label attached to e, and let $\epsilon = +1$ (resp. -1) if the orientation of e corresponds (resp. does not correspond) to the orientation of $\{0\} \times I$. Then let the braided surface S over the neighborhood of e be the braided surface which has a presentation $\sigma_i^{\epsilon} \times I$ and the image of the double point curve of $p_1(S)$ by the projection to D_2 is e. Since Γ is as in Fig. 1.1 around each vertex, S can be extended naturally over a neighborhood of each vertex. See [3, 6, 9] for more details. Thus we can construct a simple braided surface S over D_2 such that the original chart is a chart of S.

The boundary of a simple surface braid S consists of trivial closed m-braid. Consider a natural embedding of $D_1 \times D_2$ in $\mathbb{R}^4$, and paste m disks to S to obtain an embedding of a closed surface in $\mathbb{R}^4$. The resulting surface is called the *closure* of S. It is known [7, 9] that any oriented surface knot is presented by the closure of a simple surface braid; thus it is presented by a chart Γ on D_2 such that $\Gamma \cap D_2 = \emptyset$. We call such a chart presenting a surface link a *surface link chart*.

In [5, 9], a surface link chart is called simply a chart. However, in this paper we distinguish a "surface link chart" from a "chart".

Two charts on D_2 of the same degree are *C-move equivalent* if they are related by a finite sequence of ambient isotopies of D_2 and C-moves (CI, CII, CIII-moves) as follows.

Let Γ and Γ' be two charts on D_2 of the same degree. Then Γ' is said to be obtained from Γ (or Γ is said to be obtained from Γ') by a *CI-move*, *CII-move* or *CIII-move* if there exists a 2-disk E in D_2 such that the loop ∂E is in general position with respect to Γ and Γ' and $\Gamma \cap (D_2 - E) = \Gamma' \cap (D_2 - E)$ and the following condition holds:
(CI) There are no black vertices in $\Gamma \cap E$ nor $\Gamma' \cap E$.

A CI-move as in Fig. 1.2 is called a CI-move of type (1), (2) or (3) respectively; see [9] for the complete set of CI-moves.
(CII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.3, where $|i - j| > 1$.
(CIII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.4, where $|i - j| = 1$.

It is shown as a minor modification of [5, 8, 9] that two simple braided surfaces of the same degree are equivalent if and only if their charts are C-move equivalent. Two surface knots are *equivalent* if there is an ambient isotopy of $\mathbb{R}^4$ which carries one to the other. Thus it follows that for two surface link charts of the same degree, their presenting surface knots are equivalent if the charts are C-move equivalent.

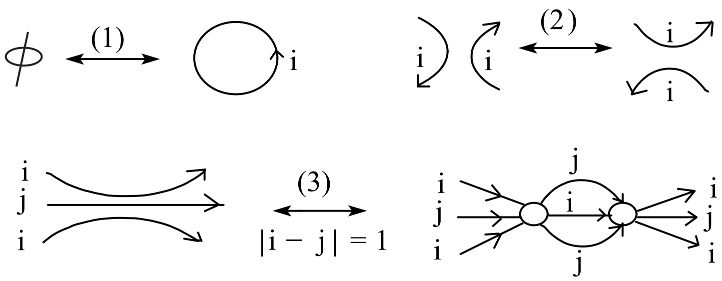

Fig. 1.2. CI-moves of types (1), (2) and (3).

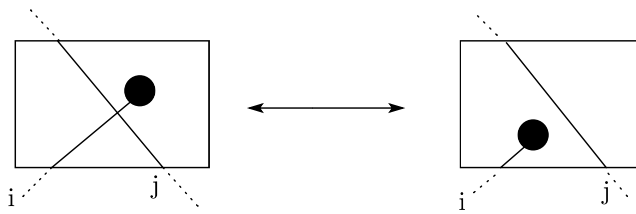

Fig. 1.3. CII-moves, where $|i - j| > 1$.

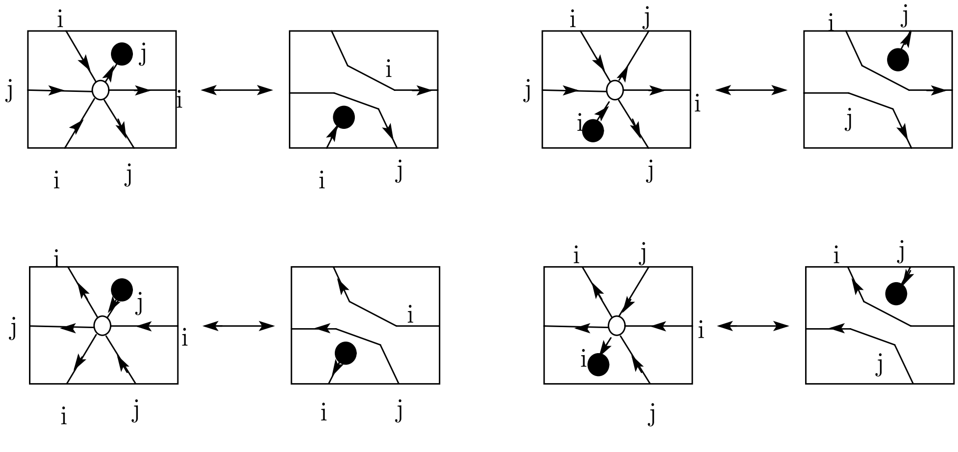

Fig. 1.4. CIII-moves, where $|i - j| = 1$.

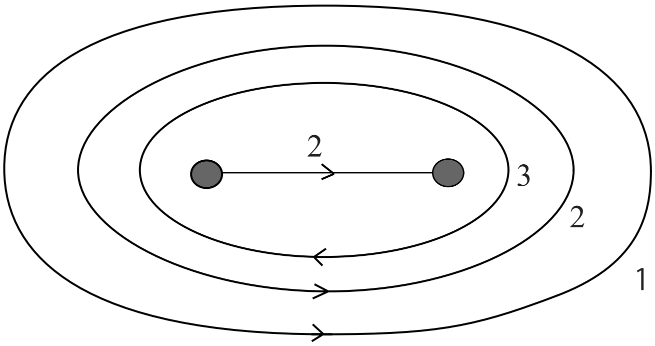

Fig. 1.5. An oval nest $O(2;\bar{3}21)$.

Throughout this paper, let us denote the oval nest with a free edge with the label i and its surrounding loops with the labels $i_1, i_2, \dots, i_n$ and the orientation $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ from the free edge outward by $O(i; i_1^* i_2^* \cdots i_n^*)$, where $\epsilon_j = \pm 1$ and $i_j^* = i_j$ (resp. $\bar{i}_j$) if $\epsilon_j = +1$ (resp. -1) (see Fig. 1.5). In particular, let us denote the free edge $O(i; \emptyset)$ by F_i. For $0 < i < j$, let us denote $i(i+1)\cdots j$ (resp. $\bar{i}(\overline{i+1})\cdots\bar{j}$) by $i \nearrow j$ (resp. $\bar{i} \nearrow \bar{j}$), and for $0 < j < i$, let us denote $i(i-1)\cdots j$ (resp. $\bar{i}(\overline{i-1})\cdots\bar{j}$) by $i \searrow j$ (resp. $\bar{i} \searrow \bar{j}$).

Let Γ_1 and Γ_2 be charts of the same degree in 2-disks D_1 and D_2 respectively, where $D_i = [0, 1] \times [0, 1]$ for $i = 1, 2$. Identifying D_1 with $[0, 1] \times [0, 1/2]$ and D_2 with $[0, 1] \times [1/2, 1]$, we have a new chart $\Gamma_1 \cup \Gamma_2$ in $D_1 \cup D_2 = [0, 1] \times [0, 1]$. We will call it a *split union* of Γ_1 and Γ_2, and use the notation $\Gamma_1 \cup \Gamma_2$.

Let us define the braid group relations between two sequences of integers as follows:
1. $\emptyset \sim i \cdot \bar{i} \sim \bar{i} \cdot i$, for a positive integer i,
2. $i \cdot j \sim j \cdot i$, for positive integers$i$, j with $|i - j| > 1$,
3. $i \cdot j \cdot i \sim j \cdot i \cdot j$, for positive integers i, j with $|i - j| = 1$.

In this paper, we will identify a braid $\sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \cdots \sigma_{i_n}^{\epsilon_n}$ with a sequence of integers $i_1^* i_2^* \cdots i_n^*$ with the braid group relations, where $i_j^* = i_j$ (resp. $\bar{i}_j$) if $\epsilon_j = +1$ (resp. -1). Then we have the following lemma.

Lemma 1.3. *For positive integers i, j and braids b, c such that $\bar{b} i b = \bar{c} j c$, the following oval nests are equivalent*:

$$O(i; b) \sim O(j; c). \tag{1.1}$$

Before the proof, we review the notions of a braid system of a chart and slide equivalence.

Let Γ be a chart of degree m on a 2-disk D_2. Let q_0 be a fixed point on the boundary of D_2, and $\Sigma(\Gamma)$ the set of black vertices in Γ. Let $\mathfrak{A} = (a_1, a_2, \dots, a_n)$ be a *Hurwitz arc system* with the starting point set $\Sigma(\Gamma)$ and the terminal point q_0, which is, for any i and j, $a_i \cap a_j = \{q_0\}$ and the normal vector of a_i points to a_{i+1}. For each

$i = 1, 2, \dots, n$, consider a loop c_i in $D_2 \setminus \Sigma(\Gamma)$ with the base point q_0 such that it starts from q_0 and goes along a_i, turns around the starting point of a_i (the black vertex in Γ which is at the other end of a_i) anti-clockwise and comes back along a_i to q_0. Let η_i be the element of $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$ represented by this loop c_i. The fundamental group is a free group of rank n generated by $\eta_1, \eta_2, \dots, \eta_n$. We call $\eta_1, \eta_2, \dots, \eta_n$ the *Hurwitz generators* of $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$ associated with $\mathfrak{A}$. A *braid system* $\vec{b} = (b_1, b_2, \dots, b_n)$ of the chart Γ is an ordered n-tuple of elements of B_m such that each b_i is the m-braid represented by η_i, i.e. η_i in $\pi_1(D_2 \setminus \Sigma(\Gamma), q_0)$ represents the m-braid b_i in the simple surface braid of degree m which is represented by Γ on D_2.

Two braid systems are *slide equivalent* if we can transform one to the other by applying a finite sequence of the following equivalence relations:

$$(b_1, \dots, b_i, b_{i+1}, \dots, b_n) \sim (b_1, \dots, b_{i-1}, b_{i+1}, b_{i+1}^{-1} b_i b_{i+1}, b_{i+2}, \dots, b_n).$$

Two charts of the same degree are equivalent if and only if their braid systems are slide equivalent (see [7, Chapter 17 and Section 18.10]).

Proof of Lemma 1.3. We can take a braid system of $\vec{b}$ of $O(i; b)$ to be $\vec{b} = (b^{-1}\sigma_i b, b^{-1}\sigma_i^{-1} b)$. Since $\bar{b} i b = \bar{c} j c$, we have $\vec{b} = (c^{-1}\sigma_j c, c^{-1}\sigma_j^{-1} c)$, which is a braid system of $O(j; c)$. □

By Lemma 1.3, in particular the following equivalent deformations hold. We will prove several of them using C-moves. Let i, j be positive integers and b, b', c, c' be braids. For a positive integer k, Let $k^* \in \{k, \bar{k}\}$. If $b = b'$, then

(1.2) $$O(i; b) \sim O(i; b').$$

(1.3) $$O(i; i^*) \sim O(i; \emptyset) = F_i \quad \text{(see Fig. 1.6)},$$

(1.4) $$O(i; j^*) \sim O(i; \emptyset) = F_i, \quad \text{where } |i - j| > 1 \text{ (see Fig. 1.7)},$$

(1.5) $$O(i; j) \sim O(j; \bar{i}), \quad \text{where } |i - j| = 1 \text{ (see Fig. 1.8)}.$$

If $O(i; c) \sim O(j; c')$, then

(1.6) $$O(i; cb) \sim O(j; c'b).$$

Moreover, applying a CI-move of type (2) between the outermost loop labeled j of the oval nest $O(i; b \cdot j^*)$ and the free edge F_j, we can see that

(1.7) $$O(i; b \cdot j^*) \cup F_j \sim O(i; b) \cup F_j,$$

where b is a braid.

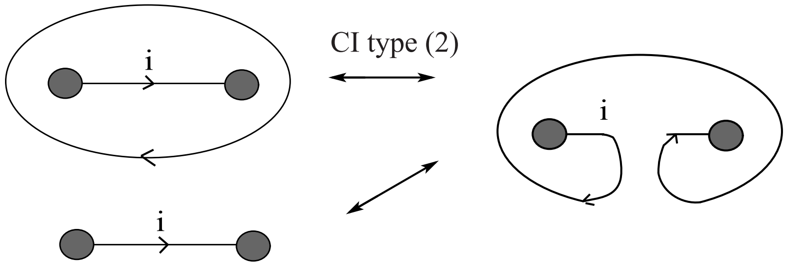

Fig. 1.6. $O(i;\bar{i}) \sim O(i;\emptyset) = F_i$.

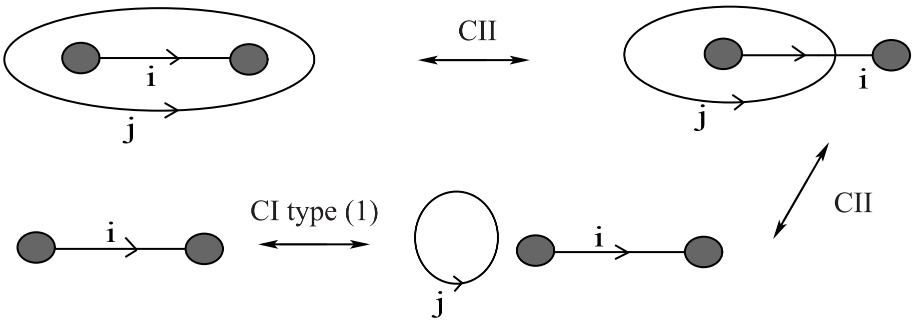

Fig. 1.7. $O(i;\,j) \sim O(i;\emptyset) = F_i$, where $|i-j| > 1$.

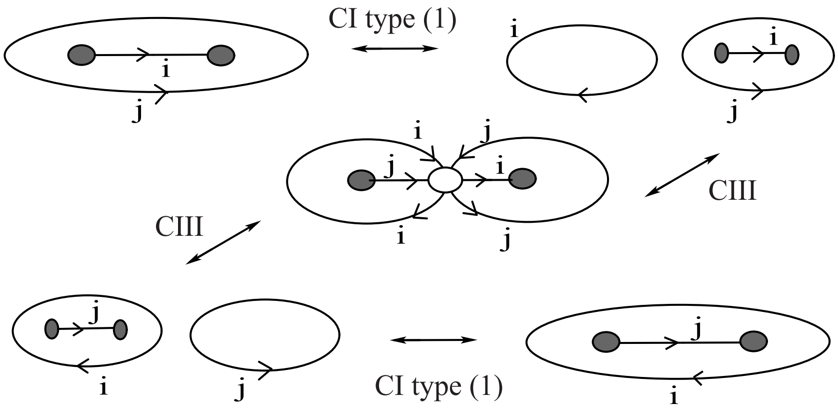

Fig. 1.8. $O(i;\,j) \sim O(j;\bar{i})$, where $|i-j| = 1$.

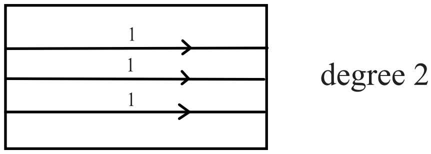

Fig. 2.1. A chart on T presenting the spun T^2-knot of a trefoil.

2. A torus-covering knot and its chart description

It is known [7, 9] that any oriented surface knot can be presented by a branched covering over the standard 2-sphere. A torus-covering knot was introduced in [13] as a new construction of a surface knot, by considering the standard torus instead of the standard 2-sphere. In this section, we give the definition of a torus-covering knot (see also [13]). The spun T^2-knot of a classical knot is a torus-covering knot. A torus-covering knot is presented by a chart on the standard torus T. We can obtain a surface link chart presenting a torus-covering knot from its chart on T ([14], see Theorem 2.2). Part of the obtained surface link chart is called a 1-handle chart. We obtain the 1-handle chart for the spun T^2-knot (Lemma 2.4).

Let T be a standard torus in $\mathbb{R}^4$, that is, the boundary of an unknotted solid torus in a 3-space in $\mathbb{R}^4$. Let us consider a tubular neighborhood $N(T)$ of T, and identify $N(T)$ with $D^2 \times S^1 \times S^1$, where D^2 is a 2-disk, and S^1 is a circle. The first S^1 corresponds to the meridian, and the second S^1 corresponds to the longitude of T. Let us identify S^1 with $I/\sim$, where $I = [0, 1]$ and $0 \sim 1$. For a manifold S in $N(T)$, let us denote by $S \cap (D^2 \times I \times I)$ the manifold in $D^2 \times I \times I$ obtained from S by cutting it at $D^2 \times S^1 \times \{0\}$ and $D^2 \times \{0\} \times S^1$.

DEFINITION 2.1. A *torus-covering knot* is a surface knot S in $\mathbb{R}^4$ such that $S \subset N(T)$ and moreover $S \cap (D^2 \times I \times I)$ is a simple braided surface.

By definition, a torus-covering knot S is presented by a chart on T. As we mentioned, for two charts of the same degree, their presenting braided surfaces are equivalent if the charts are C-move equivalent. Hence it follows that for two charts on T of the same degree, their presenting torus-covering knots are equivalent if the charts are C-move equivalent.

The *spun T^2-knot* of a classical knot K is the product of K in a 3-ball B^3 with S^1, embedded into $\mathbb{R}^4$ via the natural embedding of $B^3 \times S^1$ into $\mathbb{R}^4$ ([15, 2]). Identify S^1 with the longitude of T. Since any classical knot is equivalent to a closed braid by Alexander's Theorem, the spun T^2-knot of any K is a torus-covering knot (see [13, Propositions 2.11]); see Fig. 2.1 for example.

Now we review a theorem, which shows how to obtain a surface link chart from a chart on T ([14]). A chart is presented by a simple braided surface. A simple braided surface is presented by a motion picture consisting of isotopic transformations and hyperbolic transformations. A *motion picture* of a braided surface $S \subset B^3 \times I$ is a one-parameter family $\{\pi(S \cap (B^3 \times \{t\}))\}_{t \in I}$, where $\pi \colon B^3 \times I \to B^3$ is the projection (see [9]).

Let $\{h_t\}_{t \in [0,1]}$ be an ambient isotopy of $\mathbb{R}^3$. For a classical link L, we have an isotopy (a one-parameter family) $\{h_t(L)\}$ of classical links. We say that $h_1(L)$ is obtained from L by an *isotopic transformation*, and we use the notation that $L \to h_1(L)$ is an isotopic transformation (see [9, Section 9.1]).

Let L be a classical link in $\mathbb{R}^3$. A 2-disk B in $\mathbb{R}^3$ is called a *band* attaching to L if $L \cap B$ is a pair of disjoint arcs in ∂B. A *band set* attaching to L is a union $\mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_m$ of mutually disjoint bands $B_1, B_2, \ldots, B_m$ attaching to L. For a subset X of a space, let us denote by $\mathrm{Cl}(X)$ the closure of X. Define a link $h(L; \mathcal{B})$ by

$$h(L; \mathcal{B}) = \mathrm{Cl}((L \cup \partial\mathcal{B}) - (L \cap \mathcal{B})).$$

We say that the link $h(L; \mathcal{B})$ is obtained from L by a *hyperbolic transformation* along $\mathcal{B}$, and we use the notation that $L \to h(L; \mathcal{B})$ is a hyperbolic transformation (see [9, Section 9.1]).

For a classical m-braid c, let $\iota_k^l(c)$ be the $(m+k+l)$-braid obtained from c by adding k (resp. l) trivial strings before (resp. after) c, and put

$$\begin{aligned}
\Pi_i^m &= \sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+i}, \quad \Pi_i'^m = \sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-i},\\
\Delta_m &= \Pi_{m-1}^m \Pi_{m-2}^m \cdots \Pi_1^m, \quad \Delta_m' = \Pi_{m-1}'^m \Pi_{m-2}'^m \cdots \Pi_1'^m,\\
\Theta_m &= \sigma_m \cdot \Pi_{m-1}'^m \cdot \Pi_{m-1}^m \cdot \sigma_m \cdot \Pi_{m-2}'^m \cdot \Pi_{m-2}^m \cdots \sigma_m \cdot \Pi_1'^m \cdot \Pi_1^m \cdot \sigma_m.
\end{aligned}$$

Theorem 2.2 ([14]). *Let Γ_T be a chart of degree m on $I \times I$, obtained from a chart on T (of degree m) by cutting T by the meridian and the longitude. Let a (resp. b) be a classical m-braid presented by $\Gamma_T \cap (I \times \{0\})$ (resp. $\Gamma_T \cap (\{0\} \times I)$). Then the torus-covering knot presented by Γ_T is presented by a surface link chart Γ_S of degree $2m$ as in* Fig. 2.2. *Here H_b is a chart of degree $2m$ presenting the simple braided surface whose motion picture is as follows*:

$$\begin{aligned}
\iota_0^m(b) &\to \iota_0^m(b) \cdot (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Delta_m' \cdot \Delta_m \dot{\to} \iota_0^m(b) \cdot (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m\\
&\to (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \iota_0^m(\bar{b}^*) \cdot \Theta_m \to (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \cdot \iota_m^0(\bar{b}^*)\\
&\dot{\to} (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Delta_m' \cdot \Delta_m \cdot \iota_m^0(\bar{b}^*) \to \iota_m^0(\bar{b}^*),
\end{aligned}$$

where $\to$ is an isotopic transformation and $\dot{\to}$ is a hyperbolic transformation along bands corresponding to the m σ_m's, and $-(H_b)^*$ is the orientation-reversed mirror image of H_b, and $\bar{b}^*$ is the m-braid obtained from the classical m-braid b by taking its mirror image and reversing all the crossings.

DEFINITION 2.3. We call H_b the 1-*handle chart* of Γ_T.

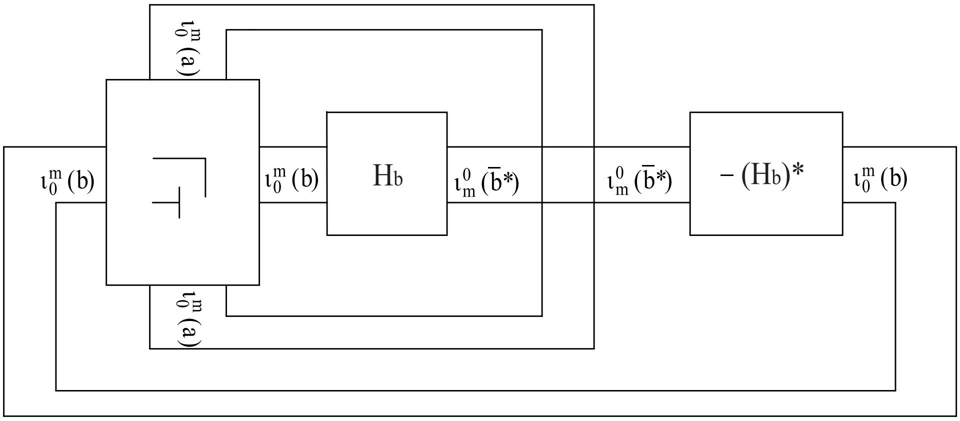

Fig. 2.2. The surface link chart Γ_S of degree $2m$.

Let us consider the spun T^2-knot of $\hat{b}$, where $\hat{b}$ denotes the closure of a classical braid b. Let us determine Γ_T on $I \times I$ to be a chart presenting the braided surface $b \times I$; then the braids presented by $\Gamma_T \cap (I \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I)$ are b and e respectively, where e is the trivial braid. The 1-handle chart of Γ_T is H_e. We obtain H_e, as follows.

Lemma 2.4. *Let e be the trivial m-braid. Then the* 1*-handle chart H_e is equivalent to the chart as follows*:

$$H_e \sim \bigcup_{k=0}^{m-1} O_k,$$

where O_k is the oval nest

$$O_k = O\left(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\right)$$

for $k = 0, 1, 2, \ldots, m-1$. Note that for $k = 0$, $O_0 = O(m; \emptyset) = F_m$.

Proof. By Theorem 2.2, H_e is a chart presenting the simple braided surface as follows:

$$\begin{aligned} e &\to (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Delta_m' \cdot \Delta_m \\ &\dot{\to} (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \\ &\dot{\to} (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Delta_m' \cdot \Delta_m \to e, \end{aligned} \tag{2.1}$$

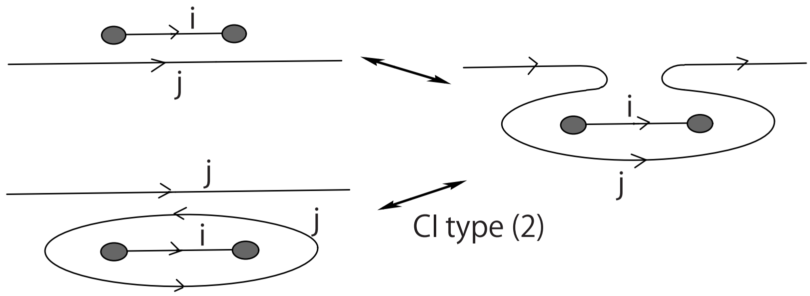

Fig. 2.3. Moving a free edge across an edge.

where $\rightarrow$ means an isotopy transformation and $\dot{\rightarrow}$ means a hyperbolic transformation along bands corresponding to the m σ_m's. Here e is the trivial $2m$-braid. Note that since (2.1) presents a simple surface braid, H_e does not have a boundary. The 1-handle chart H_e has m free edges, whose labels are all m. All the other edges have labels other than m and neither of them is connected with a black vertex. Draw H_e on $[0, 1/2] \times [0, 1]$ such that we can read the braids e, $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m$, $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$, $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m$ and e of (2.1) at $[0, 1/2] \times \{t_1\}, \ldots, [0, 1/2] \times \{t_5\}$ respectively, where $0 < t_1 < \cdots < t_5 < 1$. Let q_k $(k = 0, 1, \ldots, m-1)$ be the black vertex corresponding to the $(m-k)$-th σ_m of the braid $(\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$.

Let us denote by F_k the free edge connected with the black vertex q_k. Let us move the free edges into $[1/2, 1] \times [0, 1]$ using CI-moves of type (2) as in Fig. 2.3 to be a split sum of oval nests and H'_e, where H'_e is the chart $H_e - \left(\bigcup_{k=0}^{m-1} F_k\right)$. Since H'_e has no black vertices, we can eliminate it by a CI-move. Thus we have a split sum of oval nests.

Let $\tilde{O}_k$ $(k = 0, 1, \ldots, m-1)$ be the oval nest F_k becomes. We will see that each $\tilde{O}_k$ is equivalent to the oval nest O_k. First we will obtain $\tilde{O}_k$. It suffices to see what edges F_k crosses as it moves into $[1/2, 1] \times [0, 1]$. We have

$$\Theta_m = \sigma_m \cdot \Pi'^m_{m-1} \cdot \Pi^m_{m-1} \cdot \sigma_m \cdot \Pi'^m_{m-2} \cdot \Pi^m_{m-2} \cdots \sigma_m \cdot \Pi'^m_1 \cdot \Pi^m_1 \cdot \sigma_m.$$

The first free edge F_0 does not cross any edge. Hence $\tilde{O}_0 = F_0$. Then the second free edge F_1 crosses edges representing $\Pi'^m_1 \cdot \Pi^m_1 = \sigma_{m-1}\sigma_{m+1}$, so it becomes the oval nest $\tilde{O}_1 = O(m; m-1m+1)$. The third free edge F_2 crosses edges representing $\Pi'^m_2 \cdot \Pi^m_2 \cdot \Pi'^m_1 \cdot \Pi^m_1 = (\sigma_{m-1}\sigma_{m-2})(\sigma_{m+1}\sigma_{m+2})\sigma_{m-1}\sigma_{m+1}$. Hence it becomes the oval nest $\tilde{O}_2 = O(m; (m-1)(m-2) \cdot (m+1)(m+2) \cdot (m-1) \cdot (m+1))$. Repeating this step, we see that in general F_k crosses edges representing $\Pi'^m_k \cdot \Pi^m_k \cdot \Pi'^m_{k-1} \cdot \Pi^m_{k-1} \cdots \Pi'^m_1 \cdot \Pi^m_1$, so it becomes an oval nest $\tilde{O}_k = O\left(m; \prod_{j=0}^{k-1}((m-1 \searrow m-k+j) \cdot (m+1 \nearrow m+k-j))\right)$ for $k = 0, 1, \ldots, m-1$.

We can show that if $i+1<k$ then $(i \searrow j)(k \nearrow l)$ can be transformed to $(k \searrow l)(i \nearrow j)$ by the braid group relation 2, i.e.

$$(i \searrow j)(k \nearrow l) \sim (k \searrow l)(i \nearrow j). \tag{2.2}$$

Using (2.2), we see that $\tilde{O}_k \sim O\big(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j)\prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\big)$, which is O_k. □

3. Main theorem

An oriented surface knot is *unknotted* if it is equivalent to the connected sum of several standard tori. It is known [4] that any oriented surface knot S can be deformed to an unknotted surface knot by applying a finite number of oriented 1-handle surgeries. The *unknotting number* of S is the minimum number of oriented 1-handle surgeries necessary to deform it to be unknotted. In this section, we show Theorem 3.1 and Corollary 3.2.

Theorem 3.1. *Let S be the spun T^2-knot of a classical knot $\hat{b}$, where b is a classical m-braid $(m>1)$ such that there exists a permutation τ of degree $m-1$ which satisfies the following conditions*:
(a1) *There is an integer $r \in \{1,2,\dots,m-1\}$ such that for each $k \in \{1,2,\dots,m-1\}-\{r\}$, $\sigma_k \cdot b = b \cdot \sigma_{\tau(k)}$, and*
(a2) *For each $i, j \in \{1,2,\dots,m-1\}$, if $i \neq j$, then $\tau^i(1) \neq \tau^j(1)$. Note that then $\tau^{m-1}(1)=1$.*
Moreover assume that S is not unknotted. Then the unknotting number of S is one.

By Theorem 3.1 we have an alternative proof of the fact [10] that the spun T^2-knot of a torus (p,q)-knot has the unknotting number one.

Corollary 3.2. *The spun T^2-knot of a classical torus (p,q)-knot has the unknotting number one.*

Proof. First we show that the spun T^2-knot is not unknotted. The knot group of the spun T^2-knot of a classical torus (p,q)-knot is isomorphic to the knot group of the classical torus (p,q)-knot ([15]). Hence we can see that the spun T^2-knot is not unknotted.

We determine the braid b and the permutation τ, as follows. A classical torus (p,q)-knot is presented by the closure of the p-braid $b=(\sigma_1\sigma_2\cdots\sigma_{p-1})^q$, where p and q are coprime integers and moreover $p>1$. Let r be defined by $q \bmod p$ such that $r \in \{0,1,2,\dots,p-1\}$. Since p and q are coprime, $r \neq 0$ and it follows that $r \in \{1,2,\dots,p-1\}$. Let us define a permutation τ of degree $p-1$ by

$$\begin{pmatrix} 1 & 2 & \cdots & r-1 & r & r+1 & r+2 & \cdots & p-1 \\ p-r+1 & p-r+2 & \cdots & p-1 & p-r & 1 & 2 & \cdots & p-r-1 \end{pmatrix}.$$

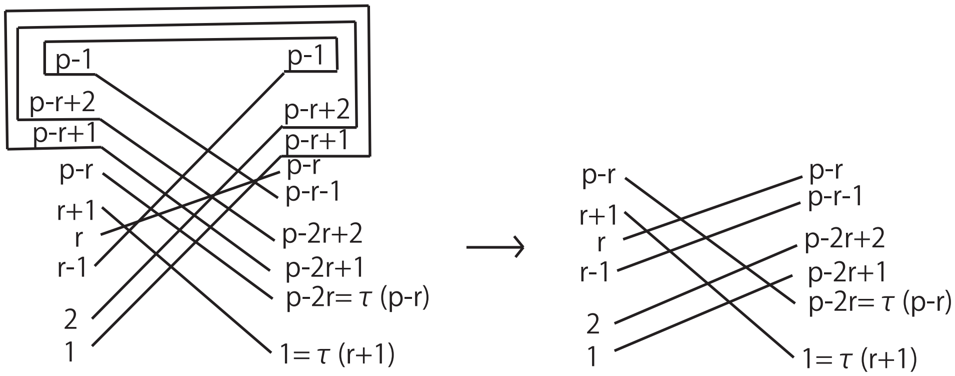

Fig. 3.1. The braid associated with τ if $r-1 < p-r-1$.

We show that Condition (a1) of Theorem 3.1 holds, as follows. If $k \neq 1$, then we can show that $\sigma_k(\sigma_1\sigma_2\cdots\sigma_{p-1}) = (\sigma_1\sigma_2\cdots\sigma_{p-1})\sigma_{k-1}$. Similarly we have $\sigma_1(\sigma_1\sigma_2\cdots\sigma_{p-1})^2 = (\sigma_1\sigma_2\cdots\sigma_{p-1})^2\sigma_{p-1}$. From these two equations, we have $\sigma_k \cdot b = b \cdot \sigma_{\tau(k)}$ for each $k \in \{1, 2, \dots, p-1\} - \{r\}$. Thus Condition (a1) holds.

Next we will show that τ satisfies Condition (a2) of Theorem 3.1. Let us define Condition (a2)$'$ as follows.

(a2)$'$ The permutation τ is associated with a classical braid c such that $\hat{c}$ is a knot, i.e. $\hat{c}$ is connected.

We can see that if the permutation τ satisfies (a2)$'$, then (a2) holds, as follows. If Condition (a2) does not hold, then $\tau^i(1) = \tau^j(1)$ for some $i, j \in \{1, 2, \dots, p-1\}$ with $i \neq j$. We can assume that $j > i$. Then we have $\tau^{j-i}(1) = 1$, where $0 < j-i < p-1$. On the other hand, if τ is associated with a classical braid c such that $\hat{c}$ is a knot, then $\tau^k(1) \neq 1$ for any k with $0 < k < p-1$. This is a contradiction.

From now on we will show that τ satisfies (a2)$'$. Since $r \in \{1, 2, \dots, p-1\}$ with $r = q \mod p$, and p and q are coprime integers, we see that $r = 1$ or p and r are coprime. If $r-1 = p-r-1$, then $p = 2r$. Since $r = 1$ or p and r are coprime, we have $r = 1$ and $p = 2$. Then $\tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is associated with the trivial braid of degree one, whose associated closed braid is a trivial knot. Hence we can assume that $r-1 \neq p-r-1$. If $r-1 < p-r-1$, then the permutation τ is associated with a braid whose diagram is as in Fig. 3.1, where we omit the crossing information. Here we have $\tau(r-j) = p-j$ for $j = 1, 2, \dots, r-1$ and $\tau(p-j) = p-r-j$ for $j = 1, 2, \dots, p-r-1$. Hence we have $\tau^2(r-j) = \tau(p-j) = p-r-j$ for $j = 1,2,\dots,r-1$, which means that the $(r-j)$-th string of the closed braid is connected with the $(p-j)$-th string, which is connected with the $(p-r-j)$-th string. Hence we can assume that there is no $(p-j)$-th string, and the $(r-j)$-th string of the closed braid is connected with the $(p-r-j)$-th string, where $j = 1,2,\dots,r-1$ (see Fig. 3.1).

Thus it suffices to show that the following permutation satisfies (a2)′:

$$\begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & p-r \\ p-2r+1 & p-2r+2 & \cdots & p-r & 1 & 2 & \cdots & p-2r \end{pmatrix}. \tag{3.1}$$

Similarly, if $r-1 > p-r-1$, then we have $\tau^2(r-j) = \tau(p-j) = p-r-j$ for $j = 1, 2, \ldots, p-r-1$. Hence it suffices to show that the following permutation satisfies (a2)′:

$$\begin{pmatrix} 1 & 2 & \cdots & 2r-p & 2r-p+1 & 2r-p+1 & \cdots & r \\ p-r+1 & p-r+2 & \cdots & r & 1 & 2 & \cdots & p-r \end{pmatrix}. \tag{3.2}$$

If $r-1 < p-r-1$ (resp. $r-1 > p-r-1$), then $p-r > r$ (resp. $r > p-r$). Hence together with $1 \leq r \leq p-1$, we can see that $p-r > 1$ (resp. $r > 1$). Thus the permutation (3.1) (resp. (3.2)) is associated with the m-braid $c = (\sigma_1\sigma_2 \cdots \sigma_{m-1})^n$, where $m = p-r$ (resp. r) is the degree of (3.1) (resp. (3.2)) with $m > 1$, and $n = m - \tau(1) + 1$. Since for (3.1) (resp. (3.2)) we have $m = p-r$ (resp. r) and $\tau(1) = p-2r+1$ (resp. $p-r+1$), it follows that $(m,n) = (p-r, r)$ (resp. $(m,n) = (r, 2r-p)$). Note that in both cases $n > 0$. Since $r = 1$ or p and r are coprime, together with $m > 1$ and $n > 0$, it follows that in both cases $n = 1$ or m and n are coprime. If $n = 1$, then $\hat{c}$ ($c = \sigma_1\sigma_2 \cdots \sigma_{m-1}$) is a trivial knot, and if m and n are coprime, then $\hat{c}$ ($c = (\sigma_1\sigma_2 \cdots \sigma_{m-1})^n$) is a torus (m, n)-knot. Thus τ satisfies (a2)′, and it follows that τ satisfies (a2). Therefore the spun T^2-knot has the unknotting number one by Theorem 3.1. □

Proof of Theorem 3.1. We show that the unknotting number of S is one. Let Γ_S be a surface link chart presenting S. An *unknotted chart* is a chart presented by a configuration consisting of free edges ([5]). An unknotted oriented surface knot is presented by an unknotted chart ([5]). For an oriented surface knot, adding a free edge to the surface link chart corresponds to a nice 1-handle surgery, which is an oriented 1-handle surgery ([6]). Thus it suffices to see that the surface link chart obtained from Γ_S by adding a free edge is equivalent to an unknotted chart.

We will determine Γ_S by [14] (see Theorem 2.2). The chart Γ_T on $I \times I$ presents the braided surface $b \times I$; thus the braids presented by $\Gamma_T \cap (I \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I)$ are b and e respectively (see Section 2). By Lemma 2.4 and (3.26) of Lemma 3.3, we can assume that the 1-handle chart H_e is as follows:

$$H_e = \bigcup_{k=0}^{m-1} O_k,$$

where

$$O_k = O(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)). \tag{3.3}$$

Let us define and O'_k as follows:

$$O'_k = O(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)\cdot b). \tag{3.4}$$

The oval nest O'_k is obtained from O_k by adding loops describing b around it. By [14] (see Theorem 2.2), the surface link chart Γ_S obtained from Γ_T is as follows:

$$\Gamma_S = \bigcup_{i=0}^{m-1} O_i \cup \bigcup_{i=0}^{m-1} O'_i. \tag{3.5}$$

Remark that Γ_S is a ribbon chart of degree $2m$ (see [5, 9]).

We will show that the surface link chart Γ_S can be deformed to an unknotted chart by adding a free edge.

STEP 1. We show that

$$O_{m-k-1} \cup O_{m-k} \cup F_{2m-k} \sim O_{m-k-1} \cup F_k \cup F_{2m-k},$$

for $k \in \{1, 2, \ldots, m-1\}$.

By (3.3) and (1.7), we have

$$\begin{aligned}
&O_{m-k} \cup F_{2m-k}\\
&= O(k; (\overline{k+1} \nearrow \overline{m})(m+1 \nearrow 2m-k)) \cup F_{2m-k}\\
&\sim O(k; (\overline{k+1} \nearrow \overline{m})(m+1 \nearrow 2m-k-1)) \cup F_{2m-k}.
\end{aligned}$$

Let us denote $O(k; (\overline{k+1} \nearrow \overline{m})(m+1 \nearrow 2m-k-1))$ by $\tilde{O}_{m-k}$. By (1.7) we have

$$O(k+1; \emptyset) \cup O(k; \overline{k+1}) \sim O(k+1; \emptyset) \cup O(k; \emptyset).$$

Hence we have

$$O(k+1; c) \cup O(k; \overline{k+1}\cdot c) \sim O(k+1; c) \cup O(k; c) \tag{3.6}$$

for a braid c by (1.6). By (3.3) and (3.6) we have

$$\begin{aligned}
&O_{m-k-1} \cup \tilde{O}_{m-k}\\
&= O(k+1; (\overline{k+2} \nearrow \overline{m})(m+1 \nearrow 2m-k-1))\\
&\quad \cup O(k; (\overline{k+1}) \cdot (\overline{k+2} \nearrow \overline{m})(m+1 \nearrow 2m-k-1))\\
&\sim O(k+1; (\overline{k+2} \nearrow \overline{m})(m+1 \nearrow 2m-k-1))\\
&\quad \cup O(k; (\overline{k+2} \nearrow \overline{m})(m+1 \nearrow 2m-k-1))\\
&= O_{m-k-1} \cup O(k; (\overline{k+2} \nearrow \overline{m})(m+1 \nearrow 2m-k-1)).
\end{aligned}$$

By (1.4) we see that

$$O(k;(\overline{k+2}\nearrow\overline{m})(m+1\nearrow 2m-k-1))\sim O(k;\emptyset)=F_k.$$

Thus we have

$$O_{m-k-1}\cup O_{m-k}\cup F_{2m-k}\sim O_{m-k-1}\cup F_k\cup F_{2m-k}.$$

STEP 2. Similarly, we show that

$$O'_{m-k-1}\cup O'_{m-k}\cup F_{\tau(k)}\sim O'_{m-k-1}\cup F_{2m-k}\cup F_{\tau(k)},$$

for $k\in\{1,2,\dots,m-1\}-\{r\}$.

By (a1), $b^{-1}\sigma_k b=\sigma_{\tau(k)}$ for $k\in\{1,2,\dots,m-1\}-\{r\}$. Hence we have

$$O(k;b)\sim F_{\tau(k)} \tag{3.7}$$

for $k\in\{1,2,\dots,m-1\}-\{r\}$ by Lemma 1.3.

Similarly to Step 1, using (3.27) of Lemma 3.3, we have

$$O_{m-k-1}\cup O_{m-k}\cup F_k\sim O_{m-k-1}\cup F_k\cup F_{2m-k} \tag{3.8}$$

for $k\in\{1,2,\dots,m-1\}$. By (3.8) and (1.6), we have

$$O'_{m-k-1}\cup O'_{m-k}\cup O(k;b)\sim O'_{m-k-1}\cup O(k;b)\cup O(2m-k;b). \tag{3.9}$$

By (3.7), $O(k;b)\sim F_{\tau(k)}$ for $k\in\{1,2,\dots,m-1\}-\{r\}$. On the other hand, by (1.4) and $2m-k>(m-1)+1$, we have $O(2m-k;b)\sim O(2m-k;\emptyset)=F_{2m-k}$. Hence together with (3.9), we see that

$$O'_{m-k-1}\cup O'_{m-k}\cup F_{\tau(k)}\sim O'_{m-k-1}\cup F_{2m-k}\cup F_{\tau(k)}$$

for $k\in\{1,2,\dots,m-1\}-\{r\}$.

STEP 3. Let us denote Step 1 as follows:

$$\phi_l\colon O_{l-1}\cup O_l\cup F_{m+l}\to O_{l-1}\cup F_{m-l}\cup F_{m+l}$$

for $l\in\{1,2,\dots,m-1\}$, and Step 2 as

$$\psi_l\colon O'_{l-1}\cup O'_l\cup F_{\tau(m-l)}\to O'_{l-1}\cup F_{m+l}\cup F_{\tau(m-l)}$$

for $l\in\{1,2,\dots,m-1\}-\{m-r\}$.

We introduce several notations to make things easy to see. Let us define F^l, F'^l and F''^l as follows:

$$F^l := F_{m-l}, \tag{3.10}$$

$$F'^l := F_{m+l}, \tag{3.11}$$

$$F''^l := F_{\tau(m-l)} \tag{3.12}$$

for $l \in \{1, 2, \ldots, m-1\}$. Moreover, for an integer s, let us define τ_s to be

$$\tau_s := m - \tau^{-s}(r). \tag{3.13}$$

Step 1 is written as follows:

$$\phi_l \colon O_{l-1} \cup O_l \cup F'^l \to O_{l-1} \cup F^l \cup F'^l, \tag{3.14}$$

for $l \in \{1, 2, \ldots, m-1\}$, and Step 2 is

$$\psi_l \colon O'_{l-1} \cup O'_l \cup F''^l \to O'_{l-1} \cup F'^l \cup F''^l, \tag{3.15}$$

for $l \in \{1, 2, \ldots, m-1\} - \{m-r\}$. Since by definition (3.13) $m-r = \tau_0$, Step 2 holds true for $l \in \{1, 2, \ldots, m-1\} - \{\tau_0\}$.

From now on we show that Γ_S can be deformed to an unknotted chart by adding a free edge F_r. Let us define charts $I_0, I_1, \ldots, I_{2m-4}$ of degree $2m$. First, define I_0 as follows:

$$I_0 := \Gamma_S \cup F_r,$$

which is by (3.5) as follows:

$$\begin{aligned} I_0 = O_0 &\cup O_{\tau_1} \cup O_{\tau_2} \cup \cdots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \\ &\cup O'_0 \cup O'_{\tau_1} \cup O'_{\tau_2} \cup \cdots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \\ &\cup F_r. \end{aligned} \tag{3.16}$$

Note that by (a2), $\{\tau_0, \tau_1, \ldots, \tau_{m-2}\} = \{1, 2, \ldots, m-1\}$. For $n = 1, 2, \ldots, m-2$, let us define I_{2n} as follows:

$$\begin{aligned} I_{2n} := O_0 &\cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \cup F^{\tau_1} \cup F^{\tau_2} \cup \cdots \cup F^{\tau_n} \\ &\cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \cup F'^{\tau_1} \cup F'^{\tau_2} \cup \cdots \cup F'^{\tau_n} \\ &\cup F_r. \end{aligned} \tag{3.17}$$

And for $n = 0, 1, 2, \ldots, m-3$, let us define I_{2n+1} as follows:

$$I_{2n+1} := (I_{2n} - O'_{\tau_{n+1}}) \cup F'^{\tau_{n+1}}. \tag{3.18}$$

We will show that I_{2n+1} (resp. I_{2n+2}) is obtained from I_{2n} (resp. I_{2n+1}) by applying Steps 2 (resp. Steps 1) for $n = 0, 1, \ldots, m-3$.

When we have I_{2n} ($n = 0, 1, \ldots, m-3$), there is an integer $l_0 < \tau_{n+1}$ such that for any l with $l_0 < l < \tau_{n+1}$, $O'_l \not\subset I_{2n}$ and $O'_{l_0} \subset I_{2n}$. Note that such an l_0 exists, for $0 < \tau_{n+1}$ and $O'_0 \subset I_{2n}$ for every $n \in \{0, 1, \ldots, m-3\}$. Since $r = \tau^0(r) = \tau(m-(m-\tau^{-1}(r))) = \tau(m-\tau_1)$, by the definition of F''^l (3.12) we have

$$F_r = F''^{\tau_1}. \tag{3.19}$$

For $n = 0$, by (3.16) we have $l_0 = \tau_1 - 1$, and by (3.19) we see that

$$I_0 \supset F''^{\tau_1} \cup O'_{\tau_1 - 1} \cup O'_{\tau_1}. \tag{3.20}$$

By the definitions (3.10) and (3.13), we have

$$F_{\tau^{-s}(r)} = F_{m-(m-\tau^{-s}(r))} = F^{m-\tau^{-s}(r)} = F^{\tau_s},$$

and by the definitions (3.12) and (3.13), we have

$$F_{\tau^{-s}(r)} = F_{\tau(\tau^{-(s+1)}(r))} = F''^{m-\tau^{-(s+1)}(r)} = F''^{\tau_{s+1}}.$$

Hence we have

$$F^{\tau_s} = F''^{\tau_{s+1}}, \tag{3.21}$$

for each s. By the definition of I_{2n} (3.17) and (3.21) we have

$$\begin{aligned} I_{2n} = {} & O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{m-2}} \cup O_{\tau_0} \cup F''^{\tau_2} \cup F''^{\tau_3} \cup \cdots \cup F''^{\tau_{n+1}} \\ & \cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{m-2}} \cup O'_{\tau_0} \cup F'^{\tau_1} \cup F'^{\tau_2} \cup \cdots \cup F'^{\tau_n} \\ & \cup F_r \end{aligned} \tag{3.22}$$

for $n = 1, 2, \ldots, m-3$. By (3.22) and (3.19), we can see that if $F'^l \subset I_{2n}$, then $F''^l \subset I_{2n}$. So together with (3.20), we have

$$\begin{aligned} I_{2n} \supset {} & F''^{l_0+1} \cup F''^{l_0+2} \cup \cdots \cup F''^{\tau_{n+1}-1} \cup F''^{\tau_{n+1}} \\ & \cup O'_{l_0} \cup F'^{l_0+1} \cup F'^{l_0+2} \cup \cdots \cup F'^{\tau_{n+1}-1} \cup O'_{\tau_{n+1}} \end{aligned} \tag{3.23}$$

for $n = 0, 1, \ldots, m-3$. By (3.22), we can see that if $F'^l \subset I_{2n}$, then $l \in \{\tau_1, \tau_2, \ldots, \tau_n\}$. Hence $l_0+1, l_0+2, \ldots, \tau_{n+1}-1 \in \{\tau_1, \tau_2, \ldots, \tau_n\}$. By (a2) and $n \leq m-3$, none of $l_0+1, l_0+2, \ldots, \tau_{n+1}-1, \tau_{n+1}$ is τ_0. So we can apply Steps 2 (3.15) and its inverses to I_{2n} to deform $O'_{\tau_{n+1}}$ to $F'^{\tau_{n+1}}$. The result is I_{2n+1} by the definition of I_{2n+1} (3.18):

$$\psi_{l_0+1} \circ \cdots \circ \psi_{\tau_{n+1}-1} \circ \psi_{\tau_{n+1}} \circ \psi^{-1}_{\tau_{n+1}-1} \circ \cdots \circ \psi^{-1}_{l_0+2} \circ \psi^{-1}_{l_0+1}(I_{2n}) = I_{2n+1} \tag{3.24}$$

for $n = 0, 1, \dots, m-3$.

By (3.16) and (3.17), we see that if $O'_l \subset I_{2n}$, then $O_l \subset I_{2n}$, and if $F'^l \subset I_{2n}$, then $F^l \subset I_{2n}$. Hence, by the definition of I_{2n+1} (3.18),

$$\begin{aligned} I_{2n+1} \supset O_{l_0} \cup F^{l_0+1} \cup F^{l_0+2} \cup \cdots \cup F^{\tau_{n+1}-1} \cup O_{\tau_{n+1}} \\ \cup F'^{l_0+1} \cup F'^{l_0+2} \cup \cdots \cup F'^{\tau_{n+1}} \end{aligned}$$

for $n = 0, 1, \dots, m-3$, where l_0 is the same integer used in deforming I_{2n} to I_{2n+1}. And by the definitions (3.16), (3.17) and (3.18) we have

$$\begin{aligned} I_{2n+2} &= (I_{2n} - O'_{\tau_{n+1}} - O_{\tau_{n+1}}) \cup F'^{\tau_{n+1}} \cup F^{\tau_{n+1}} \\ &= (I_{2n+1} - O_{\tau_{n+1}}) \cup F^{\tau_{n+1}} \end{aligned}$$

for $n = 0, 1, \dots, m-3$. Similarly to (3.24), we can deform I_{2n+1} to I_{2n+2} by applying Steps 1 (3.14) and its inverses and deforming $O_{\tau_{n+1}}$ to $F^{\tau_{n+1}}$:

$$\phi_{l_0+1} \circ \cdots \circ \phi_{\tau_{n+1}-1} \circ \phi_{\tau_{n+1}} \circ \phi^{-1}_{\tau_{n+1}-1} \circ \cdots \circ \phi^{-1}_{l_0+2} \circ \phi^{-1}_{l_0+1}(I_{2n+1}) = I_{2n+2} \tag{3.25}$$

for $n = 0, 1, \dots, m-3$.

Thus, repeating Steps 2 (3.24) and Steps 1 (3.25) alternately $m-2$ times each, we have

$$\begin{aligned} I_{2(m-2)} = O_0 \cup O_{\tau_0} \cup \bigcup_{n=1}^{m-2} F^{\tau_n} \\ \cup O'_0 \cup O'_{\tau_0} \cup \bigcup_{n=1}^{m-2} F'^{\tau_n} \\ \cup F_r. \end{aligned}$$

By (a2), we have $\{\tau_1, \tau_2, \dots, \tau_{m-2}\} = \{1, 2, \dots, m-1\} - \{\tau_0\} = \{1, 2, \dots, m-1\} - \{m-r\}$. Hence together with (3.10) and (3.11) we have

$$I_{2(m-2)} = O_0 \cup O_{m-r} \cup O'_0 \cup O'_{m-r} \cup \bigcup_{k \neq m, 2m-r} F_k,$$

where

$$O_{m-r} \sim O(2m-r; (\overline{2m-r-1} \searrow \overline{m})(m-1 \searrow r))$$

by (3.27) of Lemma 3.3. On the other hand, by definition $O_0 = F_m$. Hence, we have free edges of all labels except $2m-r$, using which and (1.7) we can deform the oval nest O_{m-r} to the free edge F_{2m-r}.

Therefore $\Gamma_S \cup F_r$ can be deformed to a chart containing $\bigcup_{k=1}^{2m-1} F_k$, using which and (1.7) we can deform $\Gamma_S \cup F_r$ to have only free edges, which is an unknotted chart. $\square$

Lemma 3.3. *The oval nest of* Lemma 2.4

$$O_k = O\left(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\right)$$

for $k = 1, 2, \dots, m-1$, *is equivalent to the following*:

$$O_k \sim O(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)) \tag{3.26}$$

$$\sim O(m+k; (\overline{m+k-1} \searrow \overline{m})(m-1 \searrow m-k)). \tag{3.27}$$

Proof. First, we will show that the braid $\prod_{j=0}^{k-1}(m-1 \searrow m-k+j)$ is equivalent to $\prod_{j=0}^{k-1}(m-k+j \searrow m-k)$, i.e.

$$\prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \sim \prod_{j=0}^{k-1}(m-k+j \searrow m-k). \tag{3.28}$$

For positive integers l, i_1, i_2 with $l \geq i_2 > i_1$, we have $(l \searrow i_1)i_2 \sim (i_2-1)(l \searrow i_1)$. Hence we can see that

$$(l \searrow i_1)(l \searrow i_2) \sim (l-1 \searrow i_2-1)(l \searrow i_1). \tag{3.29}$$

By (3.29), we see that

$$\begin{aligned}
&\prod_{j=0}^{k-1}(m-1 \searrow m-k+j)\\
&= (m-1 \searrow m-k) \cdot \prod_{j=1}^{k-1}(m-1 \searrow m-k+j)\\
&\sim \prod_{j=1}^{k-1}(m-2 \searrow m-k+j-1) \cdot (m-1 \searrow m-k)\\
&\sim \cdots\\
&\sim \prod_{j=s-1}^{k-1}(m-s \searrow m-k+j-(s-1)) \prod_{j=k-(s-1)}^{k-1}(m-k+j \searrow m-k)\\
&= (m-s \searrow m-k) \prod_{j=s}^{k-1}(m-s \searrow m-k+j-(s-1)) \prod_{j=k-(s-1)}^{k-1}(m-k+j \searrow m-k)
\end{aligned}$$

$$
\begin{aligned}
&\sim \prod_{j=s}^{k-1}(m-s-1 \searrow m-k+j-s)\cdot(m-k+(k-s)\searrow m-k)\\
&\quad\cdot \prod_{j=k-(s-1)}^{k-1}(m-k+j\searrow m-k)\\
&= \prod_{j=s}^{k-1}(m-(s+1)\searrow m-k+j-s)\cdot \prod_{j=k-s}^{k-1}(m-k+j\searrow m-k)\\
&\sim \cdots\\
&\sim (m-k)\prod_{j=1}^{k-1}(m-k+j\searrow m-k)\\
&= \prod_{j=0}^{k-1}(m-k+j\searrow m-k),
\end{aligned}
$$

which is (3.28). Similarly, we have another equivalence relation:

$$
\prod_{j=0}^{k-1}(m+1\nearrow m+k-j)\sim \prod_{j=0}^{k-1}(m+k-j\nearrow m+k). \tag{3.30}
$$

Note that for positive integers l, i_1, i_2 with $l \le i_2 < i_1$, we can easily show that $(l \nearrow i_1)(l \nearrow i_2) \sim (l+1 \nearrow i_2+1)(l \nearrow i_1)$.

Using (1.4), we can show that if $m-1 > i$, then

$$
O(m;(i\searrow j)\cdot c)\sim O(m;c) \tag{3.31}
$$

for a braid c. Similarly we can show that if $m+1 < i$, then

$$
O(m;(i\nearrow j)\cdot c)\sim O(m;c). \tag{3.32}
$$

By (3.28) and (3.30), we have

$$
\begin{aligned}
O_k &= O\left(m;\prod_{j=0}^{k-1}(m-1\searrow m-k+j)\prod_{j=0}^{k-1}(m+1\nearrow m+k-j)\right)\\
&\sim O\left(m;\prod_{j=0}^{k-2}(m-k+j\searrow m-k)\cdot(m-1\searrow m-k)\right.\\
&\qquad\left.\cdot\prod_{j=0}^{k-2}(m+k-j\nearrow m+k)\cdot(m+1\nearrow m+k)\right).
\end{aligned}
$$

For $j = 0, 1, \ldots, k-2$, we have $(m+k-j) - ((m-1)+1) = k - j > 0$. Hence $m + k - j > (m-1) + 1$. By (2.2), we have

$$O_k \sim O\Bigg(m; \prod_{j=0}^{k-2}(m-k+j \searrow m-k) \cdot \prod_{j=0}^{k-2}(m+k-j \nearrow m+k) \cdot (m-1 \searrow m-k)(m+1 \nearrow m+k)\Bigg).$$

By (3.31) and (3.32), we have

$$O_k \sim O(m; (m-1 \searrow m-k)(m+1 \nearrow m+k)). \tag{3.33}$$

Now we will show that

$$O(m; (m-1 \searrow m-k) \cdot c) \sim O(m-k; (\overline{m-k+1} \nearrow \overline{m}) \cdot c), \tag{3.34}$$

where c is a braid. For positive integers i_1, i_2 with $i_1 > i_2$, by (1.5) and (1.6) we have $O(i_1; (i_1 - 1 \searrow i_2)) \sim O\big(i_1 - 1; \overline{i_1} \cdot (i_1 - 2 \searrow i_2)\big)$, which is equivalent to $O(i_1 - 1; (i_1 - 2 \searrow i_2) \cdot \overline{i_1})$ by (2.2). Thus we have

$$O(i_1; (i_1 - 1 \searrow i_2)) \sim O(i_1 - 1; (i_1 - 2 \searrow i_2) \cdot \overline{i_1}). \tag{3.35}$$

Using (3.35) and (1.6), we can see that

$$\begin{aligned}
&O(m; (m-1 \searrow m-k)) \\
&\sim O(m-1; (m-2 \searrow m-k) \cdot \overline{m}) \\
&\sim \cdots \\
&\sim O(m-s; (m-s-1 \searrow m-k) \cdot (\overline{m-s+1} \nearrow \overline{m})) \\
&\sim O(m-s-1; (m-s-2 \searrow m-k) \cdot (\overline{m-s}) \cdot (\overline{m-s+1} \nearrow \overline{m})) \\
&= O(m-s-1; (m-s-2 \searrow m-k) \cdot (\overline{m-s} \nearrow \overline{m})) \\
&\sim \cdots \\
&\sim O(m-k; (\overline{m-k+1} \nearrow \overline{m})).
\end{aligned}$$

Hence by (1.6), we have (3.34).

By (3.33) and (3.34), we have

$$\begin{aligned}
O_k &\sim O(m; (m-1 \searrow m-k)(m+1 \nearrow m+k)) \\
&\sim O(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)),
\end{aligned}$$

which is (3.26).

By (3.33) and (2.2), we can see that

$$\begin{aligned} &O(m; (m-1 \searrow m-k)\cdot(m+1 \nearrow m+k)) \\ &\sim O(m; (m+1 \nearrow m+k)\cdot(m-1 \searrow m-k)). \end{aligned} \tag{3.36}$$

And similarly to (3.34), we can see that

$$O(m; (m+1 \nearrow m+k)\cdot c) \sim O(m+k; (\overline{m+k-1} \searrow \overline{m})\cdot c), \tag{3.37}$$

for a braid c. Hence by (3.36) and (3.37) we have the other equivalence relation (3.27):

$$O_k \sim O(m+k; (\overline{m+k-1} \searrow \overline{m})(m-1 \searrow m-k)). \qquad \square$$

ACKNOWLEDGEMENTS. The author would like to thank the referee for the helpful and kind advice. The author is supported by JSPS Research Fellowships for Young Scientists.

References

[1] J.S. Birman: Braids, Links, and Mapping Class Groups, Annals of Mathematics Studies **82**, Princeton Univ. Press, Princeton, NJ, 1974.

[2] J. Boyle: *Classifying* 1*-handles attached to knotted surfaces*, Trans. Amer. Math. Soc. **306** (1988), 475–487.

[3] J.S. Carter and M. Saito: Knotted Surfaces and Their Diagrams, Mathematical Surveys and Monographs **55**, Amer. Math. Soc., Providence, RI, 1998.

[4] F. Hosokawa and A. Kawauchi: *Proposals for unknotted surfaces in four-spaces*, Osaka J. Math. **16** (1979), 233–248.

[5] S. Kamada: *Surfaces in* $\mathbf{R}^4$ *of braid index three are ribbon*, J. Knot Theory Ramifications **1** (1992), 137–160.

[6] S. Kamada: 2*-dimensional braids and chart descriptions*; in Topics in Knot Theory (Erzurum, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **399**, Kluwer Acad. Publ., Dordrecht, 1993, 277–287.

[7] S. Kamada: *A characterization of groups of closed orientable surfaces in* 4*-space*, Topology **33** (1994), 113–122.

[8] S. Kamada: *An observation of surface braids via chart description*, J. Knot Theory Ramifications **5** (1996), 517–529.

[9] S. Kamada: Braid and Knot Theory in Dimension Four, Mathematical Surveys and Monographs **95**, Amer. Math. Soc., Providence, RI, 2002.

[10] T. Kanenobu and Y. Marumoto: *Unknotting and fusion numbers of ribbon* 2*-knots*, Osaka J. Math. **34** (1997), 525–540.

[11] A. Kawauchi: A Survey of Knot Theory, translated and revised from the 1990 Japanese original by the author, Birkhäuser, Basel, 1996.

[12] J.M. Montesinos: *On twins in the four-sphere*, I, Quart. J. Math. Oxford Ser. (2) **34** (1983), 171–199.

[13] I. Nakamura: *Surface links which are coverings over the standard torus*, Algebr. Geom. Topol. **11** (2011), 1497–1540.

[14] I. Nakamura: *Braiding surface links which are coverings over the standard torus*, J. Knot Theory Ramifications **21** (2012), 1250011, 25 pp.
[15] C. Livingston: *Stably irreducible surfaces in S^4*, Pacific J. Math. **116** (1985), 77–84.
[16] L. Rudolph: *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan

Current address:
Department of Mathematics
Gakushuin University
1-5-1 Mejiro Toshima-ku Tokyo, 171-8588
Japan
e-mail: inasa@math.gakushuin.ac.jp