<table>
<thead>
<tr>
<th>Title</th>
<th>UNKNOTTING THE SPUN $T^2$-KNOT OF A CLASSICAL TORUS KNOT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nakamura, Inasa</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 49(4) P.875-P.899</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-12</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/23425">https://doi.org/10.18910/23425</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/23425</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>

*Osaka University Knowledge Archive: OUKA*

http://ir.library.osaka-u.ac.jp/dspace/

Osaka University
UNKNOTTING THE SPUN $T^2$-KNOT OF A CLASSICAL TORUS KNOT

INASA NAKAMURA

(Received May 11, 2009, revised February 15, 2011)

Abstract

We show that for the closure of a classical braid which satisfies certain conditions, the spun $T^2$-knot of the classical knot has the unknotting number one. This gives an alternative proof of the fact that the spun $T^2$-knot of a classical torus knot has the unknotting number one.

0. Introduction

A surface knot is the image of a smooth embedding of a closed connected surface into the Euclidean 4-space $\mathbb{R}^4$. Kanenobu and Marumoto [10] showed that the spun 2-knot of a classical torus knot has the unknotting number one. Hence it follows that the spun $T^2$-knot of a classical torus knot has the unknotting number one. Here, the spun $T^2$-knot of a classical knot $K$ is the product of $K$ in a 3-ball $B^3$ with a circle $S^1$, embedded into $\mathbb{R}^4$ via the natural embedding of $B^3 \times S^1$ into $\mathbb{R}^4$ ([15, 2]). In this paper, we show that for the closure of a classical braid which satisfies certain conditions, the spun $T^2$-knot of the classical knot has the unknotting number one (Theorem 3.1). Theorem 3.1 gives an alternative proof of the above-mentioned fact that the spun $T^2$-knot of a classical torus knot has the unknotting number one (Corollary 3.2). The proof of Theorem 3.1 is shown by a diagrammatic method, by using a surface link chart presenting the spun $T^2$-knot.

A surface link chart is a sort of finite graph in a 2-disk with some additional data ([5, 8, 9]). Any oriented surface knot is presented by a surface link chart ([7, 8, 9]). An unknotted surface knot is presented by an unknotted chart ([5, 9]). It is known [4] that any oriented surface knot $S$ can be deformed to an unknotted surface knot by applying 1-handle surgeries along a finite number of mutually disjoint oriented 1-handles. The unknotting number of $S$ is the minimum number of such 1-handles necessary to deform $S$ to be unknotted. A free edge is an edge in a chart such that the end points are vertices of degree one. Applying a 1-handle surgery to an oriented surface knot $S$ along a nice 1-handle is presented by adding a free edge to a surface link chart presenting $S$ ([6]).
Theorem 3.1 is shown as follows. First, we obtain a surface link chart presenting the spun $T^2$-knot. Then we add a free edge to the chart, and deform it to an unknotted chart (a configuration consisting of free edges) by equivalence relations. We obtain a surface link chart presenting the spun $T^2$-knot, as follows. We showed in [14] (see Theorem 2.2) how to obtain a surface link chart which presents a surface knot in the form of a branched covering over the standard torus. We call such a surface knot a torus-covering knot ([13], see Definition 2.1). Since a spun $T^2$-knot is a torus-covering knot, we can obtain a surface link chart presenting the spun $T^2$-knot by [14].

This paper is organized as follows. In Section 1, we review a braided surface and its chart description, and prepare several notations. In Section 2, we review the definition of a torus-covering knot and Theorem 2.2. In Section 3, we show Theorem 3.1 and Corollary 3.2.

1. A braided surface and its chart description

A braided surface was defined in [16, 7, 9]. A surface braid is a braided surface with some boundary condition, and a notion of a chart was introduced [5, 9] to present a simple surface braid. Equivalent simple surface braids have distinct chart presentations. The notion of C-move equivalence between two charts of the same degree was introduced [5, 8, 9] to give the equivalence class of the chart which represents the equivalence class of a simple surface braid. The notion of a chart can be easily extended to a chart presenting a simple braided surface. In this section, we review a braided surface, and extend the notion of a chart description to a simple braided surface. We review the fact that any oriented surface knot is presented by the closure of a simple surface braid ([7, 9]); thus it is presented by a chart. In order to present a certain chart called an “oval nest”, we introduce a notation, and we prepare several equivalence relations between oval nests.

**Definition 1.1.** A compact and oriented 2-manifold $S$ embedded in a bidisk $D_1 \times D_2$ properly and locally flatly is called a **braided surface** of degree $m$ if $S$ satisfies the following conditions:

(i) $p_2|_{S}: S \to D_2$ is a branched covering map of degree $m$, 
(ii) $\partial S$ is a closed $m$-braid in $D_1 \times \partial D_2$, where $D_1$, $D_2$ are 2-disks, and $p_2: D_1 \times D_2 \to D_2$ is the projection to the second factor.

Two braided surfaces are **equivalent** if there is a fiber-preserving ambient isotopy of $D_1 \times D_2$ rel $D_1 \times \partial D_2$ which carries one to the other. A braided surface $S$ is called **simple** if $\#(S \cap p_2^{-1}(x)) = m - 1$ or $m$ for each $x \in D_2$. A braided surface $S$ is called a **surface braid** if $\partial S$ is the trivial closed braid. A surface braid $Q_m \times D_2$ is called **trivial**, where $Q_m$ is a set of $m$ interior points of $D_1$.

When a simple braided surface $S$ is given, we obtain a graph on $D_2$, as follows. Identify $D_1$ with $I \times I$, where $I = [0, 1]$. Consider the singular set $\text{Sing}(p_1(S))$ of the
image of $S$ by the projection $p_1$ to $I \times D_2$. Perturbing $S$ if necessary, we can assume that $\text{Sing}(p_1(S))$ consists of double point curves, triple points, and branch points. Moreover we can assume that the singular set of the image of $\text{Sing}(p_1(S))$ by the projection to $D_2$ consists of a finite number of double points such that the preimages belong to double point curves of $\text{Sing}(p_1(S))$. Thus the image of $\text{Sing}(p_1(S))$ by the projection to $D_2$ forms a finite graph $\Gamma$ on $D_2$ such that the degree of its vertex is either 1, 4 or 6. An edge of $\Gamma$ corresponds to a double point curve, and a vertex of degree 1 (resp. 6) corresponds to a branch point (resp. triple point).

For such a graph $\Gamma$ obtained from a simple braided surface $S$, we give orientations and labels to the edges of $\Gamma$, as follows. Let us consider a path $\rho$ in $D_2$ such that $\rho \cap \Gamma$ is a point $P$ of an edge $e$ of $\Gamma$. Then $S \cap p_2^{-1}(\rho)$ is a classical $m$-braid with one crossing in $p_2^{-1}(\rho)$ such that $P$ corresponds to the crossing of the $m$-braid. Let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the $m$-braid group $B_m$. Let $\sigma_1^\epsilon$ ($i \in \{1, 2, \ldots, m-1\}, \epsilon \in \{+1, -1\}$) be the presentation of $S \cap p_2^{-1}(\rho)$. Then label the edge $e$ by $i$, and moreover give $e$ an orientation such that the normal vector of $\rho$ corresponds (resp. does not correspond) to the orientation of $e$ if $\epsilon = +1$ (resp. $-1$). We call such an oriented and labeled graph a chart of $S$.

In general, we define a chart on $D_2$ as follows.

**Definition 1.2.** Let $m$ be a positive integer. A finite graph $\Gamma$ on a 2-disk $D_2$ is called a chart of degree $m$ if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2$ consists of a finite number of vertices of degree 1.
(ii) Every edge is oriented and labeled by an element of $\{1, 2, \ldots, m-1\}$.
(iii) Every vertex has degree 1, 4, or 6.
(iv) The adjacent edges around each vertex in $\text{Int}(D_2)$ are oriented and labeled as shown in Fig. 1.1, where we depict a vertex of degree 1 by a black vertex, and a vertex of degree 6 by a white vertex.

In a chart, an edge without end points is called a loop. An edge whose end points are black vertices is called a free edge. A configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an oval nest.
A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple braided surface presented by the chart. A chart presents a simple braided surface. In particular, a chart $\Gamma$ such that $\Gamma \cap \partial D_2 = \emptyset$ presents a simple surface braid.

When a chart $\Gamma$ on $D_2$ is given, we can reconstruct a simple braided surface $S$ over $D_2$ as follows. Let $m$ be the degree of $\Gamma$, and let $N(\Gamma)$ be a neighborhood of $\Gamma$ in $D_2$. Let us consider a trivial braided surface $S = Q_m \times (D_2 - N(\Gamma))$ over $D_2 - N(\Gamma)$, where $Q_m$ is a set of $m$ interior points of $D_1$. We extend $S$ over a neighborhood of each edge as follows. Identify a neighborhood of an edge $e$ with $I \times I$ such that $e$ is identified with $\{1/2\} \times I$. Let $i$ be the label attached to $e$, and let $\varepsilon = +1$ (resp. $-1$) if the orientation of $e$ corresponds (resp. does not correspond) to the orientation of $\{0\} \times I$.

Then let the braided surface $S$ over the neighborhood of $e$ be the braided surface which has a presentation $\sigma_i^j \times I$ and the image of the double point curve of $p_1(S)$ by the projection to $D_2$ is $e$. Since $\Gamma$ is as in Fig. 1.1 around each vertex, $S$ can be extended naturally over a neighborhood of each vertex. See [3, 6, 9] for more details. Thus we can construct a simple braided surface $S$ over $D_2$ such that the original chart is a chart of $S$.

The boundary of a simple surface braid $S$ consists of trivial closed $m$-braid. Consider a natural embedding of $D_1 \times D_2$ in $\mathbb{R}^4$, and paste $m$ disks to $S$ to obtain an embedding of a closed surface in $\mathbb{R}^4$. The resulting surface is called the closure of $S$. It is known [7, 9] that any oriented surface knot is presented by the closure of a simple surface braid; thus it is presented by a chart $\Gamma$ on $D_2$ such that $\Gamma \cap D_2 = \emptyset$. We call such a chart presenting a surface link a surface link chart.

In [5, 9], a surface link chart is called simply a chart. However, in this paper we distinguish a “surface link chart” from a “chart”.

Two charts on $D_2$ of the same degree are C-move equivalent if they are related by a finite sequence of ambient isotopies of $D_2$ and C-moves (CI, CII, CIII-moves) as follows.

Let $\Gamma$ and $\Gamma'$ be two charts on $D_2$ of the same degree. Then $\Gamma'$ is said to be obtained from $\Gamma$ (or $\Gamma$ is said to be obtained from $\Gamma'$) by a CI-move, CII-move or CIII-move if there exists a 2-disk $E$ in $D_2$ such that the loop $\partial E$ is in general position with respect to $\Gamma$ and $\Gamma'$ and $\Gamma \cap (D_2 - E) = \Gamma' \cap (D_2 - E)$ and the following condition holds:

(CI) There are no black vertices in $\Gamma \cap E$ nor $\Gamma' \cap E$.

A CI-move as in Fig. 1.2 is called a CI-move of type (1), (2) or (3) respectively; see [9] for the complete set of CI-moves.

(CII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.3, where $|i - j| > 1$.

(CIII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.4, where $|i - j| = 1$.

It is shown as a minor modification of [5, 8, 9] that two simple braided surfaces of the same degree are equivalent if and only if their charts are C-move equivalent. Two surface knots are equivalent if there is an ambient isotopy of $\mathbb{R}^4$ which carries one to the other. Thus it follows that for two surface link charts of the same degree, their presenting surface knots are equivalent if the charts are C-move equivalent.
Fig. 1.2. CI-moves of types (1), (2) and (3).

Fig. 1.3. CII-moves, where $|i - j| > 1$.

Fig. 1.4. CIII-moves, where $|i - j| = 1$. 
Throughout this paper, let us denote the oval nest with a free edge with the label $i$ and its surrounding loops with the labels $i_1, i_2, \ldots, i_n$ and the orientation $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ from the free edge outward by $O(i; i_1^* i_2^* \cdots i_n^*)$, where $\epsilon_j = \pm 1$ and $i_j^* = i_j$ (resp. $\bar{i}_j$) if $\epsilon_j = +1$ (resp. $-1$) (see Fig. 1.5). In particular, let us denote the free edge $O(i; \emptyset)$ by $F_i$. For $0 < i < j$, let us denote $i(i+1) \cdots j$ (resp. $i(i+1) \cdots \bar{j}$) by $i \nearrow j$ (resp. $\bar{i} \nearrow \bar{j}$), and for $0 < j < i$, let us denote $i(i-1) \cdots j$ (resp. $\bar{i}(\bar{i}-1) \cdots \bar{j}$) by $i \searrow j$ (resp. $\bar{i} \searrow \bar{j}$).

Let $\Gamma_1$ and $\Gamma_2$ be charts of the same degree in 2-disks $D_1$ and $D_2$ respectively, where $D_i = [0, 1] \times [0, 1]$ for $i = 1, 2$. Identifying $D_1$ with $[0, 1] \times [0, 1/2]$ and $D_2$ with $[0, 1] \times [1/2, 1]$, we have a new chart $\Gamma_1 \cup \Gamma_2$ in $D_1 \cup D_2 = [0, 1] \times [0, 1]$. We will call it a split union of $\Gamma_1$ and $\Gamma_2$, and use the notation $\Gamma_1 \cup \Gamma_2$.

Let us define the braid group relations between two sequences of integers as follows:

1. $\emptyset \sim i \cdot \bar{i} \sim \bar{i} \cdot i$, for a positive integer $i$,
2. $i \cdot j \sim j \cdot i$, for positive integers $i$, $j$ with $|i - j| > 1$,
3. $i \cdot j \cdot i \sim j \cdot i \cdot j$, for positive integers $i$, $j$ with $|i - j| = 1$.

In this paper, we will identify a braid $\sigma_{i_1}^* \sigma_{i_2}^* \cdots \sigma_{i_n}^*$ with a sequence of integers $i_1^* i_2^* \cdots i_n^*$ with the braid group relations, where $i_j^* = i_j$ (resp. $\bar{i}_j$) if $\epsilon_j = +1$ (resp. $-1$). Then we have the following lemma.

**Lemma 1.3.** For positive integers $i$, $j$ and braids $b$, $c$ such that $\tilde{b}ib = \tilde{c}jc$, the following oval nests are equivalent:

\[ O(i; b) \sim O(j; c). \]  

(1.1)

Before the proof, we review the notions of a braid system of a chart and slide equivalence.

Let $\Gamma$ be a chart of degree $m$ on a 2-disk $D_2$. Let $q_0$ be a fixed point on the boundary of $D_2$, and $\Sigma(\Gamma)$ the set of black vertices in $\Gamma$. Let $\mathcal{A} = (a_1, a_2, \ldots, a_n)$ be a Hurwitz arc system with the starting point set $\Sigma(\Gamma)$ and the terminal point $q_0$, which is, for any $i$ and $j$, $a_i \cap a_j = \{q_0\}$ and the normal vector of $a_i$ points to $a_{i+1}$. For each
Let \( k, i = 1, 2, \ldots, n \), consider a loop \( c_i \) in \( D_2 \setminus \Sigma(\Gamma) \) with the base point \( q_0 \) such that it starts from \( q_0 \) and goes along \( a_i \), turns around the starting point of \( a_i \) (the black vertex in \( \Gamma \) which is at the other end of \( a_i \)) anti-clockwise and comes back along \( a_i \) to \( q_0 \). Let \( \eta_i \) be the element of \( \pi_1(D_2 \setminus \Sigma(\Gamma), q_0) \) represented by this loop \( c_i \). The fundamental group is a free group of rank \( n \) generated by \( \eta_1, \eta_2, \ldots, \eta_n \). We call \( \eta_1, \eta_2, \ldots, \eta_n \) the Hurwitz generators of \( \pi_1(D_2 \setminus \Sigma(\Gamma), q_0) \) associated with \( \mathcal{A} \). A braid system \( \tilde{b} = (b_1, b_2, \ldots, b_n) \) of the chart \( \Gamma \) is an ordered \( n \)-tuple of elements of \( B_m \) such that each \( b_i \) is the \( m \)-braid represented by \( \eta_i \), i.e. \( \eta_i \) in \( \pi_1(D_2 \setminus \Sigma(\Gamma), q_0) \) represents the \( m \)-braid \( b_i \) in the simple surface braid of degree \( m \) which is represented by \( \Gamma \) on \( D_2 \).

Two braid systems are slide equivalent if we can transform one to the other by applying a finite sequence of the following equivalence relations:

\[
(b_1, \ldots, b_i, b_{i+1}, \ldots, b_n) \sim (b_1, \ldots, b_{i-1}, b_{i+1}, b_i^{-1} b_{i+1} b_i b_{i+1}, b_{i+2}, \ldots, b_n).
\]

Two charts of the same degree are equivalent if and only if their braid systems are slide equivalent (see [7, Chapter 17 and Section 18.10]).

Proof of Lemma 1.3. We can take a braid system of \( \tilde{b} \) of \( O(i; b) \) to be \( \tilde{b} = (b^{-1} \sigma_i b, b^{-1} \sigma_i^{-1} b) \). Since \( \tilde{b} b \tilde{b} = \tilde{c} j \tilde{c} \), we have \( \tilde{b} = (c^{-1} \sigma_j c, c^{-1} \sigma_j^{-1} c) \), which is a braid system of \( O(j; c) \).

By Lemma 1.3, in particular the following equivalent deformations hold. We will prove several of them using C-moves. Let \( i, j \) be positive integers and \( b, b', c, c' \) be braids. For a positive integer \( k \), Let \( k^* \in \{k, \overline{k}\} \). If \( b = b' \), then

1. \( O(i; b) \sim O(i; b') \).
2. \( O(i; i^*) \sim O(i; \emptyset) = F_i \) (see Fig. 1.6),
3. \( O(i; j^*) \sim O(i; \emptyset) = F_j \), where \( |i - j| > 1 \) (see Fig. 1.7),
4. \( O(i; j) \sim O(j; \overline{i}) \), where \( |i - j| = 1 \) (see Fig. 1.8).

If \( O(i; c) \sim O(j; c') \), then

5. \( O(i; cb) \sim O(j; c'b) \).

Moreover, applying a CI-move of type (2) between the outermost loop labeled \( j \) of the oval nest \( O(i; b \cdot j^*) \) and the free edge \( F_j \), we can see that

6. \( O(i; b \cdot j^*) \cup F_j \sim O(i; b) \cup F_j \),

where \( b \) is a braid.
Fig. 1.6. $O(i; i) \sim O(i; \emptyset) = F_i$.

Fig. 1.7. $O(i; j) \sim O(i; \emptyset) = F_i$, where $|i - j| > 1$.

Fig. 1.8. $O(i; j) \sim O(j; i)$, where $|i - j| = 1$. 
2. A torus-covering knot and its chart description

It is known [7, 9] that any oriented surface knot can be presented by a branched covering over the standard 2-sphere. A torus-covering knot was introduced in [13] as a new construction of a surface knot, by considering the standard torus instead of the standard 2-sphere. In this section, we give the definition of a torus-covering knot (see also [13]). The spun $T^2$-knot of a classical knot is a torus-covering knot. A torus-covering knot is presented by a chart on the standard torus $T$. We can obtain a surface link chart presenting a torus-covering knot from its chart on $T$ ([14], see Theorem 2.2). Part of the obtained surface link chart is called a 1-handle chart. We obtain the 1-handle chart for the spun $T^2$-knot (Lemma 2.4).

Let $T$ be a standard torus in $\mathbb{R}^4$, that is, the boundary of an unknotted solid torus in a 3-space in $\mathbb{R}^4$. Let us consider a tubular neighborhood $N(T)$ of $T$, and identify $N(T)$ with $D^2/\mathbb{Z}$, where $D^2$ is a 2-disk, and $\mathbb{Z}$ is a circle. The first $\mathbb{Z}$ corresponds to the meridian, and the second $\mathbb{Z}$ corresponds to the longitude of $T$. Let us identify $\mathbb{Z}$ with $I/\sim$, where $I = [0, 1]$ and $0 \sim 1$. For a manifold $S$ in $N(T)$, let us denote by $S \cap (D^2 \times I \times I)$ the manifold in $D^2 \times I \times I$ obtained from $S$ by cutting it at $D^2 \times \{0\} \times \mathbb{Z}$ and $D^2 \times \{1\} \times \mathbb{Z}$.

**Definition 2.1.** A torus-covering knot is a surface knot $S$ in $\mathbb{R}^4$ such that $S \subset N(T)$ and moreover $S \cap (D^2 \times I \times I)$ is a simple braided surface.

By definition, a torus-covering knot $S$ is presented by a chart on $T$. As we mentioned, for two charts of the same degree, their presenting braided surfaces are equivalent if the charts are C-move equivalent. Hence it follows that for two charts on $T$ of the same degree, their presenting torus-covering knots are equivalent if the charts are C-move equivalent.

The spun $T^2$-knot of a classical knot $K$ is the product of $K$ in a 3-ball $B^3$ with $\mathbb{Z}$, embedded into $\mathbb{R}^4$ via the natural embedding of $B^3 \times \mathbb{Z}$ into $\mathbb{R}^4$ ([15, 2]). Identify $\mathbb{Z}$ with the longitude of $T$. Since any classical knot is equivalent to a closed braid by Alexander’s Theorem, the spun $T^2$-knot of any $K$ is a torus-covering knot (see [13, Propositions 2.11]); see Fig. 2.1 for example.

Now we review a theorem, which shows how to obtain a surface link chart from a chart on $T$ ([14]). A chart is presented by a simple braided surface. A simple braided surface is presented by a motion picture consisting of isotopic transformations
and hyperbolic transformations. A motion picture of a braided surface \( S \subset B^3 \times I \) is a one-parameter family \( \{ \tau(S \cap (B^3 \times \{ t \})) \}_{t \in I} \), where \( \tau : B^3 \times I \to B^3 \) is the projection (see [9]).

Let \( \{ h_t \}_{t \in [0,1]} \) be an ambient isotopy of \( \mathbb{R}^3 \). For a classical link \( L \), we have an isotopy (a one-parameter family) \( \{ h_t(L) \} \) of classical links. We say that \( h_1(L) \) is obtained from \( L \) by an isotopic transformation, and we use the notation that \( L \to h_1(L) \) is an isotopic transformation (see [9, Section 9.1]).

Let \( L \) be a classical link in \( \mathbb{R}^3 \). A 2-disk \( B \) in \( \mathbb{R}^3 \) is called a band attaching to \( L \) if \( L \cap B \) is a pair of disjoint arcs in \( \partial B \). A band set attaching to \( L \) is a union \( \mathcal{B} = B_1 \cup B_2 \cup \cdots \cup B_m \) of mutually disjoint bands \( B_1, B_2, \ldots, B_m \) attaching to \( L \).

For a subset \( X \) of a space, let us denote by \( \text{Cl}(X) \) the closure of \( X \). Define a link \( h(L; \mathcal{B}) \) by

\[
h(L; \mathcal{B}) = \text{Cl}((L \cup \partial \mathcal{B}) - (L \cap \mathcal{B})).
\]

We say that the link \( h(L; \mathcal{B}) \) is obtained from \( L \) by a hyperbolic transformation along \( \mathcal{B} \), and we use the notation that \( L \to h(L; \mathcal{B}) \) is a hyperbolic transformation (see [9, Section 9.1]).

For a classical \( m \)-braid \( c \), let \( l_i^k(c) \) be the \((m + k + l)\)-braid obtained from \( c \) by adding \( k \) (resp. \( l \)) trivial strings before (resp. after) \( c \), and put

\[
\begin{align*}
\Pi_i^m &= \sigma_m \cdots \sigma_{m+i}, \\
\Delta^m &= \Pi_m - 1 \cdot \Pi_{m-1} \cdot \cdots \cdot \Pi_1, \\
\Theta^m &= \sigma_m \cdot \Pi_{m-1} \cdot \cdots \cdot \Pi_1.
\end{align*}
\]

**Theorem 2.2** ([14]). Let \( \Gamma_T \) be a chart of degree \( m \) on \( I \times I \), obtained from a chart on \( T \) (of degree \( m \)) by cutting \( T \) by the meridian and the longitude. Let \( \Gamma \) (resp. \( b \)) be a classical \( m \)-braid presented by \( \Gamma_T \cap (I \times \{0\}) \) (resp. \( \Gamma_T \cap (\{0\} \times I) \)). Then the torus-covering knot presented by \( \Gamma_T \) is presented by a surface link chart \( \Gamma_B \) of degree \( 2m \) as in Fig. 2.2. Here \( H_b \) is a chart of degree \( 2m \) presenting the simple braided surface whose motion picture is as follows:

\[
\begin{align*}
l_0^m(b) \to l_0^m(b) \cdot (\Delta_m)^{-1} \cdot \Delta_m \cdot \Delta_m &\to l_0^m(b) \cdot (\Delta_m)^{-1} \cdot \Delta_m \cdot \Theta_m \\
&\to (\Delta_m^{-1} \cdot \Delta_m) \cdot l_0^m(\tilde{b}^*) \cdot \Theta_m &\to (\Delta_m^{-1} \cdot \Delta_m) \cdot l_0^m(\tilde{b}^*) \to l_0^m(\tilde{b}^*),
\end{align*}
\]

where \( \to \) is an isotopic transformation and \( \tilde{\cdot} \) is a hyperbolic transformation along bands corresponding to the \( m \) \( \sigma_m \)'s, and \( -(H_b)^* \) is the orientation-reversed mirror image of \( H_b \), and \( \tilde{b}^* \) is the \( m \)-braid obtained from the classical \( m \)-braid \( b \) by taking its mirror image and reversing all the crossings.

**Definition 2.3.** We call \( H_b \) the 1-handle chart of \( \Gamma_T \).
Let us consider the spun $T^2$-knot of $\hat{b}$, where $\hat{b}$ denotes the closure of a classical braid $b$. Let us determine $\Gamma_T$ on $I \times I$ to be a chart presenting the braided surface $b \times I$; then the braids presented by $\Gamma_T \cap (I \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I)$ are $b$ and $e$ respectively, where $e$ is the trivial braid. The 1-handle chart of $\Gamma_T$ is $H_e$. We obtain $H_e$, as follows.

**Lemma 2.4.** Let $e$ be the trivial $m$-braid. Then the 1-handle chart $H_e$ is equivalent to the chart as follows:

$$H_e \sim \bigcup_{k=0}^{m-1} O_k,$$

where $O_k$ is the oval nest

$$O_k = O \left( m; \prod_{j=0}^{k-1} (m - 1 - m - k + j) \prod_{j=0}^{k-1} (m + 1 - m + k - j) \right)$$

for $k = 0, 1, 2, \ldots, m - 1$. Note that for $k = 0$, $O_0 = O(m; \emptyset) = F_m$.

**Proof.** By Theorem 2.2, $H_e$ is a chart presenting the simple braided surface as follows:

$$e \rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m$$

$$\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$$

$$\rightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \rightarrow e,$$

Fig. 2.2. The surface link chart $\Gamma_S$ of degree $2m$. 

[Diagram of the surface link chart $\Gamma_S$ of degree $2m$.]
where → means an isotopy transformation and → means a hyperbolic transformation along bands corresponding to the $m$ $\sigma_m$’s. Here $e$ is the trivial $2m$-braid. Note that since (2.1) presents a simple surface braid, $H_e$ does not have a boundary. The 1-handle chart $H_e$ has $m$ free edges, whose labels are all $m$. All the other edges have labels other than $m$ and neither of them is connected with a black vertex. Draw $H_e$ on $[0, 1/2] \times [0, 1]$ such that we can read the braids $e, (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Delta_m', (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m, (\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$ and $e$ of (2.1) at $[0, 1/2] \times \{t_1\}, \ldots, [0, 1/2] \times \{t_5\}$ respectively, where $0 < t_1 < \cdots < t_5 < 1$. Let $q_k$ ($k = 0, 1, \ldots, m - 1$) be the black vertex corresponding to the $(m - k)$-th $\sigma_m$ of the braid $(\Delta_m')^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$.

Let us denote by $F_k$ the free edge connected with the black vertex $q_k$. Let us move the free edges into $[1/2, 1] \times [0, 1]$ using CI-moves of type (2) as in Fig. 2.3 to be a split sum of oval nests and $H'_e$, where $H'_e$ is the chart $H_e - \bigcup_{k=0}^{m-1} F_k$. Since $H'_e$ has no black vertices, we can eliminate it by a CI-move. Thus we have a split sum of oval nests.

Let $\tilde{O}_k$ ($k = 0, 1, \ldots, m - 1$) be the oval nest $F_k$ becomes. We will see that each $\tilde{O}_k$ is equivalent to the oval nest $O_k$. First we will obtain $\tilde{O}_k$. It suffices to see what edges $F_k$ crosses as it moves into $[1/2, 1] \times [0, 1]$. We have

$$\Theta_m = \sigma_m \cdot \Pi_{m-1}^m \cdot \Pi_{m-1}^m \cdot \sigma_m \cdot \Pi_{m-2}^m \cdot \Pi_{m-2}^m \cdots \sigma_m \cdot \Pi_1^m \cdot \Pi_1^m \cdot \sigma_m.$$ 

The first free edge $F_0$ does not cross any edge. Hence $\tilde{O}_0 = F_0$. Then the second free edge $F_1$ crosses edges representing $\Pi_1^m \cdot \Pi_1^m = \sigma_{m-1} \sigma_{m+1}$, so it becomes the oval nest $\tilde{O}_1 = O(m; m - 1m + 1)$. The third free edge $F_2$ crosses edges representing $\Pi_2^m \cdot \Pi_1^m \cdot \Pi_1^m = (\sigma_{m-1} \sigma_{m-2}) (\sigma_{m+1} \sigma_{m+2}) \sigma_{m-1} \sigma_{m+1}$. Hence it becomes the oval nest $\tilde{O}_2 = O(m; (m - 1)(m - 2) \cdot (m + 1)(m + 2) \cdot (m - 1) \cdot (m + 1))$. Repeating this step, we see that in general $F_k$ crosses edges representing $\Pi_k^m \cdot \Pi_k^m \cdot \Pi_{k-1}^m \cdot \Pi_{k-1}^m \cdots \Pi_1^m \cdot \Pi_1^m$, so it becomes an oval nest $\tilde{O}_k = O(m; \prod_{j=0}^{k-1}((m - 1) \cdot (m - k + j) \cdot (m + 1) \cdot (m + k - j)))$ for $k = 0, 1, \ldots, m - 1$.

---

**Fig. 2.3. Moving a free edge across an edge.**
We can show that if \( i + 1 < k \) then \((i \not\sim j)(k \not\sim l)\) can be transformed to \((k \not\sim l)(i \not\sim j)\) by the braid group relation 2, i.e.

\[
(i \not\sim j)(k \not\sim l) \sim (k \not\sim l)(i \not\sim j).
\]

Using (2.2), we see that \( O_k \sim O(m; \prod_{j=0}^{l-1}(m-1 \not\sim m-k+j) \prod_{j=0}^{l-1}(m+1 \not\sim m+k-j)) \), which is \( O_k \).

3. Main theorem

An oriented surface knot is unknotted if it is equivalent to the connected sum of several standard tori. It is known [4] that any oriented surface knot \( S \) can be deformed to an unknotted surface knot by applying a finite number of oriented 1-handle surgeries. The unknotting number of \( S \) is the minimum number of oriented 1-handle surgeries necessary to deform it to be unknotted. In this section, we show Theorem 3.1 and Corollary 3.2.

**Theorem 3.1.** Let \( S \) be the spun \( T^2 \)-knot of a classical knot \( \hat{b} \), where \( b \) is a classical \( m \)-braid \((m > 1)\) such that there exists a permutation \( \tau \) of degree \( m-1 \) which satisfies the following conditions:

(a1) There is an integer \( r \in \{1, 2, \ldots, m-1\} \) such that for each \( k \in \{1, 2, \ldots, m-1\} - \{r\} \), \( \sigma_k \cdot b = b \cdot \sigma_{\tau(k)} \), and

(a2) For each \( i, j \in \{1, 2, \ldots, m-1\} \), if \( i \neq j \), then \( \tau^i(1) \neq \tau^j(1) \). Note that then \( \tau^{m-1}(1) = 1 \).

Moreover assume that \( S \) is not unknotted. Then the unknotting number of \( S \) is one.

By Theorem 3.1 we have an alternative proof of the fact [10] that the spun \( T^2 \)-knot of a torus \((p, q)\)-knot has the unknotting number one.

**Corollary 3.2.** The spun \( T^2 \)-knot of a classical torus \((p, q)\)-knot has the unknotting number one.

Proof. First we show that the spun \( T^2 \)-knot is not unknotted. The knot group of the spun \( T^2 \)-knot of a classical torus \((p, q)\)-knot is isomorphic to the knot group of the classical torus \((p, q)\)-knot ([15]). Hence we can see that the spun \( T^2 \)-knot is not unknotted.

We determine the braid \( b \) and the permutation \( \tau \), as follows. A classical torus \((p, q)\)-knot is presented by the closure of the \( p \)-braid \( b = (\sigma_1 \sigma_2 \cdots \sigma_{p-1})^q \), where \( p \) and \( q \) are coprime integers and moreover \( p > 1 \). Let \( r \) be defined by \( q \mod p \) such that \( r \in \{0, 1, 2, \ldots, p-1\} \). Since \( p \) and \( q \) are coprime, \( r \neq 0 \) and it follows that \( r \in \{1, 2, \ldots, p-1\} \). Let us define a permutation \( \tau \) of degree \( p-1 \) by

\[
\left( \begin{array}{cccccc}
1 & 2 & \cdots & r-1 & r & r+1 \\
p-r+1 & p-r+2 & \cdots & p-1 & p-r & 1 \\
& & & \tau & & \\
& & & & & 
\end{array} \right).
\]
We show that Condition (a1) of Theorem 3.1 holds, as follows. If \( k \neq 1 \), then we can show that \( \sigma_k(\sigma_1\sigma_2\cdots\sigma_{p-1}) = (\sigma_1\sigma_2\cdots\sigma_{p-1})\sigma_{k-1} \). Similarly we have \( \sigma_1(\sigma_1\sigma_2\cdots\sigma_{p-1})^2 = (\sigma_1\sigma_2\cdots\sigma_{p-1})^2\sigma_{p-1} \). From these two equations, we have \( \sigma_k \cdot b = b \cdot \sigma_{\tau(k)} \) for each \( k \in \{1, 2, \ldots, p-1\} - \{r\} \). Thus Condition (a1) holds.

Next we will show that \( \tau \) satisfies Condition (a2) of Theorem 3.1. Let us define Condition (a2)' as follows.

(a2)' The permutation \( \tau \) is associated with a classical braid \( c \) such that \( \hat{c} \) is a knot, i.e. \( \hat{c} \) is connected.

We can see that if the permutation \( \tau \) satisfies (a2)', then (a2) holds, as follows. If Condition (a2) does not hold, then \( \tau^i(1) = \tau^j(1) \) for some \( i, j \in \{1, 2, \ldots, p-1\} \) with \( i \neq j \). We can assume that \( j > i \). Then we have \( \tau^{j-i}(1) = 1 \), where \( 0 < j-i < p-1 \).

On the other hand, if \( \tau \) is associated with a classical braid \( c \) such that \( \hat{c} \) is a knot, then \( \tau^k(1) \neq 1 \) for any \( k \) with \( 0 < k < p-1 \). This is a contradiction.

From now on we will show that \( \tau \) satisfies (a2)'. Since \( r \in \{1, 2, \ldots, p-1\} \) with \( r = q \) mod \( p \), and \( p \) and \( q \) are coprime integers, we see that \( r = 1 \) or \( p \) and \( r \) are coprime. If \( r - 1 = p - r - 1 \), then \( p = 2r \). Since \( r = 1 \) or \( p \) and \( r \) are coprime, we have \( r = 1 \) and \( p = 2 \). Then \( \tau = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \), which is associated with the trivial braid of degree one, whose associated closed braid is a trivial knot. Hence we can assume that \( r - 1 \neq p - r - 1 \). If \( r - 1 < p - r - 1 \), then the permutation \( \tau \) is associated with a braid whose diagram is as in Fig. 3.1, where we omit the crossing information. Here we have \( \tau(r - j) = p - j \) for \( j = 1, 2, \ldots, r - 1 \) and \( \tau(p - j) = p - r - j \) for \( j = 1, 2, \ldots, p - r - 1 \). Hence we have \( \tau^2(r - j) = \tau(p - j) = p - r - j \) for \( j = 1, 2, \ldots, r - 1 \), which means that the \( (r-j) \)-th string of the closed braid is connected with the \( (p-j) \)-th string, which is connected with the \( (p-r-j) \)-th string. Hence we can assume that there is no \( (p-j) \)-th string, and the \( (r-j) \)-th string of the closed braid is connected with the \( (p-r-j) \)-th string, where \( j = 1, 2, \ldots, r - 1 \) (see Fig. 3.1).
Thus it suffices to show that the following permutation satisfies (a2):

\[
(3.1) \begin{pmatrix}
1 & 2 & \cdots & r & r+1 & r+2 & \cdots & p-r \\
p-2r+1 & p-2r+2 & \cdots & p-r & 1 & 2 & \cdots & p-2r
\end{pmatrix}.
\]

Similarly, if \( r-1 > p-r-1 \), then we have \( \tau^{2}(r-j) = \tau(p-j) = p-r-j \) for \( j = 1, 2, \ldots, p-r-1 \). Hence it suffices to show that the following permutation satisfies (a2):

\[
(3.2) \begin{pmatrix}
1 & 2 & \cdots & 2r-p & 2r-p+1 & 2r-p+1 & \cdots & r \\
p-r+1 & p-r+2 & \cdots & r & 1 & 2 & \cdots & p-r
\end{pmatrix}.
\]

If \( r-1 < p-r-1 \) (resp. \( r-1 > p-r-1 \)), then \( p-r > r \) (resp. \( r > p-r \)). Hence together with \( 1 \leq r \leq p-1 \), we can see that \( p-r > 1 \) (resp. \( r > 1 \)). Thus the permutation \((3.1)\) (resp. \((3.2)\)) is associated with the \( m \)-braid \( c = (\sigma_{1}\sigma_{2}\cdots\sigma_{m-1})^{n} \), where \( m = p-r \) (resp. \( r \)) is the degree of \((3.1)\) (resp. \((3.2)\)) with \( m > 1 \), and \( n = m - \tau(1) + 1 \). Since for \((3.1)\) (resp. \((3.2)\)) we have \( m = p-r \) (resp. \( r \)) and \( \tau(1) = p-2r+1 \) (resp. \( p-r+1 \)), it follows that \((m,n) = (p-r,r)\) (resp. \((m,n) = (r,2r-p)\)). Note that in both cases \( n > 0 \). Since \( r = 1 \) or \( p \) and \( r \) are coprime, together with \( m > 1 \) and \( n > 0 \), it follows that in both cases \( n = 1 \) or \( m \) and \( n \) are coprime. If \( n = 1 \), then \( \hat{\tau} (c = \sigma_{1}\sigma_{2}\cdots\sigma_{m-1}) \) is a trivial knot, and if \( m \) and \( n \) are coprime, then \( \hat{\tau} (c = (\sigma_{1}\sigma_{2}\cdots\sigma_{m-1})^{n}) \) is a torus \((m,n)\)-knot. Thus \( \tau \) satisfies \((a2)'\), and it follows that \( \tau \) satisfies \((a2)\). Therefore the spun \( T^{2}\)-knot has the unknotting number one by Theorem 3.1.

Proof of Theorem 3.1. We show that the unknotting number of \( S \) is one. Let \( \Gamma_{S} \) be a surface link chart presenting \( S \). An unknotted chart is a chart presented by a configuration consisting of free edges ([5]). An unknotted oriented surface knot is presented by an unknotted chart ([5]). For an oriented surface knot, adding a free edge to the surface link chart corresponds to a nice 1-handle surgery, which is an oriented 1-handle surgery ([6]). Thus it suffices to see that the surface link chart obtained from \( \Gamma_{S} \) by adding a free edge is equivalent to an unknotted chart.

We will determine \( \Gamma_{S} \) by [14] (see Theorem 2.2). The chart \( \Gamma_{T} \) on \( I \times I \) presents the braided surface \( b \times I \); thus the braids presented by \( \Gamma_{T} \cap (I \times \{0\}) \) and \( \Gamma_{T} \cap (\{0\} \times I) \) are \( b \) and \( e \) respectively (see Section 2). By Lemma 2.4 and (3.26) of Lemma 3.3, we can assume that the 1-handle chart \( H_{e} \) is as follows:

\[
H_{e} = \bigcup_{k=0}^{m-1} O_{k},
\]

where

\[
(3.3) \quad O_{k} = O(m-k; (m-k+1) \not\div \overline{m})(m+1 \not\div m+k).
\]
Let us define and $O'_k$ as follows:

$$O'_k = O(m - k; (m - k + 1 \notdiv m)(m + 1 \notdiv m + k) \cdot b).$$

The oval nest $O'_k$ is obtained from $O_k$ by adding loops describing $b$ around it. By [14] (see Theorem 2.2), the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is as follows:

$$\Gamma_S = \bigcup_{i=0}^{m-1} O_i \cup \bigcup_{i=0}^{m-1} O'_i.$$  

Remark that $\Gamma_S$ is a ribbon chart of degree $2m$ (see [5, 9]).

We will show that the surface link chart $\Gamma_S$ can be deformed to an unknotted chart by adding a free edge.

**Step 1.** We show that

$$O_{m-k-1} \cup O_{m-k} \cup F_{2m-k} \sim O_{m-k-1} \cup F_k \cup F_{2m-k},$$

for $k \in \{1, 2, \ldots, m - 1\}$.

By (3.3) and (1.7), we have

$$O_{m-k} \cup F_{2m-k}$$

$$= O(k; (k + 1 \notdiv m)(m + 1 \notdiv 2m - k)) \cup F_{2m-k}$$

$$\sim O(k; (k + 1 \notdiv m)(m + 1 \notdiv 2m - k - 1)) \cup F_{2m-k}.$$  

Let us denote $O(k; (\overline{k + 1} \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1))$ by $\hat{O}_{m-k}$. By (1.7) we have

$$O(k + 1; \emptyset) \cup O(k; \overline{k + 1}) \sim O(k + 1; \emptyset) \cup O(k; \emptyset).$$

Hence we have

$$O(k + 1; c) \cup O(k; \overline{k + 1} \cdot c) \sim O(k + 1; c) \cup O(k; c)$$

for a braid $c$ by (1.6). By (3.3) and (3.6) we have

$$O_{m-k-1} \cup \hat{O}_{m-k}$$

$$= O(k + 1; (k + 2 \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1))$$

$$\cup O(k; (k + 1) \cdot (k + 2 \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1))$$

$$\sim O(k + 1; (k + 2 \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1))$$

$$\cup O(k; (k + 2 \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1))$$

$$= O_{m-k-1} \cup O(k; (k + 2 \notdiv \overline{m})(m + 1 \notdiv 2m - k - 1)).$$
By (1.4) we see that
\[ O(k; (k + 2 \cdot m)(m + 1 \cdot 2m - k - 1)) \sim O(k; \emptyset) = F_k. \]

Thus we have
\[ O_{m-k-1} \cup O_{m-k} \cup F_{2m-k} \sim O_{m-k-1} \cup F_k \cup F_{2m-k}. \]

STEP 2. Similarly, we show that
\[ O'_{m-k-1} \cup O'_{m-k} \cup F_{\tau(k)} \sim O'_{m-k-1} \cup F_{2m-k} \cup F_{\tau(k)}, \]
for \( k \in \{1, 2, \ldots, m-1\} \setminus \{r\}. \)

By (a1), \( b^{-1}\sigma_kb = \sigma_{\tau(k)} \) for \( k \in \{1, 2, \ldots, m-1\} \setminus \{r\}. \) Hence we have
\[ O(k; b) \sim F_{\tau(k)} \]
for \( k \in \{1, 2, \ldots, m-1\} \setminus \{r\} \) by Lemma 1.3.

Similarly to Step 1, using (3.27) of Lemma 3.3, we have
\[ O_{m-k-1} \cup O_{m-k} \cup F_k \sim O_{m-k-1} \cup F_k \cup F_{2m-k} \]
for \( k \in \{1, 2, \ldots, m-1\}. \) By (3.8) and (1.6), we have
\[ O'_{m-k-1} \cup O'_{m-k} \cup O(k; b) \sim O'_{m-k-1} \cup O(k; b) \cup O(2m-k; b). \]

By (3.7), \( O(k; b) \sim F_{\tau(k)} \) for \( k \in \{1, 2, \ldots, m-1\} \setminus \{r\}. \) On the other hand, by (1.4) and \( 2m-k > (m-1)+1, \) we have \( O(2m-k; b) \sim O(2m-k; \emptyset) = F_{2m-k}. \) Hence together with (3.9), we see that
\[ O'_{m-k-1} \cup O'_{m-k} \cup F_{\tau(k)} \sim O'_{m-k-1} \cup F_{2m-k} \cup F_{\tau(k)} \]
for \( k \in \{1, 2, \ldots, m-1\} \setminus \{r\}. \)

STEP 3. Let us denote Step 1 as follows:
\[ \phi_l: O_{l-1} \cup O_l \cup F_{m+l} \to O_{l-1} \cup F_{m-l} \cup F_{m+l} \]
for \( l \in \{1, 2, \ldots, m-1\}, \) and Step 2 as
\[ \psi_l: O'_{l-1} \cup O'_l \cup F_{\tau(m-l)} \to O'_{l-1} \cup F_{m+l} \cup F_{\tau(m-l)} \]
for \( l \in \{1, 2, \ldots, m-1\} \setminus \{m-r\}. \)
We introduce several notations to make things easy to see. Let us define $F^l$, $F^d$ and $F^{rd}$ as follows:

\begin{align}
F^l &:= F_{m-l}, \\
F^d &:= F_{m+l}, \\
F^{rd} &:= F_{r(m-l)}
\end{align}

for $l \in \{1, 2, \ldots, m-1\}$. Moreover, for an integer $s$, let us define $\tau_s$ to be

\begin{equation}
\tau_s := m - \tau^{-s}(r).
\end{equation}

Step 1 is written as follows:

\begin{equation}
\phi_l : O_{l-1} \cup O_l \cup F^d \to O_{l-1} \cup F^l \cup F^d,
\end{equation}

for $l \in \{1, 2, \ldots, m-1\}$, and Step 2 is

\begin{equation}
\psi_l : O'_{l-1} \cup O'_l \cup F^{rd} \to O'_{l-1} \cup F^d \cup F^{rd},
\end{equation}

for $l \in \{1, 2, \ldots, m-1\} - \{m-r\}$. Since by definition (3.13) $m-r = \tau_0$, Step 2 holds true for $l \in \{1, 2, \ldots, m-1\} - \{\tau_0\}$.

From now on we show that $\Gamma_S$ can be deformed to an unknotted chart by adding a free edge $F_r$. Let us define charts $I_0, I_1, \ldots, I_{2m-4}$ of degree $2m$. First, define $I_0$ as follows:

\begin{equation}
I_0 := \Gamma_S \cup F_r,
\end{equation}

which is by (3.5) as follows:

\begin{equation}
I_0 = O_0 \cup O_{\tau_1} \cup \cdots \cup O_{\tau_{2m-2}} \cup O_{\tau_0} \cup O'_{\tau_1} \cup \cdots \cup O'_{\tau_{2m-2}} \cup O'_{\tau_0} \cup F_r.
\end{equation}

Note that by (a2), $\{\tau_0, \tau_1, \ldots, \tau_{m-2}\} = \{1, 2, \ldots, m-1\}$. For $n = 1, 2, \ldots, m-2$, let us define $I_{2n}$ as follows:

\begin{equation}
I_{2n} := O_0 \cup O_{\tau_{n+1}} \cup \cdots \cup O_{\tau_{2n}} \cup O_{\tau_0} \cup F_{\tau_1} \cup F_{\tau_2} \cup \cdots \cup F_{\tau_n} \cup O'_{\tau_{n+1}} \cup \cdots \cup O'_{\tau_{2n}} \cup O'_{\tau_0} \cup F'_{\tau_1} \cup F'_{\tau_2} \cup \cdots \cup F'_{\tau_n} \cup F_r.
\end{equation}

And for $n = 0, 1, 2, \ldots, m-3$, let us define $I_{2n+1}$ as follows:

\begin{equation}
I_{2n+1} := (I_{2n} - O'_{\tau_{n+1}}) \cup F'_{\tau_{n+1}}.
\end{equation}
We will show that $I_{2n+1}$ (resp. $I_{2n+2}$) is obtained from $I_{2n}$ (resp. $I_{2n+1}$) by applying Steps 2 (resp. Steps 1) for $n = 0, 1, \ldots, m - 3$.

When we have $I_{2n}$ ($n = 0, 1, \ldots, m - 3$), there is an integer $l_0 < \tau_{n+1}$ such that for any $l$ with $l_0 < l < \tau_{n+1}$, $O'_l \not\subseteq I_{2n}$ and $O'_{l_0} \subseteq I_{2n}$. Note that such an $l_0$ exists, for $0 < \tau_{n+1}$ and $O'_{l_0} \subseteq I_{2n}$ for every $n \in \{0, 1, \ldots, m - 3\}$. Since $r = \tau^0(r) = \tau(m - (m - \tau^{-1}(r))) = \tau(m - \tau_1)$, by the definition of $F^{rl}$ (3.12) we have

$$F_r = F^{rl_0}.$$  

For $n = 0$, by (3.16) we have $l_0 = \tau_1 - 1$, and by (3.19) we see that

$$I_0 \supset F^{rl_0} \cup O'_{\tau_1} \cup O'_{\tau_1}.  

By the definitions (3.10) and (3.13), we have

$$F_{\tau^m(r)} = F_{m-(m-\tau^m(r))} = F^{m-\tau^m(r)} = F^r,$$

and by the definitions (3.12) and (3.13), we have

$$F_{\tau^m(r)} = F_{\tau(\tau^{-1}(r))} = F^{\tau^{-1}(r)} = F^{rl_{\tau_1}}.$$

Hence we have

$$F^r = F^{rl_{\tau_1}},$$

for each $s$. By the definition of $I_{2n}$ (3.17) and (3.21) we have

$$I_{2n} = O_0 \cup O_{\tau_1} \cup O_{\tau_2} \cup \cdots \cup O_{\tau_{n-1}} \cup O_{\tau_n} \cup F^{rl_0} \cup F^{rl_2} \cup \cdots \cup F^{rl_{\tau_1}}$$

$$\cup O'_{\tau_1} \cup O'_{\tau_2} \cup \cdots \cup O'_{\tau_{n-1}} \cup O'_{\tau_n} \cup F^{rl_0} \cup F^{rl_2} \cup \cdots \cup F^{rl_{\tau_1}}$$

$$\cup F_r,$$

for $n = 1, 2, \ldots, m - 3$. By (3.22) and (3.19), we can see that if $F^{rl} \subseteq I_{2n}$, then $F^{rl} \subseteq I_{2n}$. So together with (3.20), we have

$$I_{2n} \supset F^{rl_0 + 1} \cup F^{rl_0 + 2} \cup \cdots \cup F^{rl_{\tau_1} - 1} \cup F^{rl_{\tau_1}}$$

$$\cup O'_{l_0} \cup F^{rl_0 + 1} \cup F^{rl_0 + 2} \cup \cdots \cup F^{rl_{\tau_1} - 1} \cup O'_{\tau_{n+1}}$$

for $n = 0, 1, \ldots, m - 3$. By (3.22), we can see that if $F^{rl} \subseteq I_{2n}$, then $l \in \{\tau_1, \tau_2, \ldots, \tau_n\}$. Hence $l_0 + 1, l_0 + 2, \ldots, \tau_{n+1} - 1 \in \{\tau_1, \tau_2, \ldots, \tau_n\}$. By (a2) and $n \leq m - 3$, none of $l_0 + 1, l_0 + 2, \ldots, \tau_{n+1} - 1, \tau_{n+1}$ is $\tau_0$. So we can apply Steps 2 (3.15) and its inverses to $I_{2n}$ to deform $O'_{\tau_{n+1}}$ to $F^{\tau_{n+1}}$. The result is $I_{2n+1}$ by the definition of $I_{2n+1}$ (3.18):

$$\psi_{l_0 + 1} \circ \cdots \circ \psi_{\tau_{n+1}} \circ \psi_{\tau_{n+1}^{-1}} \circ \cdots \circ \psi_{l_0 + 2} \circ \psi_{l_0 + 1}(I_{2n}) = I_{2n+1}.$$
for \( n = 0, 1, \ldots, m - 3 \).

By (3.16) and (3.17), we see that if \( O'_t \subseteq I_{2n} \), then \( O_t \subseteq I_{2n} \), and if \( F'_l \subseteq I_{2n} \), then \( F_l \subseteq I_{2n} \). Hence, by the definition of \( I_{2n+1} \) (3.18),
\[
I_{2n+1} \supset O_{l_0} \cup F_{l_0+1} \cup F_{l_0+2} \cup \cdots \cup F_{t_{n+1}-1} \cup O_{t_{n+1}}
\]
\[
\cup F_{t_{n+1}+1} \cup F_{t_{n+2}} \cup \cdots \cup F_{t_{2n+1}}
\]
for \( n = 0, 1, \ldots, m - 3 \), where \( l_0 \) is the same integer used in deforming \( I_{2n} \) to \( I_{2n+1} \).

And by the definitions (3.16), (3.17) and (3.18) we have
\[
I_{2n+2} = (I_{2n} - O_{t_{n+1}} - O_{t_{n+1}}) \cup F_{t_{n+1}} \cup F_{t_{n+2}}
\]
\[
= (I_{2n+1} - O_{t_{n+1}}) \cup F_{t_{n+1}}
\]
for \( n = 0, 1, \ldots, m - 3 \). Similarly to (3.24), we can deform \( I_{2n+1} \) to \( I_{2n+2} \) by applying Steps 1 (3.14) and its inverses and deforming \( O_{t_{n+1}} \) to \( F_{t_{n+1}} \):

\[
\phi_{l_{n+1}} \circ \cdots \circ \phi_{t_{n+1}-1} \circ \phi_{t_{n+1}} \circ \phi_{t_{n+1}}^{-1} \circ \cdots \circ \phi_{l_{n+2}} \circ \phi_{l_{n+1}+1}(I_{2n+1}) = I_{2n+2}
\](3.25)

for \( n = 0, 1, \ldots, m - 3 \).

Thus, repeating Steps 2 (3.24) and Steps 1 (3.25) alternately \( m - 2 \) times each, we have
\[
I_{2(m-2)} = O_0 \cup O_{l_0} \cup \bigcup_{n=1}^{m-2} F_{t_n}
\]
\[
\cup O'_0 \cup O'_{l_0} \cup \bigcup_{n=1}^{m-2} F_{t_n'}
\]
\[
\cup F_r.
\]

By (a2), we have \( \{r_1, r_2, \ldots, r_{m-2}\} = \{1, 2, \ldots, m-1\} - \{t_0\} = \{1, 2, \ldots, m-1\} - \{m-r\} \).

Hence together with (3.10) and (3.11) we have
\[
I_{2(m-2)} = O_0 \cup O_{m-r} \cup O'_0 \cup O'_{m-r} \cup \bigcup_{k \neq m, 2m-r} F_k,
\]
where
\[
O_{m-r} \sim O(2m - r; (2m - r - 1) \setminus m)(m - 1 \setminus r)
\]
by (3.27) of Lemma 3.3. On the other hand, by definition \( O_0 = F_m \). Hence, we have free edges of all labels except \( 2m - r \), using which and (1.7) we can deform the oval nest \( O_{m-r} \) to the free edge \( F_{2m-r} \).

Therefore \( \Gamma_S \cup F_r \) can be deformed to a chart containing \( \bigcup_{k=1}^{2m-1} F_k \), using which and (1.7) we can deform \( \Gamma_S \cup F_r \) to have only free edges, which is an unknotted chart.
Lemma 3.3. The oval nest of Lemma 2.4

\[ O_k = O \left( m; \prod_{j=0}^{k-1} (m - 1 \setminus m - k + j) \prod_{j=0}^{k-1} (m + 1 \setminus m + k - j) \right) \]

for \( k = 1, 2, \ldots, m - 1 \), is equivalent to the following:

\[
\begin{align*}
(3.26) & \quad O_k \sim O(m - k; (m - k + 1 \setminus m)(m + 1 \setminus m + k)) \\
(3.27) & \quad \sim O(m + k; (m + k - 1 \setminus m)(m - 1 \setminus m - k)).
\end{align*}
\]

Proof. First, we will show that the braid \( \prod_{j=0}^{k-1} (m - 1 \setminus m - k + j) \) is equivalent to \( \prod_{j=0}^{k-1} (m - k + j \setminus m - k) \), i.e.

\[
\prod_{j=0}^{k-1} (m - 1 \setminus m - k + j) \sim \prod_{j=0}^{k-1} (m - k + j \setminus m - k).
\]

(3.28)

For positive integers \( l, i_1, i_2 \) with \( l \geq i_2 > i_1 \), we have \( (l \setminus i_1)i_2 \sim (i_2 - 1)(l \setminus i_1) \). Hence we can see that

\[
(l \setminus i_1)(l \setminus i_2) \sim (l - 1 \setminus i_2 - 1)(l \setminus i_1).
\]

(3.29)

By (3.29), we see that

\[
\prod_{j=0}^{k-1} (m - 1 \setminus m - k + j)
= (m - 1 \setminus m - k) \cdot \prod_{j=1}^{k-1} (m - 1 \setminus m - k + j)
\sim \prod_{j=1}^{k-1} (m - 2 \setminus m - k + j - 1) \cdot (m - 1 \setminus m - k)
\sim \ldots
\]

\[
\sim \prod_{j=s-1}^{k-1} (m - s \setminus m - k + j - (s - 1)) \prod_{j=k-(s-1)}^{k-1} (m - k + j \setminus m - k)
= (m - s \setminus m - k) \prod_{j=i}^{k-1} (m - s \setminus m - k + j - (s - 1)) \prod_{j=k-(s-1)}^{k-1} (m - k + j \setminus m - k)
\]

\]

UNKNOTTING SPUN T^2-KNOT
\[
\sim \prod_{j=s}^{k-1} (m - s - 1 \backslash m - k + j - s) \cdot (m - k + (k - s) \backslash m - k)
\]
\[
\cdot \prod_{j=k-s}^{k-1} (m - k + j \backslash m - k)
\]
\[
= \prod_{j=s}^{k-1} (m - (s + 1) \backslash m - k + j - s) \cdot \prod_{j=k-s}^{k-1} (m - k + j \backslash m - k)
\]
\[
\sim \ldots
\]
\[
\sim (m - k) \prod_{j=1}^{k-1} (m - k + j \backslash m - k)
\]
\[
= \prod_{j=0}^{k-1} (m - k + j \backslash m - k),
\]

which is (3.28). Similarly, we have another equivalence relation:

\[
(3.30) \quad \prod_{j=0}^{k-1} (m + 1 \not\nearrow m + k - j) \sim \prod_{j=0}^{k-1} (m + k - j \not\nearrow m + k).
\]

Note that for positive integers \(l, i_1, i_2\) with \(l \leq i_2 < i_1\), we can easily show that \((l \nearrow i_1)(l \not\nearrow i_2) \sim (l + 1 \not\nearrow i_2 + 1)(l \nearrow i_1)\).

Using (1.4), we can show that if \(m - 1 > i\), then

\[
(3.31) \quad O(m; (i \backslash j) \cdot c) \sim O(m; c)
\]

for a braid \(c\). Similarly we can show that if \(m + 1 < i\), then

\[
(3.32) \quad O(m; (i \not\nearrow j) \cdot c) \sim O(m; c).
\]

By (3.28) and (3.30), we have

\[
O_k = O \left( m; \prod_{j=0}^{k-1} (m - 1 \backslash m - k + j) \prod_{j=0}^{k-1} (m + 1 \not\nearrow m + k - j) \right)
\]
\[
\sim O \left( m; \prod_{j=0}^{k-2} (m - k + j \backslash m - k) \cdot (m - 1 \backslash m - k) \prod_{j=0}^{k-2} (m + k - j \not\nearrow m + k) \cdot (m + 1 \not\nearrow m + k) \right).
\]
For \( j = 0, 1, \ldots, k - 2 \), we have \( (m + k - j) - ((m - 1) + 1) = k - j > 0 \). Hence \( m + k - j > (m - 1) + 1 \). By (2.2), we have

\[
\mathcal{O}_k \sim O \left( m; \prod_{j=0}^{k-2} (m - k + j \setminus m - k) \cdot \prod_{j=0}^{k-2} (m + k - j \setminus m + k) \cdot (m - 1 \setminus m - k)(m + 1 \setminus m + k) \right).
\]

By (3.31) and (3.32), we have

\[
(3.33) \quad \mathcal{O}_k \sim O(m; (m - 1 \setminus m - k)(m + 1 \setminus m + k)).
\]

Now we will show that

\[
(3.34) \quad O(m; (m - 1 \setminus m - k) \cdot c) \sim O(m - k; (m - k + 1 \setminus m) \cdot c),
\]

where \( c \) is a braid. For positive integers \( i_1, i_2 \) with \( i_1 > i_2 \), by (1.5) and (1.6) we have

\[
O(i_1; (i_1 - 1 \setminus i_2)) \sim O(i_1 - 1; \overline{i_1} \cdot (i_1 - 2 \setminus i_2),
\]

which is equivalent to \( O(i_1 - 1; (i_1 - 2 \setminus i_2) \cdot \overline{i_1}) \) by (2.2). Thus we have

\[
(3.35) \quad O(i_1; (i_1 - 1 \setminus i_2)) \sim O(i_1 - 1; (i_1 - 2 \setminus i_2) \cdot \overline{i_1}).
\]

Using (3.35) and (1.6), we can see that

\[
O(m; (m - 1 \setminus m - k)) \\
\sim O(m - 1; (m - 2 \setminus m - k) \cdot \overline{m}) \\
\sim \ldots \\
\sim O(m - s; (m - s - 1 \setminus m - k) \cdot (m - s + 1 \setminus \overline{m})) \\
\sim O(m - s - 1; (m - s - 2 \setminus m - k) \cdot (m - s) \cdot (m - s + 1 \setminus \overline{m})) \\
= O(m - s - 1; (m - s - 2 \setminus m - k) \cdot (m - s + 1 \setminus \overline{m})) \\
\sim \ldots \\
\sim O(m - k; (m - k + 1 \setminus \overline{m})).
\]

Hence by (1.6), we have (3.34).

By (3.33) and (3.34), we have

\[
\mathcal{O}_k \sim O(m; (m - 1 \setminus m - k)(m + 1 \setminus m + k)) \\
\sim O(m - k; (m - k + 1 \setminus \overline{m})(m + 1 \setminus m + k)),
\]

which is (3.26).
By (3.33) and (2.2), we can see that

\[
O(m; (m - 1 \searrow m - k) \cdot (m + 1 \nearrow m + k)) \\
\sim O(m; (m + 1 \nearrow m + k) \cdot (m - 1 \searrow m - k)).
\]

(3.36)

And similarly to (3.34), we can see that

\[
O(m; (m + 1 \nearrow m + k) \cdot c) \sim O(m + k; (m + k - 1 \searrow m) \cdot c),
\]

(3.37)

for a braid \(c\). Hence by (3.36) and (3.37) we have the other equivalence relation (3.27):

\[
O_k \sim O(m + k; (m + k - 1 \searrow m)(m - 1 \searrow m - k)).
\]

Acknowledgements. The author would like to thank the referee for the helpful and kind advice. The author is supported by JSPS Research Fellowships for Young Scientists.

References


Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan

Current address:
Department of Mathematics
Gakushuin University
1-5-1 Mejiro Toshima-ku Tokyo, 171-8588
Japan
e-mail: inasa@math.gakushuin.ac.jp