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LOCALLY CONFORMAL KÄHLER STRUCTURES ON COMPACT SOLVMANIFOLDS

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Abstract

Let (M, g, J) be a compact Hermitian manifold and Ω the fundamental 2-form of (g, J). A Hermitian manifold (M, g, J) is said to be locally conformal Kähler if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The purpose of this paper is to investigate a relation between a locally conformal Kähler structure and the adapted differential operator on compact solvmanifolds.

Introduction

Let (M, g, J) be a $2n$-dimensional compact Hermitian manifold. We denote by Ω the fundamental 2-form, that is, the 2-form defined by $\Omega(X, Y) = g(X, JY)$. A Hermitian manifold (M, g, J) is said to be *locally conformal Kähler* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The closed 1-form ω is called *Lee form*. In the case $\omega = df$, $(M, e^{-f} g, J)$ is Kähler. The main non-Kähler examples of locally conformal Kähler manifolds are Hopf manifolds [15], Inoue surfaces [14] and generalized Kodaira–Thurston manifolds [5] (cf. [6]). Note that Inoue surfaces and generalized Kodaira–Thurston manifolds have a structure of a compact solvmanifold. In this paper, we investigate locally conformal Kähler structures on a compact solvmanifold $\Gamma\backslash G$ with a left-invariant complex structure, where G is a simply-connected solvable Lie group and Γ is a lattice of G, that is, a discrete co-compact subgroup.

An n-dimensional complex manifold M is said to be *complex parallelizable* if it admits holomorphic vector fields $\{X_1, \dots, X_n\}$ which are linearly independent at every point. Wang [16] proved that a compact complex parallelizable manifold M is biholomorphic to a homogeneous space $D\backslash G$, where G is a simply connected complex Lie group and D is a discrete subgroup of G. Abbena–Grassi [1] proved that a non-toral compact complex parallelizable manifold has no locally conformal Kähler structures. The author in [12] proved that if a compact nilmanifold M with a left-invariant complex structure has a locally conformal Kähler structure, then M is biholomorphic to a Kodaira–Thurston manifold or a generalized Kodaira–Thurston manifold.

A locally conformal Kähler manifold (M, g, J) is said to be *a generalized Hopf manifold* if the Lee form ω is parallel with respect to metric g. Vaisman [15] proved

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that a generalized Hopf manifold has a structure of a principal S^1-bundle over a compact Sasakian manifold. Hopf manifolds and generalized Kodaira–Thurston manifolds are generalized Hopf manifolds. Tricerri [14] constructed a locally conformal Kähler structure on Inoue surfaces with non-parallel Lee form ω. In Section 3, we prove that

Main Theorem. *Let* $(\Gamma\backslash G, g, J)$ *be a locally conformal Kähler solvmanifold of* $\dim \Gamma\backslash G \geq 4$ *with a left-invariant complex structure and* ω *the Lee form. We assume that* $(\Gamma\backslash G, g, J)$ *satisfies the following conditions*:

1. *There exists a left-invariant closed* 1*-form* ω_0 *such that the Lee form* ω *is cohomologous to* ω_0.
2. *The fundamental* 2*-form* Ω *of* (g, J) *can be written as* $\Omega = -\omega \wedge \eta + d\eta$, *where* $\eta = -\omega \circ J$.

Then $\Gamma\backslash G$ *has a left-invariant locally conformal Kähler structure with the parallel Lee form* ω_0, *in particular,* $\Gamma\backslash G$ *is a generalized Hopf manifold.*

In Section 4, we see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem.

Hasegawa [7] proved that a complex structure on a 4-dimensional compact solvmanifold is induced from a left-invariant complex structure on the solvable Lie group and classified 4-dimensional compact solvmanifolds with a complex structure. By this classification, we have a classification of 4-dimensional locally conformal Kähler solvmanifolds in Section 4.

1. Preliminaries

Let $(M = \Gamma\backslash G, g, J)$ be a compact locally conformal Kähler solvmanifold such that J is left-invariant. In this section, we see that the locally conformal Kähler metric g induces a left-invariant locally conformal Kähler metric and consider its fundamental 2-form on the Lie algebra $\mathfrak{g}$ of G.

Let Ω be the fundamental 2-form of (g, J). There exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. We now assume that the associated Lee form satisfies the condition 1 in Main Theorem. Then there exists a left-invariant closed 1-form ω_0 such that $\omega_0 - \omega = df$. This assumption holds for completely solvmanifolds [9]. Then we define a left-invariant 2-form Ω_0 by

$$\Omega_0(X, Y) = \int_M (e^f \Omega)(X, Y)\, d\mu$$

for left-invariant vector fields X, Y, where $d\mu$ is the volume element induced by a bi-invariant volume element on G. Belgun [3] proved that Ω_0 is J-invariant and it is the fundamental 2-form of a Hermitian structure $(\langle\ ,\ \rangle, J)$ on the solvable Lie algebra $\mathfrak{g}$ of G such that $d\Omega_0 = \omega_0 \wedge \Omega_0$.

DEFINITION 1.1. Let $\mathfrak{g}$ be a Lie algebra and α a closed 1-form on $\mathfrak{g}$. We define the adapted algebraic complex $\left(\bigwedge \mathfrak{g}^*, d_\alpha\right)$ with the differential operator d_α:

$$d_\alpha \beta := \alpha \wedge \beta + d\beta$$

for $\beta \in \wedge^p \mathfrak{g}^*$. Note that $d_\alpha^2 = 0$ because α is closed. A p-form β is called *α-closed* if $d_\alpha \beta = 0$. It is called *α-exact* if there exists a $(p-1)$-form γ such that $\beta = d_\alpha \gamma$.

On the above Hermitian structure $(\langle\ ,\ \rangle, J)$ on $\mathfrak{g}$, since ω_0 is a closed 1-form, the fundamental 2-form Ω_0 is $-\omega_0$-closed. We say that $(\langle\ ,\ \rangle, J)$ on $\mathfrak{g}$ is *locally conformal Kähler*. In Section 4, we consider a relation between a property of Lee form and the adapted differential operator on 4-dimensional solvmanifolds.

We next show that Ω_0 is $-\omega_0$-exact if the second condition in Main Theorem also holds true.

Proposition 1.2 (cf. [3]). *Let Ω_0 be the 2-form on $\mathfrak{g}$ above. If $\Omega = -\omega \wedge \eta + d\eta$, then Ω_0 is $-\omega_0$-exact. In addition, if $\eta = -\omega \circ J$, then $\eta_0 = -k\omega_0 \circ J$, where $k = \int_M e^f\, d\mu$.*

Proof. Since $\Omega = -\omega \wedge \eta + d\eta$ and $\omega = \omega_0 - df$, we see that

$$\begin{aligned} e^f \Omega &= e^f(-\omega \wedge \eta + d\eta) \\ &= e^f(-\omega_0 \wedge \eta + df \wedge \eta + d\eta) \\ &= -\omega_0 \wedge e^f \eta + d(e^f) \wedge \eta + e^f\, d\eta \\ &= -\omega_0 \wedge e^f \eta + d(e^f \eta). \end{aligned}$$

Then we get

$$\begin{aligned} \Omega_0(X, Y) &= \int_M (-\omega_0 \wedge e^f \eta + d(e^f \eta))(X, Y)\, d\mu \\ &= -\omega_0(X) \int_M (e^f \eta)(Y)\, d\mu + \omega_0(Y) \int_M (e^f \eta)(X)\, d\mu + \int_M d(e^f \eta)(X, Y)\, d\mu, \end{aligned}$$

for $X, Y \in \mathfrak{g}$.

Now, we define a 1-form η_0 on $\mathfrak{g}$ by $\eta_0(X) = \int_M (e^f \eta)(X)\, d\mu$ for $X \in \mathfrak{g}$. Since $d\mu$ is the right-invariant volume element, its Lie derivative $L_X\, d\mu$ along a left-invariant vector fields X is zero. Then, for any function F on M, we see that

$$(XF)\, d\mu = (L_X F)\, d\mu = L_X(F\, d\mu) = di(X)F\, d\mu + i(X)\, d(F\, d\mu) = di(X)F\, d\mu,$$

where $i(X)$ is the interior product with the vector field X. Therefore, by Stokes's theorem, we have

$$\begin{aligned}&\int_M d(e^f\eta)(X,Y)\,d\mu\\&=\int_M X((e^f\eta)(Y))\,d\mu-\int_M Y((e^f\eta)(X))\,d\mu-\int_M(e^f\eta)([X,Y])\,d\mu\\&=\int_M di(X)(e^f\eta(Y))\,d\mu-\int_M di(Y)(e^f\eta(X))\,d\mu-\eta_0([X,Y])\\&=d\eta_0(X,Y).\end{aligned}$$

Thus we see that

$$\begin{aligned}\Omega_0(X,Y)&=-\omega_0(X)\int_M(e^f\eta)(Y)\,d\mu+\omega_0(Y)\int_M(e^f\eta)(X)\,d\mu+\int_M d(e^f\eta)(X,Y)\,d\mu\\&=-\omega_0(X)\eta_0(Y)+\omega_0(Y)\eta_0(X)+d\eta_0(X,Y)\\&=(-\omega_0\wedge\eta_0+d\eta_0)(X,Y).\end{aligned}$$

Similarly, if $\eta=-\omega\circ J$, then we have

$$\begin{aligned}\eta_0(X)&=\int_M(e^f\eta)(X)\,d\mu=-\int_M(e^f\omega)(JX)\,d\mu=-\int_M(e^f\omega_0-e^f\,df)(JX)\,d\mu\\&=-\omega_0(JX)\int_M e^f\,d\mu+\int_M d(e^f)(JX)\,d\mu\\&=-\omega_0(JX)\int_M e^f\,d\mu+\int_M(JX)(e^f)\,d\mu\\&=-\left(\int_M e^f\,d\mu\right)\omega_0(JX).\end{aligned}$$

Thus we see that $\eta_0=-k\omega_0\circ J$, where $k=\int_M e^f\,d\mu$. □

By Proposition 1.2, we see that $\Omega_0=-\omega_0\wedge k\eta_0'+k\,d\eta_0'=k(-\omega_0\wedge\eta_0'+d\eta_0')=kd_{-\omega_0}\eta_0'$, where $\eta_0'=(1/k)\eta_0=-\omega_0\circ J$. Then the locally conformal Kähler structure (Ω_0,J) induces the locally conformal Kähler structure $(\Omega_0'=d_{-\omega_0}(-\omega_0\circ J),J)$ on the solvable Lie algebra $\mathfrak{g}$. Therefore, by replacing Ω_0 by Ω_0', in order to prove Main Theorem, it is enough to show that if $(\mathfrak{g},\Omega_0=d_{-\omega_0}(-\omega_0\circ J),J)$ is a locally conformal Kähler solvable Lie algebra, then the Lee form ω_0 is parallel with respect to $\langle\ ,\ \rangle$ (see Section 3).

2. The adjoint operators and the inner product

Let $(\langle\ ,\ \rangle,J)$ be the locally conformal Kähler structure as we consider in Section 1 on solvable Lie algebra $\mathfrak{g}$, namely Ω_0 can be written as $\Omega_0=d_{-\omega_0}\eta_0=-\omega_0\wedge\eta_0+d\eta_0$, where ω_0 is a closed 1-form on $\mathfrak{g}$ and $\eta_0 = -\omega_0 \circ J$. In this section, we investigate the properties of $(\mathfrak{g}, \langle\ ,\ \rangle, J)$.

Let γ be the canonical isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$ induced by the inner product $\langle\ ,\ \rangle$. Put $A = \gamma(\omega_0)$. By this normalization, we may assume that $\langle A, A\rangle = 1$.

We easily see that an abelian Lie algebra of dimension equal to or more than 4 has no locally conformal Kähler structures. From now on, we assume that $\mathfrak{g}$ is not abelian.

Since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. We take the descending central series for $[\mathfrak{g}, \mathfrak{g}]$: $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supset \cdots \supset \mathfrak{n}^{(r)} \supset \mathfrak{n}^{(r+1)} = 0$, where $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ $(i \geq 1)$ and $\mathfrak{n}^{(r)} \neq 0$. We easily see that $\mathfrak{n}^{(r)}$ is contained in the center $Z(\mathfrak{n})$ of $\mathfrak{n}$. Note that, for $X \in \mathfrak{g}$, $\mathrm{ad}(X)\mathfrak{n}^{(i)} \subset \mathfrak{n}^{(i)}$ for each i. Then we get

Lemma 2.1. *$J\mathfrak{n}^{(r)} \subset [\mathfrak{g}, \mathfrak{g}]^{\perp}$, where $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ is the orthogonal component of $[\mathfrak{g}, \mathfrak{g}]$.*

Proof. For $Z \in \mathfrak{n}^{(r)}$ and $X \in [\mathfrak{g}, \mathfrak{g}]$, we see that

$$\langle JZ, X\rangle = \Omega_0(X, Z) = (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Z) = 0,$$

because ω_0 is closed and $Z \in Z(\mathfrak{n})$. ☐

Lemma 2.2. *For $Z, Z' \in \mathfrak{n}^{(r)}$, $[JZ, Z'] = [JZ', Z]$ and $[JZ, JZ'] = 0$.*

Proof. The Nijenhuis tensor $\mathbf{N}_J$ vanishes, because J is integrable. Since $Z, Z' \in \mathfrak{n}^{(r)}$, we see that

$$\begin{aligned} 0 = \mathbf{N}_J(Z, Z') &= [Z, Z'] + J[JZ, Z'] + J[Z, JZ'] - [JZ, JZ'] \\ &= \qquad J\{[JZ, Z'] + [Z, JZ']\} - [JZ, JZ']. \end{aligned}$$

Note that $[JZ, Z'], [Z, JZ'] \in \mathfrak{n}^{(r)}$. It follows that $J\{[JZ, Z'] + [Z, JZ']\} \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$ by Lemma 2.1. Thus we have $[JZ, Z'] + [Z, JZ'] = 0$ and $[JZ, JZ'] = 0$. ☐

By Lemmas 2.1 and 2.2, we have

Proposition 2.3. *For $U, V, W \in \mathfrak{n}^{(r)}$,*

$$\langle \mathrm{ad}(JU)V, W\rangle + \omega_0(JV)\langle U, W\rangle = \langle V, \mathrm{ad}(JU)W\rangle + \omega_0(JW)\langle U, V\rangle.$$

Proof. Since $\Omega_0 = -\omega_0 \wedge \eta_0 + d\eta_0$ and ω_0 is closed 1-form, we see that

$$\begin{aligned} \langle \mathrm{ad}(JU)V, W\rangle &= \Omega_0(J\,\mathrm{ad}(JU)V, W) = -\Omega_0(\mathrm{ad}(JU)V, JW) \\ &= (\omega_0 \wedge \eta_0 - d\eta_0)(\mathrm{ad}(JU)V, JW) \\ &= -\omega_0(JW)\eta_0(\mathrm{ad}(JU)V) - d\eta_0(\mathrm{ad}(JU)V, JW). \end{aligned} \tag{2.1}$$

Since $\eta_0 = -\omega_0 \circ J$, we get

$$\begin{aligned} -\omega_0(JW)\eta_0(\mathrm{ad}(JU)V) &= \omega_0(JW)\, d\eta_0(JU, V) \\ &= \omega_0(JW)\{\Omega_0(JU, V) + \omega_0 \wedge \eta_0(JU, V)\} \\ &= \omega_0(JW)\langle U, V\rangle - \omega_0(JW)\omega_0(JU)\omega_0(JV). \end{aligned} \tag{2.2}$$

From the derivation conditions and Lemma 2.2, we get

$$\begin{aligned} d\eta_0(\mathrm{ad}(JU)V, JW) &= d\eta_0(JU, [V, JW]) - d\eta_0(V, [JU, JW]) \\ &= d\eta_0(JU, [V, JW]) \\ &= d\eta_0(JU, [W, JV]) \\ &= d\eta_0([JU, W], JV) + d\eta_0(W, [JU, JV]) \\ &= d\eta_0([JU, W], JV) = -d\eta_0(JV, [JU, W]). \end{aligned}$$

It follows that

$$\begin{aligned} &-d\eta_0(\mathrm{ad}(JU)V, JW) \\ &= d\eta_0(JV, [JU, W]) \\ &= \Omega_0(JV, [JU, W]) + \omega_0 \wedge \eta_0(JV, [JU, W]) \\ &= \langle V, [JU, W]\rangle - \omega_0(JV)\, d\eta_0(JU, W) \\ &= \langle V, [JU, W]\rangle - \omega_0(JV)\{\Omega_0(JU, W) + \omega_0 \wedge \eta_0(JU, W)\} \\ &= \langle V, [JU, W]\rangle - \omega_0(JV)\langle U, W\rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW). \end{aligned} \tag{2.3}$$

From (2.1) and (2.2)), (2.3), we have

$$\begin{aligned} \langle \mathrm{ad}(JU)V, W\rangle &= \omega_0(JW)\langle U, V\rangle - \omega_0(JW)\omega_0(JU)\omega_0(JV) \\ &\quad + \langle V, [JU, W]\rangle - \omega_0(JV)\langle U, W\rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW), \\ \langle \mathrm{ad}(JU)V, W\rangle + \omega_0(JV)\langle U, W\rangle &= \langle V, \mathrm{ad}(JU)W\rangle + \omega_0(JW)\langle U, V\rangle. \end{aligned}$$ □

It is well-known that if a solvable Lie group G admits a lattice Γ, then the solvable Lie algebra $\mathfrak{g}$ of G is unimodular. We define

DEFINITION 2.4. A solvable Lie algebra $\mathfrak{g}$ is called strongly unimodular if, for $X \in \mathfrak{g}$, $\mathrm{tr}\, \mathrm{ad}(X)|_{\mathfrak{n}^{(i)}} = 0$ for each i.

Benson–Gordon [4] proved that if a solvable Lie group G admits a lattice Γ, then the solvable Lie algebra $\mathfrak{g}$ of G is strongly unimodular.

We take an orthonormal frame $\{Z_1, \dots, Z_m\}$ of $\mathfrak{n}^{(r)}$ and consider the strongly unimodular conditions of $\mathrm{ad}(JZ_i)$ from $\mathfrak{n}^{(r)}$ to $\mathfrak{n}^{(r)}$ for each i.

For each i, let

$$\operatorname{ad}(JZ_i)(Z_1, \ldots, Z_m) = (Z_1, \ldots, Z_m)\begin{pmatrix} a_{11}(i) & \cdots & a_{1m}(i) \\ \vdots & \vdots & \vdots \\ a_{m1}(i) & \cdots & a_{mm}(i) \end{pmatrix}.$$

By Proposition 2.3, we see that

$$\begin{gathered} \langle \operatorname{ad}(JZ_i)Z_j, Z_k\rangle + \omega_0(JZ_j)\langle Z_i, Z_k\rangle = \langle Z_j, \operatorname{ad}(JZ_i)Z_k\rangle + \omega_0(JZ_k)\langle Z_i, Z_j\rangle, \\ a_{kj}(i) + \delta_{ik}\omega_0(JZ_j) = a_{jk}(i) + \delta_{ij}\omega_0(JZ_k). \end{gathered}$$

It follows that, in the case of $j, k \neq i$,

$$a_{kj}(i) = a_{jk}(i)$$

and, in the case of $j = i$, $k \neq i$,

$$a_{ki}(i) = a_{ik}(i) + \omega_0(JZ_k).$$

Then we get

$$(a_{jk}(i)) = A_i + \begin{array}{c} \text{line } i \\ \begin{pmatrix} & \omega_0(JZ_1) & \\ 0 & \vdots & 0 \\ & \omega_0(JZ_m) & \end{pmatrix} \end{array}, \tag{2.4}$$

where A_i is an $(m \times m)$-symmetric matrix.

Moreover, put

$$A_i = \begin{array}{cc} & \text{line } i \\ \text{row } i & \begin{pmatrix} & & a_1^i & & \\ & * & \vdots & * & \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & & \vdots & & \\ & * & a_m^i & * & \end{pmatrix} \end{array},$$

for each i. Since $[JZ_i, Z_j] = [JZ_j, Z_i]$, we get

$$a_{jj}(i) = \langle \operatorname{ad}(JZ_i)Z_j, Z_j\rangle = \langle \operatorname{ad}(JZ_j)Z_i, Z_j\rangle = a_i^j$$

for $j \neq i$. Then A_i can be written as

$$A_i = \begin{array}{rc} & \begin{array}{ccccc} & & \text{line } i & & \end{array} \\ \begin{array}{r} \\ \\ \text{row } i \\ \\ \\ \end{array} & \left(\begin{array}{ccccc} a_i^1 & & a_1^i & & \\ & \ddots & \vdots & * & \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & & \vdots & \ddots & \\ & * & a_m^i & & a_i^m \end{array}\right) \end{array}.$$

Thus (2.4) can be expressed as follows:

$$(a_{jk}(i)) = \begin{array}{rc} & \begin{array}{ccccc} & & \text{line } i & & \end{array} \\ \begin{array}{r} \\ \\ \\ \\ \text{row } i \\ \\ \\ \\ \\ \end{array} & \left(\begin{array}{ccccc} a_i^1 & & a_1^i + \omega_0(JZ_1) & & \\ & * & & & \\ & \ddots & \vdots & * & \\ * & & & & \\ a_1^i & \cdots & a_i^i + \omega_0(JZ_i) & \cdots & a_m^i \\ & & & * & \\ & & \vdots & \ddots & \\ & * & & * & \\ & & a_m^i + \omega_0(JZ_m) & & a_i^m \end{array}\right) \end{array}. \tag{2.5}$$

From the strongly unimodular condition of $\mathrm{ad}(JZ_i)|_{\mathfrak{n}^{(r)}}$, we have

Proposition 2.5. *For each* i, $\left(\sum_j a_i^j\right) + \omega_0(JZ_i) = 0$.

Then we have

Corollary 2.6. $\sum_i \omega_0(JZ_i)^2 = 1$.

Proof. For each i, we see that

$$\begin{aligned} 1 = \langle Z_i, Z_i \rangle &= \Omega_0(JZ_i, Z_i) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(JZ_i, Z_i) \\ &= -\omega_0(JZ_i)\eta_0(Z_i) - \eta_0([JZ_i, Z_i]) \\ &= \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \end{aligned}$$

because $\eta_0 = -\omega_0 \circ J$. From (2.5),

$$\begin{aligned} 1 &= \langle Z_i, Z_i\rangle = \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \\ 1 &= \omega_0(JZ_i)^2 + \omega_0 \circ J\left(\sum_j (a^i_j + \omega_0(JZ_j))Z_j\right), \\ 1 &= \omega_0(JZ_i)^2 + \sum_j a^i_j \omega_0(JZ_j) + \sum_j \omega_0(JZ_j)^2. \end{aligned} \tag{2.6}$$

Take the sum of (2.6) for i, we get

$$\begin{aligned} \sum_i 1 &= \sum_i \omega_0(JZ_i)^2 + \sum_{i,j} a^i_j \omega_0(JZ_j) + \sum_{i,j} \omega_0(JZ_j)^2, \\ m &= \sum_i \omega_0(JZ_i)^2 + \sum_i \left(\sum_j a^j_i\right)\omega_0(JZ_i) + m\sum_i \omega_0(JZ_i)^2. \end{aligned}$$

By Proposition 2.5, we have

$$\begin{aligned} m &= \sum_i \omega_0(JZ_i)^2 + \sum_i(-\omega_0(JZ_i))\omega_0(JZ_i) + m\sum_i \omega_0(JZ_i)^2, \\ m &= \sum_i \omega_0(JZ_i)^2 - \sum_i \omega_0(JZ_i)^2 + m\sum_i \omega_0(JZ_i)^2, \\ m &= m\sum_i \omega_0(JZ_i)^2, \end{aligned}$$

which implies that $\sum_i \omega_0(JZ_i)^2 = 1$. □

Corollary 2.7. $A = \sum_i \omega_0(JZ_i)JZ_i \in J\mathfrak{n}^{(r)}$.

Proof. Since $A = \gamma(\omega_0)$, it can be given by

$$A = \sum_i \omega_0(JZ_i)JZ_i + B,$$

where $B \in (J\mathfrak{n}^{(r)})^\perp \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. From $\langle A, A\rangle = 1$, we have

$$1 = \langle A, A\rangle = \sum_i \omega_0(JZ_i)^2\langle JZ_i, JZ_i\rangle + \langle B, B\rangle = \sum_i \omega_0(JZ_i)^2 + \langle B, B\rangle.$$

By Corollary 2.6, we have $\langle B, B\rangle = 0$, which implies that $B = 0$. □

3. Structure on the solvable Lie algebra

We use same notation introduced in Section 2. In this section, we prove that JA is in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ and have Main Theorem.

We see $JA = -\sum_i \omega_0(JZ_i)Z_i \in \mathfrak{n}^{(r)}$ by Corollary 2.7. Put $Z_0 = JA$. Note that $\eta_0(Z_0) = -\omega_0 \circ J(JA) = 1$ and $\eta_0(A) = -\omega_0 \circ J(A) = 0$. We get

Lemma 3.1. *For* $X \in \mathfrak{g}$, $d\eta_0(A, X) = 0$ *and* $d\eta_0(Z_0, X) = 0$.

Proof. Since $\eta_0 = -\omega_0 \circ J$, we see that

$$\begin{aligned} d\eta_0(A, X) &= \Omega_0(A, X) + \omega_0 \wedge \eta_0(A, X) = \langle A, JX \rangle + \eta_0(X) = \omega_0(JX) + \eta_0(X) \\ &= 0. \end{aligned}$$

Similarly, we get

$$\begin{aligned} d\eta_0(Z_0, X) &= \Omega_0(Z_0, X) + \omega_0 \wedge \eta_0(Z_0, X) = -\langle X, JZ_0 \rangle - \omega_0(X) = \langle X, A \rangle - \omega_0(X) \\ &= 0. \end{aligned}$$ □

Then we see that

Proposition 3.2. *For* $U \in \mathfrak{n}^{(r)}$, $\mathrm{ad}(A)U = 0$.

Proof. By a straightforward computation, we see that

$$\begin{aligned} \langle \mathrm{ad}(A)U, \mathrm{ad}(A)U \rangle &= \Omega_0(J \circ \mathrm{ad}(A)U, \mathrm{ad}(A)U) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \mathrm{ad}(A)U, \mathrm{ad}(A)U) \\ &= -\omega_0(J \circ \mathrm{ad}(A)U)\eta_0(\mathrm{ad}(A)U) + d\eta_0(J \circ \mathrm{ad}(A)U, \mathrm{ad}(A)U). \end{aligned}$$

By Lemma 3.1 and the derivation conditions,

$$\begin{aligned} \langle \mathrm{ad}(A)U, \mathrm{ad}(A)U \rangle &= \omega_0(J \circ \mathrm{ad}(A)U)\, d\eta_0(A, U) - d\eta_0(\mathrm{ad}(A)U, J \circ \mathrm{ad}(A)U) \\ &= -d\eta_0(\mathrm{ad}(A)U, J \circ \mathrm{ad}(A)U) \\ &= -d\eta_0(A, [U, J \circ \mathrm{ad}(A)U]) + d\eta_0(U, [A, J \circ \mathrm{ad}(A)U]) \\ &= d\eta_0(U, [A, J \circ \mathrm{ad}(A)U]). \end{aligned}$$

Now, since $JA = Z_0$ and $\mathrm{ad}(A)U \in \mathfrak{n}^{(r)}$, we see that

$$[A, J \circ \mathrm{ad}(A)U] = [-JZ_0, J \circ \mathrm{ad}(A)U] = 0$$

by Lemma 2.2. It follows that $d\eta_0(U, [A, J \circ \mathrm{ad}(A)U]) = 0$, which implies that $\mathrm{ad}(A)U = 0$. □

Therefore we have

Theorem 3.3. $Z_0 \in Z(\mathfrak{g})$.

Proof. Let $X \in \mathfrak{g}$. By Lemma 3.1, we see that

$$\begin{aligned}\langle \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0\rangle &= \Omega_0(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0)\\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0)\\ &= -\omega_0(J \circ \mathrm{ad}(X)Z_0)\eta_0(\mathrm{ad}(X)Z_0) + d\eta_0(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0)\\ &= \omega_0(J \circ \mathrm{ad}(X)Z_0)\, d\eta_0(X, Z_0) + d\eta_0(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0)\\ &= d\eta_0(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0).\end{aligned}$$

Moreover, from the derivation conditions and $Z_0 \in Z([\mathfrak{g}, \mathfrak{g}])$,

$$\begin{aligned}\langle \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0\rangle &= d\eta_0(J \circ \mathrm{ad}(X)Z_0, \mathrm{ad}(X)Z_0) = -d\eta_0(\mathrm{ad}(X)Z_0, J \circ \mathrm{ad}(X)Z_0)\\ &= -d\eta_0(X, [Z_0, J \circ \mathrm{ad}(X)Z_0]) + d\eta_0(Z_0, [X, J \circ \mathrm{ad}(X)Z_0])\\ &= -d\eta_0(X, [Z_0, J \circ \mathrm{ad}(X)Z_0]).\end{aligned}$$

Now, since $\mathrm{ad}(X)Z_0 \in \mathfrak{n}^{(r)}$, we see that

$$[Z_0, J \circ \mathrm{ad}(X)Z_0] = [\mathrm{ad}(X)Z_0, JZ_0] = [A, \mathrm{ad}(X)Z_0] = 0$$

by Lemma 2.2 and Proposition 3.2. It follows that $d\eta_0(X, [Z_0, J\,\mathrm{ad}(X)Z_0]) = 0$, which implies that $\mathrm{ad}(X)Z_0 = 0$ for $X \in \mathfrak{g}$. □

From Theorem 3.3, we have

Corollary 3.4. $\mathrm{ad}(A) \circ J = J \circ \mathrm{ad}(A)$.

Proof. Let $X \in \mathfrak{g}$. Since J is integrable, we see that

$$\begin{aligned}0 = \mathbf{N}_J(A, X) &= [A, X] + J[JA, X] + J[A, JX] - [JA, JX]\\ 0 &= [A, X] + J[Z_0, X] + J[A, JX] - [Z_0, JX].\end{aligned}$$

From $Z_0 \in Z(\mathfrak{g})$, $[A, X] + J[A, JX] = 0$. Then we have our claim. □

Corollary 3.5. *Lee form* ω_0 *is parallel, if and only if* Ω_0 *is* $\mathrm{ad}(A)$*-invariant.*

Proof. Let ∇ be the Riemannian connection of $\langle\ ,\ \rangle$. For $X, Y \in \mathfrak{g}$, we see that

$$\begin{aligned}2(\nabla_X\omega_0)(Y) &= -2\omega_0(\nabla_X Y) = -2\langle A, \nabla_X Y\rangle \\ &= -\langle [A, X], Y\rangle - \langle X, [A, Y]\rangle - \langle A, [X, Y]\rangle \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]) - \omega_0([X, Y]) \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]),\end{aligned}$$

because ω_0 is a closed 1-form. From Corollary 3.4, we get

$$2(\nabla_X\omega_0)(Y) = -\Omega_0([A, JX], Y) - \Omega_0(JX, [A, Y]) = (\mathrm{ad}^*(A)\Omega_0)(JX, Y).$$

Thus we have our claim. □

Proof of Main Theorem. By Corollary 3.5, to prove Main Theorem, it is enough to show that Ω_0 is ad(A)-invariant.

Let $X, Y \in \mathfrak{g}$. By a straightforward computation, we see that

$$\begin{aligned}(\mathrm{ad}^*(A)\Omega_0)(X, Y) &= -\Omega_0([A, X], Y) - \Omega_0(X, [A, Y]) \\ &= (\omega_0 \wedge \eta_0 - d\eta_0)([A, X], Y) + (\omega_0 \wedge \eta_0 - d\eta_0)(X, [A, Y]) \\ &= -\omega_0(Y)\eta_0([A, X]) - d\eta_0([A, X], Y) \\ &\quad + \omega_0(X)\eta_0([A, X]) - d\eta_0(X, [A, Y]).\end{aligned}$$

By Lemma 3.1, we get $\eta_0([A, X]) = -d\eta_0(A, X) = 0$ and $\eta_0([A, Y]) = 0$. Moreover, from the derivation conditions,

$$d\eta_0([A, X], Y) + d\eta_0(X, [A, Y]) = d\eta_0(A, [X, Y]) = 0,$$

which implies that Ω_0 is ad(A)-invariant. This completes the proof of Main Theorem. □

4. Examples

In this section, we give examples of Main Theorem and consider locally conformal Kähler structures on 4-dimensional compact solvmanifolds. Note that a compact Kähler solvmanifold is a finite quotient of a complex tours which has a structure of a complex tours bundle over a complex tours ([2], [8]).

Hasegawa [7] classified a 4-dimensional compact solvmanifold and proved that any complex structure on such solvmanifold is induced from a left-invariant complex structure on Lie group. By this classification, we see that a 4-dimensional locally conformal Kähler solvmanifold is biholomorphic to Kodaira–Thurston manifold, Secondary Kodaira–Thurston manifold or Inoue surfaces.

We see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem:

EXAMPLE 4.1 (Kodaira–Thurston manifold [5]). Let G be a 3-dimensional nilpotent Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ. Let $\mathfrak{g}$ be the nilpotent Lie algebra corresponding to G. $\mathfrak{g} \times \mathbb{R}$ is given by

$$\mathfrak{g} \times \mathbb{R} = \operatorname{span}\{X, Y, Z, A \colon [X, Y] = Z\}.$$

Let $\{x, y, z, \omega\}$ be the dual base of $\{X, Y, Z, A\}$:

$$dx = dy = d\omega = 0, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle\ ,\ \rangle$ on $\Gamma\backslash G \times S^1$ such that $\{X, Y, Z, A\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z, JX = Y$. Then $(\Gamma\backslash G \times S^1, \langle\ ,\ \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega} z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$-exact. We easily see that Lee form ω is parallel.

EXAMPLE 4.2 (Secondary Kodaira–Thurston manifold (cf. [15])). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(x \sin t + y \cos t) & \frac{1}{2}(x \cos t - y \sin t) & z \\ 0 & \cos t & \sin t & x \\ 0 & -\sin t & \cos t & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ:

$$\Gamma = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(u \sin 2\pi s + v \cos 2\pi s) & \frac{1}{2}(u \cos 2\pi s - v \sin 2\pi s) & w \\ 0 & \cos 2\pi s & \sin 2\pi s & u \\ 0 & -\sin 2\pi s & \cos 2\pi s & v \\ 0 & 0 & 0 & 1 \end{pmatrix} : s, u, v, w \in \mathbb{Z} \right\}.$$

Let $\mathfrak{g}$ be the solvable Lie algebra corresponding to G:

$$\mathfrak{g} = \mathrm{span}\{A, X, Y, Z \colon [A, X] = Y,\ [A, Y] = -X,\ [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge y, \quad dy = \omega \wedge x, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle\ ,\ \rangle$ on $\Gamma\backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z$, $JX = Y$. Then $(\Gamma\backslash G, \langle\ ,\ \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega} z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$-exact. We easily see that Lee form ω is parallel.

A locally conformal Kähler structure on Inoue surfaces are different from one on solvmanifolds as Main Theorem:

EXAMPLE 4.3 (Inoue surface S^0 [11], [14]). Let $B \in \mathrm{SL}(3, \mathbb{Z})$ be a unimodular matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\beta \neq \bar{\beta}$, and eigenvectors $(a_1, a_2, a_3), (b_1, b_2, b_3)$ of α, β, respectively. Then we define a structure of the group on $\mathbb{H} \times \mathbb{C} = \{(x + \sqrt{-1}\alpha^t, z) \colon x, t \in \mathbb{R}, z \in \mathbb{C}\}$ as follows:

$$(x + \sqrt{-1}\alpha^t, z) \cdot (x' + \sqrt{-1}\alpha^{t'}, z') = (\alpha^t x' + x + \sqrt{-1}\alpha^{t+t'}, \beta^t z' + z).$$

It can be expressed by

$$G = \left\{ \begin{pmatrix} \alpha^t & 0 & 0 & x \\ 0 & \beta^t & 0 & z \\ 0 & 0 & \bar{\beta}^t & \bar{z} \\ 0 & 0 & 0 & 1 \end{pmatrix} \colon t, x \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

Thus we easily see that the Lie group G is solvable and it admits a lattice Γ:

$$\Gamma = \left\{ \begin{pmatrix} \alpha^s & 0 & 0 & \\ 0 & \beta^s & 0 & P \begin{pmatrix} u \\ w_1 \\ w_2 \end{pmatrix} \\ 0 & 0 & \bar{\beta}^s & \\ 0 & 0 & 0 & 1 \end{pmatrix} \colon s, u, w_1,\ w_2 \in \mathbb{Z} \right\},$$

where $P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{C})$. Let $\mathfrak{g}$ be the solvable Lie algebra corresponding to G:

$$\mathfrak{g} = \mathrm{span}\{A, X, Y_1, Y_2 \colon [A, X] = -2rX,\ [A, Y_1] = rY_1 + \theta Y_2,\ [A, Y_2] = rY_2 - \theta Y_1\},$$

where $\alpha = e^{-2r}$ and $\beta = e^{r+\sqrt{-1}\theta}$. Let $\{\omega, x, y_1, y_2\}$ be the dual base of $\{A, X, Y_1, Y_2\}$:

$$d\omega = 0, \quad dx = 2r\omega \wedge x, \quad dy_1 = -r\omega \wedge y_1 + \theta\omega \wedge y_2, \quad dy_2 = -r\omega \wedge y_2 - \theta\omega \wedge y_1.$$

We define a left-invariant metric $\langle\ ,\ \rangle$ on $\Gamma\backslash G$ such that $\{A, X, Y_1, Y_2\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = X$, $JY_1 = Y_2$. Then $(\Gamma\backslash G, \langle\ ,\ \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega = -\omega \wedge x - y_1 \wedge y_2.$$

Note that the fundamental 2-form Ω is not $-2r\omega$-exact. We see that the Lee form ω is not parallel: $(\nabla_X\omega)X \neq 0$.

EXAMPLE 4.4 (Inoue surface S^+ [5], [11], [14]). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}y & \frac{1}{2}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

We can construct a lattice Γ on G (cf. [13]). Let $\mathfrak{g}$ be the solvable Lie algebra corresponding to G:

$$\mathfrak{g} = \operatorname{span}\{A, X, Y, Z : [A, X] = X,\ [A, Y] = -Y,\ [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge x, \quad dy = \omega \wedge y, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle\ ,\ \rangle$ on $\Gamma\backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Y$, $JZ = X$. Then $(\Gamma\backslash G, \langle\ ,\ \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge y - z \wedge x.$$

Note that the fundamental 2-form Ω is not $-\omega$-exact. We see that the Lee form ω is not parallel: $(\nabla_Y\omega)Y \neq 0$.

We mention that an Inoue surface S^- is not of the form $\Gamma\backslash G$, but it is a double covering space of an Inoue surface S^+ (cf. [10]). Then an Inoue surface S^- has a locally conformal Kähler structure.

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