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LOCALLY CONFORMAL KÄHLER STRUCTURES ON COMPACT SOLVMANIFOLDS

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Abstract

Let (M, g, J) be a compact Hermitian manifold and Ω the fundamental 2-form of (g, J) . A Hermitian manifold (M, g, J) is said to be locally conformal Kähler if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The purpose of this paper is to investigate a relation between a locally conformal Kähler structure and the adapted differential operator on compact solvmanifolds.

Introduction

Let (M, g, J) be a $2n$ -dimensional compact Hermitian manifold. We denote by Ω the fundamental 2-form, that is, the 2-form defined by $\Omega(X, Y) = g(X, JY)$. A Hermitian manifold (M, g, J) is said to be *locally conformal Kähler* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The closed 1-form ω is called *Lee form*. In the case $\omega = df$, $(M, e^{-f}g, J)$ is Kähler. The main non-Kähler examples of locally conformal Kähler manifolds are Hopf manifolds [15], Inoue surfaces [14] and generalized Kodaira–Thurston manifolds [5] (cf. [6]). Note that Inoue surfaces and generalized Kodaira–Thurston manifolds have a structure of a compact solvmanifold. In this paper, we investigate locally conformal Kähler structures on a compact solvmanifold $\Gamma \backslash G$ with a left-invariant complex structure, where G is a simply-connected solvable Lie group and Γ is a lattice of G , that is, a discrete co-compact subgroup.

An n -dimensional complex manifold M is said to be *complex parallelizable* if it admits holomorphic vector fields $\{X_1, \dots, X_n\}$ which are linearly independent at every point. Wang [16] proved that a compact complex parallelizable manifold M is biholomorphic to a homogeneous space $D \backslash G$, where G is a simply connected complex Lie group and D is a discrete subgroup of G . Abbena–Grassi [1] proved that a non-toral compact complex parallelizable manifold has no locally conformal Kähler structures. The author in [12] proved that if a compact nilmanifold M with a left-invariant complex structure has a locally conformal Kähler structure, then M is biholomorphic to a Kodaira–Thurston manifold or a generalized Kodaira–Thurston manifold.

A locally conformal Kähler manifold (M, g, J) is said to be a *generalized Hopf manifold* if the Lee form ω is parallel with respect to metric g . Vaisman [15] proved

that a generalized Hopf manifold has a structure of a principal S^1 -bundle over a compact Sasakian manifold. Hopf manifolds and generalized Kodaira–Thurston manifolds are generalized Hopf manifolds. Tricerri [14] constructed a locally conformal Kähler structure on Inoue surfaces with non-parallel Lee form ω . In Section 3, we prove that

Main Theorem. *Let $(\Gamma \backslash G, g, J)$ be a locally conformal Kähler solvmanifold of $\dim \Gamma \backslash G \geq 4$ with a left-invariant complex structure and ω the Lee form. We assume that $(\Gamma \backslash G, g, J)$ satisfies the following conditions:*

1. *There exists a left-invariant closed 1-form ω_0 such that the Lee form ω is cohomologous to ω_0 .*
2. *The fundamental 2-form Ω of (g, J) can be written as $\Omega = -\omega \wedge \eta + d\eta$, where $\eta = -\omega \circ J$.*

Then $\Gamma \backslash G$ has a left-invariant locally conformal Kähler structure with the parallel Lee form ω_0 , in particular, $\Gamma \backslash G$ is a generalized Hopf manifold.

In Section 4, we see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem.

Hasegawa [7] proved that a complex structure on a 4-dimensional compact solvmanifold is induced from a left-invariant complex structure on the solvable Lie group and classified 4-dimensional compact solvmanifolds with a complex structure. By this classification, we have a classification of 4-dimensional locally conformal Kähler solvmanifolds in Section 4.

1. Preliminaries

Let $(M = \Gamma \backslash G, g, J)$ be a compact locally conformal Kähler solvmanifold such that J is left-invariant. In this section, we see that the locally conformal Kähler metric g induces a left-invariant locally conformal Kähler metric and consider its fundamental 2-form on the Lie algebra \mathfrak{g} of G .

Let Ω be the fundamental 2-form of (g, J) . There exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. We now assume that the associated Lee form satisfies the condition 1 in Main Theorem. Then there exists a left-invariant closed 1-form ω_0 such that $\omega_0 - \omega = df$. This assumption holds for completely solvmanifolds [9]. Then we define a left-invariant 2-form Ω_0 by

$$\Omega_0(X, Y) = \int_M (e^f \Omega)(X, Y) d\mu$$

for left-invariant vector fields X, Y , where $d\mu$ is the volume element induced by a bi-invariant volume element on G . Belgun [3] proved that Ω_0 is J -invariant and it is the fundamental 2-form of a Hermitian structure $(\langle \cdot, \cdot \rangle, J)$ on the solvable Lie algebra \mathfrak{g} of G such that $d\Omega_0 = \omega_0 \wedge \Omega_0$.

DEFINITION 1.1. Let \mathfrak{g} be a Lie algebra and α a closed 1-form on \mathfrak{g} . We define the adapted algebraic complex $(\bigwedge \mathfrak{g}^*, d_\alpha)$ with the differential operator d_α :

$$d_\alpha \beta := \alpha \wedge \beta + d\beta$$

for $\beta \in \bigwedge^p \mathfrak{g}^*$. Note that $d_\alpha^2 = 0$ because α is closed. A p -form β is called α -closed if $d_\alpha \beta = 0$. It is called α -exact if there exists a $(p-1)$ -form γ such that $\beta = d_\alpha \gamma$.

On the above Hermitian structure $(\langle \cdot, \cdot \rangle, J)$ on \mathfrak{g} , since ω_0 is a closed 1-form, the fundamental 2-form Ω_0 is $-\omega_0$ -closed. We say that $(\langle \cdot, \cdot \rangle, J)$ on \mathfrak{g} is *locally conformal Kähler*. In Section 4, we consider a relation between a property of Lee form and the adapted differential operator on 4-dimensional solvmanifolds.

We next show that Ω_0 is $-\omega_0$ -exact if the second condition in Main Theorem also holds true.

Proposition 1.2 (cf. [3]). *Let Ω_0 be the 2-form on \mathfrak{g} above. If $\Omega = -\omega \wedge \eta + d\eta$, then Ω_0 is $-\omega_0$ -exact. In addition, if $\eta = -\omega \circ J$, then $\eta_0 = -k\omega_0 \circ J$, where $k = \int_M e^f d\mu$.*

Proof. Since $\Omega = -\omega \wedge \eta + d\eta$ and $\omega = \omega_0 - df$, we see that

$$\begin{aligned} e^f \Omega &= e^f (-\omega \wedge \eta + d\eta) \\ &= e^f (-\omega_0 \wedge \eta + df \wedge \eta + d\eta) \\ &= -\omega_0 \wedge e^f \eta + d(e^f \eta) \wedge \eta + e^f d\eta \\ &= -\omega_0 \wedge e^f \eta + d(e^f \eta). \end{aligned}$$

Then we get

$$\begin{aligned} \Omega_0(X, Y) &= \int_M (-\omega_0 \wedge e^f \eta + d(e^f \eta))(X, Y) d\mu \\ &= -\omega_0(X) \int_M (e^f \eta)(Y) d\mu + \omega_0(Y) \int_M (e^f \eta)(X) d\mu + \int_M d(e^f \eta)(X, Y) d\mu, \end{aligned}$$

for $X, Y \in \mathfrak{g}$.

Now, we define a 1-form η_0 on \mathfrak{g} by $\eta_0(X) = \int_M (e^f \eta)(X) d\mu$ for $X \in \mathfrak{g}$. Since $d\mu$ is the right-invariant volume element, its Lie derivative $L_X d\mu$ along a left-invariant vector fields X is zero. Then, for any function F on M , we see that

$$(XF) d\mu = (L_X F) d\mu = L_X(F d\mu) = di(X)F d\mu + i(X) d(F d\mu) = di(X)F d\mu,$$

where $i(X)$ is the interior product with the vector field X . Therefore, by Stokes's theorem, we have

$$\begin{aligned}
 & \int_M d(e^f \eta)(X, Y) d\mu \\
 &= \int_M X((e^f \eta)(Y)) d\mu - \int_M Y((e^f \eta)(X)) d\mu - \int_M (e^f \eta)([X, Y]) d\mu \\
 &= \int_M di(X)(e^f \eta(Y)) d\mu - \int_M di(Y)(e^f \eta(X)) d\mu - \eta_0([X, Y]) \\
 &= d\eta_0(X, Y).
 \end{aligned}$$

Thus we see that

$$\begin{aligned}
 \Omega_0(X, Y) &= -\omega_0(X) \int_M (e^f \eta)(Y) d\mu + \omega_0(Y) \int_M (e^f \eta)(X) d\mu + \int_M d(e^f \eta)(X, Y) d\mu \\
 &= -\omega_0(X) \eta_0(Y) + \omega_0(Y) \eta_0(X) + d\eta_0(X, Y) \\
 &= (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Y).
 \end{aligned}$$

Similarly, if $\eta = -\omega \circ J$, then we have

$$\begin{aligned}
 \eta_0(X) &= \int_M (e^f \eta)(X) d\mu = - \int_M (e^f \omega)(JX) d\mu = - \int_M (e^f \omega_0 - e^f df)(JX) d\mu \\
 &= -\omega_0(JX) \int_M e^f d\mu + \int_M d(e^f)(JX) d\mu \\
 &= -\omega_0(JX) \int_M e^f d\mu + \int_M (JX)(e^f) d\mu \\
 &= - \left(\int_M e^f d\mu \right) \omega_0(JX).
 \end{aligned}$$

Thus we see that $\eta_0 = -k\omega_0 \circ J$, where $k = \int_M e^f d\mu$. □

By Proposition 1.2, we see that $\Omega_0 = -\omega_0 \wedge k\eta'_0 + k d\eta'_0 = k(-\omega_0 \wedge \eta'_0 + d\eta'_0) = kd_{-\omega_0}\eta'_0$, where $\eta'_0 = (1/k)\eta_0 = -\omega_0 \circ J$. Then the locally conformal Kähler structure (Ω_0, J) induces the locally conformal Kähler structure $(\Omega'_0 = d_{-\omega_0}(-\omega_0 \circ J), J)$ on the solvable Lie algebra \mathfrak{g} . Therefore, by replacing Ω_0 by Ω'_0 , in order to prove Main Theorem, it is enough to show that if $(\mathfrak{g}, \Omega_0 = d_{-\omega_0}(-\omega_0 \circ J), J)$ is a locally conformal Kähler solvable Lie algebra, then the Lee form ω_0 is parallel with respect to $\langle \cdot, \cdot \rangle$ (see Section 3).

2. The adjoint operators and the inner product

Let $(\langle \cdot, \cdot \rangle, J)$ be the locally conformal Kähler structure as we consider in Section 1 on solvable Lie algebra \mathfrak{g} , namely Ω_0 can be written as $\Omega_0 = d_{-\omega_0}\eta_0 = -\omega_0 \wedge \eta_0 + d\eta_0$,

where ω_0 is a closed 1-form on \mathfrak{g} and $\eta_0 = -\omega_0 \circ J$. In this section, we investigate the properties of $(\mathfrak{g}, \langle \cdot, \cdot \rangle, J)$.

Let γ be the canonical isomorphism from \mathfrak{g}^* to \mathfrak{g} induced by the inner product $\langle \cdot, \cdot \rangle$. Put $A = \gamma(\omega_0)$. By this normalization, we may assume that $\langle A, A \rangle = 1$.

We easily see that an abelian Lie algebra of dimension equal to or more than 4 has no locally conformal Kähler structures. From now on, we assume that \mathfrak{g} is not abelian.

Since \mathfrak{g} is solvable, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. We take the descending central series for $[\mathfrak{g}, \mathfrak{g}]$: $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supset \dots \supset \mathfrak{n}^{(r)} \supset \mathfrak{n}^{(r+1)} = 0$, where $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ ($i \geq 1$) and $\mathfrak{n}^{(r)} \neq 0$. We easily see that $\mathfrak{n}^{(r)}$ is contained in the center $Z(\mathfrak{n})$ of \mathfrak{n} . Note that, for $X \in \mathfrak{g}$, $\text{ad}(X)\mathfrak{n}^{(i)} \subset \mathfrak{n}^{(i)}$ for each i . Then we get

Lemma 2.1. $J\mathfrak{n}^{(r)} \subset [\mathfrak{g}, \mathfrak{g}]^\perp$, where $[\mathfrak{g}, \mathfrak{g}]^\perp$ is the orthogonal component of $[\mathfrak{g}, \mathfrak{g}]$.

Proof. For $Z \in \mathfrak{n}^{(r)}$ and $X \in [\mathfrak{g}, \mathfrak{g}]$, we see that

$$\langle JZ, X \rangle = \Omega_0(X, Z) = (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Z) = 0,$$

because ω_0 is closed and $Z \in Z(\mathfrak{n})$. □

Lemma 2.2. For $Z, Z' \in \mathfrak{n}^{(r)}$, $[JZ, Z'] = [JZ', Z]$ and $[JZ, JZ'] = 0$.

Proof. The Nijenhuis tensor \mathbf{N}_J vanishes, because J is integrable. Since $Z, Z' \in \mathfrak{n}^{(r)}$, we see that

$$\begin{aligned} 0 &= \mathbf{N}_J(Z, Z') = [Z, Z'] + J[JZ, Z'] + J[Z, JZ'] - [JZ, JZ'] \\ &= J\{[JZ, Z'] + [Z, JZ']\} - [JZ, JZ']. \end{aligned}$$

Note that $[JZ, Z'], [Z, JZ'] \in \mathfrak{n}^{(r)}$. It follows that $J\{[JZ, Z'] + [Z, JZ']\} \in [\mathfrak{g}, \mathfrak{g}]^\perp$ by Lemma 2.1. Thus we have $[JZ, Z'] + [Z, JZ'] = 0$ and $[JZ, JZ'] = 0$. □

By Lemmas 2.1 and 2.2, we have

Proposition 2.3. For $U, V, W \in \mathfrak{n}^{(r)}$,

$$\langle \text{ad}(JU)V, W \rangle + \omega_0(JV)\langle U, W \rangle = \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)\langle U, V \rangle.$$

Proof. Since $\Omega_0 = -\omega_0 \wedge \eta_0 + d\eta_0$ and ω_0 is closed 1-form, we see that

$$\begin{aligned} \langle \text{ad}(JU)V, W \rangle &= \Omega_0(J \text{ad}(JU)V, W) = -\Omega_0(\text{ad}(JU)V, JW) \\ (2.1) \quad &= (\omega_0 \wedge \eta_0 - d\eta_0)(\text{ad}(JU)V, JW) \\ &= -\omega_0(JW)\eta_0(\text{ad}(JU)V) - d\eta_0(\text{ad}(JU)V, JW). \end{aligned}$$

Since $\eta_0 = -\omega_0 \circ J$, we get

$$\begin{aligned}
 (2.2) \quad & -\omega_0(JW)\eta_0(\text{ad}(JU)V) = \omega_0(JW) d\eta_0(JU, V) \\
 & = \omega_0(JW)\{\Omega_0(JU, V) + \omega_0 \wedge \eta_0(JU, V)\} \\
 & = \omega_0(JW)\langle U, V \rangle - \omega_0(JW)\omega_0(JU)\omega_0(JV).
 \end{aligned}$$

From the derivation conditions and Lemma 2.2, we get

$$\begin{aligned}
 d\eta_0(\text{ad}(JU)V, JW) &= d\eta_0(JU, [V, JW]) - d\eta_0(V, [JU, JW]) \\
 &= d\eta_0(JU, [V, JW]) \\
 &= d\eta_0(JU, [W, JV]) \\
 &= d\eta_0([JU, W], JV) + d\eta_0(W, [JU, JV]) \\
 &= d\eta_0([JU, W], JV) = -d\eta_0(JV, [JU, W]).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 (2.3) \quad & -d\eta_0(\text{ad}(JU)V, JW) \\
 &= d\eta_0(JV, [JU, W]) \\
 &= \Omega_0(JV, [JU, W]) + \omega_0 \wedge \eta_0(JV, [JU, W]) \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV) d\eta_0(JU, W) \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV)\{\Omega_0(JU, W) + \omega_0 \wedge \eta_0(JU, W)\} \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV)\langle U, W \rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW).
 \end{aligned}$$

From (2.1) and (2.2)), (2.3), we have

$$\begin{aligned}
 \langle \text{ad}(JU)V, W \rangle &= \omega_0(JW)\langle U, V \rangle - \omega_0(JW)\omega_0(JU)\omega_0(JV) \\
 &\quad + \langle V, [JU, W] \rangle - \omega_0(JV)\langle U, W \rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW), \\
 \langle \text{ad}(JU)V, W \rangle + \omega_0(JV)\langle U, W \rangle &= \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)\langle U, V \rangle. \quad \square
 \end{aligned}$$

It is well-known that if a solvable Lie group G admits a lattice Γ , then the solvable Lie algebra \mathfrak{g} of G is unimodular. We define

DEFINITION 2.4. A solvable Lie algebra \mathfrak{g} is called strongly unimodular if, for $X \in \mathfrak{g}$, $\text{tr ad}(X)|_{\mathfrak{n}^{(i)}} = 0$ for each i .

Benson–Gordon [4] proved that if a solvable Lie group G admits a lattice Γ , then the solvable Lie algebra \mathfrak{g} of G is strongly unimodular.

We take an orthonormal frame $\{Z_1, \dots, Z_m\}$ of $\mathfrak{n}^{(r)}$ and consider the strongly unimodular conditions of $\text{ad}(JZ_i)$ from $\mathfrak{n}^{(r)}$ to $\mathfrak{n}^{(r)}$ for each i .

For each i , let

$$\mathrm{ad}(JZ_i)(Z_1, \dots, Z_m) = (Z_1, \dots, Z_m) \begin{pmatrix} a_{11}(i) & \cdots & a_{1m}(i) \\ \vdots & \vdots & \vdots \\ a_{m1}(i) & \cdots & a_{mm}(i) \end{pmatrix}.$$

By Proposition 2.3, we see that

$$\begin{aligned} \langle \mathrm{ad}(JZ_i)Z_j, Z_k \rangle + \omega_0(JZ_j)\langle Z_i, Z_k \rangle &= \langle Z_j, \mathrm{ad}(JZ_i)Z_k \rangle + \omega_0(JZ_k)\langle Z_i, Z_j \rangle, \\ a_{kj}(i) + \delta_{ik}\omega_0(JZ_j) &= a_{jk}(i) + \delta_{ij}\omega_0(JZ_k). \end{aligned}$$

It follows that, in the case of $j, k \neq i$,

$$a_{kj}(i) = a_{jk}(i)$$

and, in the case of $j = i, k \neq i$,

$$a_{ki}(i) = a_{ik}(i) + \omega_0(JZ_k).$$

Then we get

$$(2.4) \quad (a_{jk}(i)) = A_i + \begin{pmatrix} \text{line } i & & \\ & \omega_0(JZ_1) & \\ & \vdots & \\ & \omega_0(JZ_m) & \\ & & 0 \end{pmatrix},$$

where A_i is an $(m \times m)$ -symmetric matrix.

Moreover, put

$$A_i = \text{row } i \begin{pmatrix} & & \text{line } i & & \\ & & a_1^i & & \\ & * & \vdots & * & \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & & \vdots & & \\ & * & a_m^i & * & \end{pmatrix},$$

for each i . Since $[JZ_i, Z_j] = [JZ_j, Z_i]$, we get

$$a_{jj}(i) = \langle \mathrm{ad}(JZ_i)Z_j, Z_j \rangle = \langle \mathrm{ad}(JZ_j)Z_i, Z_j \rangle = a_i^j$$

for $j \neq i$. Then A_i can be written as

$$A_i = \text{row } i \begin{pmatrix} & \text{line } i \\ a_i^1 & a_1^i & & * \\ & \ddots & \vdots & \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & & \vdots & \ddots & \\ & * & a_m^i & & a_i^m \end{pmatrix}.$$

Thus (2.4) can be expressed as follows:

$$(2.5) \quad (a_{jk}(i)) = \text{row } i \begin{pmatrix} & \text{line } i \\ a_i^1 & a_1^i + \omega_0(JZ_1) & & & \\ & * & & \vdots & * \\ & \ddots & & \vdots & \\ * & & & a_i^i + \omega_0(JZ_i) & \cdots & a_m^i \\ a_1^i & \cdots & & & * & \\ & & & \vdots & \ddots & \\ & * & a_m^i + \omega_0(JZ_m) & * & & a_i^m \end{pmatrix}.$$

From the strongly unimodular condition of $\text{ad}(JZ_i)|_{\mathfrak{n}^\vee}$, we have

Proposition 2.5. *For each i , $(\sum_j a_i^j) + \omega_0(JZ_i) = 0$.*

Then we have

Corollary 2.6. $\sum_i \omega_0(JZ_i)^2 = 1$.

Proof. For each i , we see that

$$\begin{aligned} 1 &= \langle Z_i, Z_i \rangle = \Omega_0(JZ_i, Z_i) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(JZ_i, Z_i) \\ &= -\omega_0(JZ_i)\eta_0(Z_i) - \eta_0([JZ_i, Z_i]) \\ &= \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \end{aligned}$$

because $\eta_0 = -\omega_0 \circ J$. From (2.5),

$$\begin{aligned}
 1 &= \langle Z_i, Z_i \rangle = \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \\
 (2.6) \quad 1 &= \omega_0(JZ_i)^2 + \omega_0 \circ J \left(\sum_j (a_j^i + \omega_0(JZ_j))Z_j \right), \\
 1 &= \omega_0(JZ_i)^2 + \sum_j a_j^i \omega_0(JZ_j) + \sum_j \omega_0(JZ_j)^2.
 \end{aligned}$$

Take the sum of (2.6) for i , we get

$$\begin{aligned}
 \sum_i 1 &= \sum_i \omega_0(JZ_i)^2 + \sum_{i,j} a_j^i \omega_0(JZ_j) + \sum_{i,j} \omega_0(JZ_j)^2, \\
 m &= \sum_i \omega_0(JZ_i)^2 + \sum_i \left(\sum_j a_j^i \right) \omega_0(JZ_i) + m \sum_i \omega_0(JZ_i)^2.
 \end{aligned}$$

By Proposition 2.5, we have

$$\begin{aligned}
 m &= \sum_i \omega_0(JZ_i)^2 + \sum_i (-\omega_0(JZ_i))\omega_0(JZ_i) + m \sum_i \omega_0(JZ_i)^2, \\
 m &= \sum_i \omega_0(JZ_i)^2 - \sum_i \omega_0(JZ_i)^2 + m \sum_i \omega_0(JZ_i)^2, \\
 m &= m \sum_i \omega_0(JZ_i)^2,
 \end{aligned}$$

which implies that $\sum_i \omega_0(JZ_i)^2 = 1$. □

Corollary 2.7. $A = \sum_i \omega_0(JZ_i)JZ_i \in J\mathfrak{n}^{(r)}$.

Proof. Since $A = \gamma(\omega_0)$, it can be given by

$$A = \sum_i \omega_0(JZ_i)JZ_i + B,$$

where $B \in (J\mathfrak{n}^{(r)})^\perp \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. From $\langle A, A \rangle = 1$, we have

$$1 = \langle A, A \rangle = \sum_i \omega_0(JZ_i)^2 \langle JZ_i, JZ_i \rangle + \langle B, B \rangle = \sum_i \omega_0(JZ_i)^2 + \langle B, B \rangle.$$

By Corollary 2.6, we have $\langle B, B \rangle = 0$, which implies that $B = 0$. □

3. Structure on the solvable Lie algebra

We use same notation introduced in Section 2. In this section, we prove that JA is in the center $Z(\mathfrak{g})$ of \mathfrak{g} and have Main Theorem.

We see $JA = -\sum_i \omega_0(JZ_i)Z_i \in \mathfrak{n}^{(r)}$ by Corollary 2.7. Put $Z_0 = JA$. Note that $\eta_0(Z_0) = -\omega_0 \circ J(JA) = 1$ and $\eta_0(A) = -\omega_0 \circ J(A) = 0$. We get

Lemma 3.1. *For $X \in \mathfrak{g}$, $d\eta_0(A, X) = 0$ and $d\eta_0(Z_0, X) = 0$.*

Proof. Since $\eta_0 = -\omega_0 \circ J$, we see that

$$\begin{aligned} d\eta_0(A, X) &= \Omega_0(A, X) + \omega_0 \wedge \eta_0(A, X) = \langle A, JX \rangle + \eta_0(X) = \omega_0(JX) + \eta_0(X) \\ &= 0. \end{aligned}$$

Similarly, we get

$$\begin{aligned} d\eta_0(Z_0, X) &= \Omega_0(Z_0, X) + \omega_0 \wedge \eta_0(Z_0, X) = -\langle X, JZ_0 \rangle - \omega_0(X) = \langle X, A \rangle - \omega_0(X) \\ &= 0. \end{aligned} \quad \square$$

Then we see that

Proposition 3.2. *For $U \in \mathfrak{n}^{(r)}$, $\text{ad}(A)U = 0$.*

Proof. By a straightforward computation, we see that

$$\begin{aligned} \langle \text{ad}(A)U, \text{ad}(A)U \rangle &= \Omega_0(J \circ \text{ad}(A)U, \text{ad}(A)U) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(A)U, \text{ad}(A)U) \\ &= -\omega_0(J \circ \text{ad}(A)U)\eta_0(\text{ad}(A)U) + d\eta_0(J \circ \text{ad}(A)U, \text{ad}(A)U). \end{aligned}$$

By Lemma 3.1 and the derivation conditions,

$$\begin{aligned} \langle \text{ad}(A)U, \text{ad}(A)U \rangle &= \omega_0(J \circ \text{ad}(A)U) d\eta_0(A, U) - d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U) \\ &= -d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U) \\ &= -d\eta_0(A, [U, J \circ \text{ad}(A)U]) + d\eta_0(U, [A, J \circ \text{ad}(A)U]) \\ &= d\eta_0(U, [A, J \circ \text{ad}(A)U]). \end{aligned}$$

Now, since $JA = Z_0$ and $\text{ad}(A)U \in \mathfrak{n}^{(r)}$, we see that

$$[A, J \circ \text{ad}(A)U] = [-JZ_0, J \circ \text{ad}(A)U] = 0$$

by Lemma 2.2. It follows that $d\eta_0(U, [A, J \circ \text{ad}(A)U]) = 0$, which implies that $\text{ad}(A)U = 0$. \square

Therefore we have

Theorem 3.3. $Z_0 \in Z(\mathfrak{g})$.

Proof. Let $X \in \mathfrak{g}$. By Lemma 3.1, we see that

$$\begin{aligned}
 \langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle &= \Omega_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\
 &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\
 &= -\omega_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) + d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\
 &= \omega_0(J \circ \text{ad}(X)Z_0, d\eta_0(X, Z_0)) + d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\
 &= d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0).
 \end{aligned}$$

Moreover, from the derivation conditions and $Z_0 \in Z([\mathfrak{g}, \mathfrak{g}])$,

$$\begin{aligned}
 \langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle &= d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) = -d\eta_0(\text{ad}(X)Z_0, J \circ \text{ad}(X)Z_0) \\
 &= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]) + d\eta_0(Z_0, [X, J \circ \text{ad}(X)Z_0]) \\
 &= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]).
 \end{aligned}$$

Now, since $\text{ad}(X)Z_0 \in \mathfrak{n}^{(r)}$, we see that

$$[Z_0, J \circ \text{ad}(X)Z_0] = [\text{ad}(X)Z_0, JZ_0] = [A, \text{ad}(X)Z_0] = 0$$

by Lemma 2.2 and Proposition 3.2. It follows that $d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]) = 0$, which implies that $\text{ad}(X)Z_0 = 0$ for $X \in \mathfrak{g}$. \square

From Theorem 3.3, we have

Corollary 3.4. $\text{ad}(A) \circ J = J \circ \text{ad}(A)$.

Proof. Let $X \in \mathfrak{g}$. Since J is integrable, we see that

$$\begin{aligned}
 0 = \mathbf{N}_J(A, X) &= [A, X] + J[JA, X] + J[A, JX] - [JA, JX] \\
 0 &= [A, X] + J[Z_0, X] + J[A, JX] - [Z_0, JX].
 \end{aligned}$$

From $Z_0 \in Z(\mathfrak{g})$, $[A, X] + J[A, JX] = 0$. Then we have our claim. \square

Corollary 3.5. *Lee form ω_0 is parallel, if and only if Ω_0 is $\text{ad}(A)$ -invariant.*

Proof. Let ∇ be the Riemannian connection of $\langle \cdot, \cdot \rangle$. For $X, Y \in \mathfrak{g}$, we see that

$$\begin{aligned} 2(\nabla_X \omega_0)(Y) &= -2\omega_0(\nabla_X Y) = -2\langle A, \nabla_X Y \rangle \\ &= -\langle [A, X], Y \rangle - \langle X, [A, Y] \rangle - \langle A, [X, Y] \rangle \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]) - \omega_0([X, Y]) \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]), \end{aligned}$$

because ω_0 is a closed 1-form. From Corollary 3.4, we get

$$2(\nabla_X \omega_0)(Y) = -\Omega_0([A, JX], Y) - \Omega_0(JX, [A, Y]) = (\text{ad}^*(A)\Omega_0)(JX, Y).$$

Thus we have our claim. \square

Proof of Main Theorem. By Corollary 3.5, to prove Main Theorem, it is enough to show that Ω_0 is $\text{ad}(A)$ -invariant.

Let $X, Y \in \mathfrak{g}$. By a straightforward computation, we see that

$$\begin{aligned} (\text{ad}^*(A)\Omega_0)(X, Y) &= -\Omega_0([A, X], Y) - \Omega_0(X, [A, Y]) \\ &= (\omega_0 \wedge \eta_0 - d\eta_0)([A, X], Y) + (\omega_0 \wedge \eta_0 - d\eta_0)(X, [A, Y]) \\ &= -\omega_0(Y)\eta_0([A, X]) - d\eta_0([A, X], Y) \\ &\quad + \omega_0(X)\eta_0([A, Y]) - d\eta_0(X, [A, Y]). \end{aligned}$$

By Lemma 3.1, we get $\eta_0([A, X]) = -d\eta_0(A, X) = 0$ and $\eta_0([A, Y]) = 0$. Moreover, from the derivation conditions,

$$d\eta_0([A, X], Y) + d\eta_0(X, [A, Y]) = d\eta_0(A, [X, Y]) = 0,$$

which implies that Ω_0 is $\text{ad}(A)$ -invariant. This completes the proof of Main Theorem. \square

4. Examples

In this section, we give examples of Main Theorem and consider locally conformal Kähler structures on 4-dimensional compact solvmanifolds. Note that a compact Kähler solvmanifold is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus ([2], [8]).

Hasegawa [7] classified a 4-dimensional compact solvmanifold and proved that any complex structure on such solvmanifold is induced from a left-invariant complex structure on Lie group. By this classification, we see that a 4-dimensional locally conformal Kähler solvmanifold is biholomorphic to Kodaira–Thurston manifold, Secondary Kodaira–Thurston manifold or Inoue surfaces.

We see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem:

EXAMPLE 4.1 (Kodaira–Thurston manifold [5]). Let G be a 3-dimensional nilpotent Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ . Let \mathfrak{g} be the nilpotent Lie algebra corresponding to G . $\mathfrak{g} \times \mathbb{R}$ is given by

$$\mathfrak{g} \times \mathbb{R} = \text{span}\{X, Y, Z, A : [X, Y] = Z\}.$$

Let $\{x, y, z, \omega\}$ be the dual base of $\{X, Y, Z, A\}$:

$$dx = dy = d\omega = 0, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G \times S^1$ such that $\{X, Y, Z, A\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z, JX = Y$. Then $(\Gamma \backslash G \times S^1, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega}z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$ -exact. We easily see that Lee form ω is parallel.

EXAMPLE 4.2 (Secondary Kodaira–Thurston manifold (cf. [15])). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(x \sin t + y \cos t) & \frac{1}{2}(x \cos t - y \sin t) & z \\ 0 & \cos t & \sin t & x \\ 0 & -\sin t & \cos t & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ :

$$\Gamma = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(u \sin 2\pi s + v \cos 2\pi s) & \frac{1}{2}(u \cos 2\pi s - v \sin 2\pi s) & w \\ 0 & \cos 2\pi s & \sin 2\pi s & u \\ 0 & -\sin 2\pi s & \cos 2\pi s & v \\ 0 & 0 & 0 & 1 \end{pmatrix} : s, u, v, w \in \mathbb{Z} \right\}.$$

Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y, Z : [A, X] = Y, [A, Y] = -X, [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge y, \quad dy = \omega \wedge x, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z, JX = Y$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega}z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$ -exact. We easily see that Lee form ω is parallel.

A locally conformal Kähler structure on Inoue surfaces are different from one on solvmanifolds as Main Theorem:

EXAMPLE 4.3 (Inoue surface S^0 [11], [14]). Let $B \in \text{SL}(3, \mathbb{Z})$ be a unimodular matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\beta \neq \bar{\beta}$, and eigenvectors $(a_1, a_2, a_3), (b_1, b_2, b_3)$ of α, β , respectively. Then we define a structure of the group on $\mathbb{H} \times \mathbb{C} = \{(x + \sqrt{-1}\alpha^t, z) : x, t \in \mathbb{R}, z \in \mathbb{C}\}$ as follows:

$$(x + \sqrt{-1}\alpha^t, z) \cdot (x' + \sqrt{-1}\alpha^{t'}, z') = (\alpha^t x' + x + \sqrt{-1}\alpha^{t+t'}, \beta^t z' + z).$$

It can be expressed by

$$G = \left\{ \begin{pmatrix} \alpha^t & 0 & 0 & x \\ 0 & \beta^t & 0 & z \\ 0 & 0 & \bar{\beta}^t & \bar{z} \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

Thus we easily see that the Lie group G is solvable and it admits a lattice Γ :

$$\Gamma = \left\{ \begin{pmatrix} \alpha^s & 0 & 0 & P \begin{pmatrix} u \\ w_1 \\ w_2 \end{pmatrix} \\ 0 & \beta^s & 0 & \\ 0 & 0 & \bar{\beta}^s & \\ 0 & 0 & 0 & 1 \end{pmatrix} : s, u, w_1, w_2 \in \mathbb{Z} \right\},$$

where $P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})$. Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y_1, Y_2 : [A, X] = -2rX, [A, Y_1] = rY_1 + \theta Y_2, [A, Y_2] = rY_2 - \theta Y_1\},$$

where $\alpha = e^{-2r}$ and $\beta = e^{r+\sqrt{-1}\theta}$. Let $\{\omega, x, y_1, y_2\}$ be the dual base of $\{A, X, Y_1, Y_2\}$:

$$d\omega = 0, \quad dx = 2r\omega \wedge x, \quad dy_1 = -r\omega \wedge y_1 + \theta\omega \wedge y_2, \quad dy_2 = -r\omega \wedge y_2 - \theta\omega \wedge y_1.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y_1, Y_2\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = X$, $JY_1 = Y_2$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega = -\omega \wedge x - y_1 \wedge y_2.$$

Note that the fundamental 2-form Ω is not $-2r\omega$ -exact. We see that the Lee form ω is not parallel: $(\nabla_X \omega)X \neq 0$.

EXAMPLE 4.4 (Inoue surface S^+ [5], [11], [14]). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}y & \frac{1}{2}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

We can construct a lattice Γ on G (cf. [13]). Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y, Z : [A, X] = X, [A, Y] = -Y, [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge x, \quad dy = \omega \wedge y, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Y$, $JZ = X$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge y - z \wedge x.$$

Note that the fundamental 2-form Ω is not $-\omega$ -exact. We see that the Lee form ω is not parallel: $(\nabla_Y \omega)Y \neq 0$.

We mention that an Inoue surface S^- is not of the form $\Gamma \backslash G$, but it is a double covering space of an Inoue surface S^+ (cf. [10]). Then an Inoue surface S^- has a locally conformal Kähler structure.

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