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LOCALLY CONFORMAL KÄHLER STRUCTURES ON COMPACT SOLVMANIFOLDS

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Abstract

Let (M, g, J) be a compact Hermitian manifold and Ω the fundamental 2-form of (g, J) . A Hermitian manifold (M, g, J) is said to be locally conformal Kähler if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The purpose of this paper is to investigate a relation between a locally conformal Kähler structure and the adapted differential operator on compact solvmanifolds.

Introduction

Let (M, g, J) be a $2n$ -dimensional compact Hermitian manifold. We denote by Ω the fundamental 2-form, that is, the 2-form defined by $\Omega(X, Y) = g(X, JY)$. A Hermitian manifold (M, g, J) is said to be *locally conformal Kähler* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The closed 1-form ω is called *Lee form*. In the case $\omega = df$, $(M, e^{-f}g, J)$ is Kähler. The main non-Kähler examples of locally conformal Kähler manifolds are Hopf manifolds [15], Inoue surfaces [14] and generalized Kodaira–Thurston manifolds [5] (cf. [6]). Note that Inoue surfaces and generalized Kodaira–Thurston manifolds have a structure of a compact solvmanifold. In this paper, we investigate locally conformal Kähler structures on a compact solvmanifold $\Gamma \backslash G$ with a left-invariant complex structure, where G is a simply-connected solvable Lie group and Γ is a lattice of G , that is, a discrete co-compact subgroup.

An n -dimensional complex manifold M is said to be *complex parallelizable* if it admits holomorphic vector fields $\{X_1, \dots, X_n\}$ which are linearly independent at every point. Wang [16] proved that a compact complex parallelizable manifold M is biholomorphic to a homogeneous space $D \backslash G$, where G is a simply connected complex Lie group and D is a discrete subgroup of G . Abbena–Grassi [1] proved that a non-toral compact complex parallelizable manifold has no locally conformal Kähler structures. The author in [12] proved that if a compact nilmanifold M with a left-invariant complex structure has a locally conformal Kähler structure, then M is biholomorphic to a Kodaira–Thurston manifold or a generalized Kodaira–Thurston manifold.

A locally conformal Kähler manifold (M, g, J) is said to be a *generalized Hopf manifold* if the Lee form ω is parallel with respect to metric g . Vaisman [15] proved

that a generalized Hopf manifold has a structure of a principal S^1 -bundle over a compact Sasakian manifold. Hopf manifolds and generalized Kodaira–Thurston manifolds are generalized Hopf manifolds. Tricerri [14] constructed a locally conformal Kähler structure on Inoue surfaces with non-parallel Lee form ω . In Section 3, we prove that

Main Theorem. *Let $(\Gamma \backslash G, g, J)$ be a locally conformal Kähler solvmanifold of $\dim \Gamma \backslash G \geq 4$ with a left-invariant complex structure and ω the Lee form. We assume that $(\Gamma \backslash G, g, J)$ satisfies the following conditions:*

1. *There exists a left-invariant closed 1-form ω_0 such that the Lee form ω is cohomologous to ω_0 .*
2. *The fundamental 2-form Ω of (g, J) can be written as $\Omega = -\omega \wedge \eta + d\eta$, where $\eta = -\omega \circ J$.*

Then $\Gamma \backslash G$ has a left-invariant locally conformal Kähler structure with the parallel Lee form ω_0 , in particular, $\Gamma \backslash G$ is a generalized Hopf manifold.

In Section 4, we see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem.

Hasegawa [7] proved that a complex structure on a 4-dimensional compact solvmanifold is induced from a left-invariant complex structure on the solvable Lie group and classified 4-dimensional compact solvmanifolds with a complex structure. By this classification, we have a classification of 4-dimensional locally conformal Kähler solvmanifolds in Section 4.

1. Preliminaries

Let $(M = \Gamma \backslash G, g, J)$ be a compact locally conformal Kähler solvmanifold such that J is left-invariant. In this section, we see that the locally conformal Kähler metric g induces a left-invariant locally conformal Kähler metric and consider its fundamental 2-form on the Lie algebra \mathfrak{g} of G .

Let Ω be the fundamental 2-form of (g, J) . There exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. We now assume that the associated Lee form satisfies the condition 1 in Main Theorem. Then there exists a left-invariant closed 1-form ω_0 such that $\omega_0 - \omega = df$. This assumption holds for completely solvmanifolds [9]. Then we define a left-invariant 2-form Ω_0 by

$$\Omega_0(X, Y) = \int_M (e^f \Omega)(X, Y) d\mu$$

for left-invariant vector fields X, Y , where $d\mu$ is the volume element induced by a bi-invariant volume element on G . Belgun [3] proved that Ω_0 is J -invariant and it is the fundamental 2-form of a Hermitian structure $(\langle \cdot, \cdot \rangle, J)$ on the solvable Lie algebra \mathfrak{g} of G such that $d\Omega_0 = \omega_0 \wedge \Omega_0$.

DEFINITION 1.1. Let \mathfrak{g} be a Lie algebra and α a closed 1-form on \mathfrak{g} . We define the adapted algebraic complex $(\wedge \mathfrak{g}^*, d_\alpha)$ with the differential operator d_α :

$$d_\alpha \beta := \alpha \wedge \beta + d\beta$$

for $\beta \in \wedge^p \mathfrak{g}^*$. Note that $d_\alpha^2 = 0$ because α is closed. A p -form β is called α -closed if $d_\alpha \beta = 0$. It is called α -exact if there exists a $(p - 1)$ -form γ such that $\beta = d_\alpha \gamma$.

On the above Hermitian structure $(\langle \cdot, \cdot \rangle, J)$ on \mathfrak{g} , since ω_0 is a closed 1-form, the fundamental 2-form Ω_0 is $-\omega_0$ -closed. We say that $(\langle \cdot, \cdot \rangle, J)$ on \mathfrak{g} is *locally conformal Kähler*. In Section 4, we consider a relation between a property of Lee form and the adapted differential operator on 4-dimensional solvmanifolds.

We next show that Ω_0 is $-\omega_0$ -exact if the second condition in Main Theorem also holds true.

Proposition 1.2 (cf. [3]). *Let Ω_0 be the 2-form on \mathfrak{g} above. If $\Omega = -\omega \wedge \eta + d\eta$, then Ω_0 is $-\omega_0$ -exact. In addition, if $\eta = -\omega \circ J$, then $\eta_0 = -k\omega_0 \circ J$, where $k = \int_M e^f d\mu$.*

Proof. Since $\Omega = -\omega \wedge \eta + d\eta$ and $\omega = \omega_0 - df$, we see that

$$\begin{aligned} e^f \Omega &= e^f (-\omega \wedge \eta + d\eta) \\ &= e^f (-\omega_0 \wedge \eta + df \wedge \eta + d\eta) \\ &= -\omega_0 \wedge e^f \eta + d(e^f) \wedge \eta + e^f d\eta \\ &= -\omega_0 \wedge e^f \eta + d(e^f \eta). \end{aligned}$$

Then we get

$$\begin{aligned} \Omega_0(X, Y) &= \int_M (-\omega_0 \wedge e^f \eta + d(e^f \eta))(X, Y) d\mu \\ &= -\omega_0(X) \int_M (e^f \eta)(Y) d\mu + \omega_0(Y) \int_M (e^f \eta)(X) d\mu + \int_M d(e^f \eta)(X, Y) d\mu, \end{aligned}$$

for $X, Y \in \mathfrak{g}$.

Now, we define a 1-form η_0 on \mathfrak{g} by $\eta_0(X) = \int_M (e^f \eta)(X) d\mu$ for $X \in \mathfrak{g}$. Since $d\mu$ is the right-invariant volume element, its Lie derivative $L_X d\mu$ along a left-invariant vector fields X is zero. Then, for any function F on M , we see that

$$(XF) d\mu = (L_X F) d\mu = L_X(F d\mu) = di(X)F d\mu + i(X) d(F d\mu) = di(X)F d\mu,$$

where $i(X)$ is the interior product with the vector field X . Therefore, by Stokes's theorem, we have

$$\begin{aligned} & \int_M d(e^f \eta)(X, Y) d\mu \\ &= \int_M X((e^f \eta)(Y)) d\mu - \int_M Y((e^f \eta)(X)) d\mu - \int_M (e^f \eta)([X, Y]) d\mu \\ &= \int_M di(X)(e^f \eta(Y)) d\mu - \int_M di(Y)(e^f \eta(X)) d\mu - \eta_0([X, Y]) \\ &= d\eta_0(X, Y). \end{aligned}$$

Thus we see that

$$\begin{aligned} \Omega_0(X, Y) &= -\omega_0(X) \int_M (e^f \eta)(Y) d\mu + \omega_0(Y) \int_M (e^f \eta)(X) d\mu + \int_M d(e^f \eta)(X, Y) d\mu \\ &= -\omega_0(X)\eta_0(Y) + \omega_0(Y)\eta_0(X) + d\eta_0(X, Y) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Y). \end{aligned}$$

Similarly, if $\eta = -\omega \circ J$, then we have

$$\begin{aligned} \eta_0(X) &= \int_M (e^f \eta)(X) d\mu = - \int_M (e^f \omega)(JX) d\mu = - \int_M (e^f \omega_0 - e^f df)(JX) d\mu \\ &= -\omega_0(JX) \int_M e^f d\mu + \int_M d(e^f)(JX) d\mu \\ &= -\omega_0(JX) \int_M e^f d\mu + \int_M (JX)(e^f) d\mu \\ &= -\left(\int_M e^f d\mu\right)\omega_0(JX). \end{aligned}$$

Thus we see that $\eta_0 = -k\omega_0 \circ J$, where $k = \int_M e^f d\mu$. □

By Proposition 1.2, we see that $\Omega_0 = -\omega_0 \wedge k\eta'_0 + k d\eta'_0 = k(-\omega_0 \wedge \eta'_0 + d\eta'_0) = kd_{-\omega_0}\eta'_0$, where $\eta'_0 = (1/k)\eta_0 = -\omega_0 \circ J$. Then the locally conformal Kähler structure (Ω_0, J) induces the locally conformal Kähler structure $(\Omega'_0 = d_{-\omega_0}(-\omega_0 \circ J), J)$ on the solvable Lie algebra \mathfrak{g} . Therefore, by replacing Ω_0 by Ω'_0 , in order to prove Main Theorem, it is enough to show that if $(\mathfrak{g}, \Omega_0 = d_{-\omega_0}(-\omega_0 \circ J), J)$ is a locally conformal Kähler solvable Lie algebra, then the Lee form ω_0 is parallel with respect to \langle, \rangle (see Section 3).

2. The adjoint operators and the inner product

Let (\langle, \rangle, J) be the locally conformal Kähler structure as we consider in Section 1 on solvable Lie algebra \mathfrak{g} , namely Ω_0 can be written as $\Omega_0 = d_{-\omega_0}\eta_0 = -\omega_0 \wedge \eta_0 + d\eta_0$,

where ω_0 is a closed 1-form on \mathfrak{g} and $\eta_0 = -\omega_0 \circ J$. In this section, we investigate the properties of $(\mathfrak{g}, \langle \cdot, \cdot \rangle, J)$.

Let γ be the canonical isomorphism from \mathfrak{g}^* to \mathfrak{g} induced by the inner product $\langle \cdot, \cdot \rangle$. Put $A = \gamma(\omega_0)$. By this normalization, we may assume that $\langle A, A \rangle = 1$.

We easily see that an abelian Lie algebra of dimension equal to or more than 4 has no locally conformal Kähler structures. From now on, we assume that \mathfrak{g} is not abelian.

Since \mathfrak{g} is solvable, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. We take the descending central series for $[\mathfrak{g}, \mathfrak{g}]$: $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supset \dots \supset \mathfrak{n}^{(r)} \supset \mathfrak{n}^{(r+1)} = 0$, where $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{(i)}]$ ($i \geq 1$) and $\mathfrak{n}^{(r)} \neq 0$. We easily see that $\mathfrak{n}^{(r)}$ is contained in the center $Z(\mathfrak{n})$ of \mathfrak{n} . Note that, for $X \in \mathfrak{g}$, $\text{ad}(X)\mathfrak{n}^{(i)} \subset \mathfrak{n}^{(i)}$ for each i . Then we get

Lemma 2.1. $J\mathfrak{n}^{(r)} \subset [\mathfrak{g}, \mathfrak{g}]^\perp$, where $[\mathfrak{g}, \mathfrak{g}]^\perp$ is the orthogonal component of $[\mathfrak{g}, \mathfrak{g}]$.

Proof. For $Z \in \mathfrak{n}^{(r)}$ and $X \in [\mathfrak{g}, \mathfrak{g}]$, we see that

$$\langle JZ, X \rangle = \Omega_0(X, Z) = (-\omega_0 \wedge \eta_0 + d\eta_0)(X, Z) = 0,$$

because ω_0 is closed and $Z \in Z(\mathfrak{n})$. □

Lemma 2.2. For $Z, Z' \in \mathfrak{n}^{(r)}$, $[JZ, Z'] = [JZ', Z]$ and $[JZ, JZ'] = 0$.

Proof. The Nijenhuis tensor \mathbf{N}_J vanishes, because J is integrable. Since $Z, Z' \in \mathfrak{n}^{(r)}$, we see that

$$\begin{aligned} 0 &= \mathbf{N}_J(Z, Z') = [Z, Z'] + J[JZ, Z'] + J[Z, JZ'] - [JZ, JZ'] \\ &= J\{[JZ, Z'] + [Z, JZ']\} - [JZ, JZ']. \end{aligned}$$

Note that $[JZ, Z'], [Z, JZ'] \in \mathfrak{n}^{(r)}$. It follows that $J\{[JZ, Z'] + [Z, JZ']\} \in [\mathfrak{g}, \mathfrak{g}]^\perp$ by Lemma 2.1. Thus we have $[JZ, Z'] + [Z, JZ'] = 0$ and $[JZ, JZ'] = 0$. □

By Lemmas 2.1 and 2.2, we have

Proposition 2.3. For $U, V, W \in \mathfrak{n}^{(r)}$,

$$\langle \text{ad}(JU)V, W \rangle + \omega_0(JV)\langle U, W \rangle = \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)\langle U, V \rangle.$$

Proof. Since $\Omega_0 = -\omega_0 \wedge \eta_0 + d\eta_0$ and ω_0 is closed 1-form, we see that

$$\begin{aligned} \langle \text{ad}(JU)V, W \rangle &= \Omega_0(J \text{ad}(JU)V, W) = -\Omega_0(\text{ad}(JU)V, JW) \\ (2.1) \quad &= (\omega_0 \wedge \eta_0 - d\eta_0)(\text{ad}(JU)V, JW) \\ &= -\omega_0(JW)\eta_0(\text{ad}(JU)V) - d\eta_0(\text{ad}(JU)V, JW). \end{aligned}$$

Since $\eta_0 = -\omega_0 \circ J$, we get

$$\begin{aligned}
 (2.2) \quad -\omega_0(JW)\eta_0(\text{ad}(JU)V) &= \omega_0(JW) d\eta_0(JU, V) \\
 &= \omega_0(JW)\{\Omega_0(JU, V) + \omega_0 \wedge \eta_0(JU, V)\} \\
 &= \omega_0(JW)\langle U, V \rangle - \omega_0(JW)\omega_0(JU)\omega_0(JW).
 \end{aligned}$$

From the derivation conditions and Lemma 2.2, we get

$$\begin{aligned}
 d\eta_0(\text{ad}(JU)V, JW) &= d\eta_0(JU, [V, JW]) - d\eta_0(V, [JU, JW]) \\
 &= d\eta_0(JU, [V, JW]) \\
 &= d\eta_0(JU, [W, JV]) \\
 &= d\eta_0([JU, W], JV) + d\eta_0(W, [JU, JV]) \\
 &= d\eta_0([JU, W], JV) = -d\eta_0(JV, [JU, W]).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 (2.3) \quad &-d\eta_0(\text{ad}(JU)V, JW) \\
 &= d\eta_0(JV, [JU, W]) \\
 &= \Omega_0(JV, [JU, W]) + \omega_0 \wedge \eta_0(JV, [JU, W]) \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV) d\eta_0(JU, W) \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV)\{\Omega_0(JU, W) + \omega_0 \wedge \eta_0(JU, W)\} \\
 &= \langle V, [JU, W] \rangle - \omega_0(JV)\langle U, W \rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW).
 \end{aligned}$$

From (2.1) and (2.2)), (2.3), we have

$$\begin{aligned}
 \langle \text{ad}(JU)V, W \rangle &= \omega_0(JW)\langle U, V \rangle - \omega_0(JW)\omega_0(JU)\omega_0(JV) \\
 &\quad + \langle V, [JU, W] \rangle - \omega_0(JV)\langle U, W \rangle + \omega_0(JV)\omega_0(JU)\omega_0(JW), \\
 \langle \text{ad}(JU)V, W \rangle + \omega_0(JV)\langle U, W \rangle &= \langle V, \text{ad}(JU)W \rangle + \omega_0(JW)\langle U, V \rangle. \quad \square
 \end{aligned}$$

It is well-known that if a solvable Lie group G admits a lattice Γ , then the solvable Lie algebra \mathfrak{g} of G is unimodular. We define

DEFINITION 2.4. A solvable Lie algebra \mathfrak{g} is called strongly unimodular if, for $X \in \mathfrak{g}$, $\text{tr ad}(X)|_{\mathfrak{n}^{(i)}} = 0$ for each i .

Benson–Gordon [4] proved that if a solvable Lie group G admits a lattice Γ , then the solvable Lie algebra \mathfrak{g} of G is strongly unimodular.

We take an orthonormal frame $\{Z_1, \dots, Z_m\}$ of $\mathfrak{n}^{(r)}$ and consider the strongly unimodular conditions of $\text{ad}(JZ_i)$ from $\mathfrak{n}^{(r)}$ to $\mathfrak{n}^{(r)}$ for each i .

For each i , let

$$\text{ad}(JZ_i)(Z_1, \dots, Z_m) = (Z_1, \dots, Z_m) \begin{pmatrix} a_{11}(i) & \cdots & a_{1m}(i) \\ \vdots & \vdots & \vdots \\ a_{m1}(i) & \cdots & a_{mm}(i) \end{pmatrix}.$$

By Proposition 2.3, we see that

$$\begin{aligned} \langle \text{ad}(JZ_i)Z_j, Z_k \rangle + \omega_0(JZ_j)\langle Z_i, Z_k \rangle &= \langle Z_j, \text{ad}(JZ_i)Z_k \rangle + \omega_0(JZ_k)\langle Z_i, Z_j \rangle, \\ a_{kj}(i) + \delta_{ik}\omega_0(JZ_j) &= a_{jk}(i) + \delta_{ij}\omega_0(JZ_k). \end{aligned}$$

It follows that, in the case of $j, k \neq i$,

$$a_{kj}(i) = a_{jk}(i)$$

and, in the case of $j = i, k \neq i$,

$$a_{ki}(i) = a_{ik}(i) + \omega_0(JZ_k).$$

Then we get

$$(2.4) \quad (a_{jk}(i)) = A_i + \begin{pmatrix} \text{line } i & & \\ \omega_0(JZ_1) & & \\ \vdots & & \\ \omega_0(JZ_m) & & \mathbf{0} \end{pmatrix},$$

where A_i is an $(m \times m)$ -symmetric matrix.

Moreover, put

$$A_i = \text{row } i \begin{pmatrix} & \text{line } i & \\ & a_1^i & \\ * & \vdots & * \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & \vdots & & & \\ * & a_m^i & & & * \end{pmatrix},$$

for each i . Since $[JZ_i, Z_j] = [JZ_j, Z_i]$, we get

$$a_{jj}(i) = \langle \text{ad}(JZ_i)Z_j, Z_j \rangle = \langle \text{ad}(JZ_j)Z_i, Z_j \rangle = a_i^j$$

for $j \neq i$. Then A_i can be written as

$$A_i = \text{row } i \begin{pmatrix} & \text{line } i & & & \\ a_i^1 & a_1^i & & * & \\ & \ddots & \vdots & & \\ a_1^i & \cdots & a_i^i & \cdots & a_m^i \\ & & \vdots & \ddots & \\ & * & a_m^i & & a_i^m \end{pmatrix}.$$

Thus (2.4) can be expressed as follows:

$$(2.5) \quad (a_{jk}(i)) = \text{row } i \begin{pmatrix} & \text{line } i & & & \\ a_i^1 & & a_1^i + \omega_0(JZ_1) & & \\ & * & & \vdots & * \\ & \ddots & & \vdots & \\ a_1^i * & \cdots & a_i^i + \omega_0(JZ_i) & \cdots & a_m^i \\ & & \vdots & \ddots & * \\ & * & a_m^i + \omega_0(JZ_m) & * & a_i^m \end{pmatrix}.$$

From the strongly unimodular condition of $\text{ad}(JZ_i)|_{\mathfrak{n}^\varphi}$, we have

Proposition 2.5. For each i , $(\sum_j a_i^j) + \omega_0(JZ_i) = 0$.

Then we have

Corollary 2.6. $\sum_i \omega_0(JZ_i)^2 = 1$.

Proof. For each i , we see that

$$\begin{aligned} 1 &= \langle Z_i, Z_i \rangle = \Omega_0(JZ_i, Z_i) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(JZ_i, Z_i) \\ &= -\omega_0(JZ_i)\eta_0(Z_i) - \eta_0([JZ_i, Z_i]) \\ &= \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \end{aligned}$$

because $\eta_0 = -\omega_0 \circ J$. From (2.5),

$$\begin{aligned}
 1 &= \langle Z_i, Z_i \rangle = \omega_0(JZ_i)^2 + \omega_0(J[JZ_i, Z_i]), \\
 (2.6) \quad 1 &= \omega_0(JZ_i)^2 + \omega_0 \circ J \left(\sum_j (a_j^i + \omega_0(JZ_j))Z_j \right), \\
 1 &= \omega_0(JZ_i)^2 + \sum_j a_j^i \omega_0(JZ_j) + \sum_j \omega_0(JZ_j)^2.
 \end{aligned}$$

Take the sum of (2.6) for i , we get

$$\begin{aligned}
 \sum_i 1 &= \sum_i \omega_0(JZ_i)^2 + \sum_{i,j} a_j^i \omega_0(JZ_j) + \sum_{i,j} \omega_0(JZ_j)^2, \\
 m &= \sum_i \omega_0(JZ_i)^2 + \sum_i \left(\sum_j a_j^i \right) \omega_0(JZ_i) + m \sum_i \omega_0(JZ_i)^2.
 \end{aligned}$$

By Proposition 2.5, we have

$$\begin{aligned}
 m &= \sum_i \omega_0(JZ_i)^2 + \sum_i (-\omega_0(JZ_i))\omega_0(JZ_i) + m \sum_i \omega_0(JZ_i)^2, \\
 m &= \sum_i \omega_0(JZ_i)^2 - \sum_i \omega_0(JZ_i)^2 + m \sum_i \omega_0(JZ_i)^2, \\
 m &= m \sum_i \omega_0(JZ_i)^2,
 \end{aligned}$$

which implies that $\sum_i \omega_0(JZ_i)^2 = 1$. □

Corollary 2.7. $A = \sum_i \omega_0(JZ_i)JZ_i \in J\mathfrak{n}^{(r)}$.

Proof. Since $A = \gamma(\omega_0)$, it can be given by

$$A = \sum_i \omega_0(JZ_i)JZ_i + B,$$

where $B \in (J\mathfrak{n}^{(r)})^\perp \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. From $\langle A, A \rangle = 1$, we have

$$1 = \langle A, A \rangle = \sum_i \omega_0(JZ_i)^2 \langle JZ_i, JZ_i \rangle + \langle B, B \rangle = \sum_i \omega_0(JZ_i)^2 + \langle B, B \rangle.$$

By Corollary 2.6, we have $\langle B, B \rangle = 0$, which implies that $B = 0$. □

3. Structure on the solvable Lie algebra

We use same notation introduced in Section 2. In this section, we prove that JA is in the center $Z(\mathfrak{g})$ of \mathfrak{g} and have Main Theorem.

We see $JA = -\sum_i \omega_0(JZ_i)Z_i \in \mathfrak{n}^{(r)}$ by Corollary 2.7. Put $Z_0 = JA$. Note that $\eta_0(Z_0) = -\omega_0 \circ J(JA) = 1$ and $\eta_0(A) = -\omega_0 \circ J(A) = 0$. We get

Lemma 3.1. *For $X \in \mathfrak{g}$, $d\eta_0(A, X) = 0$ and $d\eta_0(Z_0, X) = 0$.*

Proof. Since $\eta_0 = -\omega_0 \circ J$, we see that

$$\begin{aligned} d\eta_0(A, X) &= \Omega_0(A, X) + \omega_0 \wedge \eta_0(A, X) = \langle A, JX \rangle + \eta_0(X) = \omega_0(JX) + \eta_0(X) \\ &= 0. \end{aligned}$$

Similarly, we get

$$\begin{aligned} d\eta_0(Z_0, X) &= \Omega_0(Z_0, X) + \omega_0 \wedge \eta_0(Z_0, X) = -\langle X, JZ_0 \rangle - \omega_0(X) = \langle X, A \rangle - \omega_0(X) \\ &= 0. \end{aligned} \quad \square$$

Then we see that

Proposition 3.2. *For $U \in \mathfrak{n}^{(r)}$, $\text{ad}(A)U = 0$.*

Proof. By a straightforward computation, we see that

$$\begin{aligned} \langle \text{ad}(A)U, \text{ad}(A)U \rangle &= \Omega_0(J \circ \text{ad}(A)U, \text{ad}(A)U) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(A)U, \text{ad}(A)U) \\ &= -\omega_0(J \circ \text{ad}(A)U)\eta_0(\text{ad}(A)U) + d\eta_0(J \circ \text{ad}(A)U, \text{ad}(A)U). \end{aligned}$$

By Lemma 3.1 and the derivation conditions,

$$\begin{aligned} \langle \text{ad}(A)U, \text{ad}(A)U \rangle &= \omega_0(J \circ \text{ad}(A)U) d\eta_0(A, U) - d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U) \\ &= -d\eta_0(\text{ad}(A)U, J \circ \text{ad}(A)U) \\ &= -d\eta_0(A, [U, J \circ \text{ad}(A)U]) + d\eta_0(U, [A, J \circ \text{ad}(A)U]) \\ &= d\eta_0(U, [A, J \circ \text{ad}(A)U]). \end{aligned}$$

Now, since $JA = Z_0$ and $\text{ad}(A)U \in \mathfrak{n}^{(r)}$, we see that

$$[A, J \circ \text{ad}(A)U] = [-JZ_0, J \circ \text{ad}(A)U] = 0$$

by Lemma 2.2. It follows that $d\eta_0(U, [A, J \circ \text{ad}(A)U]) = 0$, which implies that $\text{ad}(A)U = 0$. □

Therefore we have

Theorem 3.3. $Z_0 \in Z(\mathfrak{g})$.

Proof. Let $X \in \mathfrak{g}$. By Lemma 3.1, we see that

$$\begin{aligned} \langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle &= \Omega_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\ &= (-\omega_0 \wedge \eta_0 + d\eta_0)(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\ &= -\omega_0(J \circ \text{ad}(X)Z_0)\eta_0(\text{ad}(X)Z_0) + d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\ &= \omega_0(J \circ \text{ad}(X)Z_0) d\eta_0(X, Z_0) + d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) \\ &= d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0). \end{aligned}$$

Moreover, from the derivation conditions and $Z_0 \in Z([\mathfrak{g}, \mathfrak{g}])$,

$$\begin{aligned} \langle \text{ad}(X)Z_0, \text{ad}(X)Z_0 \rangle &= d\eta_0(J \circ \text{ad}(X)Z_0, \text{ad}(X)Z_0) = -d\eta_0(\text{ad}(X)Z_0, J \circ \text{ad}(X)Z_0) \\ &= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]) + d\eta_0(Z_0, [X, J \circ \text{ad}(X)Z_0]) \\ &= -d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]). \end{aligned}$$

Now, since $\text{ad}(X)Z_0 \in \mathfrak{n}^{(r)}$, we see that

$$[Z_0, J \circ \text{ad}(X)Z_0] = [\text{ad}(X)Z_0, JZ_0] = [A, \text{ad}(X)Z_0] = 0$$

by Lemma 2.2 and Proposition 3.2. It follows that $d\eta_0(X, [Z_0, J \circ \text{ad}(X)Z_0]) = 0$, which implies that $\text{ad}(X)Z_0 = 0$ for $X \in \mathfrak{g}$. \square

From Theorem 3.3, we have

Corollary 3.4. $\text{ad}(A) \circ J = J \circ \text{ad}(A)$.

Proof. Let $X \in \mathfrak{g}$. Since J is integrable, we see that

$$\begin{aligned} 0 = \mathbf{N}_J(A, X) &= [A, X] + J[JA, X] + J[A, JX] - [JA, JX] \\ 0 &= [A, X] + J[Z_0, X] + J[A, JX] - [Z_0, JX]. \end{aligned}$$

From $Z_0 \in Z(\mathfrak{g})$, $[A, X] + J[A, JX] = 0$. Then we have our claim. \square

Corollary 3.5. *Lee form ω_0 is parallel, if and only if Ω_0 is $\text{ad}(A)$ -invariant.*

Proof. Let ∇ be the Riemannian connection of (\cdot, \cdot) . For $X, Y \in \mathfrak{g}$, we see that

$$\begin{aligned} 2(\nabla_X \omega_0)(Y) &= -2\omega_0(\nabla_X Y) = -2\langle A, \nabla_X Y \rangle \\ &= -\langle [A, X], Y \rangle - \langle X, [A, Y] \rangle - \langle A, [X, Y] \rangle \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]) - \omega_0([X, Y]) \\ &= -\Omega_0(J[A, X], Y) - \Omega_0(JX, [A, Y]), \end{aligned}$$

because ω_0 is a closed 1-form. From Corollary 3.4, we get

$$2(\nabla_X \omega_0)(Y) = -\Omega_0([A, JX], Y) - \Omega_0(JX, [A, Y]) = (\text{ad}^*(A)\Omega_0)(JX, Y).$$

Thus we have our claim. \square

Proof of Main Theorem. By Corollary 3.5, to prove Main Theorem, it is enough to show that Ω_0 is $\text{ad}(A)$ -invariant.

Let $X, Y \in \mathfrak{g}$. By a straightforward computation, we see that

$$\begin{aligned} (\text{ad}^*(A)\Omega_0)(X, Y) &= -\Omega_0([A, X], Y) - \Omega_0(X, [A, Y]) \\ &= (\omega_0 \wedge \eta_0 - d\eta_0)([A, X], Y) + (\omega_0 \wedge \eta_0 - d\eta_0)(X, [A, Y]) \\ &= -\omega_0(Y)\eta_0([A, X]) - d\eta_0([A, X], Y) \\ &\quad + \omega_0(X)\eta_0([A, X]) - d\eta_0(X, [A, Y]). \end{aligned}$$

By Lemma 3.1, we get $\eta_0([A, X]) = -d\eta_0(A, X) = 0$ and $\eta_0([A, Y]) = 0$. Moreover, from the derivation conditions,

$$d\eta_0([A, X], Y) + d\eta_0(X, [A, Y]) = d\eta_0(A, [X, Y]) = 0,$$

which implies that Ω_0 is $\text{ad}(A)$ -invariant. This completes the proof of Main Theorem. \square

4. Examples

In this section, we give examples of Main Theorem and consider locally conformal Kähler structures on 4-dimensional compact solvmanifolds. Note that a compact Kähler solvmanifold is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus ([2], [8]).

Hasegawa [7] classified a 4-dimensional compact solvmanifold and proved that any complex structure on such solvmanifold is induced from a left-invariant complex structure on Lie group. By this classification, we see that a 4-dimensional locally conformal Kähler solvmanifold is biholomorphic to Kodaira–Thurston manifold, Secondary Kodaira–Thurston manifold or Inoue surfaces.

We see that Kodaira–Thurston manifolds and secondary Kodaira–Thurston manifolds are examples of Main Theorem:

EXAMPLE 4.1 (Kodaira–Thurston manifold [5]). Let G be a 3-dimensional nilpotent Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ . Let \mathfrak{g} be the nilpotent Lie algebra corresponding to G . $\mathfrak{g} \times \mathbb{R}$ is given by

$$\mathfrak{g} \times \mathbb{R} = \text{span}\{X, Y, Z, A : [X, Y] = Z\}.$$

Let $\{x, y, z, \omega\}$ be the dual base of $\{X, Y, Z, A\}$:

$$dx = dy = d\omega = 0, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G \times S^1$ such that $\{X, Y, Z, A\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z, JX = Y$. Then $(\Gamma \backslash G \times S^1, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega}z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$ -exact. We easily see that Lee form ω is parallel.

EXAMPLE 4.2 (Secondary Kodaira–Thurston manifold (cf. [15])). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(x \sin t + y \cos t) & \frac{1}{2}(x \cos t - y \sin t) & z \\ 0 & \cos t & \sin t & x \\ 0 & -\sin t & \cos t & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

The Lie group G admits a lattice Γ :

$$\Gamma = \left\{ \begin{pmatrix} 1 & -\frac{1}{2}(u \sin 2\pi s + v \cos 2\pi s) & \frac{1}{2}(u \cos 2\pi s - v \sin 2\pi s) & w \\ 0 & \cos 2\pi s & \sin 2\pi s & u \\ 0 & -\sin 2\pi s & \cos 2\pi s & v \\ 0 & 0 & 0 & 1 \end{pmatrix} : s, u, v, w \in \mathbb{Z} \right\}.$$

Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y, Z : [A, X] = Y, [A, Y] = -X, [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge y, \quad dy = \omega \wedge x, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Z, JX = Y$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge z - x \wedge y = d_{-\omega}z = d_{-\omega}(-\omega \circ J).$$

Hence the fundamental 2-form Ω_0 is $-\omega$ -exact. We easily see that Lee form ω is parallel.

A locally conformal Kähler structure on Inoue surfaces are different from one on solvmanifolds as Main Theorem:

EXAMPLE 4.3 (Inoue surface S^0 [11], [14]). Let $B \in \text{SL}(3, \mathbb{Z})$ be a unimodular matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\beta \neq \bar{\beta}$, and eigenvectors $(a_1, a_2, a_3), (b_1, b_2, b_3)$ of α, β , respectively. Then we define a structure of the group on $\mathbb{H} \times \mathbb{C} = \{(x + \sqrt{-1}\alpha^t, z) : x, t \in \mathbb{R}, z \in \mathbb{C}\}$ as follows:

$$(x + \sqrt{-1}\alpha^t, z) \cdot (x' + \sqrt{-1}\alpha^{t'}, z') = (\alpha^t x' + x + \sqrt{-1}\alpha^{t+t'}, \beta^t z' + z).$$

It can be expressed by

$$G = \left\{ \left(\begin{array}{cccc} \alpha^t & 0 & 0 & x \\ 0 & \beta^t & 0 & z \\ 0 & 0 & \bar{\beta}^t & \bar{z} \\ 0 & 0 & 0 & 1 \end{array} \right) : t, x \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

Thus we easily see that the Lie group G is solvable and it admits a lattice Γ :

$$\Gamma = \left\{ \left(\begin{array}{cccc} \alpha^s & 0 & 0 & P \begin{pmatrix} u \\ w_1 \\ w_2 \end{pmatrix} \\ 0 & \beta^s & 0 & \\ 0 & 0 & \bar{\beta}^s & \\ 0 & 0 & 0 & 1 \end{array} \right) : s, u, w_1, w_2 \in \mathbb{Z} \right\},$$

where $P = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \end{pmatrix} \in \text{GL}(3, \mathbb{C})$. Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y_1, Y_2 : [A, X] = -2rX, [A, Y_1] = rY_1 + \theta Y_2, [A, Y_2] = rY_2 - \theta Y_1\},$$

where $\alpha = e^{-2r}$ and $\beta = e^{r+\sqrt{-1}\theta}$. Let $\{\omega, x, y_1, y_2\}$ be the dual base of $\{A, X, Y_1, Y_2\}$:

$$d\omega = 0, \quad dx = 2r\omega \wedge x, \quad dy_1 = -r\omega \wedge y_1 + \theta\omega \wedge y_2, \quad dy_2 = -r\omega \wedge y_2 - \theta\omega \wedge y_1.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y_1, Y_2\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = X, JY_1 = Y_2$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega = -\omega \wedge x - y_1 \wedge y_2.$$

Note that the fundamental 2-form Ω is not $-2r\omega$ -exact. We see that the Lee form ω is not parallel: $(\nabla_X \omega)X \neq 0$.

EXAMPLE 4.4 (Inoue surface S^+ [5], [11], [14]). Let G be a 4-dimensional solvable Lie group given by

$$G = \left\{ \left(\begin{array}{cccc} 1 & -\frac{1}{2}y & \frac{1}{2}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{array} \right) : t, x, y, z \in \mathbb{R} \right\}.$$

We can construct a lattice Γ on G (cf. [13]). Let \mathfrak{g} be the solvable Lie algebra corresponding to G :

$$\mathfrak{g} = \text{span}\{A, X, Y, Z : [A, X] = X, [A, Y] = -Y, [X, Y] = Z\}.$$

Let $\{\omega, x, y, z\}$ be the dual base of $\{A, X, Y, Z\}$:

$$d\omega = 0, \quad dx = -\omega \wedge x, \quad dy = \omega \wedge y, \quad dz = -x \wedge y.$$

We define a left-invariant metric $\langle \cdot, \cdot \rangle$ on $\Gamma \backslash G$ such that $\{A, X, Y, Z\}$ is an orthonormal frame and a left-invariant complex structure J by $JA = Y, JZ = X$. Then $(\Gamma \backslash G, \langle \cdot, \cdot \rangle, J)$ is a locally conformal Kähler manifold with the fundamental 2-form given by

$$\Omega_0 = -\omega \wedge y - z \wedge x.$$

Note that the fundamental 2-form Ω is not $-\omega$ -exact. We see that the Lee form ω is not parallel: $(\nabla_Y \omega)Y \neq 0$.

We mention that an Inoue surface S^- is not of the form $\Gamma \backslash G$, but it is a double covering space of an Inoue surface S^+ (cf. [10]). Then an Inoue surface S^- has a locally conformal Kähler structure.

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